A CONCISE DEFINITION OF A MODEL CATEGORY

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Abstract. This short note gives a concise definition of a model category using the notion of a weak factorization system. In justifying this definition, we prove that in the presence of the two weak factorization systems, it follows formally that the class of weak equivalences $\mathcal{W}$ is closed under retracts. Hence, the only explicit additional requirement needed for the class $\mathcal{W}$ is that it satisfies the 2 of 3 property.

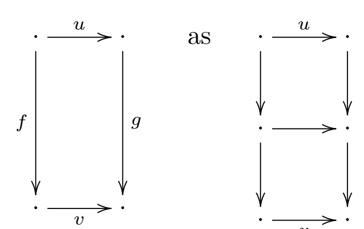
1. Weak Factorization Systems

Recall a *weak factorization system* on a category $\mathcal{K}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

(I) Every $f \in \mathcal{K}^{\mathbf{2}}$ can be factored as $f = rl$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

(II) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$. That is, $\mathcal{R}$ is precisely the class of maps which have the right lifting property with respect to all maps in $\mathcal{L}$, and $\mathcal{L}$ is the class of maps that have the left lifting property with respect to all maps in $\mathcal{R}$.

Loosely speaking, a weak factorization system is *functorial* if there is a functorial factorization of any square



with the top two vertical arrows in $\mathcal{L}$ and the bottom two vertical arrows in $\mathcal{R}$. The vertical composites in the right diagram should be f and g respectively and the middle arrow should depend functorially on u and v.

There are many equivalent ways describe this precisely (and some incorrect ones, such as the definition given in the first edition of [Hov99]). I recently encountered the following particularly nice definition in [Gar]:

Let $\mathbf{2} = \{0 \leq 1\}$ and $\mathbf{3} = \{0 \leq 1 \leq 2\}$ denote ordinals, regarded as categories. There are three injective functors $d^0, d^1, d^2 : \mathbf{2} \to \mathbf{3}$, with the superscript corresponding to the object of $\mathbf{3}$ not contained in the image. These injections induce functors $d_0, d_1, d_2 : \mathcal{K}^{\mathbf{3}} \to \mathcal{K}^{\mathbf{2}}$. A *functorial factorization* is a functor $F : \mathcal{K}^{\mathbf{2}} \to \mathcal{K}^{\mathbf{3}}$ such that $d_1 F = \mathrm{id}$. A weak factorization system is *functorial* if the components of the factorizations given by F fall into the appropriate classes (the top vertical arrows in $\mathcal{L}$, the bottom ones in $\mathcal{R}$).

Date: September 3, 2009.

For more details, including proofs that various definitions of weak factorization systems are equivalent, see [Rie].

2. Model Structure

With these notions, we can give a particularly concise definition of model structure on a category.

Definition 2.1. A *model structure* on a complete and cocomplete category $\mathcal{K}$ consists of three classes of morphisms $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{W}$ such that the following two properties hold.

[wfs] $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are two weak factorization systems on $\mathcal{K}$.

[2 of 3] $\mathcal{W}$ satisfies the 2 of 3 property: if two out of three of f, g, and gf are in $\mathcal{W}$, so is the third.

Remark 2.2. Some authors, including [Hov99], want the weak factorization systems to be functorial, which requires the obvious modification of the above definition.

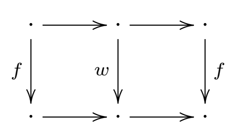
Remark 2.3. It is well known that both classes of a weak factorization systems are closed under retracts; in particular, this must be true of $\mathcal{C}$ and $\mathcal{F}$. So to show that the above definition agrees with the usual (e.g., as in [Hov99]), we must only show that $\mathcal{W}$ is necessarily closed under retracts. We record this fact in the following lemma.

Lemma 2.4. *If* $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ *is a model structure on* $\mathcal{K}$ *in the sense of Definition 2.1, then* $\mathcal{W}$ *is closed under retracts.*

We give two proofs of this fact: the first in the case where $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are functorial weak factorization systems and the second in general. The first proof was obvious to the author, though it most likely appears in the literature somewhere. The second very clever argument can be found in the appendices of [Joy].

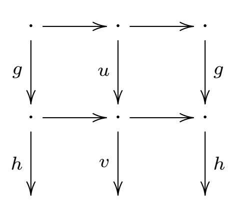
Proof 1. Let $w \in \mathcal{W}$ and suppose

(2.5)



is a retract diagram. Applying the functorial factorization from $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ to this diagram, we obtain

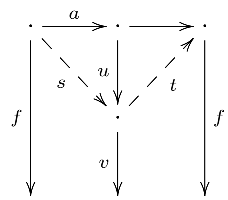
(2.6)



with $u, g \in \mathcal{C} \cap \mathcal{W}$, and $v, h \in \mathcal{F}$ such that $w = vu$ and $f = hg$. The horizontal composites of (2.5) are the identity arrow at f in $\mathcal{K}^{\mathbf{2}}$; therefore, because the factorization is functorial, the middle horizontal composite of (2.6) is also an identity. Hence, h is a retract of v. By the 2 of 3 property, v is a weak equivalence, so

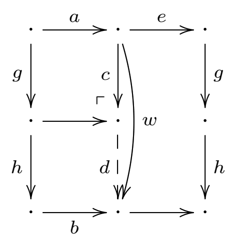
$v \in \mathcal{F} \cap \mathcal{W}$ and hence $h \in \mathcal{F} \cap \mathcal{W}$ by Remark 2.3. Thus $f = hg \in \mathcal{W}$ by the 2 of 3 property. □

Proof 2. Given a retract diagram (2.5) with $w \in \mathcal{W}$, suppose first that $f \in \mathcal{F}$. Factor w as $w = vu$ using either weak factorization system; as before by the 2 of 3 property, $u \in \mathcal{C} \cap \mathcal{W}$ and $v \in \mathcal{F} \cap \mathcal{W}$. We obtain arrows s and t as shown

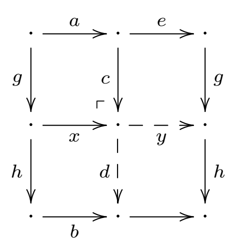


where s is simply the composite ua and t is a solution to the lifting problem given by $u \in \mathcal{C} \cap \mathcal{W}$ and $f \in \mathcal{F}$. The top triangles commute, so $ts = 1$, which means that f is a retract of v. As $v \in \mathcal{F} \cap \mathcal{W}$, f is a well, and we are done in this case.

Now return to the situation of (2.5) with no additional hypotheses on f. Factor f as $f = hg$ with $g \in \mathcal{C} \cap \mathcal{W}$ and $h \in \mathcal{F}$ and construct the indicated pushout



The left class of a weak factorization system is closed under pushouts, so $c \in \mathcal{C} \cap \mathcal{W}$. The arrows w and bh form a cone over the pushout diagram, so there is a unique morphism d as shown such that $dc = w$. By the 2 of 3 property, $d \in \mathcal{W}$. Similarly, ge and the identity form a cone over the pushout diagram, so there is a unique morphism y shown below



such that yx is the identity. The lower two squares now display h as a retract of d. As $d \in \mathcal{W}$ and $h \in \mathcal{F}$, the above argument shows that $h \in \mathcal{W}$. But g was already in $\mathcal{W}$, so by the 2 of 3 property $f = hg \in \mathcal{W}$. Thus, $\mathcal{W}$ is closed under retracts. □

References

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