EE364a Homework 7 additional problems

Suggestions for exercise 9.30

We recommend the following to generate a problem instance:

```
n = 100;
m = 200;
randn('state',1);
A=randn(m,n);
```

Of course, you should try out your code with different dimensions, and different data as well.

In all cases, be sure that your line search *first* finds a step length for which the tentative point is in $\mathbf{dom}\, f$; if you attempt to evaluate f outside its domain, you'll get complex numbers, and you'll never recover.

To find expressions for $\nabla f(x)$ and $\nabla^2 f(x)$, use the chain rule (see Appendix A.4); if you attempt to compute $\partial^2 f(x)/\partial x_i \partial x_j$, you will be sorry.

To compute the Newton step, you can use `vnt=-H\g`.

Suggestions for exercise 9.31

For 9.31a, you should try out $N = 1$, $N = 15$, and $N = 30$. You might as well compute and store the Cholesky factorization of the Hessian, and then back solve to get the search directions, even though you won't really see any speedup in Matlab for such a small problem. After you evaluate the Hessian, you can find the Cholesky factorization as `L=chol(H,'lower')`. You can then compute a search step as `-L'\(L\g)`, where `g` is the gradient at the current point. Matlab will do the right thing, *i.e.*, it will first solve `L\g` using forward subsitution, and then it will solve `-L'\(L\g)` using backward substitution. Each substitution is order n^2.

To fairly compare the convergence of the three methods (*i.e.*, $N = 1$, $N = 15$, $N = 30$), the horizontal axis should show the approximate total number of flops required, and not the number of iterations. You can compute the approximate number of flops using $n^3/3$ for each factorization, and $2n^2$ for each solve (where each 'solve' involves a forward subsitition step and a backward subsitution step).

Additional exercises

1. *Three-way linear classification.* We are given data

$$x^{(1)}, \ldots, x^{(N)}, \quad y^{(1)}, \ldots, y^{(M)}, \quad z^{(1)}, \ldots, z^{(P)},$$

three nonempty sets of vectors in $\mathbf{R}^n$. We wish to find three affine functions on $\mathbf{R}^n$,

$$f_i(z) = a_i^T z - b_i, \quad i = 1, 2, 3,$$

that satisfy the following properties:

$$\begin{array}{ll} f_1(x^{(j)}) > \max\{f_2(x^{(j)}), f_3(x^{(j)})\}, & j = 1, \ldots, N, \\ f_2(y^{(j)}) > \max\{f_1(y^{(j)}), f_3(y^{(j)})\}, & j = 1, \ldots, M, \\ f_3(z^{(j)}) > \max\{f_1(z^{(j)}), f_2(z^{(j)})\}, & j = 1, \ldots, P. \end{array}$$

In words: f_1 is the largest of the three functions on the x data points, f_2 is the largest of the three functions on the y data points, f_3 is the largest of the three functions on the z data points. We can give a simple geometric interpretation: The functions f_1, f_2, and f_3 partition $\mathbf{R}^n$ into three regions,

$$\begin{array}{rcl} R_1 & = & \{z \mid f_1(z) > \max\{f_2(z), f_3(z)\}\}, \\ R_2 & = & \{z \mid f_2(z) > \max\{f_1(z), f_3(z)\}\}, \\ R_3 & = & \{z \mid f_3(z) > \max\{f_1(z), f_2(z)\}\}, \end{array}$$

defined by where each function is the largest of the three. Our goal is to find functions with $x^{(j)} \in R_1$, $y^{(j)} \in R_2$, and $z^{(j)} \in R_3$.

Pose this as a convex optimization problem. You may not use strict inequalities in your formulation.

Solve the specific instance of the 3-way separation problem given in `sep3way_data.m`, with the columns of the matrices `X`, `Y` and `Z` giving the $x^{(j)}$, $j = 1, \ldots, N$, $y^{(j)}$, $j = 1, \ldots, M$ and $z^{(j)}$, $j = 1, \ldots, P$. To save you the trouble of plotting data points and separation boundaries, we have included the plotting code in `sep3way_data.m`. (Note that `a1`, `a2`, `a3`, `b1` and `b2` contain arbitrary numbers; you should compute the correct values using `cvx`.)

2. *Efficient numerical method for a regularized least-squares problem.* We consider a regularized least squares problem with smoothing,

$$\text{minimize} \quad \sum_{i=1}^{k} (a_i^T x - b_i)^2 + \delta \sum_{i=1}^{n-1} (x_i - x_{i+1})^2 + \eta \sum_{i=1}^{n} x_i^2,$$

where $x \in \mathbf{R}^n$ is the variable, and $\delta, \eta > 0$ are parameters.

(a) Express the optimality conditions for this problem as a set of linear equations involving x. (These are called the normal equations.)

(b) Now assume that $k \ll n$. Describe an efficient method to solve the normal equations found in (2a). Give an approximate flop count for a general method that does not exploit structure, and also for your efficient method.

(c) *A numerical instance.* In this part you will try out your efficient method. We'll choose $k = 100$ and $n = 2000$, and $\delta = \eta = 1$. First, randomly generate A and b with these dimensions. Form the normal equations as in (2a), and solve them using a generic method. Next, write (short) code implementing your efficient method, and run it on your problem instance. Verify that the solutions found by the two methods are nearly the same, and also that your efficient method is much faster than the generic one.

Note: You'll need to know some things about Matlab to be sure you get the speedup from the efficient method. Your method should involve solving linear equations with tridiagonal coefficient matrix. In this case, both the factorization and the back substitution can be carried out very efficiently. The Matlab documentation says that banded matrices are recognized and exploited, when solving equations, but we found this wasn't always the case. To be sure Matlab knows your matrix is tridiagonal, you can declare the matrix as sparse, using `spdiags`, which can be used to create a tridiagonal matrix. You could also create the tridiagonal matrix conventionally, and then convert the resulting matrix to a sparse one using `sparse`.

One other thing you need to know. Suppose you need to solve a group of linear equations with the same coefficient matrix, *i.e.*, you need to compute $F^{-1}a_1, ..., F^{-1}a_m$, where F is invertible and a_i are column vectors. By concatenating columns, this can be expressed as a single matrix

$$\left[F^{-1}a_1 \ \cdots \ F^{-1}a_m\right] = F^{-1}\left[a_1 \ \cdots \ a_m\right].$$

To compute this matrix using Matlab, you should collect the righthand sides into one matrix (as above) and use Matlab's backslash operator: `F\A`. This will do the right thing: factor the matrix F once, and carry out multiple back substitutions for the righthand sides.