PART II

Representation Theory

Recall that an element v of a finite dimensional vector space V has a column vector representation, if for a basis $\beta = \{u_1, u_2, \ldots, u_n\}$, which spans the vector space, there exists a set of scalars $a_1, a_2, \ldots, a_n$ such that v is equivalent to the sum

$$v = \sum_{i=1}^{n} a_i u_i. \tag{4.88}$$

If this is the case, the coordinate vector of v in the β basis is

$$[v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}. \tag{4.89}$$

From here, one can delve into the subject of linear algebras and study how linear transformations or operators, such as $T : V \to V$, act on the vector v. In inspecting these linear transformations, there is a natural correspondence between linear transformation of bases vectors u_i and an $n \times n$ ordered grid of tuples generated from the linear map, i.e. matrices.

Matrices are a thoroughly useful object to characterize the set of linear transformations of a vector space. Naturally, we expect that the vector space of the Lie algebra to also benefit from such a treatment. However, the analogy which we have set up is quite crude. First, a group is not a field, it lacks a second multiplication operation required by a field. Therefore how such linear transformations would appear in the study of Lie groups is not obvious. In order to study such objects, we would need a vector space of operators which preserve the group multiplication. This is precisely the aim of *representations*.

Definition 28 (Group Representations) Suppose $\mathcal{G}$ is a group, not necessarily Lie, a representation R is a set of non–singular matrices

$$\mathsf{R}(\mathcal{G}) = \left\{ \forall g \in \mathcal{G} \middle| \mathsf{D}(g) \in \mathrm{Mat}_n(\mathbb{F}) \right\} \tag{4.90}$$

such that elements of R satisfy

$$\mathsf{D}(g_1 \star g_2) = \mathsf{D}(g_1) \cdot \mathsf{D}(g_2) \tag{4.91}$$
$$\mathsf{D}(g \star e) = \mathsf{D}(g) = \mathsf{D}(e \star g) \tag{4.92}$$

with $\mathsf{D}(e)$ identified as the identity matrix.

Definition 29 (Lie Algebra Representations) Suppose $\mathfrak{g}$ is a Lie algebra, a representation, S, is a set of matrices

$$\mathsf{S}(\mathcal{G}) = \left\{ \forall X \in \mathfrak{g} \middle| \mathsf{d}(X) \in \mathrm{Mat}_n(\mathbb{F}) \right\} \tag{4.93}$$