Alberto Santini

Topological groupoids

Kandidatproject

Institut for Matematiske Fag,
Københavns Universitet.
Lektor: Jesper Michael Møller

Contents

CHAPTER

ONE

BASIC CONCEPTS

1.1 Introduction to categories

There are basically two ways of approaching groupoids. The first one is algebraically, considering them as a particular generalization of the algebraic structure of group. The second one, that is the one we'll use and need to prove our results is the category theoretical approach. To do this, we'll introduce some basic aspects of category theory, with in mind the fact that we'll always use it as tool for our topological aim.
Categories have been introduced by Saunders Mac Lane ([3]) and Samuel Eilenberg as a way to generalize particular dual properties of groups and lattices, but then they have proven useful as a generalization of almost all algebraic structures mathematicians usually deal with. Their basic constituents are *objects* and *morphisms* (sometimes called arrows) that satisfy certain axioms, as explained in what follows.

Definition 1.1.1. A **category** is a quadruple $\mathbf{A} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Hom}, \mathrm{Id}, \circ)$ where

- $\mathrm{Obj}(\mathbf{A})$ is a class, whose members are called the **objects** of $\mathbf{A}$.
- For each $A, B \in \mathrm{Obj}(\mathbf{A})$, $\mathrm{Hom}_{\mathbf{A}}(A, B)$ is a set whose elements are called **morphisms** and are denoted as $A \xrightarrow{f} B$; we also call A the **domain** and B the **codomain** of f.
- For each $A \in \mathrm{Obj}(\mathbf{A})$, $\mathrm{Id}_{\mathbf{A},A}$ is a morphism $A \xrightarrow{\mathrm{Id}_{\mathbf{A},A}} A$, called the **identity** on A.

- For each $A, B, C, \in \mathrm{Obj}(\mathbf{A})$ and each $A \xrightarrow{f} B, B \xrightarrow{g} C$, $g \circ f$ is a morphism $A \xrightarrow{g \circ f} C$; $\circ$ is called a **composition law**.

such that the following holds:

C1 The composition$\circ$ is associative: given $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$ we have $(f \circ g) \circ h = f \circ (g \circ h)$.

C2 The identities $\mathrm{Id}_{\mathbf{A},A}$ actually act as identities for the composition: given $A \xrightarrow{f} B$ we have that $f \circ \mathrm{Id}_{\mathbf{A},A} = f$ and $\mathrm{Id}_{\mathbf{A},B} \circ f = f$.

C3 The various sets $\mathrm{Hom}_{\mathbf{A}}(A, B)$ are pairwise disjoint.

We will sometimes refer to

$$\mathrm{Hom}_{\mathbf{A}} := \bigsqcup_{A,B \in \mathrm{Obj}(\mathbf{A})} \mathrm{Hom}_{\mathbf{A}}(A, B) \tag{1.1.1}$$

$$\mathrm{Hom}_{\mathbf{A}}(A, -) := \bigsqcup_{B \in \mathrm{Obj}(\mathbf{A})} \mathrm{Hom}_{\mathbf{A}}(A, B) \tag{1.1.2}$$

$$\mathrm{Hom}_{\mathbf{A}}(-, B) := \bigsqcup_{A \in \mathrm{Obj}(\mathbf{A})} \mathrm{Hom}_{\mathbf{A}}(A, B) \tag{1.1.3}$$

$$\tag{1.1.4}$$

Example 1.1.2. Examples of categories are:

- **Set** where the objects are the sets and the morphisms are functions between them.

- **Pos** where the objects are partially ordered sets and the morphisms are the isomorphisms between posets (i.e. order-preserving maps).

- **Lin** where the objects are finite-dimensional K-vector spaces and the morphisms are the linear maps between them.

- **Grp** where the objects are groups and the morphisms are group homomorphisms.

- Every monoid M is also a category with just an object, M itself and whose morphisms are the elements of M (so that the identity is the identity of M and the composition is just the multiplication on M).

- **Top** where the objects are topological spaces and the morphisms are continuous functions between them.

- Given a topological space X, we define the category $\mathbf{P}\,X$; the objects are the points of X and the morphisms are the paths between them (a path from $x \in X$ to $y \in X$ is a continuous function $p_{x,y} : [0,1] \to X$ such that $p_{x,y}(0) = x$ and $p_{x,y}(1) = y$). The identity for $x \in X$ is the null-path — i.e. the constant function $p_{x,x}(t) = x \ \forall t \in [0,1]$ — and the composition is the concatenation of paths.

Definition 1.1.3. Given two categories $\mathbf{A}$ and $\mathbf{B}$ we say that $\mathbf{B}$ is a **subcategory** of $\mathbf{A}$ if

- $\mathrm{Obj}(\mathbf{B}) \subseteq \mathrm{Obj}(\mathbf{A})$.

- Given two objects $A, B \in \mathrm{Obj}(\mathbf{B})$ we have that $\mathrm{Hom}_{\mathbf{B}}(A, B) \subseteq \mathrm{Hom}_{\mathbf{A}}(A, B)$.

- Composition in $\mathbf{B}$ is the same as the one in $\mathbf{A}$.

- For each $A \in \mathrm{Obj}(\mathbf{B})$ we have $\mathrm{Id}_{\mathbf{B},A} = \mathrm{Id}_{\mathbf{A},A}$.

$\mathbf{B}$ is said to be **wide** if it has exactly the same objects as $\mathbf{A}$, while it is said to be **full** if each couple of objects of $\mathbf{B}$ has the same morphisms both in $\mathbf{A}$ and $\mathbf{B}$.

Definition 1.1.4. Given two morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$, we say that they are respectively a **co-retraction** and a **retraction** if $g \circ f = \mathrm{Id}_{\mathbf{A},A}$. Sometimes we say that g is a left-inverse of f and f is a right-inverse of g. The following theorem will show that, fixed a morphism that has both a left and right inverse, it is legitimate to simply refer to its inverse — without specifying if left or right. We will call such a morphism **invertible** or an **isomorphism**.

Theorem 1.1.5. *Given $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$, $B \xrightarrow{h} A$, if $g \circ f = \mathrm{Id}_{\mathbf{A},B}$ and $f \circ h = \mathrm{Id}_{\mathbf{A},A}$, then $g = h$.*

Proof.

$$g = g \circ \mathrm{Id}_{\mathbf{A},A} = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{Id}_{\mathbf{A},B} \circ h = h \tag{1.1.5}$$

□

Definition 1.1.6. Two objects $A, B \in \mathrm{Obj}(\mathbf{A})$ are called **isomorphic** if there exists an isomorphism $A \xrightarrow{f} B$. It's trivial to prove that the relation "*is isomorphic to*" is an equivalence relation on $\mathrm{Obj}(\mathbf{A})$, if this is a set.

Exercise 1.1.7. An object $O_i \in \mathrm{Obj}(\mathbf{C})$ is said to be **initial** if $|\,\mathrm{Hom}_{\mathbf{C}}(O_i, A)| = 1$ for all $A \in \mathrm{Obj}(\mathbf{C})$. Analogously $O_f \in \mathrm{Obj}(\mathbf{C})$ is called **final** if $|\,\mathrm{Hom}_{\mathbf{C}}(A, O_f)| = 1$ for all $A \in \mathrm{Obj}(\mathbf{C})$. An object that is both initial and final is called a **zero object**.

- Prove that all initial objects are isomorphic — as are all final objects.
- Prove that **Set** and **Top** have initial and final objects but no zero object.
- Prove that **Grp** has a zero object.

Proof.

- Let $A, B \in \mathrm{Obj}(\mathbf{C})$ be two initial objects. By hypothesis there are unique morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{A}$, so there is a morphism $A \xrightarrow{g \circ f} A$. But since A is initial $\mathrm{Hom}_{\mathbf{C}}(A, A)$ has only one element, so we must have that $g \circ f = \mathrm{Id}_{\mathbf{C},A}$ and analogously $f \circ g = \mathrm{Id}_{\mathbf{C},B}$, so A and B are isomorphic. An entirely similar reasoning may be applied to final objects.
- Since functions can have empty domains, but can't have empty codomains it's clear that the only initial object in **Set** is $\emptyset$. On the other hand, the only object that allow only one map to be constructed such that it has the object as codomain, must have only one element and so the final objects are the singletons. Hence it's clear then there is no zero object in **Set**. Reasoning in the exact same way one can find that the empty space is the only initial object of **Top** and that one-point space are its only final objects.
- Consider any group $G = \{e_G\}$. By the very definition, there is just one homomorphism from G to any other group H (and it is $e_G \mapsto e_H$) and one homomorphism from H to G (and it is $h \mapsto e_G$ for all $h \in H$).

$\square$

Exercise 1.1.8. A morphism $A \xrightarrow{f} B$ of $\mathbf{C}$ is called a **monomorphism** if

$$\forall C \in \mathrm{Obj}(\mathbf{C})\ \forall g, h \in \mathrm{Hom}_{\mathbf{C}}(C, A) \quad f \circ g = f \circ h \ \Rightarrow\ g = h \tag{1.1.6}$$

Analogously it is called an **epimorphism** if

$$\forall C \in \mathrm{Obj}(\mathbf{C})\ \forall g,h \in \mathrm{Hom}_{\mathbf{C}}(B,C) \quad g \circ f = h \circ f \ \Rightarrow\ g = h \tag{1.1.7}$$

We write monomorphisms as $A \overset{f}{\rightarrowtail} B$ and epimorphisms as $A \overset{f}{\twoheadrightarrow} B$.

- Prove that a morphism of **Set** is a monomorphism if and only if it is injective.
- Prove that a morphism of **Set** is an epimorphism if and only if it is surjective.
- Prove that every isomorphism is both a monomorphism and an epimorphism.
- Give an example of a category that has a morphism that is both an epimorphism and a monomorphism, but not an isomorphism.

Proof.

- Given $A \overset{f}{\rightarrowtail} B$ and $a, a' \in A$ with $a \neq a'$, consider a singleton $\{s\} \in \mathrm{Obj}(\mathbf{Set})$ and the functions

$$f_a, f_{a'} : \{s\} \to A \quad : \quad f_a(x) = a \ \wedge\ f_{a'}(x) = a' \tag{1.1.8}$$

  Since f is a monomorphism, we have that $a \neq a' \ \Rightarrow\ f \circ f_a \neq f \circ f_{a'}$ and so

$$f(a) = f(f_a(x)) \neq f(f_{a'}(x)) = f(a') \tag{1.1.9}$$

  On the other hand, if f is injective, given $g, h : C \to A$ with $g \neq h$, let $c \in C$ be a point such that $g(c) \neq h(c)$. Then by injectivity $f(g(c)) \neq f(h(c))$ and so $f \circ g \neq f \circ h$.

- Given $A \overset{f}{\twoheadrightarrow} B$, consider a set with two elements $\{s,t\} \in \mathrm{Obj}(\mathbf{Set})$ and the functions

$$B \xrightarrow{g,h} \{s,t\} \tag{1.1.10}$$

$$g(x) = s \quad \forall x \in B \tag{1.1.11}$$

$$h(x) = \begin{cases} s & \text{if } x \in f(A) \\ t & \text{if } x \notin f(A) \end{cases} \tag{1.1.12}$$

Now notice that $\forall a \in A$ we have that $g(f(a)) = s$ and $h(f(a)) = s$, so $g \circ f = h \circ f$ and since f is an epimorphism this implies that $g = h$, that is to say that $\forall x \in B \ \ x \in f(A)$ and so f is injective. On the other hand, given the injection $A \xrightarrow{f} B$, consider $C \in \mathrm{Obj}\,\mathbf{C}$ and the morphisms $B \xrightarrow{g,h} C$ such that $g \circ f = h \circ f$. By surjectivity, each $b \in B$ can be written as $f(a)$ for some $a \in A$. So we have that

$$g(b) = g(f(a)) = (g \circ f)(a) = (h \circ f)(a) = h(f(a)) = h(b) \ \ \forall b \in B \qquad (1.1.13)$$

ans so $g = h$, that is to say f is an epimorphism.

- Given an isomorphism $A \xrightarrow{f} B$, let $B \xrightarrow{g} A$ be its inverse. Given $C \in \mathrm{Obj}(\mathbf{C})$ and two morphisms $C \xrightarrow{h,j} A$ such that $f \circ h = f \circ j$ we have that

$$h = (g \circ f) \circ h = g \circ (f \circ h) = g \circ (f \circ j) = (g \circ f) \circ j = j \qquad (1.1.14)$$

and so f is a monomorphism. In a totally analogous way one proves that it is also an epimorphism.

- Here the inclusions of smaller objects into bigger objects (in sufficiently *good* categories) seem to be good candidates of such morphisms, since they are injective but not surjective. The problem arises when one wants to prove that they are epimorphism, since two functions that are equal when restricted to smaller objects are not equal in general. This last observation, however is a precious hint, since there is a famous case in which equality on restriction implies equality. Consider the category **Top** of topological spaces and take $\mathbb{R}, \mathbb{Q} \in \mathrm{Obj}(\mathbf{Top})$ where those spaces are taken with the usual euclidean topology on $\mathbb{R}$ and the induced topology on $\mathbb{Q}$. Notice that, by the very definition, the isomorphisms of **Top** are the homeomorphic maps between spaces. So it's clear that the canonical inclusion $i : \mathbb{Q} \to \mathbb{R}$ is not an isomorphism. However, let's show that it is both a monomorphism and an epimorphism. Given a space $X \in \mathrm{Obj}(\mathbf{Top})$ and two morphisms (continuous maps) $f, g : X \to \mathbb{Q}$ such that $i \circ f = i \circ g$ we have that by injectivity

$$(i \circ f)(x) = (i \circ g)(x) \ \Rightarrow \ f(x) = g(x) \ \ \forall x \in X \qquad (1.1.15)$$

and so i is a monomorphism. On the other hand, given the continuous maps $f, g : \mathbb{R} \to X$ such that $f \circ i = g \circ i$, we have that

$$\forall q \in \mathbb{Q} \quad (f \circ i)(q) = (g \circ i)(q) \;\Rightarrow\; f(i(q)) = g(i(q)) \;\Rightarrow\; f(q) = g(q) \tag{1.1.16}$$

so f, g have the same values on the rationals. But $\mathbb{Q}$ is a dense subset of $\mathbb{R}$ and f, g are continuous; it's a very well known fact that $f \restriction \mathbb{Q} = g \restriction \mathbb{Q} \;\Rightarrow\; f = g$ and so i is an epimorphism. In case one doesn't trust common knowledge, it is also easy to prove this result. Given $x \in \mathbb{R}$ consider a sequence $\langle x_n \;:\; n \in \mathbb{N} \rangle \subset \mathbb{Q}$ with the Cauchy property and such that $\lim_n x_n = x$. Since $f(x_n) = g(x_n)\ \forall n \in \mathbb{N}$ and f, g are continuous, they can be *put into* the limit and so

$$f(x) = \lim_n f(x_n) = \lim_n g(x_n) = g(x) \tag{1.1.17}$$

$\square$

1.2 Groupoids and the fundamental groupoid

In this section, we introduce the main tool we will use to study relevant properties of topological spaces: groupoids and in particular the *fundamental groupoid*, that can be regarded roughly as the groupoid of the paths of a space.

Definition 1.2.1. We say that a category $\mathbf{A}$ is a **groupoid** if $\mathrm{Obj}(\mathbf{A})$ is a set and if all its morphisms are isomorphisms. A **subgroupoid** of $\mathbf{A}$ is a subcategory $\mathbf{B}$ of $\mathbf{A}$ such that $\mathbf{B}$ is itself a groupoid — i.e. for every morphism of $\mathbf{B}$, its inverse is also a morphism of $\mathbf{B}$.

Definition 1.2.2. Consider the space $\mathbf{P}\,X$ of example 1.1.2. Remember that its morphisms were the paths between two points: given $x, y \in X$ we have that

$$\mathrm{Hom}_{\mathbf{P}\,X}(x, y) = \{a : [0,1] \to X \text{ continuous } \;:\; a(0) = x \;\wedge\; a(1) = y\} \tag{1.2.1}$$

We now introduce an equivalence relation in $\mathrm{Hom}_{\mathbf{P}\,X}(x, y)$. We will say that two paths a, b are **homotopic rel endpoints** x, y if there exists an **homotopy of paths** between them. Such a homotopy is a continuous function $F : [0,1]^2 \to X$ such that

$$F(0,t) = a(t) \ \wedge \ F(1,t) = b(t) \ \forall t \in [0,1] \tag{1.2.2}$$
$$F(s,0) = x \ \wedge \ F(s,1) = y \ \forall s \in [0,1] \tag{1.2.3}$$

It is trivial to check that this actually is an equivalence relation; we will denote it by $\sim$. So, intuitively, $a \sim b$ if we can construct a continuous family of paths from x to y such that the first of those paths is just a and the last is b.

Exercise 1.2.3. Prove that $\sim$ preserves the operation of concatenation of paths.

Proof. Given $x, y, z \in X$ and $a, b \in \mathrm{Hom}_{\mathbf{P}X}(x,y)$, $c, d \in \mathrm{Hom}_{\mathbf{P}X}(y,z)$ such that $a \sim b$ through homotopy F and $c \sim d$ through homotopy G, then we have that $a + c \sim b + d$ and the homotopy is $H : [0,1]^2 \to X$ given by

$$H(s,t) = \begin{cases} F(s,2t) & \text{if } t \in [0, \frac{1}{2}[\\ G(s,2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \tag{1.2.4}$$

Also, if we call 0_x the constant path at x, $0_x = [0,1] \times \{x\}$, given any path a from x to y we have that $a + (-a) \sim 0_x$. In fact, a homotopy between them is $H : [0,1]^2 \to X$ given by:

$$H(s,t) = \begin{cases} a((1-s)2t) & \text{if } t \in [0, \frac{1}{2}[\\ a(s(2t-1)) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \tag{1.2.5}$$

□

Definition 1.2.4. We may now define the **fundamental groupoid** of a space X as a groupoid πX with $\mathrm{Obj}(\pi X) = X$ and $\mathrm{Hom}_{\pi X}(x,y) = \mathrm{Hom}_{\mathbf{P}X}(x,y)/\sim$. To show that this definition actually makes sense, we'll have to prove the next simple theorem.

Theorem 1.2.5. *If we call $[a]$ the equivalence class of $a \in \mathrm{Hom}_{\mathbf{P}X}(x,y)$ in $\mathrm{Hom}_{\pi X}$, we have that*

1. $-[a] = [-a]$ *for every* $x \xrightarrow{a} y$.
2. $[a] + [b] = [a+b]$ *where* $x \xrightarrow{a} y$ *and* $y \xrightarrow{b} z$, *as proven in exercise 1.2.3.*
3. $[a] + [0_y] = [0_x] + [a] = [a]$ *for every* $x \xrightarrow{a} y$.

4. $[a] + [-a] = [0_x]$ *for every* $x \xrightarrow{a} y$*, as proven in exercise 1.2.3.*

Proof. As we have already proved points 2 and 4, we're proving here only 1 and 3.

1. Given $a, a' \in [a]$ we want to show that $-a \sim -a'$. In fact, if $F(s,t)$ is a homotopy between a and a', then $G(s,t) = F(s,-t)$ is a homotopy between $-a$ and $-a'$.

3. Since in $\mathrm{Hom}_{\mathbf{P}X}(x,y)$ we have that $a + 0_y = 0_x + a = a$, the thesis follows from point 2.

□

Definition 1.2.6. If X is such that for every $x, y \in X$ we have that $\mathrm{Hom}_{\pi X}(x,y)$ contains only one element, then X and πX are called **1-connected**. If this property doesn't hold hold for the whole space X but holds for each of its path-connected components, we call the space and its fundamental groupoid **simply-connected**.

Example 1.2.7. If X is a convex normed vector space then it is 1-connected, since all the paths from x to y are homotopic. In fact, given $a, b \in \mathrm{Hom}_{\mathbf{P}X}(x,y)$ we have that $F(s,t) = (1-s)a(t) + sb(t)$ is well defined for the convexity (since it connects the points of path a to the points of path b via segments) and is an homomorphism.

Note 1.2.8. From now on, we'll simplify the notetion a little bit. We'll call our groupoids with simple capital letters $G, H, \ldots$ and when we'll write $a \in G$ what we mean is that we have fixed $x, y \in \mathrm{Obj}(G)$ and we are taking $a \in \mathrm{Hom}_G(x,y)$.

Definition 1.2.9. Given a groupoid G and a subset O of its objects, it's easy to construct the subgroupoid H of G that has O as objects and is full: simply, for every $x, y \in \mathrm{Obj}(H)$ we take $\mathrm{Hom}_H(x,y) := \mathrm{Hom}_G(x,y)$. If O consists of only one element $O = \{x\}$ we'll denote the corresponding full subgroupoid as $G(x)$. We call this latter and — in general — every groupoid with only one element a **group**. In particular, given a topological space X and a point $x \in X$, the group $\pi X(x)$ is called the **fundamental group** of X at x and — to adhere to the classical algebraic topological writing — we'll denote it with $\pi_1(X,x)$.

Definition 1.2.10. The notion of connectedness given in definition 1.2.6 can be generalized to any kind of groupoid. We say that G is **connected** if $\mathrm{Hom}_G(x,y)$ is non-empty for every $x, y \in \mathrm{Obj}(G)$. The following trivial theorem further characterize this definition in the case of the fundamental groupoid.

Theorem 1.2.11. *A fundamental groupoid πX is connected if and only if X is path-connected.*

Proof. This theorem simply follows from the definitions of the fundamental groupoid and of path-connected spaces. □

Definition 1.2.12. Given an object $x_0 \in \mathrm{Obj}(G)$, let's consider $O = \{y \in \mathrm{Obj}(G) : \mathrm{Hom}_G(x, y) \neq \emptyset\}$. We call the full subgroupoid $C(x_0)$ of G whose object set is O, the **component** of G containing x_0.

Observation 1.2.13. We'd like to simplify the relation between a connected groupoid G and its groups $G(x)$ (where $x \in \mathrm{Obj}(G)$), by showing that all those groups are isomorphic, so that it makes sense to refer to what we call *the* **object group** of G. We'll do this in the next theorem.

Theorem 1.2.14. *Given a connected groupoid G and $x_1, y_1, x_2, y_2 \in \mathrm{Obj}(G)$, there is a bijection*

$$\varphi : \mathrm{Hom}_G(x_1, y_1) \to \mathrm{Hom}_G(x_2, y_2) \tag{1.2.6}$$

which, in case $x_1 = y_1 \ \wedge \ x_2 = y_2$ is a group isomorphism between $G(x_1)$ and $G(x_2)$.

Proof. Let's take $x_1 \xrightarrow{a} x_2$ and $y_1 \xrightarrow{b} y_2$, which exist by the connectedness. We define:

$$\varphi : \mathrm{Hom}_G(x_1, y_1) \to \mathrm{Hom}_G(x_2, y_2) \tag{1.2.7}$$
$$\psi : \mathrm{Hom}_G(x_2, y_2) \to \mathrm{Hom}_G(x_1, y_1) \tag{1.2.8}$$

As:

$$\varphi(c) = b + c - a \tag{1.2.9}$$
$$\psi(c) = -b + c + a \tag{1.2.10}$$

Where we intend as usual that e.g. $c - a$ is $c \circ (-a)$. It is clear that $\varphi\psi = \mathrm{Id}_{\mathrm{Hom}_G(x_2, y_2)}$ and $\psi\varphi = \mathrm{Id}_{\mathrm{Hom}_G(x_1, y_1)}$, so φ is a bijection. Also, if $x_1 = y_1 \ \wedge \ x_2 = y_2$ we have that

$$\varphi(c) = a + c - a := a_{x_1}(c) \tag{1.2.11}$$

Let's show that this is a group isomorphism. For all $c_1, c_2 \in G(x)$ we have:

$$a_{x_1}(c_1) + a_{x_1}(c_2) = \tag{1.2.12}$$
$$=a + c_1 - a + a + c_2 - a = \tag{1.2.13}$$
$$=a + (c_1 + c_2) - a = \tag{1.2.14}$$
$$=a_{x_1}(c_1 + c_2) \tag{1.2.15}$$

□

Definition 1.2.15. We'd now like to generalize the notion of 1-connected groupoid even where no topological space is assumed behind it. We say that a groupoid T is a **tree groupoid** if for every two objects $x, y \in \mathrm{Obj}(T)$ there is an unique element $a_{y,x}$ in $\mathrm{Hom}_T(x, y)$. Notice that given a third object z, then for uniqueness

$$a_{z,x} = a_{y,x} + a_{z,y} \tag{1.2.16}$$

Observation 1.2.16. Given a non-empty set X, it's a trivial task to build a tree groupoid T such that $X = \mathrm{Obj}(T)$. We fix one element $a_{y,x}$ for each two $x, y \in X$ and set $\mathrm{Hom}_T(x, y) := \{a_{y,x}\}$; finally we just impose that equation 1.2.16 holds.

Observation 1.2.17. Observation 1.2.16 ensure us that a connected groupoid G can contain a wide tree subgroupid T. It's clear that, supposing we don't know the morphisms of G, given T there are infinitely many possible ways to create a connected groupoid G such that T is one if its wide tree subgroupoids. But if we fix an element $x_0 \in \mathrm{Obj}(G)$ and we know how $G(x_0)$ is made, then we'll be able to reconstruct the whole structure of G. In fact, let's call t_x the only element of $\mathrm{Hom}_T(x_0, x)$. Now, given $x, y \in \mathrm{Obj}(G)$ and $x \xrightarrow{a} y$, we can write it as

$$a = t_y + c_a - t_x \text{ for some unique } c_a \in G(x_0) \tag{1.2.17}$$

For this reason, given a third object $z \in \mathrm{Obj}(G)$ and a $y \xrightarrow{b} z$, we have that:

$$a + b = (t_z + c_b - t_y) + (t_y + c_a - t_x) = \tag{1.2.18}$$
$$= t_z + (c_b + c_a) - t_x \Rightarrow \tag{1.2.19}$$
$$\Rightarrow c_{b+a} = c_b + c_a \tag{1.2.20}$$

1.3 Functors

Functors are the main tools used to make categories interact with each other, so to link different algebraic (or topological) structures in a categorical way. At a more abstract level — and leaving out the paradoxical implication that this may have — they could even be regarded as morphisms in the category of categories!

Definition 1.3.1. Given two categories $\mathbf{A}, \mathbf{B}$, a **functor** $F : \mathbf{A} \to \mathbf{B}$ is a function that assigns to each object $A \in \mathrm{Obj}(\mathbf{A})$ an object $F(A) \in \mathrm{Obj}(\mathbf{B})$ and to each morphism $A \xrightarrow{f} A'$ of $\mathbf{A}$ a morphism $F(A) \xrightarrow{F(f)} F(A')$ of $\mathbf{B}$, such that

F1 F preserves composition, i.e. $F(f \circ g) = F(f) \circ F(g)$.

F2 F preserves identity morphisms, i.e. $F(\mathrm{Id}_{\mathbf{A},A}) = \mathrm{Id}_{\mathbf{B},F(A)}$.

Sometimes we'll simply write FA and Ff for $F(A)$ and $F(f)$.

Observation 1.3.2. From the definition of functor it follows immediately that it preserves the status of being a retraction, co-retraction or isomorphism.

Example 1.3.3. Examples of functors are:

- For every category $\mathbf{A}$ the **identity functor** $\mathrm{Id} : \mathbf{A} \to \mathbf{A}$.
- For every two categories $\mathbf{A}, \mathbf{B}$, fixed a $B \in \mathrm{Obj}(\mathbf{B})$, we obtain a **constant functor** F such that

$$F(A) = B \ \forall A \in \mathrm{Obj}(\mathbf{A}) \text{ and } F(f) = \mathrm{Id}_{\mathbf{B},B} \ \ \forall A \xrightarrow{f} A' \tag{1.3.1}$$

- Given any category from 1.1.2 (e.g. **Top**) we can remove any algebraic or topological structure from its objects via the **forgetful functor**. E.g. $F : \mathbf{Top} \to \mathbf{Set}$ with (denoting a topological space X with the more formal couple (X, ϑ) of a set of points and a topology) $F((X, \vartheta)) = X$ and $F(f) = f$.
- Given two categories $\mathbf{P}\,X, \mathbf{P}\,Y$ consider a map $f : X \to Y$; we have the functor

$$P_f : \mathbf{P}\,X \to \mathbf{P}\,Y \quad P_f(x) = f(x) \ \forall x \in X \tag{1.3.2}$$

and given $a \in \mathrm{Hom}_{\mathbf{P}\,X}(x,y)$ we take as $P_f(a)$ an arbitrary path in $\mathrm{Hom}_{\mathbf{P}\,Y}(f(x), f(y))$. Note that if $Y = X$ and $f = \mathrm{Id}_X$, then P_{Id} is the identity functor. Also, given the spaces X, Y, Z and the maps $f : X \to Y$ and $g : Y \to Z$ we have that $P_{g \circ f} = P_g \circ P_f$. Basically, this means that P is a functor from **Top** $\to$ **Cat**, where **Cat** is the category of the categories (we'll leave the logical implications of considering such a category out of our work!). Also, for observation 1.3.2 we have that if X and Y are homeomorfic through f, then $\mathbf{P}\,X$ and $\mathbf{P}\,Y$ are isomorphic through the functor P_f.

- Given a topological space X, the functor $F : \mathbf{P}\,X \to \pi X$ such that

$$F(x) = x \;\forall x \in X \text{ and } F(a) = [a] \;\forall a \in \mathbf{P}\,X \tag{1.3.3}$$

Observation 1.3.4. As we know a groupoid is a category, but we may as well define the category of groupoids, **Grpd**. Its objects are groupoids and its morphisms are functors between groupoids. So it naturally arises the question wether the application that sends a space X into its fundamental groupoid πX is a functor itself. We settle this question in the nex theorem.

Theorem 1.3.5. *The function π : **Top** $\to$ **Grpd** that sends X into πX is a functor.*

Proof. Given two spaces X, Y and a continuous map $f : X \to Y$, let's consider the corresponding functor P_f as defined in example 1.3.3. Now given $x, x' \in X$ and two paths $a, b : [0,1] \to X$ from x to x' within the same homotopy class through homotopy $F : [0,1]^2 \to X$, it's easy to check that $f \circ F$ is a homotopy between $f \circ a$ and $f \circ b$. So we have create a function $\pi f : \pi X \to \pi Y$ defined as

$$\pi f([a]) = [f \circ a] \tag{1.3.4}$$

which is a morphism between the groupoids πX and πY. Now we have successfully created our π : **Top** $\to$ **Grpd**: given a space X it gives its fundamental groupoid πX and given a continuous $f : X \to Y$ it gives the morphism πf. $\square$

Observation 1.3.6. Theorem 1.3.5 and observation 1.3.2 gives us a ready-to-use class of isomorphic fundamental groupoids: in fact, if two spaces are homeomorfic, their fundamental groupoids are isomorphic.

Definition 1.3.7. Given a topological space X and a point $x \in X$ we call **pointed space** the couple (X, x). Given two pointed spaces $(X, x), (Y, y)$ we call a **pointed map** a continuous $f : X \to Y$ such that $f(x) = y$. This gives us the category $\mathbf{Top_{pt}}$, with objects the pointed spaces and morphisms the pointed maps between them.

Theorem 1.3.8. *The function $\pi_1 : \mathbf{Top_{pt}} \to \mathbf{Grp}$ that sends (X, x) into $\pi_1(X, x)$ is a functor.*

Proof. The proof is analogous at that of theorem 1.3.5. Given a pointed space (X, x) we associate it with its fundamental group $\pi_1(X, x)$ and given a pointed map $f : X \to Y$ we associate it to a morphism $\pi_1 f : \pi_1(X, x) \to \pi_1(Y, y)$ again defined as

$$\pi_1 f([a]) = [f \circ a] \tag{1.3.5}$$

Since the composition of a loop a around x with the pointed map f gives a loop around y, we have actually defined a functor π_1. $\square$

Definition 1.3.9. We would now like to show that the fundamental groupoid of the product of two spaces is isomorphic to *the product of the fundamental groupoids* of the two spaces. It may sound quite intuitive what this product of groupoids should be, nonetheless we will give here the general definition of **product of two categories** $\mathbf{A}, \mathbf{B}$ as the category $\mathbf{A} \times \mathbf{B}$ whose objects class is

$$\mathrm{Obj}(\mathbf{A}) \times \mathrm{Obj}(\mathbf{B}) \tag{1.3.6}$$

and — for each two objects $(A_1, B_1) := C, (A_2, B_2) := D$ — its morphisms are:

$$\mathrm{Hom}_{\mathbf{A}\times\mathbf{B}}(C, D) = \mathrm{Hom}_{\mathbf{A}}(A_1, A_2) \times \mathrm{Hom}_{\mathbf{B}}(B_1, B_2) \tag{1.3.7}$$

Also the composition of a morphism $(a_1, b_1) := c \in \mathbf{A} \times \mathbf{B}$ with a morphism $(a_2, b_2) := d \in \mathbf{A} \times \mathbf{B}$ is defined component-wise as the morphism

$$d \circ c = (a_2 \circ a_1, b_2 \circ b_1) \tag{1.3.8}$$

while the inverse is simply

$$c^{-1} = (a_1^{-1}, b_1^{-1}) \tag{1.3.9}$$

Luckily enough, with this definition all the good properties of functors one would expect continue to hold. For example, given a category $\mathbf{C}$ and the functors $f_A : \mathbf{C} \to \mathbf{A}$ and $f_B : \mathbf{C} \to \mathbf{B}$, then there is an unique functor $f : \mathbf{C} \to \mathbf{A} \times \mathbf{B}$ such that composed with the canonical projections it gives back the functors f_A and f_B. Now we are ready to prove the following theorem.

Exercise 1.3.10. Given a category $\mathbf{C}$, we call its **dual** the category $\mathbf{C}^{\mathbf{op}}$ such that

$$\mathrm{Obj}(\mathbf{C}^{\mathbf{op}}) = \mathrm{Obj}(\mathbf{C}) \tag{1.3.10}$$

$$\mathrm{Hom}_{\mathbf{C}^{\mathbf{op}}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(B, A) \tag{1.3.11}$$

Furthermore, if we call $f^* \in \mathrm{Hom}_{\mathbf{C}^{\mathbf{op}}}(A, B)$ the morphism corresponding to $f \in \mathrm{Hom}_{\mathbf{C}}(B, A)$, then the composition law in $\mathbf{C}^{\mathbf{op}}$ is

$$g^* \circ f^* = (f \circ g)^* \tag{1.3.12}$$

Given two categories $\mathbf{C}, \mathbf{D}$, we say that $\Gamma : \mathbf{C} \to \mathbf{D}$ is a **contravariant functor** if

$$f \in \mathrm{Hom}_{\mathbf{C}}(A, B) \ \Rightarrow \ \Gamma(f) \in \mathrm{Hom}_{\mathbf{D}}(\Gamma(B), \Gamma(A)) \ \ \forall A, B \in \mathrm{Obj}(\mathbf{C}) \tag{1.3.13}$$

$$\Gamma(g \circ f) = \Gamma(f) \circ \Gamma(g) \ \ \forall A, B, C \in \mathrm{Obj}(\mathbf{C}) \ \forall A \xrightarrow{f} B, B \xrightarrow{g} C \tag{1.3.14}$$

Prove that the contravariant functors $\mathbf{C} \to \mathbf{D}$ are determined by the functors $\mathbf{C}^{\mathbf{op}} \to \mathbf{D}$.

Proof. Given the functor $F : \mathbf{C}^{\mathbf{op}} \to D$, we define $\Gamma : \mathbf{C} \to \mathbf{D}$ as

$$\Gamma(A) = F(A) \ \ \forall A \in \mathrm{Obj}(\mathbf{C}) = \mathrm{Obj}(\mathbf{C}^{\mathbf{op}}) \tag{1.3.15}$$

$$\Gamma(f) = F(f^*) \ \ \forall A \xrightarrow{f} B \tag{1.3.16}$$

Now, applying the definition of dual category and of functor (see definition 1.3.1), we have first of all that

$$\forall f \in \mathrm{Hom}_{\mathbf{C}}(A, B) \ \ f* \in \mathrm{Hom}_{\mathbf{C}^{\mathbf{op}}}(B, A) \ \Rightarrow \tag{1.3.17}$$

$$\Rightarrow F(f*) \in \mathrm{Hom}_{\mathbf{D}}(F(B), F(A)) \ \Rightarrow \tag{1.3.18}$$

$$\Rightarrow \Gamma(f) \in \mathrm{Hom}_{\mathbf{D}}(\Gamma(B), \Gamma(A)) \tag{1.3.19}$$

that is the first property of contravariance. The second property is soon proved; in fact

$$\Gamma(g \circ f) = F((g \circ f)^*) = F(f^* \circ g^*) = F(f*) \circ F(g*) = \Gamma(f) \circ \Gamma(g) \tag{1.3.20}$$

□

Theorem 1.3.11. *Given the topological spaces X_1, X_2 and $X = X_1 \times X_2$, then πX is isomorphic to $\pi X_1 \times \pi X_2$.*

Proof. Consider the morphisms from the fundamental groupoid of the product space to the fundamental groupoids of the spaces

$$\pi p_1 : \pi X \to \pi X_1 \tag{1.3.21}$$

$$\pi p_2 : \pi X \to \pi X_2 \tag{1.3.22}$$

defined as the morphisms induced by the canonical (topological) projections from the product space X to the spaces X_1, X_2. As seen in definition 1.3.9 these gives an unique functor $f : \pi X \to \pi X_1 \times \pi X_2$ such that, composed with the canonical (categorical) projections it gives back πp_1 and πp_2. As it's natural to imagine, such f is

$$f([a]) = ([p_1 \circ a], [p_2 \circ a]) \tag{1.3.23}$$

Now we are ready to construct the isomorphism from πX to $\pi X_1 \times \pi X_2$: fixed two points $x = (x_1, x_2), y = (y_1, y_2) \in X$ we want to show that f induces a bijection

$$\mathrm{Hom}_{\pi X}(x, y) \to \mathrm{Hom}_{\pi X_1}(x_1, y_1) \times \mathrm{Hom}_{\pi X_2}(x_2, y_2) \tag{1.3.24}$$

In fact, given two paths $a, b \in \mathrm{Hom}_{\pi X}(x, y)$ such that $f([a]) = f([b])$ we have that

$$f([a]) = f([b]) \;\Rightarrow\; ([p_1 \circ a], [p_2 \circ a]) = ([p_1 \circ b], [p_2 \circ b]) \;\Rightarrow\; \ldots \tag{1.3.25}$$

Now, if we call F_1 the homotopy between $p_1 \circ a$ and $p_1 \circ b$ and F_2 the homotopy between $p_2 \circ a$ and $p_2 \circ b$, the function $F : [0, 1]^2 \to X$ with

$$F(s, t) = (F_1(s, t), F_2(s, t)) \tag{1.3.26}$$

is a homotopy between a and b and then

$$\dots \Rightarrow [a] = [b] \tag{1.3.27}$$

so the induced map is injective. To prove that it is also a surjection, consider any element (a_1, a_2) in $\mathrm{Hom}_{\pi X_1}(x_1, y_1) \times \mathrm{Hom}_{\pi X_2}(x_2, y_2)$. The element $a = (a_1, a_2)$ is in $\mathrm{Hom}_{\pi X}(x, y)$ and it's trivial to see that, by definition, $f([a]) = ([a_1], [a_2])$. □

Theorem 1.3.12. *In an analogous way, one can prove that given the pointed spaces (X_1, x_1) and (X_2, x_2), then*

$$\pi_1(X_1, x_1) \times \pi_1(X_2, x_2) \tag{1.3.28}$$

is isomorphic to

$$\pi_1(X_1 \times X_2, (x_1, x_2)) \tag{1.3.29}$$

1.4 Pushouts

Pushouts are fundamental for our aim, since they are the object that behaves best, among the ones we've encountered by far. In fact, they bring with themselves two desirable properties: commutativity of diagrams (i.e. paths of arrows only depend on their endpoints) and universality (i.e. uniqueness up to isomorphism that is to say, using a more philosophical language, taht we can forget about particular instances of objects and focus on their ontology).

Definition 1.4.1. Given a category $\mathbf{A}$ and $A_1, A_2 \in \mathrm{Obj}(\mathbf{A})$ we call a **coproduct** of A_1 and A_2 a diagram of the type

$$A_1 \xrightarrow{i_1} A \xleftarrow{i_2} A_2 \tag{1.4.1}$$

for some $A \in \mathrm{Obj}(\mathbf{A})$, such that for any diagram $A_1 \xrightarrow{j_1} A' \xleftarrow{j_2} A_2$ — again with $A' \in \mathrm{Obj}(\mathbf{A})$ — there is a unique $A \xrightarrow{a} A'$ such that

$$a \circ i_1 = j_1 \ \wedge \ a \circ i_2 = j_2 \tag{1.4.2}$$

This condition is expressed in a more concise way with the diagram:

$$
\begin{tikzcd}
 & A' & \\
A_1 \arrow[r, "i_1"'] \arrow[ur, "j_1"] & A \arrow[u, dashed, "a"] & A_2 \arrow[l, "i_2"] \arrow[ul, "j_2"']
\end{tikzcd}
\tag{1.4.3}
$$

We will usually write that $A = A_1 \sqcup A_2$ and say that A is the coproduct of A_1 and A_2.

Note 1.4.2. From now on, when we'll show diagrams in which what is an object, what is a morphism and at which Hom class it belongs to is clear, we'll omit to specify it.

Definition 1.4.3. Given a category $\mathbf{A}$ and $A_0, A_1, A_2 \in \mathrm{Obj}(\mathbf{A})$, we call a **pushout** of i_1 and i_2 a diagram

$$
\begin{tikzcd}
A_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & A_1 \arrow[d, "u_1"] \\
A_2 \arrow[r, "u_2"'] & A
\end{tikzcd}
\tag{1.4.4}
$$

Such that it is commutative — i.e. $u_1 \circ i_1 = u_2 \circ i_2$ — and that if there is another commutative diagram

$$
\begin{tikzcd}
A_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & A_1 \arrow[d, "v_1"] \\
A_2 \arrow[r, "v_2"'] & A'
\end{tikzcd}
\tag{1.4.5}
$$

then there exists an unique morphism $A \xrightarrow{a} A'$ such that

$$
a \circ u_1 = v_1 \ \wedge \ a \circ u_2 = v_2 \tag{1.4.6}
$$

Again there is a more concise notation for this (and, from no on, we will only use such kind of notations):

$$\begin{tikzcd} A_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & A_1 \arrow[d, "u_1"] \arrow[ddr, bend left, "v_1"] & \\ A_2 \arrow[r, "u_2"'] \arrow[drr, bend right, "v_2"'] & A \arrow[dr, dashed, "a"] & \\ & & A' \end{tikzcd} \tag{1.4.7}$$

We will usually write $A = A_1 \sqcup_{i_1,i_2} A_2$ and say that A is the pushout of i_1 and i_2.

Example 1.4.4. Examples of pushouts are:

- Given $A_1, A_1 \in \mathrm{Obj}(\mathbf{Set})$ with $A_0 = A_1 \cap A_2$ and i_1, i_2 canonical inclusions, the pushout of A_1 and A_2 is $A = A_1 \cup A_2$ where u_1, u_2 are again canonical inclusions.
- If $A_1, A_2 \in \mathrm{Obj}(\mathbf{Top})$ and $A_0 \leq A_1$ where i_1 is the canonical inclusion and i_2 is a continuous map (called the attaching map), then $A = A_1 \cup_{i_2} A_2$ is the **adjunction space** whose set of points is the sets of the classes of $A_1 \sqcup A_2$ according to the relation of identification of x with $i_2(x)$ for every $x \in A$ and whose topology is the one induced by the same relation. One can intuitively imagine that A_1 is being *glued* on A_2 via the map i_2.
- A special case of the above is when $(A_1, x), (A_2, y) \in \mathbf{Top_{pt}}$, $A = \{x\}$ and i_2 simply identifies x with y. In this case the adjunction space turns out to be just the wedge sum of A_1 and A_2.

Definition 1.4.5. Given two diagrams

$$\begin{tikzcd} A \arrow[r, "m_{AB}"] \arrow[d, "m_{AC}"'] & B \arrow[d, "m_{BD}"] \\ C \arrow[r, "m_{CD}"'] & D \end{tikzcd} \tag{1.4.8}$$

$$\begin{tikzcd} B \arrow[r, "m_{BE}"] \arrow[d, "m_{BD}"'] & E \arrow[d, "m_{EF}"] \\ D \arrow[r, "m_{DF}"'] & F \end{tikzcd} \tag{1.4.9}$$

we call a **composite** (or, better, an horiziontal composite) of them the diagram:

$$\begin{tikzcd} A \arrow[r, "m_{AB}"] \arrow[d, "m_{AC}"'] & B \arrow[r, "m_{BE}"] \arrow[d, "m_{BD}"] & E \arrow[d, "m_{EF}"] \\ C \arrow[r, "m_{CD}"'] & D \arrow[r, "m_{DF}"'] & F \end{tikzcd} \tag{1.4.10}$$

Note 1.4.6. Until now, we've proved all our statements, in order to get the reader accustomed with categorical methods. From now on, we'll omit some technical proof of minor importance. Most of them can be found in [2].

Theorem 1.4.7. *The composite of two pushouts is a pushout.*

Definition 1.4.8. Given the following diagram:

$$\begin{tikzcd} A_0 \arrow[rr, "i_1"] \arrow[dr, "a_0"] \arrow[dd, "i_2"'] & & A_1 \arrow[dr, "a_1"] \arrow[dd, "u_1"] & \\ & B_0 \arrow[rr, "j_1"] \arrow[dd, "j_2"] & & B_1 \arrow[dd, "v_1"] \\ A_2 \arrow[rr, "u_2"] \arrow[dr, "a_2"] & & A \arrow[dr, "a"] & \\ & B_2 \arrow[rr, "v_2"'] & & B \end{tikzcd} \tag{1.4.11}$$

We call $\underline{A}$ the back square and $\underline{B}$ the front square and suppose they are commutative. If the diagram is itself commutative, we call it a morphism between $\underline{A}$ and $\underline{B}$. The name is not casual; in fact, notice that there is a trivial identity morphism from $\underline{A}$ into itself and that the composition of such morphisms is a morphism itself. So we can consider the category $\mathbf{C}_\square$ whose objects are commutative squares and whose morphisms are the ones just defined.

Theorem 1.4.9. *Given $\underline{A}, \underline{B} \in \mathrm{Obj}(\mathbf{C}_\square)$, if $\underline{B}$ is a pushout and there is a retraction from $\underline{B}$ to $\underline{A}$, then $\underline{A}$ is also a pushout.*

1.5 Further properties of the fundamental groupoid

The main result of this section will be to calculate the fundamental groupoid of a space that can be written as the union of two open subspaces, in terms of the fundamental groupoids of those subspaces. We'll also give a way to turn a functor into another one, while respecting its inner structure, by using *natural transformations.*

Definition 1.5.1. Given a topological space X and the subspaces $X_0, X_1, X_2 \leq X$ such that $X_0 = X_1 \cap X_2$ and $X = \mathring{X}_1 \cup \mathring{X}_2$, we call the square $\underline{X}$ of X the following square

$$\begin{tikzcd} X_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & X_1 \arrow[d, "u_1"] \\ X_2 \arrow[r, "u_2"'] & X \end{tikzcd} \tag{1.5.1}$$

Note that, by basic properties of continuous maps, $\underline{X}$ is a pushout.

Definition 1.5.2. In the same notations of the previous definition, consider a set $A \subseteq X$. It is called a **representative** in X_0, X_1 and X_2 if A intersect each path-connected component of X_0, X_1 and X_2.

Theorem 1.5.3. *In the same notations of the previous definitions, we call $\pi(X, A)$ the groupoid of all the paths in X with endpoints in A and we call $\pi i_1, \pi i_2, \pi u_1, \pi u_2$ the maps induced by i_1, i_2, u_1, u_2 on the appropriate groupoids; so that we have:*

$$\pi(X_0, A) \xrightarrow{\pi i_1} \pi(X_1, A) \tag{1.5.2}$$

$$\pi(X_0, A) \xrightarrow{\pi i_2} \pi(X_2, A) \tag{1.5.3}$$

$$\pi(X_1, A) \xrightarrow{\pi u_1} \pi(X, A) \tag{1.5.4}$$

$$\pi(X_2, A) \xrightarrow{\pi u_2} \pi(X, A) \tag{1.5.5}$$

Then the following square is a pushout in **Grpd***:*

$$\begin{tikzcd}
\pi(X_0, A) \arrow[r, "\pi i_1"] \arrow[d, "\pi i_2"'] & \pi(X_1, A) \arrow[d, "\pi u_1"] \\
\pi(X_2, A) \arrow[r, "\pi u_2"'] & \pi(X, A)
\end{tikzcd} \tag{1.5.6}$$

Observation 1.5.4. In the special case when X_0 is path connected, theorem 1.5.3 can be used to reconstruct the fundamental group $\pi_1(X, x_0)$ for all $x_0 \in X_0$. But what if X_0 is not path-connected? In this case, the following definitions and the next theorem come in handy.

Definition 1.5.5. Given two categories $\mathbf{C}$ and $\mathbf{D}$ and two functors $F, G : \mathbf{C} \to \mathbf{D}$ we call a **natural transformation** from F to G a function

$$N : \mathrm{Obj}(\mathbf{C}) \to \mathrm{Hom}_{\mathbf{D}} \tag{1.5.7}$$

that sends the object $A \in \mathrm{Obj}(\mathbf{C})$ to the morphism $N_A \in \mathrm{Hom}_{\mathbf{D}}(F(A), G(A))$ in such a way that for all $B \in \mathrm{Obj}(\mathbf{C})$ and all $A \xrightarrow{f} B$ the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "N_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "N_B"'] & G(B)
\end{tikzcd} \tag{1.5.8}$$

commutes.

Example 1.5.6. As an example of natural transformation consider the categories **Grp** and **Set**. Let F, S : **Grp** $\to$ **Set** be — respectively — the forgetful functor and the squaring functor (that maps $G \xrightarrow{f} H$ to $G^2 \xrightarrow{f^2} H^2$). Also, let $*_G : G^2 \to G$ be the group operation for every group $G \in \mathrm{Obj}(\mathbf{Grp})$. Since for every group homomorphism $f : G \to H$, by definition it holds that

$$f(g *_G g') = f(g) *_H f(g') \tag{1.5.9}$$

then we have that the function N that maps $G \mapsto *_G$ is a natural transformation.

Definition 1.5.7. Let $\mathbf{C}, \mathbf{D}$ be two categories such that $\mathrm{Obj}(C)$ is a set. We define the **functor category** $\mathbf{D}^{\mathbf{C}}$ as the category whose objects are the functors $F : \mathbf{C} \to \mathbf{D}$ and the morphisms are the natural transformations between them. Here the composition is done componentwise: given the transformations N between F and G, M between G and H we define the transformation MN between F and H as the transformation such that

$$(MN)_A = M_A N_A \quad \forall A \in \mathrm{Obj}(\mathbf{C}) \tag{1.5.10}$$

Definition 1.5.8. Given two categories $\mathbf{C}$ and $\mathbf{D}$ and two functors $F, G : \mathbf{C} \to \mathbf{D}$, a natural transformation N from F to G is said to be a **natural equivalence** (or a **homotopy**) if N_A is an isomorphism for all $A \in \mathrm{Obj}(\mathbf{C})$. The reason why we call it an homotopy is explained by the fact that we can give a more *homotopy-fashioned* definition of natural equivalence in the sense that it resembles more closely to the definition of homotopy between paths. It is in fact easy to check that N can be considered as a functor $N : \mathbf{C} \times \mathbf{I} \to \mathbf{D}$ where $\mathbf{I}$ is the tree groupoid for $\{0, 1\}$ such that

$$F(A) = N(A, 0) \;\wedge G(A) = N(A, 1) \quad \forall A \in \mathrm{Obj}(\mathbf{C}) \tag{1.5.11}$$

F is usally called the initial functor and G the final functor of N. It is also easy to check that the natural equivalence is an equivalence relation and so it makes sense to write $F \simeq G$ whenever there is a natural equivalence with initial functor F and final functor G. In this case, F and G are said to be homotopic.

Observation 1.5.9. It's quite easy (but a little bit technical) to see that the fundamental groupoid and the fundamental group are preserved by natural equivalence. This means that given $X, Y \in \mathrm{Obj}(\mathbf{Top})$, if $X \simeq Y$ then $\pi X \simeq \pi Y$ and given $(X, x), (Y, y) \in \mathrm{Obj}(\mathbf{Top_{pt}})$, if $(X, x) \simeq (Y, y)$ then $\pi_1(X, x) \simeq \pi_1(Y, y)$.

Exercise 1.5.10. Two categories $\mathbf{C}, \mathbf{D}$ are said to be **equivalent** if there are two functors

$$F : \mathbf{C} \to \mathbf{D}, \quad G : \mathbf{D} \to \mathbf{C} \tag{1.5.12}$$

such that $F \circ G \simeq \mathrm{Id}_{\mathbf{D}}$ and $G \circ F \simeq \mathrm{Id}_{\mathbf{C}}$. Now, consider a field K and the category $\mathbf{Lin_b}$ as defined in example 1.1.2, with a small difference: its objects are couples

$(V, \mathscr{B}_V)$ where V is a finite-dimensional vector spaces over K and $\mathscr{B}_V$ is a base for V; its morphisms are linear maps between the spaces. Let $\mathbf{L}$ be the category defined as follows: its objects are positive natural numbers and for every $m, n \in \mathbb{N}^+$ the morphisms between n and m are the $n \times m$ matrices with entries in K, composed by matrix multiplication (when possible). Prove that $\mathbf{Lin_b}$ and $\mathbf{L}$ are equivalent.

Proof. Let's consider the following two functors

- The functor $F : \mathbf{L} \to \mathbf{Lin_b}$ is such that

$$\forall n \in \mathrm{Obj}(\mathbf{L}) = \mathbb{N} \quad F(n) = (K^n, \mathscr{E}_n) \tag{1.5.13}$$

  where $\mathscr{E}_n$ is the canonical base of K^n

$$\begin{aligned} &\forall n, m \in \mathrm{Obj}(\mathbf{L}) = \mathbb{N}^+, \ \forall M \in \mathrm{Hom}_{\mathbf{L}}(n, m) = K^{n \times m} \\ &F(M) = \varphi : (K^n, \mathscr{E}_n) \to (K^m, \mathscr{E}_m) \end{aligned} \tag{1.5.14}$$

  where φ is the linear map associated with M via the canonical bases.

- The functor $G : \mathbf{Lin_b} \to \mathbf{L}$ is such that

$$\forall (V, \mathscr{B}_V) \in \mathrm{Obj}(\mathbf{Lin_b}) \quad G(V) = \dim(V) = |\mathscr{B}_V| \tag{1.5.15}$$

  and

$$\begin{aligned} &\forall (V, \mathscr{B}_V), (W, \mathscr{B}_W) \in \mathrm{Obj}(\mathbf{Lin_b}), \ \forall \varphi : V \to W \\ &F(\varphi) = M \in K^{\dim(V) \times \dim(W)} \end{aligned} \tag{1.5.16}$$

  where M is the matrix associated with the linear map φ, via the bases $\mathscr{B}_V$ and $\mathscr{B}_W$.

Now consider that, for every space $(V, \mathscr{B}_V)$ with $\dim(V) = n$ the map $F \circ G$ sends

$$V \mapsto n \mapsto K^n \tag{1.5.17}$$

but since two spaces with the same dimension are isomorphic, this mean that $(F \circ G) \restriction \mathrm{Obj}(\mathbf{L}) \simeq \mathrm{Id}_{\mathbf{Lin_b}} \restriction \mathrm{Obj}(\mathbf{Lin_b})$. Now for the morphisms: for every linear map

$\varphi : (V, \mathscr{B}_V) \to (W, \mathscr{B}_W)$, $F \circ G$ sends

$$\varphi \mapsto M \mapsto \psi \tag{1.5.18}$$

where M is the matrix associated with φ via the bases $\mathscr{B}_V$ and $\mathscr{B}_W$ and $\psi : K^n \to K^m$ is the linear application associated with M via the canonical bases. Since for every two space with the same dimension is always possible to find an isomorphism between them that sends a fixed base into another fixed base, we have proved that $F \circ G \simeq \mathrm{Id}_{\mathbf{Lin_b}}$. Similarly, for all $n \in \mathbb{N}^+$, the map $G \circ F$ sends

$$n \mapsto (K^n, \mathscr{E}_n) \mapsto m \tag{1.5.19}$$

where $n = \dim(K^n) = m$. Also, for every $n, m \in \mathbb{N}^+$ and $M \in K^{n\times m}$, $G \circ F$ sends

$$M \mapsto \varphi \mapsto M' \tag{1.5.20}$$

where $\varphi : (K^n, \mathscr{E}_n) \to (K^m, \mathscr{E}_m)$ is associated to M via the canonical bases and $M' \in K^{n\times m}$ is the matrix associated with φ again via the canonical bases and hence, from basic linear algebra theorems, it follows that $M = M'$. So we have finally proved that $G \circ F = \mathrm{Id}_{\mathbf{L}}$ and this finally means that $\mathbf{L}$ and $\mathbf{Lin_b}$ are equivalent. □

Definition 1.5.11. In the notations of exercise, if two topological spaces X and Y are equivalent, they are usually said to be **homotopic spaces** through an **homotopy equivalence**, or to have the same homotopy type. A space that has the same homotopic type of a point — that is, of a space consisting of only one point — is called **contratible**.

Theorem 1.5.12. *A space X is contractible if and only if the identity map $\mathrm{Id}_X : X \to X$ is homotopic to a costant map.*

Proof. Suppose that X is contractible, since it is homotopic to $\{y\}$ via the equivalence $f : X \to \{y\}$ and let $g : \{y\} \to X$ be its inverse. By definition of homotopy we have that $\mathrm{Id}_X \simeq g \circ f$ and since both f and g are trivially constant, the identity is homotopic to a constant map. On the other hand if there exists a map

$$f : X \to X \quad : \quad \forall x \in X \quad f(x) = x_0 \tag{1.5.21}$$

that is homotopic to the identity, then we can consider the space $\{x_0\}$, the canonical inclusion $i : \{x_0\} \to X$ and the map $g : X \to \{x_0\}$ — which is the only map from X into $\{x_0\}$. Since we have that $g \circ i = \mathrm{Id}_{\{x_0\}}$ and $i \circ g \simeq \mathrm{Id}_X$, we realized the definition of homoopty between X and $\{x_0\}$ and so X is contractible. □

Definition 1.5.13. Given two categories **C** and **D** and a subcategory **C'** of **C**, we say that a homotopy N between F and G is a **homotopy rel C'** if N is constant on **C'**, where $F, G : \mathbf{C} \to \mathbf{D}$ are functors. We will simply write $F \simeq G$ *rel* **C'**.

Definition 1.5.14. Given a category **C** and one of its subcategories **C'**, we say that **C'** is a **deformation retract** of **C** if, called $i : \mathbf{C'} \to \mathbf{C}$ the canonical inclusion, it holds that

$$\exists r : \mathbf{C} \to \mathbf{C'} \text{ functor } : (i \circ r) \simeq \mathrm{Id}_{\mathbf{C}} \ rel \ \mathbf{C'} \tag{1.5.22}$$

In this case, r is called a **deformation retraction** from **C** to **C'**.

Theorem 1.5.15. *In the same notations of the previous definitions, consider a set $A' \subset A \cap X_1$ representative in X_1 and let $A_1 = A' \cup A - X_1$. Then there exists a pushout*

$$\begin{tikzcd}
\pi(X_0, A) \arrow[r, "\pi i_1"] \arrow[d, "\pi i_2"'] & \pi(X_1, A) \arrow[r, "r"] & \pi(X_1, A_1) \arrow[d, "\pi u_1"] \\
\pi(X_2, A) \arrow[r, "\pi u_2"'] & \pi(X, A) \arrow[r, "r'"'] & \pi(X, A_1)
\end{tikzcd} \tag{1.5.23}$$

such that both r and r' are deformation retractions.

1.6 The fundamental group of the circle

In this section, we'll take a short brake to prove a classical result of algebraic topology: the computation of the fundamental group of the circle S^1.

Theorem 1.6.1. *Consider the unit circle S^1 in the complex plane (i.e. $\mathbb{R}^2$ where we* *identify* (a, b) *with* $a + ib \in \mathbb{C}$*). Then*

$$\pi_1(S^1, 1) \simeq \mathbb{Z} \tag{1.6.1}$$

Proof. Let:

$$X_1 = S^1 - \{i\} \tag{1.6.2}$$
$$X_2 = S^1 - \{-i\} \tag{1.6.3}$$
$$X_0 = X_1 \cap X_2 \tag{1.6.4}$$
$$A = \{-1, 1\} \tag{1.6.5}$$
$$A_1 = \{1\} \tag{1.6.6}$$

So, X_1 and X_2 are just an unitary circle without a point, which is homeomorphic to $(0, 1)$ and so simply connected. X_0 is the unitary circle with two *holes* so is just the sum of two simply connected components. Now just notice that being simply connected means that any two paths with the same endpoints are equivalent, that is to say that the fundamental groupoid of X_1 and X_2 has only one morphism for each couple of objects, so it is the tree groupoid **I**. Meanwhile, X_0 has two simply connected components, so in an analogous fashion, its fundamental groupoid is just the trivial discrete groupoid with two elements (say $\{0, 1\}$), that we denote by **J**. So, by theorem 1.5.15, we have the following pushout (denoting with **K** the trivial group with just one element — say $\{0\}$)

$$\begin{tikzcd} \mathbf{J} \arrow[r] \arrow[d] & \mathbf{K} \arrow[d] \\ \mathbf{I} \arrow[r, "g"'] & \pi_1(S^1, 1) \end{tikzcd} \tag{1.6.7}$$

where g is completely defined by just fixing one value. Now, by definition of pushout, we know that for every morphism $f : \mathbf{I} \to \mathbb{Z}$ (where $\mathbb{Z}$ is regarded as a group), there is a unique $h : \pi_1(S^1, 1) \to \mathbb{Z}$ such that $h \circ g = f$. But since g is defined by one value (say $g(\iota) = \varphi$), this means that there a unique h such that $h(\varphi) = f(\iota)$ and we can suppose without loss of generality that $f(\iota) = 1 \in \mathbb{Z}$ (just by choosing the appropriate

f). Now consider the morphism

$$k : \mathbb{Z} \to \pi_1(S^1, 1) \text{ with } k(n) = n\varphi \tag{1.6.8}$$

This morphism satisfies the relation we imposed on h and so, by uniqueness, it must hold that $k \circ h = \mathrm{Id}_{\pi_1(S^1,1)}$. So k is the desired isomorphism. □

Exercise 1.6.2. Find the fundamental group of the torus, $S^1 \times S^1$

Proof. From theorems 1.6.1 and 1.3.12 it immediately follows that it is $\mathbb{Z} \times \mathbb{Z}$. In case one doesn't know that this group is abelian, it may be interesting to watch the animated proof of this fact by the University of Hannover.[1] □

1.7 Fibrations of groupoids and cofibrations of spaces

We'll now give a characterization of what inclusion of spaces we are willing to consider as "well-behaved": we'll call them cofibrations (well, an homotopist may disagree on their well-behavedness but we're homologist, aren't we?).

Note 1.7.1. In the following we will swap very often between groupoids as categories and as objects of the category **Grpd**. To make the process not too tedious, let's decide that — given two groupoids A, B — whenever we say

- "a morphism $f : A \to B$" we mean that $f \in \mathrm{Hom}_{\mathbf{Grpd}}(A, B)$
- "an object x of A" we mean that $x \in \mathrm{Obj}(A)$
- "an element a in A" — or simply $a \in A$ — we mean that $a \in \mathrm{Hom}_A$; if we say that a has starting point x we mean that $a \in \mathrm{Hom}_A(x, -)$, while if we say it has ending point x we mean that $a \in \mathrm{Hom}_A(-, x)$.
- given $a, b \in A$ if the ending point of a is the starting point of b we let $b + a := b \circ a$

Definition 1.7.2. Given two groupoids E, B and a morphism $p : E \to B$, we say that it is a **fibration** if

$$\forall x \in \mathrm{Obj}(E)\ \forall b \in \mathrm{Hom}_B(p(x), -)\ \ \exists e \in \mathrm{Hom}_A(x, -) \quad : \quad p + e = b \tag{1.7.1}$$

[1] Watch the video at http://www.youtube.com/watch?v=nLcr-DWVEto.

Exercise 1.7.3. If E, B are groups, prove that the fibrations from E to P are all and only the surjections.

Proof. Let $\mathrm{Obj}(E) = \{x\}$ and $\mathrm{Obj}(B) = \{y\}$, then there's only one way of choosing x and $p(x)$ and so we can just write $p(e)$ for $(p \circ e)(x)$ and b for $b(x)$ and those values completely define the two maps. So, the condition for being a fibration becomes

$$\forall b \in B \;\; \exists e \in E \quad : \quad p(e) = b \tag{1.7.2}$$

that is the same condition for p to be surjective. □

Definition 1.7.4. Given the groupoids B, E, the morphism $p : E \to B$ and an object u of B we denote with $p^{-1}[u]$ the subgroupoid of E that has

- $\mathrm{Obj}(p^{-1}[u]) = \{x \in \mathrm{Obj}(E) \quad : \quad p(x) = u\}$
- $\mathrm{Hom}_{p^{-1}[u]} = \{e \in E \quad : \quad p + e = 0_u\}$

Theorem 1.7.5. *Given the groupoids B, E and the fibration $p : E \to B$, for each $b \in \mathrm{Hom}_B(u, v)$ there exsits a function*

$$\underline{b} : \pi_0 p^{-1}[u] \to \pi_0 p^{-1}[v] \tag{1.7.3}$$

where $\pi_0 G$ is the set of the components of a groupoid G (cfr. definition 1.2.12), such that

- *if b is a zero object, then $\underline{b}$ is the identity*
- *$\underline{c + b} = \underline{c} + \underline{b}$ whenever it is possible to compose c and b*
- *$\underline{b}$ is bijective for all $b \in B$*

Definition 1.7.6. In the notations of definition 1.7.4, we express the condition realized via the previous theorem by saying that the groupoid B **acts** on the family of sets

$$\{\pi_0 p^{-1}[u] \;:\; u \in \mathrm{Obj}(B)\} \tag{1.7.4}$$

Observation 1.7.7. In definition 1.2.2 we defined the concept of homotopy between paths within a topological space. This concept can be extended to the one of homotopy between functions from a space X to a space Y. Given two functions $f, g : X \to Y$ an **homotopy** between f and g is a map

$$F : X \times [0, 1] \to Y \tag{1.7.5}$$

such that

$$F(x, 0) = f(x) \ \wedge \ F(x, 1) = g(x) \ \ \forall x \in X \tag{1.7.6}$$

Definition 1.7.8. Given two topological spaces A, X and a map $i : A \to X$, we say that i is a **cofibration** if for all the spaces Y, all maps $f : X \to Y$ and all homotopies $F : A \times [0, 1] \to Y$ such that

$$F(a, 0) = (f \circ i)(a) \ \ \forall a \in A \tag{1.7.7}$$

there is an homotopy $G : X \times [0, 1] \to Y$ such that

$$G(x, 0) = f(x) \ \ \forall x \in X \tag{1.7.8}$$

and

$$(G \circ (i \times \mathrm{Id}_{[0,1]}))(x, t) = F(x, t) \ \ \forall x \in X \ \forall t \in [0, 1] \tag{1.7.9}$$

When $A \subseteq X$, we say that i extends $f \circ i$ to f and $G \circ (i \times \mathrm{Id}_{[0,1]})$ to F; we also say that A and X have the **homotopy extension property**.

Observation 1.7.9. The choice of the names fibration and cofibration for two things that are apparently quite different is not at all casual. The next definitions and theorems will be devoted to expose the link between those two construtctions

Definition 1.7.10. If we consider the maps $X \to Y$ as points of the space Y^X and the homotopies between them as paths in the same space, we can use definition 1.2.2 to find some sort of *homotopies of homotopies*. So we say that two homotopies

$$F, G : X \times [0, 1] \to Y \tag{1.7.10}$$

from $f : X \to Y$ to $g : X \to Y$ are **homotopic rel endmaps** f, g if they are homotopic rel endpoints f, g in Y^X. Explicitely, that is to say that there is a function

$H : X \times [0,1]^2 \to Y$ such that

$$H(x,s,0) = F(x,s) \;\wedge\; H(x,s,1) = G(x,s) \;\; \forall (x,s) \in X \times [0,1] \tag{1.7.11}$$

$$H(x,0,t) = f(x) \;\wedge\; H(x,1,t) = g(x) \;\; \forall (x,t) \in X \times [0,1] \tag{1.7.12}$$

We call the fundamental groupoid πY^X of Y^X the **track groupoid** of X, Y. Notice that by definition, the components of this groupoid (that we should call $\pi_0 \pi Y^X$ according to the nomenclature of theorem 1.2.2, but that we'll just call $[X,Y]$ for short) are the class of homotopy of the maps in Y^X.

Definition 1.7.11. Given the maps $i : A \to X, h : Y \to Z$, consider the functions

$$i^* : \pi Y^X \to \pi Y^A \tag{1.7.13}$$

$$h_* : \pi Y^X \to \pi Z^X \tag{1.7.14}$$

that act on objects (maps) as

$$i^*(f) = f + i \tag{1.7.15}$$

$$h_*(f) = h + f \tag{1.7.16}$$

and on morphisms (homotopy classes) as

$$i^*([F]) = [F \circ (i \times \mathrm{Id}_{[0,1]})] \tag{1.7.17}$$

$$h_*([F]) = [h \circ F] \tag{1.7.18}$$

It can be easily checked that i^* and h_* can be considered both as morphisms in **Grpd** and as functors from a track groupoid to an other and furthermore, they preserve homotopy. i^* and h_* are called the **induced morphisms** of i and h.

Theorem 1.7.12. *Given the spaces A and X and the cofibration $i : A \to X$, then the induced morphsim $i^* : \pi Y^X \to \pi Y^A$ is a fibration, for every space Y.*

Proof. Given a map $f : X \to Y$, consider any homoptopy F in πY^A of initial map $f + i$. Since i is a cofibration, this means that there is an homotopy G in πY^X with initial map f and such that

$$G + (i \times \mathrm{Id}_{[0,1]}) = F \tag{1.7.19}$$

But then $i^* + G = F$ and so i^* is a fibration. $\square$

CHAPTER

TWO

PRODUCTS AND QUOTIENTS OF GROUPOIDS

2.1 Universal morphisms and universal groupoids

The main purpose of this section is to find out a way to change the objects of a groupoid and end up with another nice groupoid, by the mean of a morphism, where by “nice” we mean that the morphism we used has the unviversal property.

Definition 2.1.1. Given a groupoid G, a set $X \neq \emptyset$ and a function $\sigma : \mathrm{Obj}(G) \to X$, fixed two elements $x, x' \in X$ and a number $n \in \mathbb{N}^+$, we define a **word** of length n from x to x' as a sequence

$$a = \langle a_1, \ldots, a_n \rangle \tag{2.1.1}$$

where $a_i \in \mathrm{Hom}_G(x_i, x_i')$ $\forall i \leq n$, such that

1. $x_i' \neq x_{i+1}$ $\forall i < n$
2. $\sigma(x_i') = \sigma(X_{i+1})$ $fai < n$
3. $\sigma(x_1) = x,\ \sigma(x_n') = x'$
4. $a_i \neq \mathrm{Id}_{G,x_i}$ $\forall i \leq n$

Definition 2.1.2. Given the elements of the previous definition and using a multiplicative notation for groupoids, we construct a groupoid U, called **the universal**

groupoid, with

$$\mathrm{Obj}(U) = X \tag{2.1.2}$$

and

$$\forall x, x' \in X \;\; \mathrm{Hom}_U(x, x') = \left\{ \langle a_1, \dots, a_n \rangle \text{ from } x \text{ to } x', \; \forall n \in \mathbb{N}^+ \right\} \tag{2.1.3}$$

where we adopt the convention that, whenever $x = x'$ we also include the *empty word* $\langle\rangle_x$ in $\mathrm{Hom}_U(x, x')$ and we assume that if $x \neq y$ then $\langle\rangle_x \neq \langle\rangle_y$. We define the composition (that is, multiplication) in this groupoid by induction on the length of the words, according to the following rules:

1. $\forall a \in \mathrm{Hom}_U(x, x') \;\; \langle\rangle_{x'} \cdot a := a$ and $a \cdot \langle\rangle_x := a$
2. $\forall a \in \mathrm{Hom}_U(x, x'), \; \forall b \in \mathrm{Hom}_U(x', x'')$ with

$$a = \langle a_1, \dots, a_n \rangle \quad \wedge \quad a_i \in \mathrm{Hom}_G(x_i, x_i') \; \forall i \in \{1, \dots, n\} \tag{2.1.4}$$

$$b = \langle b_1, \dots, b_m \rangle \quad \wedge \quad b_i \in \mathrm{Hom}_G(y_i, y_i') \; \forall i \in \{1, \dots, m\} \tag{2.1.5}$$

we define

$$b \cdot a := \begin{cases} \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle & \text{if } y_1 \neq x_n' \\ \langle a_1, \dots, a_{n-1}, b_1 \cdot a_n, b_2, \dots, b_m \rangle & \text{if } y_1 = x_n' \;\wedge\; b_1 \cdot a_n \neq \mathrm{Id}_{G, x_n} \\ \langle a_1, \dots, a_{n-1}, b_2, \dots, b_m \rangle \text{ (by induction)} & \text{otherwise} \end{cases} \tag{2.1.6}$$

That each element has an inverse is easily shown, since the empty word behaves as an identity and given a word $\langle a_1, \dots, a_n \rangle$ we have that

$$\langle a_1, \dots, a_n \rangle \cdot \langle a_n^{-1}, \dots, a_1^{-1} \rangle = \langle\rangle_x \tag{2.1.7}$$

$$\langle a_n^{-1}, \dots, a_1^{-1} \rangle \cdot \langle a_1, \dots, a_n \rangle = \langle\rangle_{x'} \tag{2.1.8}$$

Also, with some patience, it's possible to show that associativity holds. Finally, since U depends by both G and σ, we will sometimes denote it as $U_\sigma(G)$.

Definition 2.1.3. Given a groupoid G, we define a morphism of groupoids $\overline{\sigma} : G \to U_\sigma(G)$ that is the identity on the objects and acts on the morphisms in the following way:

$$\sigma(a) = \begin{cases} \langle a \rangle & \text{if } x \neq x' \text{ or } x = x' \text{ and } a \neq \mathrm{Id}_{G,x} \\ \langle \rangle_x & \text{if } x = x' \text{ and } a = \mathrm{Id}_{G,x} \end{cases} \tag{2.1.9}$$

Definition 2.1.4. Given two groupoids G, H and a morphism $f : G \to H$, we say that f is **universal** if the following square is a pushout:

$$\begin{tikzcd} \mathrm{Obj}(G) \arrow[r, "f_{\mathrm{Obj}}"] \arrow[d, "i_G"'] & \mathrm{Obj}(H) \arrow[d, "i_H"] \\ G \arrow[r, "f"'] & H \end{tikzcd} \tag{2.1.10}$$

where here $\mathrm{Obj}(G)$ and $\mathrm{Obj}(H)$ are seen as the full groupoids on the sets $\mathrm{Obj}(G)$ and $\mathrm{Obj}(H)$ with only identities as morphisms and so f_{Obj} is the naturally induced morphism between those two groupoids, while i_G and i_H are the naturally induced inclusions.

Theorem 2.1.5. *The mophism $\overline{\sigma}$ of definition 2.1.3 is universal.*

2.2 Free groupoids and free product of groupoids

Here we'll give a way to produce a new groupoids from two (or many) given ones, by generalizing the notion of *free product* between groups. In particular, we will see that the "old" algebraic notion of free product between groups is just our coproduct, as we defined it in 1.4.1.

Definition 2.2.1. We define a **graph** Γ as a couple made by a set of objects or vertices $\mathrm{Obj}(\Gamma)$ and, for all $x, y \in \mathrm{Obj}(\Gamma)$ a set of oriented edges $\mathrm{Edg}_\Gamma(x, y)$ that are mutually disjoint. A vertex x such that

$$\mathrm{Edg}_\Gamma(x, y) = \mathrm{Edg}_\Gamma(y, x) = \emptyset \quad \forall y \in \mathrm{Obj}(\Gamma) \tag{2.2.1}$$

is called a discrete vertex. If all vertices are discrete, the graph is called a **discrete graph**.

Definition 2.2.2. Given two graphs Γ, Δ a graph morphism $f : \Gamma \to \Delta$ is an application that sends vertices of Γ in vertices of Δ and edges of Γ in edges of Δ. As it's easy to check, graphs with those morphisms form a category, **Gph**.

Definition 2.2.3. It's quite clear that graphs are *almost* categories (they only lack the "identity edges" in general), so we'd like to give some categorical-sounding definition for them. Given two graphs Γ, Δ, if $\Gamma \subseteq \Delta$ and the inclusion is a morphism we say that Γ is a **subgraph** of Δ. It is called **full** if $\mathrm{Obj}(\Gamma) = \mathrm{Obj}(\Delta)$ and **wide** if

$$\forall x, y \in \mathrm{Obj}(\Gamma) \;\; \mathrm{Edg}_\Gamma(x, y) = \mathrm{Edg}_\Delta(x, y) \tag{2.2.2}$$

Observation 2.2.4. Notice that every category and in particular every groupoid G can be viewed as a graph Γ with $\mathrm{Obj}(\Gamma) = \mathrm{Obj}(G)$ and $\mathrm{Edg}(\Gamma) = \mathrm{Hom}(G)$. Analogously a functor between groupoids can be viewed as a graph morphism. Notice finally that there's no need to consider the whole groupoid as a graph, but we can simply take part of it (i.e. a subset of its objects and their morphisms) and so talk about a graph *in* a groupoid. Of course, in general, this won't be a subgroupoid.

Definition 2.2.5. Given a groupoid G and a graph Γ in G, the subgroupoid **generated** by Γ is the smallest subgroupoid of G that contains Γ.

Definition 2.2.6. A groupoid G is said to be **free** if there is a graph Γ in G such that $\mathrm{Obj}(\Gamma) = \mathrm{Obj}(G)$ and whenever is given a groupoid H and a graph morphism $f : \Gamma \to H$ there is only one way to extend f into a morphism of groupoids $G \to H$. In particular, we say that G is free *on* Γ.

Theorem 2.2.7. *Given two groupoids G, H there exists a groupoid K and two morphisms $f_1 : G \to K$ and $f_2 : H \to K$ such that for every groupoid L and for every morphisms $g : G \to L$ and $h : H \to L$ such that f and g are the same on* $\mathrm{Obj}(G) \cap \mathrm{Obj}(H)$*, there is an unique morphism $k : K \to L$ such that $k \circ f_1 = g$ and $k \circ f_2 = h$.*

Proof. Consider the set

$$X = \mathrm{Obj}(G) \cup \mathrm{Obj}(H) \tag{2.2.3}$$

and the function

$$\sigma : \mathrm{Obj}(G) \sqcup \mathrm{Obj}(H) \to X \tag{2.2.4}$$

obtained as the union of the two canonical inclusions $\mathrm{Obj}(G) \to X$ and $\mathrm{Obj}(H) \to X$. Of course σ is a surjection and — in particular — is a bijection if $\mathrm{Obj}(G) \sqcup \mathrm{Obj}(H) \leftrightarrow \mathrm{Obj}(G) \cup \mathrm{Obj}(H)$, that is $\mathrm{Obj}(G) \cap \mathrm{Obj}(H) = \emptyset$. Now let $K = U_\sigma(G \sqcup H)$ and consider $\overline{\sigma} : G \sqcup H \to K$ as defined in 2.1.3. So, we can reduce the thesis in proving that f_1 and f_2 are universal; but this is clear, since the diagram

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[dr, "f_1"'] & G \sqcup H \arrow[d, "\overline{\sigma}"] & H \arrow[l, "i_H"'] \arrow[dl, "f_2"] \\ & K & \end{tikzcd} \tag{2.2.5}$$

implies that $f_1 = \overline{\sigma} \circ i_G$ and $f_2 = \overline{\sigma} \circ i_H$, where i_G and i_H are the canonical inclusions of respectively G and H in $G \sqcup H$. $\square$

Definition 2.2.8. The group K of theorem 2.2.7 is called the **free product** of G and H and is denoted as $G * H$.

Observation 2.2.9. If G, H are groups, their free product is just their coproduct as can be clearly seen by noticing that f_1 and f_2 have the universal property if and only if the following square is a pushout — a property that anyway holds in general for groupoids:

$$\begin{tikzcd} \mathrm{Obj}(G) \cap \mathrm{Obj}(H) \arrow[r, "i_H"] \arrow[d, "i_G"'] & H \arrow[d, "f_2"] \\ G \arrow[r, "f_1"'] & K \end{tikzcd} \tag{2.2.6}$$

Exercise 2.2.10. Many of the notions introduced in this chapter have a nice algebraic interpretation. For example, if we consider in particular groups, given a group G and a graph Γ in G, the subgroup H of G generated by Γ — that, as we defined it is the smallest subgroup of G that contains Γ — can be seen as the group whose elements are all the possible finite multiplications of elements (that is, morphisms) Γ and their inverses. Furthermore, given a set X, we call a **group action** of G over X a function

$a : G \times X \to X$ such that

$$a(e, x) = x \quad \forall x \in X \tag{2.2.7}$$

$$a(gh, x) = a(g, a(h, x)) \quad \forall x \in X, \ \forall g, h \in G \tag{2.2.8}$$

where $e \in G$ is the unit element of G. Now, given subgroups $G_1, \ldots, G_n$ of G such that all of them generates G (that is, the subgroup generated by them is the whole G), suppose that G acts on X and that there are $X_1, \ldots, X_n \subseteq X$ and an element $x \in X \setminus \bigcup_{i=1}^{n} X_i$ such that, for all $g \in G_i$ with $g \neq e$ and for all $i \in \{1, \ldots, n\}$

1. $a(g, X_j) \subseteq X_i$ if $j \neq i$
2. $a(g, x) \in X_i$

Show that G is the free product $G_1 * \ldots * G_n$.

Proof. For what we said, any element $g \in G$ can be written as a product of elements of the various G_i:

$$g = g_1 \cdot \ldots \cdot g_m \text{ with } g_i \in G_{n_i} \text{ for some } n_i \tag{2.2.9}$$

Since in such representation we can "collapse" consecutive elements if they belong to the same G_{n_i} (or if one of them is the unit element) my multiplying them and taking the result, we can suppose without loss of generality that g is in the form

$$g = g_1 \cdot \ldots \cdot g_k \tag{2.2.10}$$

where each $g_i \in G_{n_i}$ for some n_i, $g_i \neq e$ and $G_{n_i} \neq G_{n_{i+1}}$ for all $i \in \{1, \ldots, k\}$. As noticed in theorem 2.2.7 (generalizing from the case where $n = 2$) there is a surjective homomorphism

$$\sigma : \bigsqcup_{i=1}^{n} G_i \to G \tag{2.2.11}$$

Since we want to prove that $G = G_1 * \ldots * G_n$, that is $G = \bigsqcup_{i=1}^{n} G_i$, we have to show that σ is actually an isomoprhism, so that it is also injective as an homomorphism of groups. This means we have to prove that is kernel is trivial: $\ker \sigma = \{e\}$. Now, given a g as in (2.2.10), by the second hypothesis we have that

$$a(g_k, x) \in X_{n_k} \tag{2.2.12}$$

But then, by the first hypothesis and applying the second again, we get

$$a(g_{k-1}, a(g_k, x)) \in X_{n_{k-1}} \tag{2.2.13}$$

and so on, until we obtain

$$a(g_1, a(\dots, a(g_k, x))) \in X_{n_1} \tag{2.2.14}$$

But now notice that $a(e, x) = x$ and for how we chose x this means that $a(e, x) \notin G_{n_1}$ and so it's impossible that $\sigma(g) = \sigma(g_1 \cdot \ldots \cdot g_n) = e$ for all the $g \neq e$. This proves that $\ker \sigma$ is trivial. □

2.3 Quotient groupoids

Again we'll give a generalization of a well-known group theoretic notion, the one of quotient group. In particular, we'll give the notion of normal subgroupoid and translate the classical algebraic definition in terms of an annihilating morphism.

Definition 2.3.1. Given a groupoid G, a subgroupoid N is said to be **normal** in G if it's wide and for all $x, y \in \mathrm{Obj}(G)$ and $a \in \mathrm{Hom}_G(x, y)$

$$a_c(N(x)) = N(y) \tag{2.3.1}$$

where a_c is the conjugacy function $x \mapsto axa^{-1}$. The condition (2.3.1) is expressed for short as

$$aN(x)a^{-1} = N(y) \tag{2.3.2}$$

Definition 2.3.2. Given two groupoids G, H and a graph Γ of G a morphism $f : G \to H$ is said to **annihilate** Γ if $f(\Gamma)$ is a discrete subgroupoid of H.

Theorem 2.3.3. *Let G be a groupoid and N be a totally disconnected normal subgroupoid of G. Then there is a groupoid M and an universal morphism $p : G \to M$ such that p annihilates N (regarded as a subgraph of G).*

Definition 2.3.4. The groupoid M of theorem 2.3.3 is called a **quotient groupoid** of G and is denoted by G/N.

Observation 2.3.5. There are a lot of similarities between quotient groups in the classic algebraic sense and quotient groupoids. For instance, if given two groupoids G, H and a morphism $f : G \to H$ we call $\ker f$ the largest subgroupoid that is annihilated by f, it holds that the canonical morphism

$$G/\ker f \to \operatorname{im} f \tag{2.3.3}$$

is an isomorphism.

2.4 Adjunction spaces

The last tool we're going to need are adjunction spaces and we study some of their most important properties, that we need to prove Van Kampen's theorem.

Observation 2.4.1. Recall from example 1.4.4 the definition of adjunction space and that we showed, in the same notation of the exercise, that the following square is a pushout:

$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "i"'] & A_1 \arrow[d, "I"] \\ A_2 \arrow[r, "F"'] & A_1 \cup_f A_2 \end{tikzcd} \tag{2.4.1}$$

Where we used the shorter and more meaningful notation

$$i := i_2 \quad f := i_1 \quad I := v_1 \quad F := v_2 \tag{2.4.2}$$

and in the hypothesis that i is a cofibration.

Definition 2.4.2. First of all, we extend the notion of deformation retraction in a topological space. Given $B \subseteq A_2$, called $i_B : B \to A_2$ the canonical inclusion, we say that a retraction r is a **deformation retraction** if $i \circ r$ is homotopic to Id_{A_2}. We then say that B is a **deformation retract** of A_2. It's clear from the definitions of retraction and homotopy that B is a deformation retract of A_2 if there is a continuous map

$$R : [0,1] \times A_2 \to A_2 \quad : \quad R(0,-) = \mathrm{Id}_{A_2} \quad \wedge \quad R(1, A_2) = B \tag{2.4.3}$$

called the **retracting homotopy**.

Definition 2.4.3. Given two spaces A_0 and A_1 and a map $f : A_0 \to A_1$, consider $g : A_0 \times \{0\} \to A_1$ trivially defined by

$$g(x,0) = f(x) \tag{2.4.4}$$

We then define the **mapping cylinder** of f as the adjunction space

$$M(f) = A_1 \cup_g (A_0 \times [0,1]) \tag{2.4.5}$$

Theorem 2.4.4. *Given B which lies midway between A_0 and A_2 ($A_0 \subseteq B \subseteq A_2$) if B is a deformation retract of A_2 then $A_1 \cup_f B$ is a deformation retract of $A_1 \cup_f A_2$.*

Proof. Let $R : [0,1] \times A_2 \to A_2$ be the retracting homotopy that shows B as a deformation retract of A_2. Define the functions

$$\mathcal{F} : [0,1] \times A_2 \to A_1 \cup_f A_2 \quad : \quad \mathcal{F}(t,-) = F \circ R(t,-) \tag{2.4.6}$$

$$\mathcal{I} : [0,1] \times B \to A_1 \cup_f A_2 \quad : \quad \mathcal{I}(t,-) = I \tag{2.4.7}$$

then by definition $\mathcal{F}(t,-) \circ i = \mathcal{I}(t,-) \circ f$. So the following square commutes

$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "i"'] & A_1 \arrow[d, "\mathcal{I}"] \\ A_2 \arrow[r, "\mathcal{F}"'] & A_1 \cup_f A_2 \end{tikzcd} \tag{2.4.8}$$

and so there is an unique homotopy $H : [0,1] \times (A_1 \cup_f A_2) \to A_1 \cup_f A_2$ such that

$$H(t,-) \circ F = \mathcal{F}(t,-) \quad \wedge \quad H(t,-) \circ I = \mathcal{I}(t,-) \tag{2.4.9}$$

It's easy to check that by definition of $\mathcal{F}$ and $\mathcal{I}$ we have

$$H(0,-) = \mathrm{Id}_{A_1 \cup_f A_2} \quad \wedge \quad H(1, A_1 \cup_f A_2) = A_1 \cup_f B \tag{2.4.10}$$

and so we produced a retracting homotopy showing $A_1 \cup_f B$ as a deformation retract of $A_1 \cup_f A_2$. $\square$

Corollary 2.4.5. *In the previous theorem, if we take B to be just A_0 we obtain that whenever A_0 is a deformation retract of A_2, then A_1 is a deformation retract of $A_1 \cup_f A_2$.*

Corollary 2.4.6. *A map $f : A_0 \to A_1$ is a homotopy equivalence if and only if A_0 is a deformation retract of $M(f)$.*

Theorem 2.4.7. *Given $A_1' \subseteq A_1$ and $A_2' \subseteq A_2$ such that $A_0 \subseteq A_2'$ and $f(A_0) \subseteq A_1'$, if we call g the restriction of f to all and only the elements that are sent in A_1', then we have that $A_1' \cup_g A_2'$ is a subspace of $A_1 \cup_f A_2$.*

Corollary 2.4.8. *In the notations used throughout this chapter $M(f) \cup A_2$ is a subspace of $M(F)$. So call $j : M(f) \cup A_2 \to M(F)$ the inclusion and $q : M(F) \to A_1 \cup_f A_2$ the canonical deformation retraction. Then, if the inclusion of A_0 into A_2 is a cofibration, the map*

$$p = q \circ j : M(f) \cup A_2 \to A_1 \cup_f A_2 \tag{2.4.11}$$

is a homotopy equivalence rel A_1.

2.5 Van Kampen theorem for the fundamental groupoid

We're now about to prove the main result of our work: we will generalize what we did in section 1.5 to a more general case and will so prove *Van Kampen's theorem.*

Theorem 2.5.1 (Van Kampen's theorem). *We're in the usual setup of adjunction spaces*

$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "i"'] & A_1 \arrow[d, "I"] \\ A_2 \arrow[r, "F"'] & A_1 \cup_f A_2 \end{tikzcd} \tag{2.5.1}$$

and we have a set $C \subseteq A_2$ that is representative in A_2 and A_0 and a set $D \subseteq A_1$ representative in A_1 such that the restriction g of f to $A_2 \cap C$ has values in D; we call $B = D \cup_g C$ and $A = D \cup C$. We switch from the square (2.5.1) to the square of fundamental groupoids, constrained by C and D:

$$\begin{tikzcd} \pi(A_0, C) \arrow[r, "f"] \arrow[d, "i"'] & \pi(A_1, D) \arrow[d, "I"] \\ \pi(A_2, C) \arrow[r, "F"'] & \pi(A_1 \cup_f A_2, B) \end{tikzcd} \tag{2.5.2}$$

and we claim that this square commutes if and only if the morphism (induced by the one built in corollary 2.4.8)

$$p_{ind} : \pi(M(f) \cup A_1, A) \to \pi(A_1 \cup_f A_2, B) \tag{2.5.3}$$

is a homotopy equivalence between these two fundamental groupoids.

Proof. The main idea is to take appropriate spaces such that it will be possible to apply theorem 1.5.15. So we choose:

$$X = M(f) \cup A_2 \tag{2.5.4}$$

$$X_1 = M(f) \tag{2.5.5}$$

$$X_2 = X - A_1 \tag{2.5.6}$$

$$X_0 = X_1 \cap X_2 \tag{2.5.7}$$

$$Y = D \cup (C - A_0) \tag{2.5.8}$$

that satisfies the required hypothesis, since $\mathring{X}_1$ and $\mathring{X}_2$ cover X. So, by theorem 1.5.15 we have that p sends $D \cup_g C$ into $D \cup (C - A_0)$ bijectively. In our new simplified notations, this means that there is a bijection $B \leftrightarrow Y$. Now notice that D is representative in A_1 and so it is in the mapping cylinder $X_1 = M(f)$ of $f : A_0 \to A_1$. Also, $Y \supseteq D$, so it meets at least the same path-connected components that D meets and so it is representative in the same sets. Finally, C is representative in A_0 so of course is $C \cap A_0$ and each of its points can be sent into a point of D via the segments that make up the mapping cylinder $M(f)$. By construction of the mapping cylinder if we call s_c the segment that sends $c \in C \cap A_0$ down the mapping cylinder to a point of D, we have that $p_{ind}([s_c]) = \mathrm{Id}_{[f(c)]}$. We can compose the two horizontal arrows in the square given by 1.5.15 in (1.5.23) and obtain the pushout

$$\begin{tikzcd}
\pi(X_0, A) \arrow[r] \arrow[d] & \pi(X_1, Y) \arrow[d] \\
\pi(X_2, A) \arrow[r] & \pi(X, Y)
\end{tikzcd} \tag{2.5.9}$$

then, if we call for short $Q = A_1 \cup_f A_2$ we can extend this square via p and canonical isomorphisms induced by the homotopy equivalences (using the fact that so is p_{ind} when restricted to $p_{\simeq} : \pi(X_1, Y) \to \pi(A_1, D)$) into

$$\begin{tikzcd}
\pi(A_0, C) \arrow[rr] \arrow[dd] \arrow[dr, "\simeq"] & & \pi(A_1, D) \arrow[dd] & \\
& \pi(X_0, A) \arrow[rr] \arrow[dd] & & \pi(X_1, Y) \arrow[ul, "p_{\simeq}"'] \arrow[dd] \\
\pi(A_2, C) \arrow[rr] \arrow[dr, "\simeq"'] & & \pi(Q, B) & \\
& \pi(X_2, A) \arrow[rr] & & \pi(X, Y) \arrow[ul, "p_r"']
\end{tikzcd} \tag{2.5.10}$$

where p_r is the restriction of p_{ind} to $\pi(X, Y)$. As it's easy to check, because of the properties of s_c this box commutes. As it's clear from the picture, to determine an isomorphism from the "new" front square to the "old" back square is sufficient and necessary (s.a.n.) that p_r is an isomorphism itself. On the objects we don't have any problem since p_r is bijective. So we can affirm that the s.a.n. condition is that $p_r : \pi(X, Y) \to \pi(Q, B)$ is an homotopy equivalence. But now notice that we can remove the restricting constraint and affirm that the it's the same to request the whole $p_{ind} : \pi(X, A) \to \pi(Q, B)$ to be an homotopy equivalence, since trivially $\pi(X, Y)$ is a deformation retract of $\pi(X, A)$ because a path in $\pi(X, Y)$ is a path in $\pi(X, A)$ and so we have an isomorphism of groupoids that is given by the inclusion in one way and by the retraction in the other. So, bringing all the ideas together, we know that (2.5.9)

— that is the front square of the box — is a pushout and so (2.5.1) — that is the back square of the box — is a pushout too if we can establish an isomorphism between these two square and we proved that this is the case if and only if p_{ind} is a homotopy equivalence. □

Corollary 2.5.2. *If the inclusion of A_0 in A_2 is a cofibration, then 2.5.2 is a pushout.*

Proof. It immediately follows from corollary 2.4.8. □

Corollary 2.5.3. *If A_1 and A_2 are such that they are closed subsets of $A_1 \cup A_2$ and the inclusion of $A_1 \cap A_2$ is cofibered in A_1, then by taking B representative in A_1, A_2 and $A_1 \cap A_2$ and letting f be just the inclusion, by theorem 2.5.1 it just follows that the following squares (where all the arrows are naturally induced by the inclusions) is a pushout:*

$$\begin{tikzcd}
\pi(A_1 \cap A_2, B) \arrow[r] \arrow[d] & \pi(A_1, B) \arrow[d] \\
\pi(A_2, B) \arrow[r] & \pi(A_1 \cup A_2, B)
\end{tikzcd} \tag{2.5.11}$$

Corollary 2.5.4. *In the same hypothesis of corollary 2.5.3, if $\pi(A_1 \cap A_2, B)$ is discrete then*

$$\pi(A_1 \cup A_2, B) \simeq \pi(A_1, B) * \pi(A_2, B) \tag{2.5.12}$$

Proof. Just notice that from () being a pushout it follows that the canonical morphism from $\pi(A_1, B) \sqcup \pi(A_2, B)$ to $\pi(A_1 \cup A_2, B)$ is universal. □

Corollary 2.5.5. *In the same hypothesis of corollary 2.5.2, if $\pi(A_1, D)$ and $\pi(A_2, C)$ are discrete, $C \subseteq Y$ and $f(C) = B = D$ then the morphism F of (2.5.1) is universal.*

Proof. In this case the hypothesis we imposed and the fact that (2.5.1) is a pushout just fulfill the definition of universal morphism. □

INDEX

BIBLIOGRAPHY

[1] J. Adamek, H. Herrlich, and G.E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Dover Publications, 2009.

[2] R. Brown. *Topology and groupoids*. www.groupoids.org, 2006.

[3] S. MacLane. Groups, categories and duality. *Proceedings of the National Academy of Sciences of the United States of America*, 34(6):263, 1948.