Russell, His Paradoxes, and Cantor's Theorem: Part I

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Abstract

In these articles, I describe Cantor's power-class theorem, as well as a number of logical and philosophical paradoxes that stem from it, many of which were discovered or considered (implicitly or explicitly) in Bertrand Russell's work. These include Russell's paradox of the class of all classes not members of themselves, as well as others involving properties, propositions, descriptive senses, class–intensions, and equivalence classes of coextensional properties. Part I focuses on Cantor's theorem, its proof, how it can be used to manufacture paradoxes, Frege's diagnosis of the core difficulty, and several broad categories of strategies for offering solutions to these paradoxes.

1. Introduction

In 1961, W. V. Quine described a philosophical method he dubbed 'The Ways of Paradox'. It begins with a seemingly well–reasoned argument leading to an apparently absurd conclusion. It continues with careful scrutiny of the reasoning involved. If we are ultimately unwilling to accept the conclusion as justified, the process may end with the conclusion that, 'some tacit and trusted pattern of reasoning must be made explicit and be henceforward avoided or revised' ('Ways' 11). This method perfectly describes Bertrand Russell's philosophical work, especially from 1901 to 1910, while composing *Principia Mathematica*.

Russell is most closely associated with the class–theoretic antinomy bearing his name: the class of all those classes that are not members of themselves would appear to be a member of itself if and only if it is not. This is one of a large collection of paradoxes Russell discovered or considered that shaped his subsequent philosophy. Many, if not most, stem from violations of Cantor's powerclass theorem, the result that every class must have more subclasses than members. Together, they led Russell to be increasingly wary, not only of implicit reasoning involving class existence, but also of the very practice of taking apparent reference to mathematical, abstract, or logically complex 'things' at face value.
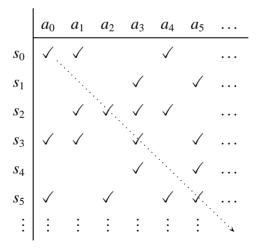
In this, the first article in a series of two, we discuss Cantor's powerclass theorem, and how it can be used to generate paradoxes. We then summarize a number of paradoxes thereby generated, either explicitly or implicitly considered by Russell himself. We conclude with a brief summary of the various kinds of solutions they might be given. In the sequel article, the impact of these paradoxes on Russell's own philosophy, and his views about their proper solution, are explored in more detail.

2. Cantor's Powerclass Theorem, Russell's Paradox and Frege's Lesson

Cantor's powerclass theorem, also known as the powerset theorem or just 'Cantor's theorem', is the widely-accepted result that every class or collection of things can be divided

into more subgroups or subclasses than it has members. The 'powerclass' of a class is the class of all its subclasses, so the theorem asserts that the powerclass of a class is always larger in size (cardinality) than the class itself. Georg Cantor established this result in 1891 with the following argument. Every class c has at least as many subclasses as members, since for each member a, the class of a alone is a subclass of c. The core of Cantor's argument involves showing that there cannot be *equally* many subclasses as members. Suppose, for *reductio ad absurdum*, that there were. In that case, the members and subclasses could be paired off so there would be a one–one function, f, mapping each subclass s of c to a distinct member of c, which we can call $f(s)$. Some members might be in the subclass they are mapped from, others not. If $s = \{a\}$, for a given subclass s and member a, and a happens to be the object $f(s)$ that s is mapped onto, then a is a member of its corresponding class, but not so if $a \neq f(s)$. Consider then the class, w, consisting precisely of the members of c that are *not* members of the subclasses that map to them. As w is itself a subset of c, it must be included in the mapping. Hence, there's some member r of c such that $f(w) = r$. Consider now whether or not $r \in w$. We defined w as the class of all members a of c that are not in the class s such that $a = f(s)$, so in the case of r, which is $f(w)$, $r \in w$ just in case $r \notin w$, which is a contradiction. Cantor concluded that there can be no such one–one function from subclasses of c to members, and, therefore, that there must be more subclasses.

If the class in question is finite or denumerable, Cantor's *reductio* reasoning can be represented in tabular form.[1] Arrange the members, $a_0, a_1, a_2, a_3, \ldots$ of c horizontally, and arrange the subclasses, $s_0, s_1, s_2, s_3 \ldots$, vertically utilizing the ordering of their corresponding members, so that $f(s_0) = a_0$, $f(s_1) = a_1$, etc. Place a checkmark where the chart intersects for the row of a subclass and the column for any member of it. For example:

	a_0	a_1	a_2	a_3	a_4	a_5	$\ldots$
s_0	✓	✓			✓		$\ldots$
s_1				✓		✓	$\ldots$
s_2		✓	✓	✓	✓		$\ldots$
s_3	✓	✓		✓		✓	$\ldots$
s_4				✓		✓	$\ldots$
s_5	✓		✓		✓	✓	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The problematic subclass, w, is generated by moving along the chart, diagonally, from the upper left, downwards and to the right, and including precisely those members of c that do *not* have checkmarks where the diagonal passes through their columns. In the example, this would include a_1 and a_4, in whose column no checkmark is found along the diagonal, but not the others. Notice, however, that as w itself should be represented by a

row, where the diagonal passes through it, there ought to be a checkmark just in case there is no checkmark, which is clearly impossible. Cantor's method of proof here is therefore called a *diagonal argument* or *diagonalization*.

This reasoning is validated within most forms of set theory, and is difficult to counter. However, it is not completely incontrovertible. In particular, the supposition that w corresponds to a well-defined subclass of c might be open to doubt, since it is defined in terms of a function whose domain is c's powerclass, and perhaps there is a vicious circle in this if w is to be included in that very range. More on this below.

Russell's initial reaction to Cantor's theorem was to regard it as guilty of error.[2] Cantor himself concluded from the theorem that there was no greatest cardinal number, since for any number of things, the number of their subclasses would be greater. Russell and others have regarded this as paradoxical (and indeed the problem here has sometimes been called 'Cantor's paradox' or 'the paradox of the greatest cardinal'). Certain classes – such as the universal class containing everything, or the class of all classes – it would seem, cannot have more subclasses than members, because all their subclasses *are* members. Indeed, at first blush, it scarcely seems possible that any collection could be larger in size (cardinality) than such huge classes as the universal class or class of all classes. Russell attempted to contravene the alleged impossibility of mapping each subclass of the class of all classes to a member by mapping each subclass to *itself*, i.e., letting $f(s)$ be s itself. Cantor's diagonal class w is then the class of all classes of classes not included in *themselves*. This class too is mapped to itself, and a contradiction results by asking if it is a member of itself. Drop the assumption that w need only contain those classes *of classes* that are not members of themselves, and this becomes Russell's paradox in its famous form. Russell was explicit in many places that Cantor's theorem was his inspiration.[3] Russell soon communicated it to Giuseppe Peano and Gottlob Frege, whose logical systems it rendered inconsistent.

Cantor's diagonalization method generalizes beyond mappings involving classes or sets. Given certain assumptions about the nature of properties (or predicates, attributes, universals, etc.), it establishes that the number of properties applicable (or not) to a certain logical kind of thing must always exceed the number of things of that kind. If we reconceive $a_0, a_1, a_2, a_3, \ldots$ in the chart as the things of the kind in question, and $s_0, s_1, s_2, s_3, \ldots$ as properties applicable to them, and view the checkmarks as indicating which things instantiate which properties, we are prompted to ask whether or not there is such a property as *not instantiating the corresponding property in the mapping*. If so, it should be included in the mapping, but then the object that corresponds to it instantiates it if and only if it does not. Here, however, the supposition that there must be such a property, merely because we seem able to describe its instantiation conditions, is even more open to doubt.

Gottlob Frege's reaction to the inconsistency in his logical system, published in an appendix to volume II of his *Grundgesetze*, usefully illustrates the matter. Rather than 'properties', Frege spoke of what he called 'concepts', understood as a kind of function from objects to truth-values. Thinking of these functions extensionally, Frege equated concepts satisfied by all and only the same objects.[4] Frege traced the presence of Russell's paradox in his system to his Basic Law V, which could be written:[5]

$$(\dot{\alpha}\, F(\alpha) = \dot{\alpha}\, G(\alpha)) \equiv (x)(F(x) \equiv G(x))$$

This states that the extension of the concept $F(\ )$, or class of all Fs, is identical to the extension of the concept $G(\ )$, or the class of all Gs, if and only if, all and only Fs are

Gs. Frege understood the notation '$\dot{\alpha}(\ldots\alpha\ldots)$' as representing a 'second-level function', a function that takes a concept as argument and returns an object as value. Originally, it was to stand for the function that takes a concept as argument, and returns as value its corresponding class, or extension. In the appendix, Frege argues that the left-to-right half of this biconditional must come out false, *regardless* of what second-level function '$\dot{\alpha}(\ldots\alpha\ldots)$' is taken to represent. While not explicitly presented as such, the reasoning is straightforwardly Cantorian. Rewrite 's_0, s_1, s_2, ...' in the chart above as '$F_0(\ )$, $F_1(\ )$, $F_2(\ )$, ...' for different concepts, and rewrite 'a_0, a_1, a_2, ...' as '$\dot{\alpha}F_0(\alpha)$, $\dot{\alpha}F_1(\alpha)$, $\dot{\alpha}F_2(\alpha)$, ...', and the connection becomes clear. If it were possible to map concepts to objects to yield distinct objects for distinct (i.e. non-coextensive) concepts, then, by what amounts to diagonalization, we could always produce a contradiction: just consider the concept of *being an object in this mapping that does not fall under the concept from which it's mapped*, i.e., the concept of being an x such that:[6]

$$(\exists F)(x = \dot{\alpha}F(\alpha) \mathbin{\&} \sim F(x))$$

The object that results by applying the function $\dot{\alpha}(\ldots\alpha\ldots)$ to *this* concept would be such as to fall under that concept if and only if it does not. This is equally true whether '$\dot{\alpha}(\ldots\alpha\ldots)$' is interpreted to yield the 'extension' of the concept to which it is applied, or whether it yields some *other* kind of object that would be different for different or non-coextensive concepts.

Let us call the result that it is impossible to generate a mapping from concepts, properties or (propositional) functions to objects that results in distinct objects for different or non-coextensive properties, 'Frege's lesson'. We might be tempted simply to take this lesson in stride, except that there are many cases in which it *seems* possible to generate a distinct object for distinct properties. In such cases, one must either react by explaining why the initial impression that a distinct object can be generated for each property was mistaken, or explain how this might be possible without diagonalization leading to contradiction.

3. A Plethora of Paradoxes

Russell felt the impact of Frege's lesson early.[7] In a Sept. 1902 letter to Frege, Russell despaired that 'from Cantor's proposition that any class contains more subclasses than objects we can elicit constantly new contradictions'.[8] It is worth listing several examples.

The first we have already discussed:

Intuition 1. Obviously for any two properties that are not coextensive it is possible to generate distinct objects: their extensions or corresponding classes!

Diagonalization result: Russell's class paradox. Consider the property an extension has just in case it does not have its defining property (or, equivalently, is not a member of itself). This property has its own distinct extension, but that extension has that property just in case it does not.

Many philosophers believe that properties or concepts can be considered objects or logical subjects in their own right. Those that do must ponder the following:[9]

Intuition 2. Obviously for any two properties (whether coextensive or not) it is possible to generate distinct objects, viz., the properties themselves.

Diagonalization result: Russell's predication paradox. Consider the property a property has just in case it does not instantiate itself. Does it, as an object, instantiate itself? It does just in case it does not.

Russell's early ontology included 'propositions' understood as mind–independent complex entities, the bearers of truth or falsity. Many other philosophers believe in similar intensional entities, though with widely varying details and vocabularies (e.g., thoughts, states-of-affairs, possible facts, belief-contents, etc.) Consider:

Intuition 3. There are as many propositions as there are properties thereof. For each property of propositions, one can generate a distinct proposition, such as the proposition that every proposition has that property, or the proposition that all propositions with that property are true.

Diagonalization result: the propositions paradox. Fasten on any one of these mappings: take the latter. Consider the property, ϕ, a proposition in this mapping has when it does not have the property of propositions of which it asserts all instances are true. For example, the proposition *all atomic propositions are true* is not itself an atomic proposition, so it has ϕ; whereas the proposition *all true propositions are true* is itself true, so it does not have ϕ. Consider then the proposition *all propositions with ϕ are true*: does it have ϕ? It does just in case it does not.[10]

Frege, famously, but also many other philosophers, including Russell prior to 'On Denoting', believe in special abstract 'semantic' objects: senses, meanings, individual concepts, denoting complexes, and so on. At least *some* (and perhaps all) of these entities are understood as picking out their 'referents' or 'denotations' in virtue of their unique possession of some property or other.

Intuition 4. There are as many descriptive senses as there are properties. For each property, we can generate a descriptive sense that picks out as denotation whatever object (if any) uniquely holds that property. While the object picked out may be the same for descriptive senses generated from distinct properties, the descriptive sense itself would be different for different properties.

Diagonalization result: the Russellian descriptive Sense Paradox. Consider the property, H, which a descriptive sense has when it lacks the property in virtue of which it presents a denotation, if any. The sense *the author of Waverly* did not write *Waverly*; hence, it has H. On the other hand, the sense *the self-identical thing* is a self-identical thing, so it lacks H. Now consider the sense *the H*; does it have H? It does just in case it does not.[11]

By a slight variation, we could consider an old fashioned 'intension' understood as a semantic entity that represents the entire class of things having a certain property, rather than just *the* thing having it, as above. Early Russell called these 'concepts of a class',[12] but I shall call them 'class-intensions' instead.

Intuition 5. There are as many class-intensions as there are properties. For each property, there is a class-intension that represents the class of things having that property. While the corresponding extension or class may be the same for different class-intensions, the class-intensions themselves are different for different properties.

Diagonalization result: The Russellian class-intension paradox. Consider the property, K, a class-intension has when it lacks the property it uses to collect together its class, if any. The class-intension *(all) teaspoons* is not a teaspoon; hence, it has K. The other hand, the class-intension *(all) class-intensions* is a class-intension, so it lacks K. Now consider the class intension *(all) class-intensions having K*; does it have K? It does just in case it does not.

(If class-intensions simply *are* properties, then this paradox collapses into the predication paradox above.)

We needn't necessarily generate a distinct intension for each property defining a class; it's enough to generate *one* corresponding to the equivalence class of coextensive properties, since it still would hold that we would get different ones for non-coextensive properties. Indeed, a version of the paradox could be formulated dealing with that equivalence class itself:

Intuition 6. We can map properties to equivalence classes of properties where the associated equivalence relation is coextensionality. For any two non-coextensive properties, the equivalence class to which they would be mapped would be different. The property of *having a heart* would be mapped to the same equivalence class as the property of *having a kidney*, but not to the same equivalence class as the property of *being a featherless biped*; though the latter would be mapped to the same equivalence class as *being human*.

Diagonalization result: the Russellian property equivalence class paradox. Consider the property ψ that an equivalence class of coextensive properties has just in case it doesn't have any (or, if you prefer, *all*, since they're coextensive) of the properties it contains. Now, consider the equivalence class of all properties coextensive with ψ: does it have ψ? If it does have ψ, then it doesn't have ψ, since it's one of the properties in the equivalence class. If it does not have ψ, then it must have at least one property coextensive with ψ, in which case, it must have ψ as well – so we get a contradiction either way.

Other examples could be given, but the above suffice to establish the general pattern of how Cantor's theorem, or, more specifically, 'Frege's lesson', generates Russell-style paradoxes almost *ad nauseam*.[13] One need only mention a category of entity – most likely, an abstract entity – that can be correlated or related to properties (or classes) in a systematic way[14] and where the identity conditions are fine-grained enough that the entities correlated with non-coextensive properties can be distinguished.

4. Kinds of Solutions

In a 1905 paper entitled 'On Some Difficulties in the Theory of Transfinite Numbers and Order Types', largely dedicated to Russell's paradox, Russell identified three broad approaches for finding a solution. It is fair to say that most contemporary approaches can still be seen as falling under one of these categories, though we shall discuss some exceptions below. Responses to the other Cantorian paradoxes can be sorted under roughly the same headings. The three categories, as Russell dubbed them, are (i) the theory of limitation of size, (ii) the 'zigzag theory', and (iii) the no classes theory. We discuss these in turn.

4.1. THEORIES OF LIMITATION OF SIZE

Consider, again, those classes that led Russell to suspect an error in Cantor's proof: the universal class, and the class of all classes. Cantor himself called such things 'inconsistent multiplicities' (in a letter to Dedekind) meaning that their size is too large for them to be considered 'one thing'. Axiomatic set theories now prevalent among mathematicians, such as Zermelo–Frænkel (ZF) set theory, also disavow the existence of a universal set, or set of all sets. This is keeping with an 'iterative' conception of a set, whereupon sets are thought to be built up out of successive applications of powerset and union operations.[15] More complicated theories, such as von Neumann–Bernays–Gödel (NBG) set theory, distinguish sets from 'proper classes', where the latter are considered too large to be members of any set or class. Here, although there may be a class of all sets, it is a proper class, and hence not a member of itself, nor are those subclasses which are also proper classes members. Insofar as it has a 'powerclass' at all, it would only contain *sets* that are subclasses of it, not all subclasses whatever. Even here, then, there is no class of *all* classes, both improper and proper.

These theories escape contradiction by denying that a distinct object can be generated for *every* property or characteristic of sets (or at least classes, in the case of NBG). Those

properties that are true of 'too many things' have no class (or no distinctive class) associated with them. There are only as many classes as there are properties that are not too widespread. This general line of avoiding inconsistency is perhaps clearer in the case of the 'Limitation of Size'–based set theory developed more recently by George Boolos ('Saving Frege' and elsewhere), formulated in a second-order logic, where Frege's Basic Law V is replaced with (New V):

$$(\grave{\alpha}\, F(\alpha) = \grave{\alpha}\, G(\alpha)) \equiv ((\mathit{Big}(F)\, \&\, \mathit{Big}(G)) \lor (x)(F(x) \equiv G(x)))$$

This commits us to as many classes (Boolos calls them 'subtensions') as there are non-big properties. Boolos defines a big property as one that is instantiated by as many things as there are things – though not necessarily by all things. Whether one adopts Boolos's proposal or a related one, the condition of not being a member of oneself, '$x \in x$', is thought to represent a property that holds of too many things to have a distinctive class associated with it, and hence, according to this approach, there is no such class as that which would be involved in Russell's class paradox. It also denies the existence of a universal class or class of all classes, thereby escaping the worries Russell initially entertained about Cantor's theorem.

The limitation–of–size approach has not been pursued much, or as directly, with regard to the other paradoxes listed in the previous section. Indeed, it is not entirely clear how to extend this approach to them in a plausible way. We shall return to this issue in the sequel article.

4.2. THE ZIGZAG THEORY

The previous approach requires that, contrary to our 'intuitions', it is untrue, after all, that we can generate a new object for *every* property (or every subclass) of our original group of things; for those that apply to too many things, there is no distinct associated object of the type suggested by the 'intuition'. The zigzag approach works differently. It grants the 'intuition' that for every property – or at least, for every *unexceptional* property – we can generate a distinct object. However, it denies that for every grammatically well–formed condition, we have the kind of unexceptional property to which the intuition correctly applies. In particular, the conditions used to specify the would-be diagonal subclasses or properties are thought to be exceptional or problematic in some sense, and that this undermines the diagonal reasoning behind Cantor's theorem.

Recall that Cantor's argument begins by assuming that a one–one mapping exists between subclasses of c and members of c, and then uses that very mapping to define *another* subset w of c which, it is alleged, cannot be included in the mapping. This is because nothing in the mapping *could* be *it*, given how w is described. The argument concludes the mapping does not exhaust the subsets of c. One might counter by questioning whether or not just any description of a subclass of c necessarily corresponds to a genuine subclass of c. In effect, one could exploit the impossibility of w's occurring in the map in question in the other direction, arguing that there can be no such subclass as w. The so-called subclass that would be generated from reversing the arrangement of checkmarks along the diagonal is no actual subclass at all, but merely an empty description to which nothing need answer. To provide a complete solution, one would need to specify conditions under which a description of a subclass (i.e., specification of conditions for inclusion in that subclass) of a given class can or cannot be guaranteed to define a subclass.

Apart from Russell's own experiments with this approach, which we leave for the sequel, and reconstructions thereof, the most thorough examination of an approach along these lines is perhaps Quine's system NF, which takes a form similar to naïve set theory, except that the class abstraction schema:

$$(\exists \gamma)\,(x)(x \in \gamma \equiv \ldots x \ldots)$$

rather than holding for all open sentences '... x ...' not containing 'γ' free, is only allowed for instances in which the open sentence '... x ...' has certain syntactic properties. In particular, it needs to be *stratified*, i.e., a function must exist assigning natural numbers to terms flanking the membership sign wherever it occurs in '... x ...' so that the number assigned to the term left of '$\in$' is one lower than that assigned to the term on the right. In NF, Cantor's theorem is unprovable (and indeed, demonstrably false for many instances), as the diagonally generated class w in Cantor's proof would be defined by an illegitimate formula. Instead, NF embraces classes that have all their subclasses as members, including a universal class and a class of all classes. Russell's class paradox is also blocked, as it too would be defined by an illegitimate formula, which is not surprising given that it can be thought of as generated by diagonalization.

Simply taken as a solution to Russell's class paradox, the overall strategy is neutral between an interpretation according to which the problematic diagonal condition does correspond to a 'property', albeit an 'exceptional one' with no corresponding subclass, and an interpretation according to which the condition, although it can be stated in a syntactically well-formed way, does not 'comprehend' a genuine property at all. For the general approach, however, to solve some of the other paradoxes mentioned in Sec. 3, particularly the predication paradox, something more like the latter interpretation seems more promising. (Note that for the other paradoxes, admitting the property but denying a well-defined subclass won't be enough, since some other entity, or even the property itself, is involved instead.) This interpretation could then allow that for every property, there is a distinct corresponding object (itself, or some proposition about it, or some descriptive sense involving it, etc.), but deny that there are such 'diagonal' properties as non-self-instantiation, H from the descriptive sense paradox, or ϕ from the propositions paradox. Again, to be fully plausible, the theory would need to explain under what conditions the specification of the exemplification conditions for a would-be property does or does not suffice to guarantee the existence of a property so delineated.

On the other interpretation, a property is admitted, but is regarded as exceptional in some sense, and therefore does not have a unique corresponding object. It is difficult to assess which of these interpretations is right for Quine, whose nominalistic tendencies steer him away from speaking in terms of 'properties' rather than linguistic formulas. Quine states the limitation on what 'conditions' define classes in the metalanguage, and in terms of syntactic features of the open sentence used to describe a class. For those who, unlike Quine, embrace second-order logic, the requirement could instead take the form of *object language* qualifiers for sorting out those properties that determine a corresponding object from those that do not. For a theory involving which properties define classes, this tack is compatible with certain neo-logicist set theories that adopt a genericized version of Boolos's (New V), in which talk of properties too 'big' to generate classes is replaced by more neutral talk about properties that are 'bad' or non-distinct-class-generating (cf. Shapiro 65):

$$(\dot{\alpha}\,F(\alpha) = \dot{\alpha}\,G(\alpha)) \equiv ((Bad(F) \;\&\; Bad(G)) \lor (x)(F(x) \equiv G(x)))$$

To count as a 'zigzag theory', 'badness' would need to be spelled out in terms of the

internal properties of a property rather than, e.g., the range of its applicability. Again, there has been very little by way of exploration of approaches along these lines as applied to other paradoxes.[16]

4.3. THE NO CLASSES THEORY

The third, and most radical, kind of solution to these paradoxes involves eschewing the kind of would–be entity that appears to violate Cantor's theorem altogether. Thinking of the classes paradox, Russell called this approach the 'no classes theory'. Here, one would deny that there are any such 'things' as classes, and suggest that discourse apparently about classes, to the extent that it is not meaningless or confused, is not to be taken at face value. Such discourse would be meaningful precisely to the extent that it is possible to reword it in a form in which no explicit mention of a class is made. For example, the claim that *the class of sedans is a subclass of the class of cars* can be reworded simply by saying that *all sedans are cars*. The solution to Russell's class paradox comes in insisting that *certain* kinds ot talk about classes cannot be so reworded. In particular, the claim that a class is a member of itself is to be regarded as meaningless, along with derivative expressions, such as that a class is not a member of itself. Hence the description used to define Russell's paradoxical class is not meaningful, and therefore does not determine a condition or property that defines a class.

For the other paradoxes, it would be more appropriate to speak of the 'no properties theory' or the 'no propositions theory', and so on. Of course, philosophers are likely familiar already with arguments showing that there is no such 'thing' as Redness, or (false) propositions such as *Jupiter is in my pocket*. To be fully plausible, however, these approaches must make sense of the apparent discourse about these entities that seems unquestionably true, such as the claim that, 'Euclid proved the proposition that there are infinitely many primes'. It also must ensure that the paraphrase given of such discourse is not by itself enough to generate the paradoxes. Again, the suggestion is likely that while some discourse about these apparent entities can be reworded in a form in which they are not mentioned, the discourse giving rise to the paradoxes cannot. Notice that it is not enough simply not to take the entities as *sui generis*. Replacing abstract 'propositions' in favor of, e.g., classes of synonymous sentences, does not help solve the paradoxes if enough such classes are posited to violate Cantor's theorem.

Nevertheless, it is approaches of this stripe that, by and large, Russell himself gravitated towards, especially from late 1905 and afterwards (after discovering his theory of descriptions). We shall take up his views in the sequel article.

4.4. LOGICAL TYPES

Another broad kind of approach, not listed by Russell in 'Some Difficulties', though, ironically, often attributed to him, posits logical *types* of things.[17] Strategies of this sort can be seen as attempting to maintain *modified or more sophisticated forms* of the 'intuitions' listed for the paradoxes, which, in the end, are found not to be inconsistent with Cantor's theorem. Maintaining that entities and the properties applicable to them fall into distinct logical types, and in keeping with the 'intuition' behind each paradox, one might suggest that for each property applicable to entities in a given category, it is possible to generate a distinct new entity, but insist that this new entity is in a separate logical category from the entities to which the original group of properties were applicable. Hence, any property applicable (or not) to these new entities is not among the original group.

For the classes paradox, for example, it amounts to dividing classes into type 1, or classes of individuals, type 2, or classes of classes of individuals, type 3, or classes of classes of classes of individuals, and so on, where it is not simply false but meaningless to ask whether $a \in b$ unless b is of a type one higher than a. Then, we are free to postulate a distinct class for every property applicable to individuals, but this class is not one of the entities to which that property may or may not apply. Diagonalization never gets off the ground, since the properties involved in the mapping are not such as to apply, or not apply, to the entities to which they're mapped, and so no system of 'checkmarks' (to recall our visualization earlier) is appropriate.

The approach has in common with the 'no classes' theory the suggestion that expressions of the form '$x \in x$' or '$x \notin y$' are not meaningful. In this, they contrast with the limitation of size and zig-zag theories insofar as the latter regard such constructions as at least grammatically well-formed, even if they do not define classes. Despite this similarity, their explanations for their meaninglessness differ. In the 'no classes' theory, no sentence of the form '$a \in b$' is to be taken as about some entity of any type denoted by 'b'; instead, the entire sentence as a whole must be reworded into a form in which no class is mentioned, which is deemed impossible in this instance. In the kind of theory mentioned here, 'b' is an independently meaningful expression; it simply differs in what *kind* of thing it means from 'a', and it engenders nonsense to attempt to say the same things of a one would say of b.

Frege's theory of 'levels' of concepts, according to which there are objects or 'saturated entities', first-level concepts (under which objects may or may not fall), second-level concepts (within which first-level concepts may or may not fall), etc. could be used to provide a response of this stripe to Russell's predication paradox.[18] For each first-level concept (i.e., property), or concept applicable to objects, there is indeed an entity, that concept itself; but that concept is not itself an object, and the question as to whether it applies to itself is meaningless; one can only ask whether or not second-level concepts are applicable to it.

Addressing the other paradoxes with this kind of strategy would involve, for example, arguing that although a distinct proposition can be derived for each property, the resulting proposition is not of the right sort either to have or not to have that property. But this is precisely what the definition of ϕ from the propositions paradox assumes, and hence, it is poorly defined. Similarly, while distinct descriptive senses might be generated from all properties, they would not be the kind of thing to which such properties may or may not apply.

4.5. OTHER APPROACHES

Lastly, there are kinds of solutions that fall into none of the above categories. These include radical approaches as might be taken by a dialethist who simply embraces the contradictions as true, while trying to insulate their harmful effects by means of a non-explosive paraconsistent logic. Such approaches raise other philosophical issues we cannot fully explore here.

Another less radical approach, however, might stem from noting that Cantor's theorem or Frege's lesson is not automatically violated by just any function that maps properties of things to things. If the same object may result as value for non-coextensive properties as argument, then the function doesn't postulate as many objects as subclasses. One might then hope to maintain the spirit of the 'intuitions' lying behind the paradoxes, but without the supposition that the entity generated is always 'distinct'.

Notice, however, that it is not enough to allow that sometimes different properties may generate the same entity in the mapping; one must allow that sometimes non-coextensive properties may generate the same entity. With regard to classes or extensions, pushing this line of response is more or less tantamount to arguing that non-coextensive properties may have the same extension, a proposal which sounds absurd on its face. Nevertheless, Frege himself endorsed such a proposal in his appendix on Russell's paradox, and Russell himself was for a time attracted to it.[19] However, without philosophical support provided by an independent theory,[20] the supposition that non-coextensive properties may determine the same class, or, in the case of the Property Equivalence Class paradox, be included in the same equivalence class of coextensive relations seems bewildering. The situation is even worse with the paradoxes involving intensional entities. Intensions are supposed to be *finer-grained* in their identity conditions than extensions, yet to solve the descriptive sense and class-intension paradoxes we'd have to allow that *the F* and *the G* be identical *senses* in some cases even when *F* and *G* aren't even coextensive, or that two distinct classes may be generated by the same class-intension.[21] There does not seem to be much to be said in favor of such approaches.[22]

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Short Biography

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Notes

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[1] Even when the class isn't denumerable, it is often worthwhile to imagine imperfectly the resulting chart abstractly anyway, just as a heuristic. The fact that such a chart isn't technically possible doesn't invalidate the core argumentative strategy.

[2] See Russell, *Papers* v3, xxxii, and 'Mathematics and the Metaphysicians'; for discussion of the historical details, see Coffa, 'Humble Origins', and Griffin, 'Prehistory'.

[3] See e.g. *PoM* 101, *IMP* 136, *MPD* 158, *Auto* 150.

[4] Frege would invoke his sense/reference distinction to explain away apparent problems with equating coextensive concepts; see his 'Comments on Sense and Reference'.

[5] Here we allow ourselves to Russellize Frege's notation somewhat, and restrict our focus to concepts as opposed to other functions.

[6] Notice that if x is a class, and '$\dot{\alpha} F(\alpha)$' is interpreted to mean 'the class of Fs' then this precisely gives the condition for x's not being a member of itself.

[7] Indeed, he was aware of the main gist prior to reading Frege's appendix – see *PoM* 103, and his letter to Frege of 24 July 1902, in Frege, *Philosophical and Mathematical Correspondence* 139.

[8] See Frege, *Correspondence* 147. Unfortunately, Frege does not seem to have fully appreciated the importance of the paradox Russell went on to describe, and that it threatened his philosophy as much as it did Russell's; see Klement, 'Russell's Paradox in Appendix B', and *Frege and the Logic of Sense and Reference*, Chap. 6.

[9] Depending on what we interpret 'properties' here to mean, Russell discusses different interpretations of this paradox in different places. Interpreted to mean what early Russell called 'predicates', by which he meant something

like Platonic universals, it occurs in *PoM* (80, 102). Interpreted to mean what he called 'propositional functions' it occurred only later. See Klement, 'Origins' for discussion of the difference.

[10] A version of this paradox, dealing simply with classes of propositions rather than properties of propositions was formulated by Russell in *PoM* (527–28). Russell also formulated it in terms of 'propositional functions' instead of classes of propositions in a letter to Frege (see Frege, *Correspondence* 159–60). Antinomies of this form were independently rediscovered by John Myhill in the context of evaluating certain later forms of intensional logic; see, e.g., Myhill, 'Problems', Anderson, 'Semantic Antinomies', and Klement, 'The Number of Senses', and 'Does Frege Have Too Many Thoughts?' It is sometimes called 'the Russell-Myhill Antinomy'.

[11] A paradox of this form is discussed at greater length in Klement, 'Cantorian Argument'.

[12] In *PoM* (67), Russell distinguishes the 'concept of a class', *all humans*, from the 'class-concept' *human*. The difference is subtle, and we could generate a Cantorian paradox from either one, though I think that the 'class-concept' is really just the property itself, and so the resulting paradox is just the predication paradox.

[13] There are, to be sure, other important paradoxes in the same family that don't fit quite as well into the rubric provided by our previous discussion, such as the paradox of relations Russell discusses in 'Mathematical Logic' (222–23), the paradox Kaplan discusses in "A Problem in Possible World Semantics," or what is called the 'class/sense paradox' in Klement, 'The Number of Senses'. The puzzle called 'Cantor's paradox' (see Sec. 2) concerning whether or not there is a greatest cardinal number is of course another paradox related to Cantor's theorem that doesn't neatly fit this rubric.

[14] Above, we often say 'generated from' but this metaphor should not be taken too seriously.

[15] This way of describing things derives from Boolos, 'The Iterative Conception'.

[16] Though see Cocchiarella, 'Russell's Paradox of the Totality of Propositions', and Cantini, 'On a Russellian Paradox', for exceptions.

[17] As I argue in the sequel article, this attribution is contentious at best.

[18] Although, actually, I think this description of the situation is somewhat misleading, given that, for Frege, the extension of a concept 'has its being' in the concept itself, and various other suggestions in his work to the effect that the extension of a concept simply *is* the concept treated as a logical subject. From this perspective, the class paradox and predication paradox are indistinguishable for Frege, which is why, I think, Frege describes Russell's description of the predication paradox in his letter to him only as 'imprecise' rather than erroneous.

[19] This is evinced by the last minute footnote to *PoM* (p. 496), in which Russell calls it 'very likely the correct solution', as well as in manuscripts of the period (*Papers* v4, 17–37). Indeed, Russell even seems to have hoped that it might work in other cases too, mentioning it in connection with the propositions paradox in particular to Frege in a letter; see Frege's *Correspondence* 159–60. Frege's own proposed 'solution' was later found to lead to more complicated contradictions. See Quine, 'Frege's Way Out' and Landini, 'Ins and Outs' for further discussion.

[20] Notice that Boolos's (New V) system is technically a theory allowing non-coextensve 'big' properties to have the same 'class', although the explanation of this makes use of a different sort of theory, and talk of 'extensions' replaced with talk of 'subtensions'.

[21] For further discussion of related issues, see Klement, 'A Cantorian Argument' 73.

[22] Merely bearing the possibility of such 'solutions' in mind, however, forces us to be rather more careful about the precise formulation of the paradoxes. Nicholas Denyer deflected what amounts to a Fregean version of the propositions paradox formulated by Adam Rieger by pointing out that Rieger's mapping didn't necessarily generate a distinct proposition from each property; however, a fairly insignificant modification to Rieger's proposal is immune to like treatment; see Klement, 'Too Many'.

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Russell, His Paradoxes, and Cantor's Theorem: Part II

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Abstract

Sequel to Part I. In these articles, I describe Cantor's power-class theorem, as well as a number of logical and philosophical paradoxes that stem from it, many of which were discovered or considered (implicitly or explicitly) in Bertrand Russell's work. These include Russell's paradox of the class of all classes not members of themselves, as well as others involving properties, propositions, descriptive senses, class-intensions and equivalence classes of coextensional properties. Part II addresses Russell's own various attempts to solve these paradoxes, including strategies that he considered and rejected (limitation of size, the zigzag theory, etc.), as well as his own final views whereupon many purported entities that, if reified, lead to these contradictions, must not be genuine entities, but 'logical fictions' or 'logical constructions' instead.

1. Introduction

This article is a sequel to 'Russell, His Paradoxes, and Cantor's Theorem: Part I', in which various Cantorian diagonal paradoxes either discovered or considered by Bertrand Russell were outlined. These include Russell's famous class paradox involving the class of all classes not members of themselves, his predication paradox involving the property of non-self-instantiation, as well as similar paradoxes involving propositions, descriptive senses, class-intensions and equivalence classes of coextensive properties. Four different lines of solutions considered by Russell were also discussed: (i) the theory of limitation of size, (ii) the zigzag theory, (iii) logical types of things, and (iv) the 'no classes (etc.)' theory. (In what follows, it is assumed that the reader has read Part I.) In this sequel, we examine in further detail the impact of these paradoxes on Russell's own philosophy, his consideration of possible solutions of all these kinds, and his reasons for, in the end, moving towards a rejection of robust metaphysical realism about many kinds of abstract objects in favor of viewing them as 'logical fictions', or mere *façons de parler*, so that such abstract things need not be taken as included among the ultimate furniture of reality.

2. Russell's Rejected Solutions

Russell spent the bulk of his intellectual energy between 1902 and 1908 trying to find responses to paradoxes such as these that both seemed philosophically sound, and allowed his project in the foundations of mathematics to escape inconsistency and paralysis. He changed tack often, and his manuscripts from the period are filled with half-starts, rejected proposals and admissions of uncertainty.[1] It is worth discussing the major approaches he considered but rejected at least briefly.

2.1. THEORIES OF LIMITATION OF SIZE

Surprisingly, perhaps, given their prevalence today, Russell gave the least consideration to limitation of size approaches. The general suggestion for the class paradox in particular is that when a certain property or condition holds of too many things, those things do not form a class or extension as a single object, even though those things instantiating a less popular property may form such a unit. Russell had many reasons for not finding this approach very attractive. For one, it does not tell us how big is too big. Certain proposals have been made, but none without drawbacks. Some might draw the line at an infinite collection, but this cripples the use of classes in mathematics. Others might draw the line at the size of the entire universe, per Boolos's (New V), but without knowing the size of the universe, we are still left in the dark about what classes we can assume to exist. Russell insisted upon an independently *philosophically well-motivated explanation* of this.

Moreover, the conceptions of classes or sets that lend themselves to this sort of view are out of sorts with Russell's own interests. 'Sets' in ZF and similar theories, understood as structures built up iteratively by union and powerset operations beginning with the empty set, seem almost unrelated to those understood in the Boolean/Whiteheadian logical tradition as involved in all categorical judgments: the objects denoted by such phrases as 'all humans', etc. (*PoM* 67). Russell understood by a 'class' the extension of a concept, or the collection we're talking about when we make a claim about those things, all of which share a common property, but in which the truth of the content asserted depends only on the makeup of that collection. From this perspective, the supposition that classes only sometimes exist, depending on their would-be size, is tantamount to suggesting that there's a radical difference in kind between talk of 'all humans' (which are finite in number, and so not too many to form a class) and talk of 'all numbers' (which are too many), which seems implausible. Russell had assumed that, on this understanding, if there are classes at all, there should be such 'large' classes as the universal class or class of classes, which would be involved in any discourse about 'all classes' or 'all things'. Recall (from Part I) that it is such collections that initially led Russell to be skeptical about Cantor's theorem.

Another reason Russell was likely unattracted overall to this approach is that its prospects are dim for solvng the *other* Cantorian paradoxes he considered. The analogous suggestion for the predication paradox would be that a property constitutes an object or logical subject, and hence can be predicated truly or falsely of itself, only when it doesn't apply to too many (other) things. Yet it seems very arbitrary to think that there should be such an object as *Humanity*, but not such an object as *Number* (Numberhood?), and it's hard to imagine a philosophical theory that could support such a position. Indeed, even those who are attracted to a 'limitation of size' solution to Russell's class paradox usually opt for a solution of a different stripe for the predication paradox, or fail to address it at all. Similar comments apply with regard to the others. Consider the propositions paradox. The 'limitation of size' approach suggests that there is a proposition of the form *all propositions with property F are true* when, but only when, the property *F* does not apply to too many things.[2] Yet it would seem bizarre to hold that, e.g., the proposition *all propositions expressed explicitly in this paper are true* exists but not *all propositions entailed by things expressed in this paper are true*, simply because I explicitly express only a finite number of propositions, but they entail countless more. This paradox is rarely discussed, but it seems difficult to imagine that even those endorsing the limitation of size approach for the class paradox would think a similar solution would apply here. Yet, Russell himself

was convinced that, as he put it, 'the close analogy' between the paradoxes 'strongly suggests that the two must have the same solution, or at least very similar solutions' (*PoM* 527).[3]

2.2. THE ZIGZAG THEORY

As an approach to the classes paradox, the zigzag theory, rather than claiming that properties that are true of too many things don't make-up a class, instead claims that certain conditions don't define a corresponding class when they have certain intrinsic features. This is a tack Russell considered, on and off, throughout 1902–1905.[4] His attraction to the view no doubt stemmed from the possibility that it might vindicate his initial doubts about Cantor's theorem and allow for such un-Cantorian (large) collections as the universal class or class of all classes.

His early ontology included 'propositional functions' (ontological correlates of open sentences), themselves capable of occurring as logical subjects in a proposition, and his logic included quantifiers ranging over such propositional functions. Russell's zigzag approach could then make use *object language* qualifiers for sorting out those propositional functions that do determine a class ('simple' or 'predicative' functions) from those that do not ('impredicative' or 'quadratic' functions). Russell might write:[5]

$$\sim\mathrm{Quad}(\phi) \supset (\psi)[\hat{x}(\phi x) = \hat{x}(\psi x) \equiv (x)(\phi x \equiv \psi x)]$$

In this regard, Russell's approach can perhaps be better likened not to Quine's broadly 'zigzag' NF, but rather to the neo-logicist set theories such as those discussed by Shapiro (mentioned in Part I).[6] Indeed, if Russell's talk of 'impredicative' or 'quadratic' propositional functions is taken as synonymous with their notion of 'badness', Russell's principle above is equivalent to the neo-logicist genericized form of Boolos's (New V).

Rather than attempting to state a single criterion for badness or impredicativity, however, Russell's approach was more piecemeal. He proposed axioms to the effect that certain simple propositional functions were predicative (or 'good') along with certain principles to the effect that certain transformations or combinations of predicative functions must yield new predicative functions. However, it appears that Russell was unable in this way to characterize a notion of predicativity that both excluded all the cases generating paradoxes and also preserved the mathematical reasoning he needed for his logicist project. As he tried out various approaches, the axioms he found it necessary to assume grew, in his own words, 'horribly complicated and unobvious' (*DRDJ* 79), and removed from any philosophical insight into the relationship between properties and classes. Sensing a dead end, he abandoned the approach.

2.3. METAPHYSICAL TYPES

Russell, of course, did eventually endorse a kind of type theory. Although he is often read as doing so, and the matter is still highly controversial, my own interpretation is that Russell's solution to the logical paradoxes in *PM* did not involve postulating different metaphysical *kinds of entities*.[7] He did, however, flirt with approaches of this sort along the way.

Russell's deepest exploration of an approach to the classes paradox along these lines was perhaps the theory of types proferred, rather tentatively, in 1902, in Appendix B of *PoM*. The theory there differentiated between individuals, 'classes as many' of individuals, 'classes

as many' of classes as many, and so on. Even here, describing his theory as dividing entities into distinct ontological types is rather misleading given that Russell did not regard a 'class as many' as a single entity, but rather, as the name implies, a plurality of distinct entities (*DRDJ* 78). Nevertheless, the theory held that different grammatical types of expressions corresponded to different logical categories of semantic values, and that it was meaningless to place one type of expression where the other ought to go. The approach was short-lived; indeed, Russell seems to have abandoned it by the time *PoM* even appeared in print.

In general, Russell's philosophical scruples committed him throughout this period to embracing a single logical type, that of 'individuals' or 'logical subjects', encompassing all entities. His argument was that it is always inconsistent to hold that any sort of entity cannot be a logical subject of a proposition, since to do so is tantamount to endorsing a proposition of the form *A is not a logical subject*, whose very form ensures its own falsity (*PoM* 45–48).[8] Indeed, even when Russell was willing to consider metaphysical types, he held that there must be combined types, including a type subsuming all objects; a concession making this early type-theory much more complicated than usual textbook formulations (e.g., that of Hatcher), and one that might even be utilized to reintroduce the paradoxes.[9] As Russell himself admitted while offering his 1903 theory of types, '[t]he fact that a word can be framed with a wider meaning than *term* [individual, logical subject] raises grave logical problems' (*PoM* 55n).

Moreover, Russell did not at the time find the general line of approach suitable as a response to the other paradoxes, explicitly listing the propositions paradox as one it cannot solve. He noted in passing that it might be possible to meet the propositions paradox with the response that '[i]t is possible, of course, to hold that propositions themselves are of various types … But this suggestion seems harsh and highly artificial' (*PoM* 528). Nevertheless, in 1908's 'Mathematical Logic as Based on the Theory of Types', after Russell had already taken up a different sort of line of response for the class paradox, he did consider a view according to which propositions needed to be sorted into various ramified orders. Properties of propositions would similarly need to be relativized to orders based on to what order of propositions they would apply to. A first-order proposition cannot involve quantification over propositions or properties at all, a second–order proposition would involve quantification over (at most) first-order propositions (or properties), and so on. While it might then be possible to define a new proposition for each property of propositions, the order of the proposition so defined would be too high to ask whether or not it has the property it involves, which blocks the diagonalization leading to the Cantorian propositions paradox.[10] Here too, again, however, Russell's attraction to the approach was short-lived. Almost precisely around the time he wrote 'Mathematical Logic', Russell began to explore the possibility that there may be no propositions at all, apparently finding that approach more promising than the suggestion that there could be a whole hierarchy of different orders of them.[11]

3. The Retreat from Pythagoras

In *My Philosophical Development*, written in the late 1950s, Russell described the evolution of his philosophy as a 'retreat from Pythagoras'. Whereas he had once believed in a wide assortment of logical, mathematical, intensional and other abstract objects – including properties, classes, propositions, numbers, truth-functions, propositional functions, denoting concepts, etc. – as his views developed, he gradually moved towards a position stressing a 'robust sense of reality' (*IMP* 135), in which only the simplest raw material of the empirical world was considered ultimately real, and which 'swept away many apparent

entities … [to] result [in] an outlook, which is less Platonic, or least realist in the mediæval sense of the word' (*PoM* 2nd ed., xiv): the abstracta he had formerly believed in reduced to 'logical fictions' or mere *façons de parler*. This retreat was by no means overnight, and had its origins even earlier in Russell's work, but was pushed on to a large extent by the desire to find uniform resolutions to the Cantorian paradoxes.

3.1. THE PREDICATION PARADOX

The core of the position is evident already with Russell's first proposed solution to one form of the properties or predication paradox.[12] In 1903, Russell used the word 'predicate' not for anything linguistic, but for the ontological correlate of an adjective phrase, understood as a Platonic universal. He first stated, and proposed to solve, a version of the paradox involving predicates, thusly:

> If *x* be a predicate, *x* may or may not be predicable of itself. Let us assume that "not-predicable of itself" is a predicate. Then, to suppose either that this predicate is, or that it is not, predicable of itself, is self-contradictory. The conclusion, in this case, seems obvious: "not-predicable of oneself" is not a predicate. (*PoM* 102)

Russell's solution is neither to rescind the claim that predicates are individuals or logical subjects in propositions,[13] nor to hold that in general that there are no propositions, even true propositions, to the effect that a certain definite predicate is not predicable of itself, e.g.:

(1)   Redness is not red.

Instead, he adopts a kind of ontological conservatism: not everything that appears as a well-formed, independently meaningful adjectival phrase actually corresponds to a single entity. We might rephrase (1) thusly:

(2)   Redness is not predicable of itself.

Or even:

(3)   Redness has the property of not being predicable of itself.

Russell concluded that the predicate phrases in these last two formulations, unlike in the first, do not represent any single entity.

Russell is not denying that the *parts* of the complex adjectival phrase 'not predicable of itself', like 'not' or 'itself', have entities that they represent. There just isn't any *one* object meant by the *entire phrase as a whole*. It may be a unit grammatically and syntactically, but according to Russell, this is no guarantee that it corresponds to a single, identifiable constituent of the corresponding proposition or fact. The make-up of the latter, one might say, is intelligible given only the 'atoms' of meaning: the truth of (2) and (3) only need the entities involved in the truth of (1) – Redness (twice over), negation, and whatever corresponds to the copula – and not some additional complex property of non-self-instantiation. Without taking that property as a single, unified 'thing' of which some properties hold and others don't in its own right, the paradox cannot get off the ground. Although this solution is not quite as radical as some of the others discussed below, since Russell is not offering a wholescale 'no properties' view, but only a 'no complexly-defined properties' view, in many ways it is still a good indication of the trajectory of his thought, and how the paradoxes would shape his metaphysical views as a whole.

3.2. CLASS–INTENSIONS AND DESCRIPTIVE SENSES

The entities involved in what, in Part I, were called 'the class-intension paradox' and 'the descriptive sense paradox' were understood early on by Russell as two different kinds of what he called 'denoting concepts'. In 1905, Russell abandoned this earlier theory in favor of his celebrated theory of descriptions of 'On Denoting'.[14] This theory helps to block these paradoxes in a fashion that has many similarities to the earlier response to the predicates paradox. Whereas Russell had previously regarded the 'all humans' part of 'all humans are mortal' as an independently meaningful part, with a unified entity, the denoting concept *all humans*, occupying a discrete part of the proposition, Russell now puts forth a view according to which 'all humans are mortal' is interpreted to mean:

$$(x)(x \text{ is human} \supset x \text{ is mortal})$$

Here, there is no unified part of the analysans corresponding to the phrase 'all humans', and hence, for Russell, the question doesn't arise as to whether *the* thing that this phrase means, the class-intension, is, or is not, human. Yet, one can still make sense out of *most* discourse about 'all humans'; doing so still requires thinking of the word 'human' as representing something in the proposition or state-of-affairs, just not the entire phrase 'all humans'.[15]

The same considerations apply to definite descriptions, and so Russell's theory of descriptions similarly spares him from the descriptive sense paradox. According to this theory, the proposition expressed by 'the author of *Frankenstein* is a woman' is analyzed:

$$(\exists x)((y)(y \text{ authored } \textit{Frankenstein} \equiv y = x) \text{ \& } x \text{ is a woman})$$

This respects the intuition that this proposition differs in meaning from 'Mary Shelley is a woman', but does so without postulating a distinct singular entity, a descriptive sense or denoting concept, that has or lacks properties on its own. Without thinking of descriptive senses as independent entities, the descriptive sense paradox poses no threat.

3.3. RUSSELL'S NO CLASSES THEORY

Russell's solution to the class paradox is still too often described as involving the kind of theory of types that separates reality into logical divisions, where individuals, classes of individuals, classes of classes of individuals, are all taken to be different kinds of beings about which the same things cannot meaningfully be said. In fact, however, this way of describing things is no more true for Russell's mature theory of types than it was for his 1903 response to the predication paradox. Russell summarized his view around the time of 1910's *Principia Mathematica* thusly:

> I have … discovered that it is possible to give an interpretation to all propositions which verbally employ classes, without assuming that there really are such things as classes at all. … That it is meaningless … to regard a class as being or not being a member of itself, must be assumed for the avoidance of a more mathematical contradiction; but I cannot see that this could be meaningless if there were such things as classes. ('Some Explanations' 357)

Instead, his response is akin to the other solutions we've considered. Expressions that seem to represent names of classes should not be taken at syntactic face-value. For Russell, class-terms, like definite description phrases, are dubbed, 'incomplete symbols', which means that they can be interpreted as making a contribution to the make-up of the meaning of sentences in which they appear, but without any one single 'thing' constituting their meaning or semantic value in the proposition expressed or state of affairs repre-

sented. In *Principia Mathematica*, we find the following contextual definition for formulæ involving class abstracts, $\hat{z}(\ldots z \ldots)$:

$$A\{\hat{z}(\psi z)\} =_{Df.} (\exists\phi)[(z)(\phi!z \equiv \psi z) \ \& \ A\{\phi!\hat{x}\}]$$

A statement seemingly about a class can be rewritten with higher-order quantification along with whatever is involved in specifying the membership conditions of the class. The statement, 'Socrates $\in \hat{z}(z$ is human)' (or *Socrates is a member of the class of all humans*) becomes:

$$(\exists\phi)[(x)(\phi!x \equiv x \text{ is human}) \ \& \ \phi!(\text{Socrates})]$$

I.e., there is a predicative propositional function satisfied by all and only humans, and Socrates satisfies it. Discourse about 'classes' is interpreted but without the assumption that the class term '$\hat{z}(z$ is human)' has its own independent meaning. (The rigamarole involving second-order quantification is a fancy way of ensuring that, unless embedded within a further intensional context, any sentence involving a class term '$\hat{z}(\psi z)$' will remain true regardless of which of numerous coextensive properties is used.)

There is, however, an important disanalogy here between this and the solutions to other paradoxes sketched above. The theory of descriptions provides not a theory about what we, in our pre-paradox naïveté, would have taken to be a proposition *about* a descriptive sense. It rather provides us with a reinterpretation of propositions which we might have hitherto interpreted as utilizing such senses to speak about their referents. Indeed, the theory provides no reinterpretation of propositions about the senses themselves, although reflection on the new theory may convince us that no such reinterpretation is necessary. Because of this, it is not even in *danger* of providing us with a way of reconstructing the reasoning involved with the descriptive sense paradox in a way that might lead back to contradiction. However, the treatment of class-terms as incomplete symbols *does* provide a reinterpretation of what we would have previously, naïvely, have thought to be about classes. Hence, the question still arises as to whether, once we reinterpret the talk of 'classes' involved, it is possible to translate the class paradox into a form where, despite not really involving classes, contradiction still results.

It is here and only here that the theory of 'types' enters in to Russell's solution. His contextual definition requires that the class abstract appear in a position that syntactically allows a higher-order or 'propositional function' variable of the type that takes as argument values of the variable used within the class abstract (the 'z' in '$\hat{z}(\psi z)$'). In Russell's system, propositional function variables are never of the same type as the variables for their possible arguments, and hence, a statement to the effect that a class is, or is not, a member of itself is uninterpretable.[16] This is the genesis of the apparent metaphysical division between classes of different types; really, however, 'classes' are just a way of speaking, for Russell.

3.4. HIGHER–ORDER QUANTIFICATION

The question still remains, however, whether or not the system of different types of propositional function variables and higher-order quantification commits Russell to metaphysical divisions of types of entities.

Russell's precise understanding of propositional functions, and higher-order quantification generally, is an incredibly complicated topic, and controversial among Russell scholars. We shall not be able to do more than scratch the surface here.[17] Russell's own views changed. There was a period, from roughly 1902 through late 1905, during which Rus-

sell believed in a special category of entities corresponding to open sentences, entities which might be 'named' by such expressions as '$\hat{x}$ is human $\supset \hat{x}$ is mortal'. By 1906, however, Russell had come to the conclusion that 'to assume a separable ϕ in ϕx is just the same, essentially, as to assume a class defined by $\phi x \ldots$' and that by having 'treated ϕ as an entity' he 'brought back the contradiction' even after he had 'thought [he] had solved the whole thing by denying classes altogether' (*DRDJ* 78).

At the time, he had hoped that he could treat propositional functions also as a mere *façons de parler* by replacing talk of propositional functions with talk of substitutions within propositions. The suggestion centers around a four-place relation, written

$$p/a; b \, ! \, q$$

which means that q (typically, a proposition) results from the substitution of the entity b for a wherever a occurs as logical subject in (proposition) p. Rather than considering a function $\hat{x}$ *is human*, one could utilize a 'matrix' consisting of the proposition *Socrates is human* and Socrates. While maintaining only one kind of variable, the theory yielded results very similar to a simple type-theory, and Russell's paradox is excluded because there is no way to represent a matrix taking 'itself' as argument, because something like '$p/a; p/a!q$' is ungrammatical. Russell's main attraction to the theory was that it offered an explanation for what goes wrong with the paradoxes without positing different ontological types of entities.[18]

The approach didn't last. It banished thinking of propositional functions as genuine objects but at the cost of necessitating thinking of propositions that way. Hence it made the propositions paradox – along with certain variants of it particular to the substitutional theory[19] – unsolvable along similar lines. Indeed, after abandoning it, Russell's move was predictable: he concluded that propositions too should be understood as 'logical fictions' or incomplete symbols. While 'that Socrates is human' in the sentence, 'Plato believes that Socrates is human', may appear to constitute a syntactic unit and thereby suggest a unified thing it 'names', Russell now holds that only the individual words making up the clause are representative. Belief must be understood not as a dyadic relation between a believer and proposition believed, but rather as a 'multiple relation' between a believer and the various components that would make up the corresponding fact were it true (*PM* 43–44, 'Nature of Truth and Falsehood'). This 'multiple relation' theory of judgment was also not long-lived in Russell's philosophy, but it is telling that Russell never returned to a realism about 'propositions' understood as mind- and language-independent intensional entities.

However, by abandoning propositions, Russell was forced to abandon the substitutional theory along with it, which left him without a clearcut way of making sense of higher-order quantification, which he took as necessary for mathematical logic. He returned to a vocabulary of 'propositional functions', but seemed wary of thinking of open sentences as representing a distinct kind of thing about which the same things cannot be said as of individuals. In an early draft of a section on 'Types' of *PM*, Russell wrote:

> A function must be an incomplete symbol. This seems to follow from the fact that $\phi!(\phi!\hat{z})$ is nonsense. The whole difficulty lies in reconciling this with the fact that a function can be an apparent [i.e., bound] variable.

Readers of *PM* provide different answers as to whether, or how, Russell reconciled this tension. According to one popular reading, Russell returned to a realist view of 'propositional functions' as mind-independent complex entities of various metaphysical types.[20] On more sophisticated versions of this account, propositional functions are still not taken

as ontologically on par with more basic entities such as particulars and universals (see esp. Linsky, *Russell's Metaphysical Logic*, ch. 2). They are instead taken as 'derived entities', metaphysically constructed out of more basic stuff. For the construction to be possible, a propositional function cannot have as arguments anything presupposing the function itself, and hence cannot take itself as argument (cf. *PM* 48f).

According to a newer but increasingly popular account, which I favor, Russell's mature account of 'propositional functions' is that they are nothing more than open sentences, and that higher-order quantification in *Principia Mathematica* is to be understood in terms of linguistic substitutions in sentences, so that a quantified formula gets its truth-conditions in terms of the truth or falsity of the results of well-formed replacements one can make for the variable.[21] Different 'types' of variable correspond to substitutions of different kinds of expressions of various complexities (and 'order' restrictions involving what kinds of further quantifiers are allowed in the legitimate substitutions for a variable of a certain kind, to ensure that the truth conditions so generated are non-circular).[22] Ultimately, the truth or falsity of higher-order quantified statements is resolved recursively in virtue of the truth or falsity of lower-order statements, eventually terminating in basic forms such as elementary propositions, which only involve such 'entities' as simple universals and particulars. This reading has in common with the view of propositional functions as derivative entities, then, the view that in some sense, truths apparently or actually about such things depend metaphysically on the more basic stuff, and the nature of this dependence rules out such apparent statements involving a propositional function taking 'itself' as argument as having well-defined truth-conditions at all.

I cannot pretend to have fully argued for, or even fully explained, this general line of interpretation for Russell's understanding of propositional functions in *PM*, but I do think it is in keeping with the general approach to paradox-dissolution Russell had applied elsewhere. Fuller discussion must be left for another occasion. I mention here only that Russell claimed later that 'In the language of the second order, variables denote symbols, not what is symbolized' (*IMT* 192) and that 'Whitehead and I thought of a propositional function as an expression' and indeed as 'nothing but an expression' (*MPD* 62, 69). In *The Philosophy of Logical Atomism* (96), he wrote that, 'a propositional function is nothing, but, like most things one wants to talk about in logic, it does not lose its importance through that fact'.

If this reading is right, then, for Russell, considered extra-linguistically, propositional functions are nothing. Classes are nothing. (And since Russell defined relations-in-extension, numbers and other mathematical structures as classes, they too, are nothing.) Descriptive senses and/or denoting concepts are nothing. In short, every category of entity one is tempted to imagine populated in sufficient numbers so as to transgress the Cantorian lessons he (and Frege) had learned so well is a category of nothing. Ockham has triumphed, and Pythagoras has retreated. But notice that Russell's advocacy in favor of ontological parsimony is not a simple-minded attitude of 'fewer is better' or 'simplicity is preferable to complexity'. Russell was convinced that the Cantorian paradoxes were unsolvable in a uniform and non-*ad hoc* way if the reality of these would-be entities were taken at face-value. Notice, moreover, that Russell's eschewel of these abstract entities did not take a form of a straightforward reduction in which it is admitted that these are things, just not 'fundamental things'. According to Russell, thinking of propositions, classes, denoting concepts, etc. as things at all is an illusion created by the surface grammar of ordinary language. Instead, he offered a replacement locution in which terms for such 'abstract things' could be wholly eliminated in terms of logical forms and/or some stock of primitive expressions for the concrete, non-complex, beings of the empirical world and their properties and relations: the 'logical atoms' of logical atomism.

4. Conclusion

Bertrand Russell engaged in a decade-long struggle with various forms of Cantorian diagonal paradoxes. Growing from it was his logical atomism,[23] whereupon 'none of the raw material of the world has smooth logical properties, but that whatever appears to have such properties is constructed artifically in order to have them' (*PoM* 2nd ed., xi). This seems to me to be a remarkable moment in the history of philosophy, and one which, even a few generations later, we have yet to appreciate fully.

In 2009, no one would take a formulation of set theory seriously if it did not offer any kind of response to Russell's paradox.[24] Similarly, no one would take a theory of truth to be complete until it had something to say about the liar paradox and its generation from Tarski's T-schema. It strikes me as highly odd that so many philosophers feel no compunction at all about positing the reality of abstract or intensional entities such as (Fregean or quasi-Fregean) senses, concepts, intensions and propositions, without even addressing the possibility of Cantorian contradictions.[25] Nor should the investigation of such matters be viewed as an annoyance or a hindrance to discovery in the areas of intel-lectual thought where senses, concepts, intensions, and the like may be of theoretical use. It is sometimes said that the 'foundational crisis' in set theory, sparked by the recognition of the paradoxes, fueled the growth and development of set theory as a branch of math-ematics throughout the 20th century. I fear we have not given 'the ways of paradox' with regard to other areas of abstract philosophy enough of a chance to do the same. We may end up with a better sense of what these entities could or might be, or we might end up (as Russell would have us) concluding that thinking of them as 'things' at all is misguided, but that realization may (as I think Russell hoped) not come at the price of the insights engendered by the theories that originally prompted us to postulate the existence of such entities.

Even when narrowly focused on the philosophy of mathematics and set/class-theory, where awareness of the possibility of Cantorian paradoxes is still kept at the forefront of researchers' minds, there are lessons to be drawn from Russell's engaged assault on them. Russell despaired to Phillip Jourdain in 1907, no doubt thinking of the writings of Zermelo and von Neumann, that 'I have given up expecting much of solutions [of the set-theoretical paradoxes]' (*DRDJ* 54). The 'solutions' offered within their approaches, still dominant if not hegemonic today, were rejected by Russell, and for compelling reasons that have scarcely been discussed since. Russell's own proposed solution to the class-theoretic paradoxes has, I think, not been well understood or thoroughly evaluated as an alternative. Even where there is, today, movement towards the creation of a new kind of philosophically minded set-theory, as within the neo-logicist movement, Russell's name usually takes a backseat to Frege's, or others'.[26] (The Russellian notion of a logical construction or incomplete symbol provides a compelling alternative to the metaphysics of abstract entities found within this literature.) The familiarity of those works by Russell cited within the 'canon' of analytic philosophy, unfortunately, I think, gives the impres-sion that his work has already been thoroughly pillaged of its sources of philosophical inspiration. This impression is, I think, however, thoroughly mistaken.

Short Biography

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Notes

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[1] Those from 1902 through mid-1905 have been published in his *Collected Papers* v3–4, but those from 1906 through 1909 are currently only available at the Russell Archives at McMaster University.

[2] Something along these lines also seems like an unintended consequence of the view prevalent in contemporary philosophy of language discussions on which structured propositions (sometimes misleadingly called 'Russellian propositions') are thought of as set-theoretic constructs. If embedded within a set-theory like NBG, this leads to the result that there are no structured propositions about proper classes (since proper classes cannot be members of any other classes) – a result that, in my opinion, constitutes a *reductio* of their view.

[3] Even when 'limitation of size' is only used as the criterion for the conditions under which classes exist (and not as a criterion for when properties exist, as objects, or when propositions, senses, etc., exist), doubts may arise as to its formal adequacy. Without a worked-out, robust, theory about the nature of properties, their identity conditions, and so on, it is difficult to estimate, for a given property, how many other properties it is likely to be coextensive with, but it does seem clear that the answer would not be much different for different properties. It seems, for example, that the number of properties coextensive with the property of self-identity is probably the same as the number of properties coextensive with the property of non-self-identity – consider, e.g., that the negation of each property in the former group would be in the latter, and vice-versa. Assuming then, that such collections are not too large to constitute classes (not, I admit, a small assumption), one then has, for each property – even properties too large to have their own extension make up a class – an associated entity (the equivalence class of coextensive properties) which can go proxy for it in class-theoretic reasoning. As Russell was himself aware (*Papers v4* 274), it then becomes possible to give a revised definition of 'class membership' where this really means instantiating one of the properties within a given class of properties, and the result is that the size limitation is effectively undone, thereby reintroducing naïve class theory. The version of Russell's paradox that would then result would not be the original class form of Russell's paradox, but rather, what was called the property equivalence class paradox in Part I, but it would be a contradiction all the same.

[4] In his published works, the zigzag theory is only explicitly mentioned in 'On Some Difficulties', though something like it is hinted at with the discussion of 'quadratic forms' in *PoM* (104, 487); however it is discussed in many surviving manuscripts from the period: see *Papers v4* parts I–III.

[5] The example is only slightly notationally altered from one of Russell's manuscripts – see *Papers v4* 9.

[6] From the standpoint of historical accuracy, then, it is somewhat unfortunate that the most in-depth attempts to give reconstructions of Russell's work in this period, found in Cocchiarella's work, build upon Quine's work instead, although Cocchiarella's reconstructions are fascinating on their own terms.

[7] This is borne out in the recent secondary literature; see especially Landini, *Russell's Hidden Substitutional Theory*; Klement, 'Form Before Function'; and Stevens, *Russellian Origins*.

[8] He similarly dismissed Frege's distinction between objects and concepts as guilty of this kind of error (*PoM* 507–10, and in their correspondence: see Frege, *Correspondence* 134–38), no doubt anticipating Geach's observations much later ('Saying and Showing') to the effect that it does not seem possible to state Frege's theory of levels in a way that doesn't violate the theory itself.

[9] For explanation of this latter point, see, e.g., Bostock, 'Russell on 'the' in the Plural', 117–18.

[10] When discussing the need for dividing propositions into ramified orders, Russell usually discusses contingent semantic paradoxes such as the Epimenides paradox. Manuscripts of the period, however, suggest that he had Cantorian paradoxes such as the one discussed previously in mind as well; see especially 'The Paradox of the Liar'.

[11] See especially, 'On the Nature of Truth' (1907) and 'Logic in which Propositions are not Entities'.

[12] Cantorian paradoxes of 'properties' can be found in more than one form in Russell's philosophy given the distinction between concepts, predicates or universals on the one hand, and propositional functions on the other. A fuller treatment of this topic would require differentiating these versions of the paradox, and discussing their different treatment by Russell at different points in his career. For further discussion, see Klement, 'Origins'.

[13] Russell did eventually, under the influence of Wittgenstein, come to hold such a position – see *PLA* 67 – but this was not a requirement of his theory of types, and he held the opposite view even after the publication of *Principia Mathematica*: see, e.g., 'Analytic Realism', 135. Even when Russell did hold such a view, however, he did not connect it directly with the paradoxes, and continued to maintain a distinction between 'predicates', entities involved in simple predications, and 'properties' or 'propositional functions', where only the former were thought of as entities at all (e.g., *MPD* 166), a distinction which by itself solves the paradox for predicates, regardless of whether or not these entities are in a distinct type. Russell also expresses misgivings about the Wittgenstein-inspired view putting universals in a distinct type, writing in an (unpublished) 1921 letter to Moore, 'there are difficulties in this view, beginning with the fact that it cannot be stated without apparent self-contradiction', echoing his 1903 criticisms of Frege. For further discussion, see Klement, 'Form Before Function'.

[14] While Russell did often connect his work with solving the paradoxes in mathematical logic with his work on descriptions (see, e.g., *DRDJ* 79, *Auto* 150, *IMP* 136, etc.), the precise intellectual motivations for the theory of descriptions remain controversial among Russell scholars, and there isn't overt evidence that paradoxes exactly like the class-intension or descriptive sense paradoxes were a direct factor. It is striking, however, that the difficult argument Russell explicitly gave in 'On Denoting' against denoting complexes involved the problem of understanding the nature of propositions actually about denoting complexes themselves rather than what they denote, and notice that it is precisely the ability to speak of denoting complexes themselves as opposed to their denotations, i.e., to predicate properties of the complexes themselves, that is needed to make these paradoxes formulable.

[15] And it should be noted that nothing in the approach even requires that if the phrase is more complicated, such as with 'all yellow horses' or 'all predicates not predicable of themselves', that we treat the antecedent of the analysans as involving a *single* property or constituent of the proposition; perhaps Yellowness and Horseness need to be constituents of the proposition expressed by '$(x)(x$ is a horse & x is yellow $\supset x$ is tame$)$', but Yellow-Horseness needn't.

[16] A statement to the effect that a class whose 'members' are coextensive propositional functions satisfies one or more of those propositional functions is similarly uninterpretable, blocking the equivalence class version of the paradox.

[17] For fuller discussion of the development of Russell's views, see especially Klement, 'Form Before Function' and 'Russell's 1903–05 Anticipation of the Lambda Calculus'.

[18] See especially his 'The Substitutional Theory of Classes and Relations', and 'On "Insolubilia"'.

[19] For further discussion, see Landini, *Russell's Hidden Substitutional Theory*, ch. 3; Stevens, *Russellian Origins*, ch. 3.

[20] See, e.g., Hylton, *Propositions, Functions and Analysis*, Linsky, *Russell's Metaphysical Logic*, and Potter, *Reason's Nearest Kin*.

[21] I am, in effect, here suggesting that Russell's held a 'substitutional' theory of higher-order quantification; for interpretations along these lines, see Sainsbury, *Russell*, Landini, *Russell's Hidden Substitutional Theory*, Stevens, *Russellian Origins*, and Klement, 'Form Before Function'.

[22] This is in essence the hierarchy of truth and falsity of *PM* 42ff.

[23] Russell says (*PLA* 35) that logical atomism is a position that 'forced itself upon' him 'while thinking about the philosophy of mathematics' but demurs from the hardline position that one position entails the other.

[24] This is true even if that response amounted to something like embracing the contradiction while containing its ill-effects, as with certain paraconsistent logics.

[25] A similar point is made recently by Harry Deutsch in his review of J. C. King's recent book.

[26] Although on the positive side, at least there seems to be some engagement with the Russellian notion of indefinite extensibility; see, e.g., Shapiro, 'Prolegomenon'; Shapiro and Wright 'All Things'.

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