EE364a Homework 4 solutions

4.11 *Problems involving ℓ_1- and ℓ_∞-norms.* Formulate the following problems as LPs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP.

(a) Minimize $\|Ax - b\|_\infty$ (ℓ_∞-norm approximation).

(b) Minimize $\|Ax - b\|_1$ (ℓ_1-norm approximation).

(c) Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \le 1$.

(d) Minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \le 1$.

(e) Minimize $\|Ax - b\|_1 + \|x\|_\infty$.

In each problem, $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$ are given. (See §6.1 for more problems involving approximation and constrained approximation.)

Solution.

(a) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & Ax - b \succeq -t\mathbf{1}. \end{array}$$

in the variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$. To see the equivalence, assume x is fixed in this problem, and we optimize only over t. The constraints say that

$$-t \le a_k^T x - b_k \le t$$

for each k, *i.e.*, $t \ge |a_k^T x - b_k|$, *i.e.*,

$$t \ge \max_k |a_k^T x - b_k| = \|Ax - b\|_\infty.$$

Clearly, if x is fixed, the optimal value of the LP is $p^\star(x) = \|Ax - b\|_\infty$. Therefore optimizing over t and x simultaneously is equivalent to the original problem.

(b) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & Ax - b \preceq s \\ & Ax - b \succeq -s \end{array}$$

with variables $x \in \mathbf{R}^n$ and$s \in \mathbf{R}^m$. Assume x is fixed in this problem, and we optimize only over s. The constraints say that

$$-s_k \le a_k^T x - b_k \le s_k$$

for each k, *i.e.*, $s_k \geq |a_k^T x - b_k|$. The objective function of the LP is separable, so we achieve the optimum over s by choosing

$$s_k = |a_k^T x - b_k|,$$

and obtain the optimal value $p^\star(x) = \|Ax - b\|_1$. Therefore optimizing over x and s simultaneously is equivalent to the original problem.

(c) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -\mathbf{1} \preceq x \preceq \mathbf{1}, \end{array}$$

with variables $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

(d) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \preceq x \preceq y \\ & -\mathbf{1} \preceq Ax - b \preceq \mathbf{1} \end{array}$$

with variables $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$.

Another reformulation is to write x as the difference of two nonnegative vectors $x = x^+ - x^-$, and to express the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x^+ + \mathbf{1}^T x^- \\ \text{subject to} & -\mathbf{1} \preceq Ax^+ - Ax^- - b \preceq \mathbf{1} \\ & x^+ \succeq 0, \quad x^- \succeq 0, \end{array}$$

with variables $x^+ \in \mathbf{R}^n$ and $x^- \in \mathbf{R}^n$.

(e) Equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y + t \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -t\mathbf{1} \preceq x \preceq t\mathbf{1}, \end{array}$$

with variables $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, and $t \in \mathbf{R}$.

4.16 *Minimum fuel optimal control.* We consider a linear dynamical system with state $x(t) \in \mathbf{R}^n$, $t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbf{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1,$$

where $A \in \mathbf{R}^{n \times n}$ and $b \in \mathbf{R}^n$ are given. We assume that the initial state is zero, *i.e.*, $x(0) = 0$.

The *minimum fuel optimal control problem* is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t)),$$

subject to the constraint that $x(N) = x_{\rm des}$, where N is the (given) time horizon, and $x_{\rm des} \in \mathbf{R}^n$ is the (given) desired final or target state. The function $f : \mathbf{R} \to \mathbf{R}$ is the *fuel use map* for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1. \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half.

Formulate the minimum fuel optimal control problem as an LP.

Solution. The minimum fuel optimal control problem is equivalent to the LP

$$\begin{array}{ll} \mbox{minimize} & \mathbf{1}^T t \\ \mbox{subject to} & Hu = x_{\rm des} \\ & -y \preceq u \preceq y \\ & t \succeq y \\ & t \succeq 2y - \mathbf{1}, \end{array}$$

with variables $u \in \mathbf{R}^N$, $y \in \mathbf{R}^N$, and $t \in \mathbf{R}$, where

$$H = \left[\begin{array}{ccccc} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{array} \right].$$

There are several other possible LP formulations. For example, we can keep the state trajectory $x(0), \ldots, x(N)$ as optimization variables, and replace the equality constraint above, $Hu = x_{\rm des}$, with the equality constraints

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1, \qquad x(0) = 0, \qquad x(N) = x_{\rm des}.$$

In this formulation, the variables are $u \in \mathbf{R}^N$, $x(0), \ldots, x(N) \in \mathbf{R}^n$, as well as $y \in \mathbf{R}^N$ and $t \in \mathbf{R}^N$.

Yet another variation is to not use the intermediate variable y introduced above, and express the problem just in terms of the variable t (and u):

$$-t \preceq u \preceq t, \quad 2u - \mathbf{1} \preceq t, \quad -2u - \mathbf{1} \preceq t,$$

with variables $u \in \mathbf{R}^N$ and $t \in \mathbf{R}^N$.

4.29 *Maximizing probability of satisfying a linear inequality.* Let c be a random variable in $\mathbf{R}^n$, normally distributed with mean $\bar{c}$ and covariance matrix R. Consider the problem

$$\begin{array}{ll} \text{maximize} & \mathbf{prob}(c^T x \geq \alpha) \\ \text{subject to} & Fx \preceq g, \quad Ax = b. \end{array}$$

Find the conditions under which this is equivalent to a convex or quasiconvex optimization problem. When these conditions hold, formulate the problem as a QP, QCQP, or SOCP (if the problem is convex), or explain how you can solve it by solving a sequence of QP, QCQP, or SOCP feasibility problems (if the problem is quasiconvex).

Solution. Define $u = c^T x$, a scalar random variable, normally distributed with mean $\mathbf{E}\, u = \bar{c}^T x$ and variance $\mathbf{E}(u - \mathbf{E}\, u)^2 = x^T R x$. The random variable

$$\frac{u - \bar{c}^T x}{\sqrt{x^T R x}}$$

has a normal distribution with mean zero and unit variance, so

$$\mathbf{prob}(u \geq \alpha) = \mathbf{prob}\left(\frac{u - \bar{c}^T x}{\sqrt{x^T R x}} \geq \frac{\alpha - \bar{c}^T x}{\sqrt{x^T R x}} \right) = 1 - \Phi\left(\frac{\alpha - \bar{c}^T x}{\sqrt{x^T R x}} \right),$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$ is the standard normal CDF.

To maximize $\mathbf{prob}(u \geq \alpha)$, we can minimize $(\alpha - \bar{c}^T x)/\sqrt{x^T R x}$ (since Φ is increasing), *i.e.*, solve the problem

$$\begin{array}{ll} \text{maximize} & (\bar{c}^T x - \alpha)/\sqrt{x^T R x} \\ \text{subject to} & Fx \preceq g \\ & Ax = b. \end{array} \qquad (1)$$

This is not a convex optimization problem, since the objective is not concave.

The problem can, however, be solved by quasiconvex optimization provided a condtion holds. (We'll derive the condition below.) The objective exceeds a value t if and only if

$$\bar{c}^T x - \alpha \geq t \sqrt{x^T R x}$$

holds. This last inequality is convex, in fact a second-order cone constraint, provided $t \geq 0$. So now we can state the condition: There exists a feasible x for which $\bar{c}^T x \geq \alpha$. (This condition is easily checked as an LP feasibility problem.) This condition, by the way, can also be stated as: There exists a feasible x for which $\mathbf{prob}(u \geq \alpha) \geq 1/2$. Assume that this condition holds. This means that the optimal value of our original problem is at least 0.5, and the optimal value of the problem (1) is at least 0. This means that we can state our problem as

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & Fx \preceq g, \qquad Ax = b \\ & \bar{c}^T x - \alpha \geq t \sqrt{x^T R x}, \end{array}$$

where we can assume that $t \geq 0$. This can be solved by bisection on t, by solving an SOCP feasibility problem at each step. In other words: the function $(\bar{c}^T x - \alpha)/\sqrt{x^T R x}$ is quasiconcave, provided it is nonnegative.

In fact, provided the condition above holds (*i.e.*, there exists a feasible x with $\bar{c}^T x \geq \alpha$) we can solve the problem (1) via convex optimization. We make the change of variables

$$y = \frac{x}{\bar{c}^T x - \alpha}, \qquad s = \frac{1}{\bar{c}^T x - \alpha},$$

so $x = y/s$. This yields the problem

$$\begin{array}{ll} \text{minimize} & \sqrt{y^T R y} \\ \text{subject to} & F y \preceq g s \\ & A y = b s \\ & \bar{c}^T y - \alpha s = 1 \\ & s \geq 0. \end{array}$$

4.30 A heated fluid at temperature T (degrees above ambient temperature) flows in a pipe with fixed length and circular cross section with radius r. A layer of insulation, with thickness $w \ll r$, surrounds the pipe to reduce heat loss through the pipe walls. The design variables in this problem are T, r, and w.

The heat loss is (approximately) proportional to Tr/w, so over a fixed lifetime, the energy cost due to heat loss is given by $\alpha_1 Tr/w$. The cost of the pipe, which has a fixed wall thickness, is approximately proportional to the total material, *i.e.*, it is given by $\alpha_2 r$. The cost of the insulation is also approximately proportional to the total insulation material, *i.e.*, $\alpha_3 rw$ (using $w \ll r$). The total cost is the sum of these three costs.

The heat flow down the pipe is entirely due to the flow of the fluid, which has a fixed velocity, *i.e.*, it is given by $\alpha_4 Tr^2$. The constants α_i are all positive, as are the variables T, r, and w.

Now the problem: maximize the total heat flow down the pipe, subject to an upper limit $C_{\max}$ on total cost, and the constraints

$$T_{\min} \leq T \leq T_{\max}, \qquad r_{\min} \leq r \leq r_{\max}, \qquad w_{\min} \leq w \leq w_{\max}, \quad w \leq 0.1r.$$

Express this problem as a geometric program.

Solution. The problem is

$$\begin{array}{ll} \text{maximize} & \alpha_4 T r^2 \\ \text{subject to} & \alpha_1 T w^{-1} + \alpha_2 r + \alpha_3 r w \leq C_{\max} \\ & T_{\min} \leq T \leq T_{\max} \\ & r_{\min} \leq r \leq r_{\max} \\ & w_{\min} \leq w \leq w_{\max} \\ & w \leq 0.1 r. \end{array}$$

This is equivalent to the GP

$$
\begin{array}{ll}
\text{minimize} & (1/\alpha_4)T^{-1}r^{-2} \\
\text{subject to} & (\alpha_1/C_{\max})Tw^{-1} + (\alpha_2/C_{\max})r + (\alpha_3/C_{\max})rw \le 1 \\
& (1/T_{\max})T \le 1, \quad T_{\min}T^{-1} \le 1 \\
& (1/r_{\max})r \le 1, \quad r_{\min}r^{-1} \le 1 \\
& (1/w_{\max})w \le 1, \quad w_{\min}w^{-1} \le 1 \\
& 10wr^{-1} \le 1
\end{array}
$$

(with variables T, r, w).

5.1 *A simple example.* Consider the optimization problem

$$
\begin{array}{ll}
\text{minimize} & x^2 + 1 \\
\text{subject to} & (x-2)(x-4) \le 0,
\end{array}
$$

with variable $x \in \mathbf{R}$.

(a) *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.

(b) *Lagrangian and dual function.* Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^\star \ge \inf_x L(x, \lambda)$ for $\lambda \ge 0$). Derive and sketch the Lagrange dual function g.

(c) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution $\lambda^\star$. Does strong duality hold?

(d) *Sensitivity analysis.* Let $p^\star(u)$ denote the optimal value of the problem

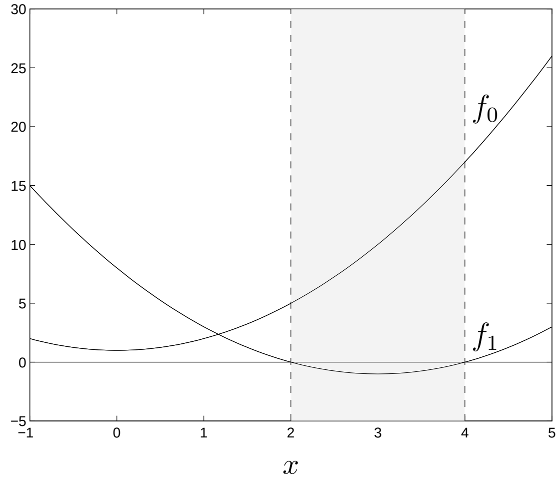
$$
\begin{array}{ll}
\text{minimize} & x^2 + 1 \\
\text{subject to} & (x-2)(x-4) \le u,
\end{array}
$$

as a function of the parameter u. Plot $p^\star(u)$. Verify that $dp^\star(0)/du = -\lambda^\star$.

Solution.

(a) The feasible set is the interval $[2, 4]$. The (unique) optimal point is $x^\star = 2$, and the optimal value is $p^\star = 5$.

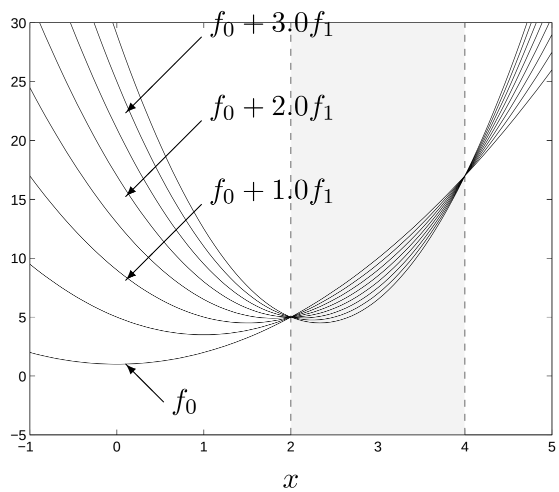
The plot shows f_0 and f_1.



(b) The Lagrangian is

$$L(x, \lambda) = (1 + \lambda)x^2 - 6\lambda x + (1 + 8\lambda).$$

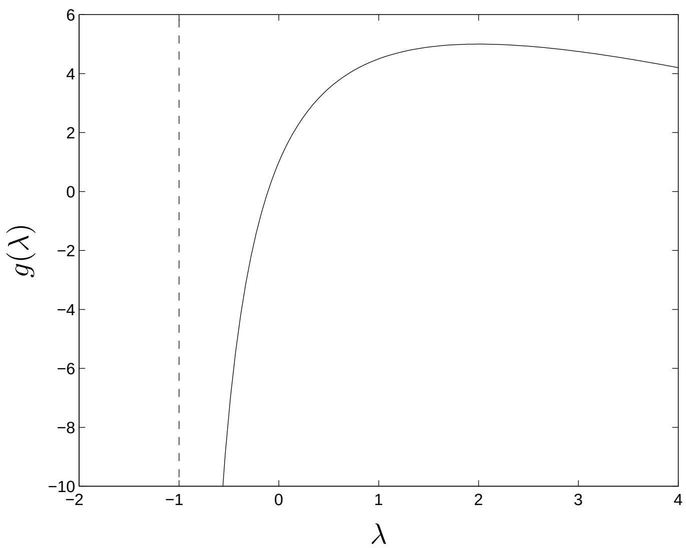
The plot shows the Lagrangian $L(x, \lambda) = f_0 + \lambda f_1$ as a function of x for different values of $\lambda \geq 0$. Note that the minimum value of $L(x, \lambda)$ over x (*i.e.*, $g(\lambda)$) is always less than $p^\star$. It increases as λ varies from 0 toward 2, reaches its maximum at $\lambda = 2$, and then decreases again as λ increases above 2. We have equality $p^\star = g(\lambda)$ for $\lambda = 2$.



For $\lambda > -1$, the Lagrangian reaches its minimum at $\tilde{x} = 3\lambda/(1 + \lambda)$. For $\lambda \leq -1$ it is unbounded below. Thus

$$g(\lambda) = \begin{cases} -9\lambda^2/(1 + \lambda) + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}$$

which is plotted below.



We can verify that the dual function is concave, that its value is equal to $p^\star = 5$ for $\lambda = 2$, and less than $p^\star$ for other values of λ.

(c) The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & -9\lambda^2/(1+\lambda) + 1 + 8\lambda \\ \text{subject to} & \lambda \geq 0. \end{array}$$

The dual optimum occurs at $\lambda = 2$, with $d^\star = 5$. So for this example we can directly observe that strong duality holds (as it must — Slater's constraint qualification is satisfied).

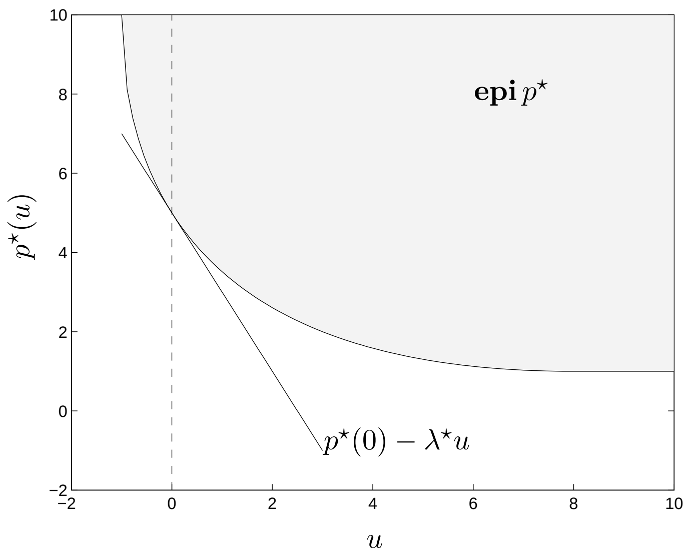
(d) The perturbed problem is infeasible for $u < -1$, since $\inf_x(x^2 - 6x + 8) = -1$. For $u \geq -1$, the feasible set is the interval

$$[3 - \sqrt{1+u}, 3 + \sqrt{1+u}],$$

given by the two roots of $x^2 - 6x + 8 = u$. For $-1 \leq u \leq 8$ the optimum is $x^\star(u) = 3 - \sqrt{1+u}$. For $u \geq 8$, the optimum is the unconstrained minimum of f_0, *i.e.*, $x^\star(u) = 0$. In summary,

$$p^\star(u) = \begin{cases} \infty & u < -1 \\ 11 + u - 6\sqrt{1+u} & -1 \leq u \leq 8 \\ 1 & u \geq 8. \end{cases}$$

The figure shows the optimal value function $p^\star(u)$ and its epigraph.



Finally, we note that $p^\star(u)$ is a differentiable function of u, and that

$$\frac{dp^\star(0)}{du} = -2 = -\lambda^\star.$$

Solutions to additional exercises

1. *Minimizing a function over the probability simplex.* Find simple necessary and sufficient conditions for $x \in \mathbf{R}^n$ to minimize a differentiable convex function f over the probability simplex, $\{x \mid \mathbf{1}^T x = 1,\ x \succeq 0\}$.

   Solution. The simple basic optimality condition is that x is feasible, *i.e.*, $x \succeq 0$, $\mathbf{1}^T x = 1$, and that $\nabla f(x)^T (y - x) \geq 0$ for all feasible y. We'll first show this is equivalent to
   $$\min_{i=1,\ldots,n} \nabla f(x)_i \geq \nabla f(x)^T x.$$
   To see this, suppose that $\nabla f(x)^T (y - x) \geq 0$ for all feasible y. Then in particular, for $y = e_i$, we have $\nabla f(x)_i \geq \nabla f(x)^T x$, which is what we have above. To show the other way, suppose that $\nabla f(x)_i \geq \nabla f(x)^T x$ holds, for $i = 1, \ldots, n$. Let y be feasible, *i.e.*, $y \succeq 0$, $\mathbf{1}^T y = 1$. Then multiplying $\nabla f(x)_i \geq \nabla f(x)^T x$ by y_i and summing, we get
   $$\sum_{i=1}^n y_i \nabla f(x)_i \geq \left(\sum_{i=1}^n y_i \right) \nabla f(x)^T x = \nabla f(x)^T x.$$
   The lefthand side is $y^T \nabla f(x)$, so we have $\nabla f(x)^T (y - x) \geq 0$.

   Now we can simplify even further. The condition above can be written as
   $$\min_{i=1,\ldots,n} \frac{\partial f}{\partial x_i} \geq \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}.$$
   But since $\mathbf{1}^T x = 1$, $x \succeq 0$, we have
   $$\min_{i=1,\ldots,n} \frac{\partial f}{\partial x_i} \leq \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i},$$
   and it follows that
   $$\min_{i=1,\ldots,n} \frac{\partial f}{\partial x_i} = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}.$$
   The right hand side is a mixture of $\partial f / \partial x_i$ terms and equals the minimum of all of the terms. This is possible only if $x_k = 0$ whenever $\partial f / \partial x_k > \min_i \partial f / \partial x_i$.

   Thus we can write the (necessary and sufficient) optimality condition as $\mathbf{1}^T x = 1$, $x \succeq 0$, and, for each k,
   $$x_k > 0 \ \Rightarrow\ \frac{\partial f}{\partial x_k} = \min_{i=1,\ldots,n} \frac{\partial f}{\partial x_i}.$$
   In particular, for k's with $x_k > 0$, $\partial f / \partial x_k$ are all equal.

2. *Complex least-norm problem.* We consider the complex least ℓ_p-norm problem
   $$\begin{array}{ll} \text{minimize} & \|x\|_p \\ \text{subject to} & Ax = b, \end{array}$$

where $A \in \mathbf{C}^{m\times n}$, $b \in \mathbf{C}^m$, and the variable is $x \in \mathbf{C}^n$. Here $\|\cdot\|_p$ denotes the ℓ_p-norm on $\mathbf{C}^n$, defined as

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

for $p \geq 1$, and $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$. We assume A is full rank, and $m < n$.

(a) Formulate the complex least ℓ_2-norm problem as a least ℓ_2-norm problem with real problem data and variable. *Hint.* Use $z = (\Re x, \Im x) \in \mathbf{R}^{2n}$ as the variable.

(b) Formulate the complex least ℓ_∞-norm problem as an SOCP.

(c) Solve a random instance of both problems with $m = 30$ and $n = 100$. To generate the matrix A, you can use the Matlab command `A = randn(m,n) + i*randn(m,n)`. Similarly, use `b = randn(m,1) + i*randn(m,1)` to generate the vector b. Use the Matlab command `scatter` to plot the optimal solutions of the two problems on the complex plane, and comment (briefly) on what you observe. You can solve the problems using the `cvx` functions `norm(x,2)` and `norm(x,inf)`, which are overloaded to handle complex arguments. To utilize this feature, you will need to declare variables to be `complex` in the `variable` statement. (In particular, you do not have to manually form or solve the SOCP from part (b).)

Solution.

(a) Define $z = (\Re x, \Im x) \in \mathbf{R}^{2n}$, so $\|x\|_2^2 = \|z\|_2^2$. The complex linear equations $Ax = b$ is the same as $\Re(Ax) = \Re b$, $\Im(Ax) = \Im b$, which in turn can be expressed as the set of linear equations

$$\begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} z = \begin{bmatrix} \Re b \\ \Im b \end{bmatrix}.$$

Thus, the complex least ℓ_2-norm problem can be expressed as

$$\begin{array}{ll} \text{minimize} & \|z\|_2 \\ \text{subject to} & \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} z = \begin{bmatrix} \Re b \\ \Im b \end{bmatrix}. \end{array}$$

(This is readily solved analytically).

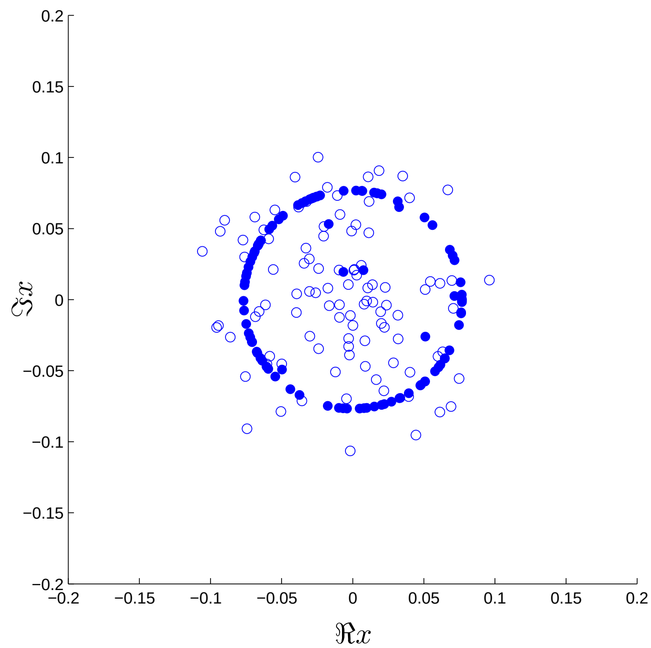
(b) Using epigraph formulation, with new variable t, we write the problem as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left\| \begin{bmatrix} z_i \\ z_{n+i} \end{bmatrix} \right\|_2 \leq t, \quad i = 1, \ldots, n \\ & \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} z = \begin{bmatrix} \Re b \\ \Im b \end{bmatrix}. \end{array}$$

This is an SOCP with n second-order cone constraints (in $\mathbf{R}^3$).

(c)
```
% complex minimum norm problem
%
randn('state',0);
m = 30; n = 100;
% generate matrix A
Are = randn(m,n); Aim = randn(m,n);
bre = randn(m,1); bim = randn(m,1);

A = Are + i*Aim;
b = bre + i*bim;

% 2-norm problem (analytical solution)
Atot = [Are -Aim; Aim Are];
btot = [bre; bim];
z_2 = Atot'*inv(Atot*Atot')*btot;
x_2 = z_2(1:100) + i*z_2(101:200);

% 2-norm problem solution with cvx
cvx_begin
  variable x(n) complex
  minimize( norm(x) )
  subject to
    A*x == b;
cvx_end

% inf-norm problem solution with cvx
cvx_begin
  variable xinf(n) complex
  minimize( norm(xinf,Inf) )
  subject to
    A*xinf == b;
cvx_end

% scatter plot
figure(1)
scatter(real(x),imag(x)), hold on,
scatter(real(xinf),imag(xinf),[],'filled'), hold off,
axis([-0.2 0.2 -0.2 0.2]), axis square,
xlabel('Re x'); ylabel('Im x');
```

The plot of the components of optimal $p = 2$ (empty circles) and $p = \infty$ (filled circles) solutions is presented below. The optimal $p = \infty$ solution minimizes the objective $\max_{i=1,\ldots,n} |x_i|$ subject to $Ax = b$, and the scatter plot of x_i shows that almost all of them are concentrated around a circle in the complex plane. This should be expected since we are minimizing the maximum magnitude of x_i, and thus almost all of x_i's should have about an equal magnitude $|x_i|$.



3. *Numerical perturbation analysis example.* Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ \text{subject to} & x_1 + 2x_2 \le u_1 \\ & x_1 - 4x_2 \le u_2, \\ & 5x_1 + 76x_2 \le 1, \end{array}$$

with variables x_1, x_2, and parameters u_1, u_2.

(a) Solve this QP, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values $x_1^\star$ and $x_2^\star$, and optimal dual variable values $\lambda_1^\star$, $\lambda_2^\star$ and $\lambda_3^\star$. Let $p^\star$ denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Hint: See §3.6 of the `CVX` users' guide to find out how to retrieve optimal dual variables. To specify the quadratic objective, use `quad_form()`.

(b) We will now solve some perturbed versions of the QP, with

$$u_1 = -2 + \delta_1, \qquad u_2 = -3 + \delta_2,$$

where δ_1 and δ_2 each take values from $\{-0.1, 0, 0.1\}$. (There are a total of nine such combinations, including the original problem with $\delta_1 = \delta_2 = 0$.) For each combination of δ_1 and δ_2, make a prediction $p^\star_{\text{pred}}$ of the optimal value of the perturbed QP, and compare it to $p^\star_{\text{exact}}$, the exact optimal value of the perturbed QP (obtained by solving the perturbed QP). Put your results in the two righthand columns in a table with the form shown below. Check that the inequality $p^\star_{\text{pred}} \leq p^\star_{\text{exact}}$ holds.

δ_1	δ_2	$p^\star_{\text{pred}}$	$p^\star_{\text{exact}}$
0	0		
0	-0.1		
0	0.1		
-0.1	0		
-0.1	-0.1		
-0.1	0.1		
0.1	0		
0.1	-0.1		
0.1	0.1		

Solution.

(a) The following Matlab code sets up the simple QP and solves it using `CVX`:

```
Q = [1 -1/2; -1/2 2];
f = [-1 0]';
A = [1 2; 1 -4; 5 76];
b = [-2 -3 1]';


cvx_begin
    variable x(2)
    dual variable lambda
    minimize(quad_form(x,Q)+f'*x)
    subject to
        lambda: A*x <= b
cvx_end
p_star = cvx_optval
```

When we run this, we find the optimal objective value is $p^\star = 8.22$ and the optimal point is $x_1^\star = -2.33$, $x_2^\star = 0.17$. (This optimal point is unique since the objective is strictly convex.) A set of optimal dual variables is $\lambda_1^\star = 1.46$, $\lambda_2^\star = 3.77$ and $\lambda_3^\star = 0.12$. (The dual optimal point is unique too, but it's harder to show this, and it doesn't matter anyway.)

The KKT conditions are

$$\begin{array}{lll}
x_1^\star + 2x_2^\star \le u_1, & x_1^\star - 4x_2^\star \le u_2, & 5x_1^\star + 76x_2^\star \le 1 \\
\lambda_1^\star \ge 0, & \lambda_2^\star \ge 0, & \lambda_3^\star \ge 0 \\
\lambda_1^\star(x_1^\star + 2x_2^\star - u_1) = 0, & \lambda_2^\star(x_1^\star - 4x_2^\star - u_2) = 0, & \lambda_3^\star(5x_1^\star + 76x_2^\star - 1) = 0, \\
2x_1^\star - x_2^\star - 1 + \lambda_1^\star + \lambda_2^\star + 5\lambda_3^\star = 0, & & \\
4x_2^\star - x_1^\star + 2\lambda_1^\star - 4\lambda_2^\star + 76\lambda_3^\star = 0. & &
\end{array}$$

We check these numerically. The dual variable $\lambda_1^\star$, $\lambda_2^\star$ and $\lambda_3^\star$ are all greater than zero and the quantities

```
A*x-b
2*Q*x+f+A'*lambda
```

are found to be very small. Thus the KKT conditions are verified.

(b) The predicted optimal value is given by

$$p^\star_{\text{pred}} = p^\star - \lambda_1^\star\delta_1 - \lambda_2^\star\delta_2.$$

The following matlab code fills in the table

```
arr_i = [0 -1 1];
delta = 0.1;
pa_table = [];
for i = arr_i
    for j = arr_i
        p_pred = p_star - [lambda(1) lambda(2)]*[i; j]*delta;
        cvx_begin
            variable x(2)
            minimize(quad_form(x,Q)+f'*x)
            subject to
                A*x <= b+[i;j;0]*delta
        cvx_end
        p_exact = cvx_optval;

        pa_table = [pa_table; i*delta j*delta p_pred p_exact]
    end
end
```

The values obtained are

δ_1	δ_2	$p^\star_{\rm pred}$	$p^\star_{\rm exact}$
0	0	8.22	8.22
0	−0.1	8.60	8.70
0	0.1	7.85	7.98
−0.1	0	8.34	8.57
−0.1	−0.1	8.75	8.82
−0.1	0.1	7.99	8.32
0.1	0	8.08	8.22
0.1	−0.1	8.45	8.71
0.1	0.1	7.70	7.75

The inequality $p^\star_{\rm pred} \leq p^\star_{\rm exact}$ is verified to be true in all cases.

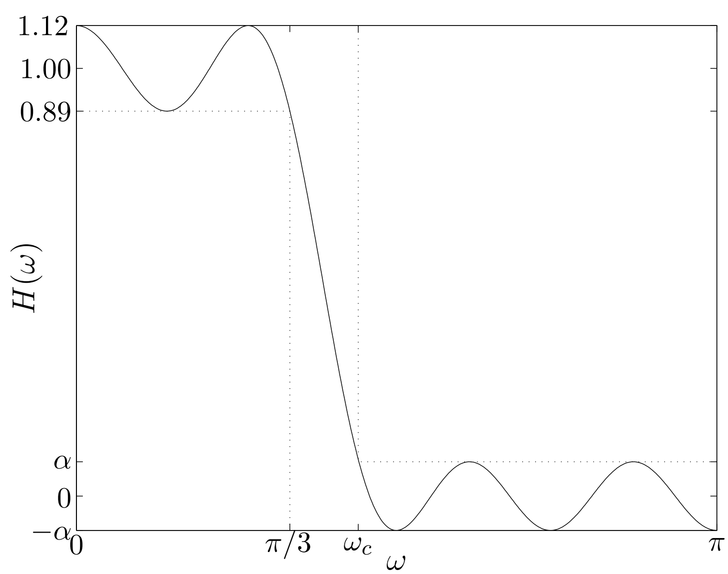
4. *FIR filter design.* Consider the (symmetric, linear phase) FIR filter described by

$$H(\omega) = a_0 + \sum_{k=1}^{N} a_k \cos k\omega.$$

The design variables are the real coefficients $a = (a_0, \ldots, a_N) \in \mathbf{R}^{N+1}$. In this problem we will explore the design of a low-pass filter, with specifications:

- For $0 \leq \omega \leq \pi/3$, $0.89 \leq H(\omega) \leq 1.12$, *i.e.*, the filter has about $\pm$1dB ripple in the 'passband' $[0, \pi/3]$.
- For $\omega_c \leq \omega \leq \pi$, $|H(\omega)| \leq \alpha$. In other words, the filter achieves an attenuation given by α in the 'stopband' $[\omega_c, \pi]$. ω_c is called the 'cutoff frequency'.

These specifications are depicted graphically in the figure below.



(a) Suppose we fix ω_c and N, and wish to maximize the stop-band attenuation, *i.e.*, minimize α such that the specifications above can be met. Explain how to pose this as a convex optimization problem.

(b) Suppose we fix N and α, and want to minimize ω_c, *i.e.*, we set the stopband attenuation and filter length, and wish to minimize the 'transition' band (between $\pi/3$ and ω_c). Explain how to pose this problem as a quasiconvex optimization problem.

(c) Now suppose we fix ω_c and α, and wish to find the smallest N that can meet the specifications, *i.e.*, we seek the shortest length FIR filter that can meet the specifications. Can this problem be posed as a convex or quasiconvex problem? If so, explain how. If you think it cannot be, briefly and informally explain why.

(d) Plot the optimal tradeoff curve of attenuation (α) versus cutoff frequency (ω_c) for $N = 7$. Is the set of achievable specifications convex? Briefly explain any interesting features, *e.g.*, flat portions, of the optimal tradeoff curve.

For this subproblem, you may sample the constraints in frequency, which means the following. Choose $K \gg N$ (perhaps $K \approx 10N$), and set $\omega_k = k\pi/K$, $k = 0, \ldots, K$. Then replace the specifications with

- For k with $0 \le \omega_k \le \pi/3$, $0.89 \le H(\omega_k) \le 1.12$.
- For k with $\omega_c \le \omega_k \le \pi$, $|H(\omega_k)| \le \alpha$.

With this approximation, the problem in part (a) becomes an LP, which allows you to solve part (d) numerically.

Solution.

(a) The first problem can be expressed as

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & f_1(a) \le 1.12 \\ & f_2(a) \ge 0.89 \\ & f_3(a) \le \alpha \\ & f_4(a) \ge -\alpha \end{array} \tag{2}$$

where

$$f_1(a) = \sup_{0 \le \omega \le \pi/3} H(\omega), \quad f_2(a) = \inf_{0 \le \omega \le \pi/3} H(\omega),$$

$$f_3(a) = \sup_{\omega_c \le \omega \le \pi} H(\omega), \quad f_4(a) = \inf_{\omega_c \le \omega \le \pi} H(\omega).$$

Problem (2) is convex in the variables a, α because f_1 and f_3 are convex functions (pointwise supremum of affine functions), and f_4 and f_5 are concave functions (pointwise infimum of affine functions).

(b) This problem can be expressed

$$\begin{array}{ll} \text{minimize} & f_5(a) \\ \text{subject to} & f_1(a) \leq 1.12 \\ & f_2(a) \geq 0.89 \end{array}$$

where f_1 and f_2 are the same functions as above, and

$$f_5(a) = \inf\{\Omega \mid -\alpha \leq H(\omega) \leq \alpha \text{ for } \Omega \leq \omega \leq \pi\}.$$

This is a quasiconvex optimization problem in the variables a because f_1 is convex, f_2 is concave, and f_5 is quasiconvex: its sublevel sets are

$$\{a \mid f_5(a) \leq \Omega\} = \{a \mid -\alpha \leq H(\omega) \leq \alpha \text{ for } \Omega \leq \omega \leq \pi\},$$

i.e., the intersection of an infinite number of halfspaces.

(c) This problem can be expressed as

$$\begin{array}{ll} \text{minimize} & f_6(a) \\ \text{subject to} & f_1(a) \leq 1.12 \\ & f_2(a) \geq 0.89 \\ & f_3(a) \leq \alpha \\ & f_4(a) \geq -\alpha \end{array}$$

where f_1, f_2, f_3, and f_4 are defined above and

$$f_6(a) = \min\{k \mid a_{k+1} = \cdots = a_N = 0\}.$$

The sublevel sets of f_6 are affine sets:

$$\{a \mid f_6(a) \leq k\} = \{a \mid a_{k+1} = \cdots = a_N = 0\}.$$

This means f_6 is a quasiconvex function, and again we have a quasiconvex optimization problem.

(d) After discretizing we can express the problem in part (a) as the LP

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & 0.89 \leq H(\omega_i) \leq 1.12 \text{ for } 0 \leq \omega_i \leq \pi/3 \\ & -\alpha \leq H(\omega_i) \leq \alpha \text{ for } \omega_c \leq \omega_i \leq \pi \end{array} \qquad (3)$$

with variables α and a. (For fixed ω_i, $H(\omega_i)$ is an affine function of a, hence all constraints in this problem are linear inequalities in a and α.) We obtain the tradeoff curve of α vs. ω_c, by solving this LP for a sequence of values of ω_c in the interval $(\pi/3, \pi]$.

Figure (1) was generated by the following matlab code.

```
clear all
N = 7;
K = 10*N;
k = [0:N]';
w = [0:K]'/K*pi;
idx= max(find(w<=pi/3));
alphas = [];

for i=idx:length(w)

    cvx_begin
    variables a(N+1,1)
    minimize( norm(cos(w(i:end)*k')*a,inf) )
    subject to
        cos(w(1:idx)*k')*a >= 0.89
        cos(w(1:idx)*k')*a <= 1.12
    cvx_end
    alphas = [alphas; cvx_optval];
end;

plot(w(idx:end),alphas,'-');
xlabel('wc');
ylabel('alpha');
```

5. *Minimum fuel optimal control.* Solve the minimum fuel optimal control problem described in exercise 4.16 of *Convex Optimization*, for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \quad x_{\mathrm{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad N = 30.$$

You can do this by forming the LP you found in your solution of exercise 4.16, or more directly using `cvx`. Plot the actuator signal $u(t)$ as a function of time t.

Solution. The following Matlab code finds the solution

```
close all
clear all

n=3;  % state dimension
N=30; % time horizon

A=[ -1 0.4 0.8; 1 0 0 ; 0 1 0];
b=[ 1 0 0.3]';
```

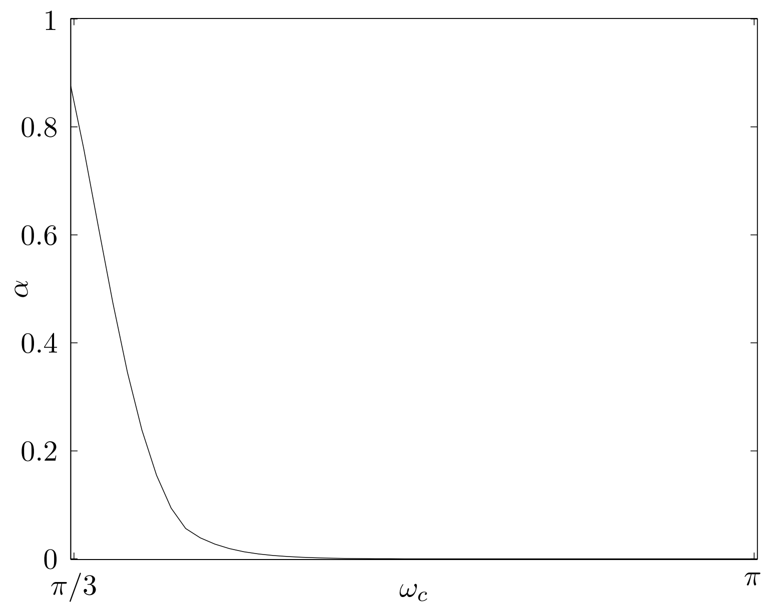


Figure 1 Tradeoff curve for problem 4d.

```
x0 = zeros(n,1);
xdes = [ 7 2 -6]';

cvx_begin
    variable X(n,N+1);
    variable u(1,N);
    minimize (sum(max(abs(u),2*abs(u)-1)))
    subject to
        X(:,2:N+1) == A*X(:,1:N)+b*u;  % dynamics
        X(:,1)== x0;
        X(:,N+1)== xdes;
cvx_end

stairs(0:N-1,u,'linewidth',2)
axis tight
xlabel('t')
ylabel('u')
```

The optimal actuator signal is shown in figure 2.

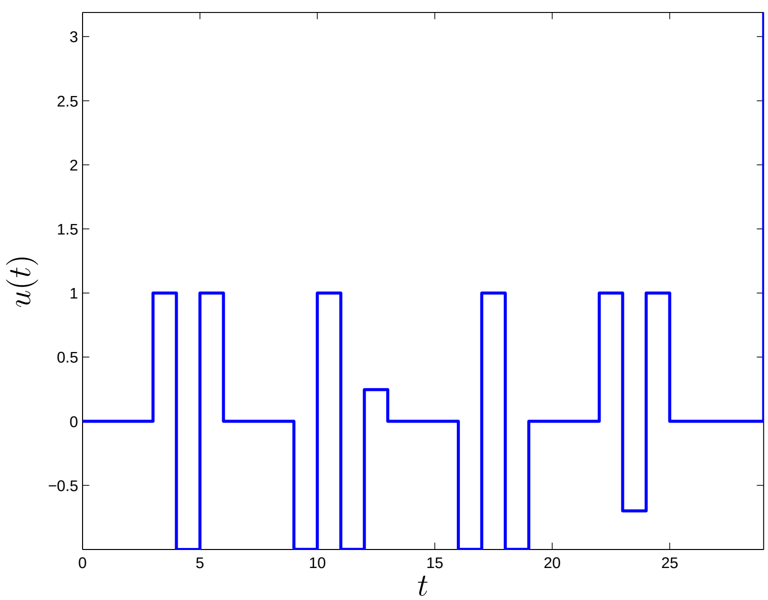


Figure 2 Minimum fuel actuator signal.