

Certain and Uncertain Inference with Indicative Conditionals*

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Abstract

This paper develops a trivalent semantics for the truth conditions and the probability of the natural language indicative conditional. Our framework rests on trivalent truth conditions first proposed by Cooper (1968) and Belnap (1973) and it yields two logics of conditional reasoning: (i) a logic C of certainty-preserving inference; and (ii) a logic U for uncertain reasoning that preserves the probability of the premises. We show systematic correspondences between trivalent and probabilistic representations of inferences in either framework, and we use the distinction between the two systems to cast light on the validity of inferences such as Modus Ponens, Or-To-If, and Conditional Excluded Middle. Specifically, the conditional behaves monotonically in C , but non-monotonically in U ; Modus Ponens is valid in C , but valid in U only for non-nested conditionals. The result is a unified account of the semantics and epistemology of indicative conditionals that can be fruitfully applied to analyzing the validity of conditional inferences.

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1 Introduction and Overview

Research on indicative conditionals (henceforth simply “conditionals”) pursues, by and large, two major projects: the semantic project of determining their *truth conditions*, and the epistemological and pragmatic project of explaining how we should *reason* with them, and when we can *assert* them. The two projects are related: Jackson (1979, p. 589) states that “we should hope for a theory which explains the assertion conditions in terms of the truth conditions”, while according to David Lewis (1976, p. 297), “assertability goes by subjective probability”, whose definition depends in part on the truth and falsity of the sentence in question (see also Adams 1965, pp. 173-174; Jackson 1979, p. 565; Leitgeb 2017, p. 278).

Ideally, we would have a unified treatment of truth conditions and probability of conditionals and, on that basis, a theory of reasoning with conditionals. Here is the standard approach. Suppose A and C are formulas of a propositional language \mathcal{L} without conditionals, and let \rightarrow denote the “if... then...” connective. Then, the probability of the sentence $A \rightarrow C$ should go by the conditional probability $p(C|A)$ (e.g., Adams 1965, 1975; Stalnaker 1970):

$$p(A \rightarrow C) = p(C | A) \quad (\text{Adams's Thesis})$$

The idea is that the conditional “if the sun is shining, Mary will go for a walk” is likely if and only if it is likely that, given sunshine, Mary goes for a walk.¹ Normative theories of conditionals often recognize Adams’s Thesis as a desideratum (e.g., Stalnaker 1970; Adams 1975). The empirical data are complex, but Adams’s Thesis is well-supported when the antecedent is *relevant* to the consequent (e.g., part of the same discourse: Over, Hadjichristidis, et al. 2007; Skovgaard-Olsen, Singmann, and Klauer 2016).

Unfortunately, David Lewis’s well-known triviality result complicates the picture. Suppose that (i) the probability of a sentence depends in the standard way on its truth conditions (i.e., the probability of A sums up the weight of the possible worlds where A is true), and (ii) the probability function is closed under conditionalization. Given (i) and (ii), Adams’s Thesis implies $p(A \rightarrow C) = p(C)$, whenever A is compat-

¹The extension of Adams’s Thesis to *arbitrary* formulas A and C , possibly involving conditionals, is known as “Stalnaker’s Thesis” (viz. Douven (2016)).

ible with both C and its negation. This is obviously a disaster since the two expressions need not be equally credible. Similar triviality results have been produced by Hájek (1989), Bradley (2000), and Milne (2003). This *reductio ad absurdum* seems to preclude a unified semantic and epistemological treatment of conditionals, at least as far as probability and probabilistic reasoning are concerned.

But this conclusion is premature: as argued by several before us (McDermott 1996; Cantwell 2006; Rothschild 2014; Lassiter 2020), we can introduce a third truth value (“neither true nor false”) and state trivalent truth conditions for natural language indicative conditionals whose probability validates Adams’s Thesis without triviality. In this paper, we make a further step toward this unification, by showing that probabilistic semantics allows us to define a logic for *certain* reasoning (i.e., where we wish to preserve certainty) as well as a structurally similar logic for *uncertain* reasoning.

In other words, we argue that *different logics of conditionals suit different epistemic endeavors*. When no conditionals are involved, certain and uncertain inference coincide: deductive logic validates all and only those inferences that preserve maximal certainty, i.e., probability 1 — and also all and only those inferences that do not increase uncertainty (e.g., Adams 1998). But conditionals complicate the picture. When premises are assumed to be certain (and we wish to preserve certainty), the inference from “if Alice goes to the party, Bob will” to “if Alice and Carol go to the party, Bob will” appears valid. Alice’s presence *ensures* Bob’s presence no matter his feelings for Carol. This picture changes when the premises are taken to be just *likely* instead of certain: Carol’s presence at the party can make Bob’s presence very unlikely if Alice’s presence does not guarantee that he will come (Lewis 1973b). Conditional reasoning from uncertain premises has non-monotonic aspects and so an adequate logic of conditionals arguably requires more than one notion of valid inference (compare Adams 1965, 1996; Santorio 2022b). The account we propose in this paper explains the difference between certain and uncertain inference by strengthening the definition of logical consequence when going from certain to uncertain reasoning.

We briefly expound the structure of our paper. The first part lays the semantic foundations. Section 2 motivates the trivalent treatment of conditionals. Section 3 argues for specific trivalent truth tables for the

indicative conditional and the Boolean connectives, based on proposals first put forward by Cooper 1968 and Belnap 1973. Section 4 defines the (non-classical) probability of trivalent sentences in analogy with defining probability in a conditional-free language.

The second part of the paper focuses on conditional reasoning. From the definition of probability in trivalent semantics, Section 5 and 6 derive two logical consequence relations for certainty-preserving inference (=the logic **C**) and for inferences that do not increase probabilistic uncertainty (=the logic **U**). We show that **C** and **U** can be characterized as preserving semantic values within trivalent logic, and in Section 7 we examine which principles of conditional logic they validate. In particular, we show that some principles such as Or-to-If or Modus Ponens with nested conditionals are controversial because they hold when we wish to preserve certainty, but not in uncertain inference.

The third part contains applications, comparisons and evaluations: Section 8 discusses nested conditionals and McGee's objection to Modus Ponens from the vantage point of our semantics and the two separate logics for certain and uncertain inference. Section 9 draws comparisons with other theories. Section 10 highlights the strengths and limits of our account. Appendix A provides proof details.

2 Truth Conditions: The Basic Idea

It is controversial whether indicative conditionals have factual truth conditions and can be treated as expressing propositions (e.g., see exchange between Jeffrey and Edgington 1991). According to the non-truth-conditional, probabilistic analysis of conditionals (Adams 1965, 1975; Edgington 1986, 1995, 2009; Over and Baratgin 2017), indicative conditionals do not express propositions; at best they have partial truth conditions. According to Adams:

[...] the term 'true' has no clear ordinary sense as applied to conditionals, particularly to those whose antecedents prove to be false [...]. In view of the foregoing remarks, it seems to us to be a mistake to analyze the logical properties of conditional statements in terms of their truth conditions. (Adams 1965, pp. 169-170)

As a result, non-truth-conditional accounts need to *stipulate* that $p(A \rightarrow C) = p(C|A)$ and develop a probabilistic theory of reasoning with conditionals on the basis of high probability preservation

(called “logic of reasonable inference” by Adams). This move yields a powerful logic for capturing core phenomena of reasoning with simple conditionals, such as their non-monotonic behavior in certain contexts, whose success is recognized by truth-conditional accounts (e.g., McGee 1989, p. 485; and more recently Ciardelli 2020, p. 544). However, by abandoning truth conditions, the probabilistic approach severs the link between semantics and epistemology. In particular, it does not cover nested conditionals and compounds of conditionals. Moreover, due to the lack of truth conditions, it does not clarify how one can argue and disagree about conditional sentences in a similar way as we do for normal, conditional-free sentences (Bradley 2012, p. 547).

Yet, even a defender of a non-truth-conditional view such as Adams (1965, p. 187) admits that we feel compelled to say that a conditional “if A , then C ” has been *verified* if we observe both A and C , and *falsified* if we observe A and $\neg C$. For example, take the sentence “if it rains, the match will be cancelled”; it seems to be true if it rains and the match is in fact cancelled, and false if the match takes place in spite of rain. Indeed, what else could be required for determining the truth or falsity of the sentence?

This “hindsight problem” (the terminology is from Khoo 2015) is a *prima facie* reason for treating conditionals as expressing propositions, and for assigning them factual truth conditions. Defenders of non-propositional accounts need to explain why facts in the actual world are *sufficient* to settle the truth and falsity of “if A , then C ” when A is true, but also why the latter is evaluated differently when A is false.

Truth-conditional accounts of indicative conditionals address this point. They come in various guises: variably strict conditionals (e.g., Stalnaker 1968), restrictor semantics (e.g., Kratzer 2012), dynamic semantics (e.g., Gillies 2009), information state semantics (e.g., Ciardelli 2020; Santorio 2022a), and many more.² Many of these accounts *emulate* Adams’s probabilistic logic of reasonable inference, or central parts thereof. For example, truth preservation in Stalnaker’s modal framework famously validates the same inference schemes as Adams’s logic

²The material conditional analysis, endorsed by Jackson and Lewis, claims that the truth conditions of the indicative and the material conditional agree, and that perceived differences are due to pragmatic, not to semantic factors (Jackson 1979; Grice 1989). This approach, however, gives up on a unified picture of truth conditions and probability in the first place. On that account, if sun were unlikely, the probability of “if the sun is shining, Mary is going for a walk” would be close to 1 regardless of Mary’s intentions, which looks unacceptable.

in their common domain. All of them, however, face the non-trivial task of modelling the probability of conditionals. While these logics offer a qualitative account of plausibility, their analysis of the *quantitative* probability of conditionals must, in the light of Lewis’s triviality result, deviate systematically from Adams’s thesis. Thus, both the truth-conditional and the non-truth-conditional approaches seem to miss out on some important aspects of conditionals.

In this paper, we propose to solve this problem by treating “if A , then C ” as a *conditional assertion*—i.e., as an assertion about C upon the supposition that A is true. Whereas, when the antecedent is false, the speaker is committed to neither truth nor falsity of the consequent. This view takes into account Adams’s observation that “true” has no clear ordinary sense when applied to indicative conditionals; it has been voiced perhaps most prominently by Quine (1950, p. 12, our emphasis):

An affirmation of the form “if p then q ” is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. *If on the other hand the antecedent turns out to have been false, our conditional affirmation is as if it had never been made.*

In other words, asserting a conditional makes an epistemic commitment only in case the antecedent turns out to be true. If it turns out to be false, the assertion is retracted: there is no factual basis for evaluating it (see also Belnap 1970, 1973). Therefore it is classified as neither true nor false. The “gappy” or “defective” truth table of Table 1 interprets this view as a partial assignment of truth values to conditionals (e.g., Reichenbach 1935; de Finetti 1936a; Adams 1975; Baratgin, Over, and Politzer 2013; Over and Baratgin 2017).³

However, without a full truth-conditional treatment, such an account is limited: it neither evaluates nested conditionals, nor Boolean compounds of conditionals. If we could complete Table 1 and provide full truth conditions in a satisfactory way, this would greatly increase

³Some accounts also use the conditional probability $p(C | A)$ as a semantic value for the conditional $A \rightarrow C$ (e.g., McGee 1989; Stalnaker and Jeffrey 1994; Sanfilippo et al. 2020). But this analysis reverses the traditional direction of the dependency between the probability and the truth conditions of a sentence: probability should depend on how often we find a sentence to be true, not vice versa.

Truth value of $A \rightarrow C$	$v(C) = 1$	$v(C) = 0$
$v(A) = 1$	1	0
$v(A) = 0$	(neither)	(neither)

Table 1: ‘Gappy’ or “defective” truth table for a conditional $A \rightarrow C$ for a (partial) valuation function in a language with conditional.

the scope and descriptive power of conditional reasoning, and facilitate the identification of theorems and valid inferences.

The obvious candidate for such truth conditions is a trivalent truth table, where the absence of commitment to the consequent C is represented by a third truth value. Instead of using partial valuations, we assign a third semantic value, $1/2$ or “indeterminate”, when the antecedent is false (see Table 2). This is a recurring idea in the literature, defended, among others, by de Finetti (1936a), Reichenbach (1944), Jeffrey (1963), Cooper (1968), Belnap (1970, 1973), Manor (1975), Farrell (1986), McDermott (1996), Olkhovikov (2002/2016), Cantwell (2008), Rothschild (2014), and Égré, Rossi, and Sprenger (2021a,b).

Truth value of $A \rightarrow C$	$v(C) = 1$	$v(C) = 1/2$
$v(A) = 1$	1	$1/2$
$v(A) = 0$	$1/2$	$1/2$

Table 2: Partial trivalent truth table for a conditional $A \rightarrow C$ for a partial valuation function in a language with conditional.

This basic idea needs to be developed in various directions. Firstly, we need to decide how to extend the truth table of Table 1 to a fully trivalent truth table for $A \rightarrow C$ where A and C can also take the value $1/2$ (=neither true nor false, indeterminate). Secondly, we need to decide how to interpret the standard Boolean connectives \wedge , \vee , \neg in the context of sentences which can take three different truth values. Doing so will allow us to deal with nested conditionals, and more generally, with arbitrary compounds of atomic sentences connected by the standard connectives and \rightarrow . Thirdly, we have to define a probability measure for trivalent sentences and a consequence relation for reasoning with certain and uncertain premises. We handle these tasks in turn in the next sections.

3 Trivalent Truth Tables

We start by extending the basic idea of Table 2 to a full trivalent truth table for $A \rightarrow C$. The two main options are displayed in Table 3 and have been proposed by Bruno de Finetti (1936a) and William Cooper (1968), respectively.⁴ In both of them the value $1/2$ can be interpreted as “neither true nor false”, “void”, or “indeterminate”. There is moreover a systematic duality between those tables: whereas de Finetti treats indeterminate antecedents like false antecedents, Cooper treats them like true ones. Thus, in de Finetti’s table the second row copies the third, whereas in Cooper’s table it copies the first.

$f \rightarrow$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	$1/2$
0	$1/2$	$1/2$	$1/2$

$f \rightarrow$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	1	$1/2$	0
0	$1/2$	$1/2$	$1/2$

Table 3: Truth tables for the de Finetti conditional (left) and the Cooper conditional (right).

Both options can be pursued fruitfully, and the choice between them primarily depends on the results which they yield. We prefer the Cooper table since it interacts more naturally with our probabilistic treatment of conditionals and the various notions of logical consequence we develop (a detailed analysis is given in Égré, Rossi, and Sprenger 2021a). However, for the arguments made in this section, which concern only simple, non-nested conditionals, there is no difference between the two.⁵

The second choice concerns the definition of the standard logical connectives of negation, conjunction, and disjunction. A natural option is given by the familiar Strong Kleene truth tables (first proposed by Łukasiewicz 1920), displayed in Table 4. Conjunction corresponds to the “minimum” of the two values, disjunction to the “maximum”, and negation to inversion of the semantic value. In particular, the trivalent analysis admits, next to the indicative conditional $A \rightarrow C$, a

⁴Belnap (1973), Olkhovikov (2002/2016), and Cantwell (2008) rediscovered Cooper’s truth table independently.

⁵Intermediate options vary the middle row, e.g., with the triple $\langle 1/2, 1/2, 0 \rangle$ (Farrell 1986), or the triple $\langle 1, 1/2, 1/2 \rangle$ (suggested by a referee). Both options give up the equivalence of $\neg(A \rightarrow C)$ and $A \rightarrow \neg C$, see Égré, Rossi, and Sprenger 2021a for details regarding the former.

Strong Kleene “material” conditional $A \supset C$, definable as $\neg(A \wedge \neg C)$, or equivalently, $\neg A \vee C$.

	f_{\neg}		f_{\wedge}	1	1/2	0		f_{\vee}	1	1/2	0
1	0	1	1	1	1/2	0	1	1	1	1	1
1/2	1/2	1/2	1/2	1/2	1/2	0	1/2	1	1/2	1/2	1/2
0	1	0	0	0	0	0	0	1	1/2	0	0

Table 4: Strong Kleene truth tables for negation, conjunction, and disjunction.

The Strong Kleene truth table for negation is standard and also yields the consequence that the conditional commutes with negation (for either the DF- or the CC-conditional): $\neg(A \rightarrow C)$ has the same truth table as $A \rightarrow \neg C$. This property squares nicely with the conditional assertion view of conditionals: when A is false, both assertions ($A \rightarrow C$ and $A \rightarrow \neg C$) are retracted, and when A is true, one of them is true when the other is false, and conversely.⁶

Unfortunately, the Strong Kleene truth tables for conjunction and disjunction have a very annoying consequence: “partitioning sentences” such as $(A \rightarrow B) \wedge (\neg A \rightarrow C)$ will always be indeterminate or false (Belnap 1973; Bradley 2002, pp. 368-370). However, a sentence such as:

If the sun shines tomorrow, John goes to the beach; and if it rains, he goes to the museum.

seems to be true (with hindsight) if the sun shines tomorrow and John goes to the beach. This intuition is completely lost in Strong Kleene semantics, regardless of whether we use the de Finetti or the Cooper table for the conditional. Even worse, “obvious truths” such as $(A \rightarrow A) \wedge (\neg A \rightarrow \neg A)$ are always classified as indeterminate.

For this reason, we endorse alternative truth tables for conjunction and disjunction, advocated by Cooper (1968) and Belnap (1973), and shown in Table 5. In these truth tables, indeterminate sentences are “truth-value neutral” in Boolean operations: true and false sentences do not change truth value when conjoined or disjoined with an indeterminate sentence. This can be motivated by observing that such

⁶Some varieties of modal semantics too imply that conditionals commute with negation: see the recent (and unrelated) semantics of Santorio (2022a) and Willer (2022). See Égré and Politzer (2013), Skovgaard-Olsen, Collins, et al. (2019), and Olivier (2019) for psycholinguistic investigations of the empirical status of negation commutation.

sentences do not add determinate content the way empirical statements do. Following Adams (1966) and Dubois and Prade (1994), we call these connectives *quasi-conjunction* and *quasi-disjunction*.

	f_{\neg}		f'_{\wedge}	1	1/2	0		f'_{\vee}	1	1/2	0
1	0	1	1	1	0	0	1	1	1	1	1
1/2	1/2	1/2	1/2	1	1/2	0	1/2	1	1/2	0	0
0	1	0	0	0	0	0	0	1	0	0	0

Table 5: Truth tables for Strong Kleene negation, paired with quasi-conjunction and quasi-disjunction as defined by Cooper (1968) and Belnap (1973).

They retain the usual properties of Boolean connectives (associativity, commutativity, the de Morgan laws, etc.), solve the problem of partitioning sentences, and have no substantial disadvantages with respect to Strong Kleene truth tables in conditional logic. Moreover, they have two non-trivial benefits. First, quasi-disjunction avoids the Linearity principle that $(A \rightarrow B) \vee (B \rightarrow A)$ cannot be false. This valid schema of two-valued logic was famously criticized by MacColl (1908), who pointed out that neither “if John is red-haired, then John is a doctor”, nor “if John is a doctor, then he is red-haired”, nor their disjunction seems acceptable in ordinary reasoning. A semantics that qualifies such expressions as systematically either true or indeterminate might thus be considered inadequate. Using quasi-conjunction and quasi-disjunction instead, $(A \rightarrow B) \vee (B \rightarrow A)$ comes out false when A is true and B is false (or vice versa).

The second benefit of quasi-conjunction and quasi-disjunction concerns the connection between conditional bets and conditional assertions. How should we evaluate the conjunction of conditional assertions like $(A \rightarrow B) \wedge (C \rightarrow D)$? The interesting case occurs when A is false, but C and D are true. Using a Dutch Book argument, McGee (1989, 496-501, Theorem 1) shows that in this case, a bet on $(A \rightarrow B) \wedge (C \rightarrow D)$ should yield a strictly positive partial return. Likewise, Sanfilippo et al. (2020, p. 156) argue that we should classify the compound bet as winning. On their account, the sentence $(A \rightarrow B) \wedge (C \rightarrow D)$ remains verified when A is false, provided C and D are true. This invites us to treat the assertion $(A \rightarrow B) \wedge (C \rightarrow D)$ as true rather than indeterminate, which quasi-conjunction enables.

For all these reasons, we adopt the Cooper truth tables for the conditional and the other connectives in the remainder of this paper. Our object-language is the language of propositional logic \mathcal{L} , supplemented with a primitive conditional connective \rightarrow , and is notated as \mathcal{L}^\rightarrow . A *Cooper valuation* is a function $v : \mathcal{L}^\rightarrow \mapsto \{0, 1/2, 1\}$ that assigns a semantic value to all sentences of \mathcal{L}^\rightarrow in agreement with the Cooper truth-tables, i.e., it interprets \neg as the strong Kleene negation, \wedge and \vee as Cooper’s quasi-conjunction and quasi-disjunction respectively, and \rightarrow as Cooper’s conditional. If we assume (as Cooper did) that atomic sentences only receive classical values (i.e., 0 and 1) in every Cooper valuation, conditional-free sentences will only receive classical values in all Cooper valuations: negations, conjunctions, and disjunctions take classical values if their sub-formulas have classical values. This assumption is not mandated by our analysis, but it is natural so long as conditional structures are the exclusive source of the third truth value in natural language.⁷ Note finally that all combinations of conditionals and conjunctions surveyed in this section validate Import-Export: $(A \wedge B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$ are extensionally equivalent formulas (Cooper 1968; Égré, Rossi, and Sprenger 2021a).

4 Probability for Trivalent Valuations

Epistemologists capture the standing of a sentence A by the *probability* of A , reflecting the agent’s evidence for and against A . When we identify sentences with sets of possible worlds, the probability of a sentence A is the cumulative credence assigned to all possible worlds where A is true.

Trivalent semantics for conditionals implements the same approach using a slight twist. As with bivalent probability, we start with a set of possible worlds W with an associated algebra \mathcal{A} , and a weight or credence function $c : \mathcal{A} \rightarrow [0, 1]$ defined on the measurable space (W, \mathcal{A}) . This function represents the subjective plausibility of a particular element of the algebra, i.e., a set of possible worlds. Our use of possible worlds is devoid of metaphysical baggage and instrumental to define credence functions, as is customary in probabilistic semantics: for us,

⁷Another source of undefinedness concerns presupposition failure, but we set aside interactions between conditionals and presupposition in this paper.

possible worlds are just Cooper valuations.⁸ Moreover, we assume that any algebra \mathcal{A} includes the singletons of worlds, i.e., for every $w \in W$, $\{w\} \in \mathcal{A}$. Finally, we assume that the credence function c is finitely additive with $c(\emptyset) = 0$, and $c(W) = 1$.

We now define a (non-classical) probability function $p : \mathcal{L}^\rightarrow \mapsto [0, 1]$, taking into account that sentences of \mathcal{L}^\rightarrow can receive three values: true, false, or indeterminate.⁹ For convenience, define

$$\begin{aligned} A_T &= \{w \in W \mid v_w(A) = 1\} & A_I &= \{w \in W \mid v_w(A) = 1/2\} \\ A_F &= \{w \in W \mid v_w(A) = 0\} \end{aligned}$$

as the sets of possible worlds where A is valued as true, false or indeterminate, relative to (Cooper) valuation functions $v_w : \mathcal{L}^\rightarrow \mapsto \{0, 1/2, 1\}$, indexed by the possible worlds they represent.

In analogy to bivalent probability, we derive the probability of a (conditional) sentence A from the (conditional) betting odds on A : how much more likely is a bet on A to be won than to be lost? For this comparison, two quantities are relevant: (1) the cumulative weight of the worlds where A is true (i.e., $c(A_T)$), and (2) the cumulative weight of the worlds where A is false, i.e., $c(A_F)$. The *decimal odds* on A are $O(A) = (c(A_T) + c(A_F))/c(A_T)$, indicating the factor by which the bettor's stake is multiplied in case A occurs and she wins the bet. Then we calculate the probability of A from the decimal odds on A by the familiar formula $p(A) = 1/O(A)$, yielding

$$p(A) := \frac{c(A_T)}{c(A_T) + c(A_F)} \quad \text{if } \max(c(A_T), c(A_F)) > 0. \text{ (Probability)}$$

Hence, the probability of a sentence corresponds to its expected semantic value, *restricted to the worlds where the sentence takes classical*

⁸Notably, this does not make the interpretation of the conditional modal or non-truth-functional: at each world w , the truth-value of $A \rightarrow C$ is given by a Cooper valuation.

⁹If you do not like to use the term "probability" in a non-classical framework, because you prefer to reserve it for standard bivalent probability, just replace it by "degree of assertability" or a similar term. This is the choice of McDermott (1996), whose definition is identical to ours. For other occurrences of that definition, see de Finetti (1936a), who pioneered it, Cantwell (2006), Rothschild (2014), and Lassiter (2020).

truth value. Additionally, we stipulate that:

$$p(A) = 1 \text{ whenever } c(A_T) + c(A_F) = 0, \quad (\text{All Indeterminate})$$

i.e., if it is certain that A takes the value $1/2$ (e.g., when A is $\perp \rightarrow \top$).

In other words, the trivalent probability of A is the ratio between the credence assigned to the worlds where A is true, and the credence assigned to the worlds where A has a classical truth value. Worlds where A takes indeterminate truth value are neglected for calculating the probability of A , except when they take up the whole space. For conditional-free sentences A and their Boolean compounds, this corresponds to the classical picture since $W = A_T \cup A_F$, or equivalently, $A_I = \emptyset$.

The idea behind (Probability) is the same that motivates classical operational definitions of probability: a sentence is assertable, or probable, *to the degree that we can rationally bet on it*, i.e., to the degree that betting on this sentence will, in the long run, provide us with gains rather than losses (e.g., Sprenger and Hartmann 2019). This is a good reason for calling the object defined by equation (Probability) a “probability”, or a measure of the plausibility of a sentence.

The structural properties of $p : \mathcal{L}^{\rightarrow} \mapsto [0, 1]$ resemble the standard axioms of probability:

- (1) $p(\top) = 1$ and $p(\perp) = 0$.
- (2) $p(A) = 1 - p(\neg A)$, provided $A_I \neq W$ (otherwise, $p(A) = p(\neg A) = 1$, by (All Indeterminate)).
- (3) $p(A \vee B) \leq p(A) + p(B)$. The equality $p(A \vee B) = p(A) + p(B)$ holds if and only if $A_T \cap B_T = \emptyset$ and $A_I = B_I$.¹⁰

Just like standard probability, our trivalent probability is not additive, but *subadditive*. Equality holds here exactly when A and B are incompatible and they take classical truth values in the same set of worlds.

The main difference to the standard picture is that the probability of a conjunction can *exceed* the probability of a conjunct. In other words, the inference of conjunction-elimination from $A \wedge B$ to B will not always preserve probability. Of course, $p(A \wedge B) \leq p(A)$ will hold as long as A and B are conditional-free sentences, but not so for conditionals. On

¹⁰The “only if” direction presupposes that $p(A) > 0$ and $p(B) > 0$.

the betting interpretation of probability, this makes sense: when A and B are false and C is true, the bet on $(A \rightarrow B) \wedge C$ yields a positive return, while the bet on $A \rightarrow B$ is called off. So we should not expect that in all circumstances $p((A \rightarrow B) \wedge C) \leq p(A \rightarrow B)$, in notable difference to bivalent probability, and some non-classical probability functions (for a survey, see Williams 2016). Exactly the same phenomenon—the failure of “and-drop” in the context of conditional reasoning—was observed in recent experiments by Santorio and Wellwood (2023). To use one of their motivating examples, while “the die will land 2 if it is even” (whose probability is $1/3$) does not entail “the die will land 1 if it is odd” (whose probability is $1/3$), their conjunction too has a probability of $1/3$ (the probability of “the die will land 1 or 2”), which is exactly what we predict in the present framework.¹¹

On this definition of probability, we obtain for conditional-free sentences $A, C \in \mathcal{L}$ that

$$p(A \rightarrow C) = \frac{c(A_T \cap C_T)}{c(A_T)} = \frac{p(A \wedge C)}{p(A)} = p(C|A)$$

(Adams’s Thesis)

as for conditional-free sentences, $p(A) = c(A_T)$, and because for bivalent A and C ,

$$\frac{c(A \rightarrow C)_T}{c(A \rightarrow C)_T + c(A \rightarrow C)_F} = \frac{c(A_T \cap C_T)}{c(A_T)}.$$

That is, instead of *postulating* Adams’s Thesis as a desideratum on the probability of a conditional, as in Stalnaker (1970) and Adams (1975, p. 3), we obtain it immediately from the semantics of trivalent conditionals, and the definition of probability as the inverse of rational betting odds.¹² The well-known triviality results by Lewis (1976) and

¹¹Santorio and Wellwood call “and-drop” the principle whereby $p(A \wedge B) < p(A)$ when $A \not\models B$, which holds for the classical definitions of probability and entailment. Convergent with our approach, Santorio and Wellwood sketch a trivalent account of their data using truth-conditions exactly equivalent to De Finetti’s for the conditional, paired with Cooper’s for the other connectives. Similarly, Ciardelli and Ommundsen (forthcoming) argue that sensible predictions for the probabilities of nested conditionals require that probability behave in a non-classical way.

¹²For more discussion of Adams’s Thesis, including experimental evidence for and against, see Stalnaker 1968; Adams 1975; Dubois and Prade 1994; Douven and Verbrugge 2010, 2013; Evans et al. 2007; Over, Hadjichristidis, et al. 2007; Égré and Cozic 2011; Over 2016; Skovgaard-Olsen, Singmann, and Klauer 2016.

others are blocked since they depend on an application of the (bivalent) Law of Total Probability, which does not hold for trivalent, non-classical probability functions (Lassiter 2020).¹³ Equipped with a definition of probability, we now proceed to characterizing logical consequence relations for certain and uncertain inference.

5 Certain Inference

For a conditional-free propositional language \mathcal{L} with only two truth values, valid inferences are supposed to preserve the truth of the premises. This is equivalent to requiring that valid inferences preserve *certainties*, i.e., probability 1 (Leblanc 1979).

In a trivalent setting, however, there is no unique notion of “truth preservation”: it can amount to preserving *strict* truth (i.e., semantic value 1), or to preserving *tolerant* truth, namely non-falsities (i.e., semantic value greater than 0), or to a combination of both (the terminology follows Cobreros et al. 2012). “Truth preservation” is thus ambiguous. On the other hand, trivalent semantics allows for a canonical extension of certainty-preserving inference to \mathcal{L}^\rightarrow : whenever all premises have probability 1, as defined in the previous section, the conclusion must have probability 1, too (Adams 1996). We call this logic **C** for “certainty-preserving inference”. Formally:

Definition 1 (Certainty Preservation or **C**-validity). *For a set of formulas $\Gamma \subseteq \mathcal{L}^\rightarrow$ and a formula $B \in \mathcal{L}^\rightarrow$, the inference from Γ to B is **C**-valid, in symbols $\Gamma \models_{\mathbf{C}} B$, if and only if for all probability functions $p : \mathcal{L}^\rightarrow \mapsto [0, 1]$: if $p(A) = 1$ for all $A \in \Gamma$, then also $p(B) = 1$.*

C is a logic that tracks reasoning with (fully) accepted sentences: given an epistemic state represented by a probability distribution, it is impossible for anyone to accept premises without committing themselves to the conclusion (compare the analogous definition of reasonable inference in Stalnaker 1975, p. 271).

¹³Bradley (2000) proposes a different triviality result: arguably, we want indicative conditionals to satisfy the Preservation Condition, such that if $p(A) > 0$ and $p(C) = 0$, then $p(A \rightarrow C) = 0$. But for this to hold in full generality, we need to posit strong logical dependencies between a conditional and its components, thus trivializing the conditional. This is indeed so for bivalent accounts, but our trivalent account implies the Preservation Condition as a theorem without having a vicious dependency between the truth values of A , C and $A \rightarrow C$.

We can show that \mathbf{C} has an equivalent characterization in trivalent logic: an inference is \mathbf{C} -valid if and only if tolerant truth is preserved in passing from Γ to B . Equivalently, we cannot assign a designated value (1 or $1/2$) to the premises without assigning it to the conclusion, too. This is the main result of this section.

Proposition 1 (Trivalent Characterization of \mathbf{C}). *For a set of formulas $\Gamma \subseteq \mathcal{L}^\rightarrow$ and a formula $B \in \mathcal{L}^\rightarrow$, the following are equivalent:*

- (1) $\Gamma \models_{\mathbf{C}} B$.
- (2) For all Cooper valuations $v : \mathcal{L}^\rightarrow \mapsto \{0, 1/2, 1\}$: if $v(A) \geq 1/2$ for all $A \in \Gamma$, then also $v(B) \geq 1/2$.

In other words, \mathbf{C} preserves truth in the (tolerant) sense that we cannot infer a false conclusion from a set of non-false premises. Equivalently, if the conclusion is false, one of the premises must be false. We have thereby got an analogous result to the equivalence between truth-preserving and certainty-preserving inference in standard propositional logic.

\mathbf{C} satisfies the classic principle $B \models_{\mathbf{C}} A \rightarrow B$, i.e., if we are certain that Bob comes to the party, then we are also certain that Bob comes to the party if Alice does. While this inference is fallacious when premises are uncertain (and known as one of the paradoxes of material implication), it is valid in any context where we have verified the premise—whether empirically or by mathematical proof.¹⁴ We also retain Conditional Proof ($A \models_{\mathbf{C}} B$ implies $\models_{\mathbf{C}} A \rightarrow B$), and some characteristic principles of deductive reasoning in \mathbf{C} , such as Modus Ponens and the Law of Identity ($\models_{\mathbf{C}} A \rightarrow A$). The logic itself is non-classical, however, it is paraconsistent ($A, \neg A \not\models_{\mathbf{C}} B$), and it satisfies various connexive principles, such as Negation Commutation, making it even negation-inconsistent ($\models_{\mathbf{C}} (A \wedge \neg A \rightarrow (A \vee \neg A) \wedge \neg(A \wedge \neg A \rightarrow (A \vee \neg A)))$).¹⁵ Some inferences such as Modus Tollens, and more generally classical

¹⁴This behavior is similar to that of the conditional developed in state space semantics by Leitgeb (2017). Note that the inference from $\neg A$ to $A \rightarrow B$ is blocked in \mathbf{C} , but valid when A is an atomic formula whose interpretation is two-valued.

¹⁵ \mathbf{C} is nearly Cooper’s propositional logic of Ordinary Discourse—except that we do not restrict \mathbf{C} to atom-classical valuations. The system is called QCC/TT in Égré, Rossi, and Sprenger (2021a), to locate it within a broader map of consequence relations and schemes for the connectives. Remarkably, Cooper himself characterized his logic proof-theoretically, making only instrumental use of trivalent semantics to establish completeness.

laws in \mathcal{L} , are only valid when restricted to conditional-free sentences with atom-classical valuations—we will get back to this in Section 7. Notably, these properties depend on interpreting the conditional via the Cooper truth table: if we had paired preservation of non-falsity with the de Finetti truth table instead, we would have lost Modus Ponens—arguably a substantial drawback for a logic that generalizes deductive logic to certain inference with conditionals.

\mathbf{C} retains Disjunctive Syllogism ($A \vee B, \neg A \models B$), but gives up Disjunction Introduction ($A \models A \vee B$). Again, $p(A) = 1$ will ensure that $p(A \vee B) = 1$ when A and B are themselves factual, conditional-free sentences. But for conditionals, we again seem to find exceptions that make intuitive sense. To use a variation on the previous example by Santorio and Wellwood, consider a die whose faces are 2-2-2-3-3-5. Then the conditional “if it lands even, it will land 2” has probability 1. But the disjunction “either it will land 2 if even, or it will land 5 if odd” only gets a probability of $2/3$ (the probability of “it will land 2 or 5”), since cases in which it lands on 3 make the first conditional void, and the second conditional false.¹⁶

Characterizing \mathbf{C} in terms of truth-preservation rather than probabilistically is not only of theoretical interest, but greatly simplifies the study of this logic: to decide theorems and valid inferences, it suffices to look at the truth tables. Section 7 studies the theorems and valid inferences in more detail and compares certain inference with \mathbf{C} to uncertain inference where instead of certainty, high probability is preserved. Notably, these properties depend on interpreting the conditional via the Cooper truth table: if we had paired preservation of non-falsity with the de Finetti truth table instead, we would have lost Modus Ponens—arguably a substantial drawback for a logic that generalizes deductive logic to certain inference with conditionals.

6 Uncertain Inference

Certain inference with conditionals is arguably monotonic: when we know B for certain, or when we suppose it as holding no matter what, we also know that B is the case under the condition that A . However, when we move to *uncertain* inference, where only high probability or

¹⁶This is a failure of “or-drop” in the sense of Santorio and Wellwood (2023), namely the rule whereby $A \not\models B$ entails $p(A) < p(A \vee B)$.

degree of assertability is preserved, things change. We may accept, assert, or find plausible B , but reject B under the condition that A . For example, the conditional “if Real Madrid faces Juventus in their next match, then Real Madrid will win” sounds highly plausible, whereas “if Real Madrid faces Juventus in their next match but most of their players are sick, then Real Madrid will win” seems much less plausible. A logic of inference with *uncertain* premises \mathbf{U} should therefore, unlike the logic \mathbf{C} , have a non-monotonic conditional, i.e., we cannot infer from $A \rightarrow C$ that $A \wedge B \rightarrow C$ for any A, B and $C \in \mathcal{L}^{\rightarrow}$.

The canonical definition of validity in a logic of uncertain inference preserves probability, as a proxy for degree of assertability (compare Adams 1975). In other words, the probability of the premise A must never exceed the probability of the conclusion B . Almost all logics of uncertain reasoning agree on this criterion for single-premise inference, which is the natural analogue of truth preservation in certain reasoning, and so we adopt it as our definition of single-premise logical consequence in uncertain reasoning:

Definition 2 (Valid Single-Premise Inference in \mathbf{U}). *For formulas $A, B \in \mathcal{L}^{\rightarrow}$: $A \models_{\mathbf{U}} B$ if and only if $p(A) \leq p(B)$ for all probability functions $p : \mathcal{L}^{\rightarrow} \mapsto [0, 1]$ based on credence functions $c : \mathcal{A} \mapsto [0, 1]$.*

Two corollaries are now immediate from Definition 2:

Corollary 1. $\models_{\mathbf{U}} B$ if and only if $p(B) = 1$ for all probability functions $p : \mathcal{L}^{\rightarrow} \mapsto [0, 1]$ based on credence functions $c : \mathcal{A} \mapsto [0, 1]$.

Corollary 2. \mathbf{C} and \mathbf{U} have the same theorems.

We can now show that the consequence relation of \mathbf{U} has, for contingent sentences A and B , equivalent characterizations in our system of certain inference \mathbf{C} , and by means of trivalent valuations:

Proposition 2 (Equivalent Characterizations of Valid Single-Premise Inference in \mathbf{U}). *For $A, B \in \mathcal{L}^{\rightarrow}$, with $\not\models_{\mathbf{C}} \neg A$ and $\not\models_{\mathbf{C}} B$, the following are equivalent:*

- (1) $A \models_{\mathbf{U}} B$.
- (2) For all Cooper valuations $v : \mathcal{L}^{\rightarrow} \mapsto \{0, 1/2, 1\}$, $v(A) \leq v(B)$. In other words, if $v(A) = 1$ then $v(B) = 1$, and if $v(A) \geq 1/2$, then $v(B) \geq 1/2$.
- (3) $A \models_{\mathbf{C}} B$ and $\neg B \models_{\mathbf{C}} \neg A$.

Condition (2) expresses that the semantic value of the conclusion must not fall below the semantic value of the premise in all possible valuations. Condition (3) expresses that $A \models_{\mathbf{C}} B$ and $\neg B \models \neg A$, i.e., from A we can infer B , and from $\neg B$ we can infer $\neg A$ in a logic of certain reasoning.¹⁷ Thus, \mathbf{U} validates fewer inferences than \mathbf{C} . The proposition states that all these conditions are equivalent to demanding that for all probability functions, the conclusion be at least as probable as the premise.

Extending this criterion to multi-premise inference is non-trivial. There are several choice points here. The first concerns the definition of probabilistic validity. Should the probability of the conclusion be at least as high as that of the *least probable* premise? Should the conclusion be at least as plausible as the *conjunction* of the premises? (Remember those definitions can differ, given the properties of conjunction in our system). Should it follow Adams's uncertainty preservation criterion, namely for the uncertainty of the conclusion to not exceed the sum of the uncertainties of the premises (Adams 1975, 1996)? A second choice point concerns whether validity should be structurally monotonic, namely insensitive to the addition of new premises, or whether the consequence relation should mirror the nonmonotonic property of the conditional.

We propose that $\Gamma \models_{\mathbf{U}} B$ if and only if for a subset $\Delta \subseteq \Gamma$ of the premises, *the probability of the conjunction of the elements of Δ never exceeds the probability of the conclusion*, regardless of the choice of the probability function. Formally:

Definition 3 (Valid Multi-Premise Inference in \mathbf{U}). *For a set of formulas $\Gamma \subseteq \mathcal{L}^{\rightarrow}$ and a formula $B \in \mathcal{L}^{\rightarrow}$: $\Gamma \models_{\mathbf{U}} B$ if and only if there is a finite subset of the premises $\Delta \subseteq \Gamma$ such that for all probability functions $p : \mathcal{L}^{\rightarrow} \mapsto [0, 1]$, $p(\bigwedge_{A \in \Delta} A) \leq p(B)$.*

Let us make two comments on this definition. Firstly, taking the probability of the conjunction of the premises will explain why Modus Ponens can be lost in uncertain inference. Secondly, defining validity by means of existential quantification over (possibly improper) *subsets* of Γ allows us to preserve the fact that a set of premises entails each

¹⁷Égré, Rossi, and Sprenger (2021a) call this logic QCC/SS \cap TT since it preserves both strict and tolerant truth value (=both strict truths and non-falsities). This is one of the logics entertained in Belnap (1973).

of its members, namely $\Gamma \models_{\mathcal{U}} A$ for any $A \in \Gamma$ (compare Dubois and Prade 1994, p. 1729; Adams 1996, p. 5).

But we could also define $\Gamma \models_{\mathcal{U}} B$ by just requiring that the conjunction of *all* members of Γ have lower probability than B , for all probability functions. This would make the consequence relation structurally non-monotonic, as it would be possible to have $A, B \not\models_{\mathcal{U}} A$, despite the fact that $A \models_{\mathcal{U}} A$ for every A . This failure of structural monotonicity aligns well, at first sight, with the non-monotonic behavior of the conditional in \mathcal{U} (i.e., $A \rightarrow C \not\models_{\mathcal{U}} (A \wedge B) \rightarrow C$). On the other hand, conditional logic is often considered a *generalization* of (monotonic) classical logic and not as a substantial departure from it (compare Adams 1965, pp. 186-87). We present the most conservative option here, but bearing in mind that for the properties of \mathcal{U} that we discuss in the sections that follow, the other choice could be made as well.

We can now extend the equivalence between probabilistic inference and a trivalent consequence relation from the single-premise to the multi-premise case. First, we need to define a consistent set of formulas in the trivalent setting:

Definition 4 (Consistent and Inconsistent Sets). *A set of formulas $\Gamma \subseteq \mathcal{L}^{\rightarrow}$ is consistent if it has no subset $\Delta \subseteq \Gamma$ such that $p(\bigwedge_{A \in \Delta} A) = 0$ in all probability functions $p : \mathcal{L}^{\rightarrow} \mapsto [0, 1]$. Equivalently, no subset $\Delta \subseteq \Gamma$ satisfies $\models_{\mathcal{C}} \neg(\bigwedge_{A \in \Delta} A)$. Γ is inconsistent if such a subset exists.*

Second, we show that multi-premise inference from consistent premise sets has an equivalent trivalent representation:

Proposition 3 (Equivalent Characterizations of Valid Multi-Premise Inference in \mathcal{U}). *For a consistent set of formulas $\Gamma \subseteq \mathcal{L}^{\rightarrow}$ and $B \in \mathcal{L}^{\rightarrow}$ with $\not\models_{\mathcal{C}} B$, the following are equivalent:*

- (1) $\Gamma \models_{\mathcal{U}} B$.
- (2) *There is a finite subset of premises $\Delta \subseteq \Gamma$ such that the semantic value of B is, for all Cooper valuations v , at least as high as the semantic value of the conjunction of the premises: $v(\bigwedge_{A_i \in \Delta} A_i) \leq v(B)$.*
- (3) *There is a finite subset of premises $\Delta \subseteq \Gamma$ such that $\bigwedge_{A_i \in \Delta} A_i \models_{\mathcal{C}} B$ and $\neg B \models_{\mathcal{C}} \neg(\bigwedge_{A_i \in \Delta} A_i)$.*

As with \mathbf{C} , the equivalence of (1) with (2) and (3) is not only attractive from a computational point of view, but it also connects probabilistic reasoning with conditionals to the trivalent semantics that defines their truth conditions in the first place.¹⁸

Proposition 3 also provides sound and complete calculi for the logic \mathbf{U} for free. For instance, since Cooper (1968) has a sound and complete Hilbert-style calculus for \mathbf{C} , this automatically translates, thanks to Proposition 3, into a sound and complete calculus for \mathbf{U} . Validity in \mathbf{U} is nothing else but the combination of two valid consequence relations in \mathbf{C} . Alternatively, still using Proposition 3, tableau- and sequent-style sound and complete axiomatizations of \mathbf{U} can be extracted from Égré, Rossi, and Sprenger (2021b).

7 Properties of \mathbf{U} and \mathbf{C}

We now evaluate the logic \mathbf{U} and the logic \mathbf{C} in terms of the inference schemes they validate, using the principles in Table 6, taken from the survey article by Égré and Rott (2021). In the table, a plus sign “+” indicates that a (meta-)inference is valid for all trivalent valuations over the atoms, and $\Gamma \models_{\mathbf{CL}} B$ means that the inference from Γ to B is classically valid. The plus sign in brackets indicates that a (meta-)inference is not valid under all trivalent valuations, but that it is valid provided schematic letters do not contain conditionals and atoms can only take value 1 or 0 (i.e., in atom-classical valuations, following the terminology of Humberstone 2011, p. 1045). A minus sign “-” indicates that a (meta-)inference is not even valid under that restriction. Finally, the connective \equiv is used in two occurrences to express identity of truth value (so $v(A \equiv B) = 1$ if $v(A) = v(B)$ and $v(A \equiv B) = 0$ otherwise).

The consideration of atom-classical valuations is to flag that \mathbf{C} and \mathbf{U} can in some cases recapture classical principles that are lost with nested conditionals in particular. As stressed by Cooper (1968, p. 314), however, \mathbf{C} , and similarly \mathbf{U} , would not be closed under Uniform Substitution if we defined validity in terms of atom-classical valuations. Consider Modus Tollens: while $p \rightarrow q, \neg q$ entails $\neg p$ in both \mathbf{C} and \mathbf{U}

¹⁸Without the restriction to consistent premise sets, Proposition 3 would not hold since \mathbf{U} satisfies, like Adams’s logic of conditionals, the *probabilistic ex falso* principle: a proposition which always obtains zero probability implies everything. If $p(A) = 0$ for all probability functions, $A \models_{\mathbf{U}} B$ holds, but without constraining the valuations of B according to (2).

Constitutive and Generally Desirable Principles in Uncertain Inference		C	U
Logical Truth	$\models A \rightarrow \top$	+	+
Law of Identity	$\models A \rightarrow A$	+	+
Stronger-Than-Material	$A \rightarrow B \models A \supset B$	(+)	(+)
Conjunctive Sufficiency	$A, B \models A \rightarrow B$	+	(+)
AND	$A \rightarrow B, A \rightarrow C \models A \rightarrow (B \wedge C)$	+	+
OR	$A \rightarrow C, B \rightarrow C \models (A \vee B) \rightarrow C$	+	(+)
Cautious Transitivity	$A \rightarrow B, (A \wedge B) \rightarrow C \models A \rightarrow C$	+	(+)
Cautious Monotonicity	$A \rightarrow B, A \rightarrow C \models (A \wedge C) \rightarrow B$	+	+
Rational Monotonicity	$A \rightarrow B, \neg(A \rightarrow \neg C) \models (A \wedge C) \rightarrow B$	+	+
Reciprocity	$A \rightarrow B, B \rightarrow A \models (A \rightarrow C) \equiv (B \rightarrow C)$	+	(+)
Conjunction Elimination	$A \wedge B \models A$	+	(+)
Disjunction Introduction	$A \models A \vee B$	(+)	(+)
Optional and Disputed Principles			
Supraclassicality (Inferences)	if $\Gamma \models_{\text{CL}} B$ then $\Gamma \models B$	-	-
Modus Ponens	$A \rightarrow B, A \models B$	+	(+)
Modus Tollens	$A \rightarrow B, \neg B \models \neg A$	(+)	(+)
Simplifying Disjunctive Antecedents	$(A \vee B) \rightarrow C \models (A \rightarrow C) \wedge (B \rightarrow C)$	(+)	(+)
Import-Export	$A \rightarrow (B \rightarrow C)$ if and only if $(A \wedge B) \rightarrow C$	+	+
Or-to-If	$\neg A \vee B \models A \rightarrow B$	+	-
Conditional Excluded Middle	$\models (A \rightarrow B) \vee (A \rightarrow \neg B)$	+	+
Connexive Principles (optional)			
Aristotle's Thesis	$\models \neg(\neg A \rightarrow A)$	+	+
Boethius's Thesis	$\models (A \rightarrow C) \rightarrow \neg(A \rightarrow \neg C)$	+	+
Undesirable Principles			
Contraposition	$A \rightarrow C \models \neg C \rightarrow \neg A$	(+)	-
Monotonicity	$A \rightarrow C \models (A \wedge B) \rightarrow C$	+	-
Transitivity	$A \rightarrow B, B \rightarrow C \models A \rightarrow C$	+	-
Generally Desirable Meta-Inferences in Uncertain Reasoning			
Supraclassicality (Laws)	if $\models_{\text{CL}} A$, then $\models A$	(+)	(+)
Left Logical Equivalence	if $A \models_{\text{CL}} B, B \models_{\text{CL}} A$, then $A \rightarrow C \models B \rightarrow C$	+	+
Right Weakening	if $B \models_{\text{CL}} C$, then $A \rightarrow B \models A \rightarrow C$	+	(+)
Rule of Conditional K	if $A_1, \dots, A_n \models_{\text{CL}} C$, then $(B \rightarrow A_1), \dots, (B \rightarrow A_n) \models (B \rightarrow C)$	+	(+)

Table 6: Overview of characteristic inferences and meta-inferences in uncertain reasoning with conditionals. +: the (meta-)inference is generally valid in C or U; (+): the (meta-)inference is valid only for conditional-free sentences with classical valuations of propositional atoms; -: the (meta-)inference is invalid.

when the interpretation of p and q is bivalent, the substitution instance $(p \rightarrow (q \rightarrow r)), \neg(q \rightarrow r)$ fails to entail $\neg p$ in both logics under the same restriction (assign q the value 0 and p the value 1, and r either 1 or 0). When validity is defined by taking account all trivalent valuations over the atoms, however, Uniform Substitution holds, and the arguments

above can be expressed as schemata using formula letters instead of atomic symbols.

The principles above the first horizontal line are generally considered to be desirable, or at least harmless, in uncertain reasoning with conditionals. The principles between the lines—e.g., Modus Ponens, Or-To-If, Import-Export, and Conditional Excluded Middle—are typically a bone of contention between theorists. We also include some tautologies that are distinctive for connexive logics. The principles at the bottom—Contraposition, Monotonicity and Transitivity—are characteristic of most monotonic logics, and logics of deductive inference in particular, but should *not* be satisfied by a logic of uncertain reasoning with non-monotonic conditionals (for compelling counterexamples, see Adams 1965). So we should expect that these principles are satisfied by **C**, but not by **U**.

Table 6 evaluates, in the rightmost columns, **C** and **U** with respect to all these principles. We cannot discuss each of them in detail, but we make some general observations. Many desirable or harmless principles are satisfied by **U** without restriction, whereas some of them only hold for conditional-free expressions over atom-classical valuations.

When we compare **U** to *classical* conditional logics (i.e., logics where all valuations are bivalent, such as Stalnaker-Lewis logics), we can consider the principles valid since making a comparison presupposes atom-classical valuations. Specifically, **U** recovers all valid inferences of System P, which is a classical benchmark for conditional logics (Adams 1975; Kraus, Lehmann, and Magidor 1990).¹⁹ Moreover, both **C** and **U** validate connexive principles such as Aristotle’s Thesis ($\neg(\neg A \rightarrow A)$) and Boethius’s Thesis ($(A \rightarrow B) \rightarrow \neg(A \rightarrow \neg B)$).

Principles that are typically considered problematic for non-monotonic conditionals—Monotonicity, Contraposition, Transitivity, (Egré and Rott 2021)—are *not* valid in **U**. These principles do not even hold when we restrict **U** to atom-classical valuations. However, they *do* (mainly) hold in our logic of certain inference **C**, where the conditional behaves monotonically. This feature is in line with our view of **C** as a generalization of classical deductive logic to a language with a conditional. In particular, Contraposition holds in **C** for atom-classical valuations of simple conditionals $p \rightarrow q$, although it does not in gen-

¹⁹Adams (1975) characterized his logic of uncertain inference by seven syntactic principles whose combination is known as System P: the Law of Identity, AND, OR, Cautious Monotonicity, Left Logical Equivalence, and Right Weakening.

eral for nested conditionals such as $p \rightarrow (q \rightarrow r)$.²⁰ In **U**, things are more straightforward: contraposition fails to preserve probability even for simple conditionals, since $p(B|A)$ can be greater than $p(\neg A|\neg B)$ for $A, B \in \mathcal{L}$.

Most interesting are optional and disputed (meta-)inference principles. Supraclassicality for inferences fails because **C** does not support Explosion, e.g., while $A \wedge \neg A \models_{\text{CL}} B$ holds for any two sentences A and B , it is not the case that $A \wedge \neg A \models_{\text{C}} B$. However, all classical laws are theorems of both **C** and **U** when restricted to atom-classical valuations. Modus Ponens and Modus Tollens hold for conditional-free sentences, but break down for nested conditionals—in line with McGee’s famous objections (see the next section for a detailed analysis). Also Simplification of Disjunctive Antecedent is preserved for atom-classical valuations only.

Import-Export holds unrestrictedly, since $A \rightarrow (B \rightarrow C)$ and $(A \wedge B) \rightarrow C$ have exactly the same truth conditions. The principle is intuitively plausible: “it appears to be a fact of English usage, confirmed by numerous examples, that we assert, deny, or profess ignorance of a compound conditional $A \rightarrow (B \rightarrow C)$ under precisely the circumstances under which we assert, deny, or profess ignorance of $(A \wedge B) \rightarrow C$ ” (McGee 1989, p. 489). Experimental evidence seems to confirm this assessment (van Wijnbergen-Huitink, Elqayam, and Over 2015). Indeed, one motivation for giving up Import-Export—e.g., in the probabilistic semantics of Sanfilippo et al. (2020)—is the pressure from Gibbard’s and Lewis’s triviality results, where Import-Export is a crucial premise (Fitelson 2015). Some accounts therefore restrict the validity of Import-Export to simple conditionals and set up an error theory of why we infer from there to the general validity of the principle. For example, Mandelkern (2020) suggests to restrict the scope of Import-Export to cases where the B in $A \rightarrow (B \rightarrow C)$ does not contain a conditional. Whether this strategy is successful is controversial (Ciardelli and Ommundsen forthcoming). By contrast, in **C** and **U** the universal validity of Import-Export does not create problems since the

²⁰Consider the Cooper valuation $v(p) = 1$ and $v(q) = 0$. This implies $v(p \rightarrow (q \rightarrow r)) = 1/2$ but $v(\neg(q \rightarrow r) \rightarrow \neg p) = 0$. By Proposition 1, this means that Contraposition fails in **C**. In the probabilistic case: consider a fair die where the sides 1 and 2 are marked in red, and the other sides in blue. Then the conditional “if it lands even, then if the outcome is greater than 2, it is blue” is certain. But the contraposed conditional “if it is not the case that if the outcome is greater than 2 the die lands blue, then it does not land even” only gets probability $1/2$ on our account.

triviality results do not apply to these logics (compare Égré, Rossi, and Sprenger 2023).

Conditional Excluded Middle (CEM) is a validity of \mathbf{C} , and is therefore valid in \mathbf{U} as well. Various analyses of indicatives endorse CEM (e.g., Stalnaker 1980; Williams 2010; Ciardelli 2020; Santorio 2022a), but there are also notable opponents (e.g., Lewis 1973b; Gillies 2009; Kratzer 2012). In \mathbf{C} , CEM is a consequence of commutation with negation, i.e., the semantic equivalence between $\neg(A \rightarrow B)$ and $A \rightarrow \neg B$, which holds in our system and is independently motivated (see footnote 3). To see this, note that $(A \rightarrow B) \vee \neg(A \rightarrow B)$ —an instance of the Law of Excluded Middle—immediately entails $(A \rightarrow B) \vee (A \rightarrow \neg B)$, that is CEM.

Finally, a crucial difference between \mathbf{C} and \mathbf{U} concerns the relation of the indicative to the material conditional $A \supset B := \neg A \vee B$. On the one hand, $A \supset B \models_{\mathbf{C}} A \rightarrow B$. Equivalently, we get the Or-to-If entailment in \mathbf{C} , namely $A \vee B \models_{\mathbf{C}} \neg A \rightarrow B$ —an instance of what Stalnaker (1975, p. 279) calls a *reasonable* inference, and which we can handle in terms of certainty preservation. To use an example from Edgington (1986, p. 191), if I am certain that it is either 8 o'clock or 11 o'clock, then I am also certain that if it is not 8 o'clock, it is 11 o'clock.

However, this inference is *invalid* when we infer the conditional from an uncertain disjunction. Edgington's point is that if I am 90% confident that it is 8 o'clock, then I am at least as confident that it is 8 or 11 o'clock, but that does not give me the same confidence that if it is not 8 then it is 11 o'clock. Thus, Or-to-If fails in \mathbf{U} , and neither does the material conditional imply the indicative conditional in \mathbf{U} , nor vice versa.

However, the simple, non-nested indicative conditional often appears to be more demanding to assert than the material conditional (e.g., Gibbard 1981; Gillies 2009). Can our account then explain this "Stronger-Than-Material" intuition? Yes—because for conditional-free sentences A and B over atom-classical valuations, $A \rightarrow B$ entails $A \supset B$ in both \mathbf{C} and in \mathbf{U} . In the context of uncertain reasoning with conditional-free statements, $p(A \rightarrow B) = p(B|A) \leq p(A \supset B)$ is a theorem. In summary, we have Or-to-If as a valid principle for reasoning from *certain* premises, but not from *uncertain* premises; nonetheless, we can explain why $A \rightarrow B$ is less acceptable than $A \supset B$ whenever antecedent and consequent are conditional-free sentences.

8 The variable status of Modus Ponens

Modus Ponens is one of our most endorsed forms of inference. However, McGee (1985) challenged its validity as a rule for rational belief in a famous counterexample that concerns the 1980 U.S. presidential elections:

If a Republican wins the election, then, if Reagan does not win, Anderson will win.

A Republican will win the election.

Therefore, if Reagan does not win the election, Anderson will.

At some point before the elections, the two premises were reasonable to believe: Ronald Reagan was predicted to win the election, and Anderson was the runner-up behind Reagan in the Republicans' primary race. By Modus Ponens we infer that if Reagan does not win, Anderson will. The logical form of that inference is: from $A \rightarrow (B \rightarrow C)$ and A , infer, by Modus Ponens, $B \rightarrow C$. However, in the polls Anderson was actually trailing both Reagan and Carter, the democrat incumbent. The sentence "if Reagan does not win the election, Anderson will" is hardly believable and the inference seems therefore unreasonable.

McGee's counterexample has generated a large literature concerning the validity of Modus Ponens. First, it is important to note that its pull depends on whether we tie logical validity to truth preservation or to probability preservation. Bledin (2015, pp. 67-68) and Punčochár and Gauker (2020, pp. 657-658), who defend Modus Ponens as a valid form of inference, argue that any context or information state that makes the premises true must also make true the conclusion. This diagnosis is shared by our logic of certain inference **C**: if we fully accept the premises (i.e., as a certainty), we are also forced to accept the conclusion. By contrast, McGee shows that Modus Ponens is an unreliable rule of inference whenever valid inference is not taken to preserve truth or certainty, but only to regulate our credences and partial beliefs (a position defended by Field 2015).

As stressed by McGee, the intuitive appeal of the counterexample depends crucially on the use of nested conditionals. In particular, Stern and Hartmann (2018) show that when the major premise of Modus Ponens is a nested conditional, the probability loss in inferring to the

conclusion can be much higher than when we apply Modus Ponens to non-nested premises. For conditional-free sentences A and B , the term

$$p(B) = p(B|A)p(A) + p(B|\neg A)(1 - p(A)) \quad (1)$$

is, by the Law of Total Probability, well controlled by the values of $p(A)$ and $p(B|A)$ —the values that represent the probability of the two premises of Modus Ponens. For example, if both values exceed .9, then $p(B) \geq .81$, so the product of the two probabilities is still a reasonably high value.

However, in the case of right-nested conditionals, the probability of the conclusion of Modus Ponens is poorly controlled:

$$p(C|B) = p(C|A \wedge B)p(A|B) + p(C|\neg A \wedge B)(1 - p(A|B)) \quad (2)$$

Suppose that premises are highly plausible, e.g. $p(A) \geq .9$ and $p(C|A \wedge B) \geq .9$, where the latter probability has been calculated by applying Import-Export and Adams’s Thesis to $A \rightarrow (B \rightarrow C)$. Then you can still assign extremely low values to three of the four probabilities on the right hand side of equation (2), and derive a very low value of $p(C|B)$. Therefore the probability loss is more pronounced in McGee’s example than when we apply Modus Ponens to simple conditionals.

Our logics mirror this diagnosis: Modus Ponens is valid in \mathbf{U} for simple conditionals with atom-classical valuations. However, \mathbf{U} does *not* validate the unrestricted form of Modus Ponens. In fact, the only countermodel to the schema $A \rightarrow B, A \models B$ is $v(A) = 1$ and $v(B) = 1/2$ (i.e., B is a conditional with false antecedent).²¹ The same kind of analysis can be applied to Modus Tollens: the schema $A \rightarrow B, \neg B \models \neg A$, is valid for atom-classical valuations, but invalid if we allow for $v(B) = 1/2$.

The fact that McGee’s Modus Ponens examples are analyzed as valid in \mathbf{C} and as invalid in \mathbf{U} is in accordance with an ambivalence regarding the validity of Modus Ponens that many modal seman-

²¹Suppose that “A Republican will win” is true if and only if Reagan or Anderson wins. The main conditional then has probability 1 (since Or-to-If is valid in \mathbf{C}), the disjunction has high probability, and the consequent has a low probability. Thus, nested Modus Ponens in McGee-type examples fails if and only if the associated Or-to-If inference fails. A trivalent countermodel to the inference in \mathbf{U} is obtained by assigning “Reagan wins” the value 1, and “Anderson wins” the value 0.

tics exhibit. Specifically, Kratzer (1986, 2012) restricts the validity of Modus Ponens to non-modal and non-conditional sentences. Khoo and Mandelkern (2019) distinguish, in a dynamic semantics framework, between two forms of Modus Ponens: one remains valid while the other falls prey to McGee’s counterexamples. Finally, like us, Santorio (2022b) considers Modus Ponens valid in certain, but invalid in uncertain inference (see also Neth 2019).

Since Import-Export features crucially in McGee’s counterexample and his impossibility theorem from the same paper (McGee 1985, pp. 465-466), philosophers and logicians have often faced a choice between both principles. Unlike the theorists who give up or restrict Import-Export, but retain (some form of) Modus Ponens (e.g., Stalnaker 1968; Lewis 1973b; Mandelkern 2020), we accept Import-Export as universally valid and restrict the validity of Modus Ponens. This account does not only give a convincing analysis of in uncertain reasoning, but also takes into account the independent reasons for retaining Import-Export that we have outlined in Section 7.

9 Comparisons

The trivalent treatment of indicative conditionals is first sketched in Reichenbach (1935) and de Finetti (1936a,b). A more detailed motivation of this approach, including an overview of the main consequence relations of interest, is given by Belnap (1970, 1973), but none of these authors provides a fully worked out account of the logic and epistemology of conditionals. The first complete trivalent account of a logic of conditionals is due to Cooper (1968), who originally created system C. However, Cooper does not connect it to the probability of conditionals. Cantwell (2008) investigates the logical consequence relation of C (=preservation of non-falsity), but uses Strong Kleene connectives for conjunction and disjunction. Moreover, his treatment of “non-bivalent probability” ends up with an altogether different probabilistic logic (Cantwell 2006).

Most similar to our approach, both in spirit and content, are the trivalent accounts developed by Dubois and Prade (1994) and McDermott (1996). However, these authors stick to de Finetti’s original truth table and (in the case of McDermott) use Strong Kleene truth tables for conjunction and disjunction. The semantic properties are thus quite

different. On the level of inferences, many features are similar, but McDermott’s logic validates Transitivity ($A \rightarrow B, B \rightarrow C$, therefore $A \rightarrow C$). While this is acceptable and even desirable in the framework of *certain* inference, it is arguably problematic when reasoning from *uncertain* premises since the probability of $p(C|A)$ is in no way controlled by $p(C|B)$ and $p(B|A)$; in fact, it can be arbitrarily low. Suppose that you live in a very sunny, dry place. Consider the sentences $A =$ “it will rain tomorrow”, $B =$ “I will work from home”, $C =$ “I will work on the balcony”. Clearly, both $A \rightarrow B$ and $B \rightarrow C$ are highly plausible, but $A \rightarrow C$ isn’t. This structural feature offers, in our view, a decisive reason to prefer our model to McDermott’s. Dubois and Prade avoid that feature, but like Adams and Cooper, they restrict their account to the flat fragment of $\mathcal{L}^{\rightarrow}$, i.e., allowing only simple, non-nested conditionals.

Inference Principle	Trivalent Logics		Bivalent Logics		
	U	MD	P	VC	C2
Stronger-Than-Material	(+)	+	+	+	+
Conjunctive Sufficiency	(+)	+	+	+	+
OR	(+)	+	+	+	+
Cautious Transitivity	(+)	+	+	+	+
Transitivity	-	+	-	-	-
Modus Ponens	(+)	(+)	+	+	+
Modus Tollens	(+)	(+)	+	+	+
Import-Export	+	+	N/A	-	-
SDA	(+)	+	N/A	-	-
Rational Monotonicity	+	(+)	N/A	+	+
Conditional Excluded Middle	+	+	N/A	-	+

Table 7: Comparison of the logic U with alternative conditional logics, restricted to inference principles where not all of the logics agree. The surveyed alternatives are System P, Lewis’ VC, Stalnaker’s C2, and McDermott’s MD.

On the side of reasoning, our logic U generalizes the benchmark account of uncertain reasoning developed in Adams’s (1975) monograph *The Logic of Conditionals*. In this book, Adams equates the probability of a conditional $A \rightarrow C$ with the conditional probability $p(C|A)$, and develops a probabilistic logic of uncertain reasoning with conditionals on that basis. The descriptive accuracy of the predictions of Adams’s logic is acknowledged both by philosophers and by psychologists of reasoning (e.g., McGee 1989, pp. 487-488; Ciardelli 2020, p. 544; Over, Hadjichristidis, et al. 2007; Over and Baratgin 2017), but due the lack of general truth conditions for compounds and Boolean combinations of conditionals, it has limited scope. Our account recovers all the infer-

ences in Adams’s logic of reasonable inference without suffering from these restrictions. Specifically, some principles that Adams needs to postulate as axioms, such as the equation $p(A \rightarrow B) = p(B|A)$ (for $A, B \in \mathcal{L}$) or the Import-Export Principle, emerge as *corollaries* of our semantics. This makes our account more unified and coherent than Adams’s.

We conclude our comparisons with a note on other truth-conditional approaches. The classical modal semantics for a conditional $A \rightarrow C$ defines it as true if C is true at the closest possible A -world (e.g., as defined by Stalnaker’s selection function or Lewisian spheres: Stalnaker 1968, 1975; Lewis 1973b,a; McGee 1989). If A is true in the actual world, the truth value of the conditional corresponds to the truth value of the consequent, as in our analysis. The fundamental difference emerges when A is false: while we assign a third truth value to the conditional, modal theorists assign a classical truth value, based on evaluating whether the consequent is true or false at the closest worlds where the antecedent is true. In other words, Stalnaker-Lewis semantics creates a disparity between the case where A is true, where truth conditions are factual, and the case where A is false, where truth conditions depend on considerations of plausibility and normality. On our approach, epistemological considerations are relevant to assertion and reasoning, but truth conditions are entirely factual.

Modern developments of modal semantics go beyond possible-world selection functions. Their common denominator is to evaluate a conditional $A \rightarrow C$ as true if C is true in all relevant contexts selected by the antecedent A (e.g., Kratzer 1986; Mandelkern 2019). Specifically, dynamic and information state semantics implement this idea by *updating* on A (e.g., Gillies 2009; Santorio 2022a). These accounts integrate the semantics of “if... then...” with the semantics of other modal operators, but integrating this framework with the probability of conditionals and uncertain reasoning is a non-trivial task (compare Goldstein and Santorio 2021; Ciardelli and Ommundsen forthcoming). The connection between truth conditions and probabilistic reasoning, and the distinction between certain and uncertain inference, is much more straightforward in our analysis.

10 Conclusions

The trivalent analysis in this paper closes the gap between the truth conditions of conditionals and their probabilistic semantics, giving us an account of reasoning in which validity depends on the epistemic status of premises. Specifically, we propose two logics that generalize the concept of valid inference to reasoning with conditionals: **C** explicates conditional reasoning with certain premises, **U** explicates conditional reasoning with uncertain premises. Although **C** is a nonclassical logic, all theorems of classical logic are also theorems of **C** when restricted to atom-classical valuations. The combination of **C** and **U** avoids Gibbard’s and Lewis’s triviality results, and provides a unified framework for conditional reasoning, in line with the observation that some inference schemes (e.g., Or-To-If, nested Modus Ponens) appear valid in certain reasoning and invalid in uncertain reasoning.

Summarizing the main features and results of our approach according to topics:

Truth Conditions The indicative conditional expresses a conditional commitment to the consequent, cancelled if the antecedent turns out false. This interpretation motivates a fully truth-functional trivalent analysis of the conditional. Following Cooper, we group indeterminate antecedents with true ones, and interpret conjunction and disjunction according to his truth tables for quasi-conjunction and -disjunction.

Probability The probability of a sentence $A \in \mathcal{L}^{\rightarrow}$ is the ratio between the weight of possible worlds where A is true, and the weight of possible worlds where A is either true or false. Adams’s Thesis $p(A \rightarrow C) = p(C|A)$ for conditional-free sentences follows as a corollary and need not be postulated as an axiom.

Certain Inference Certainty-preserving inference with conditionals is captured by the logic **C**, which preserves maximal probability. Equivalently, **C** preserves non-falsity in trivalent semantics (Proposition 1).

Uncertain Inference Reasoning with conditionals that preserves their probability or degree of uncertainty is captured by the logic **U**, which preserves probability between the conjunction of the

premises and the conclusion. Equivalently, U preserves truth *and* non-falsity for all trivalent valuations of the premises and the conclusion (Proposition 2 and 3).

Combining these semantic and epistemological elements delivers a coherent and fruitful framework. Specifically, we can use it to analyze and to explain the controversy about the validity of Modus Ponens, Or-to-If, Import-Export and other important inference principles.

More work needs to be done. The most urgent projects are to integrate this analysis with an account of Bayesian learning, to extend our analysis to a language with modal operators, such as such as “must” and “might”, and to explore the implications for the semantics and epistemology of counterfactuals. We leave these issues for further research.

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A Proofs of the Propositions

Given a model, consisting of a nonempty set of worlds W and a valuation function v , recall that $A_T, A_I, A_F \subseteq W$ denote the set of possible worlds where A is true, indeterminate, and false, respectively. Here and in the remainder, we identify possible worlds with complete valuation functions to all sentences in the language \mathcal{L}^\rightarrow .

Proposition 1 (Trivalent Characterization of **C**). *For a set of formulas $\Gamma \subseteq \mathcal{L}^\rightarrow$ and a formula $B \in \mathcal{L}^\rightarrow$, the following are equivalent:*

- (1) $\Gamma \models_{\mathbf{C}} B$.
- (2) For all Cooper valuations $v : \mathcal{L}^\rightarrow \mapsto \{0, 1/2, 1\}$: if $v(A) \geq 1/2$ for all $A \in \Gamma$, then also $v(B) \geq 1/2$.

Proof. First we show the equivalence for the single-premise case.

(2) \Rightarrow (1). (2) implies $B_F \subseteq A_F$. Suppose now that $p(A) = 1$ for some probability function p : by (Probability), this requires $c(A_F) = 0$. Because of $B_F \subseteq A_F$, and the measure properties of c , we can infer $c(B_F) \leq c(A_F)$, hence $c(B_F) = 0$ and $p(B) = 1$. This means that $A \models_{\mathbf{C}} B$.

\neg (2) \Rightarrow \neg (1). Suppose that (2) is false and that there is a model with $w \in B_F$ and $w \notin A_F$. Choose c such that $c(w) = 1$, i.e., w has maximal credence, and in particular, $c(w') = 0$, for all $w' \neq w$. Then $c(A_F) = c(B_T) = 0$, and

$$p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \frac{c(A_T)}{c(A_T) + 0} = 1, \text{ but}$$

$$p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} = \frac{0}{0 + 1} = 0,$$

contradicting (1). Hence it must be the case that $B_F \subseteq A_F$, showing (2).

Now we consider the case of more than one premise, $\Gamma = \{A_1, \dots, A_n\}$. First we note that $p(\bigwedge_{A_i \in \Gamma} A_i) = 1$ if and only if $p(A_i) = 1 \forall 1 \leq i \leq n$ (like in the classical case). Hence $A_1, \dots, A_n \models_{\mathbf{C}} B$ if and only if $\bigwedge_{A_i \in \Gamma} A_i \models_{\mathbf{C}} B$. Second, we note that for any Cooper valuation $v : \mathcal{L}^\rightarrow \mapsto \{0, 1/2, 1\}$, $v(A_i) \geq 1/2 \forall A_i \in \Gamma$ if and only if $v(\bigwedge_{A_i \in \Gamma} A_i) \geq 1/2$. Third, we can apply the proposition for the single-premise case: $\bigwedge_{A_i \in \Gamma} A_i \models_{\mathbf{C}} B$ if and only if for all Cooper valuations, $v(\bigwedge_{A_i \in \Gamma} A_i) \geq 1/2$ implies $v(B) \geq 1/2$. Taking these three observations together shows the proposition for the case of multiple premises. \square

Proposition 2 (Equivalent Characterizations of Valid Single-Premise Inference in U). *For $A, B \in \mathcal{L}^{\rightarrow}$, with $\not\models_{\mathbf{C}} \neg A$ and $\not\models_{\mathbf{C}} B$, the following are equivalent:*

- (1) $A \models_{\mathbf{U}} B$.
- (2) *For all Cooper valuations $v : \mathcal{L}^{\rightarrow} \mapsto \{0, 1/2, 1\}$, $v(A) \leq v(B)$. In other words, if $v(A) = 1$ then $v(B) = 1$, and if $v(A) \geq 1/2$, then $v(B) \geq 1/2$.*
- (3) $A \models_{\mathbf{C}} B$ and $\neg B \models_{\mathbf{C}} \neg A$.

Proof. **(2) \Leftrightarrow (3):** The implication from $v(A) \geq 1/2$ to $v(B) \geq 1/2$ is equivalent to $A \models_{\mathbf{C}} B$, by Proposition 1. The implication from $v(A) = 1$ to $v(B) = 1$ can be rephrased by contraposition as an implication from $v(B) \leq 1/2$ to $v(A) \leq 1/2$. Equivalently, if $v(\neg B) \geq 1/2$, then $v(\neg A) \geq 1/2$, and so, again by Proposition 1, $\neg B \models_{\mathbf{C}} \neg A$.

(2) \Rightarrow (1). By assumption, $A_T \subseteq B_T$ and $B_F \subseteq A_F$. Hence, $c(A_T) \leq c(B_T)$ and $c(A_F) \geq c(B_F)$. Thus, for all probability functions $p : \mathcal{L} \mapsto [0, 1]$,

$$p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} \leq \frac{c(B_T)}{c(B_T) + c(B_F)} = p(B).$$

\neg (2) $\Rightarrow \neg$ (1), Case 1: $B_F \not\subseteq A_F$. Take $w \in B_F$ (which exists since $\not\models_{\mathbf{C}} B$) with $w \notin A_F$. If $w \in B_F \cap A_T$, we are done: simply assign $c(w) = 1$ and we obtain that $p(A) = 1 > p(B) = 0$, contradicting (1). If $w \in B_F \cap A_I$, then we assign $c(w) = 1/2$, and moreover, we choose an arbitrary $w' \in A_T$ with $c(w') = 1/2$. Such a w' must exist since $\not\models_{\mathbf{C}} \neg A$. This yields a counterexample to (1):

$$p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} = \frac{1/2}{1/2 + 0} = 1$$

$$p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} \leq \frac{1/2}{1/2 + 1/2} \leq 1/2.$$

\neg (2) $\Rightarrow \neg$ (1), Case 2: $A_T \not\subseteq B_T$. Take $w \in A_T$ (which exists, since $\not\models_{\mathbf{C}} \neg A$) with $w \notin B_T$. If $w \in B_F$, then set $c(w) = 1$, yielding $p(A) = 1$ and $p(B) = 0$. So \neg (1) holds. If $A_T \cap B_F = \emptyset$, then choose a $w \in A_T \cap B_N$. Moreover, since $\not\models_{\mathbf{C}} B$, we know that there is a $w' \in B_F$. Assign the credences $c(w) = c(w') = 1/2$. Then we obtain the following

counterexample to (1):

$$p(A) = \frac{c(A_T)}{c(A_T) + c(A_F)} \geq \frac{1/2}{1/2 + 1/2} \geq 1/2$$

$$p(B) = \frac{c(B_T)}{c(B_T) + c(B_F)} = \frac{0}{0 + 1/2} = 0.$$

□

Proposition 3 (Equivalent Characterizations of Valid Multi-Premise Inference in \mathbf{U}). *For a consistent set of formulas $\Gamma \subseteq \mathcal{L}^\rightarrow$ and $B \in \mathcal{L}^\rightarrow$ with $\not\models_{\mathbf{C}} B$, the following are equivalent:*

- (1) $\Gamma \models_{\mathbf{U}} B$.
- (2) *There is a finite subset of premises $\Delta \subseteq \Gamma$ such that the semantic value of B is, for all Cooper valuations v , at least as high as the semantic value of the conjunction of the premises: $v(\bigwedge_{A_i \in \Delta} A_i) \leq v(B)$.*
- (3) *There is a finite subset of premises $\Delta \subseteq \Gamma$ such that $\bigwedge_{A_i \in \Delta} A_i \models_{\mathbf{C}} B$ and $\neg B \models_{\mathbf{C}} \neg(\bigwedge_{A_i \in \Delta} A_i)$.*

Proof. Suppose there is a subset $\Delta \subseteq \Gamma$ such that $\bigwedge_{A \in \Delta} A \models_{\mathbf{U}} B$. The consistency of the premise set Γ ensures that for no such Δ , the expression $\neg(\bigwedge_{A \in \Delta} A)$ is a theorem of \mathbf{C} . Hence we can apply Proposition 2: for this Δ , we obtain $v(\bigwedge_{A \in \Delta} A) \leq v(B)$ for all Cooper valuations v ; equivalently, $\bigwedge_{A \in \Delta} A \models_{\mathbf{C}} B$ and $\neg B \models_{\mathbf{C}} \neg(\bigwedge_{A \in \Delta} A)$.

The converse direction works in the same way, applying Proposition 2 to the conjunction of the elements of Δ in (2) or (3). □