Symmetric spectra

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1 Why do we like symmetric spectra?

An elementary theorem in homotopy theory is the Freudenthal suspension theorem: if X, Y are finite-dimensional pointed CW-complexes, $\Sigma : Top_* \to Top_*$ denotes the reduced suspension endofunctor $\Sigma X := S^1 \wedge X$, and $[X, Y]_+$ denotes the set of pointed homotopy classes of maps between X and Y, then we have that the following sequence of sets is eventually constant:

$$[X, Y]_+ \longrightarrow [\Sigma X, \Sigma Y]_+ \longrightarrow [\Sigma^2 X, \Sigma^2 Y]_+ \longrightarrow \cdots$$

$$[f] \longmapsto [\Sigma f] \longmapsto [\Sigma^2 f] \longmapsto \cdots$$

This suggests trying to form a localized category where Σ is an isomorphism. What will be the objects in this category? Usually we make our objects more difficult in return for having a nice category. One approach is to look at sequences of spaces $\{X_n\}$ and maps $\Sigma X_n \to X_{n+1}$. One can form a category of topological spectra by using these sequences as object and as morphisms the levelwise maps compatible with the suspension maps. Then there is a functor from Top_* to topological spectra mapping X to the sequence $X_i = *$ for $i < 0$ and $X_i = \Sigma^i X$ for $i \geq 0$. We can form a homotopy category of these spectra, where Σ is an isomorphism, when we invert the stable homotopy equivalences. This is the stable homotopy category. For more information, see [1].

The axioms of a generalized (reduced) cohomology theory, in particular that $H^n(f) = H^n(g)$ for homotopic f, g and $H^n(Z) \cong H^{n+1}(\Sigma Z)$ naturally, implies that a generalized cohomology theory as a functor from Top_* to $Ab^{\mathbb{Z}}$ should be closely related to the stable homotopy category. In fact, Brown's representability theorem tells us that every generalized cohomology theory is representable in a stable sense: to every generalized cohomology theory we can assign a topological spectra X such that $H^n(Z) = [Z, X_n]_+$.

This intimately ties the study of topological spectra to many topics in algebraic topology. One would like to do algebraic constructions of spectra. This makes sense since in the stable homotopy category there is a smash product (just arising from the smash product in Top_*) which is associative, commutative and unital, up to coherent isomorphism. However, a naive definition of symmetric spectra does not have such a product; Adams [1] has a unnatural handicrafted smash product of spectra, but it will not make you happy. Furthermore, there are some difficulties with associativity.

A "good" category of spectra should have the following properties:

- It should be a model category whose homotopy category is the stable homotopy category.
- It should have a closed symmetric monoidal product structure compatible with the model structure, making it a closed symmetric monodial model category.
- It should be elementary.
- It should not be hard to describe the spectra arising from generalized cohomology theories as "good" spectra.

Symmetric spectra are a "good" construction of spectra, in this sense, in the pointed simplicial set context (which we know is Quillen equivalent to Top_*). In this first lecture, we will define it and see that it has a symmetric monoidal smash product. In the second lecture, we will exhibit a model structure on symmetric spectra giving the simplicial analog of the stable homotopy category, such that it is a symmetric monoidal model category. There is an analogous definition in the topological context, which we might get to in the second lecture as well.

Most spectra constructed from generalized cohomology theories automatically come with group actions, but we just didn't use them before. For example, look at Joachim's construction of KO as a symmetric spectrum [4].

There are other approaches to "good" definitions of spectra, like S-modules. Symmetric spectra are the most elementary of these, and are defined by simply copying the definition of topological spectra in the simplicial context and adding some symmetric group actions. It is truely remarkable that a solution this elementary fixes the problem and took over 30 years to find.

There are two main sources for symmetric spectra: the original article by Mark Hovey, Brooke Shipley & Jeff Smith [2] and a book-in-progress by Stefan Schwede [3]. We will mostly be following the original article.

2 Symmetric spectra

2.1 Simplicial prerequisites

Remember that the category Δ has as objects the ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ and as morphisms the order-preserving maps. Then a simplicial set is a functor $\Delta^{op} \to Set$. This functor category we denote by $\mathcal{S}$. A series of simplicial sets is given by the representables $\Delta[n] := \Delta(-, [n])$.

A simplicial set has components X_n, which by the Yoneda lemma are given by $X_n = \mathcal{S}(\Delta[n], X)$. The elements of X_n are known as n-simplices.

A pointed simplicial set is a simplicial set X together with a distinguised 0-simplex $* \in X_0$. The category $\mathcal{S}_*$ with object pointed simplicial sets and morphisms maps of simplicial sets preserving the basepoint is called the category of pointed simplicial sets.

A simplicial set can be made into a pointed simplicial set by adding a disjoint basepoint: $X_+ := X \coprod \Delta[0]$. In fact, this gives us a functor $(-)_+ : \mathcal{S} \to \mathcal{S}_*$. This way we can define $S^0 = \Delta[0]_+$, whose importance we'll see in a minute.

The usual product of simplicial sets induces a smash product of pointed simplicial sets X, Y, which is given by:

$$X \wedge Y = X \times Y / (X \times * \cup * \times Y)$$

This smash product has the following properties:

Associativity. There is a natural isomorphism $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$.

Commutativity. There is a natural isomorphism $X \wedge Y \cong X \wedge Y$

Unit. There is a natural isomorphism $X \wedge S^0 \cong X$.

After a glance at the axioms of a symmetric monoidal category, we see that the smash product is a symmetric monoidal product, and therefore makes $\mathcal{S}_*$ into a symmetric monoidal category. In fact, it is a closed symmetric monoidal category, since $X \wedge -$ has a right adjoint $Map_{\mathcal{S}_*}(X, -)$. It is easy to verify that $Map_{\mathcal{S}_*}(X, Y)$ is given by $Map_{\mathcal{S}_*}(X, Y) = \mathcal{S}_*(X \wedge \Delta[-]_+, Y)$.

Since $\mathcal{S}_*$ is bicomplete, there is a categorical sum, $\vee$, given by $X \vee Y = X \coprod Y / (*_X \sim *_Y)$.

2.2 The definition of symmetric spectra

Let $S^1 = \Delta[1]/\partial\Delta[1]$ be the (small) simplicial circle. We denote $(S^1)^{\wedge p}$ by S^p. S^1 will take the place of the topological circle in the definition of a spectrum, which is completely analogous to the topological spectra defined in the introduction.

Definition 2.1. A spectrum is

1. A sequence of $X_0, X_1, \ldots, X_n, \ldots$ of pointed simplicial sets.
2. A map of pointed simplicial sets $\sigma : S^1 \wedge X_n \to X_{n+1}$ for each $n \geq 0$.

The maps σ are called the structure maps of the spectrum. A map of spectra $f : X \to Y$ is a sequence of maps of pointed simplicial sets $f_n : X_n \to Y_n$ such that the following diagram commutes for all $n \geq 0$:

$$\begin{tikzcd} S^1 \wedge X_n \arrow[r, "\sigma"] \arrow[d, "id\wedge f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ S^1 \wedge Y_n \arrow[r, "\sigma"'] & Y_{n+1} \end{tikzcd}$$

This is called the category of spectra and is denoted by $Sp^{\mathbb{N}}$.

An example of a spectrum is given by the sphere spectrum $\mathcal{S} = (S^0, S^1, S^2, \ldots)$ and connecting maps $\sigma : S^1 \wedge S^n \to S^{n+1}$ the canonical isomorphisms.

To define a symmetric spectrum, we need to introduce an action of symmetric groups to keep tracks of twists. Let Σ_n denote the symmetric group of permutations of the set $\{1, \ldots, n\}$. Then we have an embedding of groups $\Sigma_p \times \Sigma_q \hookrightarrow \Sigma_{p+q}$ by letting the first component work on $\{1, \ldots, p\}$ and the second component work on $\{p+1, \ldots, p+q\}$. For example, if we have $p = 2$ and $q = 3$, then $(1 \quad 2) \times (1 \quad 2 \quad 3)$ is mapped to $(1 \quad 2)(3 \quad 4 \quad 5)$.

Definition 2.2. A symmetric spectrum is

1. A sequence of $X_0, X_1, \ldots, X_n, \ldots$ of pointed simplicial sets.
2. A map of pointed simplicial sets $\sigma : S^1 \wedge X_n \to X_{n+1}$ for each $n \geq 0$.
3. A basepointed preserving left action of Σ_n on X_n such that:

$$\sigma^p = \sigma \circ (id \wedge \sigma) \circ \ldots \circ (id^{\wedge p-1} \wedge \sigma) : S^p \wedge X_n \to X_{p+n}$$

   is $\Sigma_p \times \Sigma_n$-equivariant for $p \geq 1$ and $n \geq 0$.

The maps σ are called the strucure maps of the symmetric spectrum. A map of symmetric spectra $f : X \to Y$ is a sequence of maps of pointed simplicial sets $f_n : X_n \to Y_n$ such that f_n is Σ_n-equivariant and the following diagram commutes for all $n \geq 0$:

$$\begin{tikzcd} S^1 \wedge X_n \arrow[r, "\sigma"] \arrow[d, "id\wedge f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ S^1 \wedge Y_n \arrow[r, "\sigma"'] & Y_{n+1} \end{tikzcd}$$

This is called the category of spectra and is denoted by Sp^{Σ}.

Note that only the equivariance of σ^2 and σ is needed, since the 2-cycles generate Σ_n. Furthermore, there is an obvious forgetful functor $U : Sp^{\Sigma} \to Sp^{\mathbb{N}}$ forgetting the symmetric group actions. Hence we can consider symmetric spectra as ordinary spectra carrying additional structure.

2.3 Examples

Let's list some examples of symmetric spectra:

- The first example of a symmetric spectrum is the *sphere spectrum* $\mathcal{S}$, with Σ_n acting on $S^n = (S^1)^{\wedge n}$ by permuting the factors. The isomorphism $\sigma^p : S^p \wedge S^n \to S^{n+p}$ are then clearly $\Sigma_p \times \Sigma_n$ equivariant.

- The sphere spectrum is a special case of the *symmetric suspension* $\Sigma^\infty K$ of a pointed simplicial set K. This is given by $(\Sigma^\infty K)_n = S^n \wedge K$ with $\sigma : S^1 \wedge S^n \wedge K \to S^{n+1} \wedge K$ the natural isomorphisms and the action of Σ^n given by permuting the factor of S^n in $S^n \wedge K$. Note that $\mathcal{S} = \Sigma^\infty S^0$. In terms of prolongation $\Sigma^\infty K = \mathcal{S} \wedge K$. Hence a get a functor $\Sigma^\infty : \mathcal{S}_* \to Sp^\Sigma$.

- Let A be any abelian group. Then the *Eilenberg-Mac Lane spectrum HA of A* is given by $(HA)_n = A \otimes_{\mathbb{Z}} \mathbb{Z}(S^n)$. Here $\mathbb{Z}(S^n)$ is the pointed simplicial abelian group giving by $\mathbb{Z}(S^n)_k$ the free abelian group on th non-basepoint k-simplices of S^n. The basepoint of $\mathbb{Z}(S^n)$ is 0. The maps $\sigma : S^1 \wedge (HA)_n \to (HA)_{n+1}$ maps $t \wedge (a \otimes s)$ to $a \otimes (t \wedge s)$. The action of Σ_n on $(HA)_n$ is given by permuting the n factors of the generators of $\mathbb{Z}(S^n)$.

3 Properties of the category of symmetric spectra

We'll discuss three properties of the category of symmetric spectra:

1. It is bicomplete, i.e. complete and cocomplete.
2. It can be considered as enriched in pointed simplicial sets if we change the hom-sets. More concretely, there is a closed $\mathcal{S}_*$-action of Sp^Σ.
3. It is a closed symmetric monoidal category.

3.1 Prolongation and bicompleteness

There must be some way to use the enormous amount of structure in $\mathcal{S}_*$ in Sp^Σ. One technique for this is that of prolongation. Suppose we have a functor $R : \mathcal{S}_* \to \mathcal{S}_*$ with the following properties:

1. There is a natural transformation $h : (RX) \wedge K \to R(X \wedge K)$ of bifunctors.
2. The composition $(RX) \wedge S^0 \to R(X \wedge S^0) \to R(X)$ is the unit isomorphism.
3. The following diagram commutes

$$\begin{tikzcd}
(RX \wedge K) \wedge L \arrow[r, "h \wedge L"] \arrow[d] & R(X \wedge K) \wedge L \arrow[d, "h"] \\
RX \wedge (K \wedge L) \arrow[r, "h"'] & R(X \wedge K \wedge L)
\end{tikzcd}$$

Then we can extend R to a functor $Sp^\Sigma \to Sp^\Sigma$. For a symmetric spectrum X we define the symmetric spectrum RX by setting $(RX)_n = RX_n$, and $\sigma : S^1 \wedge R(X_n) \to R(S^1 \wedge X_n) \to R(X_n)$ the composition. The action is given by applying R to the action maps. This new functor $R : Sp^\Sigma \to Sp^\Sigma$ is called the prolongation of R.

Applying prolongation to the limit and colimit functors associated to a diagram I in the bicomplete category $\mathcal{S}_*$, we see that Sp^Σ is bicomplete and $(\lim D)_n = \lim D_n$ and $(\mathrm{colim} D)_n = \mathrm{colim} D_n$.

3.2 Simplicial action on symmetric spectra

Fix a pointed simplicial set K, then $-\wedge K$ is a functor $\mathcal{S}_* \to \mathcal{S}_*$ satisfying the properties required for prolongation. Prolongation then gives us a functor $-\wedge K : Sp^\Sigma \to Sp^\Sigma$. Similarly by prolongation of the exponential pointed simplicial set, we get the power spectrum $(-)^K : Sp^\Sigma \to Sp^\Sigma$.

Using the prolongated smash product, we can now define a pointed simplicial set of maps between two symmetric spectra X, Y: $Map_{Sp^\Sigma}(X, Y) = Sp^\Sigma(X \wedge \Delta[-]_+, Y)$. Note that these pointed simplicial sets have as 0-th level the ordinary hom-set.

All these constructions interact exactly as you would expect them to interact:

Proposition 3.1. *Let X, Y be a symmetric spectrum and K, L be pointed simplicial sets, then we have*

1. *There are coherent natural isomorphisms $X \wedge (K \wedge L) \cong (X \wedge K) \wedge L$ and $X \wedge S^0 \cong X$.*
2. $(-) \wedge K \vdash (-)^K$.
3. $X \wedge (-) \vdash Map_{Sp^\Sigma}(X, -)$.

Letting $ev : X \wedge Map_{Sp^\Sigma}(X, Y) \to Y$ be the adjoint of the identity map $Map_{Sp^\Sigma}(X, Y) \to Map_{Sp^\Sigma}(X, Y)$. Similarly, using adjointness we can define the composition pairing $Map_{Sp^\Sigma}(X, Y) \wedge Map_{Sp^\Sigma}(Y, Z) \to Map_{Sp^\Sigma}(X, Z)$ the double evaluation $X \wedge Map_{Sp^\Sigma}(X, Y) \wedge Map_{Sp^\Sigma}(Y, Z) \to Z$.

These have the following properties, which are exactly the axioms of a category enriched in pointed simplicial sets, if the Map_{Sp^Σ} were the homs of Sp^Σ. In other words, there is a closed $\mathcal{S}_*$-action of Sp^Σ.

Proposition 3.2. *Let X, Y be symmetric spectra and K a pointed simplicial set, then we have*

1. *The composition pairing is associative.*
2. *The adjoint $S^0 \to Map_{Sp^\Sigma}(X, X)$ of the unit isomorphism $X \wedge S^0 \to X$ is a left and unit of the composition pairing.*
3. *There are natural isomorphisms*

$$Map_{Sp^\Sigma}(X \wedge K, Y) \cong Map_{Sp^\Sigma}(X, Y^K) \cong Map_{Sp^\Sigma}(X, Y)^K$$

3.3 Smash product of symmetric spectra

There is trick which allows you to use generalities about closed monoidal categories to construct a closed symmetric monoidal product on symmetric spectra from the smash product of pointed simplicial sets.

We start by reminding of the definition of a monoid and a module in a symmetric monoidal category. There is nothing special about this definition, if you keep in mind the category of abelian groups with the tensor product.

Definition 3.3. Let $\mathcal{C}$ be a symmetric monoidal category with symmetric monoidal product $\otimes$. Then a *commutative monoid* in $\mathcal{C}$ is an object R with maps $\mu : R \otimes R \to R$ and $\eta : e \to R$ with the following properties.

Associativity. The following diagram commutes

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] & R \otimes R \arrow[d, "\mu"] \\ R \otimes R \arrow[r, "\mu"'] & R \end{tikzcd}$$

Commutativity. If τ denotes the twist isomorphism of $\mathcal{C}$, then $\mu = \mu \circ \tau$.

Unit. The following composition is the unit isomorphisms of $\mathcal{C}$:

$$e \otimes R \xrightarrow{\eta \otimes id} R \otimes R \xrightarrow{\mu} R$$

Note that commutativity implies that a right unit axiom is not necessary.

Let R be a monoid, then a *left R-module* is an object M in $\mathcal{C}$ together with a map $m : R \otimes M \to M$ with the following properties.

Associativity. The following diagram commutes:

$$\begin{tikzcd} R \otimes R \otimes M \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes m"'] & R \otimes M \arrow[d, "m"] \\ R \otimes M \arrow[r, "m"'] & R \end{tikzcd}$$

Respects unit. The following composition is the unit isomorphism of $\mathcal{C}$:

$$e \otimes M \xrightarrow{\eta \otimes id} R \otimes M \xrightarrow{m} M$$

As an example, if $\mathcal{C}$ is the category of abelian groups, then a monoid is a ring with unit and a module over a monoid in the category of abelian groups is a module over a ring. A very useful fact about monoids and modules in any symmetric monoidal category is the following:

Proposition 3.4. *Let $\mathcal{C}$ be a symmetric monoidal category with symmetric monoidal product $\otimes$ and let R be a monoid in $\mathcal{C}$.*

If $\mathcal{C}$ is complete, then $R-mod$ is complete. If $\mathcal{C}$ is cocomplete and $R \otimes -$ preserves coequalizers, then $R - mod$ is cocomplete.

Note that any monoid is a module over itself and abusing notation, we replace μ by m from now on. Furthermore, there is a dual definition of right module over a monoid. If R is a commutative monoid then any left module is a right module and vica versa, using the multiplication:

$$M \otimes R \xrightarrow{\tau} R \otimes M \xrightarrow{m} M$$

Furthermore, since R is commutative, these actions commute, making every left or right R-module into a (R, R)-bimodule.

Theorem 3.5. *Let $\mathcal{C}$ be a symmetric monoidal category with symmetric monoidal product $\otimes$ that is bicomplete and let R be a commutative monoid such that the functor $R \otimes (-) : \mathcal{C} \to \mathcal{C}$ preserves coequalizers. Then there is a symmetric monoidal product $\otimes_R$ on the category of R-modules with R as the unit.*

If $\mathcal{C}$ is a closed symmetric monoidal category, then there is a function R-module $Hom_R(M, N)$ natural in $M, N \in \mathcal{C}$, such that $(-) \otimes_R M$ is left adjoint to $Hom_R(M, -)$.

Proof. **Symmetric monoidal product.** Because R is a commutative monoid M is naturally a (R, R)-bimodule. This means that we can consider M as a right R-module and N as a left R-module. Hence the the monoidal product $M \otimes_R N$ of two R-modules can be given as the colimit of the diagram:

$$\begin{tikzcd} M \otimes R \otimes N \arrow[r, shift left, "m \otimes id"] \arrow[r, shift right, "id \otimes m"'] & M \otimes N \end{tikzcd}$$

Since $\mathcal{C}$ is complete and $R \otimes (-)$ preserves coequalizers, we know that the category of R-modules is cocomplete, hence the coequalizer $M \otimes_R N$ exists and will again be a R-module. The associativity, unit and twist isomorphisms of $\mathcal{C}$ induce those for $R - mod$. Let's do the unit as a example: we will show that $M \otimes_R R \cong M$. This means we must show that M is a colimit of the following diagram:

$$\begin{tikzcd} M \otimes R \otimes R \arrow[r, shift left, "m\otimes id"] \arrow[r, shift right, "id\otimes m"'] & M \otimes R \end{tikzcd}$$

Consider the map $m : M \otimes R \to M$. By associativity, this map satifies $m \circ (m \otimes id) = m \circ (id \otimes m)$. What is left to show is that it is initial with this property. There exists a section $s : M \otimes R \to M \otimes R \otimes R$ given by $M \otimes R \cong M \otimes R \otimes e \to M \otimes R \otimes R$ where the last map is $id \otimes \eta$. By the unit axiom of the monoid R this satifies $(id \otimes m) \circ s = id$. Hence $f \circ (id \otimes m) = f \circ (m \otimes id)$ implies that $f = f \circ (m \otimes id) \circ s$. There exists a section $t : M \to M \otimes R$ given by $M \cong M \times e \to M \times R$ where the last map is $id \otimes \eta$.

$$\begin{tikzcd} M \otimes R \otimes R \arrow[r, shift left, "m\otimes id"] \arrow[r, shift right, "id\otimes m"'] & M \otimes R \arrow[l, bend right=50, "s"'] \arrow[r, "m"] \arrow[dr, "f"'] & M \arrow[l, bend right=50, "t"'] \arrow[d, dotted, "\tilde{f}"] \\ & & N \end{tikzcd}$$

Note that $(m \otimes id) \circ s : M \otimes R \to M \otimes R$ is the map which adds a unit on the third component, multiplies the first two. Hence it is equal to $t \circ m$, which multiplies the first two and adds a unit. Therefore we conclude that

$$f \circ t \circ m = f \circ (m \otimes id) \circ s = f$$

The fact that $m \circ t$ is an isomorphism, implies that $\tilde{f}$ is unique: $\tilde{f} \circ m = f = g \circ m$ implies that $\tilde{f} \circ m \circ t = g \circ m \circ t$ hence $\tilde{f} = g$.

Similarly, one can derive associativity and commutativity from the associativity and commutativity of the monoid and the symmetric monoidal product in $\mathcal{C}$.

Internal hom. Let $Hom_{\mathcal{C}}(M, N)$ denote the internal hom in $\mathcal{C}$. The internal hom $Hom_R(M, N)$ of left R-modules M, N is given as the limit in $\mathcal{C}$ of the diagram:

$$\begin{tikzcd} Hom_{\mathcal{C}}(M, N) \arrow[r, shift left, "m^*"] \arrow[r, shift right, "m_*"'] & Hom_{\mathcal{C}}(R \otimes M, N) \end{tikzcd}$$

where m^* is the induced map of multiplication m and m_* is the composition:

$$Hom_{\mathcal{C}}(M, N) \xrightarrow{R\otimes -} Hom_{\mathcal{C}}(R \otimes M, R \otimes N) \xrightarrow{m^*} Hom_{\mathcal{C}}(R \otimes M, N)$$

The equalizer exists and will be an R-module since $\mathcal{C}$ is complete and hence $R - mod$ is. That $- \otimes_R M$ is left adjoint to $Hom_R(M, -)$ is a consequence of the definition of $\otimes_R$ and the fact that $- \otimes X$ is left adjoint to $Hom_{\mathcal{C}}(X, -)$.

□

Let Σ be the category with objects $\bar{0} = \emptyset$ and $\bar{n} = \{1, \ldots, n\}$ for $n \geq 1$, and morphisms Σ_n as automorphisms of the sets $\bar{n}$. Thus we have $\Sigma = \coprod_{n \in \mathbb{N} \cup \{0\}} \Sigma_n$. Let $\mathcal{S}^{\mathbb{N}}$ denote the functor category $\mathcal{S}_*^{\Sigma}$. This is a sequence of pointed simplicial sets with basepoint preserving left action of Σ_n on X_n. This is simpy a simplicial set without the σ.

Note that $\mathcal{S}_*^{\Sigma}$ is a functor category, hence is bicomplete because $\mathcal{S}_*$ is.

Proposition 3.6. *There is a closed symmetric monoidal product on $\mathcal{S}_*^\Sigma$ given by:*

$$(X \otimes Y)_n = \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)$$

Its unit is the sequence $(S^0, *, *, \ldots)$ and the twist is given by $\tau(\alpha, x, y) = (\alpha\rho_{q,p}, y, x)$ where $\rho_{q,p}$ flips the first q elements of $\{1, 2, \ldots, p+q\}$ with the last p. The internal hom is given by:

$$Hom_\Sigma(X, Y) = Map_{\mathcal{S}_*^\Sigma}(X \otimes \Sigma[-]_+, Y)$$

Associativity of $\otimes$ follows from the easy to prove natural isomorphism:

$$((X \otimes Y) \otimes Z)_n = \bigvee_{p+q+r=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q \times \Sigma_r} (X_p \wedge Y_q \wedge Z_r)$$

To be able to prove things using $\otimes$, we note the following lemma:

Lemma 3.7. *There is a natural isomorphism*

$$\mathcal{S}_*^\Sigma(X \otimes Y, Z) \cong \prod_{p,q} \mathcal{S}_*^{\Sigma_p \times \Sigma_q}(X_p \wedge Y_q, Z_{p+q})$$

To make use of our earlier general facts, we note the following:

Proposition 3.8. *$\mathcal{S}$ is a* ***commutative*** *monoid in $\mathcal{S}_*^\Sigma$.*

Proof. The natural isomorphisms $m_{p,q} : S^p \wedge S^q \to S^{p+q}$ are $\Sigma_p \times \Sigma_q$-equivariant. By the lemma these assemble into a map $m : \mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ given by $m(\alpha, x, y) = \alpha \cdot x \wedge y$.

The following diagram commutes equivariantly under the action of $\Sigma_p \times \Sigma_q \times \Sigma_r$:

$$\begin{tikzcd}
S^p \wedge S^q \wedge S^r \arrow[r] \arrow[d] & S^{p+q} \wedge S^r \arrow[d] \\
S^p \wedge S^{q+r} \arrow[r] & S^{p+q+r}
\end{tikzcd}$$

By the lemma this means that m will be associative. The unit is simply the inclusion of symmetric sequences $(S^0, *, *, \ldots) \to \mathcal{S}$.

Finally the commutativity, which will be of vital importance:

$$m \circ \tau(\alpha, x, y) = \alpha\rho_{p,q} \cdot y \wedge x = \alpha \cdot x \wedge y = m(\alpha, x, y)$$

Notice the importance of the symmetric group actions in guaranteeing commutativity. □

But if $\mathcal{S}$ is a commutative monoid in $\mathcal{S}_*^\Sigma$, what are its modules? The answer is exactly what you would hope for:

Proposition 3.9. *The category of symmetric spectra is equivalent to the category of $\mathcal{S}$-modules in $\mathcal{S}_*^\Sigma$.*

Proof. We start by constructing a symmetric spectrum from a $\mathcal{S}$-module. The pairing $m : S \otimes X \to X$ is by the lemma a collection of $\Sigma_p \times \Sigma_q$-equivariant maps $m_{p,q} : S^p \wedge X_q \to X_{p+q}$. If you let $\sigma = m_{1,n} : S^1 \to X_n \to X_{n+1}$, then by the associativity axiom of an $\mathcal{S}$-module, we have that $\sigma^p = m_{p,n}$, hence they are $\Sigma_p \times \Sigma_q$ equivariant.

Conversely, if we have a symmetric spectrum, then by the lemma the $\Sigma_p \times \Sigma_q$-equivariant maps σ^p together with σ^0 the natural isomorphism $S^0 \wedge X_n \cong X_n$ assemble to a map $m : \mathcal{S} \otimes X \to X$. The definition of σ^p guarantees that m is associative and respects the unit.

Since these constructions are natural and mutually inverse, we have a equivalence of categories. □

Corollary 3.10. *The category of symmetric spectra is bicomplete and has a closed monoidal product $\wedge := \otimes_{\mathcal{S}}$ with internal hom Hom_S. That is, we have a natural isomorphism:*

$$Sp^{\Sigma}(X \wedge Y, Z) \cong Sp^{\Sigma}(X, Hom_S(Y, Z))$$

The reason that this construction fails for ordinary spectra is that S is not a commutative monoid in the category $\mathcal{S}_*^{\mathbb{N}\cup\{0\}}$, because the twist map $S^1 \wedge S^1 \to S^1 \wedge S^1$ is not the identity. The category of symmetric spectra fixes this using the symmetric actions.

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