

11. Equality constrained minimization

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- infeasible start Newton method
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Equality constrained minimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- f convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\text{rank } A = p$
- we assume p^* is finite and attained

optimality conditions: x^* is optimal iff there exists a ν^* such that

$$\nabla f(x^*) + A^T \nu^* = 0, \quad Ax^* = b$$

equality constrained quadratic minimization (with $P \in \mathbf{S}_+^n$)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

optimality condition:

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

- coefficient matrix is called KKT matrix
- KKT matrix is nonsingular if and only if

$$Ax = 0, \quad x \neq 0 \quad \implies \quad x^T P x > 0$$

- equivalent condition for nonsingularity: $P + A^T A \succ 0$

Eliminating equality constraints

represent solution of $\{x \mid Ax = b\}$ as

$$\{x \mid Ax = b\} = \{Fz + \hat{x} \mid z \in \mathbf{R}^{n-p}\}$$

- \hat{x} is (any) particular solution
- range of $F \in \mathbf{R}^{n \times (n-p)}$ is nullspace of A ($\text{rank } F = n - p$ and $AF = 0$)

reduced or eliminated problem

$$\text{minimize } f(Fz + \hat{x})$$

- an unconstrained problem with variable $z \in \mathbf{R}^{n-p}$
- from solution z^* , obtain x^* and ν^* as

$$x^* = Fz^* + \hat{x}, \quad \nu^* = -(AA^T)^{-1}A\nabla f(x^*)$$

example: optimal allocation with resource constraint

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) \\ \text{subject to} & x_1 + x_2 + \cdots + x_n = b \end{array}$$

eliminate $x_n = b - x_1 - \cdots - x_{n-1}$, *i.e.*, choose

$$\hat{x} = be_n, \quad F = \begin{bmatrix} I \\ -\mathbf{1}^T \end{bmatrix} \in \mathbf{R}^{n \times (n-1)}$$

reduced problem:

$$\text{minimize} \quad f_1(x_1) + \cdots + f_{n-1}(x_{n-1}) + f_n(b - x_1 - \cdots - x_{n-1})$$

(variables x_1, \dots, x_{n-1})

Newton step

Newton step of f at feasible x is given by (1st block) of solution of

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

interpretations

- Δx_{nt} solves second order approximation (with variable v)

$$\begin{array}{ll} \text{minimize} & \hat{f}(x + v) = f(x) + \nabla f(x)^T v + (1/2)v^T \nabla^2 f(x)v \\ \text{subject to} & A(x + v) = b \end{array}$$

- equations follow from linearizing optimality conditions

$$\nabla f(x + \Delta x_{\text{nt}}) + A^T w = 0, \quad A(x + \Delta x_{\text{nt}}) = b$$

Newton decrement

$$\lambda(x) = (\Delta x_{\text{nt}}^T \nabla^2 f(x) \Delta x_{\text{nt}})^{1/2} = (-\nabla f(x)^T \Delta x_{\text{nt}})^{1/2}$$

properties

- gives an estimate of $f(x) - p^*$ using quadratic approximation \hat{f} :

$$f(x) - \inf_{Ay=b} \hat{f}(y) = \frac{1}{2} \lambda(x)^2$$

- directional derivative in Newton direction:

$$\left. \frac{d}{dt} f(x + t \Delta x_{\text{nt}}) \right|_{t=0} = -\lambda(x)^2$$

- in general, $\lambda(x) \neq (\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x))^{1/2}$

Newton's method with equality constraints

given starting point $x \in \text{dom } f$ with $Ax = b$, tolerance $\epsilon > 0$.

repeat

1. Compute the Newton step and decrement $\Delta x_{\text{nt}}, \lambda(x)$.
 2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$.
 3. *Line search.* Choose step size t by backtracking line search.
 4. *Update.* $x := x + t\Delta x_{\text{nt}}$.
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- a feasible descent method: $x^{(k)}$ feasible and $f(x^{(k+1)}) < f(x^{(k)})$
- affine invariant

Newton's method and elimination

Newton's method for reduced problem

$$\text{minimize } \tilde{f}(z) = f(Fz + \hat{x})$$

- variables $z \in \mathbf{R}^{n-p}$
- \hat{x} satisfies $A\hat{x} = b$; $\text{rank } F = n - p$ and $AF = 0$
- Newton's method for \tilde{f} , started at $z^{(0)}$, generates iterates $z^{(k)}$

Newton's method with equality constraints

when started at $x^{(0)} = Fz^{(0)} + \hat{x}$, iterates are

$$x^{(k+1)} = Fz^{(k)} + \hat{x}$$

hence, don't need separate convergence analysis

Newton step at infeasible points

2nd interpretation of page 1–6 extends to infeasible x (*i.e.*, $Ax \neq b$)

linearizing optimality conditions at infeasible x (with $x \in \mathbf{dom} f$) gives

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax - b \end{bmatrix} \quad (1)$$

primal-dual interpretation

- write optimality condition as $r(y) = 0$, where

$$y = (x, \nu), \quad r(y) = (\nabla f(x) + A^T \nu, Ax - b)$$

- linearizing $r(y) = 0$ gives $r(y + \Delta y) \approx r(y) + Dr(y)\Delta y = 0$:

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ \Delta \nu_{\text{nt}} \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + A^T \nu \\ Ax - b \end{bmatrix}$$

same as (1) with $w = \nu + \Delta \nu_{\text{nt}}$

Infeasible start Newton method

given starting point $x \in \text{dom } f$, ν , tolerance $\epsilon > 0$, $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$.

repeat

1. Compute primal and dual Newton steps Δx_{nt} , $\Delta \nu_{\text{nt}}$.

2. *Backtracking line search* on $\|r\|_2$.

$t := 1$.

while $\|r(x + t\Delta x_{\text{nt}}, \nu + t\Delta \nu_{\text{nt}})\|_2 > (1 - \alpha t)\|r(x, \nu)\|_2$, $t := \beta t$.

3. *Update*. $x := x + t\Delta x_{\text{nt}}$, $\nu := \nu + t\Delta \nu_{\text{nt}}$.

until $Ax = b$ and $\|r(x, \nu)\|_2 \leq \epsilon$.

- not a descent method: $f(x^{(k+1)}) > f(x^{(k)})$ is possible
- directional derivative of $\|r(y)\|_2$ in direction $\Delta y = (\Delta x_{\text{nt}}, \Delta \nu_{\text{nt}})$ is

$$\left. \frac{d}{dt} \|r(y + t\Delta y)\|_2 \right|_{t=0} = -\|r(y)\|_2$$

Solving KKT systems

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} g \\ h \end{bmatrix}$$

solution methods

- LDL^T factorization
- elimination (if H nonsingular)

$$AH^{-1}A^T w = h - AH^{-1}g, \quad Hv = -(g + A^T w)$$

- elimination with singular H : write as

$$\begin{bmatrix} H + A^T Q A & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} g + A^T Q h \\ h \end{bmatrix}$$

with $Q \succeq 0$ for which $H + A^T Q A \succ 0$, and apply elimination

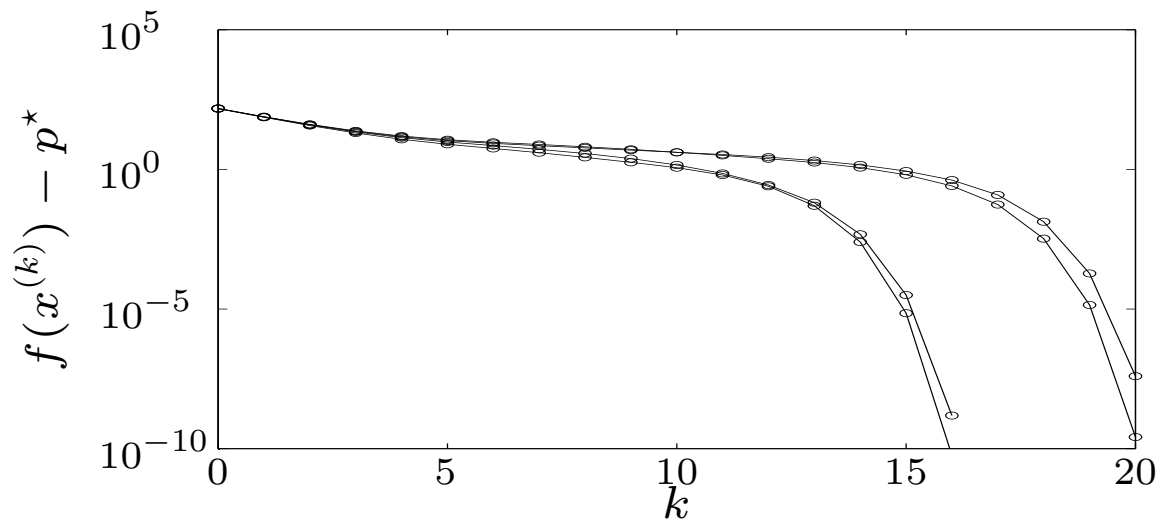
Equality constrained analytic centering

primal problem: minimize $-\sum_{i=1}^n \log x_i$ subject to $Ax = b$

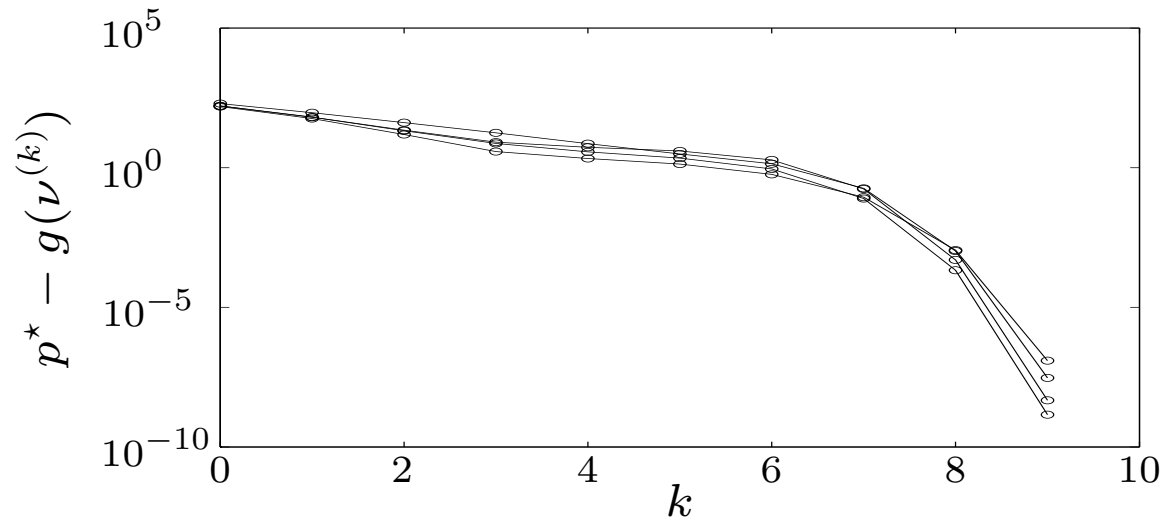
dual problem: maximize $-b^T \nu + \sum_{i=1}^n \log(A^T \nu)_i + n$

three methods for an example with $A \in \mathbf{R}^{100 \times 500}$, different starting points

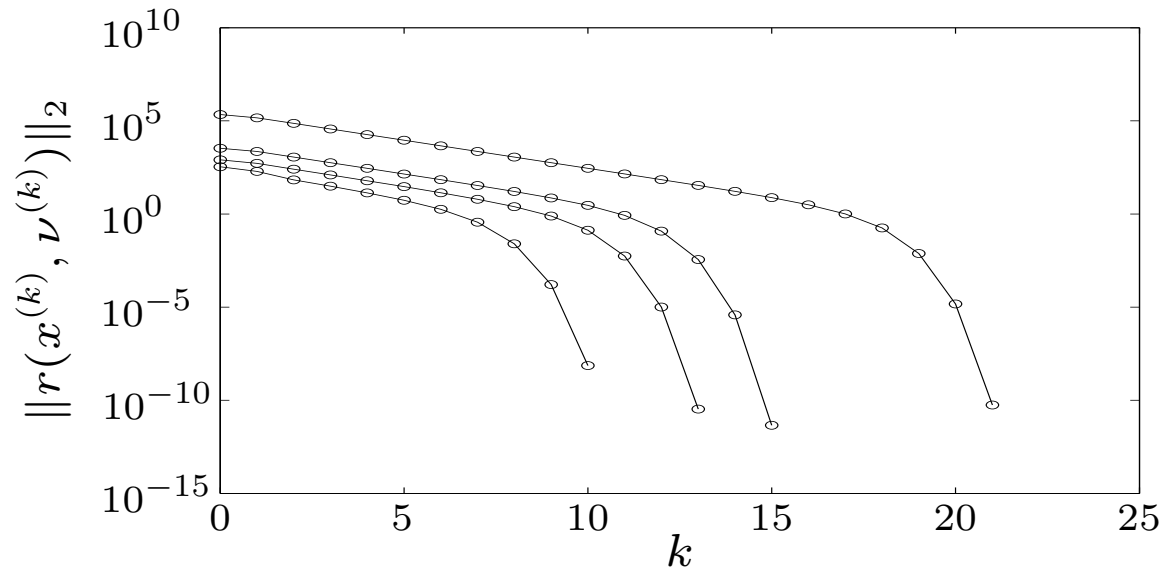
1. Newton method with equality constraints (requires $x^{(0)} \succ 0$, $Ax^{(0)} = b$)



2. Newton method applied to dual problem (requires $A^T \nu^{(0)} \succ 0$)



3. infeasible start Newton method (requires $x^{(0)} \succ 0$)



complexity per iteration of three methods is identical

1. use block elimination to solve KKT system

$$\begin{bmatrix} \mathbf{diag}(x)^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{diag}(x)^{-1} \mathbf{1} \\ 0 \end{bmatrix}$$

reduces to solving $A \mathbf{diag}(x)^2 A^T w = b$

2. solve Newton system $A \mathbf{diag}(A^T \nu)^{-2} A^T \Delta \nu = -b + A \mathbf{diag}(A^T \nu)^{-1} \mathbf{1}$

3. use block elimination to solve KKT system

$$\begin{bmatrix} \mathbf{diag}(x)^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \nu \end{bmatrix} = \begin{bmatrix} \mathbf{diag}(x)^{-1} \mathbf{1} \\ Ax - b \end{bmatrix}$$

reduces to solving $A \mathbf{diag}(x)^2 A^T w = 2Ax - b$

conclusion: in each case, solve $ADA^T w = h$ with D positive diagonal

Network flow optimization

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \phi_i(x_i) \\ \text{subject to} & Ax = b \end{array}$$

- directed graph with n arcs, $p + 1$ nodes
- x_i : flow through arc i ; ϕ_i : cost flow function for arc i (with $\phi_i''(x) > 0$)
- node-incidence matrix $\tilde{A} \in \mathbf{R}^{(p+1) \times n}$ defined as

$$\tilde{A}_{ij} = \begin{cases} 1 & \text{arc } j \text{ leaves node } i \\ -1 & \text{arc } j \text{ enters node } i \\ 0 & \text{otherwise} \end{cases}$$

- reduced node-incidence matrix $A \in \mathbf{R}^{p \times n}$ is \tilde{A} with last row removed
- $b \in \mathbf{R}^p$ is (reduced) source vector
- **rank** $A = p$ if graph is connected

KKT system

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} g \\ h \end{bmatrix}$$

- $H = \text{diag}(\phi_1''(x_1), \dots, \phi_n''(x_n))$, positive diagonal
- solve via elimination:

$$AH^{-1}A^T w = h - AH^{-1}g, \quad Hv = -(g + A^T w)$$

sparsity pattern of coefficient matrix is given by graph connectivity

$$\begin{aligned} (AH^{-1}A^T)_{ij} \neq 0 &\iff (AA^T)_{ij} \neq 0 \\ &\iff \text{nodes } i \text{ and } j \text{ are connected by an arc} \end{aligned}$$

Analytic center of linear matrix inequality

$$\begin{array}{ll} \text{minimize} & -\log \det X \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, \dots, p \end{array}$$

variable $X \in \mathbf{S}^n$

optimality conditions

$$X^* \succ 0, \quad -(X^*)^{-1} + \sum_{j=1}^p \nu_j^* A_j = 0, \quad \mathbf{tr}(A_i X^*) = b_i, \quad i = 1, \dots, p$$

Newton equation at feasible X :

$$X^{-1} \Delta X X^{-1} + \sum_{j=1}^p w_j A_j = X^{-1}, \quad \mathbf{tr}(A_i \Delta X) = 0, \quad i = 1, \dots, p$$

- follows from linear approximation $(X + \Delta X)^{-1} \approx X^{-1} - X^{-1} \Delta X X^{-1}$
- $n(n+1)/2 + p$ variables $\Delta X, w$

solution by block elimination

- eliminate ΔX from first equation: $\Delta X = X - \sum_{j=1}^p w_j X A_j X$
- substitute ΔX in second equation

$$\sum_{j=1}^p \text{tr}(A_i X A_j X) w_j = b_i, \quad i = 1, \dots, p \quad (2)$$

a dense positive definite set of linear equations with variable $w \in \mathbf{R}^p$

flop count (dominant terms) using Cholesky factorization $X = LL^T$:

- form p products $L^T A_j L$: $(3/2)pn^3$
- form $p(p+1)/2$ inner products $\text{tr}((L^T A_i L)(L^T A_j L))$: $(1/2)p^2n^2$
- solve (2) via Cholesky factorization: $(1/3)p^3$