A NEW DETERMINATION OF THE RATIO OF THE ELECTROMAGNETIC TO THE ELECTROSTATIC UNIT OF ELECTRICITY.

By E. B. Rosa and N. E. Dorsey.

"Etant donné l'intérêt qui s'attache à la détermination de la *vitesse v*, il paraît désirable que de nouvelles expériences soient entreprises. La précision des anciennes mesures peut être dépassée: *toutes les méthodes s'y prêtent*. Il y a encore à réduire quelques corrections trop incertaines; il y a à simplifier quelques mesures auxiliaires trop complexes, et par ce nouvel effort on pourra, sans aucun doute, apporter dans la mesure de *v* une précision supérieure à celle aujourd'hui acquise pour la vitesse de la lumière." (H. Abraham, Rapport présenté au Congrès international de Physique, Paris, 1900.)

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I. INTRODUCTION.

1. THE METHOD.

Although many determinations of this important quantity have been made, even the best results hitherto found differ considerably from one another, and none are sufficiently accurate to be entirely satisfactory. The earlier results were rough, due largely to imperfect apparatus. Some of the more recent determinations have been made by methods that are interesting and important in themselves, but not capable of yielding results of high accuracy. Few, if any, even of the best determinations are exact enough to justify a claim of definitely fixing the value of **v** to within one part in a thousand. Abraham, in his report to the Paris Congress of 1900, expresses a doubt whether as a result of all the determinations that have been made we can be sure of the value of **v** to within 1 part in 1,000.

One of us published[1] a determination of **v** in 1889, made at the Johns Hopkins University, using the spherical condenser which had been employed by Rowland ten years before, deriving **v** from the ratio of the electrostatic to the electromagnetic capacity of the condenser. This is probably the best of all the methods yet employed for determining the ratio of the units. By this method **v** is determined in terms of a resistance, but inasmuch as **v** is equal to the square root of the ratio of the capacities, the uncertainty due to an error in the value of the ohm is divided by two, so that if we admit an uncertainty of two parts in five thousand in the value of the international ohm, that would involve an uncertainty of only one part in five thousand in **v**. All of the methods of determining **v** which do not involve the ohm are subject to even larger uncertainties in other directions. Hence it seemed to us desirable to make a new determination of the value of **v**, using the method of capacities, and to undertake to attain a higher order of accuracy than had heretofore been realized.

[1] Phil. Mag., vol. 28, p. 315, 1889, and Am. Jour. Sci., vol. 38, p. 298, 1889. In this article I expressed the hope of repeating the determination of v the following year in order to ascertain the cause of certain systematic differences appearing in the results. A favorable opportunity, however, did not arise until fifteen years afterwards, when the superior facilities of the Bureau of Standards suggested that the time had arrived for a new and more accurate determination of v than was possible in the former work.—E. B. R.

2. SPHERICAL CONDENSERS.

Through the courtesy of Prof. Joseph S. Ames, of the Johns Hopkins University, we secured the same spherical condenser that had been employed in the experiments of 1879 and 1889, already referred to. The two balls were repolished and the two halves of the shell were reground so that they fitted together water-tight. The two electrostatic capacities of the condenser were redetermined, using first the larger ball and then the smaller ball within the shell. The electromagnetic capacity was found by means of the Wheatstone bridge, using a rotating commutator. The experiments were begun early in the summer of 1904, but owing to the difficulty of obtaining sufficiently high insulation at that time they were deferred until the following winter. Meanwhile we designed a new rotating commutator better fitted for this experiment than the one used in the preliminary work and had it constructed in the instrument shop of the Bureau.

On resuming the work in the winter of 1904–5, we were able to obtain much better measurements of the electromagnetic capacity than before, but we were unable to get a sufficiently exact determination of the capacity of the charging wire which passed through the small hole at the pole of the outer hollow sphere, touching the inner sphere. When this wire is lifted far enough to break the contact with the ball its capacity changes for two separate reasons—first, because its length within the shell has changed, and, second, because its potential is then not the same as that of the ball; and hence the field is different after the contact is broken. This difficulty we afterwards overcame completely by getting the capacity with two charging wires, one at the pole and one at the equator. First withdrawing the polar charging wire, the difference in electromagnetic capacity gave the capacity of the charging wire itself, the ball meanwhile being charged by a wire at the equator. Then replacing the polar charging wire and withdrawing that at the equator, the difference gave the capacity of the latter. This gives a determination of the correction for the capacity of the charging wire that is free from theoretical objection, and which in practice gave remarkably accurate results.

3. CYLINDRICAL CONDENSERS.

Before this difficulty due to the charging wire was overcome, however, we designed and had constructed in the instrument shop of the Bureau (early in 1905) a set of cylindrical condensers[2] which could be measured without the necessity of determining the capacity of the charging wire, in the way since described by Lord Rayleigh.[3] A pair of coaxial cylinders was mounted on a suitable base with their axes vertical, and so joined to the Maxwell bridge that the electromagnetic capacity C_1 of the inner cylinder could be measured. A second pair of cylinders of the same radii was then added to the first and the increased capacity C_2 determined. Then a third pair was added and the total capacity C_3 determined. The difference C_2-C_1 is then the capacity of the second pair of cylinders, while C_3-C_2 is the capacity of the third pair, assuming that the unknown end corrections and the capacity of the charging wire (which has remained undisturbed) have not changed during the process of building up the cylinders by adding successive sections.

4. GUARD CYLINDERS.

This method gave as good results as we had anticipated, but in the meantime we had so much improved the work with the spherical condensers that the results with the cylinders were not satisfactory. We then added a fourth pair of cylinders to the set and insulated the two end sections of the inner cylinders so that they formed guard cylinders, after the manner of the guard rings of a plate condenser. The upper guard cylinder was provided with a micrometer screw, so that the breadth of the air gap could be varied and the computed correction for the gap checked by experimental determinations with varying widths. The use of the guard cylinders necessitated a modification of the rotating commutator for charging and discharging the condensers, by adding three brushes on the opposite side of the ring, so that the middle cylinders and the guard cylinders could be charged and discharged simultaneously. Using the guard cylinders in this way we obtained results of much greater uniformity and greater apparent accuracy than we had expected.

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[2] We are under obligations to Mr. H. B. Brooks for assistance in designing this condenser. It and some other apparatus employed was built by Mr. Joseph Ludewig in the instrument shop of this Bureau.

[3] Phil. Mag., **12**, p. 97; 1906.

5. DIFFERENTIAL METHOD.

We therefore thought it wise to check the work done by the Maxwell bridge method by independent measurements, using a differential method and a differential galvanometer. The Weston Electrical Instrument Company built us for this work a very delicate differential moving coil galvanometer, with a bifilar suspension (both above and below) of fine silver wire. As this instrument could be used not only in the differential method but also in the bridge method (with the two coils in series), we were able to compare the results by the two methods very critically, as we could change over from one to the other in a moment, using the same resistances. The results agreed with great exactness. In the differential method we measured sometimes the charge and sometimes the discharge. No difference whatever could be observed.

We had now obtained a considerable number of determinations of the ratio **v**, using two different spherical condensers (the same shell in each case, but two balls of different sizes), two different capacities of cylindrical condensers, both with and without guard cylinders, and had made the measurements (both on spheres and cylinders) by two very different means, namely, the Maxwell bridge and the differential galvanometer methods. We had, moreover, varied the conditions in many ways, such as changing the speed of the commutator and so the frequency of the charge and discharge; varying the voltage of the battery; interchanging resistances; varying the galvanometers and other accessory apparatus; measuring both the charge and the discharge; and working in both summer and winter, at higher and lower temperatures (always of course correcting results to a standard temperature of 20°) and with considerable changes in atmospheric humidity. In summer this was kept low enough to get sufficiently high insulation by the use of coils of iron pipe in the room, through which cold calcium chloride brine was circulated, which condensed the surplus moisture from the atmosphere and enabled us to maintain a nearly constant temperature and a moderate humidity. In the exceptional cases when this could not be done observations were discontinued.

6. PLATE CONDENSERS.

In order to make the series of measurements more complete and to have one more chance of detecting any constant error that might possibly still remain in the mean value of v which we were obtaining, we designed and had built in the instrument shop of the Bureau a parallel plate, guard ring condenser.[4] This is not as favorable a form of condenser as the others, and possesses some disadvantages that we did not appreciate when it was designed. We have, however, obtained fair results, which confirm the value for v given by the other condensers, although the values found by the plate condenser are entitled to less weight than those obtained by the spherical condensers and the cylindrical condensers with end guard cylinders.

The value which we have found for the ratio of the electrostatic to the electromagnetic unit, resulting from all our work (carried on without interruption since November, 1904) is

$$v = 2.9963 \times 10^{10}$$

taking the dielectric constant of air as unity. Referring to vacuum this becomes

$$v = 2.9971 \times 10^{10}$$

We believe this result is correct to within 1 part in 10,000, barring any uncertainty in the value of the international ohm. This is more fully discussed below.

II. DESCRIPTION OF THE CONDENSERS.

7. SPHERICAL CONDENSER.

As already stated, the spherical condenser was loaned to the Bureau by the Johns Hopkins University. It was made in 1879 from the designs and specifications of the late Prof. H. A. Rowland. The construction of the condenser is shown in Figs. 1 and 2. It consists of a brass shell provided with leveling screws and divided along its equator so that the upper hemisphere can be removed. The interior of the shell is accurately worked to a spherical surface

[4] We are under obligations to Mr. F. S. Durston for assistance in designing this condenser and the direct reading chronograph described below. The plate condenser was built by Mr. Oscar Lange, chief mechanician of this Bureau.

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and is nickel-plated. At the pole of the upper hemisphere is a hole through which passes one of the charging wires and also the silk

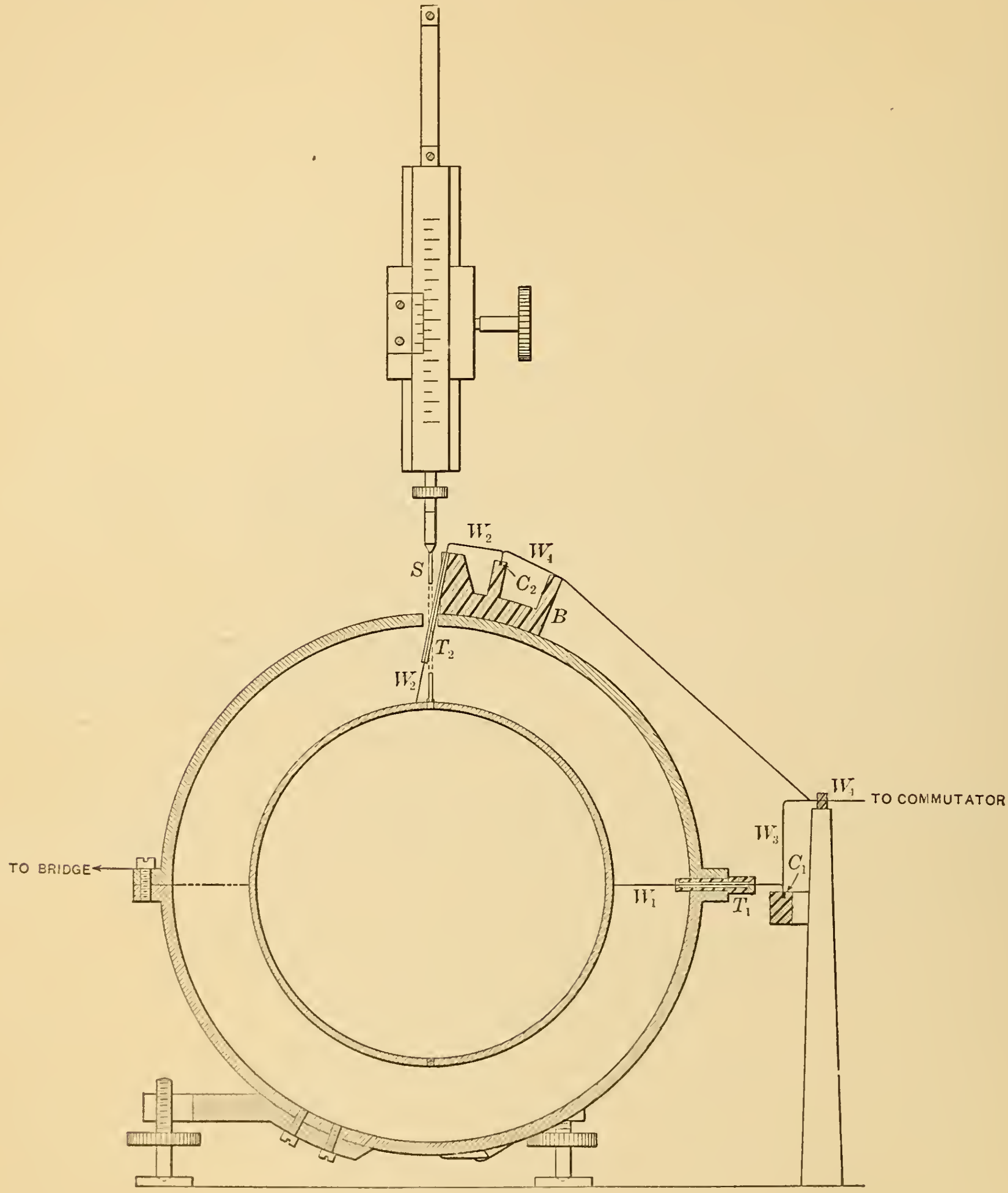

Fig. 2.—*Section of Spherical Condenser.*

The inner sphere is suspended by the silk cord S at the center of the shell. The two charging wires W_1 and W_2 are guided by the fixed tubes T_1 and T_2. The outside ends of W_1 and W_2 dip into the small mercury cups C_1 and C_2. These mercury cups were joined by the wires W_3 and W_4 to the rotating commutator.

cord by which the inner ball is suspended. In the equatorial plane of the shell is a second hole, through which we passed the second

Fig. 1.—*Spherical Condenser and Larger Ball, A.*

The ebonite block which supports the tube that serves as guide for the top charging wire is shown in place, and the upper end of the bent charging wire is shown lying in place along the top of the block and projecting out over the first notch in the ebonite; to this end is soldered a fine wire, not visible in the photograph, which is of such a length as just to dip into a small mercury cup in the top of the middle projection of the ebonite. The end of the bushing that serves as a guide for the side charging wire can be seen on the left.

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charging wire. The inner ball is either one of two carefully spun nickel-plated balls of different radii. These with the shell form two condensers of different capacities. The silk cord supporting the ball, after passing through the hole in the upper hemisphere of the shell, is attached to the lower end of a scale which can be moved vertically in guides with a rack and pinion adjustment. Fastened to one of the guides is a vernier, by means of which the position of the scale can be directly read to a tenth of a millimeter. From the upper end of the scale a strip of thin spring steel passes to one end of a lever to the other end of which is attached a counterweight. The end of the lever to which the spring steel is attached is cut to the arc of a circle with its center at the pivot of the lever. The radius of the circle being equal to the distance of the plane of the scale from the pivot of the lever, the tension upon the scale will always be in the direction of its length whatever the position of the lever, within working range.

8. ADJUSTING THE CONDENSER.

In assembling the condenser the upper rim of the lower hemisphere is first carefully leveled. The ball and the upper hemisphere are then put in place and the ball adjusted by the rack and pinion movement until its center is approximately in the plane of the rim of the lower hemisphere. The upper hemisphere is now raised and propped on blocks so that the ball can be reached along the horizontal plane between the hemispheres, care being taken that the silk suspension of the ball hangs freely through the hole in the top of the hemisphere. The horizontal position of the ball is now tested at four points by means of a brass distance piece resting upon the broad rim of the lower hemisphere. Then by carefully sliding the lower hemisphere one way or another the horizontal centering of the ball can be attained with an error probably not greater than 0.1 mm. The contact between the ball and the distance piece is determined electrically.

This adjustment being completed the upper hemisphere is lowered and fastened in place on the lower one. Then by alternately raising and lowering the ball, by the rack and pinion adjustment, so as to make contact with the top or with the bottom of the shell, contact being determined electrically, it is easy to determine the scale reading to $\frac{1}{20}$ mm corresponding to the central position of the ball. Setting the scale to this reading the condenser is ready for use.

This centering was later tested by measuring the capacity of the condenser in its adjusted position, and then displacing the ball first up and then down by known amounts. The capacity is a minimum with the ball in the center, and increases approximately as the square of the displacement in either direction. This gave a very delicate method of verifying the centering.

The warping of the floor and of the table, occasioned by variations in the humidity, causes the level of the instrument to change more or less. This of course displaces the ball from its central position slightly and so varies the capacity. In the early portion of the work the importance of this source of error was not fully appreciated, but later a delicate level was kept on the brass arm supporting the guides for the movable scale and the level of the instrument was readjusted as often as need be. The magnitude of the error that may be thus introduced is small but appreciable.

The radius of the shell is approximately 12.671 cm and of the balls 10.118 cm and 8.874 cm approximately.

9. THE CHARGING WIRES.

The method of charging the condenser is of great importance and involves the determination of a relatively large correction term. The charging can be done only by means of a wire reaching the ball through a hole in the shell. This wire produces two distinct effects, both of which must be eliminated: (1) It has a relatively large capacity which is thus added to that of the ball; (2) its presence distorts the field of force within the shell. These effects can not be eliminated by any process involving the mere withdrawal in whole or in part of the wire, or by a lowering of the ball, but must be definitely measured and allowed for. Furthermore, the capacity of the wire is greatly affected by slight changes in its position with respect to the shell, especially with respect to the edges of the hole through which it passes. Hence we must rigidly fix the position of the charging wire and then accurately determine its capacity when in this position.

In order to accomplish this we have employed two charging wires (Fig. 2). One extended diagonally through the hole in the top of the shell through which hung the silk thread supporting the ball; the second extended horizontally through a hole in the equatorial plane of the shell. In the latter hole was tightly fitted an ebonite bushing extending 12 mm outside the outer surface of the shell and 5 mm inside the inner surface. The hole through the center of this was wide at the two ends of the bushing, but for about 2 centimeters in its central portion was of such a size as just to allow a copper wire (1.3 mm in diameter) to slide with gentle friction. The enlarging of the hole at the ends of the bushing, as well as the extension of the bushing beyond the surfaces of the shell, were of course intended to improve the insulation of the charging wire. The charging wire used for this side hole consisted of the above-mentioned piece of copper wire, to the inner end of which were soldered a couple of very fine phosphor bronze wires 2 or 3 mm in length. To the outer end of the charging wire was soldered a section of fine copper wire (0.13 mm diameter) bent so that by rotating the charging wire when in position the end of this fine wire could be dipped into a minute mercury cup just large enough to receive it and a similar wire connecting permanently with the rest of the charging system. The phosphor bronze tips allow good contact to be made with the ball without appreciably displacing it from its undisturbed position. The position of the charging wire is fixed by a mark on the wire in the plane of the end of the ebonite bushing.

To the top of the shell was fastened with hard wax a block of ebonite carrying a slender ebonite tube, which extended through the hole in the shell, just clearing the cord supporting the ball and reaching within a few millimeters of the surface of the larger ball when the latter is in position. This tube was made long, on account of the necessity of using a much finer wire, than in the case of the hole in the side of the shell; the hole through this tube was of such a size that a copper wire (0.72 mm diameter) could just pass without danger of jamming. This wire also had fine phosphor bronze tips soldered to its lower end; its upper end was so bent that when the wire was pushed in the tube so that the phosphor bronze tips made good contact with the ball the bent portion was in contact with the top of the ebonite block. To the outer end of this also was soldered a section of fine copper wire of such a length as just to dip into a diminutive mercury cup when the charging wire was in place.

The two small mercury cups were permanently connected with the rest of the charging system by fine wires stretched taut between rigid supports, so as to prevent accidental variations in capacity. The position of the upper charging wire was thus uniquely determined, and its capacity could vary only by a slipping of the ebonite block waxed to the shell; this block did slowly slip as a result of the tension of the wire running from it to the rest of the apparatus, but the variation was not appreciable in a single day, and the capacity of the charging wire was redetermined every time the capacity of the condenser was measured. There was no chance for such a change in the capacity of the side charging wire.

Having definitely fixed the position of the charging wires the next thing was to measure their capacities. This was done by a method of differences. The side wire w_1 being in position and the top one w_2 removed, the electro-magnetic capacity K_1 of the commutator, leads, and ball, was measured as accurately as possible. Then w_2 was placed in position, everything else remaining as before, and the total capacity K was measured. The difference between these two capacities, $K-K_1$, consists of three terms: (1) The capacity of w_2; (2) the change in the capacity of the ball due to the redistribution of charge upon it, resulting from the presence of w_2; (3) the change in capacity of the mercury cup A and of the wire leading to it, produced by the presence of w_2. Call the sum of these three effects c_2, the effective capacity of w_2.

If we now remove w_1, leaving everything else unchanged and measure the resulting total capacity K_2, the difference between this capacity and the capacity (K) with *both* w_1 and w_2 in position will give us c_1, the effective capacity of w_1. This is made up of three terms corresponding exactly with those composing the effective capacity of w_2.

If we now remove both w_1 and w_2 and measure the remaining capacity k, which is made up of the capacity of the commutator and of the wires leading to (and including) the mercury cups C_1 and C_2, then put w_1 in place, everything else remaining unchanged, and again measure the total resulting capacity K_1, the difference of these two capacities, K_1-k, will be exactly equal to the capacity of the ball increased by what we have called c_1 the effective capacity of w_1 and which we have already measured. Thus K_1-k-c_1 is the electromag-

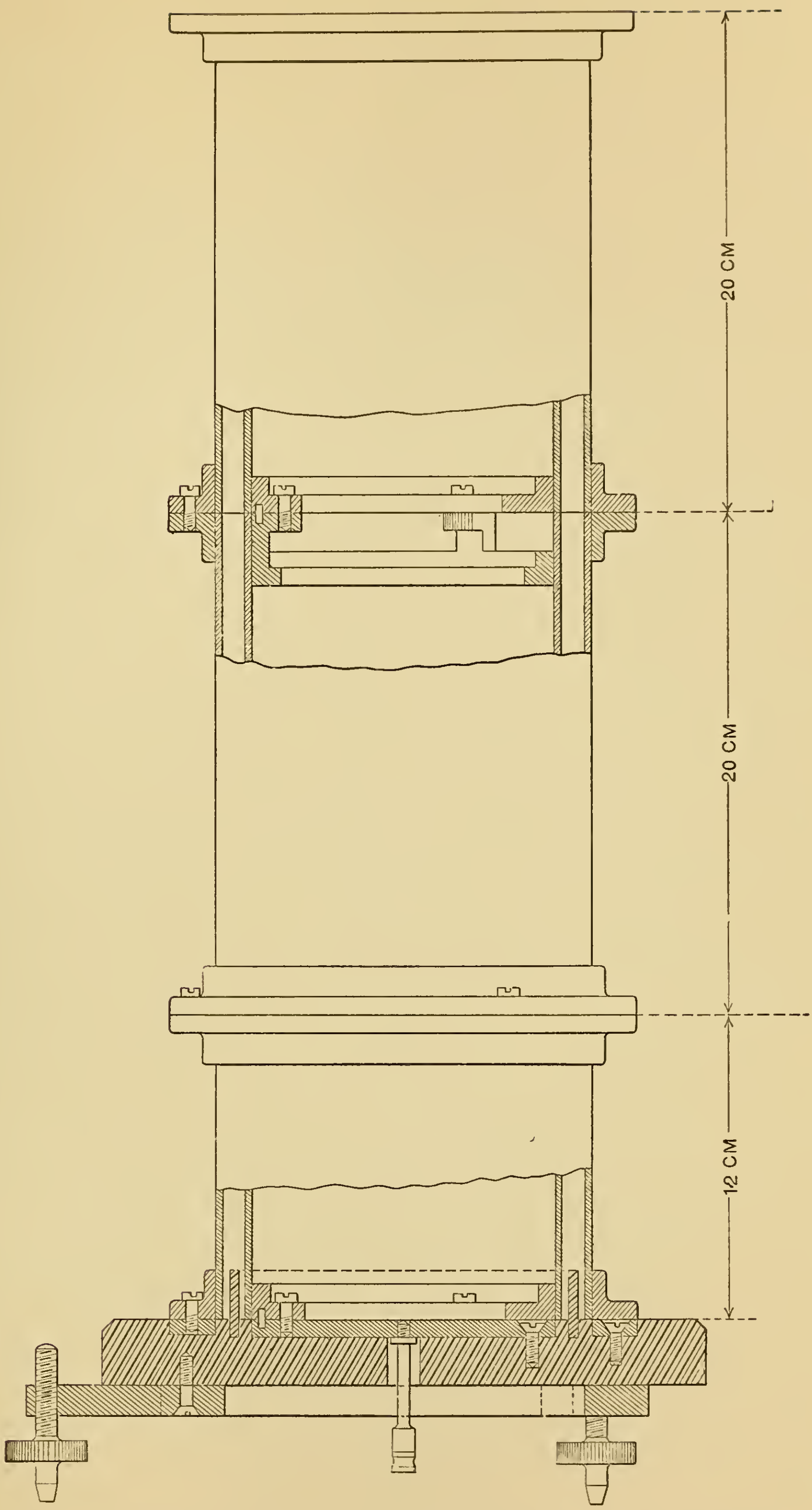

Fig. 3.—*Cylindrical Condenser of Three Sections. No Guard Cylinder.*

netic capacity C_A, or C_B, of the ball *entirely free* from all effects produced by the charging wire.[5]

10. CYLINDRICAL CONDENSER.

As stated above, the cylindrical condenser was built in the instrument shop of the Bureau. The condenser, as first constructed, consisted of three sections—a base section 12 cm long and two other sections each 20 cm long. Later a fourth section, 12 cm long, was made to serve as an upper guard cylinder. The radii of the inner cylinders are, approximately, 6.257 cm, those of the outer ones 7.241 cm. The base of the instrument is a heavy ebonite block mounted on leveling screws. In the top of the block is set a circular brass plate of slightly smaller diameter than the inner cylinder. This plate, to which the lower section of the inner cylinder is attached by screws, entirely closes the bottom of the inner cylinder. In the ebonite base, midway between the two cylinders, is set an ebonite ring, 2 mm thick and 1 cm high, thus making the leakage path between the cylinders about three times as long as it would otherwise have been. Beyond this a brass ring of a slightly larger inner diameter than the outer cylinders is set in the ebonite, and the lower section of the outer cylinder is attached to this ring by screws. Connection to the outer cylinder is made by means of one of the screws fastening it to the base. Connection to the inner cylinder is made by means of a binding post set in the middle of the under side of the ebonite base and connecting with the plate which closes the bottom of the inner cylinder. The various sections of the cylinders are interchangeable and are fastened together by three screws and three steadying pins at each joint. A brass plate of the internal diameter of the inner cylinders, provided with steadying pins and

[5] It has been stated (Abraham, Rapport, Congrès International de Physique, 1900) that Rowland, who used this condenser in his determination of the ratio of the units in 1879, failed to eliminate the capacity of the charging wire. This is only partially correct. As we have seen, the charging wire produces two effects. The portion due to its own capacity was eliminated in Professor Rowland's work, for, after charging the condenser, the charging wire was withdrawn and a second wire was introduced to discharge it. (See p. 267, Physical Papers.) It was the quantity discharged that he measured electromagnetically. On the other hand, the effect of the distortion of the field by the wire was not eliminated. This quantity, however, was very small and need not be considered when an accuracy of not more than 1 in 1,000 is sought.

Fig. 3a.—*Guard Cylinder Condenser.*

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with short projections to rest upon the upper rim of the cylinder, is used for closing the upper end of the inner cylinder when desired.

The cylinders are of brass, silver plated both inside and out. Sections of tubing of the required diameter were cut to the proper

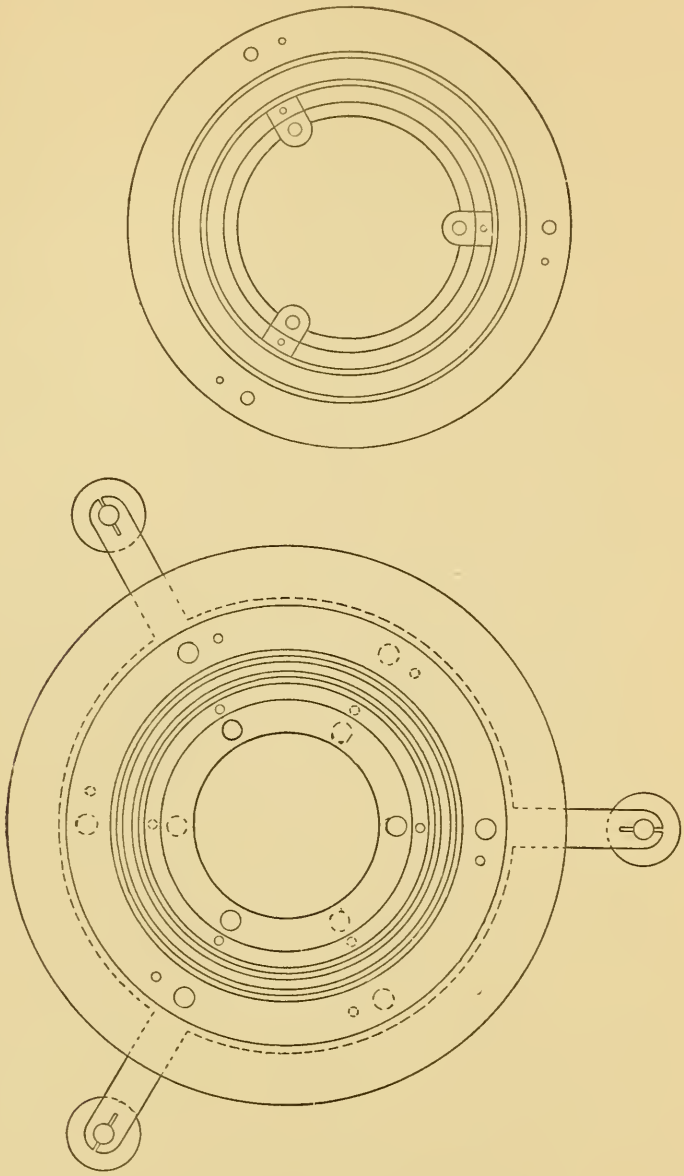

Fig. 4.—*Plan of Base of Cylindrical Condenser, showing Method of Attachment of Cylinders. Plan of End of a Section of the Condenser, showing Location of Screws and of Steadying Pins.*

lengths, and each section was then stiffened by having heavy rings soldered to it near each end, the rings being, of course, placed inside the inner cylinders and outside the outer ones; the screws and steadying pins for connecting the sections pass through the rings.

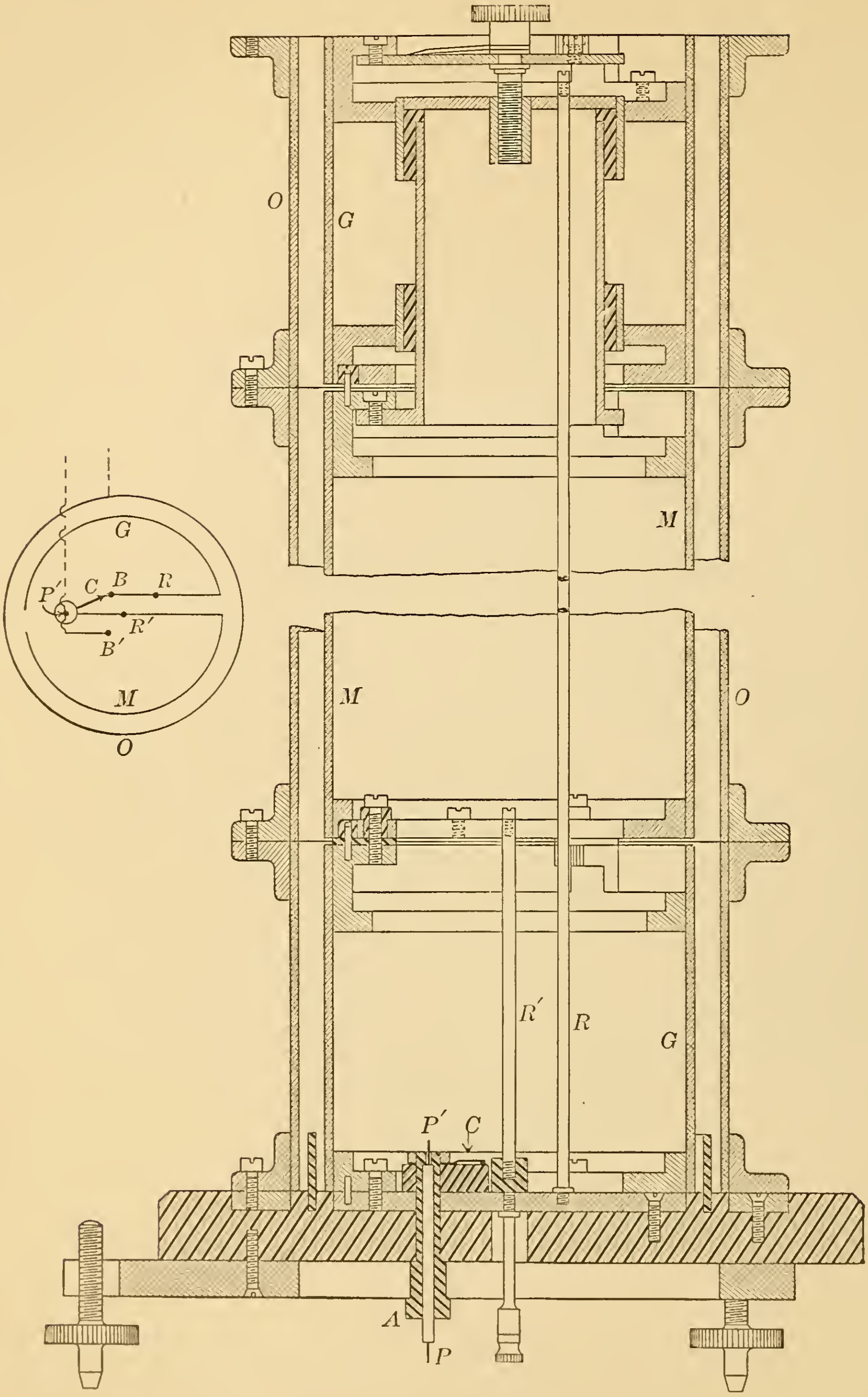

Fig. 5.—*Section of Guard Cylinder Condenser.*

The guard sections G are connected by the rod R, the upper section can be moved by the micrometer screw so as to regulate the width of the gap. The middle section M is connected to the rod R′, which in turn is connected to the spring arm C (small figure), which is rotated by the knob A so as to make contact with either B or B′, thus connecting the middle section either to the guard section or to the lead from the commutator.

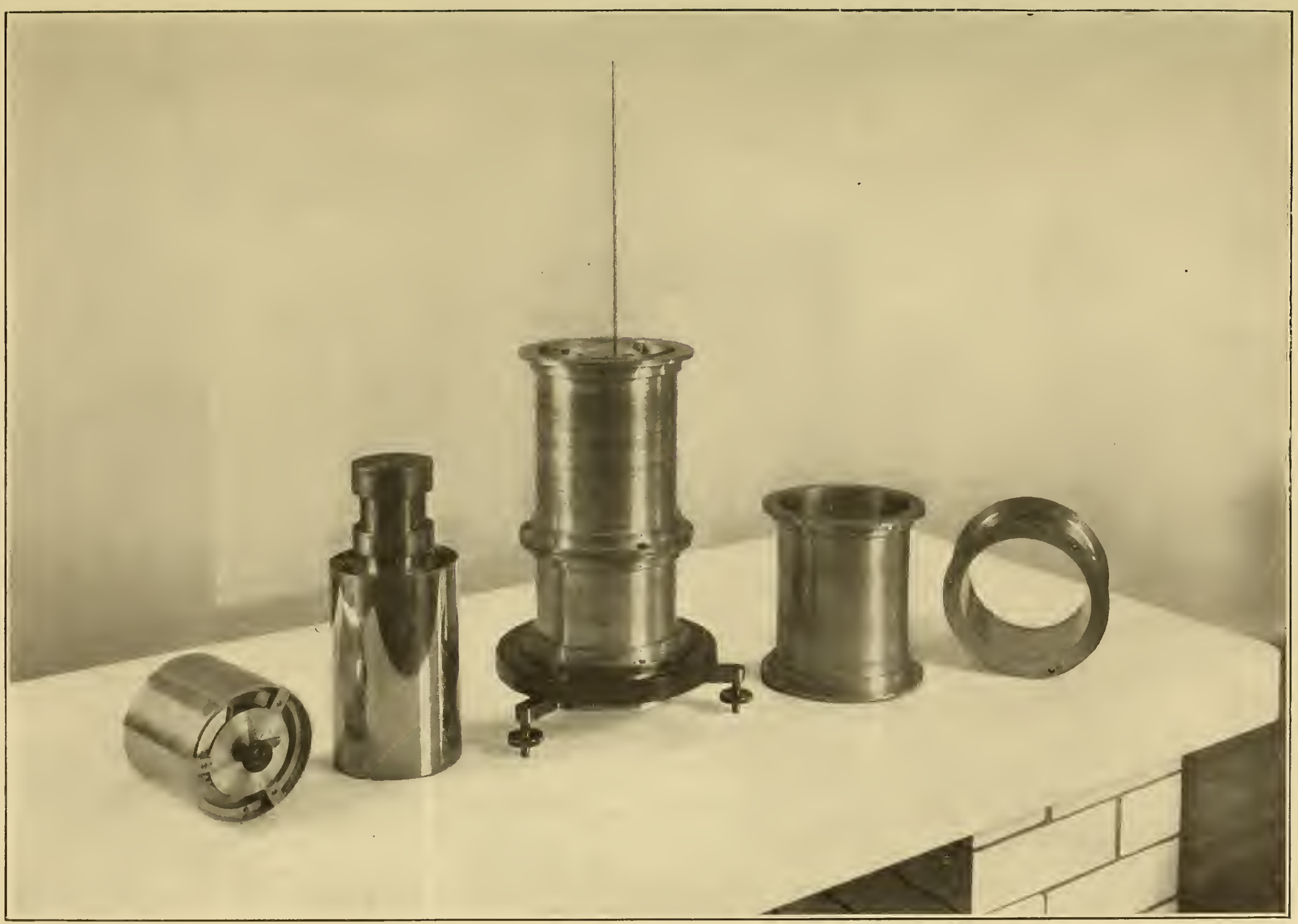

Fig. 4a.—*Cylindrical Condenser Partly Knocked Down.*

On the extreme left is shown the upper cylinder; next to it is seen inner cylinder No. 3 with the attachment which serves to guide the guard cylinder as it is raised or lowered. The corresponding outer cylinders are shown on the right. The rod extending above the lower sections serve to connect the upper and lower guard cylinders.

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The cylinders were then carefully turned to size in a precision lathe, after which they were silvered and buffed down. Each inner cylinder was then placed coaxial with its corresponding outer one, and the ends of the combinations thus formed were ground until the two ends could be closed water-tight by means of flat ground glass plates.

These cylinders were intended to be built up to different heights so that the end effects could be eliminated by a method of differences, according to the plan since outlined by Lord Rayleigh.[3]

11. THE GUARD CYLINDERS.

When it was decided to use guard cylinders, a fourth section was added. These cylinders are of course of the same style as those just described, but the attachment of the inner cylinder is peculiar. To the top of the adjoining section is rigidly fastened a smaller brass tube, attached to which, but insulated from it, is a broad, carefully turned brass ring (Fig. 5). To the top of this tube, but insulated from it, is a second broad ring closed at the top with a heavy plate, in the center of which is a hole 5 mm in diameter threaded to fit a well-made screw of ½ mm pitch. Attached to the inside of the inner guard cylinder are a pair of rings, which pass with almost a piston fit over the broad rings mentioned. This enables the guard cylinder to be moved along its axis without lateral displacement. The top of the guard cylinder is closed by a brass plate having a divided circular scale; through the center of this plate and attached to it by a collar passes the screw of ½ mm pitch of which we have spoken. In this way the cylinder can be raised or lowered by the screw and its exact position can be determined from the position of a pointer attached to the screw and moving over the scale. This is for the purpose of studying the variation of the capacity with the width of the gap between the cylinder and its guard cylinder. The guard cylinder is prevented from turning with the screw, as it is raised and lowered, by means of a steadying pin attached to the adjoining cylinder and passing through an ebonite block. The lower guard cylinder was insulated from its neighbor by means of three ebonite plates 1.6 mm thick placed between the rings attached to the inside of the cylinders. The rings were turned down until when the blocks are in place the gap between the two sections of the cylinders is 0.6 mm. The screws and steadying pins for holding these sections together are insulated from the upper section by means of ebonite bushings. The ebonite plates do not come within 3 mm of the outer face of the cylinders; the bushings for the screws and pins are still farther back.

For charging the guard-ring condenser a key of the form shown in Fig. 5 was employed.

12. SWITCH AND CHARGING WIRES.

The lead from the commutator is stretched taut and attached to an ebonite post, screwed to the table and vertically below the point P (Fig. 5). From this post it is stretched tightly to the point P, to which it is soldered, forming thus a prolongation of PP′, so that when PP′ is rotated by means of the ebonite head A the wire will be twisted about its axis, but will not be displaced with reference to other objects. P′ is connected to the button B′. The spring arm C which rotates with PP′, but is insulated from it, is connected with the rod R′, which is connected with the middle section M of the inner cylinder. The other button, B, with which C can be brought into contact, is connected with the rod R, which connects the two guard cylinders G.

Hence when C rests upon B′ the middle sections of the inner cylinder are connected through PP′ with the lead to the commutator, and the capacity measured is that of the commutator, the lead, and the condenser. On the other hand, when C rests on B the button B′ is insulated, and the middle sections of the inner cylinder are connected with the guard cylinders, and so are charged to the same potential as before, but by way of the guard-ring section of the commutator. Hence the capacity now measured is that of the commutator and leads only. The capacity of the latter must be the same as it was in the former case, for the only change that has been made is the rotation of C from B′ to B, and these portions of the system are entirely inclosed by the inner cylinder, all of which is at the same potential in both cases. They are, therefore, always uncharged, and so contribute nothing to the capacity in either case. The leakage from B′ to B and to C was frequently tested, and in all cases was found to be negligible.

13. PLATE CONDENSER.

This condenser was of the guard-ring type, and, as already stated, was built in the instrument shop of the Bureau. The details of its construction are shown in Fig. 7. The top of the instrument con-

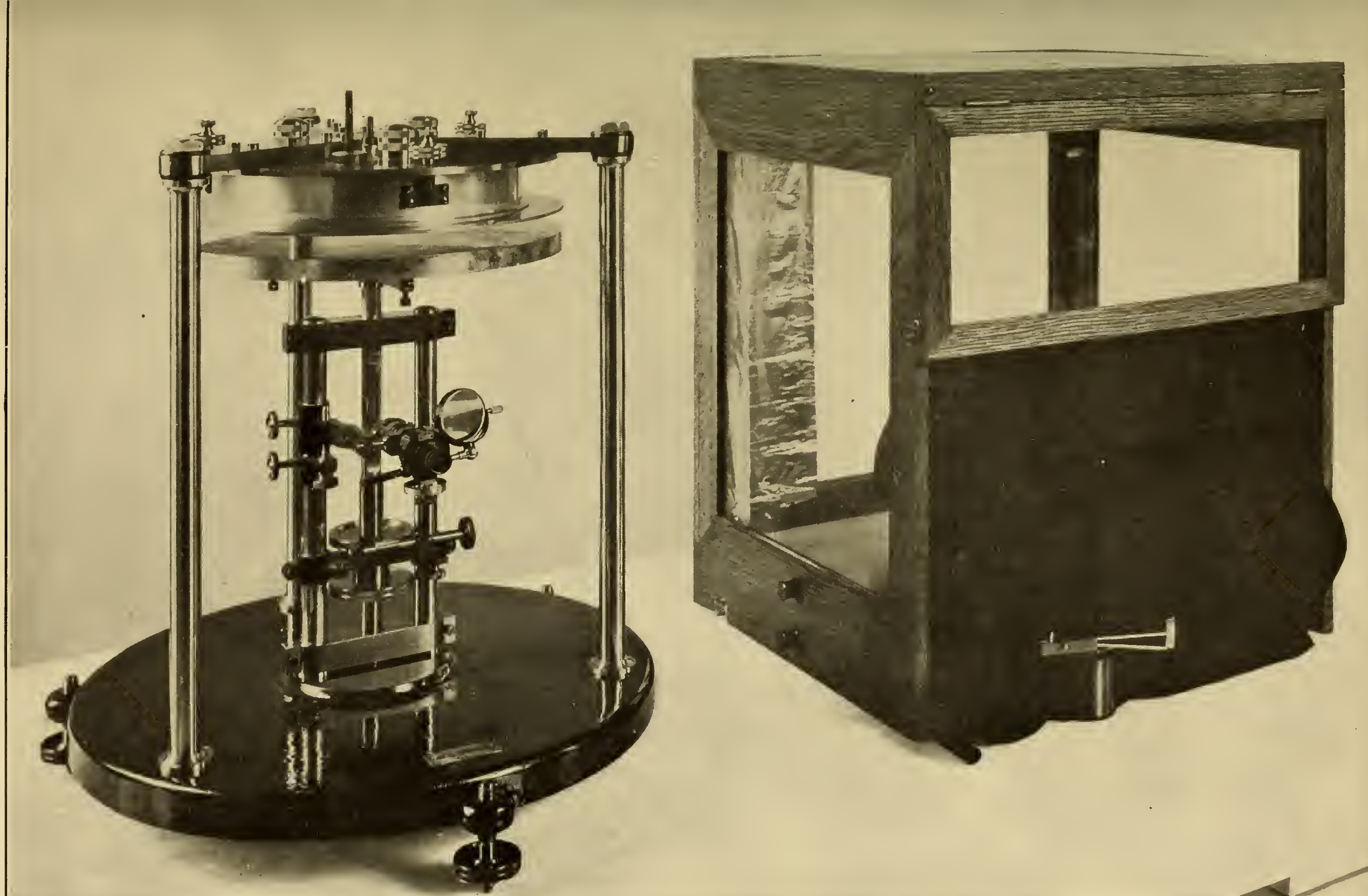

Fig. 6.—*Plate Condenser.*

To the right of the condenser is the case which fits over the condenser and within which the temperature can be maintained constant by electrical heating. Part of the front of the case is closed by a black felt curtain through which projects the eye piece of the reading microscope, and in which are two slits through which the hands can be passed for the purpose of moving the lower plate of the instrument. The back left-hand corner of the case is covered on the inside with tin foil, which is earthed so as to make the field around the leads (which pass up this corner of the case) as constant as possible. In front of the case may be seen the lever for adjusting the plates.

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sists of a brass plate 6 mm thick and 30 cm in diameter; this is supported on three steel columns by means of heavy brass blocks 3.5 cm wide and 1.2 cm thick, screwed securely to the top of the plate and extending nearly to its center. The blocks are insulated

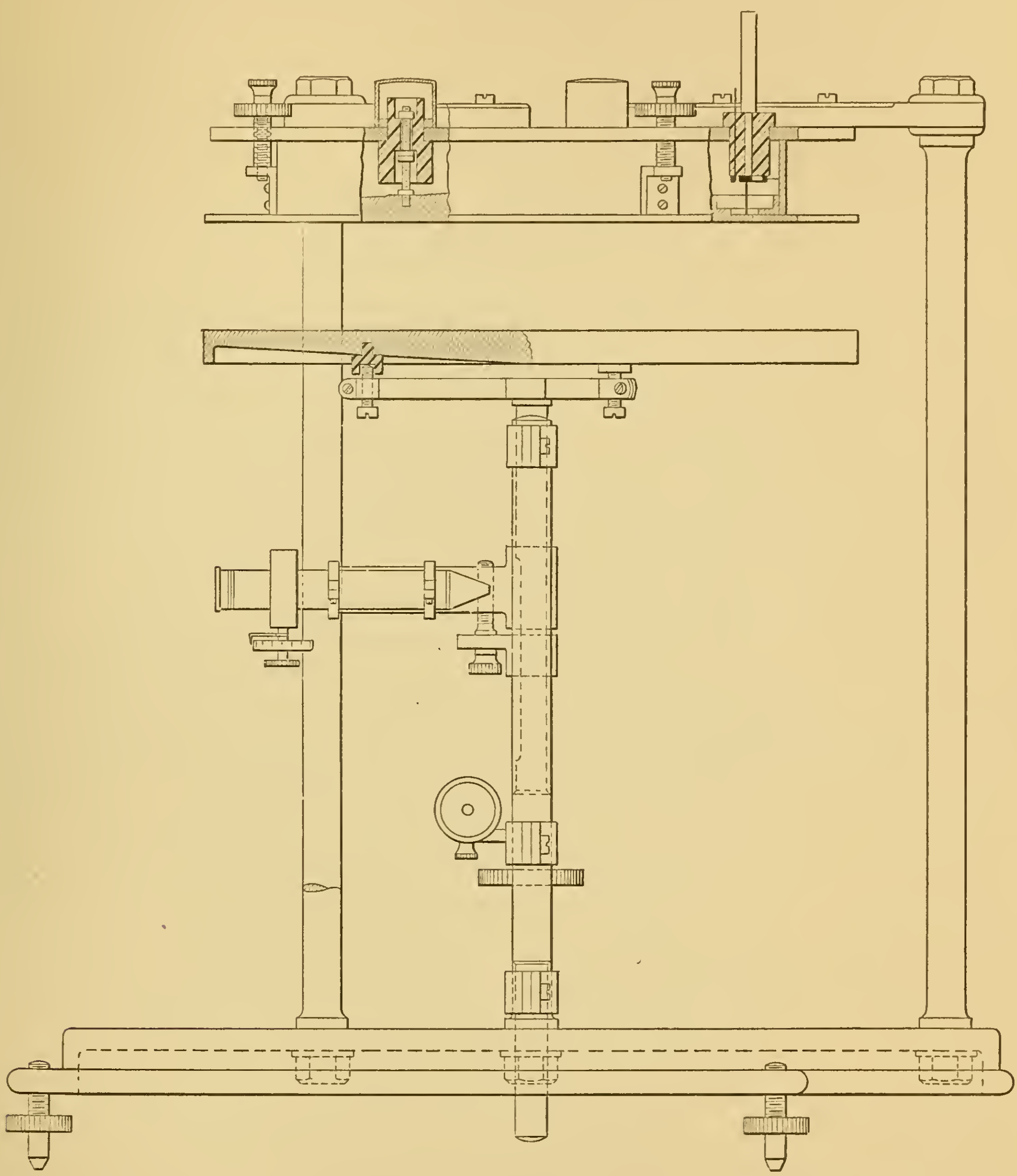

Fig. 7.—*Plate Condenser showing Manner in which the Guard Ring and Plates are Supported and Adjusted, Switch for Connecting the Collector Plate to either Guard Ring or Charging Wire, and the Microscope for Determining the Position of the Lower Plate.*

from the steel pillars by ebonite bushings. By means of three screws passing through ebonite bushings in this plate, the insulated plate of the condenser is supported below it. Lock nuts make the position of this plate definite. Three other lock-nut screws passing through the top plate support the guard ring and enable us to adjust the latter to the plane of the insulated plate. Attached to the top of the guard ring is a short cylinder having a radius of about 3 cm less than that of the ring and of such a height as to reach nearly to the top plate of the instrument; it falls short of reaching the top plate by an amount just sufficient to allow the proper adjustment of the guard ring. This cylinder, with the top plate of the instrument, completely screens the top face of the insulated plate.

14. VARYING THE CAPACITY.

The other plate is supported under these on ebonite blocks on the three leveling screws of an inverted tripod. The tripod is screwed rigidly to a steel rod 1.5 cm in diameter. This rod has at its lower end a screw of 0.5 mm pitch and passes through guides so that it can be moved slowly and accurately along its own axis. To it is fastened a fine silver scale which can be read by means of a micrometer microscope rigidly attached to the base of the instrument.

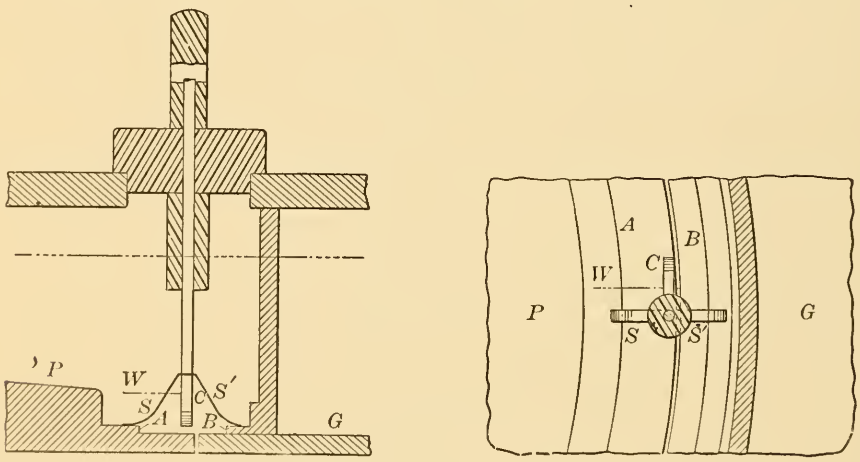

Fig. 8.—*Switch for Connecting Collector Plate P with either the Guard Ring G, or with the Wire W Connected with the Commutator.*

As shown the plate and ring are connected; by rotating the switch 90°, C makes contact with A and the plate is connected to the commutator.

By this means we can vary the distance between the plates by accurately known amounts. The entire instrument is placed inside a glass case in which the temperature can be controlled. The plates are of silvered brass and are ground plane; their backs are heavily ribbed so as to give them great rigidity. The instrument is provided with two collector plates of slightly different radii, so that two widths of guard ring gap can be employed.

15. SWITCH AND CHARGING WIRES.

The condenser is charged by means of a special key shown in Fig. 8. The wire from the commutator is stretched tightly to a stiff wire passing through an ebonite bushing set in the center of the top plate of the instrument. From the lower end of this wire, which is within the space entirely inclosed by the top plate of the condenser, the collector plate, and the guard ring, a spiral W of fine double-silk covered copper wire goes to the point C of the key. The key is situated immediately over the gap between the ring and the plate; it passes through an ebonite bushing in the top plate and is actuated by means of an ebonite handle. The guard ring and the collector plate are cut away near their edges, as shown in the figure, so that when the key is turned in the proper way and depressed the spring C makes contact with the plate, while the ends of the spring SS′ lie in the gap between A and B. In this position the plate and the guard ring are separately charged. If now the key be turned by 90°, the spring C will lie in the groove between A and B, while the other spring SS′ will touch A and B, thus connecting the plate and the guard ring; both will now be charged by means of the guard ring lead. The only portion of the lead to the collector plate which differs in the two cases is the portion of the key which is entirely surrounded by the collector plate, guard ring, and top plate of the condenser. Being in a region of uniform potential, this portion is uncharged. Hence the capacity of the main lead is the same in the two positions, and the difference in the capacities measured with the key in the two positions is exactly equal to the capacity of the collector plate of the condenser. Later a key similar to that employed for the cylinders was used.

16. ADJUSTING THE CONDENSER.

To adjust the condenser, the collector plate and guard ring are first adjusted, so as to lie approximately in a plane parallel to the top plate of the instrument and with the center of the collector plate coinciding with the center of the guard ring. When they are parallel to the top plate of the instrument they will be normal to the direction of motion of the rod carrying the lower plate. The centering of the plate is tested by measuring the width of the gap between the plate and the ring at three points by means of a micrometer microscope, which can be placed in any one of three holes in the top plate and over the gap. The coincidence of the planes of the plate and of the ring was tested by a delicate lever, whose base rested on the lower plate of the condenser and could be slid so that the shorter arm of the lever touched either the plate or the ring. In this way the planes of the plate and of the ring could be brought into coincidence to within 0.005 mm., which suffices to reduce the error due to this cause to 3 parts in 10,000 in the most unfavorable case. Having adjusted the plate and the ring, the nuts on the screws were tightly locked and the upper plate with the collector plate and guard ring attached was removed from the steel pillars supporting it, turned upside down, and the width of the gap between the ring and the plate in the plane of the face of the condenser was measured at several equidistant points. The top plate with its attachments is then replaced and the lower plate of the condenser is raised to a suitable position, and the short arm of an accurate steel square is rested on it, and the distance between its supporting rod and the pendant long arm of the square is measured at two points as far apart as possible. This is repeated with the short arm of the square in a position at right angles to its former position. These measurements afford sufficient data to enable us to determine the angle the rod makes with the plane of the plate and to correct our results for it, if necessary. The coincidence of the planes of the ring and of the plate was occasionally tested.

When in use, the various holes in the top plate of the condenser, whether closed with ebonite bushings or not, are closed with metal caps having only the openings necessary to admit the leads.

No attempt is made to measure directly the true distance between the plates. Instead of this the plates are placed close together, say, 1 mm apart, and the capacity is measured. From this and an approximate value of **v** the true electrical distance between the plates is calculated. The distance between the plates is now increased by a measured amount and the capacity is again measured. From this capacity and the total distance between the plates (the small electrical distance calculated plus the measured increase in the distance) a more approximate value of the ratio is calculated. If necessary, this better value of **v** may be used to recalculate the initial distance, and so a second approximation may be obtained. The capacity is actually measured for many different distances between the plates.

III. ELECTROSTATIC CAPACITIES.

17. SPHERICAL CONDENSER—GRAVIMETRIC METHOD.

The electrostatic capacity of a condenser bounded by two concentric spherical surfaces of radii R and r, taking the dielectric constant of the medium as unity, is

$$C=\frac{Rr}{R-r}$$

It is necessary, therefore, to determine accurately the mean interior radius of the hollow spherical shell, and the mean radii of the two spherical balls that are used in turn within the shell. The determination of these radii was made by two different methods. In the first, or gravimetric method, the volume of the shell is determined by ascertaining the mass of water it contains at a particular temperature, and the volume of the balls by finding the mass of water they displace when submerged. In the second method measurements of their diameters are made directly on the balls. On the shell accurate direct measurements can not be made. We have a check, however, on the gravimetric determination of the radius of the shell in the fact that if it were appreciably in error the results by the two balls would differ, since the error would affect the capacity when using the large ball more than when using the small one. The close agreement of the results by the two balls shows that there could be no serious error in the radius of the shell as determined by the gravimetric method.

a. **The Shell.**—The two hemispheres of the shell were screwed together in the position in which they were to be used in the determination of the electromagnetic capacity, and the joint, which was practically water-tight, was made perfect by means of a very little white lead around the joint on the outside of the shell (1904) or by means of a very little vaseline spread on the outermost edges of the flanges before the hemispheres were put together (1905). The shell thus prepared was weighed by the method of substitution. It was then filled with distilled water, care being taken to remove as completely as possible the last traces of air bubbles; this was done by

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tipping the shell rather violently from side to side and by scraping the inner surface of the filled shell with a bent copper wire. After the shell had been filled level with the top of the hole in the upper hemisphere, it was weighed by the substitution method. A little of the water was then removed by means of a pipette, and the shell was refilled as before and again weighed. Finally the shell was emptied and dried and the process repeated. The temperature of the water was determined just before the completion of each filling, and again after each weighing. From these weighings the volume is computed and from this the radius is determined. The shell and weights both being of brass the correction to vacuo in the case of the empty shell is zero. A specimen determination of the volume of the shell is given below.

June 18, 1904.

Dry Bulb 22°4 Barometer 76.36 cm; at 25°9
Wet " 20°6 −.79 " correction for temperature, humidity, etc.
75.57 " corrected pressure.

Shell filled, $t = 20°77$, weighs 26 lbs. + 8596.631 g
" empty " 26 lbs. + 98.869 "

Apparent weight of water = 8497.762 "

Approximate volume of water = 8520 cc
" " of weights = 1010 "

Resultant volume air displaced = 7510 "

Log dens. air at 22°4, 1 mm press. = $\bar{6}.1965$
Log 755.7 = 2.8783
" 7510 = 3.8756

Log weight w of air displaced = 0.9504 ∴ $w = 8.920$ g

∴ Mass of water = 8506.68 g

Log 8506.68 = 3.9297601
" D at 20°77 = $\bar{1}.9991627$

" Vol. at 20°77 = 3.9305974 ∴ Volume = 8523.097 cc
" $\frac{4}{3}\pi$ = 0.6220886
" R^3 = 3.3085088
" R = 1.1028363 ∴ $R = 12.67174$ cm at 20°77
0.00018 cor. to 20°0
$R_{20} = 12.67156$ cm

As a further check one determination of the volume of the inter-space between the larger ball (ball *A*) and the shell was made by an analogous method. This gave a value very closely agreeing with the other results.

b. **The Balls.**—A similar method was adopted for determining the radius of the balls. From the difference in their apparent weights when in air and when suspended in water, their volumes are determined and from these their mean radii are obtained. Owing to the fact that the balls are hollow they float in water; hence a sinker has to be attached in order to submerge them. In 1904 the sinker was attached to a wire harness inclosing the ball, but in 1905 it was fastened to a small hook screwed into the bottom of the ball. The weight of the ball in air was determined by the method of double weighings, and the arms of the balance were found sufficiently equal to necessitate no correction for the weighings in water. The method of reduction is shown below.

(1) *Weight of Ball B in Water.*

April 11, 1905.

Air temperature 21°.1 C Barometer 74.30 cm
Hydrograph 50 per cent −.38 " correction
 = 9.3 mm press. 73.92 " corrected pressure.

(ball *B* + sinker, both in water at 21°.05) + wire	=	54.853 g
(0.041 cc more wire + sinker) in water + wire	=	651.979 "
(ball *B* − 0.041 cc) in water at 21°.05	= −	597.126 "
Correction for 0.041 cc	= −	0.041 "
Correction for air displaced by 597 g of brass	= +	0.083 "
True weight of ball *B* in water at 21°.05	= −	597.084 "

(2) *Weight of Ball B in Air.*

Air temperature 21°.0 Barometer = 74.10 cm
Hydrograph 50 per cent −.37 " correction
 = 9.2 mm press. 73.73 " corrected pressure

Approximate volume	=	2927 cc
Vol. Weights	=	275 "
Difference	=	2652 "

Weight of ball *B* in air	=	2321.144 g
Weight of 2652 cc of air	=	3.089 "
Weight in vacuo of ball *B*	=	2324.233 "
" " water at 21°.05	=	597.084 "
Loss of weight in water	=	−2921.317 "

Log 2921.317 $= 3.4655787$

" *D* at 21°.05 $= \bar{1}.9991360$

" Vol. at 21°.05 $= 3.4664427$ ∴ Volume at 21°.05 = 2927.134 cc

" $\frac{4}{3}\pi$ $= 0.6220886$

" r^3 $= 2.8443541$

" r $= 0.9481180$ ∴ $r = 8.87397$ cm at 21°.05

0.00017 correction to 20°.0

$r = 8.87380$ cm at 20°.0

The weights used were compared with the standards of this Bureau and the necessary corrections have been applied. The thermometer used in 1904 had its corrections determined just before it was used, but owing to an oversight that used in 1905 was not standardized until 14 months after it was used. This may explain the slight difference between the two sets of results shown in the following summary; a rise of zero of 0°.11 C between the time the thermometer was used and the time it was standardized will account for the difference.

18. SUMMARY OF DIMENSIONS.

Shell; radius at 20°		Ball A; radius at 20°		Ball B; radius at 20°
1904	1905	1904	1905	1905
12.67156 cm	12.67134 cm	10.11799 cm	10.11794 cm	8.87380 cm
12.67151	12.67118	10.11811	10.11787	8.87378
12.67169	12.67149	10.11814	10.11790	8.87381
12.67159[6]	12.67140	10.11808		8.87380
	12.67151			
	12.67143			
	12.67143			
	12.67140			

It will be noticed that the 1905 values are a little lower than the values of 1904, and by nearly the same relative amount for both ball *A* and the shell.

If we assume the thermometer used in 1905 changed 0°.11 between the time it was used and the time it was standardized the values for 1905 become:

Shell = 12.67158 cm; ball *A* = 10.11806 cm; ball *B* = 8.87391 cm.

From the above values of the radii we obtain the following values for the capacities:

Ball *A* and shell as determined in 1904		C = 50.2102 cm
" " " " " " 1905		C = 50.2087 "
	Mean	= 50.2094 "
" " " " " " "	(Assuming change in thermometer)	C = 50.2098 "
Ball *B* " " " " " "		C = 29.6091 "
" " " " " " "	(Assuming change in thermometer)	C = 29.6093 "
" as determined in 1905; shell as determined in 1904		C = 29.6080 "

The last is evidently an unfair combination, involving as it does a differential error in the thermometers; but even this value of the capacity differs from the others by only 4 parts in 100,000, which corresponds to an error of 2 parts in 100,000 in **v**. Hence, if we assume that the capacities of the condenser with the two balls are as follows:

With ball *A*, C = **50.2095** cm at 20° C
" " *B*, C = **29.6092** " " "

we shall probably not be in error by more than 2 in 100,000 in the electrostatic capacity, and this error would affect the value of **v** by only 1 in 100,000.

The possible effect of the two small holes in the shell (through which the charging wires pass, and through one of which the cord for suspending the ball passes) will be discussed below.

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[6] The following values of the radii were found in 1889 (E. B. R., loc. cit.):

Ball *A* = 10.1180 cm at 17°.0 C = 10.1185 cm at 20°.0 C
Ball *B* = 8.8735 " " 16°.5 C = 8.8741 " " "
Shell = 12.6805 " " 18°.4 C = 12.6811 " " "

These differ from those values just given by 5μ for ball *A*, 3μ for ball *B*, and 96μ for the shell. The differences in the case of the balls are less than the error allowed in the former work; the large difference in the case of the shell is due to the fact that before beginning the present work a few elevations on the flanges of the hemispheres were ground down so that the two hemispheres should make good contact all around the circumference, thus reducing the mean radius of the shell.

In the above reduction D is the specific gravity of water referred to water at 4°0 C as unity as given by Chappuis.[7] If we assume that the density of water at 4°0 C is 0.999955 as given by Guillaume,[8] then the volumes as given above are too small by 45 parts in 1,000,000, the electrostatic capacities are too small by 15 parts in 1,000,000, and the value found for **v** by using the assumed capacities will be too small by 7 parts in 1,000,000.

19. SPHERICAL CONDENSERS—DIRECT MEASUREMENTS.

In order to obtain an idea as to the figuring of the balls, and to get an independent check on the dimensions determined gravimetrically each ball was measured along 50 diameters. For this pur-

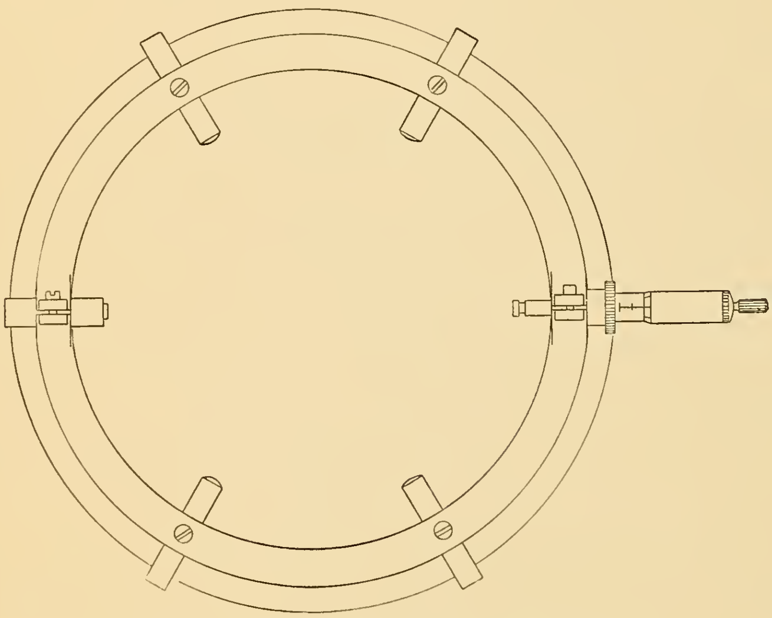

Fig. 9.—*Ring Caliper for Measuring Spheres.*

The four guides are adjusted so that the micrometer screw and its anvil lie along a diameter of the ball and the ball is just free to turn.

pose a ring caliper such as is shown in Fig. 9 was used. It was so constructed that when it and the ball to be measured were in position and resting upon a flat surface the micrometer screw and its

[7] Dilatation de l'Eau, Travaux et Memoires du Bureau International, vol. 13, pp. 1–40; 1904.

[8] Rapports Int. Cong. Paris, 1900, vol. 1, p. 99.

anvil lay in the equatorial plane, and the four guides lay along a circle of latitude of the ball. The guides were so adjusted that the ball could just turn freely, and so that the screw and its opposing anvil lay along a diameter. Diameters were measured every 18° around the equator, giving 20 measurements of 10 different diameters. Similar measurements were taken along four meridian circles differing in longitude by about 45°. Taken in this manner the diameters are not equally spaced over the surface of the sphere but are crowded at the poles and spread out at the equator, the spacing being inversely proportional to the sine of the polar angle. Hence in obtaining the mean diameter we must give each diameter a weight proportional to the sine of its polar angle. No measurement is taken nearer than 9° to the pole. In the tables below are given the differences "Diameter — End Standard" (in millimeters) for the various diameters measured; these have been corrected for a slight variation in temperature during the measurement.

The mean diameter of ball *A* at 24°.6 is equal to the length of the end standard plus the difference between it and the diameter of the ball, this difference being (as shown on the next page) 0.0609 mm the mean diameter of ball *A* is

$= 20.23132 + 0.00609 = 20.23741$ cm

0.00167 " $=$ correction to 20°.0

Mean diameter ball $A = 20.23574$ " at 20°.0

" radius " $= 10.11787$ " " "

This differs from the value found gravimetrically by about 2μ, which is very fair agreement. For this ball the range is from 0.122 to $-0.031 = 0.153$ mm. This is a rather large quantity, but will not affect the final result appreciably.

TABLE I.

Measurements of Diameters of Ball A at 24°.6; Temperature of Standard 25° C.

Meridian position	Polar distances ϕ					
	9° 171° 189° 351°	27° 153° 207° 333°	45° 135° 225° 315°	63° 117° 243° 297°	81° 99° 261° 279°	Equator
0°	0.107 0.111 0.096 0.110	0.043 0.109 0.031 0.115	0.069 0.102 0.045 0.114	0.032 0.073 0.041 0.115	0.028 0.051 0.072 0.098	0.047 0.068 0.090 0.100 0.101
45°	0.087 0.120 0.049 0.038	0.103 0.118 0.100 0.119	0.076 0.122 0.071 0.115	0.070 0.107 0.072 0.113	0.074 0.099 0.100 0.110	0.086 0.030 0.001 −0.028 −0.020
90°	0.041 0.032 0.108 0.020	0.118 0.103 0.059 0.108	0.071 0.080 0.040 0.078	0.054 0.078 0.055 0.079	0.040 0.062 0.058 0.070	0.021 0.067 0.098 0.101 0.053
135°	−0.010 0.027 0.012 −0.025	0.102 0.060 0.101 0.064	0.106 0.022 0.067 0.029	0.069 −0.020 0.049 −0.010	0.020 −0.019 −0.005 −0.024	0.012 −0.015 −0.031 0.021 0.040
Mean δ	0.0577	0.0908	0.0754	0.0611	0.0521	0.0421
sin ϕ	0.1564	0.4540	0.7071	0.8910	0.9877	1.0000
δ sin ϕ	0.0090	0.0412	0.0533	0.0544	0.0515	0.0421

$\Sigma \sin\phi = 4.1962$; $\Sigma\, \delta \sin\phi = 0.2515$

$$\therefore \frac{\Sigma\, \delta \sin\phi}{\Sigma \sin\phi} = \frac{0.2515}{4.1962} = 0.0600 \text{ mm}$$

0.0009 " = cor. to bring standard to 24°.6

0.0609 "

TABLE II.

Measurements of diameters of Ball B, at 23°.4 C. The values given in the table are the differences δ between the diameters and the length of the end standard.

Meridian positions	Polar distances ϕ					
	9° 171° 189° 351°	27° 153° 207° 333°	45° 135° 225° 315°	63° 117° 243° 297°	81° 99° 261° 279°	Equator
	+0.010	+0.007	+0.012	+0.003	−0.020	−0.022
	−0.014	−0.034	−0.053	−0.090	−0.043	−0.035
0°	+0.002	+0.011	+0.014	+0.017	−0.010	−0.047
	+0.022	−0.009	−0.051	−0.074	−0.045	−0.053
						−0.054
	+0.013	+0.020	+0.018	−0.004	0.000	−0.055
	+0.009	+0.019	−0.020	−0.022	−0.017	−0.061
45°	+0.018	+0.018	−0.007	+0.001	−0.018	−0.073
	+0.010	+0.019	+0.008	−0.020	−0.027	−0.073
						−0.054
	+0.008	+0.013	+0.015	−0.006	−0.005	−0.031
	+0.013	−0.008	−0.016	−0.040	−0.026	−0.029
90°	+0.016	+0.010	+0.012	−0.007	−0.005	−0.052
	+0.018	+0.005	−0.013	−0.032	−0.026	−0.062
						−0.052
	+0.009	+0.003	+0.010	−0.009	−0.027	−0.072
	+0.014	−0.027	−0.039	−0.054	−0.025	−0.091
135°	+0.009	+0.008	+0.010	−0.017	−0.022	−0.057
	+0.010	+0.026	−0.022	−0.039	−0.054	−0.035
						−0.028
Mean δ	+0.0104	+0.0050	−0.0076	−0.0246	−0.0231	−0.0518
sin ϕ	0.1564	0.4540	0.7071	0.8910	0.9877	1.0000
δ sin ϕ	+0.0016	+0.0023	−0.0054	−0.0219	−0.0228	−0.0518

$$\Sigma \sin\phi = 4.1962; \quad \Sigma\, \delta.\sin\phi = -0.0980$$

$$\therefore \frac{\Sigma\, \delta.\sin\phi}{\Sigma \sin\phi} = \frac{-0.0980}{4.1962} = -0.0234 \text{ mm.}$$

Hence mean diameter of ball B at 23°.4 C

= 177.5 mm end standard − 0.0234 mm

= 17.75116 − 0.00234 = 17.74882 cm

0.00109 " correction to 20°.0 C

Mean diameter of ball B = 17.74773 " at 20°.0

∴ " radius " " " = 8.87386 " " "

This agrees with the value found by the gravimetric method to within 0.6μ, which is as close an agreement as can be expected. The extreme range of the settings is from −0.091 to +0.026 = 0.117 mm. This is not large enough to affect our results appreciably.

We regard the results by the gravimetric method as much more reliable than the direct measurements, and hence take the latter only as a check on the others.

It is not possible to study the figuring of the shell with the same thoroughness as we have the balls, but by bringing the ball into contact with the top of the inner surface of the shell and then lowering it until it touches the bottom of the shell we can measure the difference between the polar diameter of the ball and the vertical diameter of the shell. By this method we find—

Vertical diameter of shell =	25.326 cm
Mean " " " =	25.343 "
Difference =	0.017 cm

We shall see later (page 476) that this error in figuring will not affect the final result.

20. CYLINDRICAL CONDENSERS—GRAVIMETRIC METHOD.

The electrostatic capacity of a condenser, consisting of two coaxial circular cylinders of radii R and r and length l, taking the dielectric constant of the medium as unity, is

$$C = \frac{l + \delta l}{2 \log_e \frac{R}{r}} \qquad (1)$$

where δl is the correction to the length due to the end effect. If end cylinders are used as guard cylinders the value of δl can be calculated, and by varying the gap between the middle cylinder and the guard cylinders the calculated value of δl can be verified experimentally. This we have done. If there are no guard cylinders, δl can be eliminated by varying the length l, *provided care is taken to make the end conditions identical for the two cases*, so that δl will have the same value in each case. This is a very important matter, and involves many precautions.

In order to calculate the capacity C with high accuracy it is of course necessary, in addition to ascertaining or eliminating δl, to determine l, R and r with corresponding accuracy. Since

$$C=\frac{l}{2\log_e\frac{R}{r}}=\frac{lr}{2(R-r)} \text{ approximately}$$

it is evidently necessary to get the difference of the radii $R-r$ with the same precision that is required for r. This makes it undesirable to have this difference very small. If it is 1 cm, this difference must be known to 0.5 micron to give C to within 1 in 20,000. As in the case of the spherical condenser, the radii were determined by two independent methods, (1) by gravimetric observations and (2) by direct measurements.

It ought to be stated that at the time these cylindrical condensers were built the Bureau instrument shop possessed no grinding machine, and the cylinders were not as accurately turned as could be wished. We had not at that time set so narrow a limit of tolerance as we came to later, and hence what was satisfactory then was not entirely satisfactory a year afterwards. We have, however, taken so large a number of measurements on the cylinders that we believe the mean values obtained are thoroughly trustworthy.

The cylinders were ground on their ends in pairs, so that when placed one within the other the two ends of the space between the cylinders could be closed water-tight by a pair of slightly greased, flat, ground-glass plates. By a procedure exactly analogous to that employed in the case of the shell the volume of the outer cylinder and the volume of the interspace between the two cylinders were determined. In every case the cylinders were closed at the top with a flat, ground-glass plate fitting water-tight and containing two holes to facilitate the filling. The filling was always such that the meniscus rising from these holes was estimated to be just sufficient to fill the holes level full. During the weighing metal caps were placed over the holes in order to reduce the evaporation. As the space to be filled was closed with glass at both top and bottom, the entrapped air bubbles could be readily seen and (with patience) removed.

By means of an end standard comparator the length of each cylinder along four generating lines was measured by the division of weights and measures of this Bureau. Though the ends of the cylinders were ground in pairs with the cylinders approximately coaxial, the inner cylinder of each pair was found to be about 7μ longer than the corresponding outer one. This is not sufficient to cause trouble from leakage, but needs to be considered in the determination of the radii. Since the capacity of concentric cylinders depends upon the *ratio* of their radii, it will be unaffected by any error in the absolute density of water, or in the thermometer, provided these errors are the same in the determination of the volume of the interspace between the cylinders as in the determination of the volume of the outer cylinder. The dimensions found were as follows:

Volume, outer cylinder No. 2 at 20°0 C
=3297.033 cc; length=19.99875 cm

Volume, interspace, cylinders No. 2 at 20°0 C

=836.897 cc, weight 2		
=836.822 " " 1	; length=19.99946 cm	
836.872 " weighted mean		

Volume, outer cylinder No. 3 at 20°0 C

=3293.167 cc	
=3293.114 "	; length=20.00718 cm
3293.140 "	

Volume interspace, cylinders No. 3 at 20°0 C

=832.110 cc	
=832.083 "	; length=20.00768 cm.
832.096 "	

From these we find:

Mean radius outer cylinder No. 2 at 20°.0 C = 7.24411 cm
" " inner " " " " = 6.25760 "
" " outer " No. 3 " " = 7.23831 "
" " inner " " " " = 6.25740 "
Capacity per unit of length of No. 2 = 3.41549 cm
" " " " " " No. 3 = 3.43350 cm.

The charge measured being that on the *inner* cylinder, the effective capacities of the cylinders are the capacities per unit length multiplied by the lengths of the *inner* cylinders. Hence we have for the capacities from formula (1), not including any corrections for guard-ring gap or end or edge effects, the following values:

Capacity of cylinders No. 2 = 68.3080 cm at 20°.0
" " " No. 3 = 68.6965 " " "

21. CYLINDRICAL CONDENSERS—DIRECT MEASUREMENTS.

As in the case of the spheres, the direct measurements were taken partly to obtain a check on the more accurate gravimetric determinations and partly to determine the irregularities in the cylinders. The diameters of the outer cylinders were measured at angular intervals of 22°.5 at eight or ten sections of the cylinders. The results are as follows:

TABLE III.

Measurements of Diameters of Outer Cylinder No. 2.

Angular position	Difference; Diameter—Standard. 1 unit=0.4617 μ									Temperature	Correction to 20° C	Difference at 20° C
	0°	22°.5	45°	67°.5	90°	112°.5	135°	157°.5	Mean			
1. At top	82.7	85.8	104.0	103.8	100.4	90.4	81.1	81.8	91.2	25°.6	−5.7 μ	36.4 μ
2. 2.5 cm from top	222.9	232.4	222.6	232.1	235.4	227.9	213.0	206.4	224.1	27°.8	−7.9 μ	95.6 μ
3. 5.0 " " "	218.4	223.2	223.2	224.8	230.2	219.0	214.4	213.3	220.8	25°.75	−5.8 μ	96.2 μ
4. 7.5 " " "	89.3	111.3	97.8	93.5	84.0	83.2	75.0	76.0	88.8	27°.8	−7.9 μ	33.1 μ
5. 10.0 " " "	60.8	68.6	68.8	55.6	49.8	44.8	41.8	49.2	54.9	26°.2	−6.3 μ	19.0 μ
6. 10 cm from bottom	62.0	68.2	61.6	48.0	45.0	43.5	44.0	49.2	52.7	27°.25	−7.4 μ	16.9 μ
7. 7.5 cm from bottom	51.6	54.5	41.9	24.4	14.7	19.6	32.8	34.0	34.2	27°.7	−7.8 μ	8.0 μ
8. 5.0 cm from bottom	66.1	69.6	51.2	41.6	27.3	16.6	21.6	47.8	42.7	28°.4	−8.5 μ	11.2 μ
9. 2.5 cm from bottom	157.2	168.6	146.6	120.2	103.2	106.0	113.6	149.8	133.2	27°.6	−7.7 μ	53.8 μ
10. At bottom	267.8	258.8	223.8	204.4	186.2	191.1	208.1	227.9	221.0	29°.25	−9.4 μ	92.6 μ

Weighting the top, bottom, and two middle values half as heavily as the others we find for the mean difference 47.5μ

Standard at 20°0	= 14.48324 cm
Diameter—Standard	= 0.00475 "
Mean diameter No. 2	= 14.48799 "
" radius "	= 7.24400 "

This differs from the value found gravimetrically by 1.1μ. Considering that the diameter varies through a range of 117μ, this agreement is all that could be expected.

TABLE IV.

Measurements of Diameters of Outer Cylinder No. 3.

Angular position	Difference; Diameter—Standard. 1 unit=0.4617μ									Temperature	Correction to 20°0 C	Difference at 20°0 C
	0°	22°5	45°	67°5	90°	112°5	135°	157°5	Mean			
1. At top	+103.0	+111.2	+121.4	+123.4	+125.8	+115.8	+108.3	+107.2	+114.5	27°85	−8.0μ	+ 44.9μ
2. 2.5 cm from top	− 39.0	− 35.2	− 27.1	− 12.8	− 9.0	− 6.6	− 17.5	− 23.4	− 21.4	24°90	−5.0μ	− 14.9μ
3. 5.0 cm from top	−131.8	−120.0	−106.0	−100.6	−105.9	−117.5	−136.2	−144.2	−120.3	28°3	−8.4μ	− 63.9μ
4. 10 cm from top	−177.6	−201.6	−193.4	−177.9	−161.0	−152.8	−150.8	−162.0	−172.1	29°2	−9.4μ	− 88.9μ
5. 10 cm from bottom ..	−165.0	−180.4	−183.0	−158.0	−149.3	− 133.9	−133.5	−145.1	−156.0	25°8	−5.9μ	− 77.9μ
6. 5.0 cm from bottom ..	−224.9	−246.3	−242.8	−233.8	−225.0	−211.6	−196.5	−206.4	−223.4	26°2	−6.3μ	−109.4μ
7. 2.5 cm from bottom ..	−219.6	−228.7	−241.0	−245.5	−221.8	−208.4	−196.6	−204.1	−220.7	25°1	−5.2μ	−107.1μ
8. At bottom .	− 10.8	− 8.6	− 4.8	− 5.9	− 3.8	+ 6.4	+ 6.8	+ 4.8	− 2.0	26°4	−6.5μ	− 7.4μ

Weighting the end sections 1, the middle sections 3 each, the others 2, we find for the mean difference − 65.8μ.

Standard at 20°0	= 14.48324 cm
Diameter—Standard	= −0.00658 "
Mean diameter No. 3	= 14.47666 cm at 20°0 C
" radius "	= 7.23833 " " "

This differs from the value found gravimetrically by only 0.2μ.

The diameter of this cylinder varies over a range of 172μ.

Having found that the sections of the inner cylinders are very nearly circular, three diameters of each of 20 sections were measured. The screw micrometer employed was graduated to 0.01 mm.

TABLE V.

Measurements of Diameters of Inner Cylinder No. 2.

	Series 1.—Micrometer reading in mm				Series 2.—Micrometer reading in mm				
Diameter at—	0°	60°	120°	Mean	0°	60°	120°	Mean	Weight
Distance from top—									
0.5 cm	0.154	0.162	0.159	0.1583	0.158	0.155	0.154	0.1557	1
1.5 "	.139	.155	.144	.1460					
2.5 "	.140	.151	.140	.1437	.148	.140	.148	.1453	3
3.5 "	.140	.152	.141	.1443					
4.5 "	.141	.150	.141	.1440					
5.5 "	.139	.150	.139	.1427	.145	.138	.143	.1420	3
6.5 "	.140	.148	.138	.1420					
7.5 "	.141	.146	.134	.1403					
8.5 "	.144	.149	.140	.1443	.145	.139	.144	.1427	3
9.5 "	.148	.150	.149	.1490					
10.5 "	.142	.145	.146	.1443					
11.5 "	.144	.144	.144	.1440	.144	.132	.139	.1387	3
12.5 "	.140	.143	.140	.1410					
13.5 "	.139	.143	.140	.1407					
14.5 "	.138	.142	.140	.1400	.144	.135	.139	.1393	3
15.5 "	.140	.144	.147	.1437					
16.5 "	.140	.140	.139	.1397					
17.5 "	.130	.137	.137	.1347	.138	.130	.130	.1327	3
18.5 "	.137	.140	.140	.1390					
19.5 "	.169	.166	.168	.1677	.165	.159	.164	.1627	1
Mean				0.14447				0.14202	

RESULTS, SERIES 1.

Mean reading on diameter	+0.1445 mm
" " " end standard	−0.0061 "
" diameter—standard	+0.1506 "
Correction from 23°.7 to 20°.0 C	−0.0032 "
(Diameter—Standard) at 20°.0	+0.1474 "

SERIES 2.

Mean reading on diameter	+0.1420 mm
" " " standard	−0.0084 "
" diameter—standard	+0.1504 "
Correction from 24°.2 to 20°.0 C	−0.0037 "
(Diameter—Standard) at 20°.0	+0.1467 "

A third set gave (Diameter—Standard) at 20°.0 = +0.1503 mm.

The mean of all three is (Diameter—Standard) at 20°.0 = 0.1481 mm.

End-Standard = 124.9962 mm at 20°.0
Diameter—Standard = 0.1481 " " "
Mean Diameter No. 2 = 125.1443 " " "
" Radius No. 2 = 6.25722 cm " "

This is 3.8 μ smaller than that found by the gravimetric method.

TABLE VI.

Measurements of Diameters of Inner Cylinder No. 3.

	Series 1 Temp. 24°.0 Micrometer reading in mm				Series 2 Temp. 24°.6 Micrometer reading in mm			
Diameter at—	0°	60°	120°	Mean	0°	60°	120°	Mean
Distance from top:								
0.5 cm	0.156	0.156	0.152	0.1547	0.159	0.160	0.155	0.1580
1.5 "	.138	.147	.133	.1393	.139	.138	.141	.1393
2.5 "	.133	.144	.130	.1357	.135	.137	.140	.1373
3.5 "	.139	.146	.138	.1410	.140	.141	.140	.1403
4.5 "	.137	.149	.141	.1423	.143	.145	.148	.1453
5.5 "	.134	.149	.140	.1410	.141	.146	.147	.1447
6.5 "	.137	.145	.140	.1407	.143	.143	.146	.1440
7.5 "	.138	.145	.140	.1410	.141	.141	.146	.1427
8.5 "	.136	.146	.141	.1410	.142	.141	.148	.1437
9.5 "	.139	.150	.144	.1443	.147	.145	.150	.1473
10.5 "	.141	.157	.149	.1490	.149	.149	.155	.1510
11.5 "	.145	.155	.151	.1503	.151	.150	.159	.1533
12.5 "	.140	.155	.150	.1483	.149	.149	.159	.1523
13.5 "	.140	.151	.149	.1467	.148	.146	.158	.1507
14.5 "	.139	.152	.147	.1460	.148	.142	.154	.1480
15.5 "	.140	.150	.144	.1447	.145	.144	.150	.1463
16.5 "	.139	.150	.148	.1457	.148	.148	.153	.1497
17.5 "	.140	.150	.147	.1457	.142	.140	.151	.1443
18.5 "	.140	.151	.148	.1463	.141	.143	.150	.1447
19.5 "	.150	.161	.157	.1560	.150	.150	.159	.1530
			Mean	0.1450			Mean	0.1468
			Correction to 20°	−.0035			Correction to 20°	.0040
			Mean reading at 20°	0.1415			Mean reading at 20°	0.1428
	Mean of both series, 0.1422 mm.							

Micrometer reading on end-standard at 20°.0 = −0.0077 mm
" " " cylinder " " = +0.1422 "
Diameter—Standard = +0.1499 "
" = 124.9962 "
Diameter of inner cylinder No. 3 = 125.1461 "
Radius " = 6.25730 cm

This is 1.0 μ smaller than the value found by the gravimetric method.

In the same manner several diameters at the upper end of the lowest section (No. 1) and at the lower end of the upper guard section (No. 4) were measured. The results of these measurements, as well as of those at the extremities of the other sections, are given in the following table:

TABLE VII.

Mean Radius at Ends of Sections.

Section	1	2	3	4
Radius at top, outer cylinder	7.24192	7.24344	7.24386	
" " bottom, " "		7.24625	7.24125	7.23774
Radius at top, inner cylinder	6.25758	6.25785	6.25782	
" " bottom, " "		6.25832	6.25774	6.25675

From these measurements it is evident that if the ends of the inner cylinders were truly circular and if the sections were always placed truly coaxially and with top end uppermost the offset at the junction of any two sections can in no case exceed 11.0 μ. Under the same conditions the offset in the case of the outer cylinders may amount to 61.2 μ. The effect of these offsets in the combinations actually used will be discussed below in Section V.

22. PLATE CONDENSER—DIMENSIONS OF THE PLATES.

The electrostatic capacity of a guard-ring condenser consisting of parallel circular plates is

$$C=\frac{A}{4\pi d}=\frac{(r+\delta r)^2}{4d} \qquad (2)$$

if the dielectric constant of the medium between the plates is unity. Here r is the radius of the collector plate; δr is a term due to the effect of the gap between the collector plate and the guard ring, depending upon the width of this gap and the values of r and of d; d is the distance between the plates. As already described, d is varied over wide limits and is determined at the time the condenser is used. On the other hand, the value of r is a constant and can be measured once for all. Similarly the width of the gap between the collector plate and the guard ring, depending only upon the radius of the plate and the internal radius of the guard ring, is a constant of the instrument.

The diameters of the two plates were measured with an end standard comparator. Nine equally spaced diameters were measured for each plate; the diameters were measured in both direct and reversed directions and two settings were made in each position. The results obtained are given in Table VIII.

TABLE VIII.

Diameters of the Plates at 20°0 C.

Diameter	Plate A	Plate B
1	20.0080 cm	20.0354 cm
2	20.0086 "	20.0359 "
3	20.0094 "	20.0363 "
4	20.0094 "	20.0363 "
5	20.0092 "	20.0357 "
6	20.0086 "	20.0350 "
7	20.0082 "	20.0346 "
8	20.0076 "	20.0345 "
9	20.0076 "	20.0347 "
Mean	20.00851 "	20.03538 "

From these measurements we see that the plates are practically ellipses of semi-axes 10.00470, 10.00380 for plate A and 10.01824, 10.01725 for plate B. For plate A, $2\sqrt{ab} = 20.00850$ cm, for plate B, $2\sqrt{ab} = 20.03538$ cm. Hence the areas of the plates are practically the same as the areas of the circles with diameters equal to the mean diameters of the plates, so we may take these mean diameters as the values of r in the expression for the capacity of the condenser.

Plate B, having been adjusted as already described, the entire top of the condenser was removed, and turned upside down, and the width of the gap between the collector plate and the guard ring (in the plane of the face of the plate) was measured at six points around the circumference of the plate. The measurements were made by means of a micrometer microscope, one division of the scale being equal to 1.708 microns. Six measurements were made at each point. The values found are given in Table IX.

TABLE IX.

Width of Guard Ring Gap, Plate B. One Unit=1.708μ, 20°2 C.

Position	Width of gap						Mean
1	121.0	120.5	122.3	118.5	121.3	120.6	120.7
2	103.4	101.2	105.0	99.2	102.9	101.4	102.2
3	100.5	102.5	104.0	105.0	102.1	102.0	102.7
4	109.1	105.2	108.0	108.8	105.5	106.7	107.2
5	88.9	94.3	90.8	97.1	89.0	92.3	92.4
6	102.9	(110.2)	102.1	107.0	101.8	102.1	103.2
1	124.7	124.0	119.1	119.1	117.4	120.2	120.9

Hence the mean width of the gap is 104.7 divisions = 178.8 microns. It is evident that the opening in the guard ring is considerably more elliptical than the plates. This width of gap corresponds to a mean radius of 10.03557 cm for the inner boundary of the guard ring.

IV. CORRECTIONS AND SOURCES OF ERROR—SPHERICAL CONDENSERS.

23. EFFECT OF SHELL NOT BEING PERFECT SPHERE.

The capacity of the spherical condenser is calculated on the hypothesis that both ball and shell are truly spherical and accurately concentric. As these conditions are never exactly fulfilled, it is necessary to consider the possible error thus introduced. We shall begin with a study of the shell. We have determined the radius, R, of a sphere having the same volume as the shell, and by determining the distance through which we raise the ball in order to bring it from contact with the bottom of the shell to contact with the top of it we have measured the difference between the vertical diameters of the balls and of the shell. This was done for both balls, and the mean of these measurements showed that the vertical diameter of the shell is less than its mean diameter R by

0.017 cm. If we assume this error is due to the fact that the flanges where the hemispheres come together have been ground down too much, so that each half is somewhat less than a complete hemisphere, we shall not be far from the truth and shall obtain a correction that is probably greater than the true one. For a perfect spherical condenser the capacity is $\frac{Rr}{R-r}=\frac{Rr}{d}$ where R and r are the radii of shell and ball respectively and d is the difference in their radii. Now any small change in R or r affects Rr very slightly, but produces a relatively large change in d; hence in the study of the effect of *small* errors, we may regard Rr as known and limit ourselves to an investigation of the factor $1/d$. We also know that

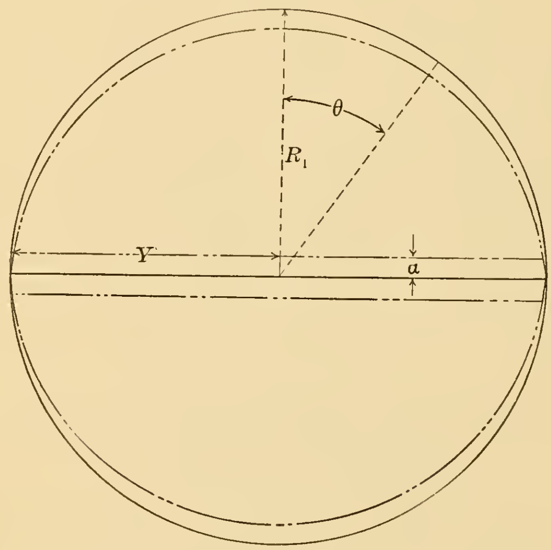

Fig. 10.

if the irregularities are slight the lines of force can depart but slightly from right lines normal to the surface. Hence from analogy with the parallel plate condenser we must give each element of $1/d$ a weight proportional to the area of the surface to which it applies. Now in the case in which d is the same for all points on a circular cone having its vertex at the center of the sphere, we may take $2\pi R^2 \sin\theta\, d\theta$ for our element of area so that for the d corresponding to the angle θ, the weight of $1/d$ will be

$$\frac{2\pi R^2 \sin\theta\, d\theta}{4\pi R^2}=\tfrac{1}{2}\sin\theta\, d\theta$$

Hence the total capacity will be

$$C = Rr\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{d_1}\, d\theta \tag{3}$$

If d_1 is independent of θ this becomes $C = \frac{Rr}{d}$ as before.

Let R_1 = true radius of curvature of the shell, assumed constant,
R = radius of a sphere *having the same volume*,
a = the amount each hemisphere has been ground off on its flange beyond the true equitorial plane,
ϵ = the amount by which the vertical radius $R_1 - a$ falls short of the mean radius R; that is,
$\epsilon = R - (R_1 - a) = R - R_1 + a$
Write $\delta = R_1 - R$
Then $\epsilon = a - \delta$ or $a = \delta + \epsilon$
But the volume of the shell is

$$V = 2\pi\int_a^{R_1} (R_1^2 - x^2)\, dx = \frac{4}{3}\pi R_1^3 - 2\pi\left(R_1^2 a - \frac{a^3}{3}\right) = \frac{4}{3}\pi R^3$$

Substituting $R_1 = R + \delta$ we find, neglecting squares, products and higher powers of ϵ and δ

$$\frac{4}{3}\pi(R^3 + 3R^2\delta) - 2\pi(\delta + \epsilon)R^2 = \frac{4}{3}\pi R^3$$

$$\therefore\ 4R^2\delta = 2R^2(\delta + \epsilon)$$

$$\therefore\ \delta = \epsilon$$

$$\text{or, } a = \delta + \epsilon = 2\epsilon$$

The distance Y along the plane of union of the two hemispheres from the center of the ball to the shell is

$$Y = \sqrt{R_1^2 - a^2} = \sqrt{R^2 + 2R\delta + \delta^2 - a^2} = \sqrt{R^2 + 2R\epsilon - 3\epsilon^2}$$

$$= R\left\{1 + \frac{2\epsilon}{R} - \frac{3\epsilon^2}{R^2}\right\}^{\frac{1}{2}} = R\left\{1 + \frac{\epsilon}{R} \cdots\right\}$$

$\therefore Y = R + \epsilon$ approximately.

That is, the horizontal radius is greater than the mean radius R by ϵ, while the vertical radius is less than the mean by the same amount. Hence, if d is the mean distance between the shell and the ball $(R-r)$ the distance between the shell and the ball along a radius of the latter making an angle θ with the vertical is:

$$d_1 = d + \epsilon - a\cos\theta$$
$$\text{or, } d_1 = d + \epsilon(1 - 2\cos\theta)$$

Substituting this value of d_1 in equation (3), we have for the true capacity of the ball

$$C = Rr\int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{d+\epsilon(1-2\cos\theta)} = Rr\int_0^1 \frac{dz}{d+\epsilon-2\epsilon z}, \text{ where } z=\cos\theta$$

$$= Rr\left[\frac{1}{2\epsilon}\log(d+\epsilon-2\epsilon z)\right]_1^o = \frac{Rr}{2\epsilon}\log\frac{d+\epsilon}{d-\epsilon}$$

$$\therefore C = \frac{Rr}{2\epsilon}\left[\begin{array}{l} \frac{\epsilon}{d} - \frac{1}{2}\left(\frac{\epsilon}{d}\right)^2 + \frac{1}{3}\left(\frac{\epsilon}{d}\right)^3 - \frac{1}{4}\left(\frac{\epsilon}{d}\right)^4 + \frac{1}{5}\left(\frac{\epsilon}{d}\right)^5 - \cdots \\ + \frac{\epsilon}{d} + \frac{1}{2}\left(\frac{\epsilon}{d}\right)^2 + \frac{1}{3}\left(\frac{\epsilon}{d}\right)^3 + \frac{1}{4}\left(\frac{\epsilon}{d}\right)^4 + \frac{1}{5}\left(\frac{\epsilon}{d}\right)^5 + \cdots \end{array}\right]$$

$$= \frac{Rr}{d}\left[1 + \frac{1}{3}\left(\frac{\epsilon}{d}\right)^2 + \frac{1}{5}\left(\frac{\epsilon}{d}\right)^4 + \cdots\right] \qquad (4)$$

But $\frac{Rr}{d} = C_0 =$ the capacity calculated on the assumption that the figuring is perfect and that the dimensions are those of perfect spheres of the same volume. Writing $\frac{\varDelta_1 C}{C_0} = \frac{C - C_0}{C_0} =$ relative increase in capacity produced by the irregularity, we have

$$\frac{\varDelta_1 C}{C_0} = \frac{1}{3}\left(\frac{\epsilon}{d}\right)^2 + \frac{1}{5}\left(\frac{\epsilon}{d}\right)^4 + \cdots$$

But $2\epsilon = 0.017$ cm

$d = 2.554$ cm for ball A

$d = 3.798$ " " " B

$$\therefore \frac{\varDelta_1 C}{C_0} = 0.0000037 = 3.7 \times 10^{-6} \text{ for ball A}$$
$$= 0.0000017 = 1.7 \times 10^{-6} \text{ " " B.}$$

This increase in the capacity due to the shell not being perfectly spherical is entirely negligible. It has been computed on the assumption that the center of the ball coincides with the center of figure of the shell. As this adjustment is never exact we shall proceed to investigate the effect of a slight displacement of the center of the ball from the center of figure of the shell.

24. EFFECT OF DISPLACEMENT OF THE BALL FROM THE CENTER.

Suppose the ball is raised by the amount β cm. Then for the upper hemisphere $d_1=d+\epsilon-(2\epsilon+\beta)\cos\theta$ and for the lower hemisphere it is $d_2=d+\epsilon-(2\epsilon-\beta)\cos\theta$.

The capacity then is

$$C=\frac{Rr}{2}\left[\int_0^{\frac{\pi}{2}}\frac{\sin\theta\,d\theta}{d+\epsilon-(2\epsilon+\beta)\cos\theta}+\int_0^{\frac{\pi}{2}}\frac{\sin\theta\,d\theta}{d+\epsilon-(2\epsilon-\beta)\cos\theta}\right]$$

Putting $\cos\theta=z$, we have

$$C=\frac{Rr}{2}\left[\int_0^1\frac{dz}{d+\epsilon-(2\epsilon+\beta)z}+\int_0^1\frac{dz}{d+\epsilon-(2\epsilon-\beta)z}\right]$$

$$=\frac{Rr}{2}\left[-\frac{1}{2\epsilon+\beta}\log\{d+\epsilon-(2\epsilon+\beta)z\}-\frac{1}{2\epsilon-\beta}\log\{d+\epsilon-(2\epsilon-\beta)z\}\right]_0^1$$

$$=\frac{Rr}{2}\left[-\frac{1}{2\epsilon-\beta}\log\frac{d-\epsilon+\beta}{d+\epsilon}-\frac{1}{2\epsilon+\beta}\log\frac{d-\epsilon-\beta}{d+\epsilon}\right]$$

$$=\frac{Rr}{2}\left[\begin{array}{l}\frac{1}{2\epsilon-\beta}\left[\begin{array}{l}\frac{\epsilon-\beta}{d}+\frac{1}{2}\left(\frac{\epsilon-\beta}{d}\right)^2+\frac{1}{3}\left(\frac{\epsilon-\beta}{d}\right)^3+\frac{1}{4}\left(\frac{\epsilon-\beta}{d}\right)^4+\cdots\\+\frac{\epsilon}{d}-\frac{1}{2}\left(\frac{\epsilon}{d}\right)^2+\frac{1}{3}\left(\frac{\epsilon}{d}\right)^3-\frac{1}{4}\left(\frac{\epsilon}{d}\right)^4+\cdots\end{array}\right]\\+\frac{1}{2\epsilon+\beta}\left[\begin{array}{l}\frac{\epsilon+\beta}{d}+\frac{1}{2}\left(\frac{\epsilon+\beta}{d}\right)^2+\frac{1}{3}\left(\frac{\epsilon+\beta}{d}\right)^3+\frac{1}{4}\left(\frac{\epsilon+\beta}{d}\right)^4+\cdots\\+\frac{\epsilon}{d}-\frac{1}{2}\left(\frac{\epsilon}{d}\right)^2+\frac{1}{3}\left(\frac{\epsilon}{d}\right)^3-\frac{1}{4}\left(\frac{\epsilon}{d}\right)^4+\cdots\end{array}\right]\end{array}\right]$$

$$=\frac{Rr}{2}\left[\begin{array}{l}\frac{1}{2\epsilon-\beta}\left\{\frac{2\epsilon-\beta}{d}-\frac{1}{2}\frac{\beta(2\epsilon-\beta)}{d^2}+\frac{1}{3}\frac{\epsilon^3+(\epsilon-\beta)^3}{d^3}-\cdots\right\}\\+\frac{1}{2\epsilon+\beta}\left\{\frac{2\epsilon+\beta}{d}+\frac{1}{2}\frac{\beta(2\epsilon+\beta)}{d^2}+\frac{1}{3}\frac{\epsilon^3+(\epsilon+\beta)^3}{d^3}+\cdots\right\}\end{array}\right]$$

$$=\frac{Rr}{2}\left[\frac{2}{d}+\frac{1}{3d^3}\left\{\frac{\epsilon^3+(\epsilon-\beta)^3}{2\epsilon-\beta}+\frac{\epsilon^3+(\epsilon+\beta)^3}{2\epsilon+\beta}\right\}+\frac{1}{4d^4}\left\{\frac{(\epsilon-\beta)^4-\epsilon^4}{2\epsilon-\beta}\right.\right.$$

$$\left.\left.+\frac{(\epsilon+\beta)^4-\epsilon^4}{2\epsilon+\beta}\right\}+\frac{1}{5d^5}\left\{\frac{(\epsilon-\beta)^5+\epsilon^5}{2\epsilon-\beta}+\frac{(\epsilon+\beta)^5+\epsilon^5}{2\epsilon+\beta}\right\}\cdots\right]$$

$$=\frac{Rr}{2}\left[\frac{2}{d}+\frac{2}{3d^3}(\epsilon^2+\beta^2)+\frac{\epsilon\beta^2}{d^4}+\frac{2(\epsilon^4+4\epsilon^2\beta^2+\beta^4)}{5d^5}+\cdots\right] \qquad (4a)$$

But $C_0=\frac{Rr}{d}$ is the capacity for concentric spheres,

$$\therefore \frac{\Delta_2 C}{C_0}=\tfrac{1}{3}\frac{\epsilon^2+\beta^2}{d^2}+\frac{\epsilon\beta^2}{2d^3}+\frac{\epsilon^4+4\epsilon^2\beta^2+\beta^4}{5d^4}+\cdots$$

If C_1 = capacity when the ball is in its central position, that is C_1 is the value of the expression (4*a*) when $\beta = 0$ (this is the same as expression 4), then

$$C-C_1=\frac{Rr}{2}\left[\frac{2\beta^2}{3d^3}+\frac{\epsilon\beta^2}{d^4}+\frac{2\beta^2(4\epsilon^2+\beta^2)}{5d^5}+\cdots\right]$$

$$\therefore \frac{\Delta_3 C}{C_1}=\frac{\beta^2}{d^2}\left[\tfrac{1}{3}+\tfrac{1}{2}\frac{\epsilon}{d}+\frac{1}{45}\frac{31\epsilon^2+9\beta^2}{d^2}+\cdots\right]$$

This quantity $\frac{\Delta_3 C}{C_1}$ can be directly measured and in this way the validity of this method of treating the problem may be checked by experiment.

We know

$$2\epsilon=0.017 \text{ cm}$$
$$d=2.554 \text{ cm for ball A}$$
$$d=3.798 \text{ cm for ball B}$$
$$\therefore \frac{\epsilon}{d}=0.00332 \text{ cm for ball A}$$
$$=0.00224 \text{ cm for ball B}$$

Hence

$$\frac{\Delta_3 C}{C_1}=\tfrac{1}{3}\frac{\beta^2}{d^2}\left[1+0.00498+\ldots\right] \text{ for ball A}$$
$$=\tfrac{1}{3}\frac{\beta^2}{d^2}\left[1+0.00336+\ldots\right] \text{ for ball B}$$

Whence we obtain the results given in Table X, which are represented by the curves on the accompanying plate, Fig. 11. The experimental values, which were obtained by displacing the ball and observing the changes in the capacity, are marked on the same plate and it is evident that these two curves represent the facts within the limits of experimental error. The average difference between the calculated and observed values is only 2 or 3 parts in 100,000.

TABLE X.

Relative Changes in Capacity of Spherical Condenser due to Displacements β of Ball from Center.

Displacement β	Value of $\frac{\Delta_3 C}{C_1}$ for	
	Ball A	Ball B
0.01 cm	0.051×10^{-4}	0.023×10^{-4}
.02 "	0.205 "	0.093 "
.04 "	0.821 "	0.371 "
.06 "	1.849 "	0.835 "
.08 "	3.287 "	1.484 "
.10 "	5.135 "	2.319 "
.12 "	7.396 "	3.340 "
.16 "	13.147 "	5.936 "
.20 "	20.540 "	9.275 "
.24 "	29.582 "	13.356 "
.30 "	46.216 "	20.868 "
.36 "	66.560 "	30.050 "

When set up the balls are probably centered to within 0.1 mm, so that when first adjusted the capacity does not differ from that calculated by more than 1 in 100,000. Owing, however, to the slight warping of the stand, the stretching of the thread supporting the ball, etc., the ball may become excentric with the shell by possibly 0.2 mm; that is, the capacity may be in error by 2 in 100,000 for ball A; but even then for ball B it will be in error by less than 1 in 100,000. Wherever possible we have endeavored to determine this correction and to apply it. During the greater portion of the time a delicate level was kept on the arm carrying the vernier by which the vertical position of the ball was determined, and the leveling screws were adjusted as need be to keep this arm level. This

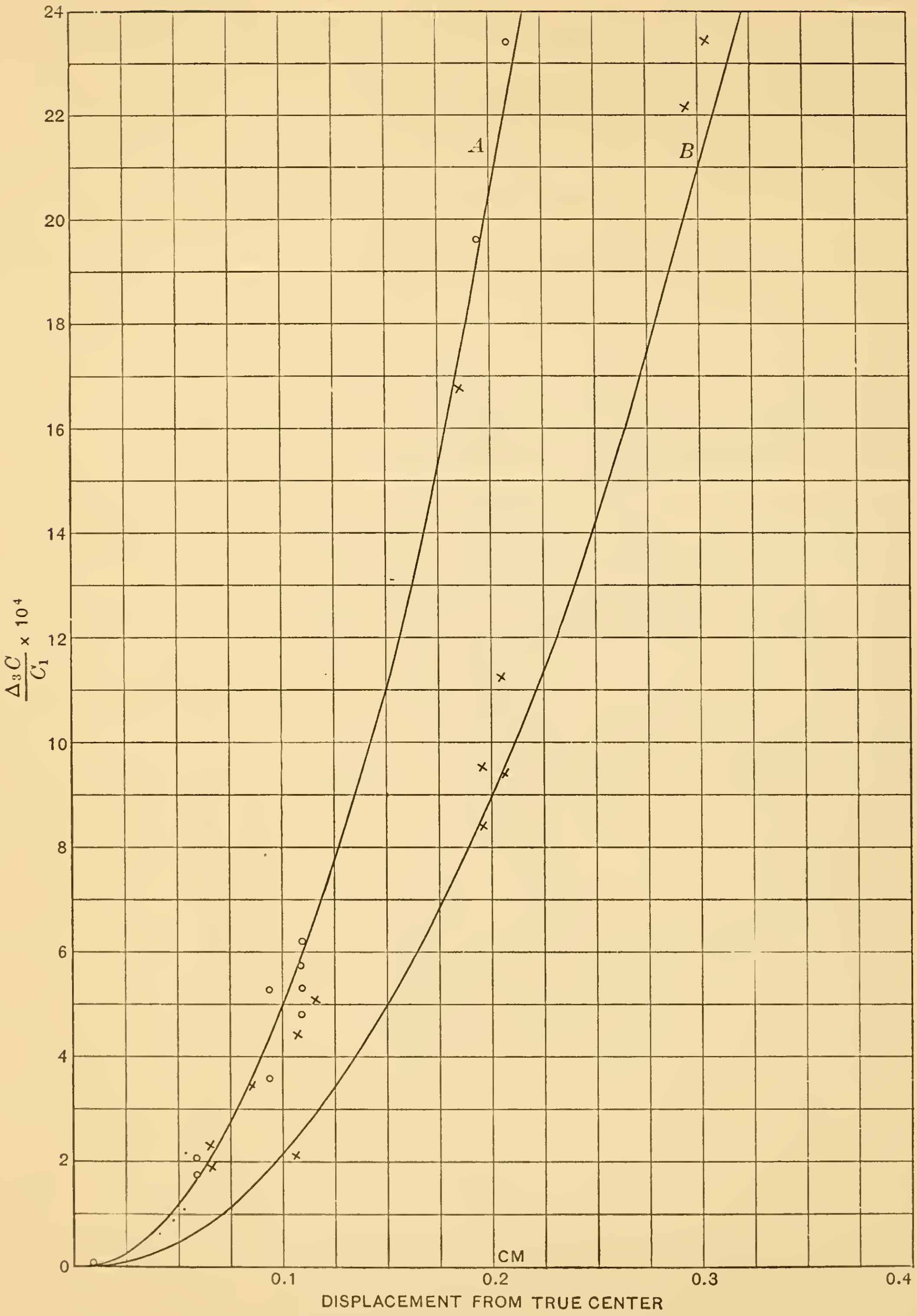

Fig. 11.—*Curve Showing the Change in the Capacity of the Spherical Condenser Due to a Displacement of the Inner Sphere.*

The upper curve applies to ball A; the lower to ball B.

largely eliminates displacement from the center due to the warping of the table and floor, but does not eliminate that due to a possible warping of the base of the instrument itself; this would tend to produce a very slight relative displacement of the shell and of the pillar from which the ball is suspended.

25. EFFECT OF HOLES IN SHELL, BUSHINGS, CORD, ETC.

There remains to be considered the effect of the holes in the shell, and of the ebonite bushings or tubes running through them, the small eye by which the silk cord is attached to the ball, and the effect of the silk cord.

The effect of the silk cord is surely less than that obtained by assuming that it is a rod passing radially from the ball to the shell (the cord does not touch the shell) without distorting the lines of force. Assuming its specific inductive capacity to be 2 and its diameter to be ½ mm, which is greater than the truth, we have for the relative increase in capacity

$$\frac{\delta C}{C}=\frac{\text{area of section of cord}}{\text{area of ball}}=\frac{\pi(0.025)^2}{4\pi(10.1)^2}=\frac{1}{653{,}000}$$
$$=1.5 \text{ parts in } 1{,}000{,}000.$$

The leakage along the cord was found to be inappreciable.

The eye in the top of the ball is 2 mm in diameter and projects 2 mm above the surface of the ball; applying the method employed on page 474 we have for the capacity

$$C=\frac{Rr_1}{4\pi r_1^2}\left(\frac{4\pi r_1^2-A}{d}+\frac{A}{d'}\right)$$

where r_1 is the *true radius* of the ball; d is the true radial distance from the surface of the ball to the shell; d' is the distance from the top of the eye to the shell; A is the sectional area of the eye.

$$\therefore d'=d-0.2 \text{ cm}$$

$$A=0.01\pi=0.0314 \text{ cm}^2$$

In calculating the capacity we used the quantity r, which is the mean radius of the ball; that is, it is the radius of the perfect sphere having a volume equal to that of the ball plus the volume of the eye, since the eye was in place when the volume was determined. Hence

$$\frac{4}{3}\pi r^3 = \frac{4\pi r_1^3}{3} + 0.2A$$

Putting $r_1 = r - \delta$ we find

$$\delta = \frac{0.2A}{4\pi r^2}$$

$$\therefore C = \frac{Rr_1}{d}\left\{1 - \frac{A}{4\pi r_1^2} + \frac{Ad}{4\pi r_1^2 d'}\right\} = \frac{Rr_1}{d}\left\{1 + \frac{\delta}{0.2}\,\frac{d-d'}{d'}\right\}$$

$$= \frac{Rr_1}{d}\left\{1 + \frac{\delta}{d'}\right\} \qquad (5)$$

The capacity we have assumed is

$$C_0 = \frac{Rr}{d_0}$$

where $d_0 = R - r = d - \delta$; substituting in (5) for r_1, d' and d their values in terms of r and d_0, we find for the true capacity, since $r_1 = r - \delta$, $d = d_0 + \delta$ and $d' = d_0 - 0.2 + \delta$,

$$C = \frac{Rr\left(1 - \frac{\delta}{r}\right)}{d_0\left(1 + \frac{\delta}{d_0}\right)}\left\{1 + \frac{\delta}{d_0\left(1 - \frac{0.2 - \delta}{d_0}\right)}\right\}$$

$$= \frac{Rr}{d_0}\left\{1 - \frac{\delta}{r} + \frac{\delta\,(0.2 - \delta)}{d_0^2}\right\}$$

$$= C_0\left(1 - \frac{\delta}{r} + \frac{0.2\delta}{d_0^2}\right) \text{ approximately.}$$

But $\delta = \frac{0.2A}{4\pi r^2}$ $\therefore \frac{\delta}{r} = \frac{0.2A}{4\pi r^3} = \frac{0.00628}{4\pi\,(10.12)^3}$ for ball A

$$= 0.0000048$$

$$\frac{0.2\delta}{d_0^2} = 0.0000015$$

Hence the capacity we have assumed is too large by only 3.3 parts in 1,000,000. For ball B the correction is still less. Hence the correction for the eye is inappreciable.

The corrections for the ebonite were determined by measuring the change produced in the electromagnetic capacity when the ebonite was removed; and the effect of the side hole was determined by measuring the change in the electromagnetic capacity produced by plugging this hole with a brass rod. From this the effect of the top hole was calculated on the assumption that the relative effects are directly proportional to the areas of the two holes. This assumption was not tested experimentally, but is surely not far wrong.

Following is a summary of the various small corrections due to the holes and ebonite:

	Ball A	Ball B
Correction to capacity $\left(\frac{\Delta C}{C}\right)$ due to side hole	-11×10^{-6}	-8×10^{-6}
" " " " " " top "	-28×10^{-6}	-20×10^{-6}
" " " " " " side ebonite	$+8\times10^{-6}$	$+5\times10^{-6}$
" " " " " " top "	$+63\times10^{-6}$	$+32\times10^{-6}$
Total	$+32\times10^{-6}$	$+9\times10^{-6}$

Hence the total correction to be added to the calculated capacity is

	Ball A	Ball B
Correction $\left(\frac{\Delta C}{C}\right)$ for density of water	$+15\times10^{-6}$	$+15\times10^{-6}$
" " " error in shell	$+4\times10^{-6}$	$+2\times10^{-6}$
" " " holes and ebonite	$+32\times10^{-6}$	$+9\times10^{-6}$
" " " eye for suspending ball	-3×10^{-6}	-2×10^{-6}
Total	$=+48\times10^{-6}$	$+24\times10^{-6}$
Correction to **v**	$=+24\times10^{-6}$	$+12\times10^{-6}$
Correction to **v**, in centimeters per second	$=0.00007\times10^{10}$	$=0.00004\times10^{10}$

V. CORRECTIONS AND SOURCES OF ERROR—CYLINDRICAL CONDENSER.

The corrections and sources of error that must be considered in the case of the cylindrical condensers are as follows:

1. Error due to a lateral displacement of the axis of the inner cylinder.

2. Error due to an inclination of the axis of the inner cylinder with respect to the outer one.

3. Errors due to the figuring of the cylinders.

4. Correction for the presence of offsets where two sections come together.

These corrections and errors apply to the cylinders however they may be used. If they are used without guard cylinders we must also determine the two following corrections:

5. Correction due to the irregular distribution of the charge upon the ends of the inner cylinder and to the manner in which the magnitude of this charge is affected by slight variations in the dimensions of the ends of the different sections, and in particular the effect of varying the length of the condenser upon this charge.

6. Correction for possible variation in the capacity between the caps closing the ends of the cylinders.

If the cylinders are used with guard cylinders we must determine one other correction:

7. Correction for the gap between the guard cylinder and the middle section of the condenser.

26. CORRECTION FOR LATERAL DISPLACEMENT OF THE AXIS OF THE INNER CYLINDER.

We shall proceed as in the case of the spheres, page 477. The distance between the surfaces at any point (Fig. 12) is

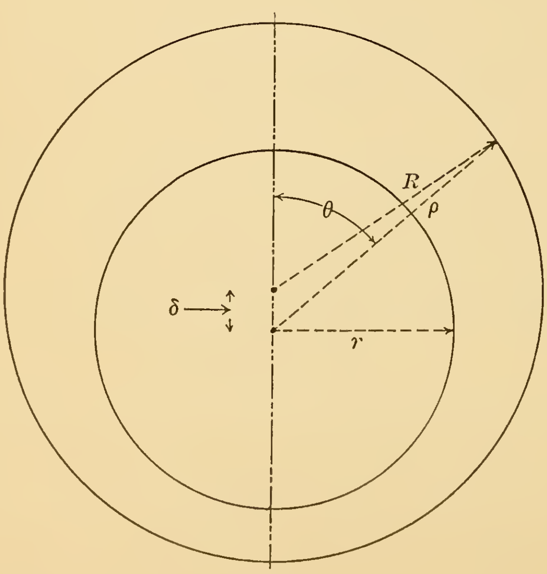

Fig. 12.

$$D_1 = \rho - r$$
$$\rho^2 + \delta^2 - 2\rho\delta\cos\theta = R^2$$

$$\therefore \rho\left\{1-\frac{\delta\cos\theta}{\rho}+\frac{\delta^2}{2\rho^2}-\frac{\delta^2\cos^2\theta}{2\rho^2}\ldots\right\}=R, \ \delta \text{ being very small.}$$

$$\therefore \rho=R+\delta\cos\theta-\frac{\delta^2\sin^2\theta}{2\rho}$$

$$=R+\delta\cos\theta-\frac{\delta^2\sin^2\theta}{2R}, \text{ approximately.}$$

$$\therefore D_1=R-r+\delta\cos\theta-\frac{\delta^2\sin^2\theta}{2R}$$

The distance for coaxial cylinders is

$$D_0=R-r$$

Hence the relative increase in the capacity of the system is

$$\frac{\Delta_1 C}{C_0}=\int_0^{2\pi}\frac{D_0-D_1}{D_0}\cdot\frac{d\theta}{2\pi}=\frac{\delta^2}{4R(R-r)} \tag{6}$$

Hence the effect varies as the square of the displacement, and is negligible when δ is very small.

In building up the condensers the distance between the inner and the outer cylinders of each section was tested at their tops, at four points around the circumference, by means of a strip of brass cut wedge shaped with a slope of 1 mm in 10 cm, and the cylinders were slightly tipped by a suitable tightening of the screws fastening them to the lower section until they were as nearly coaxial as possible. The relative displacement of the axes at the top of any section was surely never over 0.025 mm. This determines the maximum error due to this cause.

Taking $R=7.25$
$r=6.25$
$\delta=0.0025$

we find from equation (6), $\frac{\Delta_1 C}{C_0}=0.22\times10^{-6}$, which is negligible.

27. CORRECTION FOR INCLINATION OF THE AXIS OF THE INNER CYLINDER.

The error due to want of parallelism of the axes of the cylinders will be practically equal to that due to the same inclination of the two plates of a plane condenser, having the same length and distance apart, and of breadth πr, r being the radius of the inner cylinder, end effects being neglected in both cases. We shall therefore calculate the increase in the capacity of a plane condenser when the plates are inclined at an angle θ. The lines of force passing from plate A at potential V to B at potential zero (neglecting effect of edges) will be arcs of circles. The electric force f will be uniform

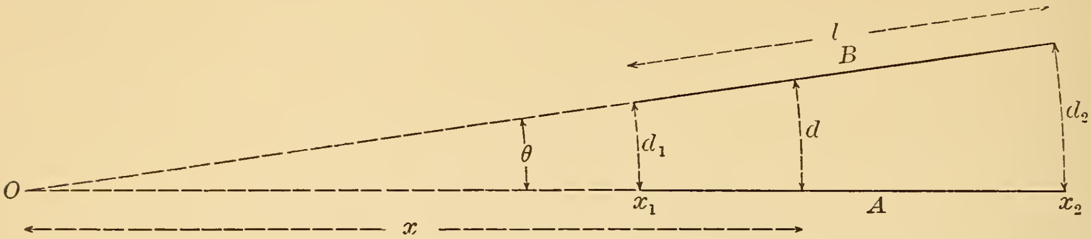

Fig. 13.

along any such line and equal to $V/\theta x$. If σ is the surface density, Q the total quantity of electricity on the plate of length l and breadth b, we have

$$f=\frac{V}{\theta x} \qquad \sigma=\frac{V}{4\pi\theta x} \qquad Q=b\int_{x_1}^{x_2}\sigma dx$$

$$\therefore\ Q=\frac{bV}{4\pi\theta}\int_{x_1}^{x_2}\frac{dx}{x}=\frac{bV}{4\pi\theta}\log\frac{x_2}{x_1}$$

The capacity then is

$$C_1=\frac{Q}{V}=\frac{b}{4\pi\theta}\log\frac{x_2}{x_1}=\frac{bl}{4\pi\delta}\log\frac{x_2}{x_1}$$

since

$$\theta=\frac{d_2}{x_2}=\frac{d_1}{x_1}=\frac{d_2-d_1}{x_2-x_1}=\frac{\delta}{l}$$

Also, since

$$\frac{x_2}{x_1}=\frac{d_2}{d_1}$$

$$C_1=\frac{bl}{4\pi\delta}\log\left(1+\frac{\delta}{d_1}\right)$$

$$=\frac{bl}{4\pi d_1}\left(1-\frac{1}{2}\frac{\delta}{d_1}+\frac{1}{3}\frac{\delta^2}{d_1^2}-\ .\ .\ .\right)$$

If d_1 is the mean distance between the plates, the point x_1 will correspond to the middle point of the plate, and the expression just obtained will be the capacity of that half of the plate for which the distance between the planes is greater than d_1. For the other half of the plate, $\log \frac{x_2}{x_1}$ becomes $\log \frac{d_1}{d_1-\delta} = -\log\left(1-\frac{\delta}{d_1}\right)$ and the expression for the capacity of this half is

$$C_2=\frac{bl}{4\pi d_1}\left(1+\frac{1}{2}\frac{\delta}{d_1}+\frac{1}{3}\frac{\delta^2}{d_1^2}+\ldots\right)$$

The sum of these, which is the capacity of the entire condenser, is

$$C=C_1+C_2=\frac{2bl}{4\pi d_1}\left(1+\frac{1}{3}\frac{\delta^2}{d_1^2}+\frac{1}{5}\frac{\delta^4}{d_1^4}+\ldots\right) \qquad (7)$$

But, if the plates were parallel and at the same mean distance, their capacity would be

$$C_0=\frac{2bl}{4\pi d_1}$$

$$\therefore \frac{\Delta_2 C}{C_0}=\frac{C-C_0}{C_0}=\frac{1}{3}\frac{\delta^2}{d_1^2}+\frac{1}{5}\frac{\delta^4}{d_1^4}+\ldots \qquad (8)$$

Hence the increase in the capacity of the coaxial cylinders due to a slight tipping of the inner with respect to the outer can not be greater than the quantity given in (8), which is obviously negligible when δ/d_1 is very small. If δ/d_1 is 0.01, $\frac{\Delta_2 C}{C_0}$ amounts to about 1 part in 30,000.

By the method of adjustment described on page 485 it is evident that the greatest possible tipping will occur when the bottom of a section is displaced 0.025 mm in one direction and the top an equal amount in the other direction. This would correspond to $\delta=$ 0.0025 cm. Substituting this in the expression for $\frac{\Delta_2 C}{C_0}$ for these cylinders, we find that the maximum error due to the relative tipping of the axis of the cylinders can not exceed about 2 in 1,000,000. Usually the tipping was much less than that here assumed.

28. CORRECTION FOR ERRORS IN FIGURING.

These errors are due (1) to conicality of the cylinders, (2) to ellipticity, (3) to other irregularities. The sections of all the cylinders are quite accurately circular, and the smaller irregularities in the surface are too slight to cause an error as large as 1 part in 100,000. The outer cylinders are, however, slightly variable in section as we pass along their length, so that there is an appreciable conicality, the effect of which needs to be investigated.

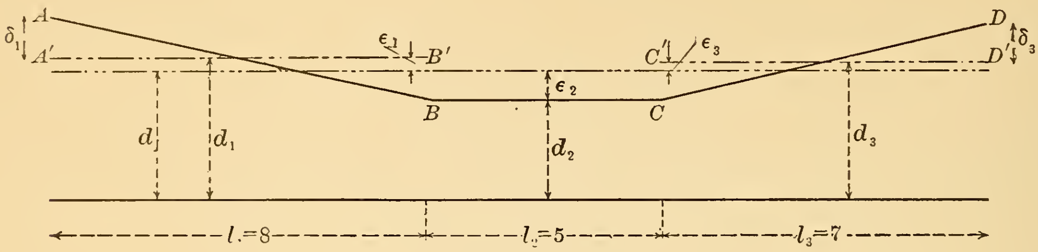

Fig. 14.

If d_1 is the mean distance between the cylinders for the first section, and δ_1 is the maximum variation of the distance from this mean for this conical portion, we can calculate the difference in the capacity between this conical section and one of uniform distance d_1 by the formula of the preceding section. This may be done for each section. Then we can calculate the difference in capacity between the several cylindrical sections of distances d_1, d_2, d_3, etc., and the single cylinder having the same mean distance d. The sum of these differences will be the correction sought. Let us take a parallel plate condenser that is made up of three sections, two of which, AB and CD, are not parallel. $AA'=\delta_1$, $DD'=\delta_3$. The parallel plates A′B′ and C′D′, which are nearly equivalent to AB and CD, are distant $d+\epsilon_1$, and $d+\epsilon_3$; and BC is distant $d-\epsilon_2$ from the lower plate.

The first section of distance $d_1=d+\epsilon_1$ has a capacity

$$C_1=\frac{A_1}{4\pi d_1}=\frac{bl_1}{4\pi}\cdot\frac{1}{d+\epsilon_1}=\frac{bl_1}{4\pi d}\left(1+\frac{\epsilon_1}{d}\right)^{-1}$$

$$\text{or, } C_1=\frac{bl_1}{4\pi d}\left\{1-\frac{\epsilon_1}{d}+\frac{\epsilon_1^2}{d^2}-\ \cdot\ \cdot\ \cdot\right\}$$

$$\text{Similarly, } C_2=\frac{bl_2}{4\pi d}\left\{1+\frac{\epsilon_2}{d}+\frac{\epsilon_2^2}{d^2}+\ \cdot\ \cdot\ \cdot\right\}\text{assuming } \epsilon_2 \text{ is negative.}$$

$$C_3=\frac{bl_3}{4\pi d}\left\{1-\frac{\epsilon_3}{d}+\frac{\epsilon_3^2}{d^2}-\ \cdot\ \cdot\ \cdot\right\}$$

Adding these equations for the total capacity of the three sections we have, since the total area A is $bl_1+bl_2+bl_3$ and $l_1\epsilon_1+l_3\epsilon_3=l_2\epsilon_2$

$$C=\frac{A}{4\pi d}\left\{1+\frac{l_1\epsilon_1^2+l_2\epsilon_2^2+l_3\epsilon_3^2}{ld^2}+\ .\ .\ .\right\}$$

$$\therefore\ \frac{\Delta_3 C}{C_0}=\frac{l_1}{l}\frac{\epsilon_1^2}{d^2}+\frac{l_2}{l}\frac{\epsilon_2^2}{d^2}+\frac{l_3}{l}\frac{\epsilon_3^2}{d^2}$$

Thus, the increase in capacity is the weighted mean of $\left(\frac{\epsilon}{d}\right)^2$ for all the sections, while the difference between the conical sections and the corresponding parallel ones is ⅓ the weighted mean of $\left(\frac{\delta}{d}\right)^2$

As an example, suppose $l_1=8$, $l_2=5$, $l_3=7$ cm

Take $\epsilon_1=+0.01$	mm	$\delta_1=0.054$ mm	$d=1$ cm
$\epsilon_2=-0.044$	"	$\delta_2=0$	
$\epsilon_3=+0.02$	"	$\delta_3=0.064$ mm	

These are small differences, but very appreciable, being of the same order of magnitude as those of our cylinders, as shown by Tables III, IV, V, and VI.

$\Sigma\left(\frac{\epsilon}{d}\right)^2$ will here be 0.0000066

$\frac{1}{3}\Sigma\left(\frac{\delta}{d}\right)^2$ " " " 0.0000087

Sum = 0.0000153 = 1.5 parts in 100,000.

Applying this method to the cases of the outer cylinders Nos. 2 and 3, we find:

Correction for No. 2 = +1.0 part in 100,000

" " No. 3 = +1.8 " " 100,000.

The mean of these is 1.4 parts in 100,000 in the electrostatic capacity, the latter being larger than if the cylinders were perfectly cylindrical. These cylinders could be reground with our present facilities so that the correction due to conicality would be not more than one-tenth as great, and hence wholly negligible in the most refined work.

29. CORRECTION FOR OFFSETS.

In the treatment of this and several succeeding problems we shall, as a first approximation, assume that we may consider each element of the cylindrical condenser as an element of a parallel plate condenser. The problem of obtaining the distribution of charge along the length of the condenser then becomes a two-dimensional problem and may be solved by the method employed by J. J. Thomson in the third chapter of "Recent Researches." The method consists in transforming the polygon defined by the section of the condenser by a plane through its axis (call this the z plane) into the real axis of what we shall call the t plane. The polygon, defined by this section, in the w plane ($w = \phi + i\psi$, where ϕ is the stream function and ψ is the potential function) is also to be transformed into the real axis of the t plane and made to correspond there point by point with the other polygon. By eliminating t from these two equations of transformation we get ϕ and ψ as function of x and y, $(x + iy = z)$, and so can obtain the distribution.

The transformation to be employed is

$$\frac{dz}{dt} = C\left(t - t_1\right)^{\frac{a_1}{\pi}-1}\left(t - t_2\right)^{\frac{a_2}{\pi}-1}\left(t - t_3\right)^{\frac{a_3}{\pi}-1} \ldots$$

where a_1, a_2, a_3 . . . . are the interior angles of the polygon at the vertices corresponding to $t = t_1$, $= t_2$, $= t_3$ . . .

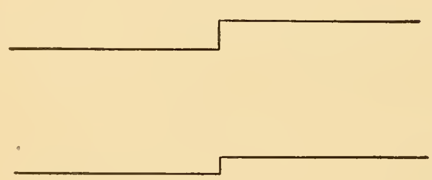

Fig. 15.

The general problem we desire to solve is that of two parallel broken planes, the two steps being in the same normal plane and of unequal height (Fig. 15). While this problem can be solved, it is easier to consider another problem, viz, the problem in which there is a step in but one of the planes; from this, by the method of images, we at once get the solution of the problem in which there are two equal steps. From these two problems we can obtain with sufficient accuracy the effect of the second step.

A condenser consisting of an infinite plane parallel to two semi-infinite planes joined by a vertical step.—The diagram in the z plane is as shown in Fig. 16. Let p and q be the distances of the infinite plane from the two semi-infinite planes. Then $q-p$ will be the height of the step.

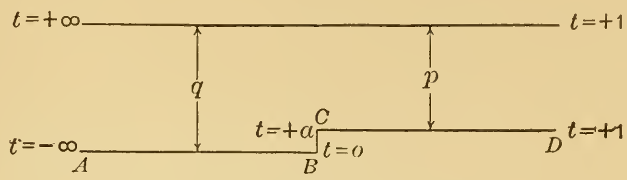

Fig. 16.

Take $t=+\infty$ at $x=-\infty$ on the infinite plate,
$t=+1$ at $x=+\infty$ on the infinite plate,
$t=0$ at B, the foot of the step; B is taken as the origin. Then at the top of the step, t will have some value a, between 0 and $+1$, to be determined from the dimensions of the apparatus. The transformation is

$$\frac{dz}{dt}=\frac{C}{t-1}\sqrt{\frac{t-a}{t}}$$

$$\therefore z=C\left[\log\left(\sqrt{t(t-a)}+t-\frac{a}{2}\right)-\sqrt{1-a}\,\log\left(\frac{\sqrt{t(t-a)}+\sqrt{1-a}}{t-1}+\frac{2-a}{2\sqrt{1-a}}\right)\right]$$

$$\therefore z=-\frac{q}{\pi}\left[\log\frac{2\sqrt{t(t-a)}+2t-a}{a}-\sqrt{1-a}\,\log\frac{2\sqrt{1-a}\sqrt{t(t-a)}-a+t(2-a)}{a(t-1)}\right]+iq$$

if the origin is taken at the point $t=0$. But, as t decreases through $+1$, z must decrease by ip,

$$\therefore \sqrt{1-a}=\frac{p}{q}$$

$$\therefore z = -\frac{q}{\pi}\log\frac{2q\sqrt{t[tq^2-(q^2-p^2)]}-(q^2-p^2)+2tq^2}{q^2-p^2}$$

$$+\frac{p}{\pi}\log\frac{2p\sqrt{t[tq^2-(q^2-p^2)]}-(q^2-p^2)+(q^2+p^2)t}{(q^2-p^2)(t-1)}+iq$$

The diagram in the w plane consists of two infinite lines (Fig. 17), and the transformation is

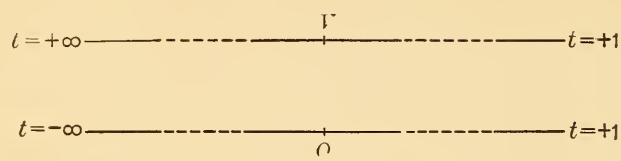

Fig. 17.

$$\frac{dw}{dt}=\frac{B}{t-1}$$

$$\therefore w=-\frac{V}{\pi}[\log(t-1)-i\pi]$$

if the potential of the upper plane is V and that of the lower one is zero. Hence the charge on a unit strip bounded by t_1 and t_2, $t_1 < t_2$, is

$$\sigma=-\frac{1}{4\pi}\frac{\partial\psi}{\partial\nu}=-\frac{1}{4\pi}\frac{\partial\phi}{\partial s}$$

$$E=\int_{s_1}^{s_2}\sigma ds=-\frac{1}{4\pi}[\phi(t_2)-\phi(t_1)]$$

$$\therefore E=\frac{V}{4\pi^2}\log\frac{t_2-1}{t_1-1}$$

Hence, the charge on the semi-infinite plane at the top of the step, and extending from $t=a$ to $t=t_1$ (t_1 nearly $=1$) is

$$E_1=\frac{V}{4\pi^2}\log\frac{1-t_1}{1-a}$$

But, for t nearly equal to but less than 1, we have

$$x_1=-\frac{q}{\pi}\log\frac{q+p}{q-p}+\frac{p}{\pi}\log\frac{4p^2}{q^2-p^2}-\frac{p}{\pi}\log(1-t_1)$$

$$\therefore E_1 = -\frac{V}{4\pi p}\left[x_1 + \frac{q}{\pi}\log\frac{q+p}{q-p} - \frac{p}{\pi}\log\frac{4q^2}{q^2-p^2}\right]$$

If q differs but slightly from p, this expression can be put in a form more suitable for computation by putting

$$q = p\,(1+a)$$

Then,

$$\frac{q+p}{q-p} = \frac{2+a}{a}$$

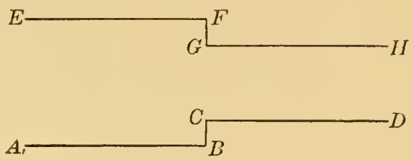

Fig. 18.

$$\frac{4q^2}{q^2-p^2} = \frac{4(1+a)^2}{a(2+a)}$$

$$\therefore E_1 = -\frac{V}{4\pi p}\left[x_1 + \frac{p}{\pi}\left\{\log\frac{2+a}{a}\cdot\frac{a(2+a)}{4(1+a)^2} + a\log\frac{2+a}{a}\right\}\right]$$

$$= -\frac{V}{4\pi p}\left[x_1 + \frac{p}{\pi}\left\{a\log\frac{2+a}{a} - \log\left(1+a-\frac{a^2}{4}\right)\right\}\right]$$

$$= -\frac{V}{4\pi p}\left[x_1 + \frac{pa}{\pi}\left\{\log\frac{2+a}{a} - 1 + \frac{3a}{4}\ldots\right\}\right]$$

By the method of images, we find that the corresponding expression for the case of two equal steps, as shown in Fig. 18, is

$$E_1' = -\frac{V}{4\pi p}\left[x_1 + \frac{pa}{\pi}\left\{\log\frac{(1+a)}{a} - 1 + \frac{3a}{2}\right\}\right]$$

in which p (the distance between GH and CD) and the height of each step (ap) have the same numerical values as before:

$$\therefore E_1 - E_1' = -\frac{V}{4\pi p}\cdot\frac{pa}{\pi}\left\{\log\frac{2+a}{a} - \log\frac{1+a}{a} - \frac{3a}{4}\right\}$$

$$= -\frac{Va}{4\pi^2}\left\{\log\left(2-\frac{a}{1+a}\right) - \frac{3a}{4}\right\}$$

$$= -\frac{Va}{4\pi^2}\left\{\log 2 - \frac{5a}{4} + \ldots\right\}$$

$$= -\frac{Va}{4\pi}(0.21)\text{ approximately.} \qquad (9)$$

Hence the effect of moving the portion EF of the plane away from AB by an amount ap is to decrease the effective length of the portion CD of the other plane by 0.21 ap. If it had been moved in the other direction the effective length would have been increased by the same absolute amount. Hence if there is a step of height a_1p in the upper plane and one of height ap in the lower one (the planes of the steps coinciding) the charge on the section CD is very nearly equal to that on a section of one of a pair of infinite parallel planes at a distance apart equal to that of the section CD and of length greater than that of CD by an amount

$$\delta l_1 = \frac{pa}{\pi}\left\{\log\frac{2+a}{a} - 1 + \frac{3a}{4}\right\} \mp \frac{pa_1}{\pi}\left(\log 2 - \frac{5a_1}{4}\right)$$

The minus sign is to be taken if the relative positions of the steps is as shown in Fig. 18.

The charge on the side of the step, between $t=o$ and $t=a$, is

$$E_2 = \frac{V}{4\pi^2}\log(1-\alpha) = -\frac{V}{4\pi^2}\log\frac{q^2}{p^2}$$

To reduce this to a form suitable for computation, when q is nearly equal to p, put $q=p(1+a)$, and we find

$$E_2 = -\frac{V}{4\pi^2}\log(1+a)^2$$
$$= -\frac{V}{2\pi^2}\left\{a - \frac{a^2}{2} + \ldots\right\} \qquad (10)$$

By the method of images we find for the case of two equal steps, as shown in Fig. 18, the expression

$$E'_2 = -\frac{V}{4\pi^2}\log\frac{q_1}{p_1}$$

$$\therefore E_2 - E'_2 = -\frac{V}{4\pi^2}\log\frac{q^2}{p^2}\cdot\frac{p_1}{q_1}$$

If we regard p as remaining fixed, then

$$p_1 = p$$
$$q_1 = q + ap = p(1+2a)$$

$$\therefore E_2 - E'_2 = -\frac{V}{4\pi^2}\log\frac{(1+a)^2}{1+2a}$$

$$= -\frac{V}{4\pi^2}(a^2 - 2a^3 + \ldots)$$

Hence the effect of moving EF away from AB by an amount ap is to decrease the charge on the side of the step by the amount, $\frac{Va^2}{4\pi^2}$, approximately. Similarly, if we regard q as remaining fixed, we must take

$$q_1 = q = p(1+a)$$
$$p_1 = p(1-a)$$

$$\therefore E_2 - E'_2 = -\frac{V}{4\pi^2}\log(1-a^2)$$

$$= +\frac{V}{4\pi^2}\left(a^2 + \frac{a^4}{2} + \ldots\right)$$

Therefore, the effect of moving GH toward CD by an amount ap is to increase the charge on the side of the step by an amount $\frac{Va^2}{4\pi^2}$, approximately. It is readily seen that these effects are exactly equal to half the effect produced by moving the entire upper plane by the same amount. Hence the addition of a step equal to βp in the upper plane will increase the charge on the side of the step in the lower plane by $\frac{\beta a V}{4\pi^2} = \frac{V}{4\pi}(0.32a\beta)$. Since this effect varies as the product of a and β, it is very small as compared with the effects given by expressions (9) and (10), if a and β are small.

The charge on the semi-infinite plane at the bottom of the step between, $t=o$ and $t=-\tau$, where τ is very large, is

$$E_3 = \frac{V}{4\pi^2}\log\left(\frac{1}{1+\tau}\right)$$
$$= -\frac{V}{4\pi^2}\log\tau, \text{ approximately.}$$

But for $t=-\tau$, τ being large, we have

$$x_{-\infty} = -\frac{q}{\pi}\log\tau + \frac{q}{\pi}\log\frac{q^2-p^2}{4q^2} + \frac{p}{\pi}\log\frac{q+p}{q-p}$$

$$\therefore E_3 = -\frac{V}{4\pi q}\left[X - \frac{q}{\pi}\log\frac{4q^2}{q^2-p^2} + \frac{p}{\pi}\log\frac{q+p}{q-p}\right]$$

where $X = -x_{-\infty}$. That is, X is the distance from B. To reduce this expression to a form more suitable for computation when q is nearly equal to p, put

$$p = q\,(1-\beta)$$

Then

$$E_3 = -\frac{V}{4\pi q}\left[X - \frac{q}{\pi}\log\frac{4}{\beta\,(2-\beta)} + \frac{q\,(1-\beta)}{\pi}\log\frac{2-\beta}{\beta}\right]$$

$$= -\frac{V}{4\pi q}\left[X - \frac{q\,\beta}{\pi}\left\{1 + \log\frac{2-\beta}{\beta} + \frac{\beta}{4} + \ldots\right\}\right]$$

Applying the method of images, we find for the case of equal steps, as shown in Fig. 18,

$$E_3' = -\frac{V'}{4\pi q'}\left[X - \frac{q'\beta'}{\pi}\left\{1 + \log\frac{2-\beta'}{\beta'} + \frac{\beta'}{4}\ldots\right\}\right]$$

where p' and q' are the half distances between the plates, V' is half the potential difference, and $p' = q'\,(1-\beta')$. If we take $2\,q' = q$, and $p' = p - \beta q$, we find

$$\beta' = 2\,\beta$$

$$\therefore E_3' = -\frac{V}{4\pi q}\left[X - \frac{q\,\beta}{\pi}\left\{1 + \log\frac{1-\beta}{\beta} + \frac{\beta}{2} + \ldots\right\}\right]$$

$$\therefore E_3 - E_3' = +\frac{V\beta}{4\pi^2}\left\{\log\frac{2-\beta}{1-\beta} - \frac{\beta}{4}\right\}$$

$$= +\frac{V\beta}{4\pi^2}\left\{\log 2 + \frac{\beta}{4} + \frac{3\,\beta^2}{8} + \ldots\right\}$$

Hence the effect of moving the portion GH of the plane nearer to CD by an amount βq is to increase the effective length of the portion AB by an amount

$$\frac{\beta q}{\pi}\left\{\log 2 + \frac{\beta}{4} + \ldots\right\}$$

Which for small values of β becomes, approximately, 0.21 βq. Hence if there is a step of $\beta_1 q$ in the upper plane and one of βq in the lower one—the planes of the steps coinciding—the charge on the section AB is very nearly equal to that of a section of one of a pair of infinite parallel planes at a distance apart equal to that of the section AB and of a length greater than that of AB by an amount

$$\delta l_3 = -\frac{q\beta}{\pi}\left\{1+\log\frac{2-\beta}{\beta}+\frac{\beta}{4}\right\} \pm \frac{q\beta_1}{\pi}\left\{\log 2+\frac{\beta_1}{4}+\ldots\right\}$$

The plus sign is to be taken if the relative position of the steps is as shown in Fig. 18.

If there is no offset in the upper plate, the total charge on the plate AD is

$$E = -\frac{V}{4\pi}\left[\begin{array}{l} \frac{x_1}{p}+\frac{a}{\pi}\left\{\log\frac{2+a}{a}-1+\frac{3a}{4}+\ldots\right\} \\ \quad +\frac{a}{\pi}\left\{2-a+\ldots\right\} \\ +\frac{X}{q}-\frac{\beta}{\pi}\left\{\log\frac{2-\beta}{\beta}+1-\frac{\beta}{4}+\ldots\right\} \end{array}\right]$$

But in this case

$$1-\beta = \frac{1}{1+a}$$

$$\therefore \beta = \frac{a}{1+a}$$

$$\therefore E = -\frac{V}{4\pi}\left[\begin{array}{l} \frac{x_1}{p}+\frac{a}{\pi}\left\{\log\frac{2+a}{a}-1+\frac{3a}{4}\ldots\right\} \\ \quad +\frac{a}{\pi}\left\{2-a\ldots\right\} \\ +\frac{X}{q}-\frac{a}{\pi}\left\{\frac{1}{1+a}\log\frac{2+a}{a}+1-\frac{5a}{4}\ldots\right\} \end{array}\right]$$

$$= -\frac{V}{4\pi}\left[\frac{x_1}{p}+\frac{X}{q}+\frac{a}{\pi}\left\{\frac{a}{1+a}\log\frac{2+a}{a}+a+\ldots\right\}\right]$$

Hence the effect of a step in one of a pair of parallel plates is to increase the magnitude of the charge by

$$\delta E = \frac{Va^2}{4\pi^2}\left\{1 + \frac{1}{1+a} \log \frac{2+a}{a} + \ . \ . \ .\right\}$$

Let us now determine the distance to which the effect of the step makes itself felt. To do this, differentiate the charge with respect to the distance to get the density, and determine the distance we must go from the step before the density becomes sensibly equal to its value at infinity.

Side CD at the top of the step:

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial t} \cdot \frac{\partial t}{\partial x} = + \frac{V}{4\pi^2} \cdot \frac{1}{1-t_1} \cdot \frac{(t_1 - 1)\pi}{q} \sqrt{\frac{t_1}{t_1 - a}}$$

$$= -\frac{V}{4\pi q}\sqrt{\frac{t_1}{t_1 - a}} \equiv \sigma_1$$

Put $t_1 = 1 - \epsilon$, then

$$\sigma_1 = -\frac{V}{4\pi q}\sqrt{\frac{1-\epsilon}{1-a-\epsilon}}$$

$$= -\frac{V}{4\pi q} \cdot \frac{1}{\sqrt{1-a}}\sqrt{1 - \epsilon + \frac{\epsilon}{1-a}}$$

$$= -\frac{V}{4\pi q} \cdot \frac{1}{\sqrt{1-a}}\left\{1 + \frac{a\ \epsilon}{2\ (1-a)}\right\}$$

$$= -\frac{V}{4\pi p}\left\{1 + \tfrac{1}{2}\ \frac{a\ \epsilon}{1-a}\right\}$$

The density at $x = \infty$ is

$$\sigma_0 = -\frac{V}{4\pi p}$$

$$\therefore \frac{\sigma_1 - \sigma_0}{\sigma_0} = \frac{a\ \epsilon}{2\ (1-a)}$$

If this ratio is equal to a very small quantity, say η, then

$$\epsilon = \frac{2\ (1-a)\eta}{a}$$

or

$$\epsilon = \frac{2p^2\eta}{q^2-p^2}$$

Hence, the distance at which $\frac{\sigma_1-\sigma_0}{\sigma_0}=\eta$ must sensibly be

$$x_1 = -\frac{q}{\pi}\log\frac{q+p}{q-p}+\frac{p}{\pi}\log\frac{4p^2}{q^2-p^2}-\frac{p}{\pi}\log\frac{2p^2\eta}{q^2-p^2}$$

$$= -\frac{q}{\pi}\log\frac{q+p}{q-p}+\frac{p}{\pi}\log\frac{2}{\eta}$$

Example:

If $q=1.0000$ cm, $q-p=0.006$ cm,
then,

$$x_1 = \frac{0.994}{\pi}\log\frac{1}{\eta} - 1.629$$

For $\eta = 10^{-5}$ this becomes

$$x_1 = 2.01 \text{ cm}$$

Hence, for such a step, the density on the plane at the top of the step and at a distance from it greater than 2 cm does not differ from its value at an infinite distance by more than 1 in 100,000.

Side AB at the bottom of the step:

$$\frac{\partial E}{\partial \tau}\cdot\frac{\partial \tau}{\partial x} = -\frac{V}{4\pi^2}\cdot\frac{1+\tau}{(1+\tau)^2}\cdot\frac{\pi(1+\tau)}{q}\sqrt{\frac{\tau}{a+\tau}}$$

$$= -\frac{V}{4\pi q}\sqrt{\frac{\tau}{a+\tau}} \equiv \sigma_3'$$

Here,

$$\sigma_0 = -\frac{V}{4\pi q}$$

$$\therefore \frac{\sigma_3-\sigma_0}{\sigma_0} = -1+\sqrt{\frac{\tau}{a+\tau}}$$

Since τ is very large we may write

$$\frac{\sigma_3-\sigma_0}{\sigma_0} = -1+\left(1+\frac{a}{\tau}\right)^{-\frac{1}{2}} = +\frac{1}{2}\frac{a}{\tau}$$

If this ratio is to be a very small quantity, say η, then

$$\frac{1}{2}\frac{a}{\tau}=\eta$$

$$\text{or,}\quad \tau=\frac{1}{2}\frac{a}{\eta}$$

Hence, the distance at which $\frac{\sigma_3-\sigma_0}{\sigma_0}=\eta$ must sensibly be

$$X\equiv -x_{-\infty}=\frac{q}{\pi}\log\frac{a}{2\eta}-\frac{q}{\pi}\log\frac{q^2-p^2}{4q^2}-\frac{p}{\pi}\log\frac{q+p}{q-p}$$

$$=\frac{q}{\pi}\log\frac{2}{\eta}-\frac{p}{\pi}\log\frac{q+p}{q-p}$$

Example:

If $q=1.000$ cm; $q-p=0.006$ cm

then, $$X=0.318\log\frac{1}{\eta}-1.398$$

For $\eta=10^{-5}$ this becomes

$$X=2.26\text{ cm}$$

Let us now apply these equations to the cases that arise in the work with the cylinders. Denoting by h the distance between the opposed surfaces at the end of the sections; by δ the excess of the radius of the upper section of the inner cylinder above that of the lower one, measured at the adjoining ends, and by δ_1, the corresponding quantity for the outer cylinders, we obtain from our measurements the following table:

TABLE XI.

Junction of sections	h		δ	δ_1
	Upper	Lower		
1 and 2	0.98793	0.98434	0.00074	0.00433
1 and 3	0.98351	0.98434	0.00016	—0.00067
2 and 3	0.98351	0.98559	—0.00011	—0.00219
2 and 4	0.98099	0.98559	—0.00110	—0.00570
3 and 4	0.98099	0.98604	—0.00107	—0.00612

The greatest correction term will evidently be that which arises from the offsets at the junction of sections 2 and 4. Taking p as the distance between the surfaces at the upper end of section 2, we have:

$$p=0.98559$$
$$q=0.98099$$
$$a=0.00111_5$$
$$\beta=0.00111_9$$
$$a_1=0.00578_2$$
$$\beta_1=0.00580_8$$

Hence, the charge on the inner cylinder of section No. 2—neglecting the face of the step—is greater than it would be on an equal length of an infinite cylinder of the same cross sectional dimensions by an amount

$$\frac{V}{4\pi p}\Delta l_1=\frac{V}{4\pi_2}\left[a\left\{\log\frac{2+a}{a}-1+\frac{3a}{4}\right\}+a_1\left\{\log 2-\frac{5a_1}{4}\right\}\right]=0.01121\frac{V}{4\pi_2}$$

Similarly, the charge on the inner cylinder of section No. 4 is greater than it would be on an equal length of an infinite cylinder of the same cross sectional dimensions by

$$\frac{V}{4\pi q}\Delta l_3=-\frac{V}{4\pi^2}\left[\beta\left\{\log\frac{2-\beta}{\beta}+1+\frac{\beta}{4}\right\}+\beta_1\left\{\log 2+\frac{\beta_1}{4}\right\}\right]=-0.01353\frac{V}{4\pi^2}$$

The charge on the side of the step is

$$\frac{V}{4\pi^2}\left[2a\left(1-\frac{a}{2}\right)+aa_1\right]=0.002235\frac{V}{4\pi^2}$$

Hence the net effect of these offsets is to increase the total charge on the inner cylinder by

$$-0.00008\frac{V}{4\pi^2}$$

which is equivalent to decreasing the length of the condenser by only 0.2μ, a quantity entirely negligible.

Hence the only case in which the presence of the offsets can produce an appreciable effect is when they occur at the gap separating the guard cylinder from the condenser proper, under which condition the charge corresponding to one of the Δl correction terms is not measured.

30. CORRECTION DUE TO IRREGULAR DISTRIBUTION OF CHARGE.

The determination of the various corrections due (1) to the irregular distribution of charge near the end of the inner cylinder, (2) to the charge on the interior of the inner cylinder and (3) to the variation of the charge on the outside of the outer cylinder as the latter is changed in height, are all parts of the same mathematical problem. Being unable to treat directly this problem for the case of circular cylinders, we shall solve two cases for systems of semi-infinite plates, viz, (1) the case of four infinitely thin semi-infinite plates (this will give us an idea of the distribution on the interior of the inner cylinder); (2) the case of a thick semi-infinite plate midway between two infinitely thin semi-infinite plates (this will give us a minimum value for the charge on the end of the inner cylinder when the latter is closed). A more interesting case is that in which the outer plates

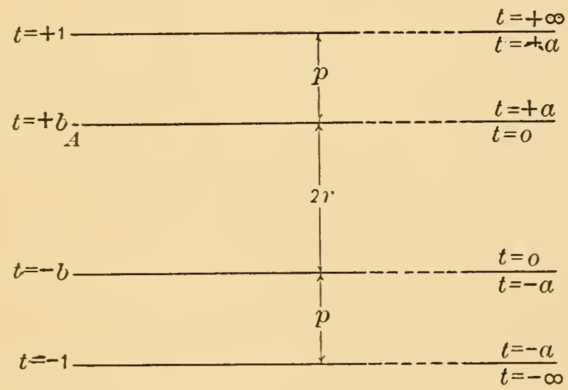

Fig. 19.

are of finite thickness, but the increased mathematical difficulties more than offset the gain.

Case I. *A condenser consisting of two mutually exterior, identical pairs of semi-infinite planes, all bounded by straight lines lying in a plane normal to the four.*—The innermost planes are both at zero potential, while the outermost ones are at the common potential V. Assume that the origin lies at the point A, Fig. 19, the y axis being normal to the planes. Assume $t=+\infty$ at the point $x=\infty$ on the upper side of the top plane; $t=+a$ at the corresponding point on the lower side; $t=+1$ at the bounding edge of the top plane; $t=+b$ at the bounding edge of the next to the top plane; $t=+a$ on the top face of the next to the top plane at $x=\infty$; $t=0$ at the corresponding point on the lower side of this plane. At the corresponding points of the other pair of planes t will have the same absolute values, but will be of the opposite sign. The values of a and b must be determined from the distances between the planes. Let p be the distance between the planes composing either pair, and $2r$ be the distance between the two innermost planes.

The transformation is

$$\frac{dz}{dt} = C\frac{(t^2-1)(t^2-b^2)}{t\,(t^2-a^2)}$$

or,
$$z = -\frac{C}{2a^2}\left[b^2 \log \frac{t^2}{b^2} + (1-a^2)(a^2-b^2) \log \frac{t^2-a^2}{a^2-b^2} - a^2(t^2-b^2) - i\pi\,(1-a^2)(a^2-b^2)\right]$$

The conditions that z shall increase by $2ir$ as t increases through zero, and by ip as t increases through a give us the equations

$$C = \frac{2ra^2}{\pi b^2} = \frac{2pa^2}{\pi(1-a^2)(a^2-b^2)}$$

These, combined with the condition that $x=o$ for $t=+1$, give the equation

$$\frac{ra^2(1-b^2)}{b^2} = 2r \log \frac{1}{b} + p \log \frac{1-a^2}{a^2-b^2} \qquad (11)$$

From the expression for C, we find

$$b^2 = \frac{a^2r(1-a^2)}{p+r-a^2r} \qquad (12)$$

The last two equations allow us to determine the values of a and of b. Replacing C by its value as given above, we find

$$z = -\frac{2r}{\pi} \log \frac{t}{b} - \frac{p}{\pi} \log \frac{t^2-a^2}{a^2-b^2} + \frac{ra^2}{\pi b^2}(t^2-b^2) + ip \qquad (13)$$

The diagram in the w plane consists of two infinite parallel lines; the values of t at the extremities of these lines are $+a$ and $-a$. The transformation is

$$\frac{dw}{dt} = B\frac{1}{t^2-a^2}$$

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$$\therefore\ w=\frac{V}{\pi}\left[\log\frac{t+a}{t-a}+i\pi\right]$$

Hence, the charge per unit length of the strip bounded by $t=t_1$ and $t=t_2$ is

$$E=\frac{V}{4\pi^2}\log\left[\frac{t_2-a}{t_2+a}\cdot\frac{t_1+a}{t_1-a}\right],\ t_1< t_2$$

The charge on the top face of the top plate is obtained by putting $t_1=1$; $t_2=t$, where t is very large. Then

$$E_1=\frac{V}{4\pi^2}\log\left[\frac{t-a}{t+a}\cdot\frac{1+a}{1-a}\right] \tag{14}$$

$$=\frac{V}{4\pi^2}\left[\log\frac{1+a}{1-a}-\frac{2a}{t}\right]$$

But, for t very large

$$t^2=\frac{b^2}{ra^2}\left[\pi x-2r\log b-p\log(a^2-b^2)\right]$$

$$\therefore\ E_1=\frac{V}{4\pi^2}\left[\log\frac{1+a}{1-a}-\frac{2a^2}{b}\sqrt{\frac{r}{\pi x-2r\log b-p\log(a^2-b^2)}}\right]$$

Hence, at a long distance from the edge, the charge on this face will be sensibly equal to

$$\frac{V}{4\pi^2}\left[\log\frac{1+a}{1-a}-\frac{2a^2}{b}\sqrt{\frac{r}{\pi x}}\right]$$

The charge on the under face of the top plate is obtained by putting $t_2=1$, $t_1=t$, where t is nearly equal to a. Then

$$E_2=\frac{V}{4\pi^2}\log\left[\frac{2a}{t-a}\cdot\frac{1-a}{1+a}\right]$$

But, for t nearly equal to, but greater than a

$$x=-\frac{2r}{\pi}\log\frac{a}{b}-\frac{p}{\pi}\log\frac{2a(t-a)}{a^2-b^2}+\frac{ra^2(a^2-b^2)}{b^2\pi}$$

$$\therefore\ E_2=+\frac{V}{4\pi p}\left[x+\frac{p}{\pi}\log\left(\frac{1-a}{1+a}\cdot\frac{4a^2}{a^2-b^2}\right)-\frac{r}{\pi}\left\{\frac{a^2(a^2-b^2)}{b^2}-\log\frac{a^2}{b^2}\right\}\right]$$

The charge on the top face of the next to the top plate is obtained by putting $t_1=b$, $t_2=t$ nearly $=a$. Then

$$E_3=\frac{V}{4\pi^2}\log\left(\frac{a-t}{2a}\cdot\frac{a+b}{a-b}\right)$$

But for t nearly equal to a

$$x=-\frac{2r}{\pi}\log\frac{a}{b}-\frac{p}{\pi}\log\frac{2a(a-t)}{(a^2-b^2)}+\frac{a^2r(a^2-b^2)}{b^2\pi}$$

$$\therefore E_3=-\frac{V}{4\pi p}\left[x+\frac{p}{\pi}\log\frac{4a^2}{(a+b)^2}-\frac{r}{\pi}\left\{\frac{a^2(a^2-b^2)}{b^2}-\log\frac{a^2}{b^2}\right\}\right]$$

The charge on the under side of the next to the top plane is obtained by putting $t_1=t_0$, a very small quantity, and $t_2=b$. Then

$$E_4=\frac{V}{4\pi^2}\log\left[\frac{a-b}{a+b}\cdot\frac{a+t_0}{a-t_0}\right]$$

$$=\frac{V}{4\pi^2}\left[\log\frac{a-b}{a+b}+\frac{2t_0}{a}\right]$$

But, for very small values of t

$$x=-\frac{2r}{\pi}\log\frac{t_0}{b}-\frac{p}{\pi}\log\frac{a^2}{a^2-b^2}-\frac{ra}{\pi}$$

$$\therefore t_0=b\left(\frac{a^2-b^2}{a^2}\right)^{\frac{p}{2r}}e^{-\frac{\pi}{2}\left(\frac{x}{r}+\frac{a^2}{\pi}\right)}$$

$$\therefore E_4=\frac{V}{4\pi^2}\left[\log\frac{a-b}{a+b}+\frac{2b}{a}\left(\frac{a^2-b^2}{a^2}\right)^{\frac{p}{2r}}e^{-\frac{\pi}{2}\left(\frac{x}{r}+\frac{a^2}{\pi}\right)}\right]$$

$$=-\frac{V}{4\pi^2}\left[\log\frac{a+b}{a-b}-\frac{2b}{a}\left(\frac{a^2-b^2}{a^2}\right)^{\frac{p}{2r}}e^{-\frac{\pi}{2}\left(\frac{x}{r}+\frac{a^2}{\pi}\right)}\right] \qquad (15)$$

From these expressions we find the total charge on the inner cylinder between its end and a long distance, x, from the end is

$$E=-\frac{V}{4\pi p}\left[x+\frac{p}{\pi}\log\frac{4a^2}{a^2-b^2}-\frac{r}{\pi}\left\{\frac{a^2(a^2-b^2)}{b^2}-\log\frac{a^2}{b^2}\right\}\right]$$

These equations are sufficient to enable us to determine the edge effect and the variations of the density with the distance from the edge, provided this distance is not too small. However, in order to study the variation of density on the upper face of the next to the top plane it is advisable to obtain a general expression for the density. It is

$$\sigma_3 = -\frac{1}{4\pi}\frac{\partial\phi}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= -\frac{V}{4\pi p}\left[\frac{(1-a^2)(a^2-b^2)}{a}\cdot\frac{t}{(1-t^2)(t^2-b^2)}\right]$$

For $t=a$, the density is uniform and equal to

$$\sigma_0 = -\frac{V}{4\pi p}$$

Hence,

$$\frac{\Delta\sigma}{\sigma_0} = \frac{\sigma_3-\sigma_0}{\sigma_0} = \frac{(1-a^2)(a^2-b^2)}{a}\cdot\frac{t}{(1-t^2)(t^2-b^2)} - 1$$

If the density is sensibly equal to σ_0, this expression must be sensibly zero. If $t^2 = a^2(1-\epsilon)$, we find

$$\frac{\sigma_3-\sigma_0}{\sigma_0} = \frac{(1-a^2)(a^2-b^2)}{a}\cdot\frac{a(1-\epsilon)^{\frac{1}{2}}}{(1-a^2+a^2\epsilon)(a^2-b^2-a^2\epsilon)} - 1$$

$$= \frac{(1-a^2)(a^2-b^2)(1-\epsilon)^{\frac{1}{2}}}{(1-a^2)(a^2-b^2)-a^2\epsilon\,[1-a^2-(a^2-b^2)]-a^4\epsilon^2} - 1$$

$$= (1-\epsilon)^{\frac{1}{2}}\left\{1+a^2\epsilon\left(\frac{1}{a^2-b^2}-\frac{1}{1-a^2}\right)+\dots\right\} - 1$$

$$= \epsilon\left\{a^2\left(\frac{1}{a^2-b^2}-\frac{1}{1-a^2}\right)-\frac{1}{2}\right\}$$

Hence, in order that

$$\frac{\sigma_3-\sigma_0}{\sigma_0} = +10^{-\eta}$$

we must have

$$t^2 = a^2\left[1 - \frac{10^{-\eta}}{a^2\left(\frac{1}{a^2-b^2} - \frac{1}{1-a^2}\right) - 0.5}\right] \qquad (16)$$

Knowing a^2 and b^2, and assuming a suitable value for η, we can find t^2, and from this can find x.

Case 2. *A condenser consisting of a thick semi-infinite plate midway between, and parallel to two infinitely thin semi-infinite parallel plates; the edges of all the plates lie in the same plane normal to the system of plates.*—The diagram in the z plane is as shown in Fig. 20. Assume

$t = +\infty$ at A	$t = -b$ at E
$= +1$ at B	$= -a$ at F
$= +a$ at C	$= -1$ at G
$= +b$ at D	$= -\infty$ at H

The transformation is

$$\frac{dz}{dt} = C\,\frac{t^2-1}{t^2-a^2}\sqrt{t^2-b^2}$$

$$\text{or, } \frac{dz}{dt} = C\left[t^2 - (1+b^2-a^2) - \frac{(1-a^2)(a^2-b^2)}{2a}\left\{\frac{1}{t-a} - \frac{1}{t+a}\right\}\right]\left(\frac{1}{\sqrt{t^2-b^2}}\right)$$

$$\therefore z = C\left[\begin{array}{l} \frac{t}{2}\sqrt{t^2-b^2} - \left(1+\frac{b^2}{2}-a^2\right)\log\left(t+\sqrt{t^2-b^2}\right) \\ -\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2-at}{(t+a)\sqrt{a^2-b^2}}\right. \\ \left. \qquad -\log\frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2+at}{(t-a)\sqrt{a^2-b^2}}\right\} \end{array}\right] + C_1 + i\,D_1$$

C_1 and D_1 being constants of integration.

If $z = 0$ at the point $t = 0$, we have

$$0 = C\left[-\left(1+\frac{b^2}{2}-a^2\right)\log b - i\left\{\left(1+\frac{b^2}{2}-a^2\right)\frac{\pi}{2} + \frac{\pi(1-a^2)\sqrt{a^2-b^2}}{2\,a}\right\}\right] + C_1 + i\,D_1$$

$$\therefore\ C_1 = C\left(1+\frac{b^2}{2}-a^2\right)\log b$$

$$D_1 = C\left[\left(1+\frac{b^2}{2}-a^2\right)\frac{\pi}{2}+\frac{\pi(1-a^2)\sqrt{a^2-b^2}}{2a}\right]$$

$$\therefore z = C\left[\begin{array}{l} \frac{t}{2}\sqrt{t^2-b^2}-\left(1+\frac{b^2}{2}-a^2\right)\log\frac{t+\sqrt{t^2-b^2}}{b} \\ -\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2-at}{(t+a)\sqrt{a^2-b^2}}\right. \\ \qquad\left.-\log\frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2+at}{(t-a)\sqrt{a^2-b^2}}\right\} \\ +\frac{i\pi}{2}\left\{\left(1+\frac{b^2}{2}-a^2\right)+\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right\} \end{array}\right]$$

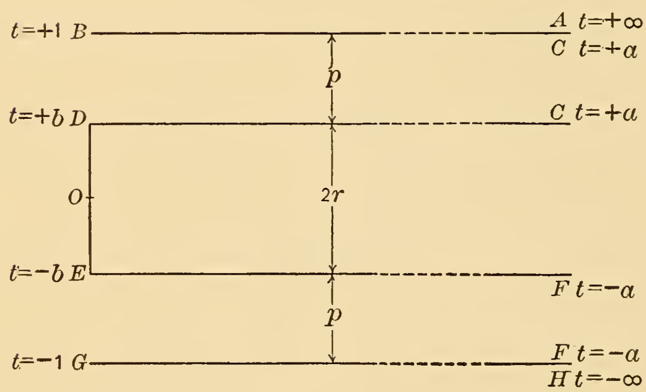

Fig. 20.

For $t=b$, we have $z=ir$

$$\therefore ir = C\left[-\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{-b(a+b)}{(a+b)\sqrt{a^2-b^2}}-\log\frac{b\,(a-b)}{-(a-b)\sqrt{a^2-b^2}}\right\}\right.$$
$$\left.+\frac{i\pi}{2}\left\{1+\frac{b^2}{2}-a^2+\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right\}\right]$$

$$= C\left[1+\frac{b^2}{2}-a^2-\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right]\frac{\pi i}{2}$$

$$\therefore r = C\left[1+\frac{b^2}{2}-a^2-\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right]\frac{\pi}{2} \qquad (17)$$

For $t=1$, $z=i\,(p+r)$

$$\therefore\ i\,(p+r)=C\left[\begin{array}{l}\frac{1}{2}\sqrt{1-b^2}-\left(1+\frac{b^2}{2}-a^2\right)\log\left(\frac{1+\sqrt{1-b^2}}{b}\right)\\ -\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{\sqrt{a^2-b^2}\sqrt{1-b^2}-b^2-a}{(1+a)\sqrt{a^2-b^2}}\right.\\ \qquad\left.-\log\frac{\sqrt{a^2-b^2}\sqrt{1-b^2}-b^2+a}{(1-a)\sqrt{a^2-b^2}}\right\}\\ +\frac{i\pi}{2}\left\{1+\frac{b^2}{2}-a^2+\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right\}\end{array}\right]$$

$$=C\left[\begin{array}{l}\frac{1}{2}\sqrt{1-b^2}-\left(1+\frac{b^2}{2}-a^2\right)\log\left(\frac{1+\sqrt{1-b^2}}{b}\right)\\ -\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\left\{\log\frac{a+b^2-\sqrt{a^2-b^2}\sqrt{1-b^2}}{(1+a)\sqrt{a^2-b^2}}\right.\\ \qquad\left.-\log\frac{a-b^2+\sqrt{a^2-b^2}\sqrt{1-b^2}}{(1-a)\sqrt{a^2-b^2}}\right\}\\ +\frac{i\pi}{2}\left\{1+\frac{b^2}{2}-a^2\right\}\end{array}\right]$$

$$\therefore\ 0=C\left[\frac{1}{2}\sqrt{1-b^2}-\left(1+\frac{b^2}{2}-a^2\right)\log\frac{1+\sqrt{1-b^2}}{b}\right.$$

$$\left.-\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\log\left(\frac{a+b^2-\sqrt{a^2-b^2}\sqrt{1-b^2}}{a-b^2+\sqrt{a^2-b^2}\sqrt{1-b^2}}\cdot\frac{(1-a)}{(1+a)}\right)\right]\qquad(18)$$

and
$$C\left[1+\frac{b}{2}-a^2\right]\frac{\pi}{2}=p+r\qquad(19)$$

As t decreases through the value a, z must decrease by ip

$$\therefore\ C\left[\frac{(1-a^2)\sqrt{a^2-b^2}}{2a}\right]\pi=p\qquad(20)$$

Substituting this value of C in (17), we obtain the relation (19) which becomes

$$\left(1+\frac{b^2}{2}-a^2\right)=\frac{p+r}{p}\left[\frac{(1-a^2)\sqrt{a^2-b^2}}{a}\right]$$

Solving this equation for b^2 we find

$$b^2=\frac{2\,(1-a^2)}{p^2a^2}\left\{\sqrt{[p^2+(1-a^2)\,(2\,pr+r^2)]^2+a^4p^2(2\,pr+r^2)}\right.$$
$$\left.-[p^2+(1-a^2)\,(2\,pr+r^2)]\right\} \quad (21)$$

This and (18) serve to determine a and b when p and r are known. Substituting the value of C in the expression for z and rearranging terms, we find

$$z=\frac{p}{\pi}\cdot\frac{a\,t\,\sqrt{t^2-b^2}}{(1-a^2)\sqrt{a^2-b^2}}-\frac{2\,(p+r)}{\pi}\log\frac{t+\sqrt{t^2-b^2}}{b}$$
$$-\frac{p}{\pi}\log\left(\frac{a\,t+b^2-\sqrt{a^2-b^2}\,\sqrt{t^2-b^2}}{a\,t-b^2+\sqrt{a^2-b^2}\,\sqrt{t^2-b^2}}\cdot\frac{t-a}{t+a}\right)+i\,(p+r)$$

The diagram in the w plane is as before. Hence, the charge per unit length of the strip between the values $t=t_1$, and $t=t_2$; $t_2>t_1$ is

$$E=\frac{V}{4\pi^2}\log\left[\frac{t_2-a}{t_2+a}\cdot\frac{t_1+a}{t_1-a}\right]$$

The total charge on unit length of the middle plate between O and C is

$$E_t=\frac{V}{4\pi^2}\log\left[\frac{t-a}{2\,a}\cdot\frac{a}{-a}\right]=\frac{V}{4\pi^2}\log\frac{a-t}{2\,a}$$

where t is less than but nearly equal to a. But, for t nearly equal to a, we find

$$x=\frac{pa^2}{\pi\,(1-a^2)}-\frac{2\,(p+r)}{\pi}\log\frac{a+\sqrt{a^2-b^2}}{b}$$
$$-\frac{p}{\pi}\log\frac{b^2}{2\,a\,(a^2-b^2)}-\frac{p}{\pi}\log\,(a-t)$$

$$\therefore E_t=-\frac{V}{4\pi p}\left[x+\frac{p}{\pi}\left\{2\log\frac{a+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}+\frac{2\,r}{p}\log\frac{a+\sqrt{a^2-b^2}}{b}-\frac{a^2}{1-a^2}\right\}\right]$$

Hence, the effective length of the thick plate is greater than its actual length by an amount

$$\delta l_t=\frac{p}{\pi}\left[2\log\frac{a+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}+\frac{2\,r}{p}\log\frac{a+\sqrt{a^2-b^2}}{b}-\frac{a^2}{1-a^2}\right] \quad (22)$$

31. EFFECT OF VARYING THE DISTANCE BETWEEN THE CLOSED ENDS OF CONCENTRIC CYLINDERS.

Lord Rayleigh[3] has shown that the force of attraction between two semi-infinite cylinders placed as shown in Fig. 21, and maintained at unit difference of potential is approximately

$$F=\Sigma\frac{(a+b)^2 J_0^2\left[\tfrac{1}{2}k(a+b)\right]}{8b^2\sinh^2 kl.\ J_0'^2(kb)}$$

the summation being with reference to k, which is given by the roots of the equation

$$J_0(kb)=0$$

Hence the values of kb are 2.404, 5.520, 8.657, etc. The upper limit for the force is obtained by putting $a=b$ in the above expression, when it becomes

$$\tfrac{1}{2}\Sigma\left[\sinh kl\right]^{-2}$$

But, the force is

$$F=\frac{dC}{dl}$$

the potential being constant and equal to unity. Hence,

$$\frac{dC}{dl}<\tfrac{1}{2}\sinh^{-2}k_1 l \qquad \text{(approximately)}$$

where $k_1=\frac{2.404}{b}$. If $b=7.25$,

$$l=12,\ \text{then}\ kl=3.98$$

$$\therefore\ \frac{dC}{dl}<0.0007$$

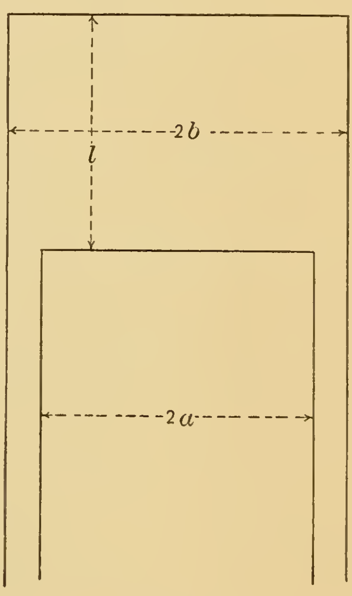

Fig. 21.

Hence, even though l should vary by 0.1 mm the capacity would be changed by but 7×10^{-6} cm, that is by but one ten-millionth part of the capacity of one section of the condenser.

32. CORRECTION FOR THE GAP BETWEEN THE GUARD CYLINDER AND THE MIDDLE CYLINDER.

Here we must take into account the fact that the guard cylinder and the middle cylinder are not continuations of one another, but there is an offset where they come together. Proceeding as before, we shall now solve the problem for a condenser of planes, and shall treat the plates as though they were of infinite thickness. The diagram in the z plane is as shown in Fig. 22.

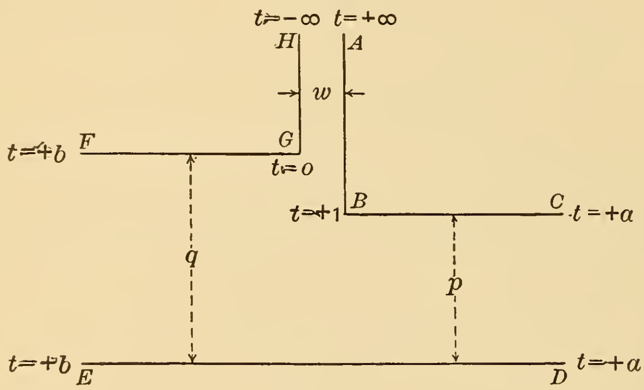

Fig. 22.

Assume $t = -\infty$ at $y = \infty$ on the HG face,

$t = 0$ at G,

$t = +b$ at $x = -\infty$ on the GF or the DE plane,

$t = +a$ at $x = +\infty$ on the BC or the DE plane,

$t = +1$ at B,

$t = +\infty$ at $y = \infty$ on the AB face.

The transformation is given by

$$\frac{dz}{dt} = C\frac{\sqrt{t(t-1)}}{(t-a)(t-b)}$$

$$\text{or, } \frac{dz}{dt} = iC\left[-1 - \frac{1}{a-b}\left\{\frac{b(1-b)}{t-b} - \frac{a(1-a)}{t-a}\right\}\right]$$

$$\therefore z = -iC\sin^{-1}(2t-1)$$

$$-\frac{iC}{a-b}\left\{\begin{array}{l}\sqrt{a(1-a)}\log\dfrac{2\sqrt{t(1-t)}\sqrt{a(1-a)}+t(1-2a)+a}{2(t-a)\sqrt{a(1-a)}}\\ -\sqrt{b(1-b)}\log\dfrac{2\sqrt{t(1-t)}\sqrt{b(1-b)}+t(1-2b)+b}{2(t-b)\sqrt{b(1-b)}}\end{array}\right\}+C_1$$

If we take B as origin, we have

$$0=-iC\left[\frac{\pi}{2}+\frac{1}{a-b}\left\{\sqrt{a(1-a)}\log\frac{1}{2\sqrt{a(1-a)}}\right.\right.$$
$$\left.\left.-\sqrt{b(1-b)}\log\frac{1}{2\sqrt{b(1-b)}}\right\}\right]+C_1$$

$$\therefore z=-iC\left\{\sin^{-1}(2t-1)-\frac{\pi}{2}\right\}$$
$$-\frac{iC}{a-b}\left\{\begin{array}{l}\sqrt{a(1-a)}\log\dfrac{2\sqrt{t(1-t)}\sqrt{a(1-a)}+t(1-2a)+a}{t-a}\\ -\sqrt{b(1-b)}\log\dfrac{2\sqrt{t(1-t)}\sqrt{b(1-b)}+t(1-2b)+b}{t-b}\end{array}\right\}$$

As t decreases through the value a, z increases by $-ip$

$$\therefore -iC=\frac{p(a-b)}{\pi\sqrt{a(1-a)}}$$

Similarly, as t decreases through the value b, z increases by $+iq$

$$\therefore -iC=\frac{q}{\pi}\frac{(a-b)}{\sqrt{b(1-b)}}$$

At G, $t=0$, $z=-w+i(q-p)$

$$\therefore -w+i(q-p)=i\pi C-\frac{\pi C}{a-b}\left\{\sqrt{a(1-a)}-\sqrt{b(1-b)}\right\}$$

$$\therefore -iC=\frac{w}{\pi}=\frac{p(a-b)}{\pi\sqrt{a(1-a)}}=\frac{q(a-b)}{\pi\sqrt{b(1-b)}}$$

Solving these equations for a and b, we find

$$a=\frac{1}{2}\left[\frac{\sqrt{[q^2-p^2+w^2]^2+4p^2w^2}+[q^2-p^2+w^2]}{\sqrt{[q^2-p^2+w^2]^2+4p^2w^2}}\right]$$

$$=\frac{1}{2}\left[1+\frac{q^2-p^2+w^2}{\sqrt{[q^2-p^2+w^2]^2+4p^2w^2}}\right]=\frac{1}{2}\left[1+\frac{q^2-p^2+w^2}{\sqrt{[q^2+p^2+w^2]^2-4q^2p^2}}\right]$$

$$b=\frac{1}{2}\left[\frac{\sqrt{[q^2-p^2-w^2]^2+4w^2q^2}+[q^2-p^2-w^2]}{\sqrt{[q^2-p^2-w^2]^2+4w^2q^2}}\right]$$

$$=\frac{1}{2}\left[1+\frac{q^2-p^2-w^2}{\sqrt{[q^2-p^2-w^2]^2+4w^2q^2}}\right]=\frac{1}{2}\left[1+\frac{q^2-p^2-w^2}{\sqrt{[q^2+p^2+w^2]^2-4q^2p^2}}\right]$$

Substituting the values of iC in the expression for z we find

$$z=\left[\begin{array}{l}\frac{W}{\pi}\left\{\sin^{-1}(2t-1)-\frac{\pi}{2}\right\}\\ +\frac{p}{\pi}\log\frac{2\sqrt{t(1-t)}\sqrt{a(1-a)}+t(1-2a)+a}{t-a}\\ -\frac{q}{\pi}\log\frac{2\sqrt{t(1-t)}\sqrt{b(1-b)}+t(1-2b)+b}{t-b}\end{array}\right]$$

The transformation from the w plane to the t plane is given by

$$\frac{dw}{dt}=\frac{B}{(t-a)(t-b)}$$

$$\therefore w=\phi+i\psi=\frac{V}{\pi}\left[\log\frac{t-b}{t-a}+i\pi\right]$$

Now the density at any point is

$$\sigma=-\frac{1}{4\pi}\frac{d\psi}{dv}=-\frac{1}{4\pi}\frac{d\phi}{ds}$$

$$=-\frac{1}{4\pi}\frac{d\phi}{dt}\cdot\frac{dt}{ds}$$

$\frac{dt}{ds}$ is positive for all these surfaces.

Hence, the charge between t_1 and t_2, $t_2>t_1$, is

$$E=\frac{V}{4\pi^2}\log\left(\frac{t_1-b}{t_1-a}\cdot\frac{t_2-a}{t_2-b}\right)$$

The charge on AB is found by putting $t_1=1$, $t_2=t$, a very large quantity, hence

$$E_1'=\frac{V}{4\pi^2}\left[\log\left(\frac{1-b}{1-a}\right)-\frac{a-b}{t}\right]$$

in the limit this becomes

$$E_1'=\frac{V}{4\pi^2}\log\frac{1-b}{1-a}$$

The charge on BC is found by putting $t_2=1$, $t_1=t$, nearly equal to a, hence

$$E_1''=\frac{V}{4\pi^2}\log\left[\frac{1-a}{1-b}\cdot\frac{a-b}{t-a}\right]$$

But, for t nearly equal to a, we have

$$z=\left[\begin{array}{l}\frac{w}{\pi}\left\{\sin^{-1}(2a-1)-\frac{\pi}{2}\right\}+\frac{p}{\pi}\log 4a(1-a)\\ \quad -\frac{q}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+a(1-2b)+b}{a-b}+\frac{p}{\pi}\log\frac{1}{t-a}\end{array}\right]$$

$$\therefore \log\frac{1}{t-a}=\frac{\pi}{p}\left[\begin{array}{l}x+\frac{w}{\pi}\left\{\frac{\pi}{2}-\sin^{-1}(2a-1)\right\}-\frac{p}{\pi}\log 4a(1-a)\\ +\frac{q}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+a(1-2b)+b}{a-b}\end{array}\right]$$

$$\therefore E_1''=\frac{V}{4\pi p}\left[\begin{array}{l}x+\frac{w}{\pi}\left\{\frac{\pi}{2}-\sin^{-1}(2a-1)-\frac{p}{\pi}\log\frac{4a(1-b)}{a-b}\right.\\ +\frac{q}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+a(1-2b)+b}{a-b}\end{array}\right]$$

Hence, the charge from A to a point on BC far from B is

$$E_1=\frac{V}{4\pi p}\left[x+\delta l_1\right]$$

where,

$$\delta l_1=\left[\begin{array}{l}\frac{w}{\pi}\left\{\frac{\pi}{2}-\sin^{-1}(2a-1)\right\}-\frac{p}{\pi}\log\frac{4a(1-a)}{a-b}\\ +\frac{q}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+a(1-2b)+b}{a-b}\end{array}\right]$$

$$\text{or, }\delta l_1=\left[\begin{array}{l}\frac{w}{\pi}\left\{\frac{\pi}{2}-\sin^{-1}\left(\frac{q^2-p^2+w^2}{\sqrt{[q^2+p^2+w^2]^2-4p^2q^2}}\right)\right\}\\ -\frac{p}{\pi}\log\frac{4p^2}{\sqrt{[q^2+p^2+w^2]^2-4p^2q^2}}\\ +\frac{q}{\pi}\log\frac{(p+q)^2+w^2}{\sqrt{[p^2+q^2+w^2]^2-4p^2q^2}}\end{array}\right] \qquad (23)$$

Similarly, the charge on HG is found by putting $t_1=-\tau$, τ being very large, $t_2=0$

then
$$E_2'=\frac{V}{4\pi^2}\left[\log\frac{a}{b}-\frac{a-b}{\tau}\right]$$

In the limit this becomes

$$E_2'=\frac{V}{4\pi^2}\log\frac{a}{b}$$

The charge between G and a point near F is found by putting $t_1=0$, $t_2=t$, t nearly $= b$, then

$$E_2''=\frac{V}{4\pi^2}\log\frac{b}{a}\cdot\frac{a-b}{b-t}$$

But, for t nearly equal to b, we have

$$x=\left[\begin{array}{l}\frac{w}{\pi}\left\{\sin^{-1}(2b-1)-\frac{\pi}{2}\right\}\\ +\frac{p}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+b(1-2a)+a}{a-b}\\ -\frac{q}{\pi}\log 4b(1-b)-\frac{q}{\pi}\log\frac{1}{b-t}\end{array}\right]$$

$$\therefore \log\frac{1}{b-t}=\frac{\pi}{q}\left[\begin{array}{l}-x+\frac{w}{\pi}\left\{\sin^{-1}(2b-1)-\frac{\pi}{2}\right\}\\ +\frac{p}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+b(1-2a)+a}{a-b}\\ -\frac{q}{\pi}\log 4b(1-b)\end{array}\right]$$

If X= distance from G to the point $(-x)$, $-x=X+w$

$$\therefore E_2''=\frac{V}{4\pi q}\left[\begin{array}{l}X+\frac{w}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}(2b-1)\right\}\\ +\frac{p}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+b(1-2a)+a}{a-b}\\ -\frac{q}{\pi}\log\frac{4a(1-b)}{a-b}\end{array}\right]$$

Hence, charge from H to a point on FG at a great distance, X, from G is

$$E_2=\frac{V}{4\pi q}[X+\delta l_2]$$

where

$$\delta l_2 = \left[\begin{array}{l} \frac{w}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}(2b-1)\right\} \\ +\frac{p}{\pi}\log\frac{2\sqrt{ab(1-a)(1-b)}+b(1-2a)+a}{a-b} - \frac{q}{\pi}\log\frac{4b(1-b)}{a-b} \end{array}\right]$$

$$= \left[\begin{array}{l} \frac{w}{\pi}\left\{\frac{\pi}{2}+\sin^{-1}\left(\frac{q^2-p^2-w^2}{\sqrt{[q^2+p^2+w^2]^2-4p^2q^2}}\right)\right\} \\ +\frac{p}{\pi}\log\frac{(p+q)^2+w^2}{\sqrt{[p^2+q^2+w^2]^2-4p^2q^2}} \\ -\frac{q}{\pi}\log\frac{4q^2}{\sqrt{[p^2+q^2+w^2]^2-4p^2q^2}} \end{array}\right] \qquad (24)$$

If we put $A \equiv \sqrt{[p^2+q^2+w^2]^2-4p^2q^2}$ and rearrange the terms, equations (23) and (24) can be put in the more simple form

$$\delta l = \left[\begin{array}{l} \frac{w}{2}-\frac{w}{\pi}\sin^{-1}\frac{w^2+q^2-p^2}{A}+\frac{p}{\pi}\log\frac{(p+q)^2+w^2}{4p^2} \\ +\frac{q-p}{\pi}\log\frac{(p+q)^2+w^2}{A} \end{array}\right] \qquad (25)$$

with the understanding that δl is the correction for the portion of the plate corresponding to p; nothing is implied as to the relative magnitudes of p and q.

33. NUMERICAL VALUES OF THE CORRECTION TERMS AND ERRORS.

1. **Cylinders without guards.**—Let us now apply these results to the cases that occur in our experimental work. We shall first consider the case in which the plates are of infinitesimal thickness.

(*a*) *At what distance from the end does the distribution of the charge on the opposed surfaces become uniform?*—Referring to equation (16), we find that, if t^2 is equal to, or greater than, the value defined by the expression

$$t^2 = a^2\left[1-\frac{10^{-5}}{a^2\left(\frac{1}{a^2-b^2}-\frac{1}{1-a^2}\right)-0.5}\right]$$

the density of the charge differs from its value at an infinite distance from the end by only 1 in 100,000.

Taking r as the radius of the inner cylinder = 6.25 cm and p as 1 cm we find

$$a^2=0.6926$$
$$b^2=0.4555$$
$$t^2=a^2\,(1-5.85\times10^{-5})$$
$$\therefore\ t^2=0.69256$$

But from equation (13) we know

$$x=-\frac{r}{\pi}\log\frac{t^2}{b^2}-\frac{p}{\pi}\log\frac{a^2-t^2}{a^2-b^2}+\frac{r}{\pi}\frac{a^2(t^2-b^2)}{b^2}$$

Hence, the distance beyond which the density differs from its value at infinity bv an amount which is less than 1 in 100,000 is

$$x=2.52\text{ cm}.$$

Hence, the distribution on the opposing surfaces at the central section of a pair of cylinders 12 cm long can not differ by an appreciable amount from its value at the center of a pair of infinitely long cylinders.

(*b*) *Effect of charge on the interior of the inner cylinder.*—If the cylinders are uncapped, there will be a certain charge on the interior of the inner one, and the magnitude of this charge will increase as the length of the cylinder is increased. It is necessary now to form an estimate of this charge. Throughout the work the lower end of the inner cylinder was always electrically closed. Hence, as the cylinder is reduced in length, the charge on its inner wall will decrease, but the charge on the upper surface of the cap on the lower end will increase. That is, this cap tends to compensate for the removal of the further end of the cylinder, and, if the cylinder is very short, it will overcompensate. Hence, the increase in the total charge on the interior of the inner cylinder as its length is increased from 12 to 32 cm is less than the charge that lies upon a length of 20 cm of the interior, between 12 and 32 cm from the end of a semi-infinite cylindrical condenser. For a series of four thin plates we find, by a slight modification of equation (15) the charge on the interior and between x_1 and x_2 centimeters from the end to be

$$E = \frac{V}{4\pi^2} \cdot \frac{2b}{a} \left(\frac{a^2 - b^2}{a^2} \right)^{-\frac{p}{2r}} e^{-\frac{a^2}{2}} \left\{ e^{-\frac{\pi x_1}{2r}} - e^{-\frac{\pi x_2}{2r}} \right\}$$

The charge per unit length upon the opposed surfaces is

$$\sigma_0 = \frac{V}{4\pi p}$$

$$\therefore \frac{E}{\sigma_0} = \frac{2bp}{\pi a} \left(\frac{a^2 - b^2}{a^2} \right)^{\frac{p}{2r}} e^{-\frac{a^2}{2}} \left\{ e^{-\frac{\pi x_1}{2r}} - e^{-\frac{\pi x_2}{2r}} \right\}$$

If $p = 1$, $r = 6.25$; then $a^2 = 0.6926$, $b^2 = 0.4555$; if we also take $x_1 = 12$, $x_2 = 32$ we find

$$\frac{E}{\sigma_0} = 0.0165 \text{ cm}$$

Hence, the variation of the charge upon the interior of the inner cylinder with the length of the cylinder, is probably inappreciable.

This conclusion is borne out by the results of experiments on the effect of capping the inner cylinder. These experiments showed that the increase in capacity, produced by capping the condenser by means of a cap fitting inside the inner cylinder and resting upon the stiffening flange near its upper end, when the condenser was 32 cm high, did not differ from that when it was 12 cm high by more than 0.0005 micro-microfarads.

(*c*) *Effect of the charge on the outside of the outer cylinder.*—In considering the system of four planes, we found that there is an appreciable charge on the outside of the outer planes even at great distances from the end. The same must be true for the cylinders. Hence, the shortening of the cylinders will tend to decrease the capacity of the remaining portion. This decrease will depend upon the magnitude of the charge originally upon the portion which has been removed, and also upon the distribution of the charge; the further the electrical center of this charge is from the end of the cylinder, the less will the removal of this portion of the outer cylinder affect the capacity of the remaining portion. Being unable to give at present a fair estimate of the actual amount by which the shortening of the cylinder by a given amount will decrease the capacity of the remaining portions, we shall limit ourselves to the calcula-

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tion of the charge on sections of the outside of the outer plane at

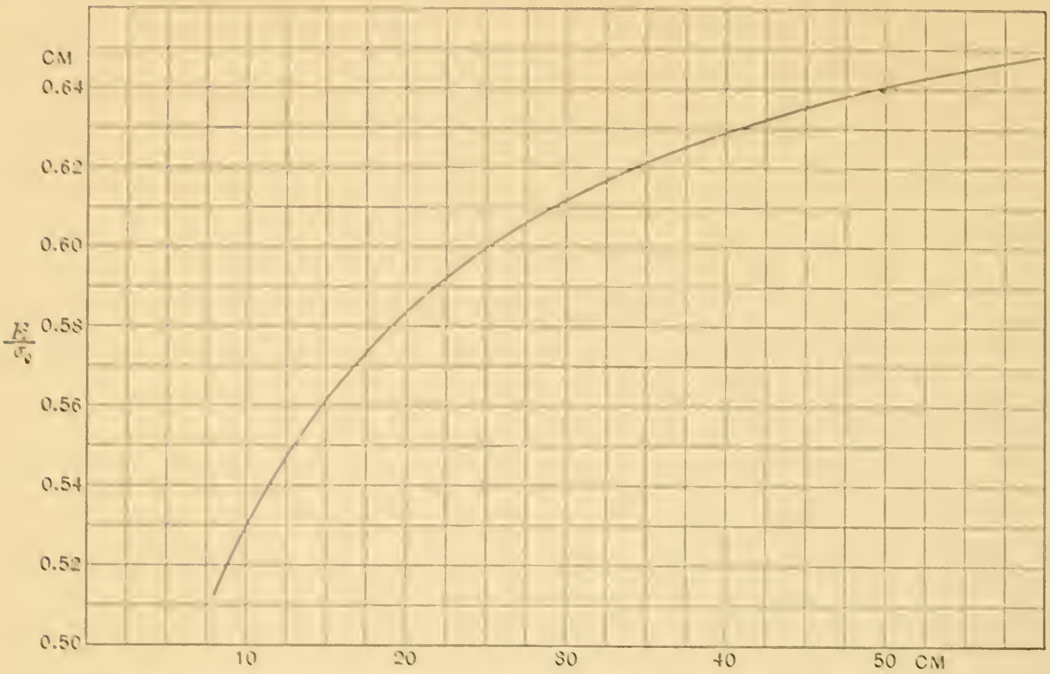

Fig. 23.—*Curve Showing Distribution of Charge on Outside of Outer Cylinder.*

different distances from the end. For this purpose we shall make use of equations (13) and (14), which, for convenience, we may rewrite as follows:

$$x = \frac{r}{\pi}\left[\frac{a^2(t^2-b^2)}{b^2} - \log\frac{t^2}{b^2}\right] - \frac{p}{\pi}\log\frac{t^2-a^2}{a^2-b^2}$$

$$\frac{E}{\sigma_0} = \frac{p}{\pi}\left[\log\frac{1+a}{1-a} - \frac{2a}{t_1} - \frac{2}{3}\left(\frac{a}{t_1}\right)^3 \cdots\right]$$

where t is greater than a. E is the total charge on a strip of the outside of the outer plane, of unit length normal to the plane of the paper, and extending from the edge of the plane to the point where t has the large value t_1; σ_0 is the density on the under surface of this plane and at a long distance from the edge. Taking $p=1$, $r=6.25$, we find $a^2=0.6926$, $b^2=0.4555$; and from these values we obtain the curve connecting the ratio $\frac{E}{\sigma_0}$ with x. This curve is given in Fig. 23, and from it we find that the charge on the section extending from $x=12$ to $x=32$ is equivalent to the charge on a length of 0.72 mm of the opposed surfaces; this is equal to 36 parts in 10,000 of the whole charge on this section. Similarly the charge between $x=32$ and $x=52$ is equal to the charge on a length of 0.26 mm of the opposed surfaces; this equals 13 parts in 10,000 of the whole charge on this section. Hence, it is very possible that the charge on the end of the inner cylinder may be increased by 10 parts in 10,000 of the charge on a 20 cm section of the condenser when the height of the cylinder is increased from 12 cm to 32 cm.

(*d*) *Correction for different diameters.*—There is a further correction due to the fact that the ends of the various sections of the cylinders have slightly different diameters. This will cause the charges on the ends of the various sections to differ. To get an idea as to the magnitude of this variation when the outer cylinder is not capped, it is necessary to study the distribution of charge on the end of a semi-infinite thick plate midway between two equal semi-infinite plates of finite thickness. This problem, however, is more difficult than the one for which the outer planes are of infinitesimal thickness, and its result must lie between that given by the latter and that obtained from the problem of a semi-infinite thick plate midway between two infinite plates. This last problem has been solved,[9] and applies approximately to the case in which the outer cylinder extends some distance beyond the ends of the inner one, and both are capped.

The expression for the excess over its true length of the effective length of a thick plate between two semi-infinite plates of infinitesimal thickness is given by equation (22). It is

$$\delta l_t = \frac{p}{\pi}\left[\frac{2r}{p}\log\frac{a+\sqrt{a^2-b^2}}{b} + 2\log\frac{a+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}} - \frac{a^2}{1-a^2}\right]$$

where $2r$ is the thickness of the plate, $2(p+r)$ is the distance between the thin plates, and a and b are functions of the ratio of r to p.

With the same notation, the expression given by J. J. Thomson for this quantity, when the outermost plates are of infinite length, is

$$\delta l_T = \frac{p}{\pi}\left[\frac{r}{p}\log\frac{2p+r}{r} + 2\log\frac{2p+r}{p}\right]$$

If we take $p=1$, $r=6.25$, the quantities a and b have the values $a^2=0.80340$, $b^2=0.68783$; while if $p=1$, $r=6.25$ (1.001), we find

[9] J. J. Thomson, Recent Researches, § 237.

$a^2=0.80355$, $b^2=0.68804$. From these we find the values in the following table:

p	r	δl_t	δl_T
1	6.25	1.110	1.896
1	6.25 (1.001)	1.110	1.896

Hence, the quantity in the brackets varies but slowly as the ratio of r to p is changed. But the charge per unit length of the plate is inversely proportional to p, hence the additional charge added depends only upon the quantity in the brackets, and so is almost independent of p. Changing the ratio of p to r by so much as 1 in 1,000 will, in our case, change the effective length of the condenser by less than 10 μ, which is only 5 parts in 100,000 of the length of one section.

This concludes the mathematical discussion of the problem of cylinders without guard cylinders; and it is evident that if both cylinders are not capped at both ends, there is a considerable error introduced by the variation in the charges on the ends of the inner cylinder, produced by the change in the distribution of the charge on the outside of the outer cylinder as the length of the condenser is varied. Even were both cylinders capped at both ends it would still be necessary to determine experimentally the effect, if any, of the varying height of the cylinder upon the capacity of the lead wire and commutator.

In this work, the lower end of the inner cylinder was always closed, but the lower end of the outer one was never closed; it was, however, only 17 cm above a large (50 x 70 cm) earthed sheet of metal, and the heavy brass ring carrying the feet of the condenser was separated from the lower end of the outer cylinder by but 2 cm of ebonite (the base of the instrument) and was also earthed. We think this adjustment very completely shielded the lower end of the inner cylinder from variations in charge produced by a change in the height of the outer cylinder. For heights above 24 cm this was tested experimentally as follows:

Sections No. 1 being in place, outer cylinder No. 4 was placed on outer cylinder No. 1, and was closed with a brass plate. The speed being adjusted and held constant by an auxiliary observer, the bridge was exactly balanced. Then, leaving everything else as before, another outer section was placed on top of the brass plate closing No. 4. On repeated trial this was found to produce an inappreciable increase in capacity, certainly not greater than 1 in 200,000 of the maximum capacity of the condenser.

Unfortunately, similar tests can not be made for the upper end of the cylinder. We have, however, made a few runs in which the upper ends of both cylinders were capped. This should completely eliminate the effect of varying the length of the outer cylinder. This being eliminated the remaining errors which we have estimated that we still have to contend with are

Lateral displacement . . .	$\frac{\Delta_1 C}{C_0} <$	$+\ 0.2 \times 10^{-6}$
Inclination of the axis . . .	$\frac{\Delta_2 C}{C_0} <$	$+\ 1.0 \times 10^{-6}$
Errors in figuring (estimated)	$\frac{\Delta_3 C}{C_0} <$	$+\ 20.0 \times 10^{-6}$
Offsets	$\frac{\Delta C}{C_0} <$	$+\ 1.0 \times 10^{-6}$
Variation in distances between the surfaces at the ends of the sections . . .	$\frac{\Delta C}{C_0} <$	50.0×10^{-6}
Variation in position of cap .	$\frac{\Delta C}{C_0} <$	1.0×10^{-6}
Total	$\frac{\Delta C}{C_0} <$	$+\ 70 \times 10^{-6}$

Hence, the actual capacity of the cylinders without guard cylinders, but closed at both ends, may exceed the calculated capacity by about 7 in 100,000. To these errors is to be added the uncertainty introduced by variations in the capacity of the commutator.

2. **Cylinders with guards.**—In this part of the work we are not concerned with the charge on the outside of the outer cylinder or upon the ends of the inner cylinder; and, each guard cylinder being electrically closed, there can be no charge on the inside of the inner cylinder. We have also seen (p. 518) that the irregular distribution of charge on the outer surface of the inner cylinder does not

extend beyond 2.52 cm from the end (to an accuracy of 1 in 100,000 in the density), hence there is no correction to be applied for this irregularity. Also, as we do not have to stop to build up the condenser, we can pass rapidly from a determination of the capacity of the condenser, leads, and commutator to that of the leads and commutator only, and back again. Thus, variations in the capacity of the commutator can produce but a slight effect in the result of even a single determination. The other corrections considered under the discussion of the correction for cylinders without guards apply here, and in addition we must determine the correction for the width of the gap between the guard cylinder and the portion of which the capacity is measured.

In equation (25) we have the expression for the correction for a gap in one electrode of an infinite plate condenser, the distance between the plates being different on the two sides of the gap. Whence we find that the charge on the plate on one side of the gap is

$$E=\frac{V}{4\pi p}(x+\delta l)$$

where

$$\delta l=\frac{w}{\pi}\left[\frac{\pi}{2}-\sin^{-1}\frac{w^2+q^2-p^2}{A}\right]+\frac{p}{\pi}\log\frac{(p+q)^2+w^2}{4p^2}$$
$$+\frac{q-p}{\pi}\log\frac{(p+q)^2+w^2}{A}$$

In this expression p is the distance between the plate under consideration and the plate ED (Fig. 22), q is the corresponding distance for the plate on the other side of the gap, w is the width of the gap, and $A\equiv\sqrt{(q^2+p^2+w^2)^2-4p^2q^2}$

If $q=p(1+\beta)$, where β is a small quantity, we may write

$$\delta l=\frac{w}{\pi}\left[\frac{\pi}{2}-\sin^{-1}\frac{w^2+q^2-p^2}{A}\right]+\frac{p\beta}{\pi}\left[1+\log\frac{(p+q)^2+w^2}{A}-\frac{\beta^2}{4}+\frac{w^2}{4p^2}\right]$$

These expressions are obtained on the hypothesis that the face ED is a continuous plane. If, however, there is an offset in the opposite plane, as shown in Figs. 24, 26, or 27, the problem is more complicated. If HGF represents the portion corresponding to the guard cylinder, then in the cases with which we are concerned the step B′ is never in the plane AB. Now, since the gap correction depends almost entirely upon the rearrangement of the tubes of induction that would end upon the portion GG′ (Fig. 24) of the plate if the gap were absent, it is evident that in calculating the correction for the gap we must take as p, in our formulæ, the distance between BC and that plane which is cut by the prolongation of AB; q will then be $p+a_1$ where a_1 is the excess of the radius of the inner cylinder of the condenser proper over the radius of the guard cylinder. The correction thus obtained is given in terms of the capacity per unit of length of the condenser for which the distance between the plates is p. If A′B′ is on the guard cylinder side of AB, p will be the distance between the plates of the condenser the capacity of which we measure; if A′B′ is on the other side of AB, p will be less than this distance by an amount a_2 (where a_2 is the excess of the radius of the outer cylinder of the condenser proper over the radius of the outer cylinder of the guard section).

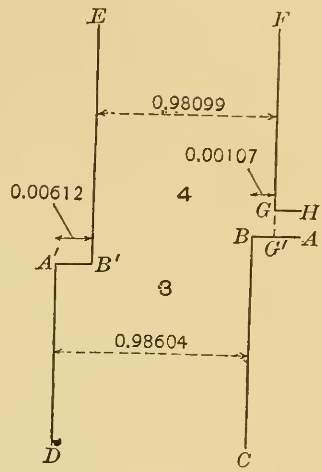

Fig. 24.

The calculated correction in this case must be multipled by $\left(1+\frac{a_2}{p}\right)$ in order to express it in terms of the capacity of unit of length of the section BC.

Now in the actual cases that occur in this work A′B′ is always displaced 0.06 cm from the plane of AB (0.06 cm is the width of the lower gap), except when the middle cylinder consists of one section only. Since the total correction due to the step A′B′ is small, and this displacement is small, no appreciable error can be introduced by assuming that the actual effect of A′B′ is the same as if there were no gap and the two steps were in the same plane. We shall make this assumption. For the upper gap the offsets were as shown in Fig. 24, $a_1=0.00107$ cm, $a_2=0.00612$ cm; distance between FG and EB′ $=0.98099$ cm; distance between BC and

A′D = 0.98604 cm. Hence for the gap formula we must take

$$p = 0.97992$$
$$q = 0.98099$$

From these we obtain Table XII, in which w = width of gap; δl = increase of capacity produced by the gap, expressed in terms of the capacity of unit of length of a condenser for which the distance between the plates is 0.97992; $\delta l'$ is the product of δl by

$$\left(\frac{0.98371}{0.97992}\right)\frac{1}{40}$$

where 0.98371 is the mean of the mean distances between the plates for the two sections, and 40 is the sum of the lengths of the two sections; hence $\delta l'$ is the correction in terms of the capacity of the total condenser formed of section 2 and section 3; the fourth column contains the differences between the value of $\delta l'$ and the value of $\delta l'$ for $w = 0$; the last column contains the product of the values in the preceding column by $\left(1+\frac{1}{4}\frac{p}{r}\right)$.

TABLE XII.

Variation of Capacity with Width of Guard Gap.

w	δl	$\delta l'$	$\delta l' - \delta l'_0$	$(\delta l' - \delta l'_0)\left(1+\frac{1}{4}\frac{p}{r}\right)$
0.00	0.00290	7.3×10^{-5}	0.0×10^{-5}	0.0×10^{-5}
.01	.00679	17.0 "	9.7 "	10.1 "
.02	.01150	28.8 "	21.5 "	21.4 "
.05	.02584	64.8 "	57.5 "	59.8 "
.10	.04941	124.0 "	116.7 "	121.3 "
.20	.09414	236.2 "	228.9 "	238.0 "
.30	.13615	341.7 "	334.4 "	347.6 "
.40	.17497	439.1 "	431.8 "	448.8 "
.50	.21072	528.8 "	521.5 "	542.1 "

The factor $\left(1+\frac{1}{4}\frac{p}{r}\right)$ is the correction introduced by J. J. Thomson[10]

[10] Phil. Trans., 181 A, p. 590; 1890.

on account of the curvature of the plates. The justification of this

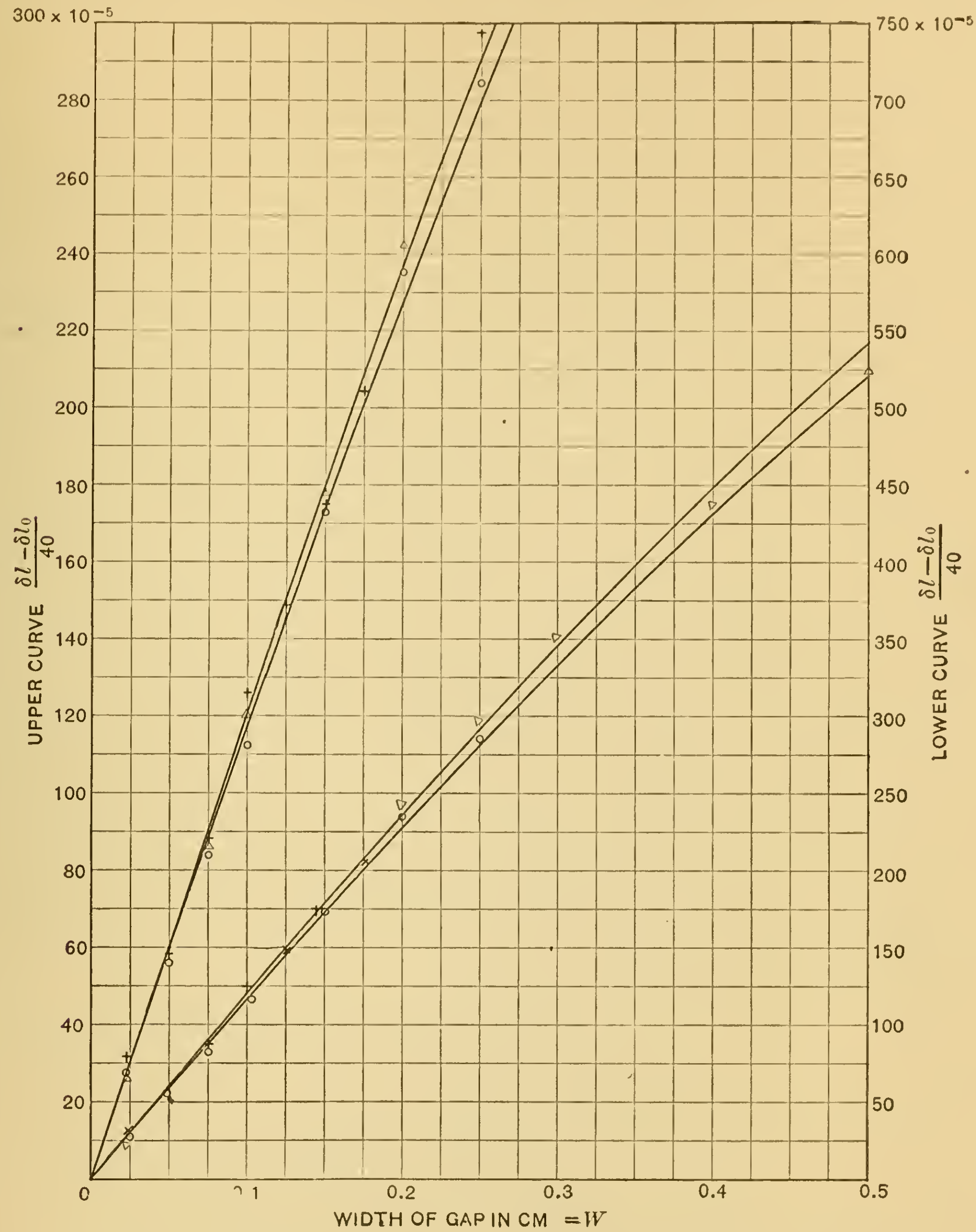

Fig. 25.—*Curve Showing Variation of Capacity with Width of Guard Gap.*

particular factor is not obvious. In Maxwell's Treatise, § 200, there is a similar factor of curvature deduced for a system of coaxial cylinders, *all of which* are at the same potential and placed end-on at a certain distance from a plane at a different potential ; and its justification is based explicitly upon the fact that the electric force in the direction of the radius of curvature is negligible except near the edges of the ends of the cylinders. None of these conditions are fulfilled in the case which we are considering, so this deduction can not be appealed to in justifying the factor.

The variation of the capacity with the width of the gap was measured for gaps of 0.25 mm to 5.0 mm, and the results of three independent series of such observations are plotted in Fig. 25, and the curves determined by the figures in the last two columns of Table XII are drawn on the same plot. From this it is evident that the experimental results lie between the two curves, and, on the whole, rather nearer the lower curve than the one which contains the $\left(1+\frac{1}{4}\frac{p}{r}\right)$ factor. The measurements are not of sufficient accuracy, however, to enable us to decide as to the correctness of the $\left(1+\frac{1}{4}\frac{p}{r}\right)$ factor. For gaps of 0.5 mm the difference in the correction as given by the two expressions is only 2×10^{-5} of the total capacity we are measuring, and so is negligible; in this work we shall (for such small gaps) assume that the simpler expression is correct.

Except in one instance, the width of the upper gap was 0.05 cm during the entire course of the work; hence, the total correction, in terms of centimeters of length of section No. 3, for the upper gap, offsets, etc., is as follows:

Correction for 0.05 cm gap, opposite face plane

$$=0.02584\times\frac{0.9809}{0.9799}=0.02587$$

Correction for offset in outer cylinder

$$=\left\{\begin{array}{l}0.21\times0.00612\\-0.32\times0.00612\times0.0017\end{array}\right\}=0.00128$$

Correction for elevation of upper end of inner section No. 3 by 0.06 cm above top of outer section No. 3

$$=0.06\times\frac{0.00612}{0.986}\times0.981=\underline{0.00036}$$

Total end correction . . . . $=0.02751$

If the condenser is composed of section No. 2 and section No. 3 its length will be 40 cm and the mean distance between the charged surfaces will be 0.98371 cm; hence, for this case the above correction will amount to $\frac{0.02751}{40} \times \frac{0.9837}{0.9809} = 69.0$ parts in 100,000 of the total.

If the condenser is composed of section No. 3 only, the correction will be $\frac{0.02751}{20} = 137.6$ parts in 100,000 of the total, if the lower gap is 0.06 cm; if the gap is 0.05 cm the correction will be $\frac{0.2745}{20} = 137.2$ parts in 100,000 of the total.

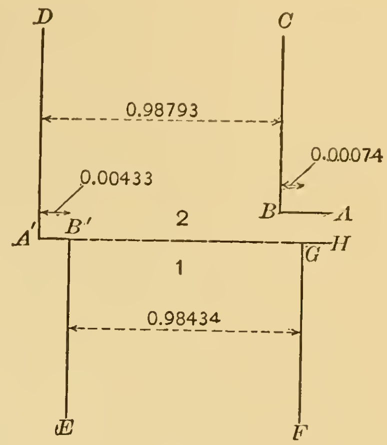

Fig. 26.

For the lower gap, also, two cases are to be considered, according as section No. 2 or section No. 3 adjoins section No. 1. Here the gap width is 0.05 cm in the former case and 0.06 cm in the latter.

With section No. 2 next to section No. 1, the dimensions are as shown in Fig. 26. From this data we find the correction for a gap of 0.06 cm to be equivalent to a length of 0.03028 cm of a condenser for which the plate distance is 0.98793. Since the mean distance for section No. 2 is 0.98651 we find the correction in terms of centimeters of length of section No. 2 to be as follows:

Correction for 0.06 cm gap, opposite face plane

$$= 0.03028 \times \frac{0.9865}{0.9879} = \quad 0.03024$$

Correction for offset in outer cylinder

$$\left\{\begin{matrix} 0.21 \times 0.00433 \\ +0.32 \times 0.00433 \times 0.00074 \end{matrix}\right\} = \quad 0.00091$$

Total correction . . . . = 0.03115

If the condenser is composed of both section No 2 and section No. 3 this correction is equal to $\frac{0.03115}{40} \times \frac{0.9837}{0.9865} = 77.6$ parts in 100,000 of the total.

With section No. 3 next to section No. 1, the dimensions are as shown in Fig. 27. From this data we find the correction for a gap of 0.05 cm to be equivalent to a length of 0.02572 cm of a condenser for which the plate distance is 0.98351 cm. Since the mean distance for section No. 3 is 0.98091 cm, we find the correction in terms of centimeters of length of section No. 3 to be as follows:

Correction for 0.05 cm gap, opposite face plane

$$=0.02572 \times \frac{0.9809}{0.9835} = \quad 0.02565$$

Correction for offset in outer cylinder

$$\left\{\begin{matrix} -0.21 \times 0.00067 \\ -0.32 \times 0.00067 \times 0.00016 \end{matrix}\right\} = -0.00014$$

Total correction . . . . = 0.02551

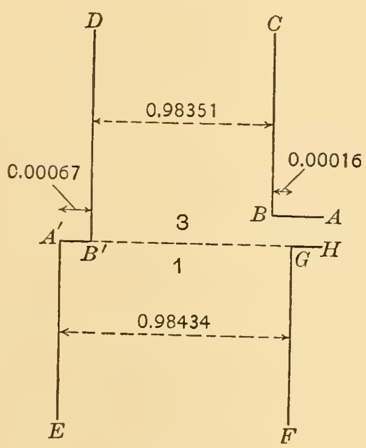

Fig. 27.

For this case the main condenser consists of section No. 3 only; hence, this correction amounts to 127.6 parts in 100,000 of the whole.

There are also offsets where section No. 2 and section No. 3 meet, but as shown on page 501 the effect of such small offsets occurring in the middle of a long condenser has the effect of increasing the effective length of the condenser by a negligible amount. Hence, when the guard cylinder condenser consists of section 2 and section 3, plus the guard cylinders, its electrostatic capacity in centimeters is the sum of the following quantities:

Sum of the two sections regarded as portions of infinite cylinders	$=137.004$
Correction for upper gap and offset $=137.0 \times 69.0 \times 10^{-5}$.	$=\quad 0.095$
Correction for lower gap and offset $=137.0 \times 77.6 \times 10^{-5}$.	$=\quad 0.106$
Total capacity at 20°0 C	**$=137.205$**

If the main condenser consists of section No. 3 only, its capacity is

Section No. 3 regarded as portion of an infinite cylinder .	$=68.696$
Correction for upper gap and offset $=68.70 \times 137.2 \times 10^{-5}$	$=\quad 0.094$
Correction for lower gap and offset $=68.70 \times 127.6 \times 10^{-5}$	$=\quad 0.088$
Total capacity at 20°0 C	**$=68.878$**

VI. CORRECTIONS AND SOURCES OF ERROR—PLATE CONDENSER.

The corrections and sources of error that must be considered in the calculation of the electrostatic capacity of the plate condenser are as follows:

(1) Stability, scale error, error in setting, temperature changes.

(2) Inclination of the normal of the movable plate to the direction of motion.

(3) Lack of parallelism of the plates.

(4) Finite width of guard ring.

(5) Correction for the width of the gap between the plate and the guard ring.

(6) Displacement of the plate from the plane of the guard ring.

34. STABILITY, SCALE ERROR, ERROR IN SETTING, TEMPERATURE CHANGES.

By means of a silk thread passing through a hole in the top of the instrument, a 50 g weight can be lowered upon the collector plate of the instrument or lifted from it. It was found that the lowering of the weight upon the plate produced no appreciable change in the electromagnetic capacity. Also, by means of a Zeiss vertical comparator, the change in the position of the movable plate when a 100 g weight was placed upon it, was measured and found to be less than 0.1μ. Hence the stability of the plates is all that is required.

The position of the lower plate was determined by means of a silver scale attached to the vertical column carrying the lower plate, and read by a fixed microscope. This scale was graduated to 0.1 mm divisions, and was calibrated by the Division of Weights and Measures of this Bureau, and the corrections, so determined, have in every case been applied.

Every setting of the plate was so made that a millimeter division of the scale coincided with the zero setting of the microscope to within two microns. Then, five settings upon the line were made with the micrometer of the microscope, and the mean taken. The microscope was by Zeiss, and each division of the divided head corresponded to one micron. Except in rare instances the extreme range in any group of settings did not exceed $0.5\ \mu$, and was usually less. The microscope mounting was very rigid. A load of 100 g on the eye-piece end of the microscope changed the reading by only 1.2 μ; on the removal of the load the reading returned to its original value to within less than 0.1 μ. Hence, we think the settings are known to within 0.2 μ at least.

The method of procedure adopted in the work with the plate condenser is based upon the assumption that the distance between the plates, for a given scale reading, is the same throughout each entire set of observations. For an instrument composed of brass, steel, and ebonite this necessitates very constant temperature conditions; and at first these conditions were not fulfilled. This rendered valueless the earlier observations. For the later work the condenser was inclosed in a glass case in which, by electrical heating, the temperature could be raised to one or two degrees above the room temperature. By frequent measurement of the resistances of two groups each of three coils of copper wire, distributed about the condenser, it was possible to detect slight changes in the temperature, and, by suitable hand regulation of the heating current, to maintain the temperature constant to within one or two tenths of a degree. This procedure sufficiently eliminates this source of error.

35. INCLINATION OF THE NORMAL OF THE MOVABLE PLATE TO THE DIRECTION OF MOTION.

The only error introduced by the inclination of the normal to the movable plate to the direction of motion is that due to the fact that the measured motion of the column carrying the plate is greater than the change in the distance between the plates in the ratio of one to the cosine of the angle of inclination. By the method of adjustment described on page 454 it is easy to determine this angle and to reduce it sufficiently to make this error negligible. If this angle is not more than one-fourth of one degree the error introduced will be only 1 in 100,000.

36. LACK OF PARALLELISM OF THE PLATES.

As we saw on page 487 (equation 7) the capacity of a rectangular section of length l, normal to the axis of inclination, and of breadth b of a pair of slightly inclined plates is

$$C=\frac{bl}{4\pi d_1}\left(1+\frac{1}{3}\frac{\delta^2}{d_1^{\,2}}+\frac{1}{5}\frac{\delta^4}{d_1^{\,4}}+\ .\ .\ .\right)$$

where d_1 is the mean distance between the plates, and δ is half of the difference between the greatest and the least distances between the plates in this section. We may write this expression in the form

$$C=\frac{bl}{4\pi d_1'}$$

where $$d_1'=d_1\left[1-\frac{1}{3}\frac{\delta^2}{d_1^2}-\frac{4}{45}\frac{\delta^4}{d_1^4}-\quad . \quad . \quad .\right]$$

We shall call d_1' the effective distance between the plates.

If we now increase the distance between the plates until d_1 becomes d_2 the effective distance will become

$$d_2'=d_2\left[1-\frac{1}{3}\frac{\delta^2}{d_2^2}-\frac{4}{45}\frac{\delta^4}{d_2^4}-\quad . \quad . \quad .\right]$$

$$\therefore d_2'-d_1'=d_2-d_1+\frac{d_2-d_1}{3\,d_2\,d_1}\delta^2+\frac{4}{45}\frac{d_2^3-d_1^3}{d_2^3\,d_1^3}\delta^4+\quad . \quad . \quad .$$

$$=d_2-d_1+e$$

Where e is the quantity defined by the equation

$$e=\frac{(d_2-d_1)\,\delta^2}{3d_2\,d_1}\left[1+\frac{4}{15}\frac{d_2^2+d_2d_1+d_1^2}{d_1^2d_2^2}\delta^2+\quad . \quad . \quad .\right] \qquad (26)$$

Hence, the effective distance that the plate has been moved is greater than the actual distance by the amount e.

By the method of procedure adopted in this work we measure d_1' and d_2-d_1, and assume that $d_2'=d_1'+d_2-d_1$. Hence the relative excess of the true value of the electrostatic capacity for the distance d_2 over its assumed value is

$$\frac{\Delta_1 C}{C_0}=-\frac{e}{d_2'}=-\frac{d_2-d_1}{3\,d_2^2d_1}\delta^2, \text{ approximately.} \qquad (27)$$

It is evident, other things being equal, that this error is greater the smaller the initial distance, d_1; and for a given value of d_1 it is greater the smaller the value of d_2, provided d_2 is not too nearly equal to d_1. In Table XIII we give the values of $\frac{\Delta_1 C}{C_0}\cdot\frac{10^6}{\delta_\mu^2}\cdot$ To get $\frac{\Delta_1 C}{C_0}$ for any value of δ, therefore, we multiply the tabular value of $\frac{\Delta_1 C}{C_0}\;\frac{10^6}{\delta_\mu^2}$ by $\delta^2 10^{-6}$, expressing δ in microns. Thus if d_1 is 0.05 and $d_2=0.15$, then $\frac{\Delta_1 C}{C_0}$ will amount to 0.0001 if $\delta=18.4\mu$, which corresponds to an inclination of but 38″. If δ were so much as 100μ the inclination would be but 3′ 26″, while the assumed capacity would be in error by nearly 3 parts in 1,000.

TABLE XIII.

Error Produced by Inclination of Plates.

$d_1=0.05$		$d_1=0.06$	
d_2	$\frac{\Delta_1 C}{C_0}\times\frac{10^6}{\delta^2_\mu}$	d_2	$\frac{\Delta_1 C}{C_0}\times\frac{10^6}{\delta^2_\mu}$
0.15	−0.296	0.16	−0.217
0.25	.213	0.26	.164
0.55	110	0.56	.088
1.05	.060	1.06	.049
2.05	.032	2.06	.026
2.55	.026	2.56	.021

The importance of this adjustment was not recognized until the conclusion of the work, and the lack of proper care in this respect explains one of the most annoying characteristics of all the work on the plates, viz, the values of **v** obtained for the shorter distances were almost invariably higher than the values found from the other condensers. If δ were 50μ, which might easily have been the case, the calculated electrostatic capacity for $d_2=0.15$ would be too high by 7.4 in 10,000 and the value found for **v** would be too high by 3.7 in 10,000, which is of the same order of magnitude as the variations actually observed.

Furthermore, while the upper plate was ground until it was very truly plane, the movable plate had a surface which was appreciably cylindrical. This error in figuring would of itself introduce an error of the kind which we have just considered.

37. CORRECTION FOR THE FINITE WIDTH OF THE GUARD RING.

J. J. Thomson[11] has shown that the charge on a strip of unit length normal to the paper and extending from the edge C (Fig. 28) of one of a pair of semi-infinite parallel plates to a point P at a distance x, is (if we consider the opposed surfaces only)

$$E=-\frac{\frac{1}{2}V}{4\pi^2}\log(t+1)$$

[11] Recent Researches, § 235.

where the distance (x) from the edge is given by the expression

$$x=\frac{p}{2\pi}[t-\log(1+t)]$$

where t lies between -1 and 0, and p is the distance between the plates. Putting in the last expression

$$t=-1(1-10^{-n})$$

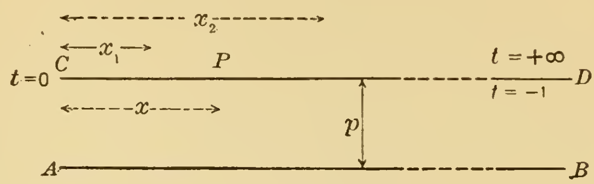

Fig. 28.

we find

$$\frac{2\pi x}{p}=2.302\,n+10^{-n}-1$$

Hence, if n is not less than 2, that is, if $\frac{2\pi x}{p}$ is not less than 3.614, the value of $\frac{2\pi x}{p}$ is given with an accuracy of at least 1 in 300 by the approximate formula

$$\frac{2\pi x}{p}=\log\left(\frac{1}{1+t}\right)-1$$

$$\text{or, } t=-1+e^{-\left(\frac{2\pi x}{p}+1\right)}$$

In the actual case with which we are at present concerned, x is not less than 5 (the width of the guard ring) and p is not greater than 2.5, hence $\frac{2\pi x}{p}$ is not less than 12.56. Hence, as a second approximation we have

$$\log\frac{1}{1+t}=\frac{2\pi x}{p}+1-e^{-\left(\frac{2\pi x}{p}+1\right)}$$

$$\therefore E=\frac{V}{4\pi p}\left[x+\frac{p}{2\pi}-\frac{p}{2\pi}e^{-\left(\frac{2\pi x}{p}+1\right)}\right]$$

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Hence, the charge on the strip included between $x=x_1$ and $x=x_2$ is

$$E=\frac{V}{4\pi p}\left[x_2-x_1+\frac{p}{2\pi}\left\{e^{-\left(\frac{2\pi x_1}{p}+1\right)}-e^{-\left(\frac{2\pi x_2}{p}+1\right)}\right\}\right]$$

That is, the effective width of the strip is greater than its true width by an amount

$$\delta l=\frac{p}{2\pi}\left\{e^{-\left(\frac{2\pi x_1}{p}+1\right)}-e^{-\left(\frac{2\pi x_2}{p}+1\right)}\right\}$$

The actual increase in the capacity of our circular plate will be less than that obtained by assuming that this entire excess of charge is situated on the extreme edge of the collector plate. Hence, the effect of the finite width of the guard ring is to increase the capacity of the plate by an amount which is less than

$$\Delta_2 C=\frac{R\delta l}{2p}\left(1+\frac{1}{4}\frac{p}{R}\right)$$

where the factor $\left(1+\frac{1}{4}\frac{p}{R}\right)$ is J. J. Thomson's correction for curvature. Hence the relative increase in capacity will be less than

$$\frac{\Delta_2 C}{C_0}=\frac{2\delta l\left(1+\frac{1}{4}\frac{p}{R}\right)}{R}$$

In the present case the width of the guard ring is 5 cm, R is 10 cm, hence

$$\frac{\Delta_2 C}{C_0}=\frac{p\left\{e^{-\left(\frac{10\pi}{p}+1\right)}-e^{-\left(\frac{30\pi}{p}+1\right)}\right\}}{10\pi}\left(1+\frac{p}{40}\right)$$

$$=0.0118\,p\left\{e^{-\frac{31.4}{p}}-e^{-\frac{94.2}{p}}\right\}\left\{1+0.025\,p\right\}$$

Giving p the greatest value used in this work (2.5 cm), we find

$$\frac{\Delta_2 C}{C_0}=1.4\times 10^{-7}$$

Hence the width of the guard ring is more than sufficient for the highest possible accuracy.

38. CORRECTION FOR GUARD RING GAP AND FOR DISPLACEMENT OF THE PLATE FROM THE PLANE OF THE GUARD RING.

The capacity of a circular plate condenser is[12]

$$C=\frac{\left[R+\delta l\left(1+\frac{1}{4}\frac{p}{R}\right)\right]^2}{4p}$$

Where R is the radius of the plate, p is the distance between the plates and δl is the edge correction when R is infinite.

On page 517 (equation 25) we found the following expression for δl for the case in which the distance between the plates is different on the two sides of the gap:

$$\delta l=\frac{w}{2}+\frac{1}{\pi}\left[p\log\frac{(p+q)^2+w^2}{4p^2}+(q-p)\log\frac{(p+q)^2+w^2}{A}-w\sin^{-1}\frac{w^2+q^2-p^2}{A}\right]$$

$$=\frac{w}{2}+\Delta l$$

where p is the distance between the plates on the collector plate side of the gap, q is the distance between the plates on the guard ring side of the gap, w is the width of the gap, and

$$A\equiv\sqrt{[p^2+q^2+w^2]^2-4p^2q^2}$$

For abbreviation we write

$$\Delta l\equiv\frac{1}{\pi}\left[p\log\frac{(p+q)^2+w^2}{4p^2}+(q-p)\log\frac{(p+q)^2+w^2}{A}-w\sin^{-1}\frac{w^2+q^2-p^2}{A}\right] \quad (28)$$

Hence,

$$C=\frac{\left(R+\frac{w}{2}\right)^2}{4p}\left\{1+\frac{wp}{8R^2}+\frac{\Delta l}{R}\left(1+\frac{1}{4}\frac{p}{R}\right)\right\}^2$$

[12] Maxwell, Electricity and Magnetism, Vol. I, § 201.

which may be written

$$C=\frac{\left(R+\frac{w}{2}\right)^2}{4d}$$

We shall call the quantity d the effective distance between the plates. If p_1 is the actual distance between the plates, the effective distance is

$$d_1=p_1\left\{1-\frac{wp_1}{4R^2}-\frac{2(\Delta l)_1}{R}\left(1+\frac{1}{4}\frac{p_1}{R}\right)\right\} \text{ approximately.}$$

Increasing the distance between the plates until it is equal to p_2

$$d_2=p_2\left\{1-\frac{wp_2}{4R^2}-\frac{2(\Delta l)_2}{R}\left(1+\frac{1}{4}\frac{p_2}{R}\right)\right\}$$

$$\therefore\ d_2-d_1=p_2-p_1+\left[\frac{2p_1(\Delta l)_1}{R}\left(1+\frac{1}{4}\frac{p_1}{R}\right)+\frac{wp_1^2}{4R^2}\right.$$
$$\left.-\frac{2p_2(\Delta l)_2}{R}\left(1+\frac{1}{4}\frac{p_2}{R}\right)-\frac{wp_2^2}{4R^2}\right]$$

By the method of procedure adopted in this work we calculate a_1, measure p_2-p_1, and assume $d_2=d_1+p_2-p_1$. Hence, the relative amount which must be added to the assumed capacity at the distance p_2 to make it equal to the true capacity is

$$\frac{\Delta_3 C}{C_0}=\left\{\frac{2(\Delta l)_2}{R}\left(1+\frac{1}{4}\frac{p_2}{R}\right)+\frac{wp_2}{4R^2}\right\}-\frac{p_1}{p_2}\left\{\frac{2(\Delta l)_1}{R}\left(1+\frac{1}{4}\frac{p_1}{R}\right)+\frac{wp_1}{4R^2}\right\} \quad (29)$$

We may write this in the form

$$\frac{\Delta_3 C}{C_0}=F(p_2)-\frac{p_1}{p_2}F(p_1)$$

where

$$F(p)\equiv\frac{2(\Delta l)}{R}\left(1+\frac{1}{4}\frac{p}{R}\right)+\frac{wp}{4R^2}$$

If $(q-p)$ is not too great, we can expand $F(p)$ in ascending powers of $(q-p)$. Omitting all after the term of the first power, we have

$$F(p)=F_0(p)+(q-p)\,F_0'(p)+\ \ldots$$

$$=\frac{2}{R\pi}\left[p\log\frac{4p^2+w^2}{4p^2}-w\sin^{-1}\frac{w}{\sqrt{4p^2+w^2}}\right]\left(1+\frac{1}{4}\frac{p}{R}\right)+\frac{wp}{4R^2}$$

$$+\frac{q-p}{R\pi}\left[\log\frac{4p^2+w^2}{w^2}\right]\left(1+\frac{1}{4}\frac{p}{R}\right)$$

and

$$\frac{\Delta_3 C}{C_0}=F_0(p_2)-\frac{p_1}{p_2}F_0(p_1)+(q-p)\left[F_0'(p_2)-\frac{p_1}{p_2}F_0'(p_1)\right] \qquad (30)$$

The values of $F_0(p)$ and of $F_0'(p)$ for several values of p, and for $w=0.0179$, and $R=10$ are given in Table XIV. These values apply directly to the results obtained from plate B.

TABLE XIV.

Correction for Displacement of Guard Ring.

p	$F_0(p)$	$F_0'(p)\times 10^{-4}$
0.05	$-\ 99\times 10^{-6}$	11.1×10^{-6}
0.15	$-\ 28$ "	18.0 "
0.25	$-\ 9$ "	21.4 "
0.55	$+\ 15$ "	26.6 "
1.05	$+\ 42$ "	31.1 "
2.05	$+\ 89$ "	36.4 "
2.55	$+112$ "	38.3 "

If $(q-p)$ is expressed in microns, the numbers in the third column are the values of $F_0'(p)$ to be substituted directly in equation 30. Hence, even if $(q-p)$ were so small as 5 μ the assumed capacity for p_2 equal to 0.25 would be too small by 11 parts in 100,000, if p_1 were equal to 0.05; for p_2 equal to 2.55 it would be too small by 30 in 100,000.

An accuracy of adjustment approaching this was attained only in the last few determinations. For all the former work the accuracy of the adjustment is unknown and can not be satisfactorily determined; the relative displacement of the collector plate and the ring was probably large.

[The electromagnetic measurements and the concluding portion of this article will be given in the next number of this Bulletin].

WASHINGTON, May 20, 1907.

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