Bolzano's Mathematical Infinite

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Abstract

Bernard Bolzano (1781-1848) is commonly thought to have attempted to develop a theory of size for infinite collections that follows the so-called part-whole principle, according to which the whole is always greater than any of its proper parts. In this paper, we develop a novel interpretation of Bolzano's mature theory of the infinite and show that, contrary to mainstream interpretations, it is best understood as a theory of infinite sums. Our formal results show that Bolzano's infinite sums can be equipped with the rich and original structure of a non-commutative ordered ring, and that Bolzano's views on the mathematical infinite are, after all, consistent.

Introduction

Bernard Bolzano (1781-1848) was a Bohemian priest with eclectic interests ranging from logic and mathematics to political and moral philosophy. One of his more famous writings is a booklet his pupil Příhonský published under the title *Paradoxien des Unendlichen* (from now on *PU* for short), Paradoxes of the Infinite. Likely contributing to its fame, this booklet was read and referred to by both Cantor and Dedekind. Perhaps because of this association, the booklet is also routinely interpreted as a text anticipating several ideas of Cantor's transfinite set theory (*cf.* Berg 1962, 1992; Šebestík 1992; Rusnock 2000), especially in sections §§29-33, in which Bolzano sketches a "calculation of the infinite". As a consequence, appraisal of the *PU* is almost exclusively conducted in terms of how much Bolzano's work on the infinite agrees with later developments in set theory. In particular, many shortcomings of Bolzano's calculation of the infinite are attributed to his adherence to the *part-whole principle*:

PW1 For any sets A, B, if $A \subsetneq B$, then $size(A) < size(B)$.

In the case of infinite sets, it is well-known that this principle contradicts the *bijection principle*, according to which the existence of a one-to-one correspondence between two sets is a necessary and sufficient condition for the equality of their sizes. One *locus classicus* for the tension between these two principles is the seventeenth-century dialogue of Galileo's *Discourses and Mathematical Demonstrations Relating to Two New Sciences* (Galileo 1958, pp. 44-45). The characters debate among themselves the example of the set of natural numbers $\mathbb{N}$, which can be put into one-to-one correspondence with the proper subset $\mathbb{N}^{(2)}$ of square natural numbers. Thus $\mathbb{N}$ and $\mathbb{N}^{(2)}$ have the same size according to the bijection principle, while the size of $\mathbb{N}^{(2)}$ is strictly smaller than that of $\mathbb{N}$ according to **PW1**. While Bolzano is commonly taken to have adopted **PW1** in the *PU*, Cantor successfully founded his theory of powers and cardinal numbers on the bijection principle. Thus, as long as Cantor's way out of Galileo's paradox is perceived as the "right" way to compute the size of infinite collections, Bolzano's alternative can only be seen as an intriguing yet fundamentally flawed attempt.

This privileged status of the bijection principle however has started to be scrutinized in recent years thanks to a renewed interest in potential alternatives to Cantor's theory of the mathematical infinite. In particular, Mancosu (2009) shows that there is a long historical tradition of thinkers and mathematicians who favored **PW1** over the bijection principle, and that recent mathematical developments in Benci and Di Nasso (2003) establish that a consistent theory of the sizes of infinite collections can be founded on **PW1** rather than on the bijection principle. This theory, called the theory of *numerosities*, is a refinement of the Cantorian theory of cardinals that allows for two sets A and B to be considered of different sizes even in the presence of a bijection between the two (Benci and Di Nasso 2003, p. 51). This directly contradicts the claim that Cantor's theory is the only viable theory of the infinite, and thus calls for a reappraisal of alternative theories that until recently had been dismissed as essentially misguided or inconsistent.

Our main goal is to offer such a reappraisal of Bolzano's mature theory of the mathematical infinite. In particular, we propose an interpretation of Bolzano's calculation of the infinite in §§29-33 of the *PU* which stresses its conceptual and mathematical independence from set theory proper, and argue that Bolzano is more interested in developing a theory of infinite sums rather than a way of measuring the sizes of infinite collections. This leads us to reassess the role that part-whole reasoning plays in Bolzano's computations and to provide a formal reconstruction of his position that underscores its coherence and originality, and is overall a more charitable appreciation of Bolzano's ideas on the infinite. In particular, we show that Bolzanian sums in our interpretation form a non-commutative ordered ring, a well-behaved algebraic structure that nonetheless vastly differs from Cantorian cardinalities.

We proceed as follows. In Section 1 we discuss several sources of what we call the received view of the *PU*, and introduce enough background to set the stage for our novel interpretation. In Section 2 we focus on Bolzano's calculation of the infinite and argue that his work is best understood as a theory of infinite sums. This leads in Sections 3 and 4 to a formal reconstruction of Bolzano's computations with infinite quantities, which aims to establish both the consistency and the originality of his position. Finally, in Section 5, we recap the main points of our formalization and discuss its implications for the interpretation of the *PU*.

1 The received view on the *PU*

Bolzano's *PU* is a short yet ambitious booklet in which the author aims to show that, when properly defined and handled, the concept of the infinite is not intrinsically contradictory, and many paradoxes having to do with the infinite in mathematics (but also in physics and metaphysics) can actually be solved. In the course of addressing the paradoxes of the infinite in mathematics, Bolzano develops what looks like a theory of transfinite quantities (§§28-29, 32-33), which is what commentators tend to focus on when appraising the contents of the *PU*.

One such commentator is, as is known (Šebestík 1992; Rusnock 2000; Ferreirós 2007), Georg Cantor. Cantor (1883) introduces Bolzano as a proponent of actual infinity, and specifically actually infinite numbers in mathematics, in contrast to Leibniz's arguments against infinite numbers:

> Still, the actual infinite such as we confront for example in the well-defined point sets or in the constitution of bodies out of point-like atoms [...] has found its most authoritative defender in Bernard Bolzano, one of the most perceptive philosophers and mathematicians of our century, who has developed his views on the topic in the beautiful and rich script *Paradoxes of the Infinite*, Leipzig 1851. The aim is to prove how the contradictions of the infinite sought for by the sceptics and peripatetics of all times do not exist at all, as soon as one makes the not always quite easy effort of taking into account the concepts of the infinite according to their true content. (Cantor 1883 in Cantor 1932, p. 179)[1]

And still:

[1] In this and all other cases for which a published English translation is not cited, the translations are the authors'. Original German:
Doch den entschiedensten Verteidiger hat das Eigentlich-unendliche, wie es uns beispielsweise in den wohldefinierten Punktmengen oder in der Konstitution der Körper aus punktuellen Atomen [...] entgegentritt, in einem höchst scharfsinnigen Philosophen und Mathematiker unseres Jahrhunderts, in Bernard Bolzano gefunden, der seine betreffenden Ansichten namentlich in der schönen und gehaltreichen Schrift: ,,Paradoxien des Unendlichen, Leipzig 1851" entwickelt hat, deren Zweck es ist, nachzuweisen, wie die von Skeptikern und Peripatetikern *aller Zeiten* im Unendlichen gesuchten Widersprüche gar nicht vorhanden sind, sobald man sich nur die freilich nicht immer ganz leichte Mühe nimmt, die Unendlichkeitsbegriffe allen Ernstes ihrem wahren Inhalte nach in sich aufzunehmen.

> Bolzano is perhaps the only one who confers a certain status to actually infinite numbers, or at least they are often mentioned [by him]; nevertheless I completely and wholly disagree with the way in which he handles them, not being able to formulate a proper definition thereof, and I consider for instance §§29-33 of that book as untenable and wrong. For a genuine definition of actually infinite numbers, the author is lacking both the general concept of *power*, and the accurate concept of *number*. It is true that the seeds of both notions appear in a few places in the form of special cases, but it seems to me he does not work his way through to full clarity and distinction, and this explains several contradictions and even a few mistakes of this worthwhile script. (*ibid.*, p. 180)[2]

Cantor's comments in many ways set the tone of how the *PU* are mainly perceived even today, namely as a rich and interesting essay that nevertheless displays some serious shortcomings. Cantor diagnoses Bolzano's mistakes as being fundamentally due to an imprecise characterization of *power* and *number*. Without entering a discussion on Cantorian powers, it is useful for us to notice how Cantor is readily reinterpreting Bolzano's text in the light of his own research. The concept and terminology of *powers* was Cantor's own, which he introduced starting from 1878 in his papers. What Cantor means is that Bolzano did not have the right notion of size for infinite sets, the right notion being Cantor's own powers, and this shortcoming causes Bolzano to go astray in §§29-33. Another aspect of Cantor's comments on the *PU* which we want to stress is that Cantor straightforwardly presents Bolzano's "calculation of the infinite" (*Rechnung des Unendlichen*, §28) as a version of his own transfinite arithmetic, albeit imprecise and imperfect.

[2]Bolzano ist vielleicht der einzige, bei dem die eigentlich-unendlichen Zahlen zu einem gewissen Rechte kommen, wenigstens ist von ihnen vielfach die Rede; doch stimme ich gerade in der Art, wie er mit ihnen umgeht, ohne eine rechte Definition von ihnen aufstellen zu können, ganz und gar *nicht* mit ihm überein und sehe beispielsweise die §§29-33 jenes Buches als haltlos und irrig an. Es fehlt dem Autor zur wirklichen Begriffsfassung bestimmt-unendlicher Zahlen sowohl der allgemeine *Mächtigkeitsbegriff*, wie auch der präzise *Anzahlbegriff*. Beide treten zwar an einzelnen Stellen ihrem Keime nach in Form von Spezialitäten bei ihm auf, er arbeitet sich aber dabei zu der vollen Klarheit und Bestimmtheit, wie mir scheint, *nicht* durch, und daraus erklären sich viele Inkonsequenzen und selbst manche Irrtümer dieser wertvollen Schrift.

All commentaries on the *PU* we were able to find seem to follow suit from Cantor in that they evaluate and interpret the *PU*, and §§29-33 in particular, against the backdrop of the development of set theory. Thus Bolzano's *PU* are about infinite sets according to editors and translators of Bolzano's text (e.g., Hans Hahn in Bolzano 1920, Donald Steele in Bolzano 1950), as well as scholars such as Berg (1992, 1962), Šebestík (1992, 2017), Lapointe (2011), Ferreirós (2007) and Rusnock (2000). We now examine the most informative of these interpretations in some detail.

Among Bolzano scholars, Jan Berg is perhaps the one that embraces a set theoretic reading of Bolzano with the most conviction. Berg (1962, p. 176) writes:

> In *PU* [...] Bolzano repudiates the notion of equivalence as sufficient condition for the identity of powers of infinite sets. [...] As a result, a number of statements follow which do not correspond to Cantor's view on this subject. E.g., if "N_0" denotes the number of natural numbers (PU 45) [§29; Berg refers to the page of the 1851 edition], then in the series: $N_0, {N_0}^2, {N_0}^3, \ldots$ each ${N_0}^m$ is said to "exceed infinitely" the preceding term ${N_0}^{m-1}$ (PU 46) [§29]. But Bolzano's comparison of the powers of infinite sets is impossible to understand, since nowhere does he offer any clear sufficient condition for the equinumerousness of infinite sets.

Berg makes the same points as Cantor, namely that Bolzano's writings in *PU* are about the *powers* of infinite *sets*, and that his reasoning is impossible to follow as he does not offer sufficient conditions for the equality of size of sets. However Berg (see, for instance, his 1962, p. 177) remains convinced that a letter[3] written by Bolzano in the last year of his life witnesses a change of heart regarding how infinite sets should be compared, moving from his rejection of one-to-one correspondence to an acceptance of it

[3] This letter, dated 9 March 1848 and intended for Bolzano's former pupil Robert Zimmermann, has been published in (Bolzano 1978, pp. 187-189). Berg is the editor for the volume and his editorial notes to the letter are a reiteration of his interpretation of Bolzano having changed his mind regarding part-whole and one-to-one correspondence for infinite collections.

as a sufficient criterion for size equality.

On the heels of this interpretation, Berg (1992, pp. 42-43) sketches what he takes to be Bolzano's theory of the infinite. In a nutshell, Berg believes that any two infinite sets of natural numbers are of the same size according to Bolzano just in case "the members are related to each other by finitely many rational operations (addition, multiplication and their inverses)" (Berg 1992). Even though Berg does not use this terminology, his interpretation seems to suggest that $\mathbb{N}$ is equinumerous with an infinite subset $S \subseteq \mathbb{N}$ whenever the bijection $f : S \to \mathbb{N}$ is primitive recursive. This is an interesting suggestion, but it would imply that, for example, $\mathbb{N} - \{1\}$ and $\mathbb{N}$ are equinumerous, while this seems to contradict Bolzano's reasoning in *PU* §29 (see Section 2 below). Moreover, Berg's interpretation of the letter is far from uncontroversial (see Rusnock 2000, pp. 194-195, Šebestík 1992, pp. 469-470, to be discussed below, and Mancosu 2009, 2016), so his interpretation of this aspect of Bolzano's work is not a foregone conclusion. We will not engage with it any more than what we have already done as the controversy has less to do with *PU* and more to do with what views about the infinite Bolzano held at the very moment of his death.

A more nuanced view is offered by Šebestík (1992, pp. 435-473). When presenting the contribution of Bolzano's *PU*, Šebestík summarizes it thus:

> For the first time, the actual infinite, whose properties cease to be contradictory to simply become paradoxical, is admitted in mathematics as a well-defined concept, having a referent and only attaching to those objects capable of enumeration or measurement, that is, to sets and quantities.[4] (Šebestík 1992, p. 435)

Šebestík also interprets the *PU* as about sets and their being infinite. Even though at p. 445 he more faithfully writes that "the infinite is first and foremost a property

[4]Original French: Pour la première fois, l'infini actuel dont les propriétés cessent d'être cotradictoires pour devenir simplement paradoxales, est admis en mathématiques à titre de concept défini, ayant une référence et attaché aux seuls objets susceptibles de dénombrement ou de mesure, c'est-à-dire aux ensembles et aux grandeurs.

of *pluralities* [our emphasis]",[5] on p. 462 he then reverts to set talk at a crucial point, namely when giving his interpretation of *PU* §33:

> [Referring to §33] It is the first and last time within the *Paradoxes of the Infinite* that Bolzano deduces from the reflexivity of the set of natural numbers to the equality of number between a set and one of its proper subsets.[6]

According to Šebestík's interpretation then, and unlike Berg's, it is not quite the case that Bolzano changed his mind regarding what criterion to use to compare the size of infinite sets *after* the *PU* and just before his death. Rather, Bolzano's views in the *PU* itself are already inconsistent, because at various points in the text Bolzano either implicitly or explicitly endorses the following views:

1. The part-whole principle, that is, the whole is greater than any of its proper parts.

2. All infinite sets can be put in one-to-one correspondence with any of their infinite subsets.

3. Every set has a definite size.

4. If two sets are in one-to-one correspondence then they have the same plurality.

It is quite telling that for 1, 3, and 4 Šebestík (1992, pp. 463-464) feels the need to add set theoretic glosses, so that 1 becomes "$card(A) < card(B)$ iff A is equivalent to a proper part of B" ('$card(A) < card(B)$ si et seulement si A est équivalent à une partie propre de B'), 3 is Every set has a "unique cardinal number" ('nombre cardinal unique') and 4 If two sets are in one-to-one correspondence then they have "the same cardinal number" ('ont le même nombre cardinal').

Thus formulated, 1-4 do indeed yield a contradiction. Consider any two infinite sets A and B such that A is a proper part of B. By 3, they each have a unique cardinality, and by 1 $card(A) < card(B)$. But also, since A and B can be put into one-to-one

[5] Original: "L'infini est d'abord et avant tout une propriété des multitudes".

[6] C'est pour la première et dernière fois que, dans les *Paradoxes de l'Infini*, Bolzano conclut de la réflexivité de l'ensemble des nombres naturels à l'égalité numérique entre un ensemble et l'un de ses sous-ensembles propres.

correspondence (by 2), they have the same cardinality, by 4, so $card(A) = card(B)$, contradicting our earlier deduction that $card(A) < card(B)$. We will give our argument as per why Šebestík's contradiction does not go through in Section 2, where we highlight that a crucial ingredient in this family of counterexamples to Bolzano's claim to internal consistency in the *PU* is largely due to the set theoretic interpretation of 4.

The last interpretation we want to consider in detail is Rusnock's (2000). Rusnock (2000, p. 193) writes that in §§21-22 Bolzano "apparently based this opinion [of the insufficiency of one-to-one correspondence for equality of size] on considerations involving parts and wholes, assuming perhaps that the multiplicity of the whole must be greater than those of its parts. (Rusnock translates with "multiplicity" what we, following (Russ 2004), translate as "plurality", namely *Vielheit.*) Rusnock (2000, *ibid.*) then continues:

> But this seems to be a mistake, even in Bolzano's own terms. For his sets (*Mengen*) are by definition invariant under rearrangements of their members, and thus the appeal to the "mode of determination" seems to be illegitimate in this context.

Rusnock then produces an example to show why Bolzano is mistaken by his own lights when embracing "considerations of parts and whole". Consider the straight line abc, where a is to the left of b and b is to the left of c; call A the set of points between a and b, B the set of points between a and c. Then it is possible to map each point of A to a point of B via a translation map that is also a one-to-one correspondence. Since a translation map only "rearranges" points from one region of space to another, then B is just a rearrangement of A. Thus, A and B should be the same "set", since Bolzano's definition of "set" (*Menge*) entails that something considered as a "set" is invariant under rearrangement of parts. Yet, because A is a proper part of B, A should be strictly smaller than B, in virtue of what from now on we call "the part-whole principle": The whole is greater than any of its proper parts. This principle then is inconsistent with Bolzano's own definition of multitude.

It is not warranted however that an example such as Rusnock's really counts as a rearrangement of parts on Bolzano's terms, essentially because it relies on a metaphorical use of the term "rearrangement" in a geometric context. This metaphorical use in turn suggests conceiving of geometric figures (points and lines) as objects that move through the two-dimensional (Euclidean) space. Yet Bolzano famously rejected metaphorical talk of motion in mathematical contexts (Bolzano 1817, Introduction), and lacking that, we are not sure there is a way of rephrasing Rusnock's example so that it really counts as a rearrangement of parts on Bolzano's terms.

On the basis of our overview, we can now distil the received view about the *PU* into two theses:

(Sets) In §§29-33, Bolzano is concerned with determining size relationships between infinite sets.

(Set-PW) Bolzano's computations in §§29-33 are, at least partially, motivated by the part-whole principle for sets.

As we have seen above, the combination of these two theses motivates a reading of Bolzano's calculation of the infinite as a pre-Cantorian transfinite arithmetic that is either mistaken or downright inconsistent because of its adherence to the part-whole principle. As it will soon become apparent, we believe however that both theses incorrectly describe §§29-33 of the *PU*. Our main claim is that the standard view's identification of Bolzanian collections with the modern notion of set, and of all instances of part-whole reasoning in the *PU* to **PW1**, is too quick. Discussing the standard interpretation of Bolzano's calculation of the infinite therefore requires a clarification of the status of collections in the *PU*, and an assessment of the role that part-whole reasoning plays in Bolzano's arguments. We will take those two issues in turn. First, we briefly recap the various notions of collections that Bolzano introduces at the beginning of the *PU*, and explain the role they play in his definition of the infinite. Second, we review sections §§20-24 of the *PU*, in which Bolzano is usually interpreted as rejecting the bijection principle in favor of something like **PW1**. We believe this will provide the reader with the necessary background for our in-depth discussion of §§29-33 in Section 2.

1.1 Bolzano's collections, multitudes, and sums

Bolzano's first goal in the *PU* is to arrive at a rigorous definition of the infinite. To that end, he relies on his logical system first developed in his *Wissenschaftslehre* (*Theory of Science*, *WL* for short). In particular, Bolzano devotes the first section of the *PU* to defining several distinct notions of collection. Without going into too much detail, we summarize here the most important definitions.

Collection The concept of collection (*Inbegriff*) applies to any and all objects which are made of parts, i.e. that are not simple. In that sense, [collection] is the most general concept as it applies to any composite object. Collections, as opposed to units (*Einheiten*, sometimes also translated as unity/unities), can be decomposed into simpler parts. Anything that is made of at least two parts is a collection. (see *PU* §3)

Multitude The concept of multitude (*Menge*) is best illustrated with Bolzano's own example of a drinking glass (*PU* §6). Consider the glass as intact, and then as shattered into pieces. What changes between these two states of the glass is the arrangement (*Anordnung*) of the pieces, although the amount of glass is the same before and after. When we consider the glass as that which remains unchanged before and after the breakage, we are considering it as a multitude. "A collection which we put under a concept so that the arrangement of its parts is unimportant (in which therefore nothing essential changes for us if we merely change this arrangement) I call a multitude." (*PU* §4)[7]

Plurality When the parts of a multitude all fall under the same concept A and are therefore considered as units of kind A (i.e. simple objects of kind A), that multitude is called a plurality (*Vielheit*) of kind A. (*ibid.*)

[7]Translations of Bolzano's *PU* are always from Russ 2004.

Sum A sum (*Summe*) is a collection such that (a) its parts can also be collections, and (b) the parts of its parts can be considered as parts of the whole sum, without the sum itself having changed (*PU* §5). Consider the glass example again. Suppose we break our glass G and it shatters in exactly three pieces, a, b and c. Then suppose a breaks also into two pieces a_1 and a_2. Then our glass G, considered as a sum, is still the same: $G = a + b + c = a_1 + a_2 + b + c$.

Quantity Bolzano defines a quantity (*Größe*) as an object that can be considered of a kind A such that any two objects M, N of kind A satisfy a certain law of trichotomy (not Bolzano's expression): either they are equal to one another ($M = N$) or "one of them presents itself as a sum which includes a part equal to the other one" (*PU* §6), that is to say, $M = N + \nu$ or $N = M + \mu$. The remaining parts μ, ν themselves also need to satisfy the condition that, for any other X of kind A, either $X = \mu$ ($X = \nu$, respectively) or one of them can be presented as a sum of which the other is just a part.

To avoid any confusion, it should be noted that the concepts of multitudes, pluralities, sums and quantities are specifications of the concept of collections, and the same object can be conceptualized as more than one kind of collection at once. Quantities are a great example. From their definition, it is clear that anything that is a quantity is also a plurality, because a quantity is a multitude (of a certain kind, say A) whose parts are also objects of kind A. At the same time, the way Bolzano expresses the trichotomy law holding of relationships between quantities suggests that a quantity is also a sum, namely, an object such that the parts of its immediate parts are also parts of the object itself, and nothing about the object changes if we consider it as made of the parts of its parts, instead of just of its own immediate parts.

Moreover, the existence of various notions of collections in Bolzano's framework is at odds with the thesis **(Sets)** of the received view, according to which Bolzano tries to develop an arithmetic of infinite sets. Indeed, it is far from clear that any of the notions described above can be straightforwardly mapped onto the modern notion of a set. Following Incurvati (2020, p. 11), we consider the concept of set as used in (philosophy of) mathematics contexts to be sufficiently individuated by the three criteria:

(Unity) A set is a single entity over and above its elements.

(Decomposition) A set can be decomposed in a unique way into its elements.

(Extensionality) Sets are identical if and only if they have exactly the same elements.

Bolzano's own definitions do not imply that his multitudes, or pluralities, or sums satisfy all three criteria at once. Since multitudes, pluralities and sums are the infinite collections Bolzano concerns himself with, the identification of his infinite collections with Cantorian infinite sets is unwarranted and far from obvious. For more on Bolzano's multitudes and sets, see Simons (1997).

Nevertheless, **(Sets)** might gain some traction from the fact that Bolzano's definition of the infinite only applies to collections, or, more precisely, to pluralities:

> [...] I shall call a plurality which is greater than every finite one, i.e., a plurality which has the property that every finite multitude represents only a part of it, an *infinite plurality.* (*PU* §9)

However, the choice of defining an infinite *plurality* as opposed to simply infinity is justified in §10, where Bolzano argues that in the use made by mathematicians, "the infinite" is always an infinite plurality:

> Therefore it [is] only a question of whether through a mere definition of what is called an infinite plurality we are in a position to determine what is [the nature of] the infinite in general. This would be the case if it should prove that, strictly speaking, there is nothing other than pluralities to which the concept of infinity may be applied in its true meaning, i.e., if it should prove that infinity is really only a property of a plurality or that everything which we have defined as infinite is only called so because, and in so far as, we discover a property in it which can be regarded as an infinite plurality. Now it seems to me that is really the case. The mathematician obviously never

> uses this word in any other sense. For generally it is nearly always quantities with whose determination he is occupied and for which he makes use of the assumption of one of those of the same kind for the unit, and then of the concept of a number. (*PU* §10)

Bolzano's target when defining infinity solely as the attribute of certain collections are the imprecise definitions of infinity given by some philosophers (Hegel and his followers are cited explicitly here) who consider the mathematical infinity Bolzano talks about to be the "bad" kind (*PU* §11), while the one true infinity is God's absolute infinity. The strategy to push against this *qualitative infinite* of the philosophers is to show that, even in the case of God, who is the unity *par excellence*, when we assign infinity to Him as one of His attributes, what we are really saying is that some other attribute of His has an infinite multitude as a component.

> What I do not concede is merely that the philosopher may know an object on which he is justified in conferring the predicate of being infinite without first having identified in some respect an infinite magnitude [*Größe*] or plurality in this object. If I can prove that even in God as that being which we consider as the most perfect unity, viewpoints can be identified from which we see in him an infinite plurality, and that it is only from these viewpoints that we attribute infinity to him, then it will hardly be necessary to demonstrate further that similar considerations underlie all other cases where the concept of infinity is well justified. Now I say we call God infinite because we concede to him powers of more than one kind that have an infinite magnitude. Thus we must attribute to him a power of knowledge that is true omniscience, that therefore comprehends an infinite multitude of truths because all truths in general etc. (*PU* §11)

With that, Bolzano considers himself to have exhaustively argued for his definition of mathematical infinity as being inextricable from the concepts of plurality and quantity and inapplicable to the one-ness of any unity, even God. Thus, we conclude that

Bolzano's insistence on defining only an infinite plurality does not lend particular credence to **(Sets)** after all. Bolzano's definition unequivocally makes of infinity a quantifying attribute which, as such, can only apply to pluralities and quantities. But his insistence on discussing only infinite pluralities should be understood as in contrast with the Hegelian infinite as an attribute of a single infinite being. Talking about infinite collections, for Bolzano, is a way of clearly setting apart the *quantitative* infinite he is interested in from the qualitative infinite of the hegelians.

1.2 Bolzano's commitment to part-whole in the *PU*

As the discussion of the received view on the *PU* made clear, one point of contention in interpreting Bolzano's work on the infinite is whether (and to what extent) the principles that guide his computations with infinite quantities mirror those later used by Cantor. While part-whole considerations play an important role in Bolzano's *WL* (in particular, §102 therein; cf. Mancosu 2016, pp. 130-131, Mancosu 2009, pp. 624-625), the discussion in Berg and Šebestík's interpretations has brought to light the issue of whether, on the whole, Bolzano's treatment of infinite quantities in the *PU* obeys the part-whole principle or not. Setting aside the issue of whether an adoption of one-to-one correspondence is implicit in Bolzano's §33 (something we will come back to in Section 2), here we review §§20-24, which are usually taken to be Bolzano's discussion of one-to-one correspondence as an insufficient criterion for size equality of infinite collections on the grounds of part-whole considerations.

Let us note first that some form of part-whole reasoning seems to be present in the very notion of "being greater/smaller than" employed in the *PU*, as this passage from §19 witnesses:

> Even with the examples of the infinite considered so far it could not escape our notice that not all infinite multitudes are to be regarded as equal to one another in respect of their plurality, but that some of them are greater (or smaller) than others, i.e., another multitude is contained as a part in one multitude (or on the contrary one multitude occurs in another as a mere

> part). (*PU* §19)

Here, Bolzano glosses the claim that some multitudes are greater than others as some containing others as a part. A similar use of the part-whole principle is to be found in §20, when Bolzano compares the size of the collection of quantities smaller than 5 and the size of the collection of those smaller than 12:

> If we take two arbitrary (abstract) quantities, e.g. 5 and 12, then it is clear that the multitude of quantities which there are between zero and 5 (or which are smaller than 5) is infinite, likewise also the multitude of quantities which are smaller than 12 is infinite. And equally certainly the latter multitude is greater since the former is indisputably only a part of it. (*PU* §20)

This suggests that Bolzano's writings commit him to upholding the part-whole principle even when it comes to the comparison of infinite quantities, because the principle is part and parcel of the definition of the order relation among quantities.

Having thus established Bolzano's commitment to part-whole, let us also show his explicit rejection of what nowadays we call one-to-one correspondence as a sufficient criterion for equality of size for infinite collections:

> I claim that two multitudes, that are both infinite, can stand in such a relationship to each other that, on the one hand, it is possible to combine each thing belonging to one multitude, with a thing of the other multitude, into a pair, with the result that no single thing in both multitudes remains without connection to a pair, and no single thing appears in two or more pairs, and also, on the other hand it is possible that one of these multitudes contains the other in itself as a mere part, so that the pluralities which they represent if we consider the members of them all as equal, i.e., as units, have the most varied relationships to one another. (*PU* §20)

In the quote above, Bolzano remarks that it is possible for two infinite multitudes to both be in a one-to-one correspondence with each other and be related as a part to its whole. This state of affairs can have the appearance of a paradox, because in the finite case checking whether two multitudes can be put into one-to-one correspondence suffices to determine whether they have the same number of terms, whereas the part-whole relation implies that one multitude must be greater than the other. Bolzano insists that the part-whole relation is what determines the greater-than relation, too:

> Therefore merely for the reason that two multitudes A and B stand in such a relation to one another that to every part a occurring in one of them A, we can seek out according to a certain rule, a part b occurring in B, with the result that all the pairs $(a + b)$ which we form in this way contain everything which occurs in A or B and contains each thing only once—merely from this circumstance we can—as we see—in no way conclude that these two multitudes are equal to one another if they are infinite with respect to the plurality of their parts (i.e., if we disregard all differences between them). But rather they are able, in spite of that relationship between them that is the same for both of them, to have a relationship of inequality in their plurality, so that one of them can be presented as a whole, of which the other is a part. (*PU* §21)

This consideration is illustrated in the preceding §20 by way of two examples, or, two versions of the same example, which considers the two intervals $(0, 5)$ and $(0, 12)$ on the real line and concludes that, since $(0, 5)$ is only a part of $(0, 12)$, $(0, 12)$ contains more quantities (or more points) than $(0, 5)$.

The reason why one has to drop the apparently successful one-to-one correspondence criterion when considering infinite quantities is that what makes one-to-one correspondence work in the finite case is precisely that one has to do with finite collections; hence at some point the process of pairing off each element from the collection with a natural number stops, whereas in the infinite case there is no last element, so the pairing-off never ends. Hence the need for a different criterion for size comparison (*PU* §22). Bolzano gives a brief explanation of how one-to-one correspondence does not suffice to reach conclusions regarding comparisons of infinite sums in §24:

> [From the proposition of §20] follows as the next consequence of it that we may not immediately put equal to one another, two sums of quantities which are equal to one another pair-wise (i.e., every one from one with every one from the other), if their multitude is infinite, unless we have convinced ourselves that the infinite plurality of these quantities in both sums is the same. That the summands determine their sums, and that therefore equal summands also give equal sums, is indeed completely indisputable, and holds not only if the multitude of these summands is finite but also if it is infinite. But because there are different infinite multitudes, in the latter case it must also be proved that the infinite multitude of these summands in the one sum is exactly the same as in the other. But by our proposition it is in no way sufficient, to be able to conclude this, if in some way one can discover for every term occurring in one sum, another equal to it in the other sum. Instead this can only be concluded with certainty if both multitudes have the same basis for their determination. (*PU* §24)

Bolzano considers here the case of a one-to-one correspondence between the terms of two infinite sums S_1 and S_2 that would map each term in S_1 to an equal term in S_2. Since the existence of a one-to-one correspondence is not enough to guarantee that S_1 and S_2 have the same number of terms, one cannot conclude that S_1 and S_2 are equal, unless the two sums also have the same "basis for their determination". This phrase does not have, to our knowledge, a standard interpretation in Bolzanian scholarship. Šebestík (1992, p. 460) does attempt an explanation of what the "determining elements" (*bestimmende Stücke*) of an object can be, according to Bolzano. However, we are not convinced that the explanation offered there extends to a notion of determination for mathematical entities. For now, we simply draw the reader's attention to the fact that Bolzano concludes his discussion of the one-to-one correspondence criterion with a methodological point about infinite *sums* which plays a crucial role in §32 and §33 (see Section 2.2 and Section 2.3 below).

To sum up, in this section we have presented what we take to be the received view on Bolzano's calculation of the infinite, and shown that it relies on the two theses **(Sets)** and **(Set-PW)**. We have argued that the existence of various notions of collections in Bolzano's framework puts some pressure on **(Sets)**, as it does not seem obvious that any of Bolzano's notions closely matches our modern notion of set. Regarding **(Set-PW)** we have shown how Bolzano appeals in §§20-24 to part-whole reasoning in the context of determining size relationships between certain infinite collections. However, we also noted that, by §24, Bolzano has pivoted from discussing sufficient criteria for the equality of size of two infinite collections to discussing sufficient criteria for the equality of two infinite sums. As we will argue in the next section, this is a crucial shift in perspective that is missed by the standard interpretation of Bolzano's calculation of the infinite. We now turn to a close analysis of the text and to our arguments in favour of a different reading of *PU* §§29-33.

2 Bolzano's calculation of the infinite

As discussed in the previous section, up to §24 Bolzano has established the following facts about infinite multitudes and pluralities:

1. Some infinite multitudes are greater than others "with respect to their plurality" (§19).

2. Two infinite multitudes can both be related as part and whole and be in a one-to-one correspondence (§20).

3. One-to-one correspondence is not sufficient to determine equality of infinite multitudes (§§21-22).

4. In the case of comparing two infinite sums, if one wants to conclude that they are equal, one needs to make sure both that there are as many summands in one as there are in the other and that each term from one sum is equal to the corresponding one in the other sum (§24).

These are the "basic rules" (*Grundregeln*, *PU* §28) which govern a proper handling of the infinite in mathematics. Bolzano is aware however that his readers might still be skeptical towards the possibility of computing with the infinite, so he explains what he means by "calculation of the infinite" in the following passage:

> Even the *concept* of a *calculation of the infinite* has, I admit, the appearance of being self-contradictory. To want to *calculate* something means to attempt a *determination of something* through numbers. But how can one determine the infinite through numbers—that infinite which according to our own definition must always be something which we can consider as a multitude consisting of infinitely many parts, i.e., as a multitude which is greater than every number, which therefore cannot possibly be determined by the statement of a mere number? But this doubtfulness disappears if we take into account that a calculation of the infinite done correctly does not aim at a calculation of that which is determinable through no number, namely not a calculation of the infinite plurality in itself, but only a determination of the *relationship* of one infinity to another. This is a matter which is feasible, in certain cases at any rate, as we shall show by several examples. (*PU* §28)

Bolzano's calculation of the infinite is minimal. He does not purport to have extended the concept of number so as to introduce infinite *numbers* (*pace* Cantor—see Section 1 above),[8] but he aims to study the relationship—that is, the ratios as well as the "greater than" relation—between two infinities whenever this can be done in a sound way, that is, in accordance with the principles he has argued for in the preceding portion of the *PU*. Armed with such principles, Bolzano can show his reader how to properly handle some apparently paradoxical results in mathematics, starting from the general theory of quantity.

[8] As Mancosu (2016, p. 163) notes, this refusal to admit infinite numbers was not unique to Bolzano's position but was shared also by Dedekind (1888) and perhaps Schröder (1873).

2.1 Computing with infinite sums

The first computations with infinite quantities are found in earnest in §29; as we will see, these quantities are always introduced and treated as *sums*.

Bolzano introduces the symbol $\overset{0}{N}$ through a symbolic equation—that is, an equation which establishes that the reference of two signs is the same (cf. definition in *Grössenlehre*, Bolzano 1975, pp. 131-132) —to stand for the *Menge* of all natural numbers. He then introduces $\overset{n}{N}$ to stand for the *Menge* of all natural numbers strictly greater than $n \in \mathbb{N}$. $\overset{1}{S}$, on the other hand (which is first introduced as $\overset{0}{S}$), is the symbol for the sum of all natural numbers.

In Bolzano's words:

> [...] if we denote the series of natural numbers by
>
> $$1, 2, 3, 4, ..., n, n + 1, \ldots \textit{ in inf.}$$
>
> then the expression
>
> $$1 + 2 + 3 + 4 + \cdots + n + (n + 1) + \ldots \textit{ in inf.}$$
>
> will be the sum of these natural numbers, and the following expression
>
> $$1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 + (n + 1)^0 + \ldots \textit{ in inf.}$$
>
> in which the single summands, $1^0, 2^0, 3^0, \ldots$ all represent mere units, represents just the number [*Menge*] of all natural numbers. If we designate this by $\overset{0}{N}$ and therefore form the merely symbolic equation
>
> $$1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 + (n + 1)^0 + \ldots \textit{ in inf.} = \overset{0}{N} \tag{1}$$
>
> and in the same way we designate the number [*Menge*] of natural numbers from $(n + 1)$ by $\overset{n}{N}$, and therefore form the equation
>
> $$(n + 1)^0 + (n + 2)^0 + (n + 3)^0 + \ldots \textit{ in inf.} = \overset{n}{N} \tag{2}$$

Then we obtain by subtraction the certain and quite unobjectionable equation

$$1^0 + 2^0 + 3^0 + \ldots + n^0 = n = \overset{0}{N} - \overset{n}{N} \tag{3}$$

This passage mentions several notions that will be central to the remainder of our analysis of Bolzano's *PU*, hence we will briefly go over them now.

First is the notion of "series" (*Reihe*), which Bolzano defines (*PU* §7) as a collection of "terms" (*Glieder*) $a, b, c, d, \ldots$ such that for each term c there is exactly another term d such that, by using the same rule for any pair c, d we can obtain (determine, *bestimmen*) c by applying said rule to d, or the inverse rule to c to obtain d instead. The natural numbers, that is, the "whole numbers" (*ganze Zahlen*) are defined as a series of objects of a certain kind A where the first term is a unit of kind A and the subsequent terms are sums obtained by adding one unit to their immediate predecessor.

The second concept we want to introduce is that of *Gliedermenge* (alternatively expressed by Bolzano as *Gliedermenge*, *Menge von Gliedern* or *Menge der Glieder*). As one can infer from *PU* §9, Bolzano considers any number series to have a *Gliedermenge*. Because a *Gliedermenge* is said to be sometimes greater, sometimes smaller, it seems reasonable to assume that this *Gliedermenge* is, if not a quantity properly said, at least something that can be quantified, i.e., treated as a quantity. In the passage we quote from §29, Bolzano introduces first the series of all natural numbers, then their sum and the *Menge* of such a sum. Given what was just said about series and *Gliedermenge* thereof, this occurrence of the word *Menge* should be read as a shorthand for *Gliedermenge* or one of its synonyms.

This occurrence of *Menge* is therefore at odds with any interpretation of Bolzano's definition of "multitude"(*Menge*) that sees it as (almost) synonymous with "set" in the modern sense. If the concept of multitude is virtually identical with that of set, then the multitude of $1 + 2 + 3 + 4 + \ldots$ *in inf.* should be just $1, 2, 3, 4, \ldots$ *in inf.* and not $1 + 1 + 1 + 1 + \ldots$ *in inf.* For the sake of preserving coherence in Bolzano's work in *PU* §§29-33 it is therefore sensible to insist that "*Gliedermenge*" is a quantitative concept.

As a consequence, since we believe that translating *Menge* here as "set", like Steele (Bolzano 1950), or "multitude", as we would have to if we were to translate *Menge* rigidly, obfuscates this quantitative aspect of the concept of "*Gliedermenge*", we prefer to respect Russ's (2004) choice and translate *Menge* as "number" when it seems to be short for *Menge der Glieder* or similar. As long as it is clear that we do not think Bolzano is introducing here genuine infinite numbers (in the sense of the German *Zahlen*), we will translate *Menge* as "number" in these contexts.

$\overset{0}{N}$ thus denotes the number (*Menge*) of all natural numbers, and for any natural number n, $\overset{n}{N}$ represents the size of the collection of all natural numbers strictly greater than n. This is all written as follows:

$$1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 + (n+1)^0 = \overset{0}{N} \tag{4}$$

$$(n+1)^0 + (n+2)^0 + \cdots = \overset{n}{N} \tag{5}$$

The 0^{th} power works in the standard way here, meaning $n^0 = 1$ for any natural number n. So for instance the size of the set of all natural numbers up to n is $1^0 + 2^0 + 3^0 + 4^0 + \cdots + n^0 = 1 + 1 + 1 \cdots + 1 = n$.

Having defined $\overset{0}{N}$ and $\overset{n}{N}$, Bolzano proceeds to show how they can be added or multiplied with one another thanks to distributivity. One then obtains a hierarchy of infinite quantities of ever-increasing *order*:

$$\begin{aligned} 1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} + \dots \ \textit{in inf.} &= (\overset{0}{N})^2 \\ 1^0.(\overset{0}{N})^2 + 2^0.(\overset{0}{N})^2 + 3^0.(\overset{0}{N})^2 + \dots \ \textit{in inf.} &= (\overset{0}{N})^3 \\ &\text{etc.} \end{aligned}$$

The notion of quantities being of different orders of infinity does not start with Bolzano and already existed in the context of infinitesimal calculus.[9] However, we will argue in Section 4 that Bolzano's computation of the product of infinite quantities is in fact very original and hence very significant for a comparison with Cantor's theory of the infinite (which we carry out in Section 5).

Having looked carefully at Bolzano's first computations with infinite sums, we now proceed to our next piece of evidence for interpreting Bolzano as primarily interested in infinite sums, namely, §32 of the *PU*.

2.2 Grandi's series

In *PU* §32, Bolzano criticizes a report by a certain M.R.S. in *Gergonne's Annales* (M.R.S. 1830) which purports to prove that the infinite sum

$$a - a + a - a + a \dots \quad (1)$$

has value $\frac{a}{2}$.

The series Bolzano focuses on is sometimes called Grandi's series after the Italian 18th century monk who first tried to compute a value for this infinite sum. Kline (1983) reports that this series was an object of great interest for mathematicians throughout the 19th century, that "caused endless dispute"(Kline 1983, pp. 307-308). It is not necessary for our summary of Bolzano's views to rehash the whole debate surrounding Grandi's series (and other divergent series) in great detail, though it is perhaps worth mentioning that Grandi's opinion, that the value of this series should be $\frac{a}{2}$, was shared also by Leibniz (Kline 1983, p. 307). Kline also reports that Leibniz's argument—which differed from Grandi's—was accepted by the Bernoulli brothers. This acceptance notwithstanding, by the time Bolzano is active there is still no clear consensus on how to treat what we would now consider divergent series. For Bolzano and his contemporaries, the question of how to assign a value to infinite sums such as Grandi's series was still a live question, one which would later lead some mathematicians (e.g., the Italian Cesàro) to define different sorts of summation.

[9]See for example the debate between Leibniz and Nieuwentijt on the existence of such higher-order infinitesimal, as presented in (Mancosu 1996, pp. 160-164).

It is therefore not surprising that one should come across a piece of writing such as M.R.S.'s. M.R.S. purports to prove that the value of Grandi's series is $\frac{a}{2}$ via an algebraic reasoning, as opposed to Leibniz's more "probabilistic" (per Kline) approach—and presumably, as opposed to Grandi's geometric approach, too. Here we quote M.R.S.'s own exposition of his proof:

> The summation of the terms of a geometric progression decreasing into the infinite can be easily deduced from the above; in fact, if one has
>
> $$x = a + aq + aq^2 + aq^3 + aq^4 + \dots,$$
>
> one can then write
>
> $$x = a + q(a + aq + aq^2 + aq^3 + \dots),$$
>
> then $x = a + qx$ or $(1 - q)x = a$, hence $x = \frac{a}{1-q}$. As per the remarks in (5), the equation
>
> $$x = a - a + a - a + a - a + \dots$$
>
> could not help in the approximation of x, as it successively gives the approximate values $a, 0, a, 0, a, 0, \dots$ among which the differences are constant; but, without resorting to Leibniz's subtle reasoning, one can immediately see that this equation comes to
>
> $$x = a - x,$$
>
> hence $x = \frac{1}{2}a$.[10] (M.R.S. 1830, pp. 363-364)

As the text shows, M.R.S.'s treatment of Grandi's series has the virtue of treating it

[10] Original French: La sommation des termes d'une prógression géométrique décroissante à l'infini se déduit bien simplement de ce qui précède; si en effet on a

$$x = a + aq + aq^2 + aq^3 + aq^4 + \dots,$$

on pourra d'abord écrire

$$x = a + q(a + aq + aq^2 + aq^3 + \dots),$$

puis $x = a + qx$ ou $(1 - q)x = a$ d'où $x = \frac{a}{1-q}$.

uniformly with other (converging) geometric series. Bolzano however is not impressed with M.R.S.'s algebraic manipulations and sees two mistakes in them. Bolzano spells out M.R.S.'s argument as follows. First, he sets

$$x = a - a + a - a + a - \ldots \textit{ in inf.} \tag{1}$$

Then, one can rewrite (1) as

$$a - (a - a + a - a + \ldots \textit{ in inf.}) \tag{2}$$

This yields $x = a - x$ and therefore $x = \frac{a}{2}$. Bolzano points out that while x is defined as $a - a + a - a + a - \ldots$ *in inf.*, the expression in (2) is not identical with it, because it does not have the same *Gliedermenge* as $a - a + a - a + a - \ldots$ *in inf.* in (1). The first a is missing so that the correct substitution ought to be the tautological $x = a + (x - a)$.

Even though Bolzano does not pause to point this out to the reader, M.R.S. is making exactly one of those mistakes Bolzano was cautioning against in §24: he has assumed equality of two quantities arising from summing up two series without checking that the two series have the same *Gliedermenge*. Note that here again Bolzano seems to be using *Menge* in a way that is closer to the meaning of "number" than to that of "set", and Russ's (2004) translation accordingly translates the term as "number". While again one should not take the translation literally, we agree with the attempt to capture a more quantitative use of *Menge* in this kind of context.

The second criticism Bolzano levels at M.R.S.'s argument is that it presupposes that $a - a + a - a + a \ldots$ refers to an actual quantity, whereas Bolzano argues that it does not. The argument Bolzano gives for this position is an example of Bolzano putting to (mathematical) use his logico-philosophical apparatus: Grandi's infinite sum is a spurious one because it does not display the sum property (*PU* §31)

$$(A + B) + C = A + (B + C) = (A + C) + B.$$

If one tries to rewrite Grandi's sum according to Bolzano's equations, the left-hand side becomes $(a - a) + (a - a) + \ldots$ *in inf.*, which according to Bolzano equals 0, whereas if one rearranges the parentheses as $a + (-a + a) + (-a + a) + (-a + a) + \ldots$ *in inf.*, one obtains a as a result. Thus indeed Grandi's expression does not satisfy Bolzano's definition of sum. Tapp (Bolzano 2012, p. 193) notes here that Bolzano's criterion is quite similar to Riemann's result (Apostol 1974, p. 197) which states that every infinite series is absolutely convergent if and only if it is preserved under permutation (an absolutely convergent series is one in which the series of the absolute values of its terms also converges). It is unfortunate though that Bolzano's criterion taken literally is too strong, as it seems to be also implying that $\overset{0}{N}$ does not designate an actual quantity (see Section 3 below).

We take this section of the *PU* as helping our case that Bolzano's work in §§29-33 should not be read as an imperfect set theory. Indeed, §32 is an example of Bolzano's principles for the computations of the infinite at work: a result published by a fellow mathematician about the computation of infinite *sums* is rejected on the basis of a violation of one of these principles. However most other commentators do not devote particular attention to §32. One notable exception is Steele, who thus summarizes §32: "Some errors in the pretended summation of $\Sigma(-1)^n a$, which is a symbol not expressing any true quantity at all" (Bolzano 1950, p. 66). Even more intriguingly, he mentions Grandi's series and the whole controversy surrounding it when introducing the historical context of the *PU* (Bolzano 1950, pp. 3-4). Yet it is as if this does not leave a trace when giving an overall appraisal of the contributions of the *PU*, or of Bolzano's contributions to mathematics and its philosophy. Bolzano is still presented as someone who *almost* anticipated Cantorian set theory, except he did not.

2.3 The sum of all squares

In the previous section, we argued that some passages of the *PU* offer textual evidence for the claim that Bolzano's work on the sizes of infinite collections should be understood as about sizes of infinite sums, that is, infinite series in modern terminology, rather than as about sizes of infinite countable sets. We now make a theoretical case as per why this interpretation is also the most charitable one.

Just following the discussion of §32, Bolzano writes that

> [...] if we wish to avoid getting onto the wrong track in our calculations with the infinite then we may never allow ourselves to declare two infinitely large quantities, which originated from the summation of the terms of two infinite series, as equal, or one to be greater or smaller than the other, because every term in the one is either equal to one in the other series, or greater or smaller than it. (*PU* §33)

So, for two infinite sums α and β, it is not the case that, say, $\alpha > \beta$ if for every term of α there is one in β that is strictly smaller.

He then continues:

> We may just as little declare such a sum as the greater just because it includes all the terms of the other and in addition many, even infinitely many, terms (which are all positive), which are absent in the other.

As an example of this principle in action, Bolzano asks us to consider the two series

$$1 + 4 + 9 + 16 + \ldots \textit{ in inf.} = \overset{2}{S}$$

and

$$1 + 2 + 3 + 4 + 5 + \ldots \textit{ in inf.} = \overset{1}{S}.$$

According to Bolzano, "no one can deny that every term of the series of all *squares*"—that is, $\overset{2}{S}$—"because it is also a natural number, also appears in the series of first powers of the natural numbers and likewise in the latter series $\overset{1}{S}$, together with all the terms of $\overset{2}{S}$ there appear many (even infinitely many) terms which are missing from $\overset{2}{S}$ because they are not square numbers." (*PU*, §33) So, the series $\overset{1}{S}$ and $\overset{2}{S}$ are such that the terms of the latter all appear in the former, and the former also includes infinitely many terms that the second series does not include. The next step in Bolzano's argument is to claim the following:

> Nevertheless $\overset{2}{S}$, the sum of all square numbers, is not smaller but is indisputably greater than $\overset{1}{S}$, the sum of the first powers of all numbers. (*PU*, §33)

Bolzano argues for this point by claiming two things: first, that "in spite of all appearance to the contrary, the *multitude of terms* [*Gliedermenge*] in both series (not considered as sums, and therefore not divisible into arbitrary multitudes of parts) is certainly the same." Second, that with the exclusion of the first term, all terms of $\overset{2}{S}$ are greater than the corresponding term in $\overset{1}{S}$. Since then the two series have the same amount of terms, but the terms of $\overset{2}{S}$ are greater than all but one of the terms in $\overset{1}{S}$, Bolzano concludes that $\overset{2}{S}$ is greater than $\overset{1}{S}$, because it is possible to termwise subtract $\overset{1}{S}$ from $\overset{2}{S}$ and one would still have a positive remainder as a result:

> But if the multitude of terms [*Menge der Glieder*] in $\overset{1}{S}$ and $\overset{2}{S}$ is the same, then it is clear that $\overset{2}{S}$ must be much greater than $\overset{1}{S}$, since, with the exception of the first term, each of the remaining terms in $\overset{2}{S}$ is definitely greater than the corresponding one in $\overset{1}{S}$. So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms as $\overset{1}{S}$, [...] (*PU*, §33)

As we can see, in §33 Bolzano repeats twice the idea that $\overset{1}{S}$ and $\overset{2}{S}$ have the same *Gliedermenge* (translated by Russ as "multitude of terms"). He is committed then to the claim

(Terms) The *Gliedermenge* in series $\overset{1}{S}$ and $\overset{2}{S}$ is the same.

This is often (see, e.g., Berg 1962, 1992; Šebestík 1992) interpreted as a sign that Bolzano was using here one-to-one correspondence to compare the size of the sets corresponding to $\overset{1}{S}$ and $\overset{2}{S}$, namely $\mathbb{N}$, the set of all natural numbers, and $\mathbb{N}^{(2)}$, the set of all squares, respectively. But if this is the case, then Bolzano is essentially violating part-whole as applied to sets, the way Šebestík suggests (cf. Section 1).

§29 and §33 taken together raise the question of how, if at all, Bolzano envisioned to generalize his notion of *Gliedermenge* from the collection of all natural numbers to any infinite subcollection thereof—or what would be a "Bolzanian enough" way of doing this.

Let us take a step back and reconsider what Bolzano does in §29. Recall that $\overset{0}{N} = 1^0 + 2^0 + 3^0 + 4^0 + \ldots$ *in inf.*, where each n^0 is one unit, as Bolzano reminds us. Assuming that $\overset{0}{N}$ is what Bolzano intended to be the *size* of $\mathbb{N}$ just in the same way as cardinals are considered to capture set size in modern set theory, the question is how to extend Bolzano's notion of size of $\mathbb{N}$ to infinite (proper) subsets of $\mathbb{N}$. Given the importance that the example of squares has in Bolzano scholarship (see our Section 1), let us try to answer the question for $\mathbb{N}^{(2)}$, specifically.

Per §29, the procedure to obtain the *Menge* (of terms, *von Gliedern*) of a series $\alpha := \alpha_1, \alpha_2, \alpha_3 \ldots$ is to first consider it as a sum

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots \textit{ in inf.},$$

and then raise each term to the power of 0. The *number* of terms in α is then identified with the value of the infinite sum $\overset{\alpha}{N} = \alpha_1^0 + \alpha_2^0 + \alpha_3^0 + \ldots$ *in inf.* This means that if we list all square numbers as $sq := 1, 4, 9, 16, 25, 36, \ldots$, the number of terms (hence the number of square numbers) should be identified with

$$\overset{sq}{N} = 1^0 + 4^0 + 9^0 + 16^0 + 25^0 + 36^0 + \ldots \textit{ in inf.}$$

Now notice that if we apply the same procedure to the series of terms of $\overset{2}{S}$, we obtain exactly the same. Since $\overset{2}{S}$ as a sum is $\overset{2}{S}$ itself, i.e.,

$$1 + 4 + 9 + 16 + 25 + 36 + \ldots \textit{ in inf.},$$

raising each term to the power of 0 yields

$$1^0 + 4^0 + 9^0 + 16^0 + 25^0 + 36^0 + \ldots \textit{ in inf.} = \overset{sq}{N}.$$

Thus the number of square numbers is the same as the number of terms in $\overset{2}{S}$. But since Bolzano endorses **(Terms)**, the number of terms in $\overset{2}{S}$ is equal to the number of terms in $\overset{1}{S}$, which is itself computed as $1^0 + 2^0 + 3^0 + \ldots$ *in inf.* $= \overset{0}{N}$. From this it immediately follows that $\overset{sq}{N}$ and $\overset{0}{N}$ have the same *Gliedermenge*. Moreover, since any term in each sum is regarded as a unit, both sums also have equal terms. Now by Bolzano's remark (*PU* §24) that "equal summands also give equal sums", we must therefore conclude that $\overset{sq}{N} = \overset{0}{N}$. But if the first one is the number of squares and the second one is the number of natural numbers, then under the standard (set theoretic) interpretation those two *sets* have the same *size*, which directly contradicts the part-whole principle. So it seems that we have reached a contradiction similar to the one highlighted by Šebestík (1992, pp. 463-464).

The first reaction would be of course to bite the bullet and accept that perhaps Bolzano did not realize that §29 and §33 would lead to a contradiction, and what is more, to a violation of part-whole. This seems to be the line that a set theoretic interpretation forces upon the reader. For, if $\overset{0}{N}$, being the *Gliedermenge* of $\overset{1}{S}$, is somehow also the size of $\mathbb{N}$, and the *Gliedermenge* of $\overset{2}{S}$ is also the size of $\mathbb{N}^{(2)}$, then of course Bolzano's remark in §33 that $\overset{1}{S}$ and $\overset{2}{S}$ have the same *Gliedermenge* cannot be reconciled with part-whole as applied to sets (**PW1**).

A second option would be to reject the generalization of the procedure of §29 to arrive at $\overset{0}{N}$ and argue that there is no analogue to $\overset{0}{N}$ for $\overset{2}{S}$. One could defend this position by pointing out that, in §28, Bolzano only commits to be able to *sometimes* compute with the infinite—not always. In particular, he does not commit to be able to determine the size of every subset of $\mathbb{N}$. We believe however that this answer is not entirely satisfactory. For one, this solution might feel *ad hoc*, because even though Bolzano may have not intended for the procedure of §29 to be applied indiscriminately to any set composed only of natural numbers, there is nothing intrinsic to the procedure itself that bars such a generalization from being carried out. Moreover, while §29 does not explicitly mention a general procedure for determining the *Gliedermenge* of an infinite sum, determining when two sums have the same *Gliedermenge* is necessary to determine whether one is greater than another, as Bolzano himself notes (see §§24 and 32). Since *Gliedermengen* are *Mengen*, multitudes, it is natural to ask whether part-whole reasoning applies to, or is even compatible with, the procedure of determining when the *Gliedermengen* of two sums are equal. In a way, then, this second option does not solve the theoretical problem raised by Bolzano's work so much as skirt around it via a "monster-barring" move.

There is a third option though, which hinges upon a closer reading of §29. Indeed, when computing quantities of the form $\overset{n}{N}$, which for him corresponds to the number of natural numbers greater than n, Bolzano does seem to apply the procedure sketched above, namely writing down the sum $(n+1)+(n+2)+(n+3)+\ldots$ *in inf.*, and then raising each term to the 0^{th} power, thus obtaining the sum $(n+1)^0+(n+2)^0+(n+3)^0+\ldots$ *in inf.*. However, if, as evidenced again in §33, the difference of two infinite sums is computed termwise, $\overset{0}{N} - \overset{n}{N}$ should be computed as:

$$(1^0 - (n+1)^0) + (2^0 - (n+2)^0) + \ldots \textit{ in inf.}$$

But each term in this sum is the difference of a unit and a unit, so it equals 0. Hence Bolzano should conclude $\overset{0}{N} - \overset{n}{N} = 0$. Instead, Bolzano writes that

$$\overset{0}{N} - \overset{n}{N} = 1^0 + 2^0 + \ldots + n^0,$$

which strongly suggests that Bolzano thinks that $\overset{0}{N} - \overset{n}{N}$ is equal to the infinite sum

$$(1^0) + (2^0) + \ldots + (n^0) + ((n+1)^0 - (n+1)^0) + ((n+2)^0 - (n+2)^0) + \ldots \textit{ in inf.}$$

But this in turn suggests that a more accurate way of representing $\overset{n}{N}$ is in fact as

$$\underbrace{\;+\quad+\ldots+\;}_{n \text{ times}} +(n+1)^0 + (n+2)^0 + \ldots \textit{ in inf.}$$

In other words, $\overset{n}{N}$ is not obtained by listing all the numbers above n in an infinite sum and raising each of them to the power of 0, but is instead obtained by erasing the first n terms from the sum corresponding to $\overset{0}{N}$. This procedure clearly changes the number of terms in the resulting sum. In order to compare $\overset{n}{N}$ to $\overset{0}{N}$, we must therefore make sure first that the two sums have the same *Gliedermenge*, which implies adding n terms to $\overset{n}{N}$ which act, quite literally, as the "ghosts of departed quantities".

This reading of Bolzano's text now gives a way out of the problem of the sum of all squares presented above. Let us consider again the example of $\mathbb{N}^{(2)}$. If we want to compute its size as a subset of $\mathbb{N}$, the way to obtain said size is first to compute that of $\mathbb{N}$, namely, $\overset{0}{N}$. We then remove from $\overset{0}{N}$ the elements whose base is not an element of $\mathbb{N}^{(2)}$, thus obtaining

$$\overset{SQ}{N} = 1^0 + \quad + \quad + \quad + 4^0 + \quad + \quad + \quad + \quad + 9^0 + \quad + \quad + \quad + \quad + \quad + \quad + 16^0 + \dots \textit{ in inf.}$$

The difference between $\overset{sq}{N}$ and $\overset{SQ}{N}$ is that, in the former, 4^0 is the second term of the sum, while it is the fourth term in $\overset{SQ}{N}$—and so on. The idea would be then that such an erasure procedure *does* change the number of elements from one set to the other, because $\mathbb{N}^{(2)}$ considered as a subset of $\mathbb{N}$ has a different size from when considered as the set underlying the sum $\overset{2}{S}$. Note that this distinction between $\overset{sq}{N}$ and $\overset{SQ}{N}$ is not available to a proponent of the received view: if $\overset{sq}{N}$ and $\overset{SQ}{N}$ are *sets*, i.e. entirely determined by their elements, then as the two sums clearly have the same terms, they should also be equal to one another. By contrast, the difference between the two sums is easy to express in our interpretation of Bolzano's computations (see next section), because $\overset{sq}{N}$ would correspond to a countable sequence with graph $\{\langle 1, 1^0\rangle, \langle 2, 4^0\rangle, \langle 3, 9^0\rangle, \dots\}$ whereas $\overset{SQ}{N}$ has graph $\{\langle 1, 1^0\rangle, \langle 2, 0\rangle, \langle 3, 0\rangle, \langle 4, 4^0\rangle, \dots\}$. Incidentally, Tapp (Bolzano 2012, p. 191) suggests a similar idea for the interpretation of §29, raising the question whether such an interpretation can actually lead to a fully-fledged coherent reading of the *PU*. Our next two sections address that question.

3 An ultrapower construction modelling Bolzano's arithmetic of the infinite

Our goal in this section is to offer a model of Bolzano's computations with infinite sums. More precisely, we interpret Bolzano's talk of infinite sums and operations between them as statements about a certain model and show that all of Bolzano's positive results as summarized in the previous section also hold in our model. Additionally, we argue that our model accurately represents Bolzano's *reasoning*, in that several of the proofs we provide closely match Bolzano's own arguments in the *PU*.

Our main idea is to associate to each infinite sum a corresponding infinite quantity. Our proposal here is closely related to the theory of numerosities (Benci and Di Nasso 2003; more recently Benci and Nasso 2019, Ch. 17), in which the numerosity of a set of natural numbers is defined as an element in an ultrapower of ℕ. However, since our focus is on assigning infinite quantities to certain infinite sums of integers, and not on assigning numerosities to sets of natural numbers, our proposal will be slightly different. Part of our model is in fact closer to the construction presented by Trlifajová (2018, pp. 20-24), which we will discuss in Section 3.4. In order to do that, we first need to outline our own proposal.

3.1 The basic framework

We start by representing Bolzano's infinite sums of integers as countable sequences of integers. Formally, we write ω^+ for the set of positive natural numbers and Z for the set of all integers, and we consider functions from $\omega^+ \to Z$. To any infinite sum $a_1 + a_2 + a_3 + \dots$ *in inf.*, we associate the function $f : i \mapsto a_i$, i.e., the function that maps each positive natural number i to the i^{th} summand of the infinite sum. As is customary, we will often identify a function $f : \omega^+ \to Z$ with the countable sequence of integers $(f(1), f(2), f(3), \dots)$. In the case of a Bolzanian sum α which has a different *Gliedermenge* because it has been obtained from another sum by erasing certain terms, we treat the erased terms as 0 and obtain the function associated to α accordingly. For example, since the sequence associated to $\overset{0}{N}$ is $(1, 1, 1, \dots)$, the sequence associated to $\overset{2}{N}$ is $(0, 0, 1, 1, \dots)$.

We consider the structure $\mathbb{Z} := (Z, +, -, 0, 1, <)$ of integers with their usual ordering and addition operation, and take an ultrapower $\mathbb{Z}_{\mathscr{U}}$ of this structure by a non-principal ultrafilter on ω^+ (i.e., a non-empty collection $\mathscr{U}$ of infinite subsets of ω^+ closed under supersets and finite intersections and such that for any $A \subseteq \omega^+$, precisely one of $A, \omega^+ \backslash A$ belongs to $\mathscr{U}$). Ultrapowers are standard constructions in mathematical logic, and a detailed presentation of their theory is beyond the scope of this paper. Instead, we refer the reader to Bell and Slomson (1974, Chs. 5, 6) for a standard introduction to ultrapowers and ultraproducts, and simply list some crucial facts below:

Lemma 3.1.

1. *Elements in the ultrapower $\mathbb{Z}_{\mathscr{U}}$ are equivalence classes of functions from ω^+ to Z. For any $f : \omega^+ \to Z$, we write its corresponding equivalence class as f^*. For any $f, g : \omega^+ \to Z$, $g^* = f^*$ if and only if f and g are equal for $\mathscr{U}$-many elements in ω^+, i.e., $\{i \in \omega^+ : f(i) = g(i)\} \in \mathscr{U}$.*

2. *There is a canonical elementary embedding of $\mathbb{Z}$ into $\mathbb{Z}_{\mathscr{U}}$, obtained by mapping any integer z to the equivalence class of the constant function $e_z : \omega^+ \to Z$ sending any $i \in \omega^+$ to z. It is customary to identify z with e_z^* and to view $\mathbb{Z}$ as an elementary substructure of $\mathbb{Z}_{\mathscr{U}}$.*

3. *Addition and subtraction are defined in $\mathbb{Z}_{\mathscr{U}}$. Given $f, g : \omega^+ \to Z$, $f^* + g^*$ is the equivalence class of the function $h : \omega^+ \to Z$ such that $h(i) = f(i) + g(i)$ for any $i \in \omega^+$. Similarly, $f^* - g^*$ is the equivalence class of the function $h : \omega^+ \to Z$ such that $h(i) = f(i) - g(i)$ for any $i \in \omega^+$.*

4. *Elements in $\mathbb{Z}_{\mathscr{U}}$ are linearly ordered. More precisely, given any $f, g : \omega^+ \to Z$, we have that $\mathbb{Z}_{\mathscr{U}} \models f < g$ if and only if $\{i \in \omega^+ : \mathbb{Z} \models f(i) < g(i)\} \in \mathscr{U}$.*

5. *Given any first-order formula $\phi(x_1, \ldots, x_n)$ and any functions $f_1, \ldots, f_n : \omega^+ \to Z$, we write $||\phi(f_1^*, \ldots, f_n^*)||$ for the set $\{i \in \omega^+ : \mathbb{Z} \models \phi(f_1(i), \ldots, f_n(i))\}$. Łoś's Theorem states that for any $\phi(x_1, \ldots, x_n)$ and any functions $f_1, \ldots, f_n$,*

$$\mathbb{Z}_{\mathscr{U}} \models \phi(f_1^*, \ldots, f_n^*) \textit{ iff } ||\phi(f_1^*, \ldots, f_n^*)|| \in \mathscr{U}.$$

6. *As a direct consequence of Łoś's Theorem, $\mathbb{Z}$ and $\mathbb{Z}_{\mathscr{U}}$ are elementarily equivalent.*

An intuitive motivation for our use of an ultrapower of $\mathbb{Z}$ can be provided along the following lines. As we have argued, we take Bolzanian infinite quantities to be infinite sums. Given an infinite sum α, we may decompose α into a sequence of partial sums $\{\alpha_n\}_{n\in\omega^+}$, where, for any positive integer n, α_n is the sum of the first n terms in α. Any such sum can be seen as providing some partial information about α, and if α were a *finite* sum with n terms, then α_n would be α itself. However, since α is infinite, there is no last term of α and no partial sum that would give us total information about α. In order to overcome this difficulty, we must try to organize the partial information given by each partial sum of the first n terms of α into a coherent whole. This is precisely the role that a non-principal ultrafilter $\mathscr{U}$ on ω^+ will play for us. One may think of $\mathscr{U}$ as a collection of properties of positive integers that describe a natural number "at infinity", distinct from all finite numbers, and providing a vantage point from which all the partial sums of α form a coherent picture. We therefore encourage the reader who may not be familiar with ultrapowers to keep the following two principles in mind:

- Properties of an infinite sum α are those that are shared by "most" partial sums of the form α_n;

- What "most" partial sums means is determined by $\mathscr{U}$. Given a set of positive integers A, the set $\{\alpha_n : n \in A\}$ contains "most" partial sums of α if and only if $A \in \mathscr{U}$.

Given a function $f : \omega^+ \to Z$, we define the *approximating sequence* of f to be the function $\sigma(f) : \omega^+ \to Z$ defined by $\sigma(f)(i) = \sum_{j=1}^{i} f(j)$ for any $i \in \omega^+$. In the case of a function f representing a Bolzanian sum α, the approximating sequence of f is simply the sequence of partial sums $(\alpha^1, \alpha^2, \dots)$ mentioned above. Our proposal consists in identifying the (possibly infinite) quantity designated by a Bolzanian sum f with $\sigma(f)^*$, i.e., with the equivalence class of its approximating sequence. To simplify notation, we will write $\mathbf{f}$ for the element $\sigma(f)^*$ in $\mathbb{Z}_{\mathscr{U}}$, but we will sometimes abuse notation and write $\mathbf{f}(i)$ for $\sigma(f)(i)$.

We are now able to represent all infinite sums and infinite quantities discussed by Bolzano, except products of infinite quantities, which we will discuss in Section 4. As outlined above, the procedure consists in turning a Bolzanian infinite sum into a countable sequence of integers, to which (the equivalence class of) an approximating sequence is then associated. Additions and order relations between infinite sums are then determined by the ultrapower. As an example, the infinite sum $1^0 + 2^0 + 3^0 + \dots$ *in inf.* is represented by the sequence $\overset{0}{N} := (1, 1, 1, \dots)$, since, according to Bolzano, each summand of this sum is a unit. Consequently, the approximating sequence of $\overset{0}{N}$ is the sequence $\sigma(\overset{0}{N}) = (1, 2, 3, \dots)$, which corresponds to the identity function on ω^+, and $\overset{\mathbf{0}}{\mathbf{N}}$ is the equivalence class of the sequence $(1, 2, 3, \dots)$. Similarly, infinite sums of the form $(n+1)^0 + (n+2)^0 + \dots$ *in inf.*, which Bolzano writes as $\overset{n}{N}$, are sums that according to him have n fewer terms than $\overset{0}{N}$. We therefore propose to model $\overset{n}{N}$ as a countable sequence in which the first n summands are 0, i.e., by the sequence $(\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 1, \dots)$. The corresponding approximating sequence $\sigma(\overset{n}{N})$ is $(\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 2, \dots)$. Equivalently, for any $i \in \omega^+$, $\sigma(\overset{n}{N})(i) = i \dot{-} n$, where $i \dot{-} n = 0$ if $i \leqslant n$ and $i - n$ otherwise.

A similar approach can be applied to represent the sums $\overset{1}{S}$ and $\overset{n}{S}$, as well as Grandi's series of the form $G_a = a - a + a - a + \dots$ *in inf.* For clarity's sake, we have collected the representation of $\overset{0}{N}, \overset{n}{N}, \overset{1}{S}, \overset{n}{S}$, and G_a in the table below:

Bolzanian Infinite Sum	Sequence Representation	approximating sequence	Corresponding Function	Infinite Quantity
$1^0 + 2^0 + \dots$ *in inf.*	$\overset{0}{N} = (1, 1, 1, 1, \dots)$	$\sigma(\overset{0}{N}) = (1, 2, 3, 4, \dots)$	$\sigma(\overset{0}{N})(i) = i$	$\overset{\mathbf{0}}{\mathbf{N}} = \sigma(\overset{0}{N})^*$
$(n+1)^0 + (n+2)^0 + \dots$ *in inf.*	$\overset{n}{N} = (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 1, \dots)$	$\sigma(\overset{n}{N}) = (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 2, 3, \dots)$	$\sigma(\overset{n}{N})(i) = i \dot{-} n$	$\overset{\mathbf{n}}{\mathbf{N}} = \sigma(\overset{n}{N})^*$
$1 + 2 + 3 + \dots$ *in inf.*	$\overset{1}{S} = (1, 2, 3, 4, \dots)$	$\sigma(\overset{1}{S}) = (1, 3, 6, 10, \dots)$	$\sigma(\overset{1}{S})(i) = \sum_{j=1}^{i} j$	$\overset{\mathbf{1}}{\mathbf{S}} = \sigma(\overset{1}{S})^*$
$1^n + 2^n + 3^n \dots$ *in inf.*	$\overset{n}{S} = (1^n, 2^n, 3^n, 4^n, \dots)$	$\sigma(\overset{n}{S}) = (1^n, (1^n + 2^n), \dots)$	$\sigma(\overset{n}{S})(i) = \sum_{j=1}^{i} j^n$	$\overset{\mathbf{n}}{\mathbf{S}} = \sigma(\overset{n}{S})^*$
$a - a + a - a + \dots$ *in inf.*	$G_a = (a, -a, a, -a, \dots)$	$\sigma(G_a) = (a, 0, a, 0, \dots)$	$\sigma(G_a)(i) = \begin{cases} a & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$	$\mathbf{G_a} = \sigma(G_a)^*$

Table 1: Representation of Bolzanian sums in $\mathbb{Z}_{\mathscr{U}}$

3.2 Modelling Bolzano's results about infinite sums

We now establish some results that echo Bolzano's own computations. We will first give proofs in our framework, then argue that those proofs are very close in spirit to Bolzano's arguments. We start with results about infinite sums of the form $\overset{\mathbf{n}}{\mathbf{N}}$ and $\overset{\mathbf{n}}{\mathbf{S}}$:

Lemma 3.2.

1. *For any natural numbers i, n, $\mathbb{Z}_{\mathscr{U}} \models i < \overset{\mathbf{n}}{\mathbf{N}}$.*
2. *For any natural number n, $\mathbb{Z}_{\mathscr{U}} \models \overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}} = n$.*
3. *For any natural number i, $\mathbb{Z}_{\mathscr{U}} \models i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}$.*
4. *For any natural numbers i, n, $\mathbb{Z}_{\mathscr{U}} \models i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}$.*

The first result asserts that all sums of the form $\overset{\mathbf{n}}{\mathbf{N}}$ are infinite, in the sense that they are greater than any finite number. The second shows that our model preserves Bolzano's part-whole intuition that certain infinite sums might have fewer terms than some others and that, as a consequence, two infinite quantities might differ by a finite quantity. Finally, the last two correspond to Bolzano's claim that some infinite quantities might be infinitely greater than some others. Note that we write $n\alpha$ as a shorthand for the sum of α with itself n times, which is defined in the ultrapower.

The proofs for all four items are all similar and can be thought of as "arguments by cofiniteness". In all cases, we show that $\mathbb{Z}_{\mathscr{U}}$ satisfies a formula ϕ by showing that $||\phi||$ is a cofinite subset of ω^+ and must therefore belong to $\mathscr{U}$ (since $\mathscr{U}$ is non-principal, it contains no finite set, so it must contain all cofinite sets).

Proof.

1. Recall that, in $\mathbb{Z}_{\mathscr{U}}$, the natural number i corresponds to (the equivalence class of) the function $e_i : m \mapsto i$. Moreover, for any natural number n, $\overset{\mathbf{n}}{\mathbf{N}}(i) = i \dot{-} n$. Thus $||i < \overset{\mathbf{n}}{\mathbf{N}}|| = \{j \in \omega^+ : i < j \dot{-} n\} = \{j \in \omega^+ : i + n < j\}$. Hence $||i < \overset{\mathbf{n}}{\mathbf{N}}||$ is a cofinite subset of ω^+ and belongs to $\mathscr{U}$, from which it follows that $\mathbb{Z}_{\mathscr{U}} \models i < \overset{\mathbf{n}}{\mathbf{N}}$.

2. Again, in $\mathbb{Z}_{\mathscr{U}}$, n is (the equivalence class of) the function $e_n : m \mapsto n$. Moreover, $\overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}}$ is (the equivalence class of) the function $f : \omega^+ \to Z$ such that

$$f(i) = \overset{\mathbf{0}}{\mathbf{N}}(i) - \overset{\mathbf{n}}{\mathbf{N}}(i) = i - (i \dot{-} n)$$

for any $i \in \omega^+$. Hence $||\overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}} = n|| = \{i \in \omega^+ : i - (i \dot{-} n) = n\} = \{i \in \omega^+ : i \geqslant n\}$. Hence $||\overset{\mathbf{n}}{\mathbf{N}} = \overset{\mathbf{0}}{\mathbf{N}} - n||$ is a cofinite subset of ω^+, and $\mathbb{Z}_{\mathscr{U}} \models \overset{\mathbf{n}}{\mathbf{N}} = \overset{\mathbf{0}}{\mathbf{N}} - n$.

3. Since $i\overset{\mathbf{0}}{\mathbf{N}} = \underbrace{\overset{\mathbf{0}}{\mathbf{N}} + \ldots + \overset{\mathbf{0}}{\mathbf{N}}}_{i \text{ times}}$, we have that $i\overset{\mathbf{0}}{\mathbf{N}}(j) = i \times j$ for any $j \in \omega^+$. On the other hand, $\overset{\mathbf{1}}{\mathbf{S}}(j) = \sum_{k=1}^{j} k$ which, by Gauss's summation theorem, is equal to $\frac{j(j+1)}{2}$. Hence

$$||i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}|| = \{j \in \omega^+ : i \times j < \sum_{k=1}^{j} k\} = \{j \in \omega^+ : i \times j < \frac{j(j+1)}{2}\} = \{j \in \omega^+ : i < \frac{j+1}{2}\}.$$

Hence $||i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}||$ is cofinite, and $\mathbb{Z}_{\mathscr{U}} \models i\overset{\mathbf{0}}{\mathbf{N}} < \overset{\mathbf{1}}{\mathbf{S}}$.

4. The argument is a simple generalization of the one above. Fix some natural numbers i and n. Then for any $k \in \omega^+$, $i\overset{\mathbf{n}}{\mathbf{S}}(k) = i \sum_{j=1}^{k} j^n$, and $\overset{\mathbf{n+1}}{\mathbf{S}}(k) = \sum_{j=1}^{k} j^{n+1}$. This means that $(\overset{\mathbf{n+1}}{\mathbf{S}} - i\overset{\mathbf{n}}{\mathbf{S}})(k) = \overset{\mathbf{n+1}}{\mathbf{S}}(k) - i\overset{\mathbf{n}}{\mathbf{S}}(k) = \sum_{j=1}^{k} (j^{n+1} - ij^n)$ for any $k \in \omega^+$. Now since $(j^{n+1} - ij^n)$ is positive for any $j > i$ and in fact assumes arbitrarily large positive values, it follows that $(\overset{\mathbf{n+1}}{\mathbf{S}} - i\overset{\mathbf{n}}{\mathbf{S}})(k)$ is positive for any large enough k. Thus $||\overset{\mathbf{n+1}}{\mathbf{S}} - \overset{\mathbf{n}}{\mathbf{S}} > 0||$ is a cofinite subset of ω^+. Now since $\mathbb{Z} \models \forall x \forall y (x - y > 0 \to y < x)$, by Łoś's Theorem we have that $\mathbb{Z}_{\mathscr{U}} \models \overset{\mathbf{n+1}}{\mathbf{S}} - i\overset{\mathbf{n}}{\mathbf{S}} > 0 \to i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}$. Hence $\mathbb{Z}_{\mathscr{U}} \models i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}$ for any natural numbers i and n.[11] $\square$

[11] A more direct proof of this result can also be given using more advanced resources from number

Let us now compare the proofs above with Bolzano's arguments in sections 29 and 33 of *PU*. Bolzano does not explicitly argue for results (1) and (3): in §29, he seems to take for granted that sums of the form $\overset{\mathbf{0}}{\mathbf{N}}$ and $\overset{\mathbf{n}}{\mathbf{N}}$ designate infinite quantities, and he simply writes that $\overset{\mathbf{1}}{\mathbf{S}}$ is "far greater than $\overset{0}{N}$". However, the same section contains the following argument for (2):

> If we designate [the number of all natural numbers] by $\overset{0}{N}$ and therefore form the merely symbolic equation
>
> $$1^0 + 2^0 + 3^0 + \ldots + n^0 + (n+1)^0 + \ldots \ \textit{in inf.} = \overset{0}{N} \tag{1}$$
>
> and in the same way we designate the number of natural numbers from $(n+1)$ $\overset{n}{N}$, and therefore form the equation
>
> $$(n+1)^0 + (n+2)^0 + (n+3)^0 + \ldots \ \textit{in inf.} = \overset{n}{N}, \tag{2}$$
>
> then we obtain by subtraction the certain and quite unobjectionable equation
>
> $$1^0 + 2^0 + 3^0 + \ldots + n^0 = n = \overset{0}{N} - \overset{n}{N} \tag{3}$$
>
> from which we therefore see how two infinite quantities $\overset{0}{N}$ and $\overset{n}{N}$ sometimes have a completely definite finite difference.

As mentioned in Section 2, we read Bolzano as arguing that subtracting $\overset{n}{N}$ from $\overset{0}{N}$ amounts to subtracting from each term i^0 after the n^{th} summand in $\overset{0}{N}$ the corresponding term i^0 in $\overset{n}{N}$. The only terms left in $\overset{0}{N} - \overset{n}{N}$ after this procedure are the first n summands in $\overset{0}{N}$, from which it follows that $\overset{0}{N} - \overset{n}{N} = n$. In our setting, $\overset{\mathbf{0}}{\mathbf{N}}$ is represented

theory. It is a standard number-theoretic fact (using for example Faulhaber's formula) that for any natural numbers k, n, $\sum_{j=1}^{k} j^n$ is a polynomial of degree $n+1$ in k, with leading term $\frac{1}{n+1}k^{n+1}$. Thus $i\overset{\mathbf{n}}{\mathbf{S}}(k)$ is a polynomial in k of degree $n+1$ with leading term $\frac{i}{n+1}k^{n+1}$, while $\overset{\mathbf{n+1}}{\mathbf{S}}(k)$ is a polynomial in k of degree $n+2$ with leading term $\frac{1}{n+2}k^{n+2}$. This means that $i\overset{\mathbf{n}}{\mathbf{S}}(k) < \overset{\mathbf{n+1}}{\mathbf{S}}(k)$ for k sufficiently large, and thus $||i\overset{\mathbf{n}}{\mathbf{S}} < \overset{\mathbf{n+1}}{\mathbf{S}}||$ is a cofinite subset of ω^+.

by (the equivalence class of) the sequence $(1, 2, 3, \dots)$, while $\overset{\mathbf{n}}{\mathbf{N}}$ is represented by the sequence $(\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 2, 3 \dots)$, and $\overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{n}}{\mathbf{N}}$ is the sequence obtained by subtracting $\overset{\mathbf{0}}{\mathbf{N}}$ from $\overset{\mathbf{n}}{\mathbf{N}}$ componentwise, i.e., the sequence $(1, 2, 3 \dots, n, n, n, \dots)$, which over $\mathscr{U}$ is equivalent to n. Similarly to Bolzano's argument, the difference between the two infinite sums $\overset{\mathbf{0}}{\mathbf{N}}$ and $\overset{\mathbf{n}}{\mathbf{N}}$ is determined by the difference between matching summands (i.e., the difference is computed componentwise) and is precisely n.

Finally, Bolzano does not explicitly argue for (4) in its full generality. In a very revealing passage in §33, however, he gives a detailed argument for the $n = 1$ instance of (4) when arguing that $\overset{2}{S}$ is infinitely greater than $\overset{1}{S}$:

> But if the multitude of terms [*Menge der Glieder*] in $\overset{1}{S}$ and $\overset{2}{S}$ is the same, then it is clear that $\overset{2}{S}$ must be much greater than $\overset{1}{S}$, since, with the exception of the *first* term, each of the remaining terms in $\overset{2}{S}$ is definitely greater than the corresponding one in $\overset{1}{S}$. So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms [*Gliederzahl*] as $\overset{1}{S}$, namely:
>
> $$0, 2, 6, 12, 20, 30, 42, 56, \dots, n(n-1), \dots \ \textit{in inf.},$$
>
> in which, with the exception of the first *two* terms, all succeeding terms are greater than the corresponding terms in $\overset{1}{S}$, so that the sum of the whole series is again indisputably greater than $\overset{1}{S}$. If we therefore subtract from this remainder the series $\overset{1}{S}$ for the second time, then we obtain as the *second* remainder a series of the same number of terms [*Gliedermenge*]
>
> $$-1, 0, 3, 8, 15, 24, 35, 48, \dots, n(n-2), \dots \ \textit{in inf.}$$
>
> in which, with the exception of the first *three* terms all the following terms are

greater than the corresponding ones in $\overset{1}{S}$, so that also this third remainder is without contradiction greater than $\overset{1}{S}$. Now since these arguments can be continued without end it is clear that the sum $\overset{2}{S}$ is infinitely greater than the sum $\overset{1}{S}$, while in general we have

$$\overset{2}{S} - m\overset{1}{S} = (1-m) + (2^2-2m) + (3^2-3m) + (4^2-4m) + \ldots + (m^2-m^2) + \ldots + n(n-m) + \ldots \textit{ in inf.}$$

In this series only a finite multitude of terms [*Menge von Gliedern*], namely the first $m-1$ are negative and the m^{th} is 0, but all succeeding ones are positive and increase indefinitely.

Let us note two features of Bolzano's argument that are shared by our interpretation. First, when determining whether one infinite sum is greater than another one, Bolzano considers which terms in the first sum are greater than the corresponding terms in the second one: this is reminiscent of the way relations between (equivalence classes of) functions are determined in an ultrapower. Moreover, Bolzano's reason to claim that $\overset{2}{S}$ is greater than $\overset{1}{S}$, $2\overset{1}{S}$, $3\overset{1}{S}$, and so on, is that in all such cases, all but finitely many terms in $\overset{2}{S}$ are strictly greater than the corresponding terms in any finite multiple of $\overset{1}{S}$. This seems very similar to the "argument by cofiniteness" that we presented above: even though the first terms of the sum $m\overset{1}{S}$ might be greater than the first terms of the sum $\overset{2}{S}$, the terms in the second sum become greater than the corresponding terms in the first one from some point onwards. In our setting, we prove that $\mathbb{Z}_{\mathscr{U}} \models \overset{\mathbf{2}}{\mathbf{S}} > m\overset{\mathbf{1}}{\mathbf{S}}$ by showing that $\overset{\mathbf{2}}{\mathbf{S}}(i) - m\overset{\mathbf{1}}{\mathbf{S}}(i) > 0$ for cofinitely many natural numbers i. To establish this, it is enough to observe, like Bolzano, that $i^2 - mi$ is positive for any $i > m$, as this implies that the sum $\sum_{j=1}^{i}(j^2 - mj)$ must be positive for i large enough. It is worth mentioning that, unlike in Bolzano's argument, our "tipping point", i.e., the value i at which $\overset{\mathbf{2}}{\mathbf{S}}(i)$ becomes strictly greater than $m\overset{\mathbf{1}}{\mathbf{S}}(i)$ is not $m+1$. This is because $\overset{\mathbf{2}}{\mathbf{S}}$ and $\overset{\mathbf{1}}{\mathbf{S}}$ are the approximating sequences of the sequences $(1, 2, 3, \ldots)$ and $(1, 4, 9, \ldots)$ respectively, while Bolzano is reasoning with the sequences of terms themselves. We therefore conclude that the general proof given for 4 closely matches Bolzano's own reasoning. In particular, our use of an ultrapower construction enables us to lift the following criterion for the inequality of two integers:

$$\forall m, n(m < n \leftrightarrow n - m > 0) \tag{1}$$

to a criterion for the inequality of two infinite sums:

$$\forall \boldsymbol{\alpha}, \boldsymbol{\beta}(\boldsymbol{\alpha} < \boldsymbol{\beta} \leftrightarrow \{i \in \omega^+ : (\boldsymbol{\beta} - \boldsymbol{\alpha})(i) > 0\} \in \mathscr{U}). \tag{2}$$

In other words, in our formalism, in order to determine whether an infinite sum $\boldsymbol{\alpha}$ is greater than another infinite sum $\boldsymbol{\beta}$, it is enough to compute their difference $\boldsymbol{\beta} - \boldsymbol{\alpha}$, which is defined termwise, and then determine whether the sum of the first i terms of $\boldsymbol{\beta} - \boldsymbol{\alpha}$ is positive for $\mathscr{U}$-many i. Our claim is that this reasoning is very close to the one displayed by Bolzano in §33. Moreover, let us note that when he argues that $\overset{2}{S}$ is greater than $m\overset{1}{S}$ for any m, because all but finitely many terms in the infinite sum $\overset{2}{S} - m\overset{1}{S}$ are positive, Bolzano can be seen as implicitly displaying a form of part-whole reasoning about *sums*, rather than sets: $m\overset{1}{S}$ is smaller than $\overset{2}{S}$ because it is contained "as a part". This is established by showing that the difference $\overset{2}{S} - m\overset{1}{S}$ is positive, and this latter fact is established in turn by noticing that all but finitely many terms in $\overset{2}{S} - m\overset{1}{S}$ are positive. Thus Bolzano can be read here as providing a criterion for when the quantity designated by a sum α is a proper part of the quantity designated by another sum β. We will come back to this point in Section 5, and we will discuss its implication for the role that part-whole reasoning plays in Bolzano's computations with the infinitely large.

3.3 Grandi's series

Finally, let us address some of Bolzano's remarks on Grandi's series. As noted above, Bolzano disagrees with the claim (attributed to M.R.S.) that the infinite sum

$$x = a - a + a - a + \dots \textit{ in inf.}$$

designates the quantity $\frac{a}{2}$. In particular, Bolzano claims that the mistake in M.R.S.'s proof is to treat the sum obtained by discarding the first term of x as $-x$. In our setting, x designates the quantity $\mathbf{G_a}$, i.e., the equivalence class of the sequence $(a, 0, a, 0, \dots)$. On the other hand, following the strategy adopted for "truncated" infinite sums like $\overset{\mathbf{n}}{\mathbf{N}}$, it seems that the infinite sum obtained by discarding the first term in x should be interpreted as the countable sequence $(0, -a, a, -a, a, \dots)$. If we write this sequence as $\overset{1}{G}_a$, we then have that $\overset{\mathbf{1}}{\mathbf{G}}_{\mathbf{a}}$ is the equivalence class of the sequence $(0, -a, 0, -a, \dots)$. But then, it follows that

$$\mathbb{Z}_{\mathscr{U}} \models \mathbf{G_a} - \overset{\mathbf{1}}{\mathbf{G}}_{\mathbf{a}} = a.$$

Indeed, for any $i \in \omega^+$, $\mathbf{G_a}(i) = a$ if i is even and 0 if i is odd, while $\overset{\mathbf{1}}{\mathbf{G}}_{\mathbf{a}}(i) = 0$ if i is even and $-a$ if i is odd. Thus $\mathbf{G_a}(i) - \overset{\mathbf{1}}{\mathbf{G}}_{\mathbf{a}}(i) = a$ for any i. Hence our interpretation agrees with Bolzano's diagnostic of the fallacy in M.R.S.'s proof:

> The series in the brackets obviously does not have the same multitude of terms [*Gliedermenge*] as the one put $= x$ at first, rather it is lacking the first a. Therefore its value, supposing it could actually be stated, would have to be denoted by $x - a$. But this would have given the identical equation
>
> $$x = a + x - a.$$

Moreover, recall that Bolzano raises a second, deeper argument against M.R.S.'s conclusion: the infinite sum x cannot designate an "actual quantity", since different ways of parsing this infinite sum yield different conclusions regarding which quantity it allegedly designates. According to Bolzano, the infinite sum

$$a - a + a - a + \dots \textit{ in inf.}$$

represents the same quantity as the sums

$$(a - a) + (a - a) + (a - a) + \dots \textit{ in inf.}$$

and

$$a + (-a + a) + (-a + a) + (-a + a) + \dots \ \textit{in inf.}$$

But the first expression simplifies as

$$0 + 0 + 0 + \dots \ \textit{in inf.},$$

while the second one simplifies as

$$a + 0 + 0 + 0 + \dots \ \textit{in inf.}.$$

Therefore, if it were a real quantity, x should be equal to both 0 and a, which is a contradiction.

What does this argument become in our interpretation? At first sight, it seems that we cannot make sense of Bolzano's claim that Grandi's series does not represent any actual quantity, since we attributed to this series the element $\mathbf{G_a}$ in $\mathbb{Z}_{\mathscr{U}}$. However, it is straightforward to verify that, depending on which subsets of ω^+ are in $\mathscr{U}$, $\mathbf{G_a}$ is computed differently in the ultrapower. Indeed, since $\mathbf{G_a}$ is the (equivalence class of) the sequence $(a, 0, a, 0, \dots)$, we have that $||\mathbf{G_a} = a|| = \{2i - 1 : i \in \omega^+\}$, while $||\mathbf{G_a} = 0|| = \{2i : i \in \omega^+\}$. Now since $\mathscr{U}$ is an ultrafilter, exactly one of $||\mathbf{G_a} = a||$ or $||\mathbf{G_a} = 0||$ belongs to $\mathscr{U}$. This implies that $\mathbb{Z}_{\mathscr{U}} \models \mathbf{G_a} = a \vee \mathbf{G_a} = 0$ regardless of our choice of ultrafilter, but the choice of $\mathscr{U}$ determines whether $\mathbb{Z}_{\mathscr{U}} \models \mathbf{G_a} = a$ or $\mathbb{Z}_{\mathscr{U}} \models \mathbf{G_a} = 0$. Thus we seem to recover at last part of Bolzano's intuition that the quantity designated by the sum $a - a + a - a + \dots$ *in inf.* is indeterminate, as it can be computed to be equal to 0 or to a.

Bolzano also argues that the sum $a - a + a - a + \dots$ *in inf.* should represent the same quantity as the sum

$$-a + (a - a) + (a - a) + \dots \ \textit{in inf.},$$

which simplifies to

$$-a + 0 + 0 + \dots \ \textit{in inf.},$$

and should therefore designate the quantity $-a$. His argument is that one may first compute Grandi's series as

$$(a - a) + (a - a) + \dots \ \textit{in inf.}$$

Using commutativity of addition an infinite number of times, swap each pair of terms in order to obtain the series

$$(-a + a) + (-a + a) + \dots \ \textit{in inf.},$$

which, by associativity is then equivalent to

$$-a + (a - a) + (a - a) + \dots \ \textit{in inf.}$$

In our setting, the infinite sum $-a + a - a + a - \dots$ *in inf.* is represented by its approximating sequence $(-a, 0, -a, 0, \dots)$. As a consequence, the infinite sums $a - a + a - a + \dots$ *in inf.* and $-a + a - a + a - \dots$ *in inf.* will be identified in $\mathbb{Z}_{\mathscr{U}}$ precisely if $\{2i : i \in \omega^+\} \in \mathscr{U}$. In fact, as shown above, in such a case both series will be identified with 0.

In light of the remarks above, it might be tempting to conclude that Bolzano's criterion for an infinite sum to represent an actual quantity, namely that the order in which the terms are summed do not change the result of the summation, could be interpreted in our framework as some kind of absoluteness of the corresponding sequences under the choice of a non-principal ultrafilter $\mathscr{U}$. However, it is straightforward to observe that Bolzano's own criterion is too strong for his purposes. Indeed, let us consider again the infinite sum $\overset{0}{N} = 1^0 + 2^0 + 3^0 + \dots$ *in inf.* If we interpret, as we have done so far, n^0 as equal to 1 for any natural number n, then this infinite sum may actually be written as

$1+1+1+\ldots$ *in inf.*, which is a special case of a geometric series of the form $\sum_{n=0}^{\infty} ar^n$ where $a = r = 1$. Similarly to Bolzano's argument for Grandi's series, we may now rewrite $\overset{0}{N}$ as

$$\begin{aligned}(1+1)+(1+1)+(1+1)+\ldots \textit{ in inf.} &= 2+2+2\ldots \textit{ in inf.}\\ &= 2(1+1+1\ldots \textit{ in inf.})\\ &= 2\overset{0}{N},\end{aligned}$$

from which we would be forced to conclude that $\overset{0}{N} - \overset{0}{N} = \overset{0}{N}$, implying that $\overset{0}{N} = 0$. Thus $\overset{0}{N}$ does not designate any infinite quantity after all, since it is equal to 0. This means that the order in which the terms in $\overset{0}{N}$ are summed determine which quantity the sum designates, which, by Bolzano's own criterion, is impossible. Of course, a Bolzanian could reply to that argument that there is a fallacy in deriving this equality, because the sum between parenthesis on the second line above does not have the same *Gliedermenge* as the original $1+1+1+\ldots$ *in inf.* Note that this response implies that changing the order in which terms are summed together, although it does not change the quantity designated by the sum, does change its *Gliedermenge*. Moreover, this answer is not entirely satisfactory. Indeed, if we assume that the right-hand side of the first equation above does not have the same *Gliedermenge* as $\overset{0}{N}$, we may therefore represent the two sums $(1{+}1){+}(1{+}1){+}(1{+}1){+}\ldots$ *in inf.* and $1{+}(1{+}1){+}(1{+}1){+}\ldots$ *in inf.* by the sequences $A_1 := (0, 2, 0, 2, 0, 2, \ldots)$ and $A_2 := (1, 0, 2, 0, 2, 0, \ldots)$ respectively. Since both sums correspond to different ways of writing $\overset{0}{N}$, we should expect that $\boldsymbol{A_1} = \boldsymbol{A_2} = \overset{\mathbf{0}}{\mathbf{N}}$. However, one quickly notices that $\sigma(A_1)(i) < \sigma(A_2)(i)$ whenever i is odd, and $\sigma(A_2)(i) < \sigma(A_1)(i)$ whenever i is even. But this immediately implies that $\mathbb{Z}_{\mathscr{U}} \models \boldsymbol{A_1} \neq \boldsymbol{A_2}$. In other words, if we interpret the two infinite sums $(1+1)+(1+1)+(1+1)+\ldots$ *in inf.* and $1+(1+1)+(1+1)+\ldots$ *in inf.* by A_1 and A_2, then in order to satisfy Bolzano's requirement that infinite associativity holds, we would need both the set of even numbers and the set of odd numbers to be in $\mathscr{U}$, which is not possible. Note however that this has little to do with our formalization: Bolzano himself seems committed to the following equalities:

$$(1+1)+(1+1)+(1+1)+\dots \textit{ in inf.} = \overset{0}{N} = 1+(1+1)+(1+1)+\dots \textit{ in inf.}$$

$$2+2+2+\dots \textit{ in inf.} = 1+2+2+\dots \textit{ in inf.},$$

but there does not seem to be any reasonable way of establishing directly the latter equality. However, let us conclude this section by noting that a weaker requirement could be imposed on infinite sums which designate actual quantities, namely that any *finite* permutation of the terms or of the order in which such terms are summed does not change the value of the sum. However, it is straightforward to verify that all sums in our formalization satisfy this criterion: any two infinite sums that differ from one another only by a finite permutation of their terms or by finitely many rearrangements of the order in which those terms are summed are represented by approximating sequences which agree on a cofinite set and are therefore identified in $\mathbb{Z}_{\mathscr{U}}$. Thus this alternative criterion is too weak to rule out Grandi's series. In short, while Bolzano's first argument against M.R.S can easily be translated in our framework, his second argument seems to prove either too much, or too little, for his purposes.

3.4 Comparisons with related work

Our central proposal is to model Bolzano's computations inside an ultrapower of the integers, and to identify the quantities designated by Bolzanian infinite sums with equivalence classes of functions from the positive integers to the integers. This idea is very close to a proposal made by Trlifajová (2018), although there are a few important differences that we must remark on. First, Trlifajová seems to be primarily interested in connecting Bolzano's ideas with some modern approaches to non-standard analysis, while we are more interested in a close reading of Bolzano's arguments and in establishing the consistency of our interpretation. Second, Trlifajová works mainly with equivalence classes of functions from ω to the real numbers. By contrast, we work with countable sequences of integers. Indeed, we believe that determining whether Bolzano's notion of a real number corresponds to our modern notion is a difficult problem. Bolzano, of course, made some significant contributions to the foundations of analysis. In particular, he developed a theory of *measurable* numbers (Bolzano 1976, Part VII) which is often seen as an attempt to define the real numbers (see e.g. Rootselaar 1964; Spalt 1991; Russ and Trlifajová 2016). Trying to model Bolzano's computations with real numbers would require us to provide a detailed discussion of Bolzano's theory of measurable numbers. Since we are primarily interested in challenging the received view according to which Bolzano's computations should be read as a flawed attempt to develop an arithmetic of the transfinite, we believe that addressing this issue would take us too far astray. Just as Bolzano's measurable numbers are beyond the scope of our goals for this paper, so are Bolzano's arguments in PU involving infinitely small quantities or infinitesimal calculus. Third, let us note that, in Trlifajová's framework, two sequences are identified if they agree on a cofinite set of natural numbers. Formally, this means that she works with a reduced power of $\mathbb{R}$ rather than an ultrapower. While we do see the appeal of using only the Fréchet filter on the natural numbers instead of a non-principal ultrafilter, we have several reasons to believe that our framework is more suitable to our purposes.

For one, only a weaker version of Łoś's theorem holds for reduced powers (see Hodges 1993, p. 445), which means that the resulting structure will not be as well-behaved as the ultrapower construction we are using. While this does not create significant technical issues at this stage, we will argue in the next section that the most accurate way of modelling Bolzano's views on the product of infinite sums is to conceive of it as an interated infinite summation. This means that one will have to work with either iterated ultrapowers, or iterated reduced powers, the general theory of which is much less developed.

Moreover, we believe that the use of a non-principal ultrafilter rather than the Fréchet filter can also be justified on interpretive grounds. Indeed, it is straightforward to verify that the reduced power $\mathbb{Z}_{\mathscr{F}}$, in which two sequences α and β are identified only if $||\alpha = \beta||$ is cofinite, does not satisfy trichotomy. For example, for the sequence $\alpha = (1, 0, 1, 0, \dots)$, we have that none of $||\alpha = 0||$, $||\alpha < 0||$ or $||0 < \alpha||$ is a cofinite set, and thus $\mathbb{Z}_{\mathscr{F}} \models \neg(\alpha < 0) \wedge \neg(\alpha > 0) \wedge \neg(\alpha = 0)$. One might argue that this is a desirable feature of a formal reconstruction of Bolzano's ideas about the infinite, since, in §28, Bolzano writes that "a determination of the relationship of one infinity to one another [...] is feasible, in certain cases at any rate ...". Nonetheless, we think that Bolzano should not be read in this passage as claiming that trichotomy may not hold in the case of infinite quantities. Indeed, as mentioned in Section 1.1, it is part of Bolzano's very definition of a quantity that it must obey the law of trichotomy. All things considered, then, we believe that a formalisation that preserves trichotomy–such as ours, using ultrapowers–is more faithful to the text than a formalization that preempts the very possibility of trichotomy for infinite quantities, such as one using the Fréchet filter.

A second related work is the recently proposed theory of numerosities (Benci and Di Nasso 2003), which we have already mentioned in the introduction to this paper. Numerosities form a positive semi-ring that is meant to capture an intuitive notion of the size of sets of natural numbers. Benci and di Nasso introduce the technical notion of a labelled set of natural numbers, i.e., a set $A \subseteq \mathbb{N}$ with an associated labelling function $\ell_A : A \to \mathbb{N}$ which is finite-to-one and represents a certain way of counting the elements of the set. One can then define the sum and product of two labelled sets in a natural way. Labelling functions allow for the representation of (disjoint unions and finite products of) subsets of $\mathbb{N}$ as *approximating sequences*, which are non-decreasing functions from $\mathbb{N} \to \mathbb{N}$. The numerosity of a set A can then be defined as the equivalence class of its approximating sequence in an ultrapower $\mathbb{N}^{\mathscr{U}}$ of $\mathbb{N}$ by a Ramsey ultrafilter $\mathscr{U}$. Benci and di Nasso show that the requirement that $\mathscr{U}$ be Ramsey guarantees that any element of $\mathbb{N}^{\mathscr{U}}$ is the numerosity of some subset of $\mathbb{N}$. They also show that for any $A, B \subseteq \mathbb{N}$, if $A \subsetneq B$ then $\mathbb{N}^{\mathscr{U}} \models \mathit{num}(A) < \mathit{num}(B)$, and that the numerosity of a disjoint sum (respectively, product) of two labelled sets is equal to the sum (respectively, product) of the numerosities of the labelled sets as computed by the ultrapower.

Numerosities share some features with our interpretation of Bolzano's computations, in particular regarding the way sums of infinite quantities are defined. However, a central motivation for the numerosity framework is to develop a theory of the size of sets of natural numbers that is consistent with what we called the set-theoretic part-whole principle **PW1**. As we will argue in Section 5, we take Bolzano's arithmetic of the infinite to be compatible with the set-theoretic part-whole principle but not motivated by it, as we do not believe that Bolzano is primarily concerned with counting sets of natural numbers but rather with developing a theory of infinite sums.

4 Higher-order infinities

4.1 The product of two infinite quantities

So far, we have shown how to interpret Bolzano's computations regarding infinite sums of the form $\overset{\mathbf{n}}{\mathbf{N}}$ and $\overset{\mathbf{n}}{\mathbf{S}}$, as well as Grandi's series. We have, however, refrained from giving an interpretation of Bolzano's computations involving products of two infinite quantities. Although our treatment of Bolzano's computations so far closely matches Trlifajová's and is consistent with numerosities, our account of Bolzanian products of infinite quantities will be quite different. Indeed, it seems at first sight that there is a natural way to define the product of two quantities in $\mathbb{Z}_{\mathscr{U}}$. Similarly to the way addition is defined, we could define the product componentwise. Formally, for any $f, g : \omega^+ \to Z$, letting $f \cdot g : \omega^+ \to Z$ be the function mapping any $i \in \omega^+$ to $f(i) \times g(i)$, we may define $f^* \cdot g^*$ as $(f \cdot g)^*$. This is the definition adopted by Benci and Di Nasso (2003) and Trlifajová (2018), and it is straightforward to check that, under this definition, the structure $(\mathbb{Z}_{\mathscr{U}}, +, \cdot, 0, 1, <)$ is an ordered commutative ring. However, we believe that this definition of the product does not satisfactorily account for Bolzano's ideas as exposed in *PU*. We will first lay out our textual evidence for this claim and then explain how our interpretation works.

Bolzano gives explicit computations of the product of two infinite quantities in only one passage towards the end of §29:

> The purely symbolic equation [(1)][12] underlying all this will surely allow the

[12]The German version of the text reads (4) here, but the context clearly suggests that this is a mistake.

derivation, through successive multiplication of both sides by $\overset{0}{N}$, of the following equations:

$$1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} + \ldots \ \textit{in inf.} = (\overset{0}{N})^2$$
$$1^0.\overset{0}{N}{}^2 + 2^0.\overset{0}{N}{}^2 + 3^0.\overset{0}{N}{}^2 + \ldots \ \textit{in inf.} = (\overset{0}{N})^3 \qquad \text{etc.}$$

from which we are convinced that there [are] also infinite quantities of so-called *higher orders*, of which one exceeds the other infinitely many times. But it also certainly follows from this [that] there are infinite quantities which have every arbitrary rational, as well as irrational, ratio $\alpha : \beta$ to one another, because, as long as $\overset{0}{N}$ denotes some infinite quantity which always remains the same, $\alpha.\overset{0}{N}$ and $\beta.\overset{0}{N}$ are likewise a pair of infinite quantities which are in the ratio $\alpha : \beta$.

Bolzano defines the product of the quantity $\overset{0}{N}$ with itself, noted $(\overset{0}{N})^2$, as the result of summing $\overset{0}{N}$ with itself $\overset{0}{N}$ many times. The equation

$$1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} + \ldots \ \textit{in inf.} = (\overset{0}{N})^2$$

is obtained from the equation

$$1^0 + 2^0 + 3^0 + \ldots n^0 + (n+1)^0 + \ldots \ \textit{in inf.} = \overset{0}{N}$$

by multiplying by $\overset{0}{N}$ on both sides. This seems to suggest that Bolzano assumes some form of distributivity of multiplication over infinite summation, which allows him to equate $(1^0 + 2^0 + 3^0 + \ldots \ \textit{in inf.}).\overset{0}{N}$ with $1^0.\overset{0}{N} + 2^0.\overset{0}{N} + 3^0.\overset{0}{N} \ldots \ \textit{in inf.}$ on the left-hand side of the equality symbol. Understood as such, $(\overset{0}{N})^2$ is an infinite sum in which all terms are infinite quantities. Quantities of the form $(\overset{0}{N})^n$ are the only example in Bolzano's text of quantities defined explicitly as infinite sums of infinite quantities. It is also worth mentioning that, even though Bolzano discusses other examples of infinite quantities being infinitely smaller or larger than one another, this is the only case in §§29-33 where some infinite quantities are explicitly referred to as being "of higher order" than some others.[13]

If we were to interpret $(\overset{0}{N})^2$ in a similar fashion as Trlifajová and Benci and Di Nasso, we would have to define the quantity $(\overset{\mathbf{0}}{\mathbf{N}})^2$ in such a way that $(\overset{\mathbf{0}}{\mathbf{N}})^2(i) = \overset{\mathbf{0}}{\mathbf{N}}(i) \cdot \overset{\mathbf{0}}{\mathbf{N}}(i) = i^2$ for all $i \in \omega^+$. However, due to the well-known fact that the sum of the first n odd numbers is always equal to n^2, the infinite sum $\overset{Odds}{S} := 1 + 3 + 5 + 7 \ldots$ *in inf.* is also represented by (the equivalence class of) the sequence $(1, 4, 9, 16, ...)$. It would therefore follow that $\mathbb{Z}_{\mathscr{U}} \models \overset{\mathbf{Odds}}{\mathbf{S}} = (\overset{\mathbf{0}}{\mathbf{N}})^2$. We should conclude that the two infinite sums $1 + 3 + 5 + \ldots$ *in inf.* and $\overset{0}{N} + \overset{0}{N} + \overset{0}{N} + \ldots$ *in inf.* actually designate the same quantity. But this seems a clear violation of Bolzano's treatment of order relationships between infinite sums. Indeed, we saw above that, in showing that $\overset{2}{S}$ was infinitely greater than $\overset{1}{S}$, Bolzano reached his conclusion by showing that the difference between matching summands in $\overset{2}{S}$ and in any finite multiple of $\overset{1}{S}$ is always positive for all but finitely many summands. In this case too, since $\overset{Odds}{S}$ and $(\overset{0}{N})^2$ have the same number of terms, we could also argue along Bolzanian lines that, for any natural number i, the difference $(\overset{0}{N})^2 - i \overset{Odds}{S}$ is given by the sum $(\overset{0}{N} - i) + (\overset{0}{N} - 3i) + (\overset{0}{N} - 5i) + \ldots$ *in inf.*, in which all summands are positive (and in fact infinite). As we have argued in Section 3.2, one can extract from Bolzano's writings

[13]The authors thank an anonymous referee for noting that an alternative interpretation of §29 is also plausible. When introducing $(\overset{0}{N})^2$ and $(\overset{0}{N})^3$, Bolzano writes that this "convinc[es us] that there are also infinite quantities of so-called *higher orders*, of which one exceeds the other infinitely many times." This can be read as meaning that whenever an infinite quantity A exceeds an infinite quantity B infinitely many times, then A is an infinite of higher order with respect to B. In other words, the definition of infinities of higher order is infinities that exceed smaller infinities by an infinitely large factor. This understanding of "higher order" is problematic, however, for at least two reasons. First, if "of higher order" simply meant "infinitely larger or smaller", then the introduction of $\overset{1}{S}$ in §29 should have sufficed to establish the existence of infinite quantities of higher-order, since Bolzano has already noted by that point that $\overset{1}{S}$ is "far greater than" $\overset{0}{N}$. Second, in the definition of "infinite" (§10), Bolzano presents the concept of infinitely smaller and infinitely greater quantities of higher order as quantities derived from, but not identical with, infinitely small and infinitely large quantities. The referee's interpretation, by contrast, would collapse the notion of infinities of higher order into that of infinities *simpliciter*, per Bolzano's definition.

a sufficient criterion for one sum α to be strictly greater than another sum β, namely when all but finitely many terms in the sum $\alpha - \beta$ are positive. We will come back to this issue at greater length in Section 5. For now, let us note that, if our interpretation is correct, we must conclude in the present case that $(\overset{0}{N})^2$ is greater than any finite multiple of $\overset{Odds}{S}$, and thus that $(\overset{0}{N})^2 \neq \overset{Odds}{S}$. The componentwise definition of the product of two quantities is therefore incompatible with Bolzano's own criterion for comparing infinite sums.

Moreover, another passage from §29 seems to explicitly contradict the "componentwise" interpretation of the product of two infinite quantities. Indeed, when introducing the sum of all natural numbers $\overset{1}{S}$, Bolzano writes:

> On the other hand if we designate the quantity which represents the *sum* of all natural numbers by $[\overset{1}{S}]$, or assert the merely symbolic equation
>
> $$1 + 2 + 3 + \ldots + n + (n + 1) + \ldots \textit{ in inf.} = [\overset{1}{S}] \tag{4}$$
>
> then we will certainly realize that $[\overset{1}{S}]$ must be far greater than $\overset{0}{N}$. But it is not so easy to determine precisely the difference between these two infinite quantities or even their (geometrical) *ratio* to one another. For if, as some people have done, we wanted to form the equation
>
> $$[\overset{1}{S}] = \frac{\overset{0}{N} \cdot (\overset{0}{N} + 1)}{2}$$
>
> then we could hardly justify it on any other ground than that for every finite multitude of terms [*Menge von Gliedern*] the equation
>
> $$1 + 2 + 3 + \ldots + n = \frac{n.(n + 1)}{2}$$
>
> holds, from which it appears to follow that for the complete infinite multitude of numbers n just becomes $\overset{0}{N}$. However it is in fact not so, because with an

infinite series it is absurd to speak of a last term which has the value $\overset{0}{N}$.

Bolzano's point here seems to be that one cannot infer from the validity of Gauss's summation theorem for finite numbers that an "infinitary" version of the summation theorem also holds for infinite quantities. His rejection of the infinite summation theorem can be given two readings, one stronger, and one weaker. On the stronger reading, Bolzano is arguing that the infinite summation theorem is false, because the only way of justifying it, namely, through an inference from the finite to the infinite, leads to a false consequence. On the weaker reading, by contrast, Bolzano is not asserting the falsity of the infinite summation theorem, but he is merely refraining from asserting its truth, because what is ostensibly the only argument to prove its truth is a defective argument.

Under the first reading, which we tend to find more natural, the componentwise definition of the product *à la* Trlifajová (2018) and Benci and Di Nasso (2003) is simply inconsistent with Bolzano's own views, as the infinite summation theorem is true in the structure $(\mathbb{Z}_{\mathscr{U}}, +, \cdot, 0, 1, <)$:

Lemma 4.1. *Let* $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)$ *be such that* $\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(i) = \overset{\mathbf{0}}{\mathbf{N}}(i) \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(i)$ *for any* $i \in \omega^+$*. Then* $\mathbb{Z}_{\mathscr{U}} \models \overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) = 2\overset{\mathbf{1}}{\mathbf{S}}$.

Proof. By definition, $||\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) = 2\overset{\mathbf{1}}{\mathbf{S}}|| = \{i \in \omega^+ : (\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1))(i) = 2\overset{\mathbf{1}}{\mathbf{S}}(i)\}$. Now for any $i \in \omega^+$, $2\overset{\mathbf{1}}{\mathbf{S}}(i) = 2 \times \frac{i(i+1)}{2} = i(i+1)$ by Gauss's summation theorem. On the other hand, $(\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1))(i) = \overset{\mathbf{0}}{\mathbf{N}}(i) \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1)(i) = i \times (i + 1)$. Thus $||\overset{\mathbf{0}}{\mathbf{N}} \cdot (\overset{\mathbf{0}}{\mathbf{N}} + 1) = 2\overset{\mathbf{1}}{\mathbf{S}}|| = \omega^+$, and therefore is contained in $\mathscr{U}$. $\square$

Since we are interested in establishing at least the consistency of Bolzano's calculation of the infinite, the stronger reading of this passage of the infinite summation theorem compels us to provide an alternative definition of the product of two Bolzanian quantities.

Moreover, we find that this conclusion also follows from the second, weaker reading mentioned above. Indeed, even if Bolzano is merely punting here on the truth of the infinite summation theorem, we find it quite revealing that he would object to the infinite summation theorem being a direct consequence of Gauss's summation theorem. Indeed, this passing from the finite to the infinite is very similar to the various "arguments by cofiniteness" that Bolzano appeals to in §§29 and 32, and which we discussed at length in the previous section. As we have noticed above, the formal setting of ultrapowers, in which operations can be defined componentwise, allows for a straightforward reconstruction of such arguments by cofiniteness, with the help of Łoś's theorem. In fact, the proof of Lemma 4.1 above proceeds precisely in the same way as the inference rejected by Bolzano: since the summation theorem holds for any $i \in \omega^+$, it transfers to the infinite quantities $\overset{\mathbf{1}}{\mathbf{S}}$ and $\overset{\mathbf{0}}{\mathbf{N}}$. Bolzano therefore seems to have two distinct attitudes with regard to these inferences from the finite to the infinite: while he uses arguments by cofiniteness when establishing results about sums and differences of infinite sums, he explicitly rejects this style of reasoning when discussing ratios of infinite sums, i.e., results about products of infinite sums. If we were to model such products componentwise, we would be allowing in our formal setting precisely the type of inference that Bolzano objects to. This seems cause enough to us to propose an alternative definition of the products of two Bolzanian sums.

4.2 Second-order infinities via an iterated ultrapower

As shown above, the componentwise interpretation of the product adopted both by Trlifajová and Benci and Di Nasso has unfortunate consequences for our project. If we want to model Bolzanian computations with the infinite as accurately as possible, we must therefore propose an alternative interpretation. Our solution springs from the observation above that the product $(\overset{0}{N})^2$ is written by Bolzano as an infinite sum in which the summands themselves are infinite quantities. Since we decided to model infinite sums of integers as functions from an index set ω^+ into the integers, we should therefore model infinite sums of possibly infinite quantities as functions from ω^+ into a structure that contains those infinite quantities, i.e., into $\mathbb{Z}_{\mathscr{U}}$.

Formally, this means that we should now work in an ultrapower of $\mathbb{Z}_{\mathscr{U}}$, i.e., in an iterated ultrapower. Letting $(\mathbb{Z}_{\mathscr{U}})^2$ denote this ultrapower, we have a straightforward embedding $\iota : \mathbb{Z}_{\mathscr{U}} \to (\mathbb{Z}_{\mathscr{U}})^2$, induced by the map sending any $f : \omega^+ \to Z$ to the map

$i \mapsto \overline{f(i)}$, where $\overline{f(i)}$ is the constant function returning $f(i)$ for any $j \in \omega^+$. Given an infinite sum of (possibly infinite) quantities in $\mathbb{Z}_{\mathscr{U}}$, say $\alpha_1 + \alpha_2 + \alpha_3 + \dots$ *in inf.*, we proceed as before by identifying this sum with the countable sequence $\alpha := (\alpha_1, \alpha_2, \alpha_3, \dots)$, and determining its quantity α as the equivalence class in the iterated ultrapower $(\mathbb{Z}_{\mathscr{U}})^2$ of the sequence $(\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots)$, where the partial sums of the first n terms in α are computed inside $\mathbb{Z}_{\mathscr{U}}$. In the case of $(\overset{0}{N})^2$, this means that we identify the infinite sum with the sequence $(\overset{0}{N})^2 := (\overset{\mathbf{0}}{\mathbf{N}}, \overset{\mathbf{0}}{\mathbf{N}}, \overset{\mathbf{0}}{\mathbf{N}}, \dots)$. The corresponding approximating sequence is then $(\overset{\mathbf{0}}{\mathbf{N}}, 2\overset{\mathbf{0}}{\mathbf{N}}, 3\overset{\mathbf{0}}{\mathbf{N}}, \dots)$, which means that $(\overset{\mathbf{0}}{\mathbf{N}})^2$ is the equivalence class of the function assigning to each $i \in \omega^+$ the quantity $i\overset{\mathbf{0}}{\mathbf{N}}$. Similarly, we could form the infinite sum $\overset{1}{S} + \overset{1}{S} + \overset{1}{S} + \dots$ *in inf.*, which corresponds to summing the quantity $\overset{1}{S}$ $\overset{0}{N}$-many times to itself. This sum is interpreted as the series $\overset{0}{N}.\overset{1}{S} := (\overset{\mathbf{1}}{\mathbf{S}}, \overset{\mathbf{1}}{\mathbf{S}}, \overset{\mathbf{1}}{\mathbf{S}}, \dots)$, with approximating sequence $(\overset{\mathbf{1}}{\mathbf{S}}, 2\overset{\mathbf{1}}{\mathbf{S}}, 3\overset{\mathbf{1}}{\mathbf{S}}, \dots)$, so $\overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}}$ is the equivalence class of the function assigning $i\overset{\mathbf{1}}{\mathbf{S}}$ to each $i \in \omega^+$.

Going one step further, we could also wonder how the product $\overset{1}{S}.\overset{0}{N}$, i.e., summing $\overset{1}{S}$-many times the quantity $\overset{0}{N}$, should be interpreted. Just as we computed $(\overset{0}{N})^2$ by taking $\overset{0}{N}$ as a unit in our summation instead of 1, it seems that, in computing $\overset{1}{S}.\overset{0}{N}$, we should take $\overset{0}{N}$ as a unit in the summation $1 + 2 + 3 + \dots$ *in inf.* which yields $\overset{1}{S}$. This suggests that summing $\overset{0}{N}$ $\overset{1}{S}$-many times with itself yields the infinite sum

$$\overset{0}{N} + 2\overset{0}{N} + 3\overset{0}{N} + \dots \textit{ in inf.}$$

According to our interpretation, this sum is represented by the sequence

$$\overset{1}{S}.\overset{0}{N} := (\overset{0}{N}, 2\overset{0}{N}, 3\overset{0}{N}, \dots),$$

whose approximating sequence is $(\overset{0}{N}, 3\overset{0}{N}, 6\overset{0}{N}, \dots)$. Hence $\overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}}$ is the equivalence class of the function that assigns $\overset{\mathbf{1}}{\mathbf{S}}(i)\overset{\mathbf{0}}{\mathbf{N}} = \frac{i(i+1)}{2}\overset{\mathbf{0}}{\mathbf{N}}$ to any $i < \omega$. More generally, given any two infinite quantities $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $\mathbb{Z}_{\mathscr{U}}$, we may define the product $\boldsymbol{\alpha}.\boldsymbol{\beta} \in (\mathbb{Z}_{\mathscr{U}})^2$ as the equivalence class of the function mapping any $i < \omega^+$ to $\boldsymbol{\alpha}(i) \times \boldsymbol{\beta}$, where $\boldsymbol{\alpha}(i) \times \boldsymbol{\beta} =$

$\underbrace{\boldsymbol{\beta}+\boldsymbol{\beta}+\ldots+\boldsymbol{\beta}}_{\boldsymbol{\alpha}(i) \text{ times}}$. The relevant definitions are summarized in the table below:

Bolzanian Infinite Sum	Sequence Representation	approximating sequence	Corresponding Function	Infinite Quantity
$\overset{0}{N}+\overset{0}{N}+\overset{0}{N}+\ldots$ *in inf.*	$(\overset{0}{N})^2 = (\overset{\mathbf{0}}{\mathbf{N}},\overset{\mathbf{0}}{\mathbf{N}},\overset{\mathbf{0}}{\mathbf{N}},\ldots)$	$\sigma(\overset{0}{N})^2) = (\overset{\mathbf{0}}{\mathbf{N}},2\overset{\mathbf{0}}{\mathbf{N}},3\overset{\mathbf{0}}{\mathbf{N}},\ldots)$	$\sigma((\overset{0}{N})^2)(i) = i\overset{\mathbf{0}}{\mathbf{N}}$	$(\overset{\mathbf{0}}{\mathbf{N}})^2 = \sigma((\overset{0}{N})^2)^*$
$\overset{1}{S}+\overset{1}{S}+\overset{1}{S}+\ldots$ *in inf.*	$\overset{0}{N}.\overset{1}{S} = (\overset{\mathbf{1}}{\mathbf{S}},\overset{\mathbf{1}}{\mathbf{S}},\overset{\mathbf{1}}{\mathbf{S}}\ldots)$	$\sigma(\overset{0}{N}.\overset{1}{S}) = (\overset{\mathbf{1}}{\mathbf{S}},2\overset{\mathbf{1}}{\mathbf{S}},3\overset{\mathbf{1}}{\mathbf{S}}\ldots)$	$\sigma(\overset{0}{N}.\overset{1}{S})(i) = i\overset{\mathbf{1}}{\mathbf{S}}$	$\overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}} = \sigma(\overset{0}{N}.\overset{1}{S})^*$
$1\overset{0}{N}+2\overset{0}{N}+3\overset{0}{N}+\ldots$ *in inf.*	$\overset{1}{S}.\overset{0}{N} = (1\overset{\mathbf{0}}{\mathbf{N}},2\overset{\mathbf{0}}{\mathbf{N}},3\overset{\mathbf{0}}{\mathbf{N}},4\overset{\mathbf{0}}{\mathbf{N}},\ldots)$	$\sigma(\overset{1}{S}.\overset{0}{N}) = (1\overset{\mathbf{0}}{\mathbf{N}},3\overset{\mathbf{0}}{\mathbf{N}},6\overset{\mathbf{0}}{\mathbf{N}},10\overset{\mathbf{0}}{\mathbf{N}},\ldots)$	$\sigma(\overset{1}{S}.\overset{0}{N})(i) = \sum_{j=1}^{i} j\overset{\mathbf{0}}{\mathbf{N}}$	$\overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}} = \sigma(\overset{1}{S}.\overset{0}{N})^*$
$\alpha(1)\beta+\alpha(2)\beta+\ldots$ *in inf.*	$\alpha.\beta = (\alpha(1)\boldsymbol{\beta},\alpha(2)\boldsymbol{\beta},\ldots)$	$\sigma(\alpha.\beta) = (\sigma(\alpha)(1)\boldsymbol{\beta},\sigma(\alpha)(2)\boldsymbol{\beta},\ldots)$	$\sigma(\alpha.\beta)(i) = \sigma(\alpha)(i)\boldsymbol{\beta}$	$\boldsymbol{\alpha}.\boldsymbol{\beta} = \sigma(\alpha.\beta)^*$

Table 2: Representation of Bolzanian products in $(\mathbb{Z}_{\mathscr{U}})^2$

This definition of the product of two infinite quantities has three important consequences. First, as evidenced already by the examples of $\overset{\mathbf{0}}{\mathbf{N}}.\overset{\mathbf{1}}{\mathbf{S}}$ and $\overset{\mathbf{1}}{\mathbf{S}}.\overset{\mathbf{0}}{\mathbf{N}}$ above, the product operation will in general not be commutative. Although this might seem as a highly non-Bolzanian feature of our setup, we remark that this does not directly contradict any of Bolzano's computations in *PU*. Moreover, contrary to the associativity and commutativity of addition, which he sees as rooted in the concept of sum and therefore a feature of the general theory of quantity, associativity and commutativity of multiplication of integers are introduced as theorems in Part III of his *Reine Zahlenlehre*, §§19-20, Bolzano 1976, pp. 62-63, instead of being part of the definition of a product. Moreover, we think that the non-commutativity of the product of two infinite quantities is itself motivated by Bolzanian considerations. Indeed, if one agrees that the correct interpretation for $\overset{0}{N}.\overset{1}{S}$ and $\overset{1}{S}.\overset{0}{N}$ are the infinite sums $\overset{1}{S}+\overset{1}{S}+\overset{1}{S}+\ldots$ *in inf.* and $\overset{0}{N}+2\overset{0}{N}+3\overset{0}{N}+\ldots$ *in inf.* respectively, then the Bolzanian strategy for comparing two infinite sums, namely computing their difference term by term, yields that $\overset{0}{N}.\overset{1}{S}-\overset{1}{S}.\overset{0}{N} = (\overset{1}{S}-\overset{0}{N})+(\overset{1}{S}-2\overset{0}{N})+(\overset{1}{S}-3\overset{0}{N})+\ldots$ *in inf.* is itself an infinite sum of positive quantities. It is therefore positive, which means that $\overset{0}{N}.\overset{1}{S}$ should be stricly greater than $\overset{1}{S}.\overset{0}{N}$.

Second, it is easy to verify that, under this definition of the product, the summation theorem does not hold in the infinite case. Indeed, in our interpretation, $\overset{\mathbf{0}}{\mathbf{N}}.(\overset{\mathbf{0}}{\mathbf{N}}+1)$ is the function mapping any $i \in \omega^+$ to $\overset{\mathbf{0}}{\mathbf{N}}(i).(\overset{\mathbf{0}}{\mathbf{N}}+1)$. Now since $\overset{\mathbf{0}}{\mathbf{N}}+1$ is (the equivalence class of) the function mapping any $j \in \omega^+$ to $j+1$, it follows that $\overset{\mathbf{0}}{\mathbf{N}}(i).(\overset{\mathbf{0}}{\mathbf{N}}+1) = i \times (\overset{\mathbf{0}}{\mathbf{N}}+1)$ maps any $j \in \omega^+$ to $i(j+1)$. On the other hand, in $(\mathbb{Z}_{\mathscr{U}})^2$, $2\overset{\mathbf{1}}{\mathbf{S}}$ maps any $i \in \omega^+$ to $\overline{2\overset{\mathbf{1}}{\mathbf{S}}(i)} = \overline{i(i+1)}$. Hence $||2\overset{\mathbf{1}}{\mathbf{S}} = \overset{\mathbf{0}}{\mathbf{N}}.(\overset{\mathbf{0}}{\mathbf{N}}+1)|| = \{i \in \omega^+ : \mathbb{Z}_{\mathscr{U}} \models \overline{i(i+1)} = i \times (\overset{\mathbf{0}}{\mathbf{N}}+1)\}$. Now for any $i, j \in \omega^+$, $\overline{i(i+1)}(j) = i(i+1)$, while $(i \times (\overset{\mathbf{0}}{\mathbf{N}}+1))(j) = i(j+1)$, hence

$\mathbb{Z}_{\mathscr{U}} \models \overline{i(i+1)} < i \times (\overset{\mathbf{0}}{\mathbf{N}} + 1)$ for all $i < \omega^+$. Therefore $(\mathbb{Z}_{\mathscr{U}})^2 \models 2\overset{\mathbf{1}}{\mathbf{S}} \neq \overset{\mathbf{0}}{\mathbf{N}}.(\overset{\mathbf{0}}{\mathbf{N}} + 1)$.

Finally, we argue that this definition of the product gives a better interpretation of Bolzano's remark that quantities like $(\overset{0}{N})^2$ are infinities of a "higher order". Indeed, our construction introduces a clear stratification between integers, infinite quantities of the first order (i.e., elements introduced in the first ultrapower $\mathbb{Z}_{\mathscr{U}}$), and infinite quantities of the second order (i.e., elements introduced in the second ultrapower $(\mathbb{Z}_{\mathscr{U}})^2$). In fact, in our interpretation, genuine second-order infinite positive quantities are always larger than any first-order infinite quantity:

Lemma 4.2. *Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{Z}_{\mathscr{U}}$ are such that $\mathbb{Z}_{\mathscr{U}} \models \boldsymbol{\alpha} > m \wedge \boldsymbol{\beta} > m$ for any integer m. Then $(\mathbb{Z}_{\mathscr{U}})^2 \models \boldsymbol{\alpha}.\boldsymbol{\beta} > \boldsymbol{\gamma}$.*

Proof. We claim that $||\boldsymbol{\alpha}.\boldsymbol{\beta} > \boldsymbol{\gamma}|| \in \mathscr{U}$. This amounts to showing that, for $\mathscr{U}$-many $j \in \omega^+$, $||\boldsymbol{\alpha}(j) \times \boldsymbol{\beta} > \overline{\boldsymbol{\gamma}(j)}|| \in \mathscr{U}$. Now suppose $\boldsymbol{\alpha}(j) > 0$ (which is true for $\mathscr{U}$-many $j \in \omega^+$). Then $k \in ||\boldsymbol{\alpha}(j) \times \boldsymbol{\beta} > \overline{\boldsymbol{\gamma}(j)}||$ if and only if $\boldsymbol{\beta}(k) > \frac{\gamma(j)}{\boldsymbol{\alpha}(j)}$, which is true for $\mathscr{U}$-many k since, letting m be the smallest integer greater than $\frac{\gamma(j)}{\boldsymbol{\alpha}(j)}$, we have that $\mathbb{Z}_{\mathscr{U}} \models \boldsymbol{\beta} > m$. □

However, an obvious drawback of modelling second order infinite quantities by iterating the ultrapower construction is that we must repeat this procedure again in order to account for third-order infinite quantities, and so on. In fact, provided we want to make sense of quantities of the form $(\overset{0}{N})^n$ for any natural number n, we must iterate our ultrapower construction countably many times. This requires us to construct models of the form $(\mathbb{Z}_{\mathscr{U}})^n$ for any n, with embeddings from each $(\mathbb{Z}_{\mathscr{U}})^n$ into $(\mathbb{Z}_{\mathscr{U}})^{n+1}$:

$$\mathbb{Z} \xrightarrow[\iota_0]{} \mathbb{Z}_{\mathscr{U}} \xrightarrow[\iota_1]{} (\mathbb{Z}_{\mathscr{U}})^2 \xrightarrow[\iota_2]{} (\mathbb{Z}_{\mathscr{U}})^3 \xrightarrow[\iota_3]{} \dots$$

Limits of iterated ultrapowers are a standard tool in mathematical logic. The direct limit $\mathbb{B}$ of this chain of ultrapowers contains quantities of arbitrarily large orders of infinity, and allows for a rigorous definition of the product $\boldsymbol{\alpha}.\boldsymbol{\beta}$ of two infinite quantities α and β. In fact, we obtain a particularly well-behaved structure:

Theorem 4.3. *The structure* $\mathbb{B} = (B, +, -, 0, 1, <, .)$ *is a non-commutative ordered ring.*

We refer the interested reader to the Appendix for a proof of this theorem as well as details about the structure $\mathbb{B}$. For now, let us simply conclude that this formal result establishes that our interpretation of Bolzanian sums yields a rich and original structure which nonetheless shares many properties with the integers.

5 Reassessing the *PU*

In the preceding sections we have touched on the following three issues:

1. Whether Bolzano's work truly was about (something like) the sets of set theory, or not. We argued that Bolzano's work in *PU* §§29-33 is best understood as being an attempt at giving solid foundations to the handling of infinite series (which correspond to Bolzano's infinite *sums*).

2. Whether part-whole reasoning plays an important role or not in Bolzano's computations. We argue that a form of part-whole reasoning about infinite *sums*, not about infinite *sets*, plays a central role in Bolzano's argument, even though Bolzano's argument does not contradict set-theoretic part-whole (**PW1** in the Introduction).

3. Whether Bolzano's relation to what we may call the "first generation" of set theorists (specifically Cantor) needs to be reassessed. We think it does.

In this section, we discuss in detail where we stand on each point in turn, making use of the formalization from Section 3 and Section 4 whenever necessary.

5.1 A theory of infinite sums

We have argued that Bolzano's primary interest in *PU* §§29-33 is in infinite sums of integers, rather than sets and their sizes. To be more specific, we wanted to illustrate that by interpreting these sections as trying (and largely failing) to anticipate Cantorian inventions, one would fundamentally misrepresent Bolzano's work. Instead of there being one notion that, like Cantor's cardinals (or powers) captures the quantitative aspect of a collection, Bolzano has rather two quantity notions associated to each of his infinite sums: the *Gliedermenge* of the corresponding series of summands, and the sum itself (which for us would be the *value*, or the result of performing the infinite addition—Bolzano's notion of sum does not allow for a distinction between a sum and its value).

These infinite sums (or the underlying series) can undergo certain transformations, which may induce a change in the *Gliedermenge*, a change in the value of the sum, or both. We saw in Bolzano's work three examples of such operations:

1. raising all the terms in a sum to the same power;
2. "erasing" some of the terms in a sum;
3. permuting terms in a sum or computing summands in a different order.

§§29 and 33 suggest that raising all natural numbers at once to the same power does not change the *Gliedermenge* of an infinite sum, but it does change its value. Indeed, $\overset{0}{N}$ and $\overset{2}{S}$ are obtained from the infinite sum $1 + 2 + 3 + \dots$ *in inf.* (i.e., $\overset{1}{S}$) by raising all terms in this sum to the 0^{th} and 2^{nd} power, respectively. Bolzano explicitly states in §33 that this operation does not change the *Gliedermenge* of the corresponding sum, which is why he is able to determine that $\overset{0}{N} < \overset{1}{S} < \overset{2}{S}$. On the other hand, the second operation, which consists in erasing some of the terms in an infinite sum, does change the *Gliedermenge* of the infinite sum in such a way that also induces a change in the overall value of the sum. Bolzano's clearest examples of this are quantities of the form $\overset{n}{N}$, which vary from $\overset{0}{N}$ only in that the first n terms of the sum are removed. Nonetheless, as we have seen above, this reasoning also appears in §32, where it plays a crucial role in Bolzano's rejection of M.R.S's identification of the infinite sum $a - a + a - a + \dots$ *in inf.* with the sum within brackets in $a-(a-a+a-a+\dots$ *in inf.*$)$. Finally, regarding the third operation, Bolzano seems to adhere to the idea that because the laws of commutativity and associativity should always hold for addition, this operation should not change the value of the sum if the sum designates any value at all. As we have shown above, Bolzano uses this criterion to argue that Grandi's series does not designate any actual quantity, but seems unaware of the fact that his argument also creates difficulties for infinite sums like $1 + 1 + 1 + 1 + \dots$ *in inf.* We have also argued that those issues should commit Bolzano to the thesis that changing the order in which terms are summed in an infinite sum also changes its *Gliedermenge*, although he does not explicitly make this point.

In our formalization of Bolzano's computations, we treat all infinite sums as countable sequences of integers, to which we associate a countable sequence of partial sums. For infinite sums which have the same *Gliedermenge* as $\overset{0}{N}$, this can be done in a straightforward way by identifying an infinite sum with its sequence of partial sums, and our ultrapower construction allows us to assign different values to such sums. For infinite sums which have a different *Gliedermenge*, like $\overset{n}{N}$ or $\overset{1}{G}_a$, we only need to make some natural choices in the way we represent them to retrieve Bolzano's results. We therefore believe to have established that Bolzano's computations in *PU* form a consistent theory of divergent infinite sums, which paint a picture of the arithmetic of the infinite largely different from our modern, set-theoretic, conception. In particular, interpreting Bolzano as developing a theory of infinite sums allows us to reassess the role that part-whole considerations play in his theory.

5.2 Part-whole reasoning in Bolzano's computations

As we have mentioned above, we do not think, *pace* Berg and Šebestík, that Bolzano's computations in *PU* §§29-33 are incompatible with his use of part-whole reasoning in §§17-24. In fact, we argue that part-whole reasoning plays a central role in Bolzano's determination of the relationship between infinite quantities. However, since, as we have argued, Bolzano is developing in §§29-33 a theory of infinite sums and not a theory of infinite (set-like) collections, we must exert caution in determining how we should understand the principle that "the whole is always greater than its proper parts". The more common interpretation of this principle (see, e.g., Mancosu 2009) is set-theoretic:

PW1 For any sets A, B, if $A \subsetneq B$, then $size(A) < size(B)$.

This formulation of the part-whole principle is, by and large, the one satisfied for labelled sets of natural numbers by numerosities as defined by Benci and Di Nasso (2003). In particular, in the numerosity structure $\langle \mathscr{N}, \leqslant \rangle$ constructed by Benci and di Nasso, the following holds:

Num For any (labelled) set of natural numbers A and any numerosity ξ, $\xi < num(A)$ if and only if there is a (labelled) set $B \subsetneq A$ such that $num(B) = \xi$.

However, a more general version of the part-whole principle, which avoids set-theoretic parlance entirely, is given by Bolzano in his *Größenlehre*. This is to be found in the definition of "greater than", which we transcribe here together with the immediately following remark, which shows that Bolzano is aware of the difficulty his definition of "less/ greater than" creates for determining relationships between quantities which may be infinitely large or infinitely small, but adopts it nonetheless:

> §27 Def. If the quantity N lets itself be considered as a whole, which includes in itself the quantity M or one that is equivalent to it as part, then we say that N is greater than M, and M is smaller than N and we write it as $N > M$ or $M < N$. Should this much be established, that M is not greater or not smaller than N; then we write in the first case $M \ngtr N$ and in the second case $M \nless N$.
>
> §28 Remark. What I here pick as definition, that each whole must be greater than its part, and the part smaller than the whole (as long as they are both quantities) some, namely already Gregory of St. Vincent and in more recent times also Schultz (in his Foundations of the pure Mathesis), do not want to concede, because of quantities which are infinitely large or infinitely small. If M is infinitely large, but m is finite, or M is finite, but m infinitely small, then people say that the whole $(M + m)$ composed from the parts m and M isn't to be truly called greater than the part M. [...][14] (Bolzano 1975, p. 237)

[14]§.27 Erkl. Wenn sich die Größe N als ein Ganzes ansehen läßt, welches die Größe M oder eine ihr gleichkommende als ein Theil in sich schließt; so sagen wir, N sey größer als M, M aber kleiner als N und schreiben dieß $N > M$ oder $M < N$. Wenn um so viel bestimmt werden soll, daß M nicht größer

The quote above clearly indicates both that Bolzano sees himself as employing some version of the part-whole principle as the criterion for size comparison between quantities, and that two quantities A and B are related as whole and part, respectively, if and only if there is a positive (non-negative, non-zero) quantity C such that $A = B + C$. Then Bolzano's definition of less-than ($<$) can be formulated as follows:

PW2 For any two quantities A, B, $A < B$ if and only if there is some positive quantity C such that $A + C = B$.

This latter principle can indeed be seen as preserving the part-whole intuition: if A is a proper part of B, then the part C of B obtained by removing A from B is non-null, and clearly its sum with A yields back B. In particular, if the operation of taking the sum of two quantities has an inverse (removing a part from a whole), then **PW2** can be rephrased as follows:

PW3 For any two quantities A, B, $A < B$ if and only if $B - A$ is positive.

Our claim is that Bolzano is endorsing **PW3** when determining order relations between infinite sums. Note that for **PW3** to apply to infinite sums, one needs first to define two things:

a) the difference $\alpha - \beta$ of two infinite sums α and β;

b) when an infinite sum α is positive.

As we have argued above, Bolzano solves those two issues in his calculation of the infinite as follows:

a) For two infinite sums α and β having the same *Gliedermenge*, their difference $\alpha - \beta$ is computed termwise: $\alpha - \beta$ is the infinite sum in which the i^{th} term is $\alpha_i - \beta_i$, i.e., the difference of the i^{th} terms of α and β respectively;

oder nicht, kleiner als N sey; so schreiben wir im ersten Falle $M \not> N$ oder im zweyten $M \not< N$.
§28 Anm. Was ich hier als Erklärung annehme, daß jedes Ganze größer als sein Theil, und der Theil kleiner als das Ganze seyn müsse, (so fern beyde Größen sind), haben Einige, nahmentlich schon Gregor v. St.Vincenz und in neuerer Zeit auch wieder Schultz (in seinen Anfangsgr. d. rei. Mathesis) in Hinsicht solcher Größen, die unendlich groß oder klein sind, nicht zugestehen wollen. Wenn M unendlich groß, m aber endlich ist, oder wenn M endlich, m aber unendlich klein ist; so behauptet man daß aus den Theilen M und m zusammengesetzte Ganze $(M + m)$ sey nicht wirklich größer als der Theil M zu nennen. [...]

b) An infinite sum α is positive if all but finitely many of its terms are positive.

Bolzano is thus able to derive from **PW3** a *sufficient* criterion for order relationships between infinite sums:

PW4 For any two infinite sums α, β, $\alpha < \beta$ if all but finitely many terms in $\beta - \alpha$ are positive.

It is worth noting once again that this criterion is exactly the version of **PW3** at play in Bolzano's proof that $\overset{2}{S}$ is infinitely greater than $\overset{1}{S}$ in §33. Moreover, Bolzano explains his reasoning in terms of part-whole relationships between sums:

> So in fact $\overset{2}{S}$ may be considered as a quantity which contains the whole of $\overset{1}{S}$ as a part of it and even has a second part which in itself is again an infinite series with an equal number of terms as $\overset{1}{S}$, namely:
>
> $$0, 2, 6, 12, 20, 30, 42, 56, \ldots, n(n-1), \ldots \ \textit{in inf.},$$
>
> in which, with the exception of the first *two* terms, all succeeding terms are greater than the corresponding terms in $\overset{1}{S}$, so that the sum of the whole series is again indisputably greater than $\overset{1}{S}$. (*PU*, §33)

We therefore conclude that the part-whole principle plays an important role in Bolzano's computations, but also that, in his calculation of the infinite, Bolzano's text should not be interpreted as displaying some instances of part-whole reasoning about sets and their proper subsets. Rather, in deriving those results, part-whole reasoning is applied to infinite sums in the precise sense of **PW4**.[15] In our formalization of Bolzano's computations,

[15]This does not mean that **PW4** is the correct interpretation of Bolzano's part-whole reasoning throughout the *PU*. As we have noted in Section 1, Bolzano is clearly committed to a form of part-whole reasoning about collections in §§19 − 24. We also thank an anonymous referee for pointing out that in the following passage from §29, Bolzano seems to endorse a form of set-theoretic part-whole principle about continuous magnitudes:

> the whole *multitude* (plurality) of quantities which lie between two given quantities, e.g. 7 and 8, although it is equal to an *infinite* [multitude] and therefore cannot be determined by any number however great, depends solely on the magnitude of the distance of those two boundary quantities from one another, i.e. on the quantity $8 - 7$, and therefore must be an equal [multitude] whenever this distance is equal.

This suggests that a more fine-grained analysis might be required in order to fully assess the role that part-whole reasoning plays in the *PU* as a whole.

we have shown that computations with infinite sums based on **PW4** could be carried out in a consistent fashion. In fact, as a simple consequence of the fact that our structure $\mathbb{B}$ is elementarily equivalent to the integers, we have that $\mathbb{B} \models \forall \alpha, \beta(\alpha < \beta \leftrightarrow \beta - \alpha > 0)$. Moreover, we have also argued that Bolzano's criterion could also be applied in a productive way to determine order relations between infinities of higher order. As a consequence, we showed how a Bolzanian product of infinite quantities could be interpreted as a non-commutative monoidal operation, i.e., a well-behaved operation which is nonetheless considerably different from the product of Cantorian cardinalities or even the product of numerosities.

Finally, let us note that, although we have argued that the correct way to interpret Bolzano's part-whole reasoning does not commit him to the set-theoretic part-whole principle (**PW1**), we nonetheless believe that **PW1** is *compatible* with Bolzano's arguments. In fact, we are now in a position to fully describe a way out for Bolzano from the apparent contradiction of §33 (cf. Section 2) that we believe is satisfactory even from a modern standpoint. Indeed, following the position sketched in Section 2, we may argue that the number (*Menge*) of natural squares is not equal to the *Gliedermenge* of the infinite sum $\overset{2}{S}$ but that it must be computed, in relation with $\overset{0}{N}$, as the *value* of the sum $\overset{SQ}{N} = 1^0 + \ + \ + 4^0 + \dots$ *in inf.* The approximating sequence of this sum is $(1, 1, 1, 2, 2, \dots)$, and it is therefore straightforward to verify that, in our model, $\mathbb{B} \models \overset{\mathbf{0}}{\mathbf{N}} - \overset{\mathbf{SQ}}{\mathbf{N}} > 0$. In other words, this interpretation avoids making Bolzano's computations inconsistent with his adherence to the principle that the whole is always greater than its proper parts. The price to pay is to argue that the existence of a one-to-one correspondence between natural numbers and squares does not imply that the two sets have the same size, even though, in the specific case of $\overset{1}{S}$ and $\overset{2}{S}$, it is instrumental in establishing that the two sums have the same *Gliedermenge*. In fact, this strategy can be generalized to any set of natural numbers. Indeed, if $A \subseteq \omega^+$, let $\chi_A : \omega^+ \to \{0, 1\}$ be the characteristic function of A, i.e., for any $n \in \omega^+$, $\chi_A(n) = 1$ if $n \in A$ and $\chi_A(n) = 0$ if $n \notin A$. We may then consider the infinite sum $\tau_A = \sum_{i=1}^{\infty} \chi_A(i)$ and identify the *number* of elements in A with $\boldsymbol{\tau_A}$. It is then straightforward to verify the following fact:

PW5 For any two $A, B \subseteq \omega^+$, if $A \subsetneq B$, then $\mathbb{B} \models \boldsymbol{\tau_A} < \boldsymbol{\tau_B}$.

Indeed, if $A \subsetneq B$, let n be the smallest number in $B \backslash A$, and observe that, for any $j \geqslant n$, $\boldsymbol{\tau_A}(j) = \sum_{i=1}^{j} \chi_A(j) < \sum_{i=1}^{j} \chi_B(j) = \boldsymbol{\tau_B}(j)$. Thus $||\boldsymbol{\tau_A} < \boldsymbol{\tau_B}||$ is cofinite, so $\mathbb{B} \models \boldsymbol{\tau_A} < \boldsymbol{\tau_B}$. In fact, this "Bolzanian" way of assigning quantities to sets of natural numbers completely coincides with how a set of natural numbers is assigned a numerosity when the structure is constructed out of an ultrapower of the natural numbers, as in Benci and Di Nasso (2003).

We stop short of arguing that this was Bolzano's position, as we do not believe that there is enough evidence in the text of *PU* to make this claim; nor are we convinced that Bolzano had a notion of sets of natural numbers and of their sizes that would allow him to conceive of the problem in those terms. Our point, however, is that Bolzano's computations with infinite sums, and his attempts to develop a general theory of a calculation of the infinite, do not, as our formalization makes clear, commit him to a rejection of the part-whole principle for sets of natural numbers.

5.3 Bolzano and early set theory

Even though our interpretation sees Bolzano as not necessarily concerned with sets and their cardinalities, this should not be seen as a claim that Bolzano's work is completely separate from, and irrelevant for, the historical development of set theory. We believe that ours is just a more cautious evaluation of the interactions between the *PU* and the early development of set theory as seen mainly in Cantor's work.

What follows is not an exhaustive comparison between Bolzano's §§29-33 and Cantorian set theory but a selective comparison on just a couple of points: the status of infinite quantities in Bolzano's and Cantor's work and the arithmetic of the infinite, respectively.

Insofar as the actual infinite in mathematics is concerned, Bolzano and Cantor are both advocates for its existence. In addition to defending the existence of the actual infinite, Bolzano provides specific examples of infinite multitudes of mathematical objects such as the multitude of all natural numbers, which is an infinitely large quantity (*PU*, §16). Infinitely large quantities exist, and they are fully legitimate objects for mathematics, meaning their relationships to one another can be computed. Although Bolzano asserts this in *PU* §28, he also makes it clear that he is not claiming to be able to express the infinite quantities themselves through numbers. The symbols $\overset{0}{N}, \overset{n}{N}, \overset{1}{S}, \overset{2}{S}$ are just shorthand for the infinite sum expressions Bolzano concludes with "… *in inf.*"—they are not separate entities, like cardinals (and ordinals) with respect to sets.[16]

Indeed, in modern set theory, ordinals are defined as canonical representatives of order types of well-ordered sets, while cardinals are canonical representatives of equivalence classes of equipollent sets (i.e., sets that can be bijected with one another). Thus, while cardinals are sets and each cardinal is the cardinal of itself, in general a set and its cardinal are two distinct entities. Whether or not Cantor himself held precisely such a view at some point during his lifetime is a complex issue that depends on how one understands the role that Cantor assigns to abstraction in his original construction of the transfinite numbers. Cantor defines the cardinal number or power of a set M to be the result of a "double act of abstraction" performed on M: first, to abstract from the nature of each individual element of M, and second, to abstract from the order of the elements relative to one another. A detailed discussion of the correct interpretation of Cantor's abstraction is beyond the scope of this paper, and we therefore refer the interested reader to Hallett (1984, pp. 119-128) and Mancosu (2016, pp. 52-59).

For our purposes, it suffices to stress that the definition of cardinal Cantor gives is such that any set, in principle, can be abstracted from twice and hence give rise to its own cardinal. Thus for instance the cardinal $\aleph_0$ can be obtained from the set of natural numbers $\mathbb{N}$ by abstracting first from the nature of each single natural number and then from the order of $\mathbb{N}$ as a whole. But one fundamental consequence of Cantor's double abstrac-

[16]Florio and Leach-Krouse (2017) have recently proposed a non-objectual interpretation of ordinals. Provided an analogous treatment can be extended to cardinals, the objectuality of cardinals as a *conceptual* difference between contemporary set theory and Bolzano's approach to infinite collections might appear less significant than what it seems to be right now. However, it would still be the case that a Cantorian definition by abstraction for cardinals certainly lends itself to a straightforward objectual interpretation, and thus our point regarding the difference in conception between Cantor and Bolzano would still hold true.

tion definition is that any set has a cardinal.[17] For Bolzano instead not all infinite strings of integers can give rise to a sum, as the case of Grandi's series witnesses, and determining which such expressions do correspond to a sum is one of the problems he tries to solve.

A second point of comparison between Cantor's and Bolzano's treatments of the infinite is the computations they perform with infinite quantities. They both strive to give a meaningful account of arithmetical operations (addition and multiplication, but also subtraction and division, or "ratios" in Bolzano's case) between transfinite cardinals and infinite sums. What this means and how they achieve it is however very different for each of them.

Cardinal multiplication is defined as taking the cardinal of the product of two sets A, B, and addition is defined as the cardinality of the disjoint union of two sets (according to Hallett (1984, p. 82) this was already Cantor's own definition). In the presence of the axiom of choice, it is an elementary fact of cardinal arithmetic that for any two infinite cardinals κ, λ, $\kappa \cdot \lambda = \kappa + \lambda = \max\{\kappa, \lambda\}$. This was already proved in the early 20th century by Hessenberg and Jourdain, who were able to generalize Cantor's result that $\aleph_0^2 = \aleph_0$ to $\aleph_\alpha \cdot \aleph_\beta = \aleph_{\max\{\alpha,\beta\}}$ (cf. Hallett 1984, pp. 79, 82). They were also able to show that for addition the same holds, namely $\aleph_\alpha + \aleph_\beta = \aleph_{\max\{\alpha,\beta\}}$. This collapse of addition and multiplication into taking the greatest of the addends in the addition case, or factors in the multiplication case, is very far from Bolzano's approach to computing with the infinite.

One important similarity between Cantor and Bolzano is that, for both of them, an actually infinite quantity, like $\overset{0}{N}$ for Bolzano or ω for Cantor, can be obtained by iterating a finite operation (adding units for Bolzano, taking successor ordinals for Cantor) on finite quantities. But they seem to conceive of this process of infinitary addition in different terms, as evidenced by the role subtraction plays in their respective systems. Cantor does not define subtraction of infinite cardinals, while, as we have seen, for Bolzano the ability

[17] Note however that if one reads Cantor as associating to any set not only its equipollence class but also a canonical well-ordered representative for it, this is actually equivalent to the Well-Ordering Principle according to which any set can be well-ordered. Therefore, if one were to reject the Well-Ordering Principle, not all sets would have a Cantorian cardinal.

to compute the difference between two infinite sums is an essential tool in determining order relationships between infinite quantities. Moreover, no two infinite cardinals can have a finite difference, in the sense that for any two infinite cardinals κ, λ, if $\kappa < \lambda$ and there is a cardinal μ such that $\kappa + \mu = \lambda$, then μ must be infinite (in fact $\mu = \lambda$). Here again Bolzano's infinities behave vastly differently, since one of his most basic results is that two infinite sums such as $\overset{0}{N}$ and $\overset{n}{N}$ have a strictly finite difference, namely n.

Similarly, in §29 we see Bolzano generate new infinities, infinities of higher order, as he claims, simply by multiplying $\overset{0}{N}$ by itself, so that $\overset{0}{N} < (\overset{0}{N})^2 < (\overset{0}{N})^3$. This is in stark contrast with Cantor's result that $\aleph_0^n = \aleph_0$, mentioned above. Moreover, we have argued that a faithful interpretation of Bolzano's criterion for inequality between infinite sums implies that the Bolzanian product of two infinite sums should be non-commutative. In fact, according to us, Bolzanian products are significantly different from products of cardinals. Bolzano does not conceive of multiplying quantities as akin to taking Cartesian products of sets. He rather seems to be extending the definition of multiplication of natural numbers that he had in his *Reine Zahlenlehre* (Bolzano 1976, p. 57), without introducing infinite numbers. Just like the product of two finite numbers $m \times n$ is defined as $\underbrace{n + \ldots + n}_{m \text{ times}}$, i.e., as obtained from the sum $m = \underbrace{1 + \ldots + 1}_{m \text{ times}}$ by replacing each unit by n, the product of two infinite quantities $\alpha.\beta$ may be obtained by writing the corresponding infinite sum for α and replacing each unit by β, as in the case of $(\overset{0}{N})^2$. Perhaps surprisingly, this latter feature of Bolzano's computation may in fact be seen as the most modern one, especially under our interpretation of the Bolzanian product. Indeed, by allowing not only his finitary operations, but also his infinitary operations (like infinite summation) to range over both finite and infinite quantities, Bolzano, just as Cantor, is able to generate a hierarchy of infinities of ever increasing order.

Conclusion

Our goal was to provide a faithful interpretation of the *PU* and especially of Bolzano's calculation of the infinite as presented in §§29-33. We argued that Bolzano's computations should not be judged as failed attempts at anticipating Cantor's transfinite arithmetic, and that Bolzano's primary interest was not in measuring the sizes of infinite collections of natural numbers, but in developing an arithmetic of infinite sums of integers. As a consequence, one should not read Bolzano as failing to anticipate Cantor's work because of his commitment to a set-theoretic version of the part-whole principle but rather as developing from part-whole considerations an original and productive way of reasoning about infinite sums. Moreover, far from shutting Bolzano out of future historiographies of set theory, this new interpretation clarifies where Bolzano's approach to the infinite stands within that history. The intentions and methods of Bolzano when computing with the infinitely large are radically different from Cantor's, yet, as we have shown, amenable to a consistent mathematical interpretation. We hope that the present work may mark only the beginning of deeper scholarly engagement with Bolzano's mathematical infinite.

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Appendix

In this appendix, we describe in more detail the ring of Bolzanian quantities $\mathbb{B}$ mentioned in Section 4.2. In particular, we show how to construct $\mathbb{B}$ as a direct limit of iterated ultrapowers, define rigorously the product of two infinite quantities, and prove Theorem 4.3.

Let us first note that a standard presentation of our construction would require us to take a direct limit of the structures:

$$\mathbb{Z} \xrightarrow[\iota_0]{} \mathbb{Z}_{\mathscr{U}} \xrightarrow[\iota_1]{} (\mathbb{Z}_{\mathscr{U}})^2 \xrightarrow[\iota_2]{} (\mathbb{Z}_{\mathscr{U}})^3 \xrightarrow[\iota_3]{} \dots$$

where for any natural number n, $(\mathbb{Z}_{\mathscr{U}})^{n+1}$ is the ultrapower of $(\mathbb{Z}_{\mathscr{U}})^n$ by $\mathscr{U}$, and each $\iota_{n+1} : (\mathbb{Z}_{\mathscr{U}})^{n+1} \to (\mathbb{Z}_{\mathscr{U}})^{n+2}$ maps (any equivalence class of) a function $f : \omega^+ \to \mathbb{Z}^n_{\mathscr{U}}$ to the function mapping any i to $\overline{f(i)}$. The inconvenience of this approach is that it requires us to introduce elements of increasing complexity in our structure, i.e., functions from ω^+ into the integers, functions from ω^+ into functions from ω^+ into the integers, and so on. However, we may present our construction differently, by drawing on the well known fact that for any sets A, B and C, there is a canonical bijection ϕ between functions from A into C^B and functions from $A \times B$ into C: given any $f : A \to C^B$, the function $\phi(f) : A \times B$ is such that $\phi(f)(a, b) = f(a)(b)$ for any $a \in A$ and $b \in B$. Instead of working with functions of higher and higher complexity, we may therefore simply work with functions of finite arity, or, equivalently functions from finite sequences of elements in ω^+ into Z. However, since we still need to identify functions using an ultrafilter $\mathscr{U}$, we also need to generalize our definition of when two n-ary functions are equivalent according to $\mathscr{U}$. This requires the following definition.

Definition 5.1. Let $\mathscr{U}$ be a non-principal ultrafilter on ω^+. For any natural number n, we define $\mathscr{U}^n$ by induction as follows:

- $\mathscr{U}^0 = \{(\omega^+)^0\}$
- $\mathscr{U}^{n+1}$ is a collection of subsets of $(\omega^+)^{n+1}$ such that for any $X \subseteq (\omega^+)^{n+1}$, $X \in \mathscr{U}^{n+1}$ if and only if $\{i \in \omega^+ : X|i \in \mathscr{U}^n\} \in \mathscr{U}$, where for any $i \in \omega^+$, $X|i$ is the set of n-tuples $\overline{j}$ in $(\omega^+)^n$ such that the $n+1$-tuples $i\overline{j} \in X$.

Note that $(\omega^+)^0$ is the set of all 0-ary sequences of elements of ω^+, i.e., contains only the empty sequence. It is also straightforward to see that, given the previous definition, $\mathscr{U}^1 = \mathscr{U}$. The following lemma will be useful later on, and is established by a straightforward induction on the natural numbers.

Lemma 5.2.

- *For any natural number n, $\mathscr{U}^n$ is an ultrafilter on $(\omega^+)^n$ which is non-principal if $n > 0$.*

- *Let m, n be two natural numbers and $X \subseteq (\omega^+)^{m+n}$. Then $X \in \mathscr{U}^{n+m}$ if and only if*

$$\{\bar{i} \in (\omega^+)^m : \{\bar{j} \in (\omega^+)^n : \overline{ij} \in X\} \in \mathscr{U}^n\} \in \mathscr{U}^m.$$

We can then define the following structures:

Definition 5.3. Let n be a natural number. We let $\mathbb{Z}_{\mathscr{U}^n} := (Z_{\mathscr{U}^n}, +, -, 0, 1)$ be the ultrapower of $\mathbb{Z}$ by $\mathscr{U}^n$. More precisely, elements in $\mathbb{Z}_{\mathscr{U}^n}$ are equivalence classes of functions from $(\omega^+)^n$ to $\mathbb{Z}$, where for any two functions $f, g : \omega^+ \to \mathbb{Z}$:

- $f^* = g^*$ iff $\{\bar{i} \in (\omega^+) : f(\bar{i}) = g(\bar{i})\} \in \mathscr{U}^n$;
- $(f + g)^* = f^* + g^*$, $(f - g)^* = f^* - g^*$;
- $f^* < g^*$ iff $\{\bar{i} \in (\omega^+) : f(\bar{i}) < g(\bar{i})\} \in \mathscr{U}^n$.

In particular, it is straightforward to verify that $\mathbb{Z}_{\mathscr{U}^0}$ is isomorphic to $\mathbb{Z}$.

Since $\mathscr{U}^n$ is an ultrafilter on $(\omega^+)^n$ for any natural number n, the previous definition is a generalization of the original construction of $\mathbb{Z}_{\mathscr{U}}$. Moreover, we have natural embeddings $\boldsymbol{\lambda_n} : \mathbb{Z}_{\mathscr{U}^n} \to \mathbb{Z}_{\mathscr{U}^{n+1}}$. In fact, those embeddings are always elementary:

Lemma 5.4. *For any $f : (\omega^+)^n \to Z$, let $\lambda_n(f) : (\omega^+)^{n+1} \to Z$ be such that for any n-tuple $\bar{i}$ and any $j \in \omega^+$, $\lambda_n(f)(\bar{i}j) = f(\bar{i})$. Then the function $\boldsymbol{\lambda_n} : \mathbb{Z}_{\mathscr{U}^n} \to \mathbb{Z}_{\mathscr{U}^{n+1}}$ defined by $\boldsymbol{\lambda_n}(f^*) = \lambda_n(f)^*$ is an elementary embedding.*

The proof of this lemma is a simple application of the Tarski-Vaught test of elementary substructures. For any natural numbers $m \leqslant n$, we let $\lambda_{m,n}$ be the composition of the embeddings $\boldsymbol{\lambda_{n-1}} \circ \boldsymbol{\lambda_{n+2}} \circ ... \circ \boldsymbol{\lambda_{m+1}} \circ \boldsymbol{\lambda_m}$. We can then define the structure $(B, +, -, 0, 1, <)$ as the direct limit of the system

$$\mathbb{Z}_{\mathscr{U}^0} \xrightarrow[\boldsymbol{\lambda_0}]{} \mathbb{Z}_{\mathscr{U}^1} \xrightarrow[\boldsymbol{\lambda_1}]{} \mathbb{Z}_{\mathscr{U}^2} \xrightarrow[\boldsymbol{\lambda_2}]{} \dots$$

We will refer to elements in B as *quantities*. By definition of the direct limit of a directed system, quantities are equivalence classes of elements in some $\mathbb{Z}_{\mathscr{U}^n}$, where for any $m \leqslant n$ and any two equivalence classes $f^* \in \mathbb{Z}_{\mathscr{U}^n}, g^* \in \mathbb{Z}_{\mathscr{U}^m}$, f^* and g^* are identified if and only if $\mathbb{Z}_{\mathscr{U}^n} \models \lambda_{m,n}(f^*) = g^*$. For any quantity $\boldsymbol{\alpha}$, we let the *order* of $\boldsymbol{\alpha}$ be the smallest natural number n such that there is some $f^* \in \boldsymbol{\alpha}$ such that $f^* \in \mathbb{Z}_{\mathscr{U}^n}$. Clearly, any $\boldsymbol{\alpha} \in B$ has a finite order n, and moreover, if $\boldsymbol{\alpha}$ has order n witnessed by some f^*, then for any natural number m, any $g^* \in \mathbb{Z}_{\mathscr{U}^{n+m}}$, and any tuples $\bar{i}$ and $\bar{j}$ of length n and m respectively, $f(\bar{i}) = g(\overline{ij})$. We may therefore abuse notation and view $\boldsymbol{\alpha}$ as a function from m-tuples of elements in ω^+ into Z for any $m \geqslant n$.

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two quantities of order m and n respectively, represented by the functions $f_{\boldsymbol{\alpha}}$ and $f_{\boldsymbol{\beta}}$ of arity m and n respectively. We define the *product* $\boldsymbol{\alpha}.\boldsymbol{\beta}$ as (the equivalence class of) the function $f_{\boldsymbol{\alpha}.\boldsymbol{\beta}} : (\omega^+)^{m+n} \to Z$ such that for any tuples $\bar{i}$ and $\bar{j}$ of length m and n respectively, $f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = f_{\boldsymbol{\alpha}}(\bar{i}) \times f_{\boldsymbol{\beta}}(\bar{j})$, i.e., $\underbrace{f_{\boldsymbol{\beta}}(\bar{j}) + \ldots + f_{\boldsymbol{\beta}}(\bar{j})}_{f_{\boldsymbol{\alpha}}(\bar{i}) \text{ times}}$. It is straightforward to verify that this operation is well-defined. Indeed, suppose $g_{\boldsymbol{\alpha}} \in \boldsymbol{\alpha}$ and $g_{\boldsymbol{\beta}} \in \boldsymbol{\beta}$ are functions of arity m and n respectively. Clearly for any m-tuple $\bar{i}$ and any n-tuple $\bar{j}$, if $f_{\boldsymbol{\alpha}}(\bar{i}) = g_{\boldsymbol{\alpha}}(\bar{i})$ and $f_{\boldsymbol{\beta}}(\bar{j}) = g_{\boldsymbol{\beta}}(\bar{j})$, then $g_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij})$. Moreover, since $f_{\boldsymbol{\alpha}}$ and $g_{\boldsymbol{\alpha}}$ are $\mathscr{U}^m$ equivalent, and $f_{\boldsymbol{\beta}}$ and $g_{\boldsymbol{\beta}}$ are $\mathscr{U}^n$ equivalent, it follows that for $\mathscr{U}^m$-many $\bar{i}$ there are $\mathscr{U}^n$-many $\bar{j}$ such that $f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = g_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij})$. Equivalently,

$$\{\bar{i} \in (\omega^+)^m : \{\bar{j} \in (\omega^+)^n : f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = g_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij})\} \in \mathscr{U}^n\} \in \mathscr{U}^m,$$

which by Lemma 5.2 implies that $\{\overline{ij} \in (\omega^+)^{m+n} : f_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij}) = g_{\boldsymbol{\alpha}.\boldsymbol{\beta}}(\overline{ij})\} \in \mathscr{U}^{m+n}$, and therefore $f^*_{\boldsymbol{\alpha}.\boldsymbol{\beta}} = g^*_{\boldsymbol{\alpha}.\boldsymbol{\beta}}$.

The next lemma establishes that the product of two quantities of order m and n is of order $m + n$. The proof is a simple application of Łoś's theorem.

Lemma 5.5. *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two quantities of order m and n respectively, and let $\boldsymbol{\gamma}$ be a quantity of order $l < m + n$. Then $\mathbb{B} \models \boldsymbol{\alpha}.\boldsymbol{\beta} \neq \boldsymbol{\gamma}$.*

Finally, we can now prove Theorem 4.3 and establish that Bolzanian sums and products form a non-commutative ordered ring.

Theorem 5.6. *The structure $\mathbb{B} = (B, +, -, 0, 1, <, .)$ is a non-commutative ordered ring.*

Proof. Note first that by construction, we have an elementary embedding from $\mathbb{Z}$ into the reduct $(B, +, -, 0, 1, <)$, which immediately implies that $\mathbb{B}$ is an ordered additive group. We therefore only need to verify the following properties:

- **Associativity**: Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ be three quantities of order l, m and n respectively. Then for any tuples $\bar{i}$, $\bar{j}$ and $\bar{k}$ of arity l, m and n respectively, we have that:

$$\begin{aligned}
\boldsymbol{\alpha}.(\boldsymbol{\beta}.\boldsymbol{\gamma})(\overline{ijk}) &= \boldsymbol{\alpha}(\bar{i}) \times (\boldsymbol{\beta}.\boldsymbol{\gamma}(\overline{jk})) \\
&= \boldsymbol{\alpha}(\bar{i}) \times (\boldsymbol{\beta}(\bar{j}) \times \boldsymbol{\gamma}(\bar{k})) \\
&= (\boldsymbol{\alpha}(\bar{i}) \times \boldsymbol{\beta}(\bar{j})) \times \boldsymbol{\gamma}(\bar{k}) \qquad \text{(by associativity of } \times \text{ in } \mathbb{Z}) \\
&= (\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})) \times \boldsymbol{\gamma}(\bar{k}) \\
&= (\boldsymbol{\alpha}.\boldsymbol{\beta}).\boldsymbol{\gamma}(\overline{ijk}).
\end{aligned}$$

- **Multiplicative identity**: Note that any integer z is represented in $\mathbb{B}$ by a quantity $\boldsymbol{z}$ of order 0, which corresponds to the set of all constant functions from finite sequences of elements in ω^+ into Z with range $\{z\}$. For any quantity $\boldsymbol{\alpha}$ of order l, we therefore have that $\boldsymbol{\alpha}.\boldsymbol{z}$ and $\boldsymbol{z}.\boldsymbol{\alpha}$ are quantities of order n such that for any l-tuple $\bar{i}$, $\boldsymbol{\alpha}.\boldsymbol{z}(\bar{i}) = \boldsymbol{\alpha}(\bar{i}) \times z$ and $\boldsymbol{z}.\boldsymbol{\alpha}(\bar{i}) = z \times \boldsymbol{\alpha}(\bar{i})$. Thus $\boldsymbol{\alpha}.\boldsymbol{z} = \boldsymbol{z}.\boldsymbol{\alpha} = \underbrace{\boldsymbol{\alpha} + ... + \boldsymbol{\alpha}}_{z \text{ times}}$. Hence in particular $\mathbf{1}.\boldsymbol{\alpha} = \boldsymbol{\alpha}.\mathbf{1} = \boldsymbol{\alpha}$.

- **Left-distributivity**: Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ be as above. Without loss of generality, assume that the order m of $\boldsymbol{\beta}$ is greater than or equal to the order n of $\boldsymbol{\gamma}$, which implies that $\boldsymbol{\beta} + \boldsymbol{\gamma}$ is also of order m. Fix an l-tuple $\bar{i}$ and an n-tuple $\bar{j}$. Note that even though $\boldsymbol{\gamma}$ is of lower order, we may still write $\boldsymbol{\gamma}(\bar{j})$. Then:

$$\begin{aligned}
\boldsymbol{\alpha}.(\boldsymbol{\beta} + \boldsymbol{\gamma})(\overline{ij}) &= \boldsymbol{\alpha}(\bar{i}) \times (\boldsymbol{\beta} + \boldsymbol{\gamma}(\bar{j})) \\
&= \boldsymbol{\alpha}(\bar{i}) \times (\boldsymbol{\beta}(\bar{j}) + \boldsymbol{\gamma}(\bar{j})) \\
&= (\boldsymbol{\alpha}(\bar{i}) \times \boldsymbol{\beta}(\bar{j})) + (\boldsymbol{\alpha}(\bar{i}) \times \boldsymbol{\gamma}(\bar{j})) \qquad \text{(by left-distributivity of } \times \text{ over } + \text{ in } \mathbb{Z}) \\
&= (\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})) + (\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})) \\
&= (\boldsymbol{\alpha}.\boldsymbol{\beta}) + (\boldsymbol{\alpha}.\boldsymbol{\gamma})(\overline{ij}).
\end{aligned}$$

- **Right-distributivity**: Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ as above, and assume the order l of $\boldsymbol{\alpha}$ is greater than or equal to the order m of $\boldsymbol{\beta}$. Let $\bar{i}$ be an l-tuple and $\bar{k}$ a n-tuple. Then:

$$\begin{aligned}
(\boldsymbol{\alpha}+\boldsymbol{\beta}).\boldsymbol{\gamma}(\overline{ik}) &= (\boldsymbol{\alpha}+\boldsymbol{\beta}(\bar{i}))\times\boldsymbol{\gamma}(\bar{k})\\
&= (\boldsymbol{\alpha}(\bar{i})+\boldsymbol{\beta}(\bar{i}))\times\boldsymbol{\gamma}(\bar{k})\\
&= (\boldsymbol{\alpha}(\bar{i})\times\boldsymbol{\gamma}(\bar{k}))+(\boldsymbol{\beta}(\bar{i})\times\boldsymbol{\gamma}(\bar{k})) \quad \text{(by right-distributivity of $\times$ over $+$ in $\mathbb{Z}$)}\\
&= (\boldsymbol{\alpha}.\boldsymbol{\gamma}(\overline{ik}))+(\boldsymbol{\beta}.\boldsymbol{\gamma}(\overline{ik}))\\
&= (\boldsymbol{\alpha}.\boldsymbol{\gamma})+(\boldsymbol{\beta}.\boldsymbol{\gamma})(\overline{ik}).
\end{aligned}$$

- **Order axiom**: Suppose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two quantities of order l and m respectively and are such that $\mathbb{B}\models 0<\boldsymbol{\alpha}$ and $\mathbb{B}\models 0<\boldsymbol{\beta}$. We claim that $\mathbb{B}\models 0<\boldsymbol{\alpha}.\boldsymbol{\beta}$. Indeed, since $\mathbb{B}\models 0<\boldsymbol{\alpha}$, we have that $\{\bar{i}\in(\omega^+)^l : 0<\boldsymbol{\alpha}(\bar{i})\}\in\mathscr{U}^l$, while it follows from $\mathbb{B}\models 0<\boldsymbol{\beta}$ that $\{\bar{j}\in(\omega^+)^m : 0<\boldsymbol{\beta}(\bar{j})\}\in\mathscr{U}^m$. Now clearly for any l-tuple $\bar{i}$ such that $0<\boldsymbol{\alpha}(\bar{i})$, if $\bar{j}$ is an m-tuple such that $0<\boldsymbol{\beta}(\bar{j})$, then $0<\boldsymbol{\alpha}(\bar{i})\times\boldsymbol{\beta}(\bar{j})$, i.e., $0<\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})$. Thus

  $$\{\bar{i}\in(\omega^+)^l : \{\bar{j}\in(\omega^+)^m : 0<\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})\}\in\mathscr{U}^m\}\in\mathscr{U}^l,$$

  which by Lemma 5.2 implies that $\{\overline{ij}\in(\omega^+)^{l+m} : 0<\boldsymbol{\alpha}.\boldsymbol{\beta}(\overline{ij})\}\in\mathscr{U}^{l+m}$, and hence $\mathbb{B}\models 0<\boldsymbol{\alpha}.\boldsymbol{\beta}$.

□

Let us conclude this appendix with a few remarks regarding the Bolzanian ring of infinite quantities $\mathbb{B}$. First, our formalization only allows us to represent infinite quantities of a finite order, i.e., infinite sums of the form $\boldsymbol{\alpha}(1)+\boldsymbol{\alpha}(2)+\boldsymbol{\alpha}(3)+\cdots$ for which there is an $n<\omega^+$ such that for all $m\geqslant n$, the order of $\boldsymbol{\alpha}(m)$ is less than or equal to the order of $\boldsymbol{\alpha}(n)$. For example, the following infinite sum is not represented by any element in $\mathbb{B}$:

$$\overset{0}{N}+(\overset{0}{N})^2+(\overset{0}{N})^3+\ldots\ \textit{in inf.}$$

Of course, if we wanted to include this sum in our model, we would have to take an ultrapower of $\mathbb{B}$ by $\mathscr{U}$ and construct another countable sequence of ultrapowers. In fact, if we wanted to close our domain of infinite quantities under taking infinite sums, we would need to keep iterating the ultrapower until the first ordinal with uncountable cofinality, i.e., until ω_1. Our structure $\mathbb{B}$, however, is more than enough to account for Bolzano's examples, and we certainly do not want to claim that the consistency of Bolzano's system requires anything like uncountable ordinals.

Second, it is quite straightforward to observe that the situation described in Lemma 4.2 generalizes to the full structure $\mathbb{B}$. Indeed, for any n, the product of any n^{th} order quantity with at least a first-order infinite quantity is always greater than or smaller than any quantity of strictly lower order. Thus, in accordance with Bolzano's original claims, multiplying infinite quantities together yields new quantities that are infinitely larger or infinitely smaller than the previous ones in a very strong sense.

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