EE364a Review Session 1

administrative info:

- office hours: tue 4-6pm, wed 4-8pm, packard 277
- review session: example problems and hw hints
- homeworks due thursdays by 5pm
- staff email: ee364a-win0708-staff@lists.stanford.edu

Combinations and hulls

$y = \theta_1 x_1 + \cdots + \theta_k x_k$ is a

- *linear combination* of x_1, . . . , x_k
- *affine combination* if $\sum_i \theta_i = 1$
- *convex combination* if $\sum_i \theta_i = 1$, $\theta_i \geq 0$
- *conic combination* if $\theta_i \geq 0$

(linear, affine, . . .) **hull** of $S = \{x_1, \ldots, x_k\}$ is a set of all (linear, affine, . . .) combinations from S

linear hull:	$\mathrm{span}(S)$
affine hull:	$\mathbf{aff}(S)$
convex hull:	$\mathbf{conv}(S)$
conic hull:	$\mathbf{cone}(S)$

example: a few simple relations:

$$\mathbf{conv}(S) \subseteq \mathbf{aff}(S) \subseteq \mathrm{span}(S), \quad \mathbf{conv}(S) \subseteq \mathbf{cone}(S) \subseteq \mathrm{span}(S).$$

example: $S = \{(1,0,0), (0,1,0), (0,0,1)\} \subseteq \mathbf{R}^3$

what is the linear hull? affine hull? convex hull? conic hull?

- **linear hull:** $\mathbf{R}^3$.
- **affine hull:** hyperplane passing through $(1,0,0), (0,1,0), (0,0,1)$.
- **convex hull:** triangle with vertices at $(1,0,0), (0,1,0), (0,0,1)$.
- **conic hull:** $\mathbf{R}^3_+$

Important rules

- **intersection**

$$S_\alpha \text{ is } \begin{pmatrix} \textbf{subspace} \\ \textbf{affine} \\ \textbf{convex} \\ \textbf{convex cone} \end{pmatrix} \text{ for } \alpha \in \mathcal{A} \Longrightarrow \bigcap_{\alpha \in \mathcal{A}} S_\alpha \text{ is } \begin{pmatrix} \textbf{subspace} \\ \textbf{affine} \\ \textbf{convex} \\ \textbf{convex cone} \end{pmatrix}$$

  example: a *polyhedron* is intersection of a finite number of halfspaces and hyperplanes.

- **functions that preserve convexity**
  examples: affine, perspective, and linear fractional functions.
  if C is convex, and f is an affine/perspective/linear fractional function, then $f(C)$ is convex and $f^{-1}(C)$ is convex.

Quantized measurements

consider the measurement setup,

$$y = 0.1\mathbf{floor}(10Ax)$$

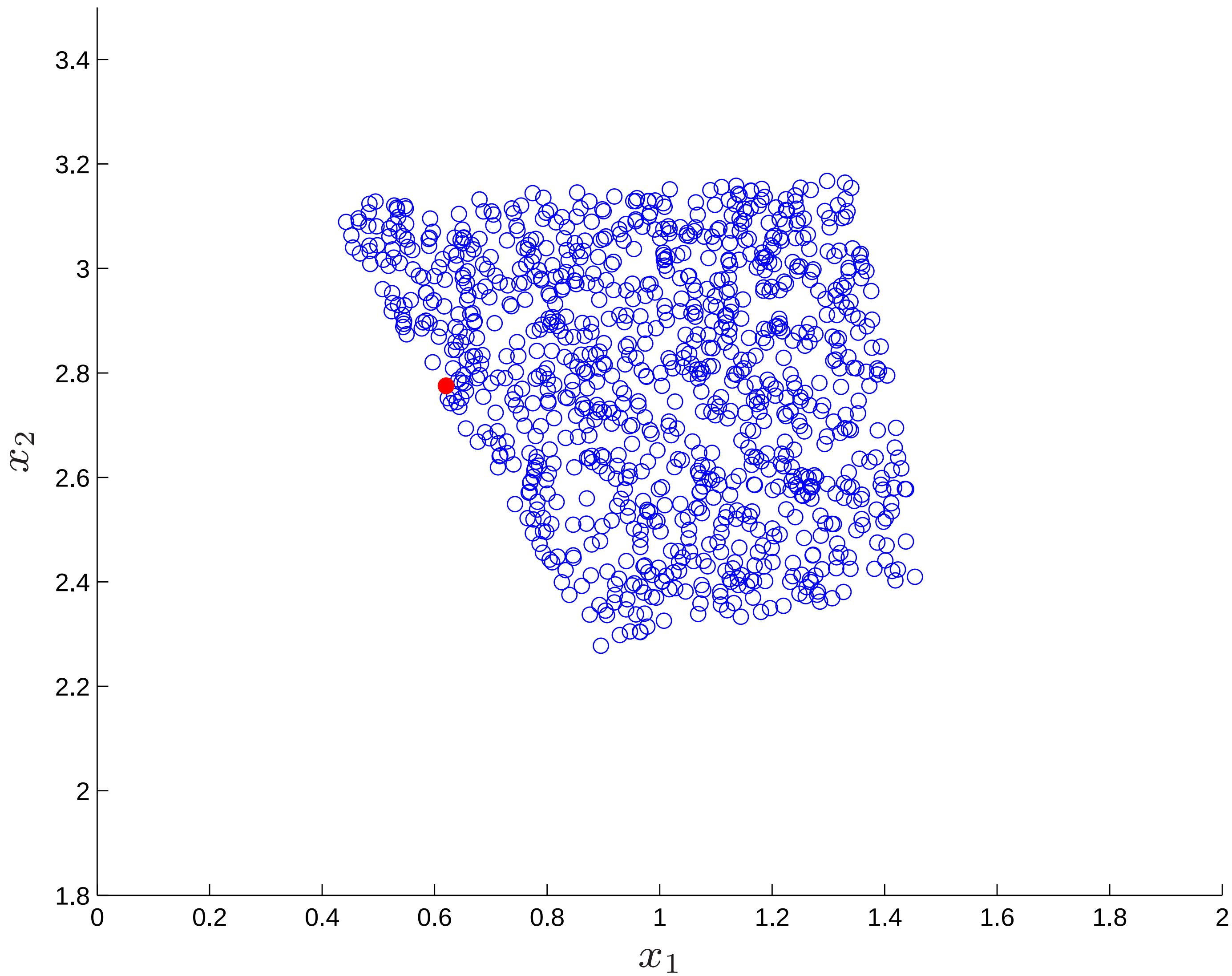
where $x \in \mathbf{R}^2$ is the input, $y \in \mathbf{R}^5$ are the measurements, and $A \in \mathbf{R}^{5\times 2}$.

- given a measurement y, we want to find the set of inputs that are consistent with the measurements. *i.e.*, the set

$$\mathcal{X} = \{x \mid 0 \leq a_i^T x - y_i \leq 0.1, i = 1, \ldots, 5\}.$$

we can explore this set by simulating, and plotting points that are inside the set. we randomly choose an $x \in \mathbf{R}^2$. if x is consistent with y, then we plot x. we repeat this a number of times. in the following plot, the blue circles represent points inside $\mathcal{X}$, and the red dot is the least squares solution, $x_{\rm ls} = A^\dagger y$.

Quantized measurements



from the simulations we suspect that $\mathcal{X}$ is a polyhedron. *i.e.*,

$$\mathcal{X} = \{x \mid Fx \leq g\}.$$

it is easy to show that,

$$F = \begin{bmatrix} -a_1^T \\ a_1^T \\ \vdots \\ -a_5^T \\ a_5^T \end{bmatrix}, \quad g = \begin{bmatrix} -y_1 \\ y_1 + 0.1 \\ \vdots \\ -y_5 \\ y_5 + 0.1 \end{bmatrix}.$$

Solution set of a quadratic inequality

let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbf{R}^n \mid x^T A x + b^T x + c \leq 0\},$$

with $A \in \mathbf{S}^n$, $b \in \mathbf{R}^n$, and $c \in \mathbf{R}$.

- show that C is convex if $A \succeq 0$.

we will show that the intersection of C with an arbitrary line $\{\hat{x} + tv \mid t \in \mathbf{R}\}$ is convex. we have,

$$(\hat{x} + tv)^T A (\hat{x} + tv) + b^T (\hat{x} + tv) + c = \alpha t^2 + \beta t + \gamma$$

where,

$$\alpha = v^T A v, \quad \beta = b^T v + 2\hat{x}^T A v, \quad \gamma = c + b^T \hat{x} + \hat{x}^T A \hat{x}.$$

the intersection of C with the line defined by $\hat{x}$ and v is the set

$$\{\hat{x} + tv \mid \alpha t^2 + \beta t + \gamma \leq 0\},$$

which is convex if $\alpha \geq 0$. This is true for any v if $A \succeq 0$.

Voronoi sets and polyhedral decomposition

let $x_0, \ldots, x_K \in \mathbf{R}^n$. consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, *i.e.*,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2, i = 1, \ldots, K\}.$$

- what kind of set is V?

answer. V is a polyhedron. we can express V as $V = \{x \mid Ax \preceq b\}$ with

$$A = 2 \begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ \vdots \\ x_K - x_0 \end{bmatrix}, \quad b = \begin{bmatrix} x_1^T x_1 - x_0^T x_0 \\ x_2^T x_2 - x_0^T x_0 \\ \vdots \\ x_K^T x_K - x_0^T x_0 \end{bmatrix}.$$

(check this!)

Conic hull of outer products

consider the set of rank-k *outer products*, defined as

$$\{XX^T \mid X \in \mathbf{R}^{n\times k},\ \ \mathbf{rank}\, X = k\}.$$

describe its conic hull in simple terms.

solution. we have $XX^T \succeq 0$ and $\mathbf{rank}(XX^T) = k$. a positive combination of such matrices can have rank up to n, but never less than k. indeed, let A and B be positive semidefinite matrices of rank k. suppose $v \in \mathcal{N}(A+B)$, then

$$(A+B)v = 0 \ \Leftrightarrow\ v^T(A+B)v = 0 \ \Leftrightarrow\ v^TAv + v^TBv = 0.$$

this implies,

$$v^TAv = 0 \ \Leftrightarrow\ Av = 0, \quad v^TBv = 0 \ \Leftrightarrow\ Bv = 0.$$

hence any vector in the $\mathcal{N}(A+B)$ must be in $\mathcal{N}(A)$, and $\mathcal{N}(B)$.

this implies that $\dim \mathcal{N}(A+B)$ cannot be greater than $\dim \mathcal{N}(A)$ or $\dim \mathcal{N}(B)$, hence a positive combination of positive semidefinite matrices can only gain rank.

it follows that the conic hull of the set of rank-k outer products is the set of positive semidefinite matrices of rank greater than or equal to k, along with the zero matrix.