EE364a Homework 4 additional problems

1. *Minimizing a function over the probability simplex.* Find simple necessary and sufficient conditions for $x \in \mathbf{R}^n$ to minimize a differentiable convex function f over the probability simplex, $\{x \mid \mathbf{1}^T x = 1,\ x \succeq 0\}$.

2. *Complex least-norm problem.* We consider the complex least ℓ_p-norm problem

$$\begin{array}{ll} \text{minimize} & \|x\|_p \\ \text{subject to} & Ax = b, \end{array}$$

where $A \in \mathbf{C}^{m\times n}$, $b \in \mathbf{C}^m$, and the variable is $x \in \mathbf{C}^n$. Here $\|\cdot\|_p$ denotes the ℓ_p-norm on $\mathbf{C}^n$, defined as

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

for $p \geq 1$, and $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$. We assume A is full rank, and $m < n$.

(a) Formulate the complex least ℓ_2-norm problem as a least ℓ_2-norm problem with real problem data and variable. *Hint.* Use $z = (\Re x, \Im x) \in \mathbf{R}^{2n}$ as the variable.

(b) Formulate the complex least ℓ_∞-norm problem as an SOCP.

(c) Solve a random instance of both problems with $m = 30$ and $n = 100$. To generate the matrix A, you can use the Matlab command `A = randn(m,n) + i*randn(m,n)`. Similarly, use `b = randn(m,1) + i*randn(m,1)` to generate the vector b. Use the Matlab command `scatter` to plot the optimal solutions of the two problems on the complex plane, and comment (briefly) on what you observe. You can solve the problems using the `cvx` functions `norm(x,2)` and `norm(x,inf)`, which are overloaded to handle complex arguments. To utilize this feature, you will need to declare variables to be `complex` in the `variable` statement. (In particular, you do not have to manually form or solve the SOCP from part (b).)

3. *Numerical perturbation analysis example.* Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ \text{subject to} & x_1 + 2x_2 \leq u_1 \\ & x_1 - 4x_2 \leq u_2, \\ & 5x_1 + 76x_2 \leq 1, \end{array}$$

with variables x_1, x_2, and parameters u_1, u_2.

(a) Solve this QP, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values $x_1^\star$ and $x_2^\star$, and optimal dual variable values $\lambda_1^\star$, $\lambda_2^\star$ and $\lambda_3^\star$. Let

$p^\star$ denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Hint: See §3.6 of the `CVX` users' guide to find out how to retrieve optimal dual variables. To specify the quadratic objective, use `quad_form()`.

(b) We will now solve some perturbed versions of the QP, with

$$u_1 = -2 + \delta_1, \qquad u_2 = -3 + \delta_2,$$

where δ_1 and δ_2 each take values from $\{-0.1, 0, 0.1\}$. (There are a total of nine such combinations, including the original problem with $\delta_1 = \delta_2 = 0$.) For each combination of δ_1 and δ_2, make a prediction $p^\star_{\text{pred}}$ of the optimal value of the perturbed QP, and compare it to $p^\star_{\text{exact}}$, the exact optimal value of the perturbed QP (obtained by solving the perturbed QP). Put your results in the two righthand columns in a table with the form shown below. Check that the inequality $p^\star_{\text{pred}} \leq p^\star_{\text{exact}}$ holds.

δ_1	δ_2	$p^\star_{\text{pred}}$	$p^\star_{\text{exact}}$
0	0		
0	-0.1		
0	0.1		
-0.1	0		
-0.1	-0.1		
-0.1	0.1		
0.1	0		
0.1	-0.1		
0.1	0.1		

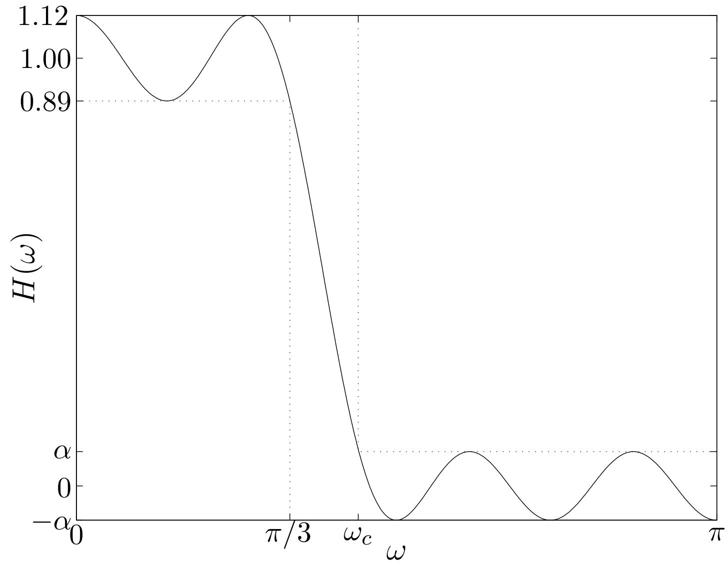
4. *FIR filter design.* Consider the (symmetric, linear phase) FIR filter described by

$$H(\omega) = a_0 + \sum_{k=1}^{N} a_k \cos k\omega.$$

The design variables are the real coefficients $a = (a_0, \ldots, a_N) \in \mathbf{R}^{N+1}$. In this problem we will explore the design of a low-pass filter, with specifications:

- For $0 \leq \omega \leq \pi/3$, $0.89 \leq H(\omega) \leq 1.12$, *i.e.*, the filter has about $\pm$1dB ripple in the 'passband' $[0, \pi/3]$.
- For $\omega_c \leq \omega \leq \pi$, $|H(\omega)| \leq \alpha$. In other words, the filter achieves an attenuation given by α in the 'stopband' $[\omega_c, \pi]$. ω_c is called the 'cutoff frequency'.

These specifications are depicted graphically in the figure below.



(a) Suppose we fix ω_c and N, and wish to maximize the stop-band attenuation, *i.e.*, minimize α such that the specifications above can be met. Explain how to pose this as a convex optimization problem.

(b) Suppose we fix N and α, and want to minimize ω_c, *i.e.*, we set the stopband attenuation and filter length, and wish to minimize the 'transition' band (between $\pi/3$ and ω_c). Explain how to pose this problem as a quasiconvex optimization problem.

(c) Now suppose we fix ω_c and α, and wish to find the smallest N that can meet the specifications, *i.e.*, we seek the shortest length FIR filter that can meet the specifications. Can this problem be posed as a convex or quasiconvex problem? If so, explain how. If you think it cannot be, briefly and informally explain why.

(d) Plot the optimal tradeoff curve of attenuation (α) versus cutoff frequency (ω_c) for $N = 7$. Is the set of achievable specifications convex? Briefly explain any interesting features, *e.g.*, flat portions, of the optimal tradeoff curve.

For this subproblem, you may sample the constraints in frequency, which means the following. Choose $K \gg N$ (perhaps $K \approx 10N$), and set $\omega_k = k\pi/K$, $k = 0, \ldots, K$. Then replace the specifications with

- For k with $0 \leq \omega_k \leq \pi/3$, $0.89 \leq H(\omega_k) \leq 1.12$.
- For k with $\omega_c \leq \omega_k \leq \pi$, $|H(\omega_k)| \leq \alpha$.

With this approximation, the problem in part (a) becomes an LP, which allows you to solve part (d) numerically.

5. *Minimum fuel optimal control.* Solve the minimum fuel optimal control problem de-

scribed in exercise 4.16 of *Convex Optimization*, for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \quad x_{\rm des} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad N = 30.$$

You can do this by forming the LP you found in your solution of exercise 4.16, or more directly using `cvx`. Plot the actuator signal $u(t)$ as a function of time t.