United States Patent

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[54] **FILTER HAVING DIRECT AND CROSS-COUPLED RESONATORS**
5 Claims, 14 Drawing Figs.

[52] U.S. Cl. .... 333/73 R, 333/73 W
[51] Int. Cl. H01p 1/20
[50] Field of Search .... 333/70, 70 S, 70 T, 73, 73 S, 73 W, 83

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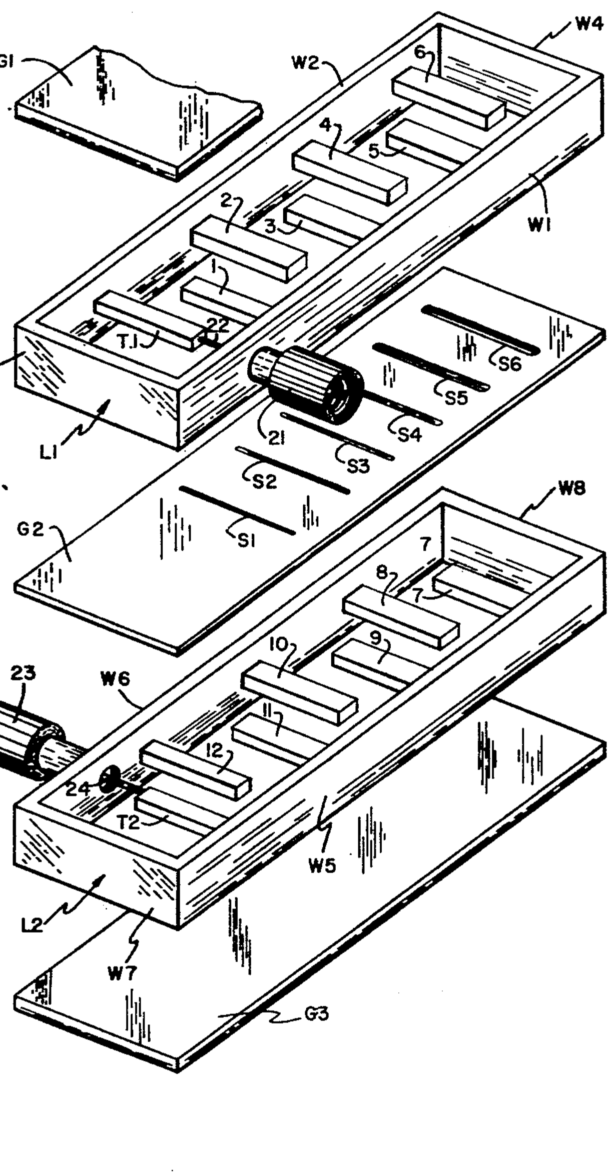
ABSTRACT: A band-pass filter is constructed from arrays of resonant elements which are coupled in a manner enabling control to be simultaneously effected over the filter's amplitude and group delay characteristics. In each array the resonators are directly coupled in consecutive order and the resonators in one array are cross coupled to resonators in the other array. The direct and cross-coupled arrangement of resonators is in accordance with a mathematical relationship which permits selection of the filter's group delay characteristic and the filter's amplitude characteristic. In accordance with that relationship, multiple paths of different lengths are provided for the wave energy travelling from the input to the output.



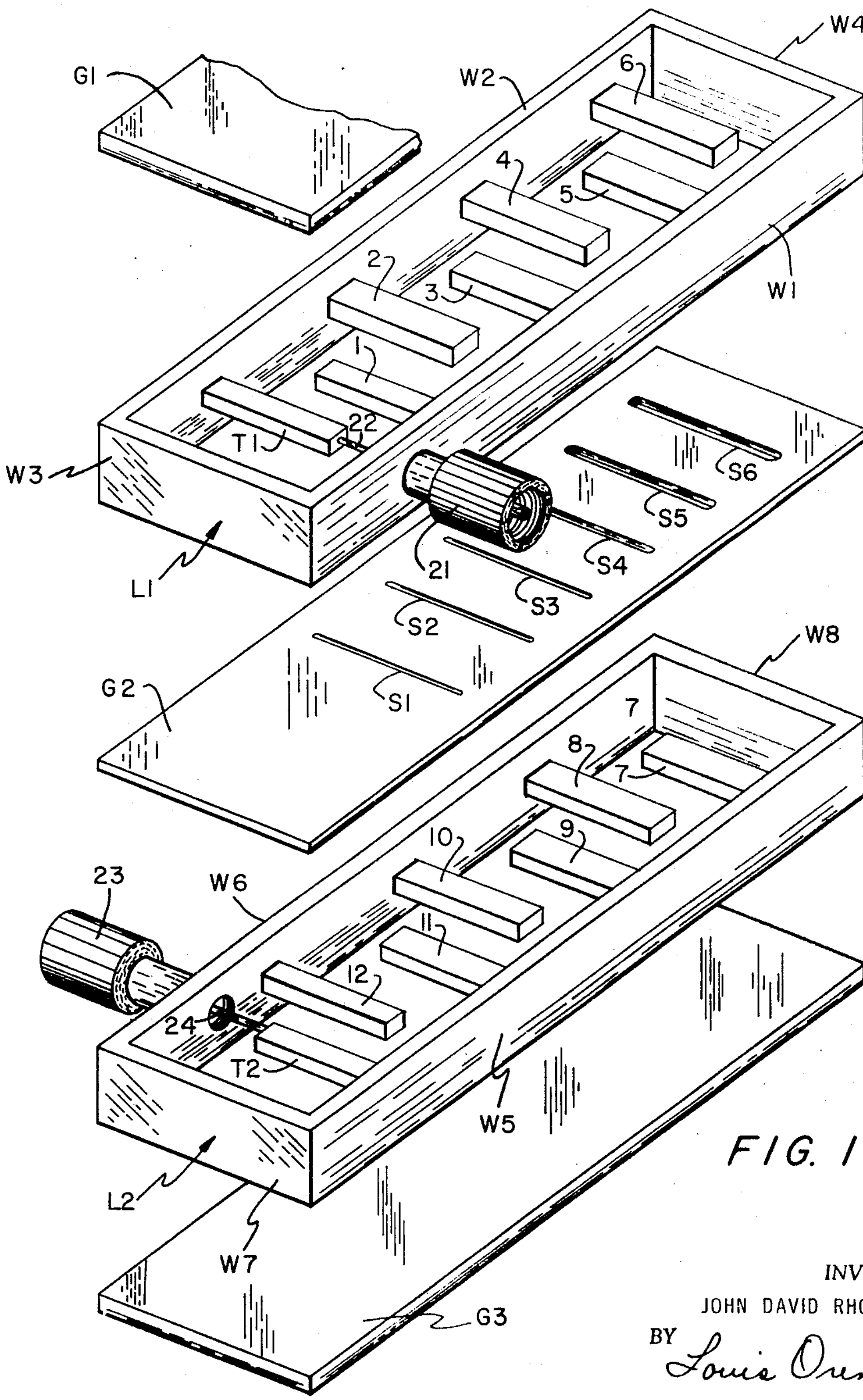


FIG. 1

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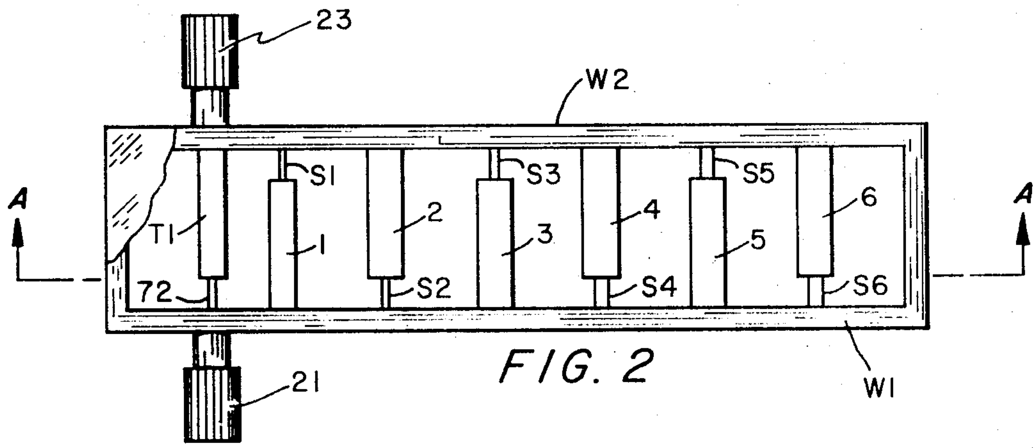

FIG. 2

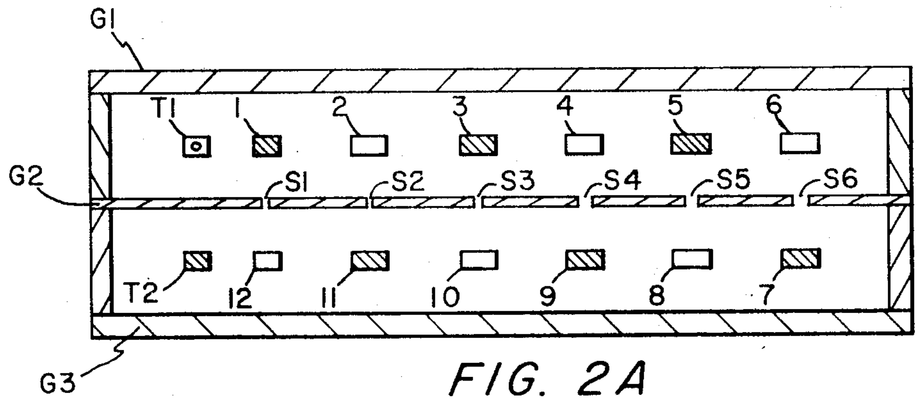

FIG. 2A

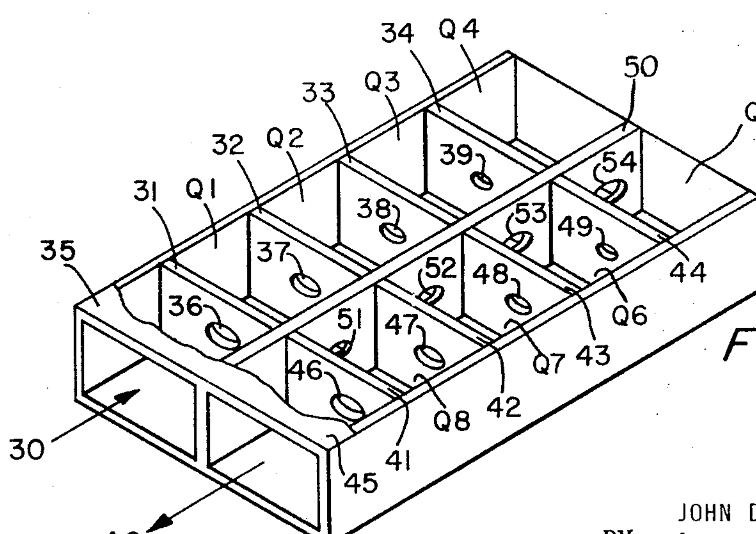

FIG. 3

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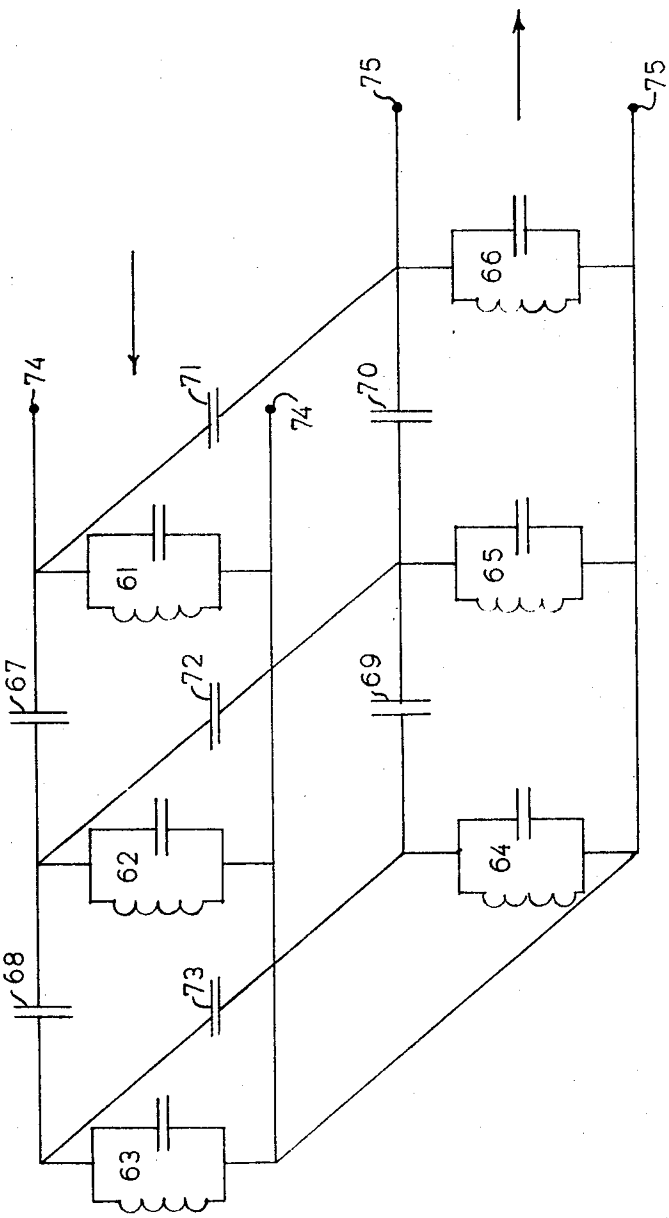

FIG 4

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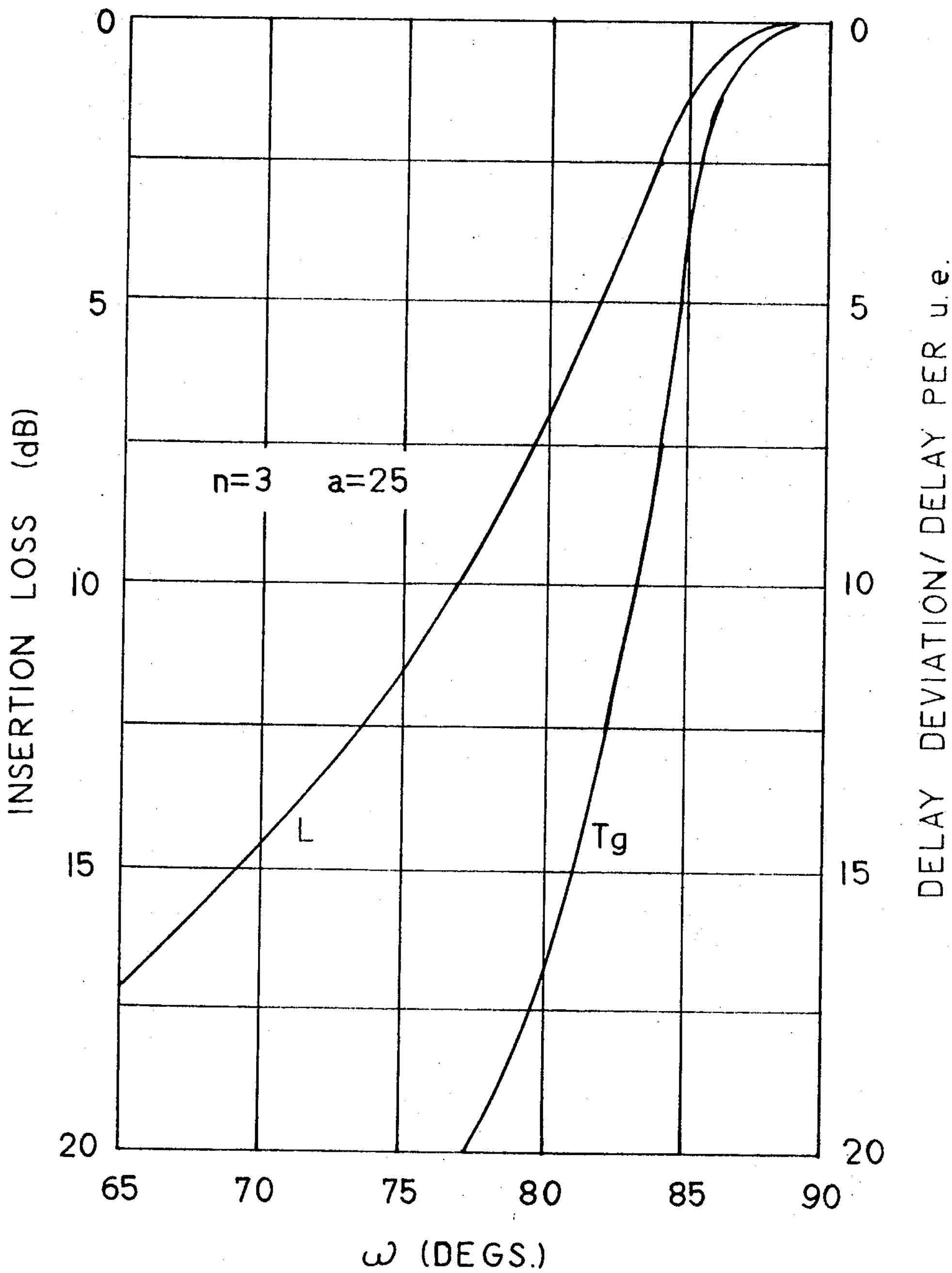

FIG. 5A

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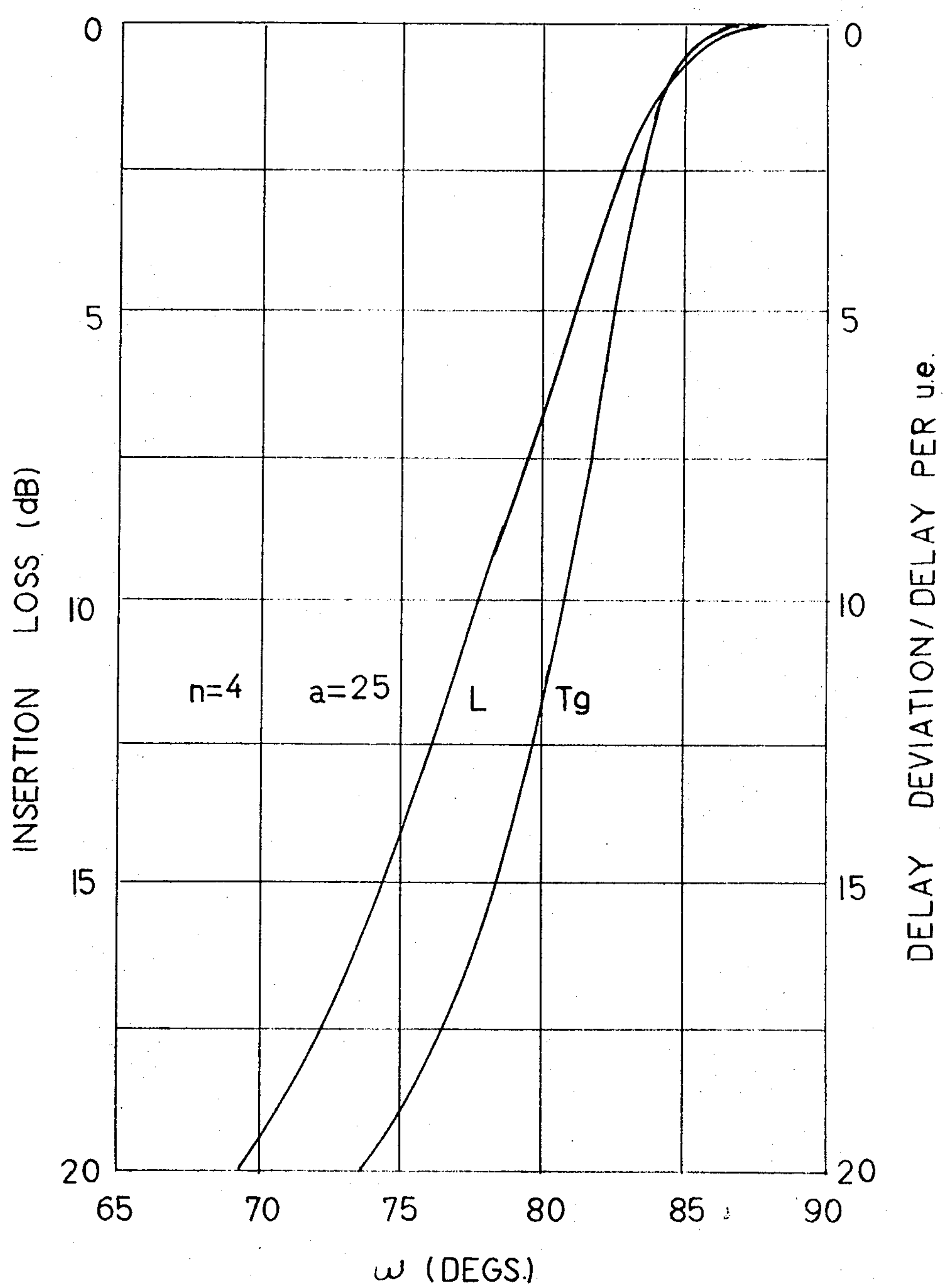

FIG. 5B

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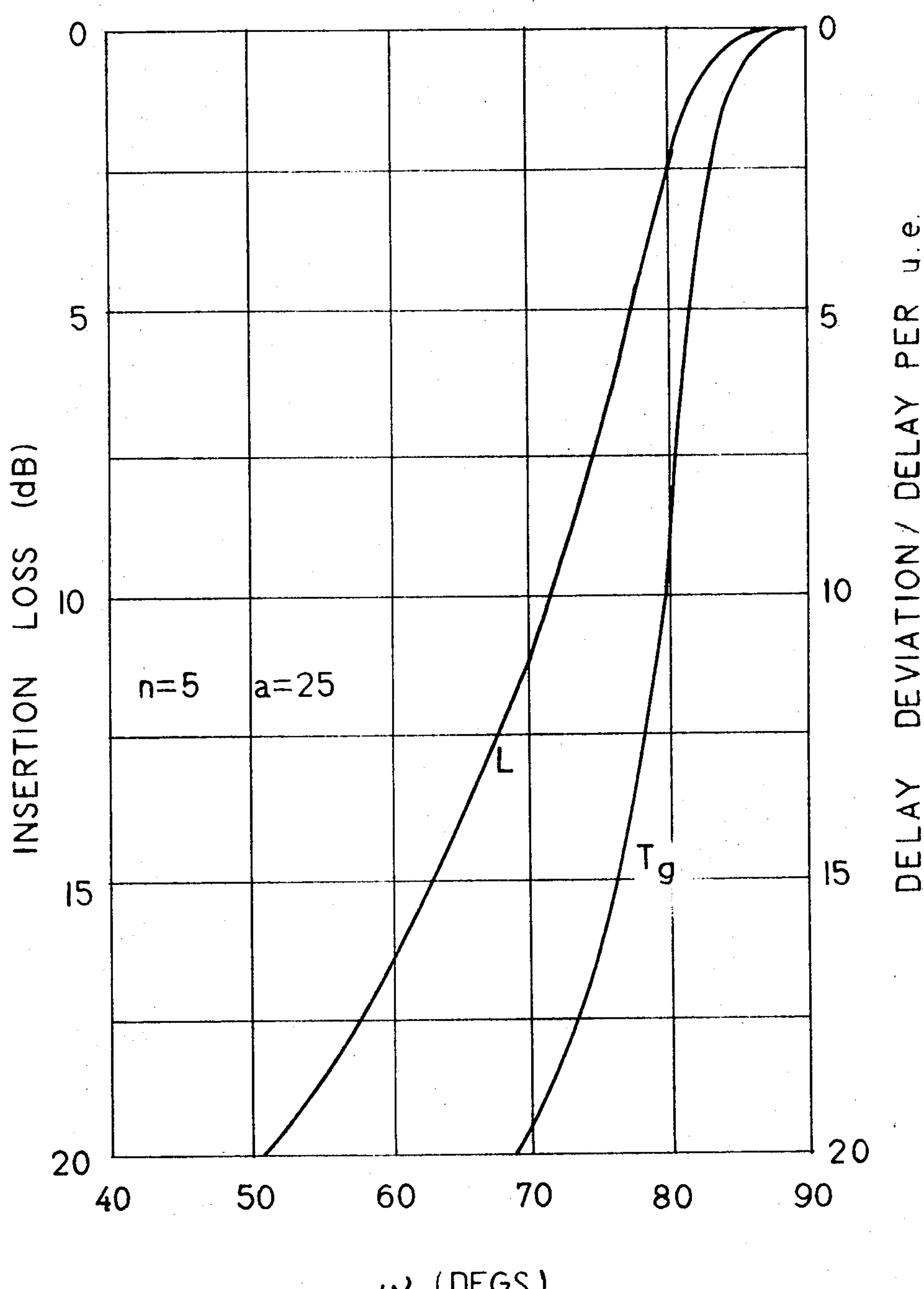

FIG. 5C

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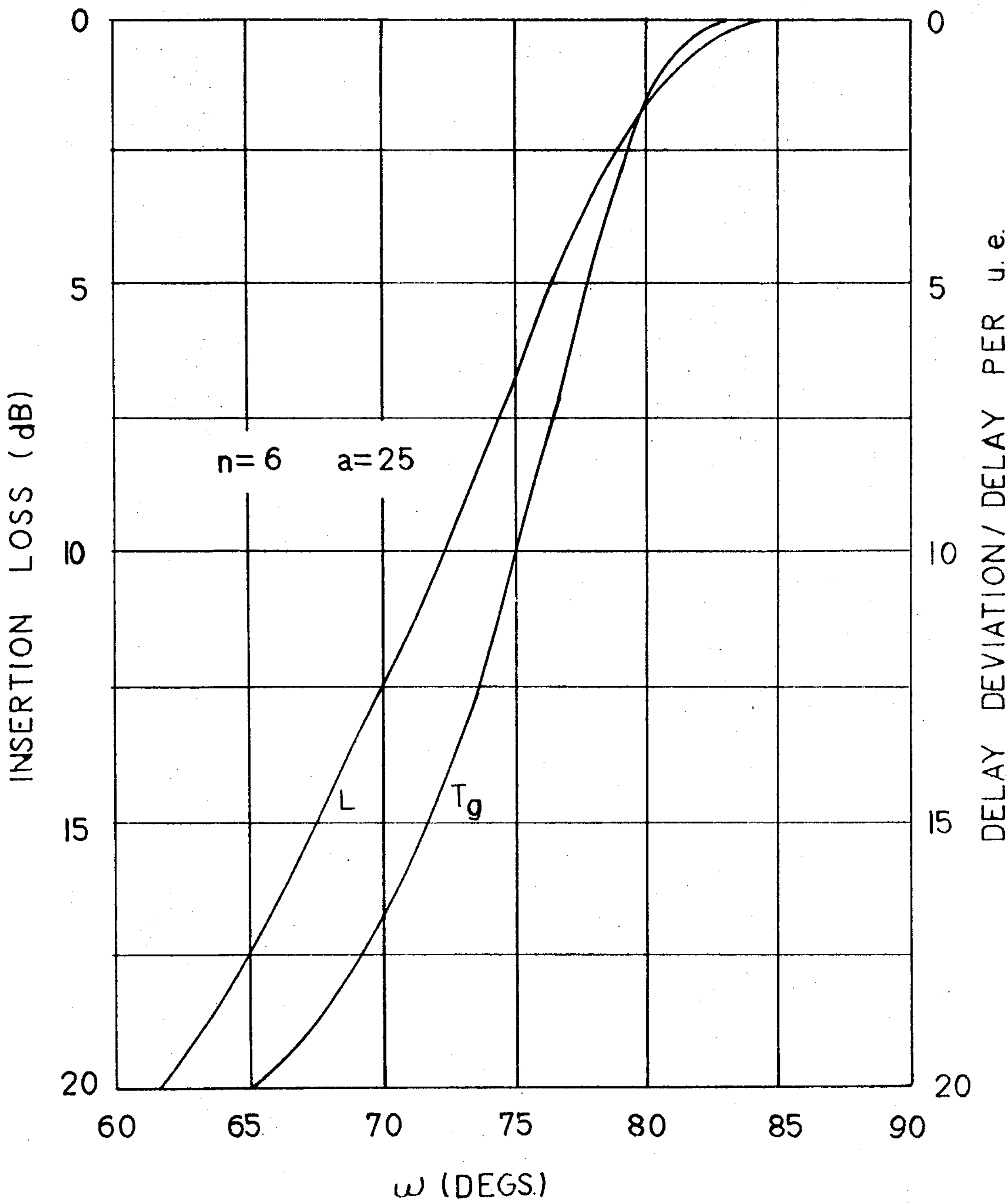

FIG. 5D

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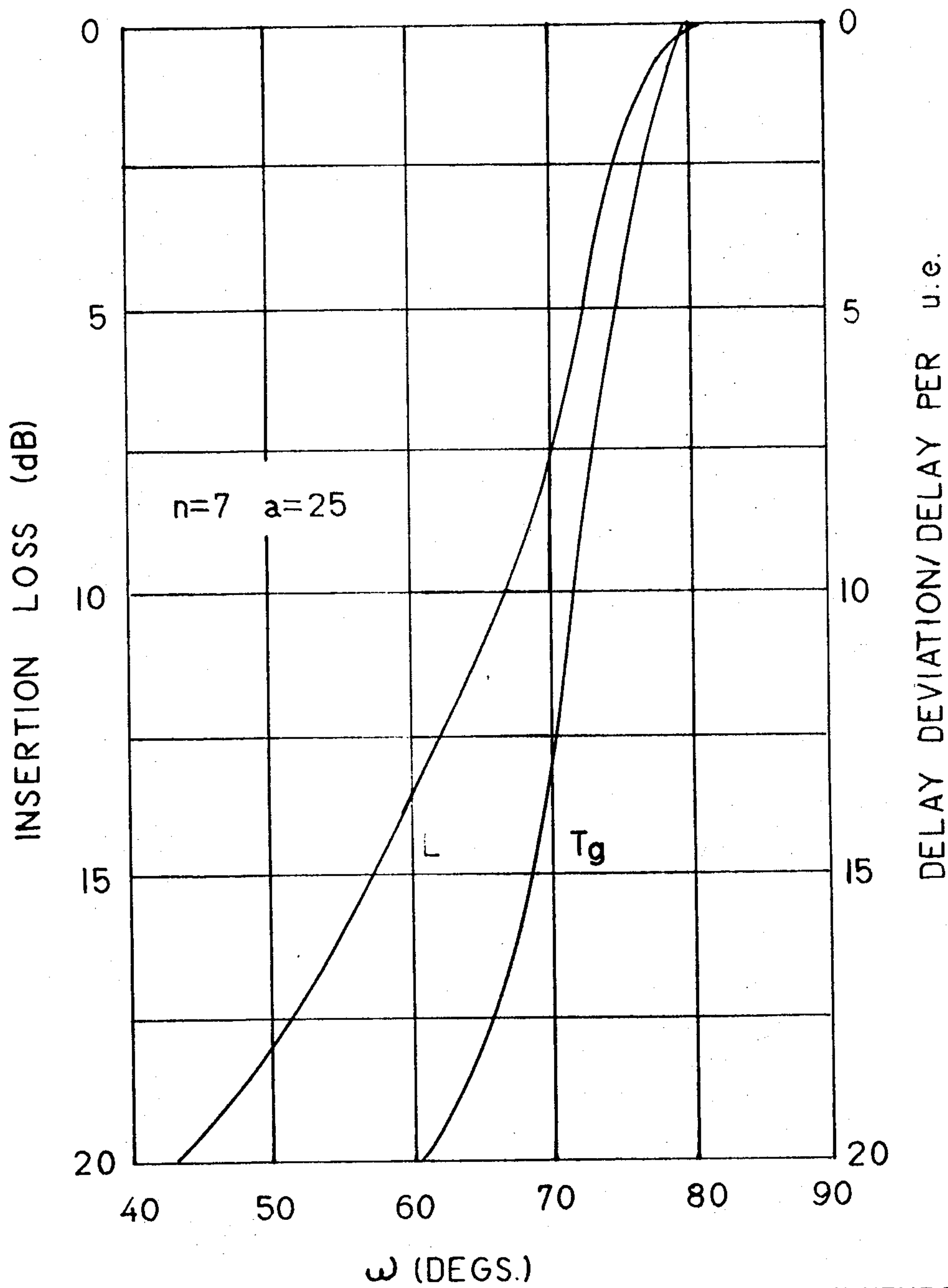

FIG. 5E

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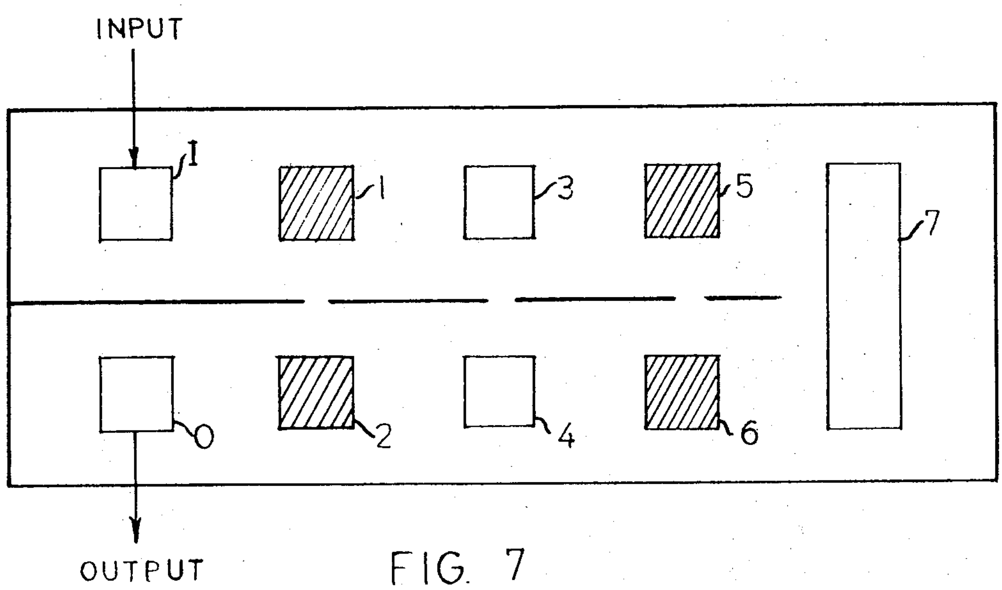

FIG. 7

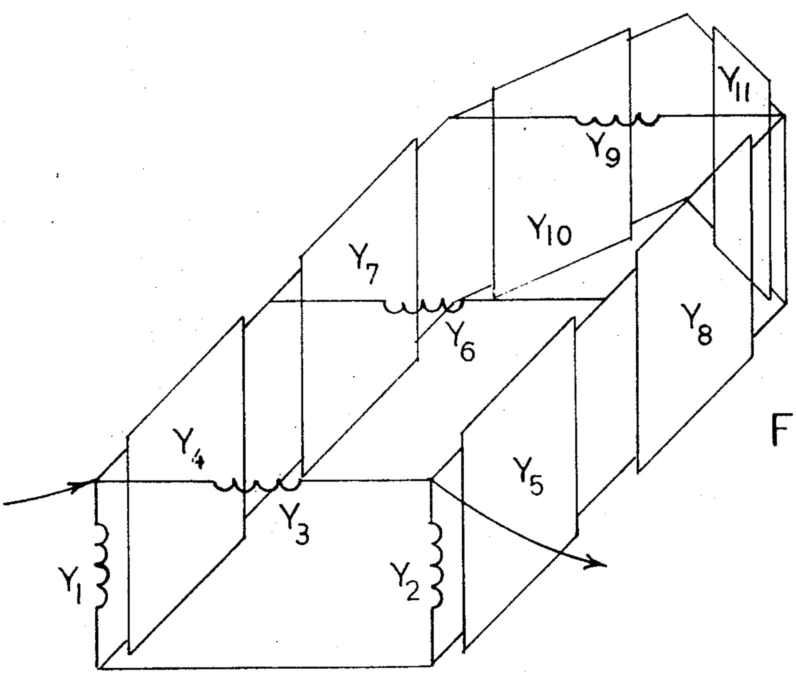

FIG. 6

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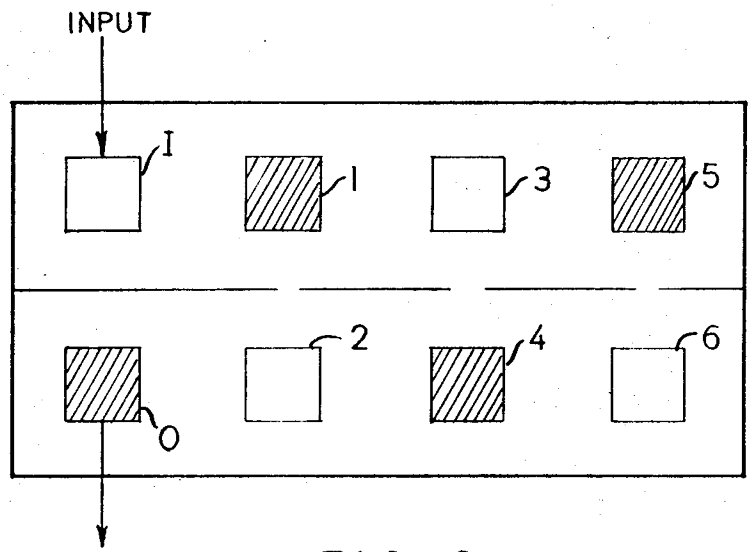

FIG. 9

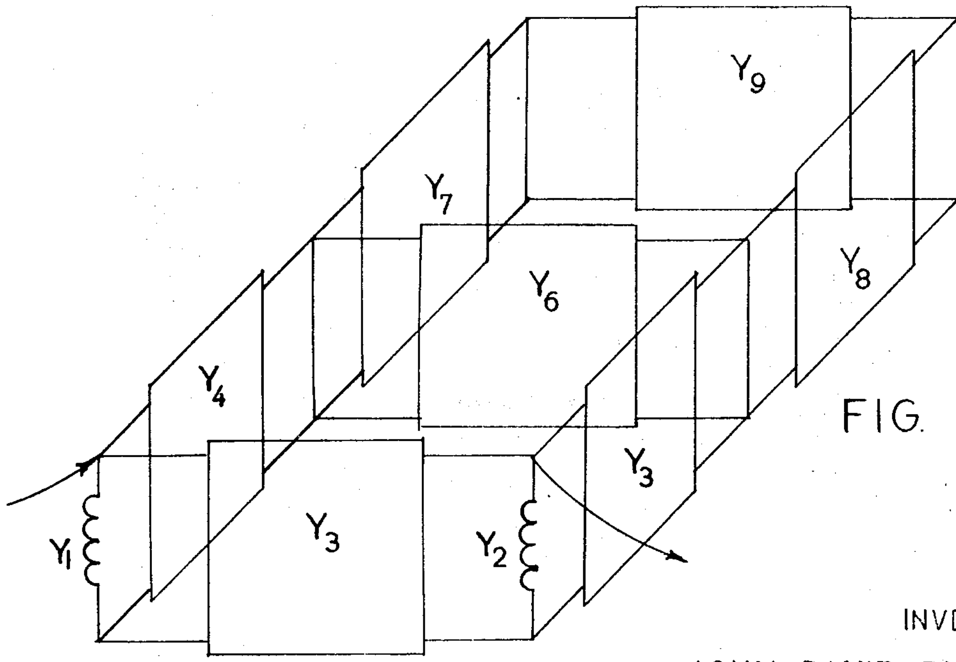

FIG. 8

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FILTER HAVING DIRECT AND CROSS-COUPLED RESONATORS

SUMMARY OF THE INVENTION

This invention relates in general to frequency selective apparatus for filtering electromagnetic waves. More particularly, the invention pertains to a band pass filter whose construction permits the simultaneous selection of a desired amplitude characteristic and a desired phase transmission characteristic.

DISCUSSION OF THE PRIOR ART

The conventional approach in realizing a desired amplitude characteristic and a desired phase transmission characteristic has been to design a filter with the desired amplitude response and then employ an appropriate all-pass network to obtain the desired phase or group delay characteristic. For example, the conventional approach, in approximating a constant delay over the pass band of a microwave filter, has been to design the filter with either a maximally flat, a Chebyshev, or an elliptic function, amplitude response and then to equalize the delay with an appropriate all-pass network. It is very difficult, using the conventional technique for a narrow band pass filter, to achieve a response with a high degree of phase linearity over the entire pass band without resorting to very complicated delay equalizer apparatus.

Qualitatively, in the conventional approach, a transfer function is constructed to meet the specified amplitude response and the transfer function is then augmented by an all-pass function to meet the required delay deviation. In the invention here disclosed, different design procedures are used. In the example presented a transfer function is constructed to approximate to a constant delay and that function is subsequently augmented by an odd polynomial, which does not affect the delay, in order to achieve the desired amplitude characteristic. The resulting network is not a cascade of an amplitude filter and an equalizer, but on the contrary realizes both amplitude and phase simultaneously in a single integrated network.

In the paper titled "Linear Phase Microwave Networks" by H. J. Carlin and G. I. Zysman, appearing in the Proceedings of the Symposium of Generalized Networks, Polytechnic Institute of Brooklyn, pp. 193—226, Apr. 1966, there is presented a technique for the construction of linear phase transfer functions which may be realized by a cascaded transmission line network. That technique is to approximate the squared magnitude of a realizable transfer function to the magnitude of a constant delay function that is not directly realizable in the form of a stepped transmission-line network. With that amplitude approximation procedure, network realizability is assured and good phase characteristics can be obtained, but the resulting amplitude response is not suitable for filter applications.

In a paper titled "Microwave Networks With Constant Delay" by J. O. Scanlan and J. D. Rhodes, I.E.E.E. Transactions on Circuit Theory, Volume CT-14, pp. 290—297, Sept. 1967, it is shown that microwave networks can be constructed with constant delay at all frequencies while simultaneously permitting a choice among a large variety of amplitude characteristics. The procedure detailed in that paper however suffers from the disadvantages that (1) it is impossible to construct a narrow band filter using a reasonable number of elements, and (2) in general, using nonredundant amplitude approximations, there does not exist a sufficient number of arbitrary parameters to permit meeting normal filter specifications.

In T. A. Abele's paper "Transmission Line Filters Approximating a Constant Delay in a Maximally Flat Sense," I.E.E.E. Transactions on Circuit Theory, Volume CT-14, pp. 298—306, Sept. 1967, it is shown that a general n^{th} ordered transfer function can be constructed to exhibit an optimum maximally flat group delay characteristic in the conventional network sense, and that the characteristic is capable of being realized by a transmission line of conventional interdigital network. The resulting amplitude characteristics however, do not approximate ideal filtering characteristics and therefore those networks have limited utility.

THE INVENTION

The invention resides in a band pass filter which simultaneously exhibits selected amplitude and phase characteristics. In an embodiment of the invention discussed herein, the amplitude characteristic has been chosen to be maximally flat about the band's center. The invention, in one form, is embodied in a pair of interdigital lines arranged with one interdigital line superposed above the other so that the digits in one line are opposite the digits in the other line. Coupling between opposed digits of the two lines is provided by slots in the ground plane plate which is disposed between and is common to the two interdigital lines. As in the conventional interdigital line, coupling exists between adjacent digits in the interdigital lines employed in the embodiment of the invention. However, a signal introduced into the input end of one interdigital line not only couples from digit to adjacent digit in that interdigital line, but also couples through the slots to the opposed digits in the other interdigital line. In a waveguide version of the invention, a first waveguide is partitioned by irises into a series of directly coupled resonant cavities and the resonant cavities in the first waveguide are cross-coupled to opposed resonant cavities in a similarly partitioned second waveguide. In a lumped constant version of the invention, parallel resonant circuits are directly coupled to form arrays and the resonant circuits in one array are cross coupled to opposed resonant circuits in the other array.

THE DRAWINGS

The invention, as to its design, construction, and mode of operation, can be better understood from the exposition which follows when it is considered in conjunction with the accompanying drawings in which:

FIG. 1 is an exploded view of a preferred embodiment of the invention employing an interdigital network,

FIG. 2 is a top plan view of the preferred embodiment with the upper ground plane plate broken away,

FIG. 2A is a sectional view of the preferred embodiment taken along the parting plane A—A in FIG. 2,

FIG. 3 depicts an embodiment of the invention employing resonant cavities,

FIG. 4 schematically depicts a lumped constant embodiment of the invention,

FIGS. 5A to 5E are graphs giving the amplitude and delay characteristics for n equal to 3, 4, 5, 6, or 7, respectively, representing results of the example design procedure,

FIG. 6 is a representation of a network resulting from the synthesis procedure for $n=7$,

FIG. 7 depicts, in schematic form, an interdigital network for $n=7$,

FIG. 8 is a representation of a network resulting from the synthesis procedure for $n=6$, and

FIG. 9 depicts, in schematic form, an interdigital network for $n=6$.

THE EXPOSITION

Referring now to FIG. 1, there is shown an exploded view of the invention realized in the form of an interdigital network. The embodiment employs a first interdigital line L1 which is superposed upon and is electrically coupled to a second interdigital line L2. The two interdigital lines are separated by the ground plane plate G2 which acts as a wall that electrically shields one line from the other line. The digits in line L1 are enclosed in a chamber formed by ground plane plates G1, G2 and a perimeter having parallel longitudinal walls W1, W2 and end walls W3 and W4. The perimeter walls and the ground plane plates, electrically, constitute a unitary structure. That is, the perimeter walls act as short circuits between the spaced ground planes. Similarly, in line L2, the digits are enclosed in a chamber formed by ground plane plates G2, G3 and a perimeter having walls W5, W6, W7, and W8. The digits in each of the interdigital lines are situated midway between their ground planes for simplicity of construction.

The digits in lines L1 and L2 are arranged in the manner of the conventional interdigital line. That is, the digits are parallel and lie in the same plane so that signal coupling exists predominantly between adjacent digits in the line. Because of the spacing and the arrangement of the digits the coupling between nonadjacent digits in the same line is so small as to be negligible. The digits are all electrically conductive bars of rectangular cross section and all the digits are of equal length, the length being approximately one-fourth of the wavelength at the filter's midband frequency. The odd-numbered digits 1, 3, 5, in line L1 each has a corresponding end short circuited to wall W1 and the even-numbered digits 2, 4, and 6 are each short circuited at the corresponding opposite end to the W2 wall. Similarly in line L2 the odd-numbered digits have corresponding ends short circuited to wall W5 and the even-numbered digits have corresponding ends short circuited to wall W6. Thus, the digits in the lines L1 and L2 alternate in having opposite ends short circuited to ground in the usual manner of an interdigital line.

The digits in line L1 are in alignment with the opposed digits of line L2. Coupling slots S1, S2, S3, S4, S5 and S6 in the ground plane shield wall G2 permit energy to couple between opposed digits in the two lines. The signal transfer between digits in different lines is here termed "cross coupling" to distinguish it from the "direct" signal coupling that occurs between adjacent digits in the same line. It should be understood that "cross coupling," in the ideal case, occurs between a digit in one line and the opposed digit only of the other line. For example, digit 3 in line L1, in the ideal case, cross couples through slot S3 with the opposed digit 10 in line L2. It is not intended that digit 3 have cross coupling through slot S3 with the digit 9 or the digit 11 in line L2.

The input signal is applied to line L1 through a coaxial connector **21** whose center conductor **22** extends through a circular aperture in the perimeter wall W2. The center conductor **22** terminates upon the tip of a digit T1. The T1 digit has one end short circuited to the perimeter wall W2 and is connected at its other end to the center conductor **22**. The digit T1 is redundant to the interdigital line L1 and serves to couple the input signal to the digit 1 by transformer action. Similarly, the output signal is obtained from line L2 through a coaxial connector **23** whose center conductor **24** is connected to one end of the transformer digit T2 in line L2. The input and output coupling to the interdigital lines by impedance transformer digits here utilized is a conventional coupling arrangement employed with interdigital structures. See, for example, page 614 et seq. of Microwave Filters, Impedance Matching Networks, and Coupling Structures by G. L. Matthaei et al., published by McGraw Hill.

In the embodiment shown in FIGS. 1 to 2A, the resonant elements are the digits in the interdigital lines. At higher microwave frequencies it may be preferable to employ cavities as the resonant elements. FIG. 3 shows an embodiment of the invention in which the resonant cavities are formed by a sequence of irises located within a hollow rectangular waveguide. A first serial array of resonant cavities Q1, Q2, Q3, Q4 is formed by the irises **31, 32, 33, 34**. The irises are spaced along the hollow rectangular waveguide **35** and partition the hollow rectangular waveguide into a tandem sequence of resonant chambers which are directly coupled in consecutive order by apertures **36, 37, 38, 39** in the irises. A second serial array of resonant cavities Q5, Q6, Q7, Q8 is similarly formed by the irises **41, 42, 43, 44** in the hollow rectangular waveguide **45**. The cavities in the second array are directly coupled, in consecutive order, by the apertures **46, 47, 48, 49** in the irises. Waveguides **35** and **45** are separated longitudinally by a common narrow wall **50**. The resonant cavities in the two arrays are disposed so that each resonant cavity in one array is opposite and aligned with a different resonant cavity in the other array. Cross coupling between opposite resonant cavities in the two arrays is provided by coupling apertures **51, 52, 53, 54** in the common narrow wall **50**.

Signal energy is coupled into the first array through input port **30** and the output of the waveguide filter is obtained at the port **40**.

The number of resonant cavities in the two arrays, the amount of direct coupling between adjacent cavities in the same array, and the amount of cross coupling between opposed resonant cavities in the two arrays principally determine the group delay and amplitude characteristics of the filter. All the cavities, in a narrow band filter, are resonant at the same frequency and as the coupling apertures vary in size, adjustments may be required to maintain the cavities at the desired resonance.

In FIG. 4 there is schematically depicted a lumped constant version of the invention having a first array of resonators **61, 62, 63** disposed in parallel and a second array of similarly disposed resonators **64, 65, 66**. Each resonator is a parallel resonant circuit formed by an inductor and a capacitor. The resonators in the first array are directly coupled in consecutive order by coupling capacitors **67, 68** and the resonators in the second array are similarly coupled by capacitors **69, 70**. The resonators of the first array are cross coupled to the resonators in the second array by cross coupling capacitors **71, 72, 73**. Assuming the input signal is applied at the terminals **74** of the first array, the filtered signal is obtained from the terminals **75** of the second array.

Having described the physical features of a lumped constant, an interdigital, and a waveguide embodiment of the invention, consideration is now given to the design, on a transmission basis of an embodiment of the invention having selected amplitude and phase characteristics. The employment of cross coupling between resonant elements in arrays of directly coupled resonators enables a wide variety of amplitude and phase requirements to be met. From among the many possible combinations of amplitude and phase requirements that can simultaneously be met, the example of an interdigital filter which over its pass band simultaneously exhibits a maximally flat group delay characteristic (i.e., linear phase) and a maximally flat amplitude characteristic has been chosen.

DESIGN PROCEDURE EXAMPLE

In this example, the design procedure is initially similar to the procedure described by Abele, cited supra. A different approach, however, is employed to derive a denominator polynomial of the transfer function which forces a maximally flat group delay about the center of the pass band (i.e., t tan$h$ $p=\infty$). It is shown by the different approach that the desired polynomial is the symmetrical Jacobi polynomial $\underline{P_n^{(\alpha,-\alpha)}t}$. See the "Handbook of Mathematical Functions," National Bureau of Standards, pp. 771—802. The numerator polynomial of the transfer function is then constructed to cause the amplitude characteristic to have the desired form. For expository purposes, the amplitude characteristic is here chosen to be maximally flat about $t=\infty$. A synthesis procedure is presented for the general n^{th} ordered transfer function and the realization is in the form of an interdigital network of relatively simple physical configuration. The realization can also be in the form of a network of resonant cavities of relatively simple configuration.

A scattering transfer function which simultaneously exhibits maximally flat amplitude and delay characteristics and which permits realization by a generalized interdigital network, must be of the form

$$S_{12}(t)=\frac{(1-t^2)^{\frac{\kappa}{2}}O_{n-\kappa}(t)}{D_n(t)} \tag{1}$$

where n is an integer such that $\underline{3\leq n\leq\alpha}$.
$\kappa=0$ for n odd
$\kappa=1$ for n even

$t = \tanh p$, where p is the normalized complex frequency variable

$D_n(t)$ is a Hurwitz polynomial of degree n

$O_{n-\kappa}(t)$ is an odd polynomial of degree $n-\kappa$ and it is necessary that

$$|S_{12}|^2 \leqq \text{ for } t = j \tan \omega$$

This is shown in my paper entitled "The Theory of Generalized Interdigital Networks" I.E.E.E. Transactions on Circuit Theory, Vol CT-16, pp. 280—288. Because of the form of $S_{12}(ta)$ only the denominator polynomial $D_n(t)$ contributes to the delay. Initially, the polynomial $D_n(t)$ is derived which gives rise to a maximally flat group delay about $t=\infty$, and then the odd polynomial $O_{n-\kappa}(t)$ is constructed such that $S_{12}(t)S_{12}(-t)$ is maximally flat about $t=\infty$.

The complex phase angle $\Psi_n(t)$ of $S_{12}(t)$ is defined as

$$\Psi_n(t) = -\tanh^{-1}\left[\frac{D_n(t)-D_n(-t)}{D_n(t)+D_n(-t)}\right] \tag{2}$$

and the corresponding group delay $T_{gn}(t)$ is defined as

$$T_{gn}(t) = -d\Psi n(t)/dp$$

which, from equation (2), reduces to

$$T_{gn}(t) = \frac{(1-t^2)}{2}\left[\frac{D'_n(t)}{D_n(t)} + \frac{D'_n(-t)}{D_n(-t)}\right] \tag{3}$$

where $D_n'(t$second $dD_n(t)/dt$ and $D_n'(-t)$ indicates the differential of $D_n(t)$ with respect to t and then the replacement of t by $-t$.

For a maximally flat group delay about $t=\infty$, equation (3) must reduce to the form

$$T_{gn}(t) = a - \frac{b}{D_n(t)D_n(-t)} \tag{4}$$

where a and b are constants.

Directly, from equation (3), one peculiarity of microwave delay functions arises with respect to frequency scaling because if the polynomial $D_n(t)$ satisfies the maximally flat delay constraint, then the polynomial $D_n(\beta t), (\beta \neq 1)$, does not result in a maximally flat delay since the factor $(1-t^2)$ in equation (3) is insensitive to frequency scaling. That property of microwave delay functions is not encountered in amplitude or delay functions of lumped networks nor in amplitude functions of microwave networks. It is necessary, therefore, in the case of microwave delay functions, to seek an alternative parameter to enable bandwidth scaling to be made. A convenient parameter for that purpose is the normalized midband delay of the filter represented by α. Under the delay normalization here adopted, α represents the delay attributed to a cascade of α unit elements of unity characteristic impedance which are one-quarter of a wavelength long at $t=\infty$; α need not be an integer.

By defining the polynomial $Q_n^{(\alpha)}(t)$ as the polynomial of degree n, dependent upon the parameter α, which results in a maximally flat group delay about $t=\infty$, then from equations (3) and (4), $Q_n^{(\alpha)}(t)$ must be the polynomial that satisfies the differential equation

$$\frac{(1-t^2)}{2}\left[\frac{Q'_n{}^{(\alpha)}(t)}{Q_n^{(\alpha)}(t)} + \frac{Q'_n{}^{(\alpha)}(-t)}{Q_n^{(\alpha)}(-t)}\right] = \alpha\left[1 - \frac{Q_n^{(\alpha)}(1)Q_n^{(\alpha)}(-1)}{Q_n^{(\alpha)}(t)Q_n^{(\alpha)}(-t)}\right] \tag{5}$$

It will now be shown that the polynomial which satisfies equation (5) is the symmetrical Jacobi polynomial $P_n^{(\alpha,-\alpha)}(t)$. For convenience, the leading coefficient of $Q_n^{(\alpha)}(t)$ is normalized to unity and we therefore have the relationship

$$\binom{2n-1}{n} Q_n^{(\alpha)}(t) = 2^{n-1} P_n^{(\alpha,-\alpha)}(t) \tag{6}$$

From the known properties of the Jacobi polynomial, we have

a. the explicit expression

$$Q_n^{(\alpha)}(t) = \sum_{m=0}^{m=n} \frac{\binom{n+\alpha}{m}\cdot\binom{\alpha-1-m}{n-m}}{2\binom{2n-1}{n}}(1+t)^m\cdot(1-t)^{n-m} \tag{7}$$

where $\binom{z}{w}$ is the binomial coefficient $\frac{z!}{(z-w)!w!}$

b. the degree varying recurrence formula

$$Q_{n+1}^{(\alpha)}(t) = tQ_n^{(\alpha)}(t) + \frac{(\alpha^2-n^2)}{(4n^2-1)}Q_{n-1}^{(\alpha)}(t) \tag{8}$$

with the initial conditions

$$Q_0^{(\alpha)}(t) = 1,\ Q_1^{(\alpha)}(t) = t+\alpha$$

c. the first ordered differential equation

$$(1-t^2)\frac{dQ^{(\alpha)}(t)}{dt} = (\alpha - nt)Q_n^{(\alpha)}(t) - \frac{(\alpha^2-n^2)}{(2n-1)}Q_{n-1}^{(\alpha)}(t) \tag{9}$$

Using properties (b) and (c), the proof that the normalized symmetrical polynomial satisfies equation (5) is as follows.

Consider the left side of equation (5), and substitute for $Q'_n{}^{(\alpha)}(t)$ from equation (9), then

$$\frac{(1-t^2)}{2}\left[\frac{Q'_n{}^{(\alpha)}(t)}{Q_n^{(\alpha)}(t)} + \frac{Q'_n{}^{(\alpha)}(-t)}{Q_n^{(\alpha)}(-t)}\right] = \alpha\left\{1 - \frac{(\alpha^2-n^2)}{2d(2n-1)}\left[\frac{Q_{n-1}^{(\alpha)}(t)}{Q_n^{(\alpha)}(t)} + \frac{Q_{n-1}^{(\alpha)}(t)}{Q_n^{(\alpha)}(-t)}\right]\right\} \tag{11}$$

From equation (8), if n is replaced by $n-1$, then

$$Q_n^{(\alpha)}(t) = tQ_{n-1}^{(\alpha)}(t) = \frac{[\alpha^2-(n-1)^2]}{(2n-1)(2n-3)}Q_{n-2}^{(\alpha)}(t) \tag{12}$$

Multiplying by $Q_{n-1}^{(\alpha)}(-t)$ and taking the even part results in

$$Q_n^{(\alpha)}(t)Q_{n-1}^{(\alpha)}(-t) + Q_n^{(\alpha)}(-t)Q_{n-1}^{(\alpha)}(t) = \frac{[\alpha^2-(n-1)^2]}{(2n-1)(2n-3)}[Q_{n-1}^{(\alpha)}(t)Q_{n-2}^{(\alpha)}(-t) + Q_{n-1}^{(\alpha)}(t)Q_{n-2}^{(\alpha)}(-t)] \tag{13}$$

Repeated application of the recurrence formula leads to the conclusion that

$$Q_n^{(\alpha)}(t)Q_{n-1}^{(\alpha)}(-t) + Q_n^{(\alpha)}(-t)Q_{n-1}^{(\alpha)}(t) = \text{constant}$$

However, the left side of equation (11) possesses a factor $(1-t^2)$ and therefore the right side of that equation must reduce to the form

$$\alpha\left[1 - \frac{Q_n^{(\alpha)}(1)Q_n^{(\alpha)}(-1)}{Q_n^{(\alpha)}(t)Q_n^{(\alpha)}(-t)}\right] \qquad \text{Q.E.D.}$$

Choosing the denominator of the scattering coefficient $S_{12}^{(t)}$ to be $Q_n^{(\alpha)}(t)$, assures that the delay of the corresponding generalized interdigital network is maximally flat about $t=\infty$.

The numerator of $S_{12}^{(t)}$ must now be determined such that the magnitude function $S_{12}(t)S_{12}(-t)$ will approximate unity in a maximally flat manner about $t=\infty$. From equation (1), we have

$$S_{12}(t)S_{12}(-t) = \frac{(1-t^2)^{\frac{\kappa}{2}}O_{n-\kappa}(t)}{Q_n^{(\alpha)}(t)Q_n^{(\alpha)}(-t)} \tag{14}$$

which must approximate unity in a maximally flat manner about $t=\infty$. As the magnitude expression contains the polynomial $Q_n^{(\alpha)}(t)Q_n^{(\alpha)}(-t)$, the properties of that polynomial must be determined. To simplify the procedure, it is convenient to define the even polynomial

$$F_n^{(\alpha)}(c) = \frac{Q_n^{(\alpha)}(t)Q_n^{(\alpha)}(-t)}{(1-t^2)^n} \tag{15}$$

where $c = \cosh p = 1/(1-t^2)1/2$ and to determine an explicit expression for $F_n^{(\alpha)}(c)$.

From the recurrence formula (8) for $Q_n^{(\alpha)}(t)$, it follows that in the expansion of $F_n^{(\alpha)}(c)$ in the form

$$F_n^{(\alpha)}(c) = \sum_{m=0}^{m=n} A_m(\alpha)c^{2m} \tag{16}$$

that $A_m(\alpha)$ is a polynomial of degree m in parameter α. Also, from the explicit expression $Q_n^{(\alpha)}(t)$ given by equation (7), it is apparent that $Q_n^{(\alpha)}(t)$ possesses a factor $(1+t)^r$ where r is any integer between 1 and n. It therefore follows that c $A_0(\alpha)=1$

and

$$A_m(\alpha) = a_m \sum_{r=0}^{r=m-1}[a^2-(n-r)^2] \tag{17}$$

for $m=1 \rightarrow n$

where a_m is constant dependent only upon n.

To determine the exact explicit expression for $F_n^{(\alpha)}(c)$ for an arbitrary α, it is now only necessary to determine this even polynomial for one specific value of α other than any integer between $-n$ and n. One such value is presented in the following theorem

Theorem

$$F_n^{(1/2)}(c)=\frac{c^{2n+1}}{2^{2n}}T_{2n+1}\left(\frac{1}{c}\right) \tag{18}$$

where $T_{2n+1}(x)$ is the Chebyshev polynomial of the first kind of degree $2n+1$.

Proof

If the delay function $T_{gn}(t)$, as defined in equation (3), is expanded in partial fraction form (see "Microwave All-Pass Networks" by J. O. Scanlan and J. D. Rhodes, I.E.E.E. Transactions on Microwave Theory and Techniques, volume MTT-16, pp. 62—79, Feb. 1968), then

$$T_{gn}(t)=\sum_{m=1}^{m=n}\frac{(1-t_i^2)}{2}\left[\frac{1}{t_i+t}+\frac{1}{t_i-t}\right] \tag{19}$$

where $t=\pm t_i$ are the poles of $T_{gn}(t)$.

From equations (5), (15), and (16) we therefore have

$$\frac{A_n(\alpha)c^{2n}}{F_n^{(\alpha)}(c)}=1-\frac{1}{\alpha}\left[\sum_{m=1}^{m=n}\frac{c_i\sqrt{c_i^2-1}}{(c_i^2-c^2)}\right] \tag{20}$$

where $c=\pm c_i$ are the zeros of $F_n^{(\alpha)}(c)$.

Equation (20) is a unique description of the magnitude polynomial of degree $2n$ in terms of the parameter α. The important point of that equation is that the function

$$\frac{A_n^{(\alpha)}c^{2n}}{F_n^{(\alpha)}(c)}$$

represents all those even, all-pole functions whose residues at any particular pole are completely independent of all other pole locations and are restricted to be of the form on the right side of equation (20) independent of the value of α. This factor leads to the consideration of the even all-pole function

$$\frac{2^{2n}}{cT_{2n+1}\left(\frac{1}{c}\right)}=\frac{2^{2n}}{c\cosh\,(2n+1)\cosh^{-1}\left(\frac{1}{c}\right)} \tag{21}$$

which, if $T_{2n+1}1/c_i=0$, reduces to

$$\frac{2^{2n}}{cT_{2n+1}\left(\frac{1}{c}\right)}=1-\frac{1}{2}\sum_{m=1}^{m=n}\frac{c_i\sqrt{c_i^2-1}}{(c_i^2-c^2)} \tag{22}$$

Comparing equations (20) and (22), we establish equation (18)

$$F_n^{\frac{1}{2}}(c)=\frac{c^{2n+1}}{2^{2n}}T_{2n+1}\left(\frac{1}{c}\right)\qquad\text{Q.E.D.}$$

From the explicit expression for $T_{2n+1}(x)$, using equations (16), (17), and (18), the explicit expression for $F_n^{(\alpha)}(c)$ for any arbitrary α is

$$F_n^{(\alpha)}(c)=\sum_{m=0}^{m=n}\frac{\binom{n}{m}^2\cdot\binom{\alpha+n}{m}\cdot\binom{\alpha-n+m-1}{m}}{\binom{2n}{m}\cdot\binom{2n}{2m}\binom{2m}{m}}2(c)^{2m} \tag{23}$$

This expression allows the group delay of the network to be expressed in a closed form and subsequently enables the amplitude expression to be written in a similar compact form once the odd polynomial $O_{n-\kappa}^{(\alpha)}(t)$ has been determined.

From equations (14) and (15), the magnitude function is

$$S_{12}(t)S_{12}(-t)=\frac{(1-c^2)[E_n^{(\alpha)}(c)]^2}{F_n^{(\alpha)}(c)} \tag{24}$$

where

$$En^{(\alpha)}(c)=\frac{O_{n-\kappa}(t)}{t(1-t^2)^{\frac{n-\kappa-1}{2}}}$$

and is an even polynomial of degree $n-1$ where n is odd, and of degree $n-2$ when n is even. We may write $E_n^{(\alpha)}(c)$ in the form

$$E_n^{(\alpha)}(c)=\sum_{m=0}^{m=r}B_m(\alpha)c^{2m} \tag{25}$$

where $r=(n-1)/2$ when n is odd and $r=(n-2)/2$ when n is even.

The r coefficient of $E_n^{(\alpha)}(c)$ must now be determined such that the magnitude function approximates unity in a maximally flat manner about $c=0$. From the explicit expression for $F_n^{(\alpha)}(c)$, it follows that $B_m^{(\alpha)}$ must be an even polynomial in α of degree $2m$. Further, $F_n^{(q)}(c)$ for q assuming any integer value between 1 and n, is the square of an even polynomial of degree $n-q$ or the square of an even polynomial of degree $n-q-1$ multiplied by the factor $(1-c^2)$, depending upon whether $n-q$ is even or odd respectively.

Therefore, a necessary constraint for the maximally flat amplitude approximation is that

$$(1-c^2)[E_n^{(q)}(c)]^2=F_n^{(q)}(c) \tag{26}$$

for $q=n-1, n-3, n-5...$

The polynomial $B_m(\alpha)$, thus, is of the form

$$B_0(\alpha)=1$$

$$B_m(\alpha)=b_m\prod_{j=0}^{j=m-1}[\alpha^2-(n-1-2j)^2] \tag{27}$$

where the b_m are coefficients dependent entirely upon n. By following a procedure similar to that used to determine the explicit expression for $F_n^{(\alpha)}(c)$, if we can find $E_n^{(\alpha)}(c)$ for a specific value of α other than $\alpha=n-1$, $n-3$, etc., then $E_n^{(\alpha)}(c)$ will be known for all α. In this case, the specific value of $n=\alpha$ is considered.

Since $F_n^{(n)}(c)=1$, we require that the polynomial in x given as

$$O(x)=\sqrt{1-c^2}\,E_n^{(n)}(c),$$

$x=\sqrt{1-c^2}$ must be the odd polynomial which approximates unity in a maximally flat manner about $x=1$. By definition, the required polynomial must be

$$O(x)=\frac{\int_0^x(1-x^2)^r dx}{\int_0^1(1-x^2)^r dx} \tag{28}$$

Consequently, the explicit expression for $E_n^{(n)}(c)$ is

$$E_n^{(n)}(c)=\sum_{m=0}^{m=r}\frac{1}{2^{2m}}\binom{2m}{m}c^{2m} \tag{29}$$

Using equations (25), (27), and (29), the explicit expression is obtained for $E_n^{(\alpha)}(c)$ for arbitrary α.

$$E_n^{(\alpha)}(c)=\sum_{m=0}^{m=r}\frac{\binom{2m}{m}\cdot\binom{n-1}{m-1}\cdot\binom{\alpha-n+2m-1}{2m-1}\cdot\binom{\alpha+n-1}{2m-1}}{2^{2m}\binom{2n-1}{2m-1}\cdot\binom{\frac{\alpha-n}{2}+m-1}{m-1}\cdot\binom{\frac{\alpha+n}{2}-1}{m-1}}C^{2m} \tag{30}$$

where $r=(n-1)/2$ for n odd and $r=(n-2)/2$ for n even. Using expressions (23) and (30), the insertion loss function L, defined for $p=j$** and the group delay T_g of the network which simultaneously realizes both a maximally flat amplitude and a maximally flat delay response are

$$L=\frac{\displaystyle\sum_{m=0}^{m=n}\frac{\binom{n}{m}^2\cdot\binom{\alpha=n}{m}\cdot\binom{\alpha-n+m-1}{m}}{\binom{2n}{m}\cdot\binom{2n}{2m}\cdot\binom{2m}{m}}(2\cos\omega)^{2m}}{\sin^2\omega\left[\displaystyle\sum_{m=0}^{m=r}\frac{\binom{2m}{m}\cdot\binom{n-1}{m-1}\cdot\binom{\alpha-n+2m-1}{2m-1}\cdot\binom{\alpha+n-1}{2m-1}}{2^{2m}\binom{2n-1}{2m-1}\cdot\binom{\frac{\alpha-n}{2}+m-1}{m-1}\binom{\frac{\alpha+n}{2}-1}{m-1}}\cos^{2m}\omega\right]} \tag{31}$$

$$T_g=\alpha\left[1-\frac{2^{2n}\binom{\alpha+n}{n}\cdot\binom{\alpha-1}{n}\cos^{2n}\omega}{\binom{2n}{n}^2\left[\displaystyle\sum_{m=0}^{m=n}\frac{\binom{n}{m}^2\binom{\alpha+n}{m}\cdot\binom{\alpha-n+m-1}{m}}{\binom{2n}{m}\cdot\binom{2n}{2m}\cdot\binom{2m}{m}}(2\cos\omega)^{2m}\right]}\right] \tag{32}$$

where $r=\frac{n-1}{2}$ for n odd

and $r=\frac{n-2}{2}$ for n even.

Up to this point, no condition has been imposed upon the values of α under which these responses may be realized by a commensurate microwave network, i.e., the constraint upon α such that $S_{12}(t)$ is a bounded real function. Obviously α must be a real constant. Furthermore, from the recurrence formula (8), it is not too difficult to establish that $Q_n^{(\alpha)}(t)$ is a Hurwitz polynomial for $\alpha\geq n-1^{(2)}$. Thus to prove that $S_{12}(t)$ is bounded real for $\alpha\geq n-1$, it would be necessary to show that the insertion loss L, defined in equation (31), is greater than or equal to unity for all real ω when $\alpha\geq n-1$. Unfortunately, there does not appear to be any simple way of expressing L to demonstrate this fact in the general n^{th} ordered case although in the low ordered cases the construction of an explicit expression for $L-1$ readily establishes this result. However, $S_{12}(t)$ is a bounded real function for all real $\alpha\geq n-1$.

In order to simplify the construction of the delay characteristic, and amplitude characteristic, and the formation of the required reflection coefficient in the low ordered cases, the three polynomials $Qn^{(\alpha)}(t), F_n^{(\alpha)}(c)$ and $E_n^{(\alpha)}(c)$ have been tabulated for $n=3\rightarrow 7$ and are here presented in tables 1, 2, and 3 respectively.

Furthermore, in order to determine the values of α and n to a first degree of approximation in the low ordered cases, typical characteristics are given in FIG. 5A through 5E for $\alpha=25$.

If $\alpha^2>>n^2$, then from equation (32) the group delay is basically a function of $\alpha\cos\omega$. Under this approximation the delay characteristic for other values of α may readily be obtained. For example, to determine the characteristic for $\alpha=47$, Ω is replaced by $\cos^{11}47/25\cos\omega$, which if ω is of the order of 90° simplifies to the approximation $(90-\omega)\rightarrow\frac{47}{25}(90-\omega)$. Additionally, the delay deviation is scaled by the factor 47/25. A similar approximation also applies to the insertion loss but since the factor $\sin^2\omega$ in equation (31) is insensitive to bandwidth scaling, an allowance must be made for this single factor. However, to determine α and n exactly to meet a specification, it will be necessary to use the algebraic expressions (31) and (32).

Synthesis Procedure

The synthesis procedure here described is valid for any general transfer function having the form of equation (1), and the following description for the maximally flat filter is ap-

TABLE 1

$$Q_n(\alpha)(t)=\sum_{m=0}^{m=n}a_m t^{n-m}$$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
3	1	α	$\frac{(2\alpha^2-3)}{5}$	$\frac{\alpha(\alpha^2-4)}{15}$				
4	1	α	$\frac{3(\alpha^2-2)}{7}$	$\frac{\alpha(2\alpha^2-11)}{21}$	$\frac{(\alpha^2-9)(\alpha^2-1)}{105}$			
5	1	α	$\frac{2(2\alpha^2-5)}{9}$	$\frac{\alpha(\alpha^2-7)}{9}$	$\frac{(\alpha^4 13\alpha^2+15)}{63}$	$\frac{\alpha(\alpha^2-16)(\alpha^2-4)}{945}$		
6	1	α	$\frac{5(\alpha^2-3)}{11}$	$\frac{2\alpha(2\alpha^2-17)}{33}$	$\frac{(2\alpha^4-32\alpha^2+45)}{99}$	$\frac{\alpha(\alpha^4-25\alpha^2+99)}{495}$	$\frac{\alpha^2-25)(\alpha^2-9)(\alpha^2-1)}{10,395}$	
7	1	α	$\frac{3(\alpha^2-7)}{13}$	$\frac{5\alpha(\alpha^2-10)}{39}$	$\frac{5\alpha(2\alpha^4-38\alpha^2+63)}{429}$	$\frac{\alpha(2\alpha^4-60\alpha^2+283)}{715}$	$\frac{4\alpha^6-165\alpha^2+1516\alpha^2-1575}{19,305}$	$\frac{\alpha(\alpha^2-36)(\alpha^2-16)(\alpha^2-4)}{135,135}$

TABLE 2

$$F_n^{(\alpha)}(c)=\sum_{m=0}^{m=n}\prod_{i=0}^{i=m}a_i\,c^{2m}$$

n	a^0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
3	1	$\frac{3(\alpha^2-9)}{5.3}$	$\frac{2(\alpha^2-4)}{3.5}$	$\frac{(\alpha^2-1)}{6}$				
4	1	$\frac{4(\alpha^2-16)}{7.4}$	$\frac{3(\alpha^2-9)}{5.7}$	$\frac{2(\alpha^2-4)}{3.9}$	$\frac{(\alpha^2-1)}{10}$			

TABLE 2—Continued

$$F_n^{(\alpha)}(c)=\sum_{m=o}^{m=n}\prod_{i=o}^{i=m} a_i c^{2m}$$

n	a^0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
5	1	$\frac{5(\alpha^2-25)}{9.5}$	$\frac{4(\alpha^2-16)}{7.9}$	$\frac{3(\alpha^2-9)}{5.12}$	$\frac{2(\alpha^2-4)}{3.14}$	$\frac{(\alpha^2-1)}{15}$		
6	1	$\frac{6(\alpha^2-36)}{11.6}$	$\frac{5(\alpha^2-25)}{9.11}$	$\frac{4(\alpha^2-16)}{7.15}$	$\frac{3(\alpha^2-9)}{5.18}$	$\frac{2(\alpha^2-4)}{3.20}$	$\frac{(\alpha^2-1)}{21}$	
7	1	$\frac{7(\alpha^2-49)}{13.7}$	$\frac{6(\alpha^2-36)}{11.13}$	$\frac{5(\alpha^2-25)}{9.18}$	$\frac{4(\alpha^2-16)}{7.22}$	$\frac{3(\alpha^2-9)}{5.25}$	$\frac{2(\alpha^2-4)}{3.27}$	$\frac{(\alpha^2-1)}{28}$

TABLE 3

$$E_n^{(\alpha)}(c)=\sum_{m=o}^{m=r} a_m c^{2m}$$

n	a_0	a_1	a_2	a_3
3	1	$\frac{(\alpha^2-4)}{10}$		
4	1	$\frac{(\alpha^2-9)}{14}$		
5	1	$\frac{(\alpha^2-16)}{18}$	$\frac{(\alpha^2-16)(\alpha^2-4)}{504}$	
6	1	$\frac{(\alpha^2-25)}{22}$	$\frac{(\alpha^2-25)(\alpha^2-9)}{792}$	
7	1	$\frac{(\alpha^2-36)}{26}$	$\frac{(\alpha^2-36)(\alpha^2-16)}{1,144}$	$\frac{(\alpha^2-36)(\alpha^2-16)(\alpha^2-4)}{61,776}$

plicable for any other case, that is for any other amplitude and phase characteristics satisfying and being specified by equation (1).

The scattering transfer coefficient of the maximally flat amplitude and delay, band pass microwave filter has been shown to be given by,

$$S_{12}(t)S_{12}(-t)=\frac{(1-c^2)\ (E_n^{(\alpha)}(c))^2}{F_n^{(\alpha)}(c)} \qquad (33)$$

$$S_{12}(t)=\frac{t(1-t^2)^{\frac{\kappa}{2}}\ P_n^{(\alpha)}(t)}{Q_n^{(\alpha)}(t)} \qquad (34)$$

$P_n^{(\alpha)}(t)$ is an even polynomial in t

$\kappa=0$ for n odd

$\kappa=1$ for n even

and

$$P_n^{(\alpha)}(t)=(1-t^2)\ \frac{n-1-\kappa}{2}\ E_n^{(\alpha)}(c)$$

Therefore, we have

$$S_{11}(t)S_{11}(-t)=1-S_{12}(t)S_{12}(-t)$$

$$=\frac{F_n^{(\alpha)}(c)-(1-c^2)\ E_n^{(\alpha)}(c))^2}{F_n^{(\alpha)}(c)} \qquad (35)$$

from which we may form $S_{11}(t)$ as

$$S_{11}(t)=\frac{N_n^{(\alpha)}(t)}{Q_n^{(\alpha)}(t)} \qquad (36)$$

Due to the form of $S_{12}(t)$, $N_n^{(\alpha)}(t)$ will be a polynomial of degree $n-1/2$ for n odd and degree $n/2$ for n even, and $N_n^{(\alpha)}(o)=Q_n^{(\alpha)}(o)$ because there is a transmission zero at the origin. Since the numerator of equation (34) must be factorized in order to obtain $N_n^{(\alpha)}(t)$, the coefficients of this polynomial will be irrational functions of α. However, closed form solutions may be obtained for $N_n^{(\alpha)}(t)$ when $n\leq 9$, that is, when $N_n^{(\alpha)}(t)$ is of degree 4 or less. The allocation of the right and left half t plane zeros of equation (35) to $N_n^{(\alpha)}(t)$ is arbitrary, but normally it is more convenient to assign the left half plane zeros to $N_n^{(\alpha)}(t)$ resulting in this polynomial being a Hurwitz polynomial. Assuming that the generator impedance is normalized to unity, we may construct the driving-point impedance function of the required resistively terminated lossless, two-port network, $Z_n^{(\alpha)}(t)$ as

$$Z_n^{(\alpha)}(t)=\frac{Q_n^{(\alpha)}(t)-N_n^{(\alpha)}(t)}{Q_n^{(\alpha)}(t)+N_n^{(\alpha)}(t)} \qquad (37)$$

Because $S_{12}(t)$ is bounded real for $\alpha\geq n-1$, $Z_n^{(\alpha)}(t)$ is a positive real function for $\alpha\geq n-1$. It has been shown by J. O. Scanlan and J. and J. D. Rhodes in "Cascade Synthesis of Distributed Networks," Proceedings of the Symposium on Generalized Networks, Polytechnic Institute of Brooklyn, pp. 227—255, Apr. 1966, that any arbitrary positive real function of this form may always be realized by a resistively terminated, lossless coupled line network by employing a cascade synthesis procedure thus realizing the zeros of $S_{12}(t)$ independently. However, in this particular case, the network would be a cascade of microwave C-type and D-type sections and unit elements and therefore the realization would not be in the most desirable physical configuration. The realization, rather, is in the form of an interdigital network in which each digit is short circuited to ground at one end and open circuited at the other end. Two different synthesis procedures are required for that realization, depending upon whether n is even or odd.

In my paper titled "The Theory of Generalized Interdigital Networks," I.E.E.E. Transactions Band-Pass Circuit Theory, Volume CT-16, pp. 280 to 288, Aug. 1969pp. a realizability theory is presented for a relatively simple form of microwave network termed the generalized interdigital network. That network consists of an n-digit line where each digit is short circuited to ground at one end and open circuited at the other. The input and output ports of the two-port network are located at the open circuited ends of any pair of digits. In "The Theory of Generalized Interdigital Networks," it is shown that a network having completely general coupling (that is, a generalized network in which any digit can couple to any other digit in the line) may always be reduced to a network in which the coupling of any digit is restricted to the immediately adjacent digits in the line and one other nonadjacent digit. If coupling were permitted only between immediately adjacent digits in the line, the network would reduce to a conventional interdigital form whose equivalent circuit would be a cascade of unit elements and a single short-circuited shunt stub as described by R. J. Wenzel in "The Exact Theory of Interdigital Band-Pass Filters and Related Coupled Structures," I.E.E.E. Transactions on Microwave Theory and Techniques, Volume MTT-13, pp. 559—575, Sept. 1965. The retention in the reduced network of cross coupling between non adjacent digits is crucial because it permits nonminimum phase transfer functions, such as the one here considered, to be realized. A complete set of realizability conditions upon the two-port admittance matrix of such a cross coupled network has been formulated requiring a variety of synthesis procedures. Two synthesis procedures are here presented, one being applicable in the case when n is odd and the second when n is even.

Synthesis Procedure for n odd

From equations (37) and (34) we may construct the transfer matrix of the lossless two-port network, which, when terminated in unity impedances, realizes the required $S_{12}(t)$.

This matrix is of the form

$$[T]=\frac{1}{P_n^{(\alpha)}(t)}\begin{bmatrix} A(t) & B(t)\\ \frac{c_0}{t}+C(t) & D(t)\end{bmatrix} \tag{38}$$

where $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are polynomials in t given by

$$2A(t)=\frac{[Q_n^{(\alpha)}(t)-N_n^{(\alpha)}(t)]}{t}-\frac{[Q_n^{(\alpha)}(-t)-N_n^{(\alpha)}(-t)]}{t}$$

$$2D(t)=\frac{[Q_n^{(\alpha)}(t)+N_n^{(\alpha)}(t)]}{t}-\frac{[Q_n^{(\alpha)}(-t)+N_n^{(\alpha)}(-t)]}{t}$$

$$2B(t)=\frac{[Q_n^{(\alpha)}(t)-N_n^{(\alpha)}(t)]}{t}+\frac{[Q_n^{(\alpha)}(-t)-N_n^{(\alpha)}(-t)]}{t}$$

$$2\left[\frac{c_0}{t}+C(t)\right]=\frac{[Q_n^{(\alpha)}(t)+N_n^{(\alpha)}(t)]}{t}+\frac{[Q_n^{(\alpha)}(-t)+N_n^{(\alpha)}(-t)]}{t} \tag{39}$$

The synthesis procedure commences with the extraction of two short-circuited shunt stubs of characteristic admittance Y_1 and Y_2, one from each end of the network to remove the transmission zero at the origin and to leave the transfer matrix

$$[T']=\begin{bmatrix}1 & 0\\ \frac{-Y_1}{t} & 1\end{bmatrix}[T]\begin{bmatrix}1 & 0\\ \frac{-Y_2}{t} & 1\end{bmatrix} \tag{40}$$

From (38), Y_1 and Y_2 are given by,

$$\left[A(t)-Y_2\frac{B(t)}{t}\right]_{t=o}=P_n^{(\alpha)}(o) \tag{41}$$

and

$$\left[D(t)-Y_1\frac{B(t)}{t}\right]_{t=o}=P_n^{(\alpha)}(o)$$

This results in $[T']$ being of the form,

$$\frac{1}{P_n^{(\alpha)}(t)}\begin{bmatrix}A'(t) & B'(t)\\ C'(t) & D'(t)\end{bmatrix} \tag{42}$$

where $A'(t)$ and $D'(t)$ are even, $B'(t)$ and $C'(t)$ are odd polynomials in t and, of course,

$$A'(t)D'(t)-B'(t)C'(t)=[P_n^{(\alpha)}(t)]^2 \tag{43}$$

and the remaining admittance matrix $[Y']$ will be

$$[Y']=\begin{bmatrix}\frac{D'(t)}{B'(t)}-\frac{Y_3}{t} & \frac{-P_n^{(\alpha)}(t)}{B'(t)}+\frac{Y_3}{t}\\ \frac{-P_n^{(\alpha)}(t)}{B'(t)}+\frac{Y_3}{t} & \frac{A'(t)}{B'(t)}-\frac{Y_3}{t}\end{bmatrix} \tag{45}$$

or

$$[Y']=\begin{bmatrix}\frac{D''(t)}{B''(t)} & \frac{-(1-t^2)P'(t)}{B''(t)}\\ \frac{-(1-t^2)P'(t)}{B''(t)} & \frac{A''(t)}{B''(t)}\end{bmatrix} \tag{46}$$

where $B''(t)=B'(t)$ and $P'(t)$ is an even polynomial in t. Converting back to the transfer matrix we have,

$$[T'']=\frac{1}{(1-t^2)P'(t)}\begin{bmatrix}A''(t) & B''(t)\\ C''(t) & D''(t)\end{bmatrix} \tag{47}$$

where

$$A''(t)D''(t)-B''(t)C''(t)=(1-t^2)^2\,(P'[t])^2$$

Two unit elements of characteristic admittance Y_4 and Y_5 are now extracted in cascade from this network, one from either end, to leave the transfer matrix $[T'']$ define by

$$(1-t^2)[T''']=\begin{bmatrix}1 & -Z_4t\\ -Y_4t & 1\end{bmatrix}\begin{bmatrix}T''\end{bmatrix}\begin{bmatrix}1 & -Z_5t\\ -Y_5t & 1\end{bmatrix} \tag{48}$$

where

$$Y_4=\frac{1}{Z_4}=\frac{D''(1)}{B''(1)}$$

and

$$Y_5=\frac{1}{Z_5}=\frac{A''(1)}{B''(1)}$$

The new transfer matrix $[T''']$ is of exactly the same form as the transfer matrix $[T']$ but of two degrees less in t. Thus, we have established a complete cycle of the synthesis procedure. This cycle is repeated until a unity transfer matrix remains.

As an example of the complete realization, the case of $n=7$ is schematically shown in FIG. 6. It is known from "The Theory of Generalized Interdigital Networks," supra, that the two-port network may be realized by a seven-digit generalized interdigital network. The characteristic admittance matrix of this interdigital line will be

$$\begin{array}{c}
\begin{array}{ccccccc}(1) & (2) & (3) & (4) & (5) & (6) & (7)\end{array}\\
\begin{bmatrix}
Y_1+Y_3+Y_4 & -Y_3 & -Y_4 & 0 & 0 & 0 & 0\\
-Y_3 & Y_2+Y_3+Y_5 & 0 & -Y_5 & 0 & 0 & 0\\
-Y_4 & 0 & Y_4+Y_6+Y_7 & -Y_6 & -Y_7 & 0 & 0\\
0 & -Y_5 & -Y_6 & Y_5+Y_6+Y_8 & 0 & Y_8 & 0\\
0 & 0 & -Y_7 & 0 & Y_7+Y_9+Y_{10} & Y_9 & Y_{10}\\
0 & 0 & 0 & Y_8 & -Y_9 & Y_8+Y_9+Y_{11} & Y_{11}\\
0 & 0 & 0 & 0 & -Y_{10} & -Y_{11} & Y_{10}+Y_{11}
\end{bmatrix}
\end{array} \tag{49}$$

The synthesis procedure continues with the extraction of a series-short-circuited stub in parallel from the network of characteristic admittance Y_3. Y_3 is chosen such that the remaining network possesses a transmission factor $(1-t^2)$ which may be realized by a cascade of two unit elements. Whence,

$$Y_3=\frac{P_n^{(\alpha)}(1)}{B'(1)}$$

where, as diagrammatically indicated in FIG. 7, digits 1, 2, 5, and 6 in the interdigital line each have a corresponding end short circuited to ground as indicated by the cross-hatching, and digits 3, 4, and 7 are each short circuited to ground at the end opposite as indicated by the absence of cross-hatching. The input and output ports are situated at the open-circuited ends of digits 1 and 2, and the open-circuited ends of the remaining digits represent the nodes as shown in FIG. 6.

Admittance scaling (see, The Theory of Generalized Interdigital Networks, supra) may now be applied to this characteristic admittance matrix to produce finite values of self-admittance to ground on all of the digits in the network. Additionally, particularly in the narrow band cases ($\alpha >> n$), a pair of redundant digits may be coupled to the input and output digits without altering the amplitude response or the delay deviation from midband delay of the terminated network. This procedure is the same as that employed in conventional interdigital filters (see, The Exact Theory of Interdigital Band-Pass Filters and Related Coupling Structures, supra.). Two unit elements of unity characteristic impedance are connected in cascade with the network, one at the input port and the other at the output port. The new augmented network will be a nine-digit generalized interdigital network with the characteristic admittance matrix,

$$
\begin{array}{cccccc}
(I) & (0) & (1) & (2) & (3) & (4) \\
\end{array}
$$

$$
\left[\begin{array}{cccccc}
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 1+Y_1+Y_3+Y_4 & -Y_3 & -Y_4 & 0 \\
0 & 1 & -Y_3 & 1+Y_2+Y_3+Y_5 & 0 & -Y_5 \\
0 & 0 & -Y_4 & 0 & Y_4+Y_6+Y_7 & -Y_6 \\
0 & 0 & 0 & -Y_5 & -Y_6 & Y_5+Y_6+Y_8 \\
 & \cdot & & & & \\
 & \cdot & & & & \\
 & \cdot & & & &
\end{array}\right.
$$

where I and 0 designate the new input and output digits which are short circuited to ground at the same end as digits 3, 4, and 7.

FIG. 7 shows, in symbolic form, a physical realization of this generalized interdigital network. As a matter of convenience, in FIG. 7 all the digits in the interdigital line are shown as uniformly spaced along the line and all the digits, with the exception of digit 7, are represented as being of uniform cross section. In the general case, the digits are not uniformly spaced along the line and differ in cross section so that the digit 3, for example, may not be directly opposite digit 4. However, in the general case, opposite digits are sufficiently aligned so that coupling through the aperture is effective. The coupling irises extend along the length of the rectangular bars and the intervening walls prevent cross coupling between diagonally adjacent digits such as digits 1 and 4. The introduction of the redundant digits also increases the separation between input and output ports.

Synthesis Procedure for n even

For n even, from equations (34) and (37), the transfer matrix of the required two-port network will be of the form

$$
\frac{1}{\sqrt{1-t^2}P_n^{(\alpha)}(t)}\begin{bmatrix} A(t) & B(t) \\ \frac{c_0}{t}+C(t) & D(t) \end{bmatrix} \qquad (51)
$$

where the polynomials $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are defined in equation (39).

The process of extracting short-circuited shunt stubs from both ends of the network is identical to the odd-ordered case, and the remaining matrix will be

$$
[T']=\frac{1}{\sqrt{1-t^2}P_n^{(\alpha)}(t)}\begin{bmatrix} A'(t) & B'(t) \\ C'(t) & D'(t) \end{bmatrix}
$$

where $A'(t)D'(t)-B'(t)C'(t)=(1-t^2)(P_n^{(\alpha)}(t))^2$

The corresponding admittance matrix is,

$$
[Y]=\begin{bmatrix} \frac{D'(t)}{B'(t)} & \frac{-\sqrt{1-t^2}P_n^{(\alpha)}(t)}{B'(t)} \\ \frac{-\sqrt{1-t^2}P_n^{(\alpha)}(t)}{B'(t)} & \frac{A'(t)}{B'(t)} \end{bmatrix} \qquad (53)
$$

In this case, we now extract a unit element of characteristic admittance Y_3 in parallel from the network such that the remaining network possesses a transmission factor $(1-t^2)3/2$, giving

$$
Y_3=\frac{P_n^{(\alpha)}(1)}{B'(1)}
$$

and the remaining admittance matrix is

$$
]Y']=\begin{bmatrix} \frac{D'(t)}{B'(t)}-\frac{Y_3}{t} & \sqrt{1-t^2}\left[\frac{P_n^{(\alpha)}(t)}{B'(t)}-\frac{Y_3}{t}\right] \\ -\sqrt{1-t^2}\left[\frac{P_n^{(\alpha)}(t)}{B'(t)}-\frac{Y_3}{t}\right] & \frac{A'(t)}{B'(t)}-\frac{Y_3}{t} \end{bmatrix} \qquad (54)
$$

or

$$
[Y']=\begin{bmatrix} \frac{D''(t)}{B''(t)} & \frac{-(1-t^2)\frac{3}{2}p'(t)}{B''(t)} \\ \frac{-(1-t^2)\frac{3}{2}p'(t)}{B''(t)} & \frac{A''(t)}{B''(t)} \end{bmatrix} \qquad (55)
$$

where $B''(t)=B'(t)$

Converting back to the transfer matrix we have

$$
[T'']=\frac{1}{(1-t^2)\frac{3}{2}p'(t)}\begin{bmatrix} A''(t) & B''(t) \\ C''(t) & D''(t) \end{bmatrix} \qquad (56)
$$

from which unit elements of characteristic admittance Y_4 and Y_5 may be extracted in cascade as in the odd-ordered case.

The complete cycle may now be repeated until the remaining network is a single-unit element. As an example of the complete realization, the case of $n=6$ is shown in FIG. 8. The six-digit generalized interdigital network which will realize this element configuration within a 1:−1 transformer possesses a characteristic admittance matrix

$$
\begin{array}{c}
\begin{array}{cccccc}
(1) & (2) & (3) & (4) & (5) & (6)
\end{array} \\
\begin{bmatrix}
Y_1+Y_3+Y_4 & -Y_3 & -Y_4 & 0 & 0 & 0 \\
-Y_3 & Y_2+Y_3+Y_5 & 0 & -Y_5 & 0 & 0 \\
-Y_4 & 0 & Y_4+Y_6+Y_7 & -Y_6 & -Y_7 & 0 \\
0 & -Y_5 & -Y_6 & Y_5+Y_6+Y_8 & 0 & -Y_8 \\
0 & 0 & -Y_7 & 0 & Y_7+Y_9 & -Y_9 \\
0 & 0 & 0 & -Y_8 & -Y_9 & Y_8+Y_9
\end{bmatrix}
\end{array} \qquad (57)
$$

where, as shown diagrammatically in FIG. 9, each digit 1, 4, and 5 has a corresponding end short circuited to ground and digits 2, 3, and 6 are each short circuited to ground at the end opposite the corresponding end. That is, considering the sequence of digits in the network to be 1, 3, 5, 6, 4, 2, the digits in the sequence alternate in having opposite ends short circuited to ground in the usual manner of an interdigital line. The input and output ports are again situated, as in the FIG. 7 embodiment, at the open-circuited ends of digits 1 and 2. Admittance scaling has been applied in the FIG. 9 embodiment to introduce a pair of redundant digits I and O at the input and output ports respectively.

The microwave band pass maximally flat linear phase filter here described exhibits insertion loss characteristics that do not differ greatly from the insertion loss characteristics of conventional microwave filters having the same number of elements, but a much higher degree of phase linearity is maintained over the entire pass band region that can be maintained by the conventional filter. The novel filter, when realized in the form of an interdigital network, is similar to conventional interdigital networks except for the coupling that is introduced between nonadjacent digits.

The invention, as is evident from the preceding exposition, can be embodied in many different forms. Whatever form the embodiment takes, it must provide multiple paths of different path lengths for the wave energy which is to be passed by the filter. In the embodiments here illustrated, the wave energy that is outside the filter's pass band is reflected back to the input whereas the wave energy in the filter's pass band proceeds from the input and appears, with slight attenuation, at the filter's output.

Because the invention can be embodied in varied structures, it is not intended that the invention be limited to the forms here illustrated or described. Rather, it is intended that the invention be delimited by the appended claims and include those structures that do not fairly depart from the essence of the invention.

What I claim is:

1. A microwave band pass filter having a flat delay characteristic over the pass band and a maximally flat amplitude characteristic, the filter comprising
   - a first series of resonant elements,
   - means for coupling an input signal to a resonant element of the first series,
   - the resonant elements in the first series being directly coupled in consecutive order,
   - a second series of resonant elements arranged with each resonant element opposite and in substantial alignment with a different resonant element in the first series,
   - means coupled to a resonant element in the second series for obtaining an output signal,
   - the resonant elements in the second series being directly coupled in consecutive order, and
   - means between the first and second series cross coupling opposed resonant elements in the two series and providing multiple paths of different lengths for wave energy in the band pass proceeding from the input to the output of the filter.

2. The microwave band pass filter according to claim 1, wherein
   - the first series of consecutively directly coupled resonant elements is formed by an interdigital line,
   - the second series of consecutively directly coupled resonant elements is formed by another interdigital line, and
   - the means between the first and second series is a ground plane plate having apertures providing cross coupling between opposed digits.

3. The microwave band pass filter according to claim 1, wherein
   - the mathematical relation between the direct coupling along the arrays and the cross coupling between opposed elements depends upon an odd polynomial $O_{n-\kappa}(t)$ and a Hurwitz polynomial $D_n(t)$, thus

$$S_{12}(t)=\frac{(1-t^2)^{\frac{\kappa}{2}}O_{n-\kappa}(t)}{D_n(t)}$$

where

$3\leq n\leq\alpha$

$S_{12}(t)$ is the scattering transfer function

$\kappa=0$ for n odd

$\kappa=1$ for n even

$t=\tanh p$, where p is the normalized complex frequency variable.

4. The microwave band pass filter according to claim 1, wherein
   - the first series of consecutively directly coupled resonant elements is formed by a sequence of resonant cavities,
   - the second series of consecutively directly coupled resonant elements is formed by a another sequence of resonant cavities having common walls separating them from the opposed resonant cavities in the first series, and
   - the cross-coupling means between opposed cavities in the first and second series are apertures in the common walls.

5. A band pass filter having a flat delay characteristic and a flat amplitude characteristic over the pass band, the filter comprising
   - a first serial array of consecutively directly coupled resonant elements;
   - means for coupling an input signal to a terminal resonant element of the first serial array;
   - a second serial array of consecutively directly coupled resonant elements arranged with each of its resonant elements opposite and in substantial alignment with a different resonant element in the first array;
   - means coupled to a terminal resonant element in the second serial array for obtaining an output signal;
   - means between the first and second arrays cross coupling opposed resonant elements in the two arrays;
   - the cross-coupling means and the resonant elements providing multiple paths of different lengths for wave energy proceeding from the input to the output of the filter;
   - and the filter being further characterized in that the mathematical relation between the direct coupling along the arrays and the cross coupling between opposed elements depends upon an odd polynomial $O_{n-\kappa}(t)$ and a Hurwitz polynomial $D_n(t)$, thus

$$S_{12}(t)=\frac{(1-t^2)^{\frac{\kappa}{2}}O_{n-\kappa}(t)}{D_n(t)}$$

where

$3\leq n\leq\alpha$

$S_{12}(t)$ is the scattering transfer function

$\kappa=0$ for n odd

$\kappa=1$ for n even

$t=\tanh p$, where p is the normalized complex frequency variable.