MODEL CATEGORIES PRIMER

$\mathscr{C}$ complete and cocomplete

Three subcategories:

$\mathscr{W}$ Weak equivalences (WE's)

$\mathscr{C}of$ Cofibrations

$\mathscr{F}ib$ Fibrations

(1) All closed under retracts of maps

(2) 2 of 3 property for $\mathscr{W}$ $(h = g \circ f)$

(3) Lifting: Consider a diagram with i a cofibration and p a fibration:

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & B \end{array}$$

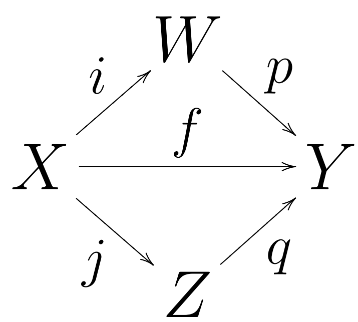
i or p acyclic (in $\mathscr{W}$) $\Longrightarrow$ lift.

i has LLP wrt p

p has RLP wrt i

(Categorical orthogonality language: Sic)

(4) Factorizations of $f\colon X \to Y$:



i a cofibration

p an acyclic fibration

j an acyclic cofibration

q a fibration.

Negotiable: factorizations functorial.

<u>Characterizations</u>:

Cofibration $\Leftrightarrow$ LLP wrt acyclic fibrations

Fibration $\Leftrightarrow$ RLP wrt acyclic cofibrations

Acyclic cofibration $\Leftrightarrow$ LLP wrt fibrations

Acyclic fibration $\Leftrightarrow$ RLP wrt cofibrations

Obsolete: "closed" model category

Cofibrant and fibrant objects

Cofibrant: $\emptyset \to X$ a cofibration.

Cofibrant approximation factorization

$$\emptyset \longrightarrow QX \xrightarrow{\pi} X$$

π an acyclic fibration.

Fibrant: $X \to *$ a fibration

Fibrant approximation factorization

$$X \xrightarrow{\iota} RX \longrightarrow *$$

ι an acyclic cofibration.

Sometimes have simplifying feature:

All objects cofibrant (sSets)

All objects fibrant (Spaces)

All objects cofibrant and fibrant ($\mathscr{C}at$)

A model structure on $\mathscr{C}at$

Weak equivalence = equivalence

Cofibration = injective on objects

Fibration = RLP wrt $* \longrightarrow \mathscr{J}$.

$*$ = trivial, $\mathscr{J}$ = two objects and an isomorphism between them.

Acyclic cof = injective equivalence

Acyclic fib = surjective equivalence

= RLP wrt the three functors

$$\emptyset \longrightarrow *, \ \partial\mathscr{I} \longrightarrow \mathscr{I}, \ \mathscr{E} \longrightarrow \mathscr{I}.$$

$\mathscr{I}$ has two objects $\partial\mathscr{I} = 0 \amalg 1$ and one arrow $0 \longrightarrow 1$, $\mathscr{E}$ has same objects and two parallel arrows $0 \longrightarrow 1$.

Factorizations of $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ through $(\mathscr{C} \times \mathscr{J}) \cup_{\mathscr{C}} \mathscr{D}$ and $\mathscr{D}^{\mathscr{J}} \times_{\mathscr{D}} \mathscr{C}$.

Topological spaces

Spaces compactly generated:
weak Hausdorff k-spaces.

h-model structure:

h-equivalence = homotopy equivalence

h-cofibration = Hurewicz cofibration
HEP = LLP wrt all $p_0\colon Y^I \to Y$

h-fibration = Hurewicz fibration
CHP = RLP wrt all $i_0\colon X \to X \times I$

General theory of h-model structures
on topologically enriched categories.

(Cole, Schwänzl-Vogt, M-Sigurdsson)

q-model structure:

q-equivalence = weak equivalence

= π_*-isomorphism

q-cofibration = "I-cell retract"
(retract of a relative I-cell complex).

$I = \{S^n \subset D^{n+1}\}$.

Relative I-cell complex:
$f\colon X \to \operatorname{colim} Y_n = Y$,
where $Y_0 = X$, $Y_{n+1} = Y_n \cup_K L$,
$K \to L$ a coproduct of maps in I.

q-fibration = Serre fibration
RLP wrt all maps in J,
$J = \{i_0\colon D^n \longrightarrow D^n \times I\}$.

m-model structure (mixed):

m-equivalence = q-equivalence

m-fibration = h-fibration

m-cofibration = determined by LLP

m-cofibrant = CW homotopy type

Theorem (Cole). Let

$$(\mathscr{W}_h, \mathscr{F}ib_h, \mathscr{C}of_h)$$

and

$$(\mathscr{W}_q, \mathscr{F}ib_q, \mathscr{C}of_q)$$

be model structures on $\mathscr{C}$ such that

$$\mathscr{W}_h \subset \mathscr{W}_q \text{ and } \mathscr{F}ib_h \subset \mathscr{F}ib_q.$$

Then there is also a model structure

$$(\mathscr{W}_q, \mathscr{F}ib_h, \mathscr{C}of_m).$$

Compactly generated model categories

A set I of maps in $\mathscr{C}$ is compact if, for domains K and relative I-cell complexes $X \longrightarrow \operatorname{colim} Y_n = Y$,

$$\operatorname{colim} \mathscr{C}(K, Y_n) \cong \mathscr{C}(K, Y).$$

Theorem. $\mathscr{C}$ bicomplete, $\mathscr{W}$ a subcategory closed under retracts and satisfying 2 out of 3 property, I and J compact sets of maps in $\mathscr{C}$. If

– any J-cell complex is acyclic and

– RLP wrt I iff (RLP wrt J) $\cap \mathscr{W}$,

then $\mathscr{C}$ is a model category with

– Fibration = RLP wrt J,

– Acyclic fibration = RLP wrt I,

– Cofibration = I-cell retract

– Acyclic cofibration = J-cell retract.

Simplicial sets

f a weak equivalence if $|f|$ is so.

$I = \{\partial\Delta_n \to \Delta_n\}$

$J = \{\Lambda^i_n \to \Delta_n\}$

cofibration = monomorphism

fibration = Kan fibration

= RLP wrt J.

$\mathscr{C}at$, Top (with q-model structure), and sSet are compactly generated.

Cofibrantly generated model categories

Compact = sequentially small. More general notion of smallness leads to more generally applicable notion of a cofibrantly generated model category, based on transfinite relative I-cell complexes. Ideas are unchanged.

Basic homotopy theory

Cylinder: Factorization of ∇

$$X \amalg X \xrightarrow{i} \text{“}X \times I\text{”} \xrightarrow{p} X$$

with $p \in \mathscr{W}$. Good if $i \in \mathscr{C}of$.
Very good if also $p \in \mathscr{F}ib$.
Very good cylinders exist.
Quillen: cylinder = good cylinder,
but then $X \times I$ in Top not allowed.

Cocylinder (= path object):
Factorization of Δ

$$X \xrightarrow{i} \text{“}X^{I}\text{”} \xrightarrow{p} X \times X$$

with $i \in \mathscr{W}$. Good if $p \in \mathscr{F}ib$.
Very good if also $i \in \mathscr{C}of$.
Very good cocylinders exist.

Left homotopy: $X \times I \to Y$
for some cylinder $X \times I$, $f \simeq_\ell g$;
(good or very good if $X \times I$ is so).

If X is cofibrant, there is a good left homotopy and $\simeq_\ell$ is an equivalence relation between maps $X \to Y$.

If X is cofibrant and Y is fibrant, there is a very good left homotopy.

Right homotopy: $X \to Y^I$
for some cocylinder Y^I, $f \simeq_r g$;
(good or very good if Y^I is so).

If X is cofibrant and Y is fibrant,

$$f \simeq_\ell g \Leftrightarrow f \simeq_r g,$$

then written $f \simeq g$.

$$\pi(X,Y) \equiv \mathscr{C}(X,Y)/(\simeq)$$

Can see $\simeq$ with any fixed good $X \times I$ or any fixed good Y^I (as classically).

<u>Whitehead</u>: $f\colon X \to Y$, X and Y fibrant and cofibrant. Then $f \in \mathscr{W}$ if and only if there exists $g\colon Y \to X$ such that $gf \simeq id_X$ and $fg \simeq \mathrm{id}_Y$.

<u>Homotopy category</u>: $\mathrm{Ho}\mathscr{C} \equiv \mathscr{C}[\mathscr{W}^{-1}]$.
$\gamma\colon \mathscr{C} \to \mathrm{Ho}\mathscr{C}$: identity on objects, $f \in \mathscr{W} \Leftrightarrow \gamma(f)$ is an isomorphism.
Morphism sets $[X, Y]$.

$$\begin{aligned}[X,Y] &\cong \pi(QRX, QRY)\\ &\cong \pi(RQX, RQY)\end{aligned}$$

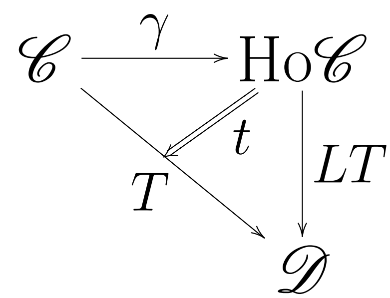
X cofibrant, Y fibrant $\Rightarrow$

$$[X,Y] = \pi(X,Y)$$

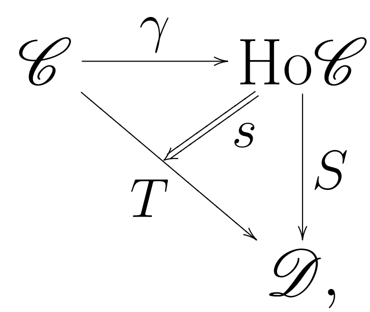
Derived functors (Dwyer–Spalinski)

$T\colon \mathscr{C} \longrightarrow \mathscr{D}$, $\mathscr{C}$ model, $\mathscr{D}$ any cat.

Left derived functor:



For any other such diagram



$$\exists!\ \tilde{s}\colon S \to LT \ \ni\ t \circ \tilde{s}\gamma = s.$$

Unique up to equivalence if it exists.

Right derived functor:

$RT\colon \mathrm{Ho}\mathscr{C} \to \mathscr{D},\ \ t\colon T \to (RT)\circ\gamma$

For any other such pair (S, s),

$$\exists!\ \tilde{s}\colon RT \to S \ \ni\ \tilde{s}\gamma \circ t = s.$$

$\mathscr{I}so(\mathscr{D}) \equiv$ isomorphisms in $\mathscr{D}$.

If $T(\mathscr{W}) \subset \mathscr{I}so(\mathscr{D})$, then

$$LT = \tilde{T}\colon \mathrm{Ho}\mathscr{C} = \mathscr{C}[\mathscr{W}^{-1}] \longrightarrow \mathscr{D}$$

is unique such that $LT \circ \gamma = T$.

$\mathscr{C}_c =$ full subcat of cofibrant objects
$\mathscr{C}_f =$ full subcat of fibrant objects
$\mathscr{C}_{cf} =$ their intersection.

If $T(\mathscr{W} \cap \mathscr{C}of) \subset \mathscr{I}so(\mathscr{D})$, then $T(\mathscr{W} \cap \mathscr{C}_c) \subset \mathscr{I}so(\mathscr{D})$ and

$$LT = \widetilde{TQ},\ TQ\colon \mathscr{C} \to \mathscr{C}_c \longrightarrow \mathscr{D},$$

with $t = T\pi$, $\pi\colon Q \to \mathrm{Id}$.

If $T(\mathscr{W} \cap \mathscr{F}ib) \subset \mathscr{I}so(\mathscr{D})$, then $T(\mathscr{W} \cap \mathscr{C}_f) \subset \mathscr{I}so(\mathscr{D})$ and

$$RT = \widetilde{TR},\ TR\colon \mathscr{C} \to \mathscr{C}_f \longrightarrow \mathscr{D},$$

with $t = T\iota$, $\iota\colon \mathrm{Id} \to R$.

$\mathscr{C}$ and $\mathscr{D}$ model categories

Quillen adjoint pair (T, U):

$$T(\mathscr{C}of_{\mathscr{C}}) \subset \mathscr{C}of_{\mathscr{D}}$$

and

$$U(\mathscr{F}ib_{\mathscr{D}}) \subset \mathscr{F}ib_{\mathscr{C}}$$

TFAE for an adjoint pair (T, U).

(T, U) is a Quillen adjoint pair.

T preserves $\mathscr{C}of$ and $\mathscr{W} \cap \mathscr{C}of$.

U preserves $\mathscr{F}ib$ and $\mathscr{W} \cap \mathscr{F}ib$.

(T, U) is a Quillen equivalence if, for $X \in \mathscr{C}_c$ and $Y \in \mathscr{D}_f$, $f\colon TX \to Y$ is a WE iff $\tilde{f}\colon X \to UY$ is a WE.

"Total" left derived functor (from TQ)

$$\mathbb{L}T = L(\gamma_{\mathscr{D}} \circ T)\colon \mathrm{Ho}\mathscr{C} \longrightarrow \mathrm{Ho}\mathscr{D}$$

$$\mathbb{L}T \circ \gamma_{\mathscr{C}} \to \gamma_{\mathscr{D}} \circ T.$$

WE on cofibrant objects.

Total right derived functor (from UR)

$$\mathbb{R}U = R(\gamma_{\mathscr{C}} \circ U)\colon \mathrm{Ho}\mathscr{D} \longrightarrow \mathrm{Ho}\mathscr{C}$$

$$\gamma_{\mathscr{C}} \circ U \to \mathbb{R}U \circ \gamma_{\mathscr{D}}.$$

WE on fibrant objects.

$(\mathbb{L}T, \mathbb{R}U)$ derived adjoint pair.

For (T, U), (T', U'), $\tau\colon T \longrightarrow T'$:

$\mathbb{L}\tau : \mathbb{L}T \to \mathbb{L}T'$ by $\mathbb{L}\tau_X = \tau_{QX}$.

2-category interpretation (Hovey)

2-category $\mathscr{C}at_{adj}$ of categories, adjunctions (T, U), and natural transformations $T \to T'$.

2-category $\mathscr{C}at_{mod}$ of model categories, Quillen adjunctions, and natural transformations.

Contravariant duality endo-2-functors D on $\mathscr{C}at_{adj}$ and $\mathscr{C}at_{mod}$ that send $\mathscr{C}$ to $\mathscr{C}^{\mathrm{op}}$, (T, U) to (U, T), τ to $\tilde{\tau}$, $\tilde{\tau}\colon U' \longrightarrow U$ the conjugate of τ.

Pseudo-2-functor

$$\mathrm{Ho}\colon \mathscr{C}at_{mod} \to \mathscr{C}at_{adj}$$

via $\mathbb{L}$ on 1-cells and 2-cells, and

$$D \circ \mathrm{Ho} = \mathrm{Ho} \circ D.$$

Characterizations of Quillen equivalences

TFAE for a Quillen adjoint pair (T, U).

(T, U) is a Quillen equivalence.

$(\mathbb{L}T, \mathbb{R}U)$ is an adjoint equivalence.

$X \to UTX \to URTX$ is a WE for $X \in \mathscr{C}_c$ and $TQUY \to TUY \to Y$ is a WE for $Y \in \mathscr{D}_f$.

T reflects WE's between cofibrant objects and $TQUY \to TUY \to Y$ is a WE for $Y \in \mathscr{D}_f$.

U reflects WE's between fibrant objects and $X \to UTX \to URTX$ is a WE for $X \in \mathscr{C}_c$.

Theorem. $(|-|, S)$ is a Quillen equivalence between sSets and Top.

Homotopy colimits and limits

$\mathbb{D}$ a "very small category":
finitely many objects,
finitely many morphisms,
strings of composable non-identity maps have bounded length.

$$\Delta\colon \mathscr{C} \longrightarrow \mathscr{C}^{\mathbb{D}}$$

$$(\mathrm{colim}, \Delta) \quad \text{or} \quad (\Delta, \lim)$$

is a Quillen adjoint pair wrt model structure on $\mathscr{C}^{\mathbb{D}}$ given by levelwise WE's and fibrations or by levelwise WE's and cofibrations.

<u>hocolim</u> or <u>holim</u> is the total left or right derived functor of colim or lim.

Any $\mathbb{D}$ works if $\mathscr{C}$ is sSets or Top.

Proper Model Categories

Consider pushout, i a cofibration:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & Y \end{array}$$

i acyclic $\Rightarrow$ j acyclic (clear)

Left proper: f acyclic $\Rightarrow$ g acyclic.

Consider pullback, p a fibration:

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

p acyclic $\Rightarrow$ q acyclic (clear)

Right proper: f acyclic $\Rightarrow$ g acyclic.

Proper = left and right proper

$\mathscr{V}$-model categories

$\mathscr{V}$ closed symmetric monoidal.

$\mathscr{C}$ $\mathscr{V}$-enriched, tensored, cotensored.

$\mathscr{C}$ and $\mathscr{V}$ model categories.

$$i\colon X \longrightarrow Y \text{ and } j\colon V \longrightarrow W$$

cofibrations in $\mathscr{C}$ and in $\mathscr{V}$.

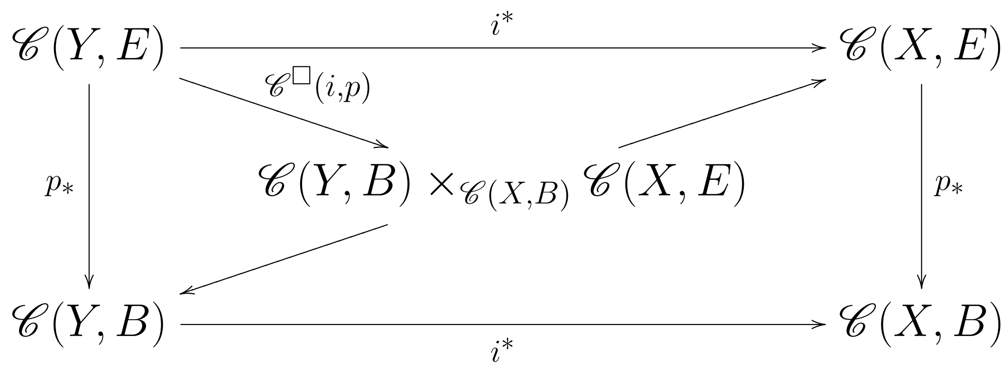
$$\begin{array}{ccccc}
X \otimes V & & \xrightarrow{\mathrm{id}\otimes j} & & X \otimes W \\
\Big\downarrow{\scriptstyle i\otimes \mathrm{id}} & & \swarrow & & \Big\downarrow{\scriptstyle i\otimes \mathrm{id}} \\
 & & (X \otimes W) \cup_{X\otimes V} (Y \otimes V) & & \\
 & \nearrow{\scriptstyle k} & & \searrow{\scriptstyle i\Box j} & \\
Y \otimes V & & \xrightarrow[\mathrm{id}\otimes j]{} & & Y \otimes W.
\end{array}$$

<u>$\mathscr{V}$-model category:</u> $i\Box j$ is a cofibration which is acyclic if i or j is acyclic.

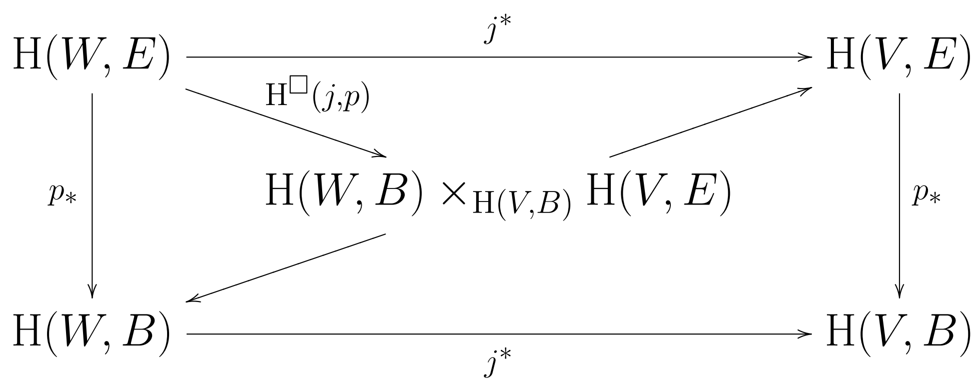
If $\mathrm{id} \otimes j$ and k are WE's, so is $i\Box j$ (left proper relevant). Similarly with roles of i and j reversed.

Equivalent conditions for $\mathscr{V}$-model.

$p\colon E \to B$ a fibration in $\mathscr{C}$.



$\mathscr{C}^{\square}(i,p)$ is a fibration which is acyclic of i or p is.



[Here $H =$ cotensor]. $\mathrm{H}^{\square}(j,p)$ is a fibration which is acyclic if j or p is.

Monoidal model structures

Given $\otimes\colon \mathscr{C}\times\mathscr{D} \to \mathscr{E}$ relating three model categories, one has the analog of the $\mathscr{V}$-model structure condition.

For cofibrations

$$i\colon X \longrightarrow Y \text{ and and } j\colon V \longrightarrow W$$

in $\mathscr{C}$ and $\mathscr{D}$,

$$i\Box j\colon (X\otimes W)\cup_{X\otimes V}(Y\otimes V) \to Y\otimes W$$

is a cofibration in $\mathscr{E}$ which is acyclic if i or j is acyclic.

Defines pairings of model categories.

Given adjoint Hom functors, there result equivalent adjoint analogues.

$\mathscr{C} = \mathscr{D} = \mathscr{E}$: this defines monoidal model categories, symmetric monoidal model categories, closed symmetric monoidal model categories.

Ho$\mathscr{C}$ then inherits a monoidal, symmetric monoidal, or closed symmetric monoidal structure.

Analogously, the case $\mathscr{E} \otimes \mathscr{C} \to \mathscr{E}$ gives $\mathscr{C}$-modules $\mathscr{E}$; $\mathscr{V}$-model structure is a special case.

Detail. If the unit S of $\mathscr{C}$ (or $\mathscr{V}$) is not cofibrant, we must also require $X \otimes QS \to X \otimes S \cong X$ to be a WE in the definitions above.

---

sSet and Top (h, q, m) are proper Cartesian monoidal model categories.

$\mathscr{C}at$ and sSets

$\mathscr{C}at$ (equivalence model structure):

– Cartesian monoidal model category

– Quillen adjoint pair

$$(\pi, \nu) \colon sSets \to \mathscr{C}at$$

$$\pi K = \text{fundamental groupoid of } K$$

Objects K_0, generating isomorphisms $y \colon d_1 y \to d_0 y$ for $y \in K_1$, relations

$$s_0 x = \mathrm{id}_x \text{ for } x \in K_0,$$

$$d_0 z d_2 z = d_1 z \text{ for } z \in K_2.$$

$$\nu \mathscr{C} = \mathrm{Nerve}(\mathscr{I}so\mathscr{C})$$

– $\mathscr{C}at$ is a simplicial model category

Enriched in sSets:

$$\mathrm{Hom}(\mathscr{C}, \mathscr{D}) = \nu(\mathscr{D}^{\mathscr{C}})$$

Tensors and cotensors via π:

$\mathscr{C} \otimes K = \mathscr{C} \times \pi K \quad H(K, C) = \mathscr{C}^{\pi K}$

Realization model structure on $\mathscr{C}at$

(Thomason): Adjunction (N, C)

$N\colon \mathscr{C}at \to$ sSets (nerve functor)

$C\colon$ sSets $\longrightarrow \mathscr{C}at$ (categorize)

CK: Objects K_0, generating maps $y\colon d_1y \to d_0y$ for $y \in K_1$, relations

$$s_0x = \mathrm{id}_x \text{ for } x \in K_0,$$

$$d_0zd_2z = d_1z \text{ for } z \in K_2.$$

πK by localization to invert y's.

(sd^2, Ex^2) endo-adjunction on sSets.

f is WE or fibration if Ex^2Nf is so.

f is a WE iff Nf is a WE iff πNf is an equivalence of groupoids and f is an H_*–isomorphism.

Theorem (Csd^2, Ex^2N) is a Quillen equivalence between $\mathscr{C}at$ and sSets.

Over and under model structures

$\mathscr{C}$ a proper model category, $B \in \mathscr{C}$.

$$\mathscr{C}/B, \quad B\backslash\mathscr{C}, \quad \mathscr{C}_B$$

Over, under, over and under cats. Proper model categories whose weak equivalences, cofibrations, fibrations are those maps which are weak equivalences, cofibrations, fibrations in $\mathscr{C}$ (on underlying total objects).

Base change functors

Assume $\mathscr{C}$, $\mathscr{C}/B$ Cartesian closed.

$f\colon A \longrightarrow B$

$$f_!\colon \mathscr{C}_A \longrightarrow \mathscr{C}_B$$

$$f^*\colon \mathscr{C}_B \longrightarrow \mathscr{C}_A$$

$$f_*\colon \mathscr{C}_A \longrightarrow \mathscr{C}_B,$$

Adjunctions $(f_!, f^*)$, (f^*, f_*).

With generic notations

$$A \xrightarrow{s} X \xrightarrow{p} A \quad \text{and} \quad B \xrightarrow{t} Y \xrightarrow{q} B$$

for objects in $\mathscr{C}_A$ and $\mathscr{C}_B$,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \longrightarrow & f_! X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ f^*Y & \longrightarrow & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[f]{} & B \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{\iota} & \mathrm{Map}_B(A, A) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \mathrm{Map}(\mathrm{id}, s)} \\ f_* X & \longrightarrow & \mathrm{Map}_B(A, X) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{Map}(\mathrm{id}, p)} \\ B & \xrightarrow[\iota]{} & \mathrm{Map}_B(A, A). \end{array}$$

First: top square a pushout.

Others: bottom square a pullback.

Formal from proper model axioms: $(f_!, f^*)$ is a Quillen adjoint pair and a Quillen equivalence if f is a WE.

For a pullback diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[f]{} & B \end{array}$$

$$j^* f_! \cong g_! i^* \quad f^* j_* \cong i_* g^*$$

$$f^* j_! \cong i_! g^* \quad j^* f_* \cong g_* i^*.$$

If (f^*, f_*) is a Quillen adjoint pair, all homotopy categories are trivial!!

Pullback

$$\begin{array}{ccc} \emptyset & \xrightarrow{\phi} & B \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle i_0} \\ B & \xrightarrow[i_1]{} & B \times I \end{array}$$

$\phi_!$ and ϕ_* take $*_\emptyset$ to $*_B$ (initial objs).

$(\phi_!, \phi^*)$ and (ϕ^*, ϕ_*) are Quillen pairs.

$$(i_0)^* \circ (i_1)_! \cong \phi_! \circ \phi^*$$

(Both take any X over B to $*_B$.)

If $(i_1)_!$ and $(i_0)^*$ were both Quillen left adjoints, we would get

$$\mathbb{L}(i_0)^* \circ \mathbb{L}(i_1)_! \cong \mathbb{L}\phi_! \circ \mathbb{L}\phi^*.$$

Since $\mathbb{L}(i_1)_!$ and $\mathbb{L}(i_0)^*$ are equivalences, this would imply that $\mathrm{Ho}\mathscr{C}_B$ is trivial.

No theory of composites $\mathbb{R}U' \circ \mathbb{L}T$!