preserved quantity, or symmetry, is the norm, and the multiplication law is matrix multiplication. Rotations in $\mathbb{R}^2$ form a *non–abelian* group as matrix multiplication is usually not commutative.

Subsection 1.1

Matrix Groups

Let $\mathrm{Mat}(n, \mathbb{F})$ denote the set of $n \times n$ matrices with entries in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Using matrix groups provides a bridge to connect the manifolds of Lie groups to matrices that one will encounter in representations later on. However, although matrix multiplication is closed and associative, and there is an obvious unit element

$$e \equiv \mathbb{I}_n \in \mathrm{Mat}(n, \mathbb{F})$$

where $\mathbb{I}_n$ is the $n \times n$ unit matrix, $\mathrm{Mat}(n, \mathbb{F})$ is not a multiplicative group because not all matrices are invertible. To account for this we define the general and special liner group.

Definition 4 The **general linear group** is a set of invertible matrices under the group multiplication operation of matrix multiplication. Compactly, this group can be expressed as

$$\mathrm{GL}(n, \mathbb{F}) = \{\mathbb{M} \in \mathrm{Mat}_n(\mathbb{F}) \mid \det(\mathbb{M}) \neq 0\}, \tag{1.3}$$

$$\mathrm{GL}^+(n, \mathbb{F}) = \{\mathbb{M} \in \mathrm{Mat}_n(\mathbb{F}) \mid \det(\mathbb{M}) > 0\}, \tag{1.4}$$

$$\mathrm{GL}^-(n, \mathbb{F}) = \{\mathbb{M} \in \mathrm{Mat}_n(\mathbb{F}) \mid \det(\mathbb{M}) < 0\}. \tag{1.5}$$

We can also be more restrictive with the matrix groups as we eventually want matrices that will be Hermitian and unitary. For those properties, we define the special linear group as follows.

Definition 5 The **special linear group**[4] is the set of invertible matrices under the group multiplication operation of matrix multiplication with unit determinant:

$$\mathrm{SL}(n, \mathbb{F}) = \{\mathbb{M} \in \mathrm{GL}(n, \mathbb{F}) \mid \det(\mathbb{M}) = 1\} \tag{1.6}$$

[4] ***Def: Special Linear Group***

From the above definitions, we immediately have closure for these groups under the determinant operation as

$$\det(\mathbb{M}_1\mathbb{M}_2) = \det(\mathbb{M}_1)\det(\mathbb{M}_2), \ \forall \mathbb{M}_1, \mathbb{M}_2 \in \mathrm{GL}(n, \mathbb{F}). \tag{1.7}$$

Now that we have closure, and since the group inversion and multiplication operations define smooth maps, it follows that the general and special linear groups are Lie groups. Then the dimension of their corresponding manifolds are:[5]

$$\dim(\mathrm{GL}(n, \mathbb{R})) = n^2, \qquad \dim(\mathrm{SL}(n, \mathbb{R})) = n^2 - 1,$$
$$\dim(\mathrm{GL}(n, \mathbb{C})) = 2n^2, \qquad \dim(\mathrm{SL}(n, \mathbb{C})) = 2n^2 - 2.$$

[5] ***Note:*** *An explanation of these dimensions of the connected pieces can be found here.*

By this point you might have noticed that $\mathrm{SL}(n, \mathbb{F})$ seems to be a subset of $\mathrm{GL}(n, \mathbb{F})$ and obeys the same group multiplication. This leads us to the ideas of *subgroups.*

Definition 6 **Subgroups** A **subgroup** H of a group G is a subset which is also a group. We write