Remark L_h being a diffeomorphism implies that the Jacobian matrix, denoted

$$J^i_j(\theta) = \frac{\partial\tilde{\theta}^i}{\partial\theta^j} \tag{4.58}$$

exists and is invertible in a neighborhood, i.e., $\det(J) \neq 0$.

Diffeomorphisms are known to generate a curve on a manifold connecting the points involved in a mapping. At each point on the curve, one can then assign a tangent vector and, more generally, a tangent space. These tangent space are also connected via a map known as the push-forward that are, in fact, induced by the existence of the left and right translation diffeomorphisms.

Definition 25 (Push–Forward) A left (and similarly right) push–forward, $L^*_h : \mathcal{T}_g(\mathcal{G}) \to \mathcal{T}_{hg}(\mathcal{G})$ is the map

$$V = v^i \frac{\partial}{\partial\theta^i} \in \mathcal{T}_g(\mathcal{G}) \mapsto \tilde{V} = \tilde{v}^i \frac{\partial}{\partial\tilde{\theta}^i} \in \mathcal{T}_{hg}(\mathcal{G}) \tag{4.59}$$

with $\tilde{v}^i = J^i_j(\theta)\ v^j$.

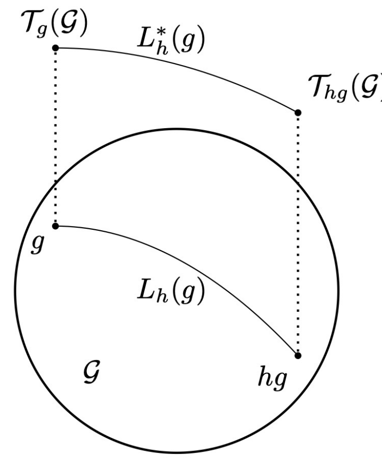

Figure 2. Visual of a push forward.

From the push–forward definition, we can define a vector field generally and connect every point in the tangent space as a push–forward of the identity element.

Definition 26 (Vector Field) A vector field is a map $\mathcal{V} : \mathcal{G} \to \mathcal{T}_g(\mathcal{G})$ such that $g \mapsto V_g$ at g. In a coordinate system

$$V(\theta) = V^i(\theta)\frac{\partial}{\partial\theta^i} \in \mathcal{T}_{g(\theta)}(\mathcal{G}) \tag{4.60}$$

is a vector field that is smooth if the components, $V^i(\theta) \in \mathbb{R}$, are each infinitely differentiable.

With regards to left and right translations, starting from a tangent vector at the identity, $\omega \in \mathcal{T}_e(\mathcal{G})$, we may define a vector field

$$\Omega(g) = L^*_g(\omega) \tag{4.61}$$

for all $g \in \mathcal{G}$. As L^*_g is smooth and invertible, $\Omega(g)$ is smooth and since the Jacobian is invertible, $\Omega(g)$ is also non-vanishing. This generalizes our result from earlier. We may now describe every point on the manifold as a push–forward of the identity.

Naturally, we want to tie in how this aids in developing a Lie algebra from Lie group. As we now have a vector space, we are just a short skip and step away from identifying the tangent space of these matrix Lie groups with the vector space needed for the Lie algebra. Continuing our program, we next introduce a basis for the tangent space. Let $\mathfrak{B} = \{\omega_a\}$ with $a = 1, \ldots, D$ be the basis for $\mathcal{T}_e(\mathcal{G})$. We obtain D independent nowhere–vanishing vector fields on $\mathcal{G}$ such that

$$\Omega_a(g) = L^*_g(\omega_a). \tag{4.62}$$

This is already a very strong constraint on $\mathcal{M}(\mathcal{G})$ because it says that there exists no other vector field on $\mathcal{M}(\mathcal{G})$ that may be constructed independently of the basis vector, i.e., all vectors on $\mathcal{M}(\mathcal{G})$ have a basis decomposition.

To see how this manifests itself on any given manifold consider the following examples.

Example The so–called **hairy ball theorem** says that any smooth vector field on $\mathcal{S}^2$ has at least two zeros. Thus $\mathcal{M}(\mathcal{G}) \neq \mathcal{S}^2$. In fact, for $\dim(\mathcal{G}) = 2$, assuming the $\mathcal{G}$ is compact,