

Inference and Simulation

EC 607, Set 04

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Prologue

Schedule

Last time

The *CEF* and least-squares regression

Today

Inference

Read MHE 3.1

Upcoming

Lab (as usual) on Friday.

Problem set 001 due April 7th

Class project, step 1 due on April 15th

Inference

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Why?

Q What's the big deal with inference?

Inference

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A We rarely know the CEF or the population (and its regression vector).

We *can* draw statistical inferences about the population using samples.

Inference

Why?

Q What's the big deal with inference?

A We rarely know the CEF or the population (and its regression vector).

We *can* draw statistical inferences about the population using samples.

Important The issue/topic of *statistical inference* is separate from *causality*.

Separate questions

1. How do we interpret the estimated coefficient $\hat{\beta}$?
2. What is the sampling distribution of $\hat{\beta}$?

Inference

Moving from population to sample

Recall The population-regression function gives us the best linear approximation to the CEF.

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We're interested in the (unknown) population-regression vector

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Inference

Moving from population to sample

Recall The population-regression function gives us the best linear approximation to the CEF.

We're interested in the (unknown) population-regression vector

$$\beta = E [\mathbf{X}_i \mathbf{X}_i']^{-1} E[\mathbf{X}_i Y_i]$$

which we estimate via the ordinary least squares (OLS) estimator[†]

$$\hat{\beta} = \left(\sum_i \mathbf{X}_i \mathbf{X}_i' \right)^{-1} \left(\sum_i \mathbf{X}_i Y_i \right)$$

[†] MHE presents a method-of-moments motivation for this derivation, where $\frac{1}{n} \sum_i \mathbf{X}_i \mathbf{X}_i'$ is our sample-based estimated for $E[\mathbf{X}_i \mathbf{X}_i']$. You've also seen others, e.g., minimizing MSE of Y_i given \mathbf{X}_i .

Inference

A classic

However you write it, this OLS estimator

$$\begin{aligned}\hat{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \\ &= \left(\sum_i \mathbf{x}_i\mathbf{x}_i'\right)^{-1} \left(\sum_i \mathbf{x}_i Y_i\right) \\ &= \beta + \left[\sum_i \mathbf{x}_i\mathbf{x}_i'\right]^{-1} \sum_i \mathbf{x}_i e_i\end{aligned}$$

is the same estimator you've been using since undergrad.

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Note I'm following *MHE* in defining $e_i = Y_i - \mathbf{X}_i'\beta$.

Inference

A classic

As you've learned, the OLS estimator

$$\hat{\beta} = \left(\sum_i \mathbf{X}_i \mathbf{X}_i' \right)^{-1} \left(\sum_i \mathbf{X}_i Y_i \right) = \beta + \left[\sum_i \mathbf{X}_i \mathbf{X}_i' \right]^{-1} \sum_i \mathbf{X}_i e_i$$

has asymptotic covariance

$$E \left[\mathbf{X}_i \mathbf{X}_i' \right]^{-1} E \left[\mathbf{X}_i \mathbf{X}_i' e_i^2 \right] E \left[\mathbf{X}_i \mathbf{X}_i' \right]^{-1}$$

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$$E [\mathbf{X}_i \mathbf{X}_i']^{-1} E [\mathbf{X}_i \mathbf{X}_i' e_i^2] E [\mathbf{X}_i \mathbf{X}_i']^{-1}$$

which we estimate by **(1)** replacing e_i with $\hat{e}_i = Y_i - \mathbf{X}_i' \hat{\beta}$ and **(2)** replacing expectations with sample means, e.g., $E [\mathbf{X}_i \mathbf{X}_i' e_i^2]$ becomes $\frac{1}{n} \sum [\mathbf{X}_i \mathbf{X}_i' \hat{e}_i^2]$.

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Standard errors of this flavor are known as heteroskedasticity-consistent (or -robust) standard errors (or Eicker-Huber-White).

Inference

Defaults

Statistical packages default to assuming homoskedasticity, *i.e.*,

$$E[e_i^2 | \mathbf{X}_i] = \sigma^2 \text{ for all } i.$$

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$$E[\mathbf{X}_i \mathbf{X}_i' e_i^2] = E[E[\mathbf{X}_i \mathbf{X}_i' e_i^2 | \mathbf{X}_i]] = E[\mathbf{X}_i \mathbf{X}_i' E[e_i^2 | \mathbf{X}_i]] = \sigma^2 E[\mathbf{X}_i \mathbf{X}_i']$$

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Now, returning to to the asym. covariance matrix of $\hat{\beta}$,

$$\begin{aligned} E[\mathbf{X}_i \mathbf{X}_i']^{-1} E[\mathbf{X}_i \mathbf{X}_i' e_i^2] E[\mathbf{X}_i \mathbf{X}_i']^{-1} &= E[\mathbf{X}_i \mathbf{X}_i']^{-1} \sigma^2 E[\mathbf{X}_i \mathbf{X}_i'] E[\mathbf{X}_i \mathbf{X}_i']^{-1} \\ &= \sigma^2 E[\mathbf{X}_i \mathbf{X}_i']^{-1} \end{aligned}$$

Inference

Defaults

Angrist and Pischke argue we should probably change our default to heteroskedasticity.

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$$\begin{aligned} E\left[(Y_i - \mathbf{X}_i'\beta)^2 \mid \mathbf{X}_i\right] \\ = E\left[\left(\{Y_i - E[Y_i \mid \mathbf{X}_i]\} + \{E[Y_i \mid \mathbf{X}_i] - \mathbf{X}_i'\beta\}\right)^2 \mid \mathbf{X}_i\right] \end{aligned}$$

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Thus, even if $Y_i \mid \mathbf{X}_i$ has constant variance, $e_i \mid \mathbf{X}_i$ is heteroskedastic.

Inference

Two notes

1. Heteroskedasticity is **not our biggest concern** in inference.

...as an empirical matter, heteroskedasticity may matter very little... If heteroskedasticity matters a lot, say, more than a 30 percent increase or any marked decrease in standard errors, you should worry about possible programming errors or other problems. (*MHE*, p.47)

2. Notice that we've **avoided "standard" stronger assumptions**, e.g., normality, fixed regressors, linear CEF, homoskedasticity.

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1. Heteroskedasticity is **not our biggest concern** in inference.

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2. Notice that we've **avoided "standard" stronger assumptions**, e.g., normality, fixed regressors, linear CEF, homoskedasticity.

Following (2): We only have large-sample, asymptotic results (consistency) rather than finite-sample results (unbiasedness).

Inference

Warning

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One practical way we can study the behavior of an estimator: **simulation**.

Note You need to make sure your simulation can actually test/respond to the question you are asking (e.g., bias vs. consistency).

Inference

Simulation

Let's compare false- and true-positive rates[†] for

1. **Homoskedasticity-assuming standard errors** ($\text{Var}[e_i | \mathbf{X}_i] = \sigma^2$)
2. **Heteroskedasticity-robust standard errors**

[†] The false-positive rate goes by many names; another common name: *type-I error rate*.

Inference

Simulation

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2. **Heteroskedasticity-robust standard errors**

Simulation outline

1. Define data-generating process (DGP).
2. Choose sample size n .
3. Set seed.
4. Run 10,000 iterations of
 - a. Draw sample of size n from DGP.
 - b. Conduct inference.
 - c. Record inferences' outcomes.

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Data-generating process

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Let's keep the disturbances well behaved.

$$Y_i = 1 + e^{0.5X_i} + \varepsilon_i$$

where $X_i \sim \text{Uniform}(0, 10)$ and $\varepsilon_i \sim N(0, 1)$.

Simulation

Data-generating process

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where $X_i \sim \text{Uniform}(0, 10)$ and $\varepsilon_i \sim N(0, 15^2)$.

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Data-generating process

$$Y_i = 1 + e^{0.5X_i} + \varepsilon_i$$

where $X_i \sim \text{Uniform}(0, 10)$ and $\varepsilon_i \sim N(0, 15^2)$.

```
library(pacman)
p_load(dplyr)
# Choose a size
n = 1000
# Generate data
dgp_df = tibble(
  ε = rnorm(n, sd = 15),
  x = runif(n, min = 0, max = 10),
  y = 1 + exp(0.5 * x) + ε
)
```

Simulation

Data-generating process

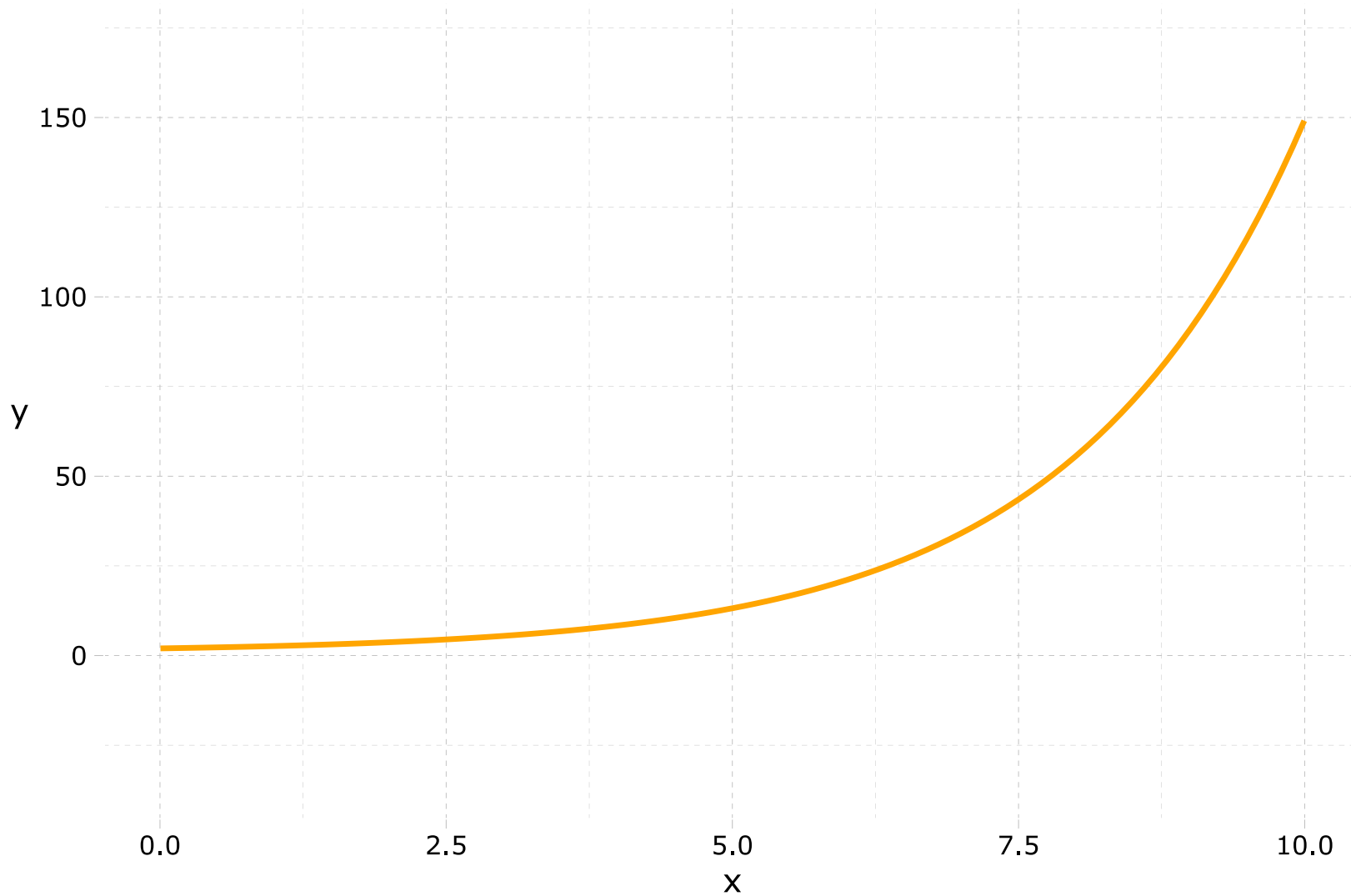
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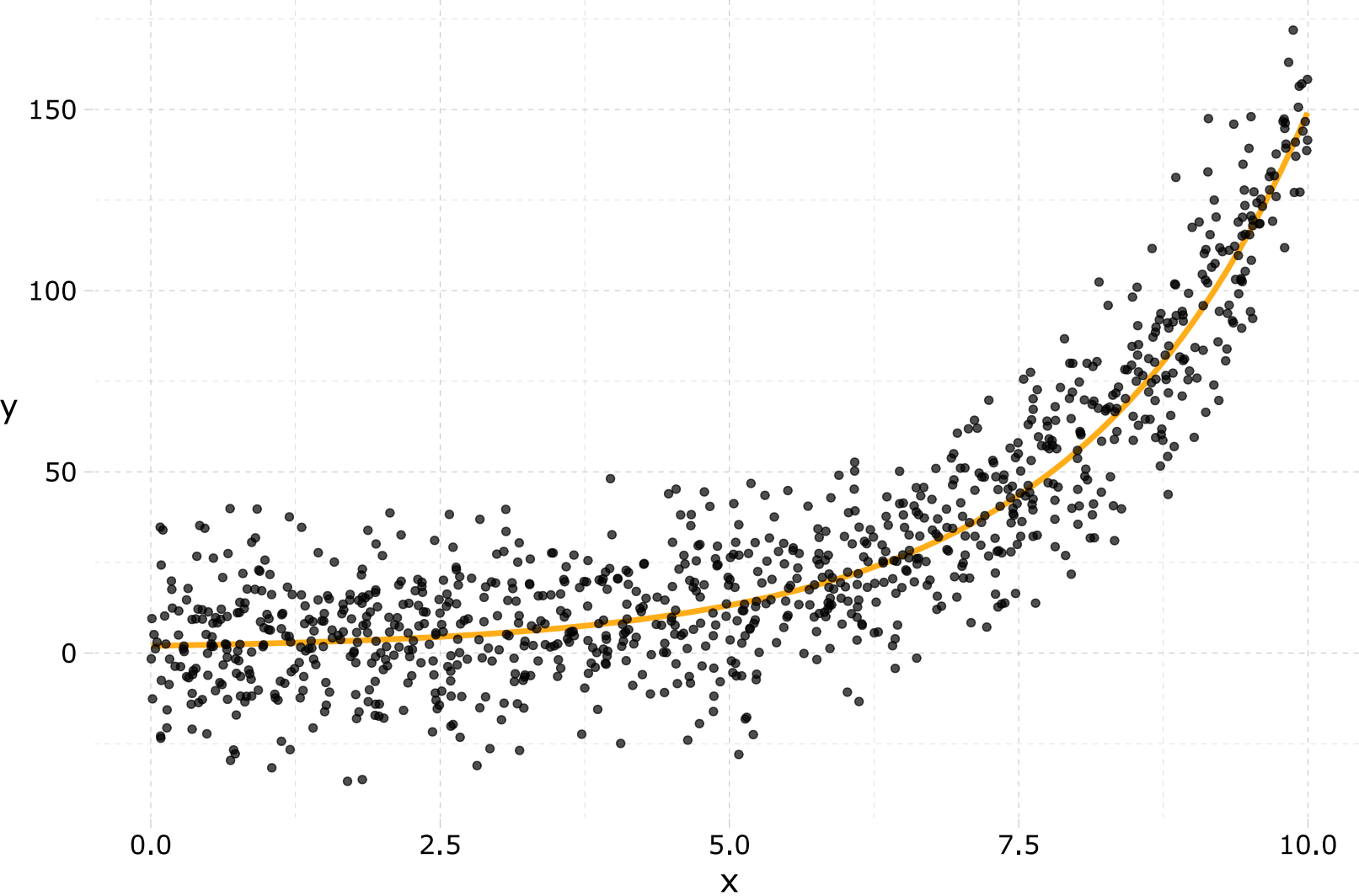
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)
```

```
#> # A tibble: 1,000 × 3
#>       ε      x      y
#>   <dbl> <dbl> <dbl>
#> 1  8.78  9.53 127.
#> 2 10.6   6.22 34.0
#> 3 -1.64  5.32 13.6
#> 4 -6.80  8.92 80.7
#> 5  9.09  1.96 12.8
#> 6 -27.3  8.84 57.0
#> 7  9.45  2.18 13.4
#> 8 -4.14  3.78  3.47
#> 9 -4.26  3.52  2.54
#> 10 -13.8  9.88 127.
#> # ... with 990 more rows
```

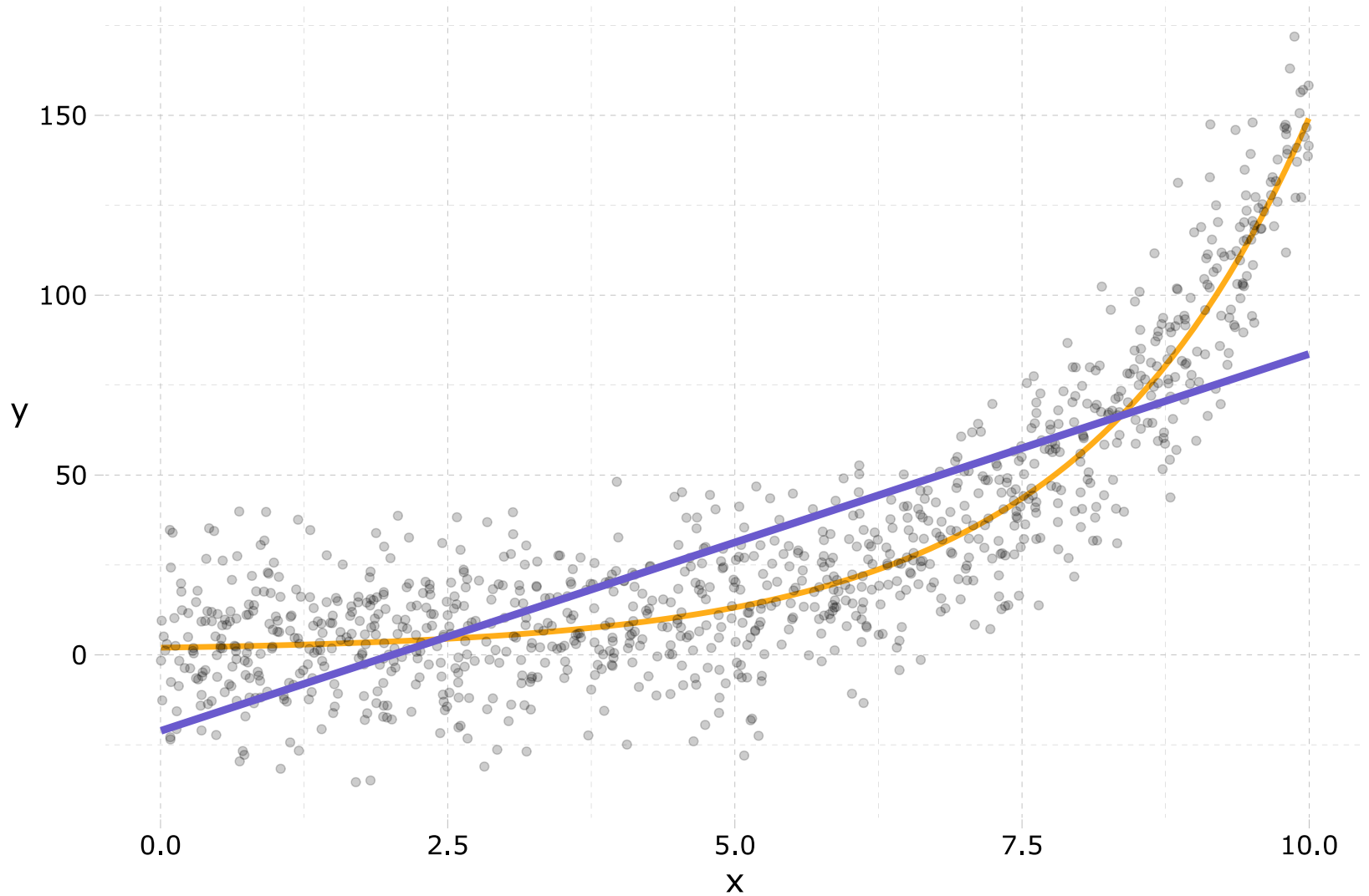
Our CEF



Our population



The population least-squares regression line



Simulation

Iterating

To make iterating easier, let's wrap our DGP in a function.

```
fun_iter = function(iter, n = 30) {  
  # Generate data  
  iter_df = tibble(  
    ε = rnorm(n, sd = 15),  
    x = runif(n, min = 0, max = 10),  
    y = 1 + exp(0.5 * x) + ε  
  )  
}
```

We still need to run a regression and draw some inferences.

Note We're defaulting to size-30 samples.

Simulation

We will use `lm_robust()` from the `estimatr` package for OLS and inference.[†]

- `se_type = "classical"` provides homoskedasticity-assuming SEs
- `se_type = "HC2"` provides heteroskedasticity-robust SEs

```
lm_robust(y ~ x, data = dgp_df, se_type = "classical") %>% tidy() %>% select(1:5)
```

```
#>           term estimate std.error statistic      p.value
#> 1 (Intercept) -21.14183  1.473496 -14.34807  1.383951e-42
#> 2             x  10.48074  0.257810  40.65294  6.560626e-214
```

```
lm_robust(y ~ x, data = dgp_df, se_type = "HC2") %>% tidy() %>% select(1:5)
```

```
#>           term estimate std.error statistic      p.value
#> 1 (Intercept) -21.14183  1.4335274 -14.74812  1.112039e-44
#> 2             x  10.48074  0.3097606  33.83495  8.788638e-168
```

[†] `lm()` works for "spherical" standard errors but cannot calculate het.-robust standard errors.

Simulation

Inference

Now add these estimators to our iteration function...

```
fun_iter = function(iter, n = 30) {  
  # Generate data  
  iter_df = tibble(  
    ε = rnorm(n, sd = 15),  
    x = runif(n, min = 0, max = 10),  
    y = 1 + exp(0.5 * x) + ε  
  )  
  # Estimate models  
  lm1 = lm_robust(y ~ x, data = iter_df, se_type = "classical")  
  lm2 = lm_robust(y ~ x, data = iter_df, se_type = "HC2")  
  # Stack and return results  
  bind_rows(tidy(lm1), tidy(lm2)) %>%  
    select(1:5) %>% filter(term == "x") %>%  
    mutate(se_type = c("classical", "HC2"), i = iter)  
}
```

Simulation

Run it

Now we need to actually run our `fun_iter()` function 10,000 times.

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There are a lot of ways to run a single function over a list/vector of values.

- `lapply()`, *e.g.*, `lapply(X = 1:3, FUN = sqrt)`
- `for()`, *e.g.*, `for (x in 1:3) sqrt(x)`
- `map()` from `purrr`, *e.g.*, `map(1:3, sqrt)`

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- `for()`, *e.g.*, `for (x in 1:3) sqrt(x)`
- `map()` from `purrr`, *e.g.*, `map(1:3, sqrt)`

We're going to go with `map()` from the `purrr` package because it easily parallelizes across platforms using the `furrr` package.

Simulation

Run it!

Run our function 10,000 times

```
# Packages  
p_load(purrr)  
# Set seed  
set.seed(12345)  
# Run 10,000 iterations  
sim_list = map(1:1e4, fun_iter)
```

Simulation

Run it!

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```

Parallelized 10,000 iterations

```
# Packages
p_load(purrr, furrr)
# Set options
set.seed(123)
# Tell R to parallelize
plan(multiprocess)
# Run 10,000 iterations
sim_list = future_map(
  1:1e4, fun_iter,
  .options = future_options(seed = T)
)
```


Simulation

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The `furrr` package (`future` + `purrr`) makes parallelization **easy and fun!** 😊

Simulation

Run it!!

Our `fun_iter()` function returns a `data.frame`, and `future_map()` returns a `list` (of the returned objects).

So `sim_list` is going to be a `list` of `data.frame` objects. We can bind them into one `data.frame` with `bind_rows()`.

```
# Bind list together  
sim_df = bind_rows(sim_list)
```

Simulation

Run it!!

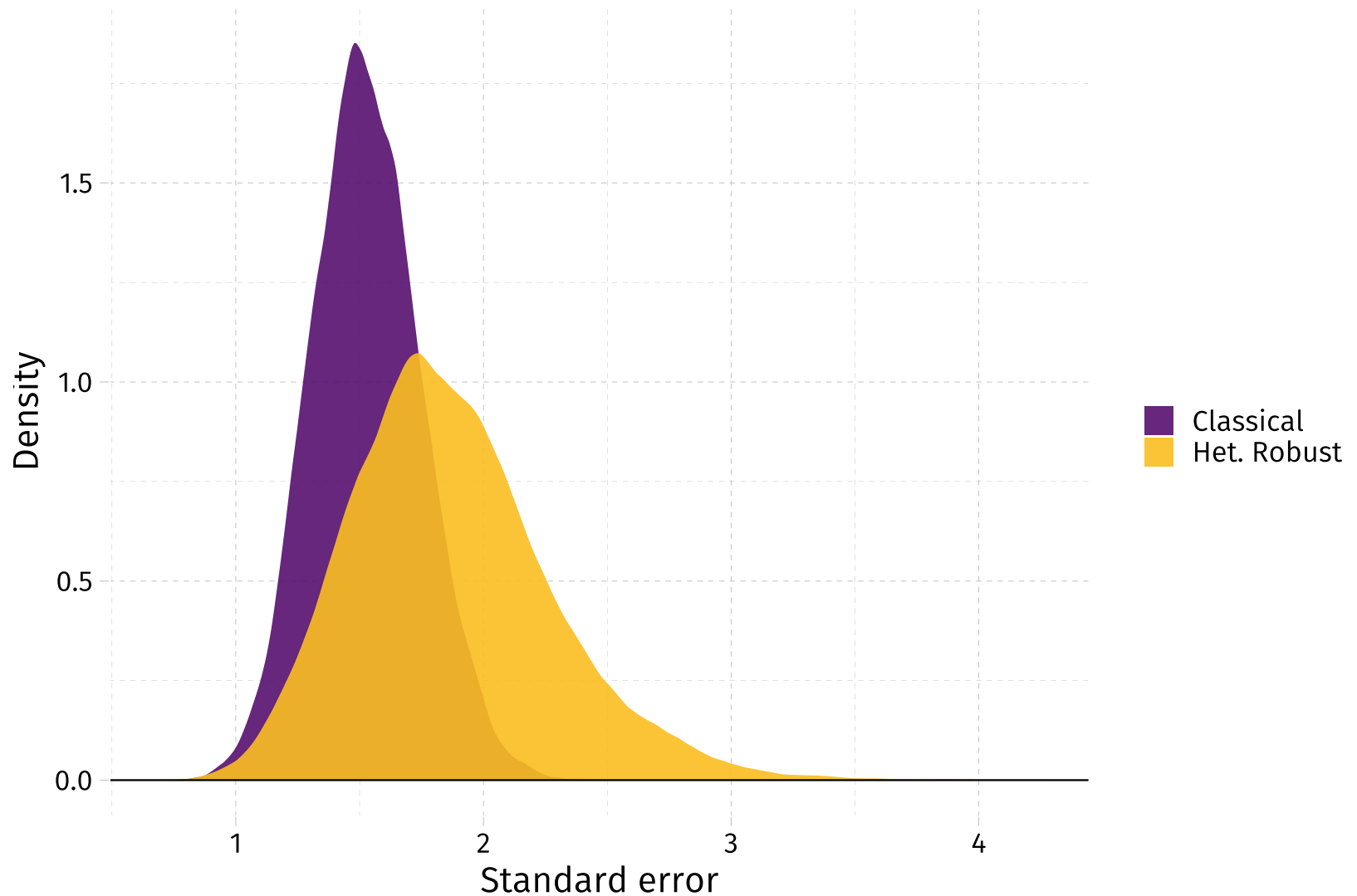
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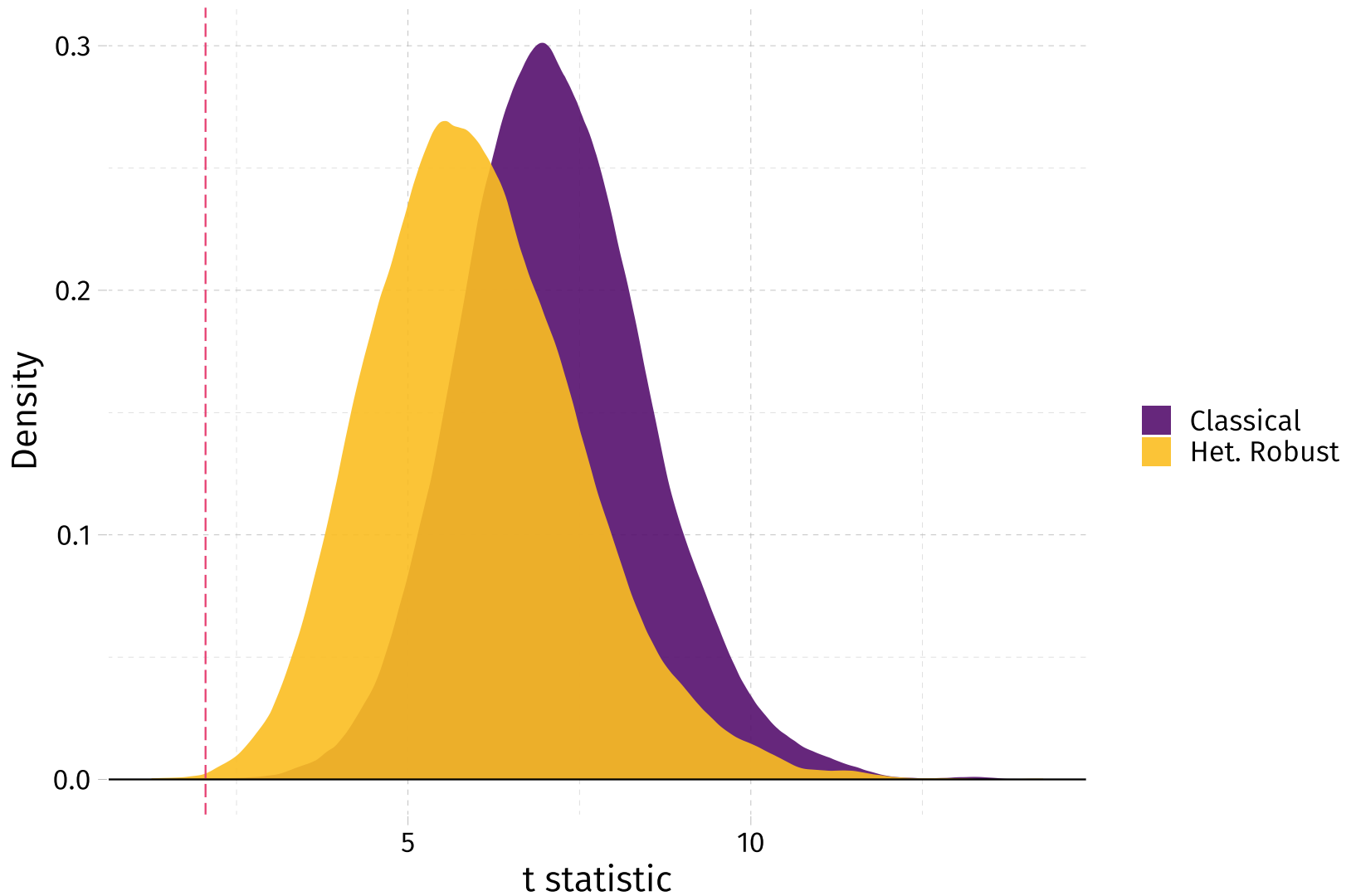
```
# Bind list together  
sim_df = bind_rows(sim_list)
```

So what are the results?

Comparing the distributions of standard errors for the coefficient on x



Comparing the distributions of t statistics for the coefficient on x



Q All of these test are for a false H_0 . How would the simulation change to enforce a *true* null hypothesis?

Simulation

Updating to enforce the null

Let's update our simulation function to take arguments γ and δ such that

$$Y_i = 1 + e^{\gamma X_i} + \varepsilon_i$$

where $\varepsilon_i \sim N(0, \sigma^2 X_i^\delta)$.

Simulation

Updating to enforce the null

Let's update our simulation function to take arguments γ and δ such that

$$Y_i = 1 + e^{\gamma X_i} + \varepsilon_i$$

where $\varepsilon_i \sim N(0, \sigma^2 X_i^\delta)$.

In other words,

- $\gamma = 0$ implies no relationship between Y_i and X_i .
- $\delta = 0$ implies homoskedasticity.

Simulation

Updating to enforce the null

Updating the function...

```
flex_iter = function(iter,  $\gamma = 0$ ,  $\delta = 1$ , n = 30) {  
  # Generate data  
  iter_df = tibble(  
    x = runif(n, min = 0, max = 10),  
     $\epsilon$  = rnorm(n, sd = 15 * x $\delta$ ),  
    y = 1 + exp( $\gamma$  * x) +  $\epsilon$   
  )  
  # Estimate models  
  lm1 = lm_robust(y ~ x, data = iter_df, se_type = "classical")  
  lm2 = lm_robust(y ~ x, data = iter_df, se_type = "HC2")  
  # Stack and return results  
  bind_rows(tidy(lm1), tidy(lm2)) %>%  
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    mutate(se_type = c("classical", "HC2"), i = iter)  
}
```

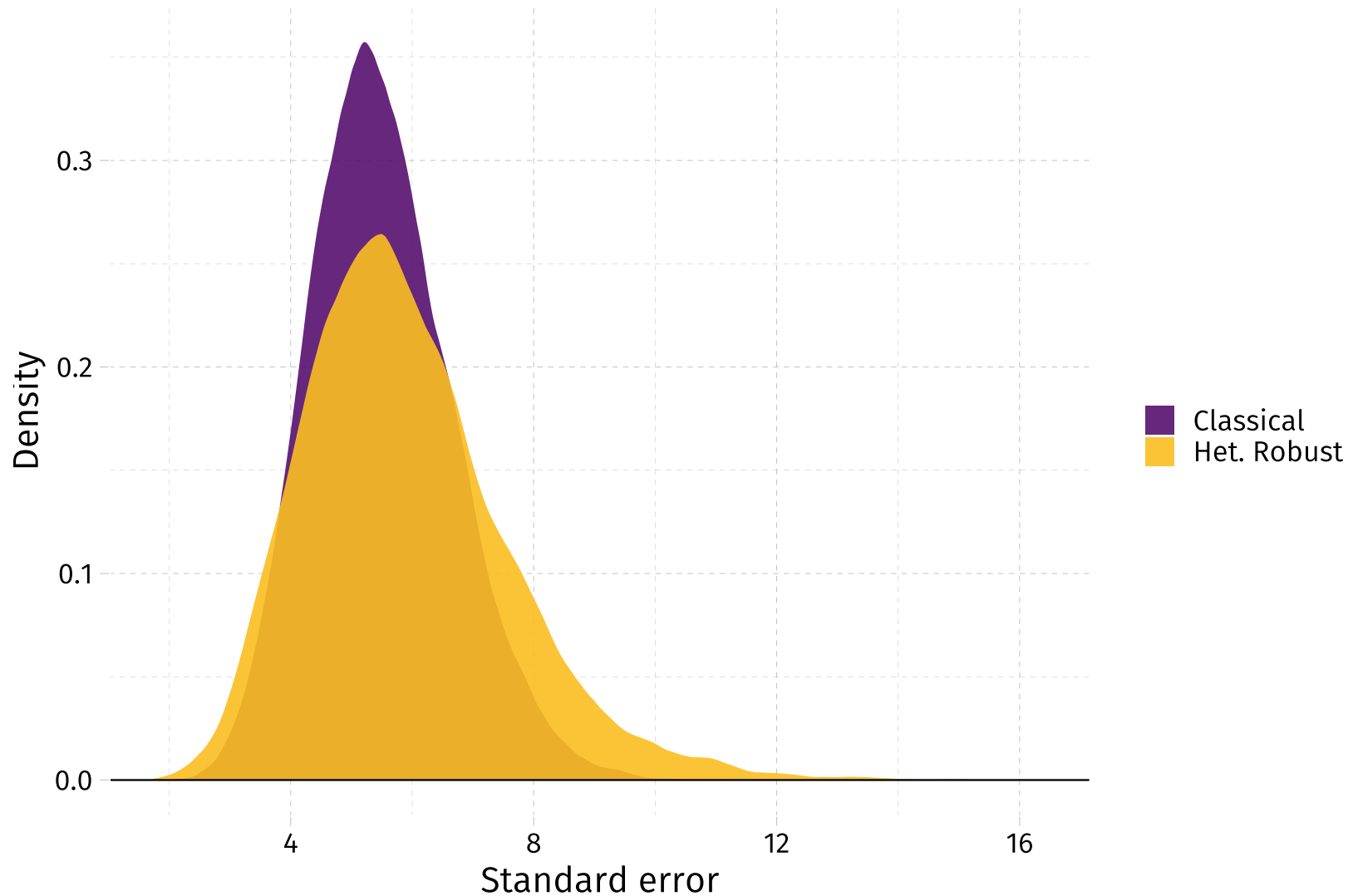
Simulation

Run again!

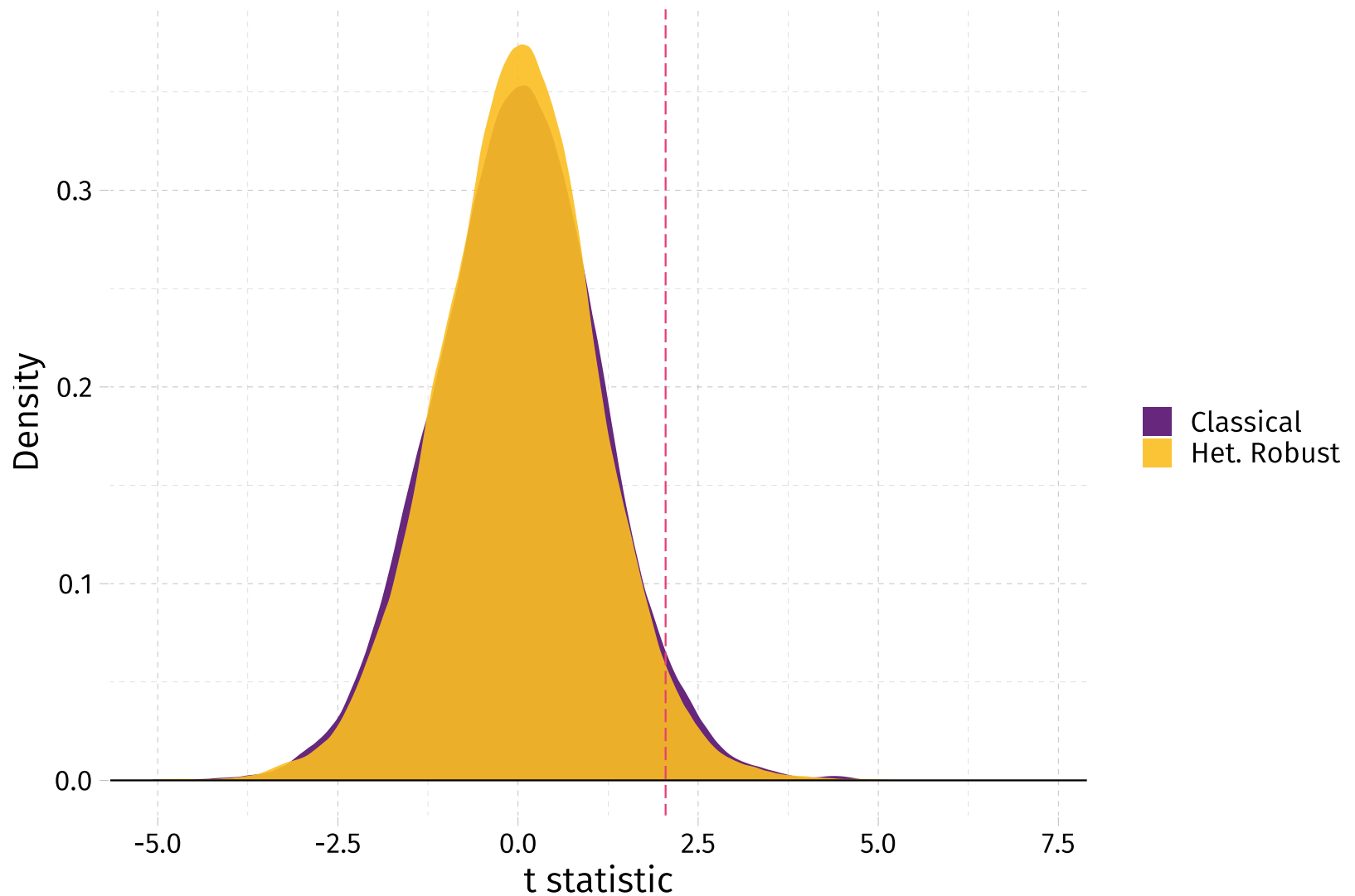
Now we run our new function `flex_iter()` 10,000 times

```
# Packages
p_load(purrr, furrr)
# Set options
set.seed(123)
# Tell R to parallelize
plan(multiprocess)
# Run 10,000 iterations
null_df = future_map(
  1:1e4, flex_iter,
  # Enforce the null hypothesis
   $\gamma = 0$ ,
  # Specify heteroskedasticity
   $\delta = 1$ ,
  .options = future_options(seed = T)
) %>% bind_rows()
```

Comparing the distributions of standard errors for the coefficient on x



Comparing the distributions of t statistics for the coefficient on x



Distributions of p -values: both methods slightly over-reject the (true) null

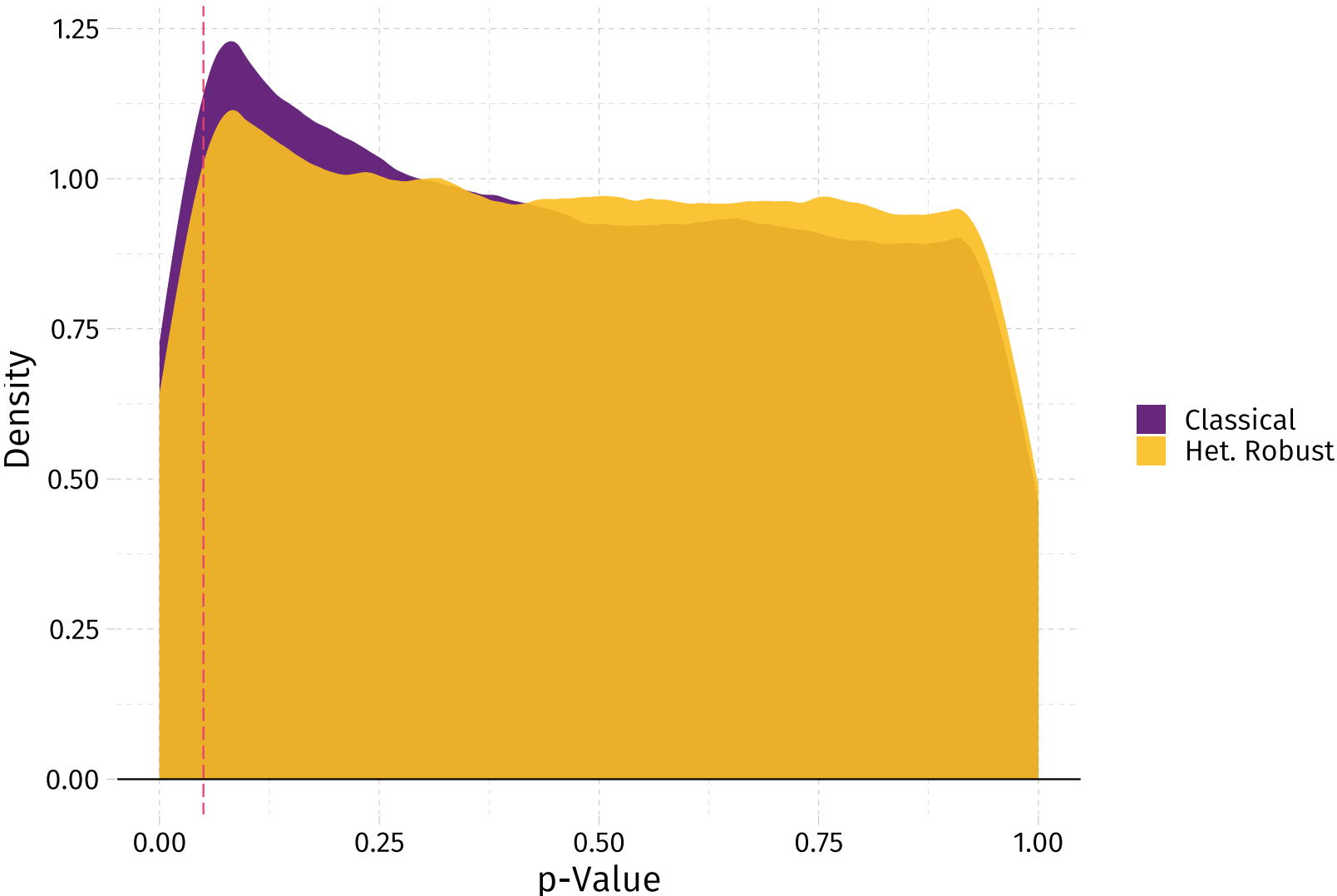


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