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Leibniz and the *Vis Viva* Controversy

*By Carolyn Iltis**

INTRODUCTION

IN 1686 GOTTFRIED WILHELM LEIBNIZ PUBLICALLY set down some thoughts on René Descartes' mechanics. In so doing he initiated the famous dispute concerning the "force" of a moving body known as the *vis viva* controversy. Two concepts, now called momentum (mv) and kinetic energy ($\frac{1}{2}mv^2$), were discussed as a single concept, "force," each differing from Newton's idea of force. One of the many underlying problems of the controversy was clarified by Roger Boscovich in 1745 and Jean d'Alembert in 1758, both of whom pointed out that *vis viva* (mv^2) and momentum (mv) were equally valid.[1]

The momentum of a body is actually the Newtonian force F acting through a time, since $v = at$ and $mv = mat = Ft$. The kinetic energy is the Newtonian force acting over a space, since $v^2 = 2as$ and $mv^2 = 2mas$ or $\frac{1}{2}mv^2 = Fs$. Although confusion over these two definitions is apparent in the various arguments of the contenders, many other sources of confusion entered into the debates. Some of these factors are clarified in the following discussion of the early years of the *vis viva* controversy.

The controversy had its roots in Descartes' law of the quantity of motion, as discussed in his *Principia philosophiae* of 1644.[2] It was Descartes' belief that God, the general cause of all motion in the universe, preserves the same quantity of motion and rest put into the world at the time of creation. The measurement of this quantity is mv, implied in the statement "we must reckon the quantity of motion in two pieces of matter as equal if one moves twice as fast as the other, and this in turn is twice as big as the first."[3] The conservation of quantity of motion is derived from God's perfection, for He is in Himself unchangeable and all His operations are performed in a perfectly constant and unchangeable manner. There thus exists an absolute quantity of motion which for the universe remains constant. When the motion in one part is diminished,

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[1] See Carolyn Iltis, "D'Alembert and the *Vis Viva* Controversy," *Studies in History and Philosophy of Science*, 1970, *1*:115–124, and "The *Vis Viva* Controversy: Leibniz to D'Alembert," doctoral dissertation, University of Wisconsin, 1967.

[2] René Descartes, *Principia philosophiae*, in *Oeuvres de Descartes*, ed. Charles Adam and Paul Tannery, 13 vols. (Paris: Cerf, 1897–1913), Vol. VIII, p. 61.

[3] *Ibid.*

that in another is increased by a like amount. Motion, like matter, once created cannot be destroyed, because the same amount of motion has remained in the universe since creation. It is evident from Descartes' application of the principle in his rules governing the collision of bodies that this quantity mv conserves only the magnitude of the quantity of motion and not its direction; that is, velocity is always treated as a positive quantity, $|v|$, rather than as a vector quantity whose direction is variable. Beginning in 1686 Leibniz wrote a series of papers objecting that the quantity which remains absolute and indestructible in nature is not quantity of motion $m|v|$ but *vis viva,* or living force, mv^2.

Shortly before this, in 1668, John Wallis, Christopher Wren, and Christiaan Huygens had presented papers to the Royal Society showing that the quantity conserved in one-dimensional collisions was not $m|v|$ but $m\mathbf{v}$, where the sign of the velocity is taken into consideration.[4] Wallis discussed hard-body inelastic collisions and Wren described elastic collisions. Huygens used rules equivalent to conservation of $m\mathbf{v}$ and mv^2 for elastic impacts. Leibniz was well acquainted with these contributions; he had discussed them in his own notes as early as 1669 and mentioned them in his *Discours de metaphysique*, published in 1686. He was thus aware of the distinction between quantity of motion $m|v|$ and the quantity later called momentum, $m\mathbf{v}$. Leibniz referred to momentum conservation as conservation of total progress (1691).[5] His arguments against Descartes beginning in 1686 were thus designed to establish the superiority of mv^2 over $m|v|$, not over $m\mathbf{v}$.

During the ensuing *vis viva* controversy several concepts were confused in the arguments between Leibniz and the Cartesians. The concepts under discussion included force, quantity of motion, momentum, quantity of progress, *vis mortua* (dead force), and *vis viva* (living force). In addition to delineating the use of these concepts in the physical examples of the contenders, I wish to make the following points:

1. Confusion existed over the use of momentum (mv) and *vis mortua*, the mass times the virtual velocity increment (mdv), in the arguments of the Cartesians.

2. The controversy was not only a dispute over the measure of "force" but also over the conservation of "force." On metaphysical grounds Leibniz was convinced that "force" was conserved in nature. He then successfully argued that mv^2 not $m|v|$ was the measure of this "force." But he implied without adequate empirical proof (except for elastic collisions) that mv^2 was also conserved in his examples. He did not use isolated *interacting* mechanical systems in his discussions of conservation of "force."

3. Leibniz's arguments are directed against the inadequacy with which Descartes'

[4] John Wallis, "A Summary Account of the General Laws of Motion," *Philosophical Transactions of the Royal Society*, 1669, *3*:864–866; Christopher Wren, "Lex Naturae de Collisione Corporum," *Phil. Trans.*, 1669, *3*:867–868; Christiaan Huygens, "Regles du mouvement dans la rencontre des corps," *Journal de sçavans*, 1669, pp. 19–24, and *Phil. Trans.*, 1669, *4*:925–928. Huygens enunciated the mv^2 law prior to Leibniz as rule 6 in the above papers: "The sum of the products of the size of each hard body multiplied by the square of its velocity is always the same before and after impact."

[5] Gottfried Wilhelm Leibniz, "Essay de dynamique sur les loix du mouvement, où il est monstré, qu'il ne se conserve pas la même quantité de mouvement, mais la même force absolue, ou bien la même quantité de l'action motrice," *Mathematische Schriften*, ed. C. I. Gerhardt, 9 vols. in 5 (Halle, 1860), Ser. II, Vol. II, pp. 215-231. English translation in Gottfried Wilhelm Leibniz, *New Essays Concerning Human Understanding*, appendix, ed. and trans. A. G. Langley (La Salle:Open Court, 1949), pp. 657–670.

measure of matter in motion, $m|v|$, described the physical world. Living force, measured by mv^2, was the essence of nature for Leibniz, an encompassing principle, basic to his whole philosophy. Thus the early *vis viva* controversy is not a pointless controversy over momentum versus kinetic energy, but a skillful attack by Leibniz against an inadequate concept, $m|v|$, and its description of the world.

LEIBNIZ'S INITIAL PAPER, 1686

Leibniz's controversy with the Cartesians over living force began in March 1686 with the publication in the *Acta Eruditorum* of his "Brevis demonstratio," or "Brief Demonstration of a Notable Error of Descartes and Others Concerning a Natural Law, According to which God is Said Always to Conserve the Same Quantity of Motion; A Law Which They Also Misuse in Mechanics."[6] In this paper and in a similar discussion in the *Discours de metaphysique*[7] of the same year, Leibniz stated that there was a difference between the concepts motive force (*motricis potentiae*) and quantity of motion $m|v|$ (*quantitas motus*) and that one cannot be estimated by the other. Leibniz, like many others, did not distinguish between mass and weight. He interchanged the Latin terms *mole*, *corpus*, and *libra* and the French terms *masse*, *pesanteur*, and *poids*. Motive force should be designated mgs or ws (weight times height), since it is this which is equivalent (except for a factor of $\frac{1}{2}$) to mv^2, which Leibniz called *vis viva*, or living force. Leibniz however did not use different words for the m in motive force and the m in mv and mv^2. Leibniz's motive force is a rudimentary form of our concept of potential energy. In modern terms his proof establishes the idea of the conversion of potential energy to kinetic energy, or more generally the basis for the work-energy theorem: $F \cdot s = \frac{1}{2}mv^2$.

Leibniz argued:

> It is reasonable that the sum of motive force [*motricis potentiae*] should be conserved [*conservari*] in nature and not be diminished—since we never see force lost by one body without being transferred to another—or augmented; a perpetual motion machine can never be successful because no machine, not even the world as a whole, can increase its force without a new impulse from without. This led Descartes, who held motive force [*vis motrix*] and quantity of motion [*quantitatem motus*] to be equivalent, to assert that God conserves [*conservari*] the same quantity of motion in the world.[8]

Leibniz's argument is based on two assumptions, both of which he claims are accepted by the Cartesians. (See Fig. 1.)

(1) "A body falling from a certain height [*altitudine*] acquires the same force [*vis*] necessary to lift it back to its original height if its direction were to carry it back and if nothing external interfered with it." "Motive force" is thus taken to be the product of the body's weight and the height from which it falls. This statement is the idea of the

[6] Gottfried Wilhelm Leibniz, "Brevis demonstratio erroris memorabilis Cartesii et aliorum circa legem naturalem, secundum quam volunt a Deo eandem semper quantitatem motus conservari; qua et in re mechanica abutuntur," *Acta Eruditorum*, 1686, pp. 161–163. A translation appears in Gottfried Wilhelm Leibniz, *Philosophical Papers and Letters*, trans. Leroy E. Loemker, 2 vols. (Chicago:Univ. of Chicago Press, 1956), Vol. I, pp. 455–463.

[7] Gottfried Wilhelm Leibniz, *Discours de metaphysique*, in *Die Philosophische Schriften von Gottfried Wilhelm Leibniz*, ed. C. I. Gerhardt, 7 vols. (Berlin, 1875–1890), Vol. IV, pp. 442, 443. English translation in Loemker, Vol. I, pp. 464–506.

[8] Leibniz, "Brevis demonstratio," p. 161; Loemker, Vol. I, p. 455.

impossibility of a perpetual motion machine. If force is neither removed (by friction) nor added to the system, it will return to its initial height. Since it cannot rise to a greater height without an external force, a perpetual motion machine cannot be constructed.[9]

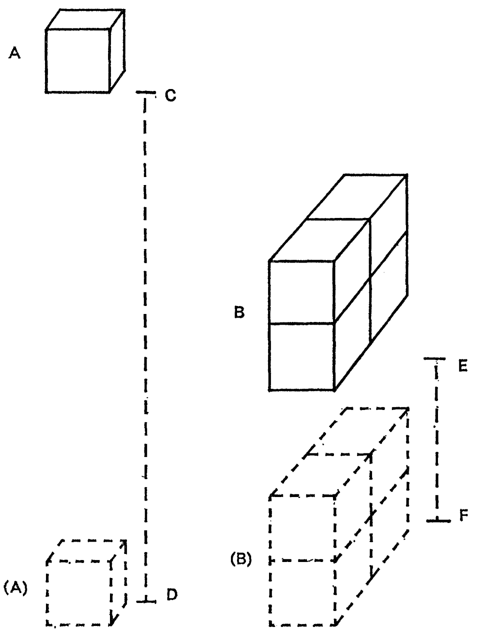

FIGURE 1. ***From* Acta Eruditorum, *March 1686, Table VI.***

(2) "The same force is necessary to raise body A of 1 pound [*libra*] to a height of 4 yards [*ulnae*] as is necessary to raise body B of 4 pounds to a height of 1 yard." In modern terms, the work done on bodies A and B will be equal: $Fs = mgs$.[10] From these two assumptions Leibniz inferred that body A of 1 pound in falling a distance $s = 4$ will acquire the same force as body B of 4 pounds falling $s = 1$:

> For in falling from C and reaching D, the body A will have there the force required to rise again to C by the first assumption; that is, it will have the force needed to raise a body of 1 pound (namely itself) to the height of 4 yards. Similarly the body B after falling from E to F will there have the force required to rise again to E, by the first assumption; that is, it will have the force sufficient to raise a body of 4 pounds (itself namely) to a height of 1 yard. Therefore, by the second assumption, the force of the body A when it arrives at D and that of the body B at F are equal.[11]

On the other hand, argued Leibniz, the Cartesian quantities of motion are not equal. For, as Galileo showed, body A in its fall will acquire twice the velocity of body B. (This is now written $2gs = v^2 - v_0^2$.) Body A, 1 pound, falling from $s = 4$, will arrive at D with a velocity 2; hence its quantity of motion mv is 2. Body B of 4 pounds falling from $s = 1$ arrives at F with velocity 1, its mv thereby being 4. Thus the quantities of

[9] This assumption had its beginnings in Jordanus' notion of *gravitas secundum situm* (gravity according to position). It is found in the writings of early-17th-century authors as the experimental observation that no system of falling weights will produce perpetual motion in any of its parts. Galileo showed that no series of inclined planes will impart to a descending body a velocity sufficient to carry it to a vertical height greater than its initial height. See Erwin Hiebert, *Historical Roots of the Conservation of Energy* (Madison:State Historical Society of Wisconsin, 1962), pp. 60, 61.

[10] The second assumption was stated by Descartes in a letter to Marin Mersenne in 1638 (*Oeuvres*, Vol. II, p. 228):

> The proof of this depends solely on the principle which is the general foundation of all statics, that no more or less force [*force*] is needed to lift a heavy body to a certain height [*hauteur*] than to lift another less heavy to a height as much greater as it is less heavy or to lift one heavier to a height as much less. As for example, that force which can lift a weight [*poids*] of 100 pounds to the height of 2 feet, can also lift one of 200 pounds to the height of 1 foot, or one of 50 to the height of 4 feet and thus of others if it is so applied to them.

[11] Leibniz, "Brevis demonstratio," p. 162; Loemker, Vol. I, p. 457.

motion are unequal, but the "motive forces" (*vis motrix*), *mgs*, as proved above, are equal.[12] Therefore, says Leibniz, the force of a body cannot be calculated by finding its quantity of motion but rather "is to be estimated from the quantity of the effect [*quantitate effectus*] it can produce, that is from the height to which it can elevate a body of given magnitude [*magnitudinus*]."

Several points are to be noted about the "Brevis demonstratio," the first paper in a long series of discussions between Leibniz and his opponents on the subject of "living force."

First of all, Leibniz has not yet introduced the term *vis viva*, that is, "living force," or its mathematical equivalent, mv^2. He does not publically speak of living force until 1695 in the well-known "Specimen dynamicum," though he uses the term in his unpublished "Essay de dynamique" in 1691.[13] In these earlier papers the discussion involves the term "motive force" (*vis motrix*), *ws*, the equivalent mv^2 being only implied by the use of the square root of the distance of fall in calculating the *mv* of bodies *A* and *B*.

Secondly, he asserts that the Cartesians were led into error by confusing the force of motion, which they estimated by the quantity of motion, with the quantity used in statics in the case of the five simple machines. In statics the tendency toward motion is estimated by the mass times the (virtual) velocity:

> Seeing that velocity and mass compensate for each other in the five common machines [*mdv*], a number of mathematicians have estimated the force of motion [*vim motricem*] by the quantity of motion, or by the product of the body and its velocity [*producto ex multiplicatione corporis in celeritatem suam*] [*mv*]. Or to speak rather in geometrical terms the forces of two bodies (of the same kind) set in motion, and acting by their mass [*mole*] as well as by their motion are said to be proportional jointly to their bodies [*corporum*] or masses [*molium*] and to their velocities [*velocitatem*].[14]

This accusation Leibniz also makes in later papers. There is no evidence that Descartes himself made this error,[15] although his followers certainly did. Quantity of motion,

[12] For the relationship $mgs = \frac{1}{2}mv^2$ implied here, Leibniz is indebted chiefly to Huygens, who used it in his derivation of the law of the compound pendulum in his *Horologium oscillatorum* (1673). Huygens also related the heights of fall of a body to the velocities acquired in proposition 8 of his *De Motu corporum ex percussione*, largely complete by 1656 but published posthumously in 1703.

[13] Leibniz first speaks of living force in his "Essay de dynamique" (1691), which was unpublished until discovered by Gerhardt in the papers at Hanover and included in the *Mathematische Schriften* (see n. 5). He also uses the term in an essay recently discovered by Pierre Costabel, described in his *Leibniz et la dynamique. Les textes de 1692* (Paris: Hermann, 1960), p. 104.

[14] Leibniz, "Brevis demonstratio," p. 162; Loemker, Vol. I, p. 455.

[15] Descartes, *Oeuvres*, Vol. II, pp. 222–246. Descartes knew that it was the commencement of movement which must be taken into account at each instant; he says, "notez que ie dis commencer à descendre, non pas simplement descendre." For a discussion of the history of the virtual velocity concept see Hiebert, *Historical Roots*, Chap. 1. Neither Leibniz nor the Cartesians used the term "virtual velocity." This was first used by Jean Bernoulli in 1717. The virtual velocity *dv* of a body is the ratio of the virtual displacement *ds* to the time element *dt*, i.e., ds/dt. Virtual displacement *ds* is the distance through which a body in equilibrium or under constraint would move if acted upon by a force which disturbs the equilibrium. Virtual velocity is the velocity the body would acquire in moving through the distance *ds*. On the use of the term, Hiebert writes (p. 53):

> Prior to the time of Varignon's *Nouvelle mecanique* of 1725, no name was attached to the principle we have been discussing [virtual work]. John Bernoulli (1667–1748) of Basel supplied an expression in 1717 in an off-hand suggestion in a letter addressed to Varignon. In this letter Bernoulli introduced the term virtual velocity [*vitesse virtuelle*] . . . . He had

later known as momentum, is not the same as the quantity formed by the product of the mass and the virtual velocity as applied to static situations. This confusion will be seen in the contributions of Abbé Catalan and Denis Papin.

Thirdly, there is a lack of clarity over what constitutes empirical proof of the conservation of "force" mv^2 over and above establishing mv^2 as a measure of "force."

Were it not for the title and the introduction quoted above, one would consider Leibniz's presentation simply to have established "motive force," or its equivalent, mv^2, as a measure of force, for he succeeds in showing that force, defined by him as *ws*, is to be estimated by the height to which it can raise a body of a given magnitude. Thus he has established a rudimentary expression for the conversion of potential energy to kinetic energy. Quantity of motion $m|v|$ is not the measure of a force so defined. However, the title states that the Cartesians have made an error in asserting that quantity of motion is *conserved.* Similarly, in the first paragraph it is stated that "it is reasonable that the sum of motive force should be *conserved* in nature," and Descartes "asserted that God *conserves* the same quantity of motion in the world." These statements imply—although Leibniz does not state this as a conclusion—that the "Brevis demonstratio" has shown that quantity of motion $m|v|$ is not conserved, whereas motive force, measured by *ws*, is conserved. The only basis for these implications concerning conservation is that the quantities of motion of bodies *A* and *B* were found to be unequal, while the motive forces *ws* of the two bodies were equal.

Three separate aspects of the establishment of conservation laws such as that of kinetic energy may be distinguished: (1) a metaphysical belief that some entity is con-

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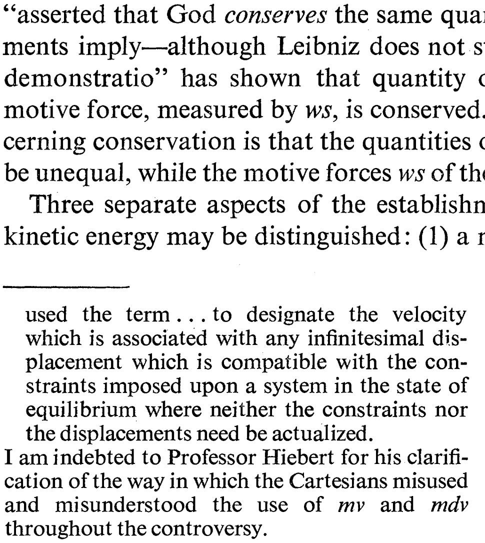
> used the term . . . to designate the velocity which is associated with any infinitesimal displacement which is compatible with the constraints imposed upon a system in the state of equilibrium where neither the constraints nor the displacements need be actualized.

I am indebted to Professor Hiebert for his clarification of the way in which the Cartesians misused and misunderstood the use of *mv* and *mdv* throughout the controversy.



The virtual velocity principle in modern notation for the case of the lever is (see figure): $F_1l_1 = F_2l_2$ or $F_1s_1 = F_2s_2$. But $F = mg$ and $ds = dv \cdot dt$. Thus $m_1gdv_1 \cdot dt_1 = m_2gdv_2 \cdot dt_2$. For the case of the lever in equilibrium, the times are equal, $dt_1 = dt_2$; hence $m_1dv_1 = m_2dv_2$, or dead force. But the *dv* are virtual velocities and not the actual velocities in the momentum expression *mv* for moving bodies. (Example mine. Leibniz's figure—Loemker, Vol. I, p. 459—with notation *l*, *m*, and *s* added.) Leibniz stated the dead force idea in relation to the lever in a supplement to the "Brevis demonstratio" written in 1695. See Loemker, Vol. I, pp. 459–460:

> The same proposition is confirmed also by the five commonly recognized mechanical powers —the lever, windlass, pulley, wedge and screw; for in all these our proposition seems to be true. For the sake of brevity, however, it will suffice to show this in the single case of the lever, or what amounts to the same thing to deduce from our rule that the distances and weights of bodies in equilibrium are in reciprocal proportion. Let us assume *AC* [see figure] to be double *BC*, and the weight *B* double the weight *A*; then I say *A* and *B* are in equilibrium. For if we assume either one to preponderate, *B* for example, and so to sink to *B'* and *A* to rise to *A'* and drop perpendiculars *A'E* and *B'D* from *A'* and *B'* to *AB*, it is clear that if *DB'* is 1 foot, *A'E* will be 2 feet and therefore that, if 2 pounds descend the distance of 1 foot, 1 pound will ascend to the height of 2 feet, and thus that, since these two are equivalent nothing is gained and the descent becomes useless, everything remaining in equilibrium as before. . . . . Even if some of these seem reconcilable with that hypothesis which estimates the product of mass by velocity, this is only accidental since the two hypotheses coincide in the case of dead forces (*potentia mortuus*) in which only the beginning or end of conatuses is actualized.

served in the universe, (2) the mathematical expression or measure of the conserved entity, and (3) the empirical proof that that particular entity is conserved in physically interacting systems. Like many other natural philosophers, Leibniz was convinced on metaphysical grounds that something was conserved in nature. This conserved entity was taken by him to be living force, *vis viva.* If living force were not conserved, the world would either lose force and run down or a perpetual motion machine would be possible.

Such a philosophical conviction is not unusual and is important in the development of other conservation laws. For example, Parmenides and the pluralists argued that "being" could neither be created nor destroyed—long before it was possible for A.-L. Lavoisier to empirically establish conservation of matter. Descartes was convinced that motion $m|v|$ was conserved in the universe before the correct empirical law was given as $m\mathbf{v}$ conservation. The caloric theory depended on the conservation of heat, before empirical evidence disproved it. J. Robert Mayer[16] and Hermann von Helmholtz were convinced of the general law of conservation of energy before compiling empirical evidence.[17] James Joule, while supplying much empirical data for the law, generalized from values so widely divergent as to be scientifically unconvincing without prior metaphysical certainty.[18] Indeed, the general conservation law which states that the total energy of the universe is conserved is a theoretical statement which cannot be verified empirically except in isolated closed systems.

Leibniz presented important mathematical arguments that mv^2 and not $m|v|$ was a correct *measure* of something conserved in nature. He did not however present convincing arguments that his measure of force was also *conserved* in the physical instances he claimed for it, with the exception of elastic collisions. In many of his other arguments Leibniz does not adequately specify a closed conservative system, since the mechanisms for transferring "force" among the parts of the system are not specified.

In the "Brevis demonstratio," if Leibniz were to establish conservation of mv^2, he would need a closed conservative system where there is a collision or a mechanical connection between the two bodies. This is not necessary for the mere establishment of the mathematical measure of a force, or the conversion of potential to kinetic energy. To establish conservation of mv^2, a mechanical method of transferring the motive force from body A to body B, such as an ideal spring, would be necessary. However, in

[16] J. Robert Mayer, "Bemerkungen über die Kräfte der unbelebten Natur," *Annalen der Chemie und Pharmacie*, 1842, *42*. See also translated excerpts in W. F. Magie, *A Source Book in Physics* (New York: McGraw Hill, 1935), p. 196.

[17] Thomas Kuhn, "Energy Conservation as an Example of Simultaneous Discovery," *Critical Problems in the History of Science* (Madison: Univ. of Wisconsin Press, 1959); see pp. 336–339 on the influence of *Naturphilosophie* in enunciating the general law of energy conservation. "In the cases of Colding, Helmholtz, Liebig, Mayer, Mohr and Seguin, the notion of an underlying imperishable force seems prior to research and almost unrelated to it. Put bluntly these pioneers seem to have held an idea capable of becoming conservation of energy for some time before they found evidence for it."

[18] Emil Meyerson, *Identity and Reality* (New York: Dover, 1962), pp. 194, 195:

> The numbers of the English physicist [Joule] vary within extraordinarily large limits; the average at which he arrives is 838 foot-pounds (for the quantity of heat capable of increasing the temperature of a pound of water by 1°F . . .); but the different experiments from which this average is drawn furnish results varying from 742 to 1,040 foot-pounds—that is by more than a third of the lowest value—and he even notes an experiment which gives 587 lbs without seeing in it any source of particularly grave experimental errors . . . it becomes really difficult to suppose that a conscientious scientist relying solely on experimental data could have been able to arrive at the conclusion that the equivalent must constitute, under all conditions, an invariable datum.

Leibniz's example the bodies fall to the ground side by side and the forces of the two falling bodies are compared merely as to equality. The effect of the ground and the possibility of a mechanical connection are ignored. Thus the implication of the title and of the introduction that the demonstration will yield information about conservation is not justified. The demonstration does successfully establish a mathematical measure of force. Leibniz's implicit identification of measure and conservation is not valid. He seems to have assumed conservation of motive force on the basis of the impossibility of perpetual motion, but his empirical demonstration of conservation is incomplete. The confounding of measure and conservation and the inattention to mechanical connections were two of the sources of confusion in the controversy with the Cartesians.[19]

THE CONTROVERSY WITH ABBÉ CATALAN

Leibniz's "Brevis demonstratio" was translated into French, and by September of the same year, 1686, it appeared in the *Nouvelles de la république des lettres*. Leibniz was immediately answered by the Cartesian Abbé Catalan in a "Courte Remarque."[20]

It has been shown, writes Catalan, that two moving bodies (*mobiles*) which are unequal in volume (for example, 1 to 4) but equal in quantity of motion (that is, 4) have velocities proportional to the reciprocal ratio of their masses (*masses*) (that is, 4 to 1). Consequently they traverse (*parcourent*), in the same time, spaces proportional to these velocities.[21] Now Galileo, he says, showed the spaces described by falling bodies to be as the squares of the times (now written $s = \frac{1}{2}gt^2$). Therefore, in the example given by Leibniz the body of 1 pound (*livre*) ascends to the height 4 in time 2 and the body of 4 pounds ascends to the height 1 in time 1. If the times are unequal, it is not surprising to find the quantities of motion unequal. But, says Catalan, if the times are made equal by suspending them to the same balance at distances reciprocal to their bulk (*grosseur*), the quantities formed by the products of their masses and distances, or masses and velocities, are equal.

Catalan here has lumped together three separate problems as one: a body's uniform traversal of space (momentum), free fall (*vis viva*), and the problem of the lever (virtual velocities). In the free-fall problem, if the times were equal, the mv would be equal only for bodies of equal weight. If the times for unequal bodies were made equal by use of a lever, the problem would be changed to a problem in statics, where virtual work or mass times the distance increment mds describes the situation. This is not the same as quantity of motion mv.

By the following February Leibniz issued a reply to Catalan,[22] answering the

[19] In another work of the year 1686, the *Discours de metaphysique*, Leibniz again refers to Descartes' error, giving the same proof and implying conservation of force in the following statements: "Our new philosophers commonly make use of the famous rule that God always *conserves* the same quantity of motion in the world.... Now it is reasonable that the same force should be conserved in the universe. . . . So these mathematicians have thought that what can be said of force can also be said of the quantity of motion." (Loemker, Vol. I, pp. 482, 483.)

[20] Abbé Catalan, "Courte Remarque de M. l'Abbé D. C. où l'on montre à M. G. G. Leibnits le paralogisme contenu dans l'objection précédente," *Nouvelles de la république des lettres*, Sept. 1686, *8*:1000–1005.

[21] *Ibid.*, p. 1002.

[22] Gottfried Wilhelm Leibniz, "Réplique à M. l'Abbé D. C. contenue dans une lettre écrite à l'auteur de ces nouvelles le 9. de Janr. 1687, touchant ce qu'a dit M. Descartes que Dieu conserve toujours dans la nature la même quantité de mouvement," *Nouv. répub. lett.*, Feb. 1687, *9*:131–144.

objection that since the two falling bodies acquire their forces in unequal times, the forces ought to be different. If the force of a body of 4 pounds having a velocity of 1 degree is transferred (*transferer*) to a body of 1 pound, according to the Cartesians the second will receive a velocity of 4 degrees to preserve (*garder*) the same quantity of motion. But, argues Leibniz, this second body should receive only a velocity of 2. And in estimating the forces that the bodies have acquired, no one (except the Abbé Catalan) will measure whether they have acquired these forces in times long or short, equal or unequal. Time has nothing to do with the measure of force (that is, *vis viva*). One can judge the present state without knowing the past. If there are two perfectly equal and identical bodies having the same velocity—the first acquiring its velocity in a collision, the second in a descent—can their forces be said to be different? This would be like saying a man is wealthier for taking more time to earn his money.[23]

Furthermore, one can change at will the time of descent by changing the line of inclination of the descent; and in an infinite number of ways two bodies can be made to descend from different heights in equal times. But a body descending from a certain height acquires the same velocity whether that descent is perpendicular and faster, or inclined and slower. Thus the distinction of time has nothing to do with the argument.[24] This was countered by Catalan in June 1687, with the observation that on an inclined plane the force necessary to lift a body is less than that necessary to lift it perpendicularly to the same height.[25]

Here again two concepts are confused. Leibniz is discussing the fall of a weight through a vertical distance (mgs), or potential energy, where the time is irrelevant. Catalan's argument is based on the idea that the applied or Newtonian force needed to push a body up an inclined plane is less than that needed to lift the body perpendicularly to the same vertical height.

In addition to the argument that force should be defined as acting through distance rather than time, Leibniz employed another tactic in the argument with Catalan.[26] He attacked Descartes' invalid third rule of motion which stated: "If [hard] body B and [hard] body C are equal in heaviness, but B moves [toward C] with slightly greater speed than C, not only do both move to the left afterwards, but B also imparts to C half the difference of their initial speeds." Considering this third rule of motion, suppose that two bodies B and C, each 1 pound, move toward each other, B with a velocity of 100 degrees and C with a velocity of 1 degree. Together their quantity of motion will be 101. Now C with its velocity of 1 can rise to 1 foot while B can rise to 10,000 feet. Thus the force of the two together before colliding would elevate 1 pound to 10,001 feet. According to Descartes' rule of motion, after the impact both move together with a speed of $50\frac{1}{2}$. By multiplying this speed by the combined weight of the two bodies the quantity of motion 101 is retained. However, in this case the force of the 2 pounds together can raise 1 pound to only $2(50\frac{1}{2})^2 = 5{,}100\frac{1}{2}$ feet. Thus, says Leibniz, almost half the force is lost without any reason and without being used elsewhere; Descartes' third rule, therefore, is wrong and with it the principle upon which it is based—conservation of $m|v|$.

[23] *Ibid.*, p. 133.

[24] *Ibid.*, p. 134.

[25] Abbé Catalan, "Remarque sur la réplique de M. L. touchant le principe mechanique de M. Descartes, contenue dans l'article VII de ces nouvelles, mois de Février, 1687," *Nouv. répub. lett.*, June 1687, *10*:577–590; see pp. 586, 587.

[26] Leibniz, "Réplique à M. l'Abbé D. C.," p. 138.

Now this example employs an actual collision of two bodies, not a mere proportionality of forces as in the first example, and again it attempts to show that $m|v|$ is not conserved. Here there is no leap from measure of force to conservation of force. Leibniz's argument seems to succeed not because of inattention to the mechanism, but because Descartes' third rule, based on $m|v|$, is itself in error. Here Leibniz is initiating a new line of argument upon which he relies in subsequent papers: that is, if Descartes' rules for colliding bodies are shown to be false, then the principle upon which they are based—conservation of quantity of motion—must also be false.[27]

THE CONTROVERSY WITH DENIS PAPIN

Another line of argument which was based on the impossibility of perpetual motion and upon the equipollence of cause and effect was followed in the discussion with Denis Papin during the period 1689–1691. In reply to Papin's paper showing that the quantities of motion in freely falling bodies are in the direct ratio of the times of motion,[28] Leibniz declares that the issue must be decided by whether or not perpetual motion can arise from the acceptance of either of the two definitions of force.[29]

He begins by clarifying the issue at stake, in order, he says, to exclude all verbal misunderstanding. Anyone is at liberty to define force as he wishes, whether as quantity of motion or as motive force. The issue is to decide which is conserved (*conservare*), whether it be the product of weight (*pondus*) and speed or the product of weight and height. This will be decided by whether or not perpetual motion can arise from the acceptance of either definition.

Taking again balls of weight 1 and 4, he allows the larger to descend from a height of 1 by means of an inclined plane (Fig. 2). When it reaches the horizontal and is moving with a velocity of 1, it meets the smaller body at rest. All of its force of 4 is now transferred to the smaller body of weight 1. Now if this body were to receive a velocity of 4, as the Cartesians would maintain in order to conserve quantity of motion, then, argues Leibniz, perpetual motion would arise. For this smaller body by virtue of its velocity of 4 could ascend an inclined plane to a height of 16 feet. Perpetual motion or an effect more powerful than its cause can arise, because in falling again to the horizontal plane it can elevate, by means of a lever, the first body of weight 4 to a height of 4 feet. Thus in the final state the first body rests at height 4 rather than height 1 as in its initial state, while the second body has been returned to its original position

[27] If, as Descartes supposes, the bodies stick together, the collision is inelastic and mv^2 is not conserved. Five years later in 1691 Leibniz stated that mv^2 was not conserved in inelastic impacts. Thus both Descartes and Leibniz are wrong from a modern point of view. If the sign of the velocity is taken into account, the final speed is $49\frac{1}{2}$: $mv + MV = 1(100) + 1(-1) = 99 = (m+M)v_f = 2v_f$; $v_f = 49\frac{1}{2}$.

[28] Denis Papin, "De Gravitatis causa et proprietatibus observationes," *Acta Eruditorum*, April 1689, pp. 183–188. In this paper Papin argues, as does Catalan, that the "force" mv of a falling body depends on the time of fall. Since falling bodies add equal increments of velocity in equal times, they also add equal quantities of motion in equal times: $v = at$; $mv = mat = Ft$ (modern terminology). Like Catalan, Papin argues incorrectly that in Leibniz's 1686 paper if the times of fall are equal the forces will be equal: "If the times are equal no more or no less force can be added or subtracted by making the space traversed longer or shorter. Thus a measure of force estimated by the spaces cannot be correct." The mv actually would be equal only for equal bodies falling in equal times. If balanced by a lever, it is an $m\,dv$ problem. See p. 187.

[29] Gottfried Wilhelm Leibniz, "De Causa gravitatis et defensio sententiae sua veris naturae legibus contra Cartesianos," *Acta Eruditorum*, May 1960, pp. 228–239.

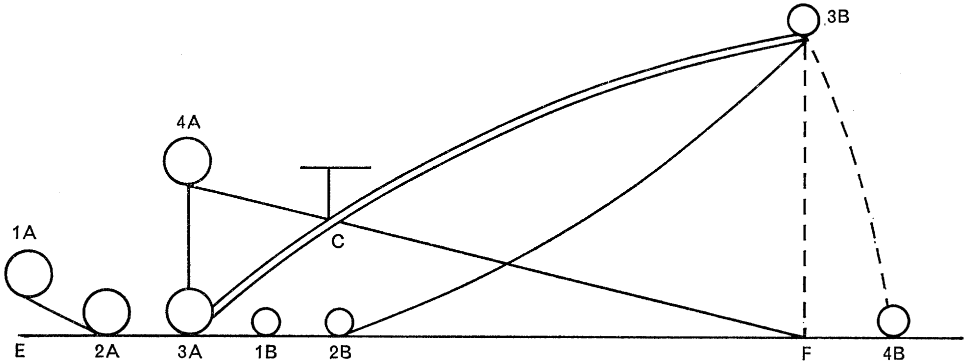


FIGURE 2. ***From* Acta Eruditorum, *May 1690.***

in the horizontal plane. No new force has been contributed or absorbed by other agents or patients. "We conclude," writes Leibniz, "against the Cartesians that quantity of motion should not always be conserved."

Denis Papin's second paper shrewdly attacks Leibniz's argument.[30] He concedes that perpetual motion is absurd and that if it could actually be demonstrated by the above example the Cartesian measure of force would be reduced to an absurdity. But he denies the possibility of actually transferring in nature *all* the "power" of body A to body B. He promises publically that if any method can be indicated by which all the moving forces of the greater body can be transferred to the smaller body at rest without the occurrence of a miracle, he will concede the victory to Leibniz. Leibniz's final reply offers some methods for transferring the "force," none of which is physically feasible.[31]

In analyzing this example several points of confusion become apparent. The purpose of Leibniz's argument is to show that $m|v|$, or quantity of motion, is not conserved. He is discussing the conservation of "force," not merely the measure of "force" in a physical experiment where the mechanism of transferring the "force" (whether defined as $m|v|$ or mv^2) is not specified. Suppose the apparatus for this thought experiment could be set up under idealized conditions. If the bodies were allowed to collide in order to transfer the force, body A would rebound slightly. Momentum mv would be conserved, but quantity of motion $m|v|$ would not, since it is not valid for

[30] Denis Papin, "Mechanicorum de viribus motricibus sententia, asserta adversus cl. GGL. objectiones," *Acta Eruditorum*, Jan. 1691, pp. 6–13.

[31] Gottfried Wilhelm Leibniz, "De Legibus naturae et vera aestimatione virium motricium contra Cartesianos. Responsio ad rationes a Dn. Papino mense Januarii proxima in *Actis* hisce p. 6. propositas," *Acta Eruditorum*, Sept. 1691, pp. 439–447; p. 443. Leibniz offers two methods of transferring all the "force" from a larger body to a smaller one at rest, claiming that additional demonstrations have been left with a friend in Florence. The first method is to divide body A into 4 parts, all equal to the size of body B, the totality retaining the velocity of body A, i.e., 1. The "power" of each of these smaller bodies is then transferred successively onto body B at rest. (If this occurs, the first collision will set body B in motion with the velocity of the first small part. But thereafter body B and the second small part of body A will be in motion with equal velocities.) Leibniz's second method is to connect bodies A and B by a sufficiently long rigid line. On this is assumed an immovable point H around which the compound is to be rotated. Point H is close enough to A and sufficiently removed from B that when A rests, B is unbound. (The details of this method are obscure, and it is not at all clear how such a device could be physically operated and still fulfill the conditions of A having an initial velocity of 1 and B having zero initial velocity.)

such a collision. But the use of collisions as a method of transferring the force does not fulfill Leibniz's conditions, because body *A* will retain some mv and mv^2. Papin's objection that all the force cannot be transferred is therefore a realistic one.

Leibniz would need to transfer the mv^2 of body *A* to body *B* by a method such as an ideal spring which does not dissipate the *vis viva*. The energy of the spring could be transferred to body *B* by releasing a catch on the spring. If such an external force is used, both quantity of motion and momentum conservation will be violated. *Vis viva* if not dissipated would be conserved. The point of Leibniz's argument is to show that neither quantity of motion nor momentum are conservation principles which are as general as *vis viva*. Later in 1691 Leibniz argued that any dissipated *vis viva* went into the small parts of a body's matter and was not lost for the universe; he had no empirical proof of this, however. The argument with Papin serves to illustrate the view that Leibniz's main effort was directed toward establishing the superiority of *vis viva* over quantity of motion $m|v|$ as a universal conservation principle. Conservation of living force encompasses a wider range of phenomena than quantity of motion.[32]

VIS VIVA AS A PHILOSOPHICAL PRINCIPLE

Leibniz was anxious to establish a broad and absolute conservation principle which would form a basis for his philosophical system. At the root of his controversy with Descartes and his followers lies not a mere mathematical dispute as to the measure of "force," $m|v|$, or mv^2, but a fundamental disagreement as to the very nature of force itself. As early as 1686 in the *Discours de metaphysique* Leibniz first elaborated on the content of the difference between motive force, equivalent to *vis viva*, and quantity of motion.[33] Here he presented an argument which was to become the spearhead of his attack on Cartesianism and to become the basis of his own philosophy of monadology:

> Force is something different from size, from form, or from motion, and the whole meaning of body is not exhausted in its extension together with its modifications. Motion,

[32] A text of Leibniz written in 1691 has been recently discovered, edited, and discussed by Pierre Costabel (*Leibniz et la dynamique*; n. 13). In regard to content this Leibniz text is very similar to the two papers written against the ideas of Papin but presents the argument in the form of logical definitions, axioms, and propositions. Proposition 4 is the same as discussed above, and the principles upon which the conclusions are based are the impossibility of perpetual mechanical motion and the requirement that the total cause equal the complete effect and the same quantity of force be conserved. Again all transfer of force is by substitution of a body in one state of motion and position for a body of a force, equal to that of the first. The possibility of physical transfer is not discussed except to say that one can imagine certain techniques for the execution of these transfers. Propositions identical to the conclusions in the other two papers are proved by use of the axioms and definitions. Proposition 8 reads: "When the forces are equal the quantities of motion are not always equal and vice versa." While this proposition is valid, the mechanical conservative system for the validity of proposition 9 is not specified: "The same quantity of motion is not always conserved." The similarity of this 1692 paper to Leibniz's 1690 paper against Papin is not mentioned by Costabel.

[33] Leibniz, *Discours de metaphysique*, Loemker, Vol. I, p. 487:

> If there were nothing in bodies but extended mass, and nothing in motion but change of place, and if everything should and could be deduced solely from the definitions by geometric necessity, it would follow, as I have elsewhere shown, that the smallest body in colliding with the greatest body at rest, would impart to it its own velocity, without losing any of this velocity itself; and it would be necessary to accept a number of other such rules which are entirely contrary to the formation of a system. But the decree of the divine wisdom to conserve always the same total force and the same total direction has provided for this.

> if we regard only its exact and formal meaning, is not something entirely real . . . But the force or the proximate cause of these changes [in the places of bodies] is something more real, and there are sufficient grounds for attributing it to one body rather than to another.[34]

What is real in nature for Leibniz is primitive force or striving, and this was developed by him in the succeeding years as the essence of the monad. Motion and extension, the essence of nature for Descartes, are to Leibniz merely relations and not realities at all.

A significant statement of the problem of the controversy is given in his "Essay de dynamique," written about 1691 but not published until 1860.[35] Here the problem is given as a search for an estimate of force as a mathematically absolute or positive quantity which can never be taken as null or negative in the impact of elastic bodies. This paper draws together the principle of conservation of relative velocity, conservation of momentum mv, which Leibniz calls quantity of progress and which does take into account the sign of the velocity, and conservation of living force mv^2. It presents the solution of elastic impact problems as the simultaneous solution of any two of these equations: "Although I put together these three equations for the sake of beauty and harmony, nevertheless two of them might suffice for our needs. For taking any two of these equations we can infer the remaining one."[36]

Although this paper remained unpublished until 1860, its ideas appeared in a paper of Leibniz's follower Jean Bernoulli (1727).[37] Leibniz wrote in his "Essay de dynamique" that after some philosophers abandoned the opinion that quantity of motion is preserved in the concourse of bodies, they did not recognize the conservation of anything absolute to hold in its place. However, our minds look for such a conservation and many find themselves unable to give up the axiom without finding another to which to subscribe.[38] He continues, "It is . . . plain that [the] conservation [of quantity of progress] does not correspond to that which is demanded of something absolute. For it may happen that the velocity, quantity of motion and force of bodies being very considerable, their progress is null. This occurs when the two opposed bodies have their quantities of motion equal."[39] But in the equation for the conservation of living force, the negative and positive velocities have the same square, and, writes Leibniz,

> . . . these different directions produce nothing more. And it is also for that reason that this equation gives something absolute, independent of the progressions from a certain side. The question here concerns only the estimating of masses and velocities, without troubling ourselves from what side these velocities arise. And it is this which satisfies at the same time the rigor of the mathematicians and the wish of the philosophers—the experiments and reasons drawn from different principles.[40]

Leibniz knew that the conservation of *vis viva* did not hold for inelastic and semi-elastic collisions: "But this loss of the total force . . . or this failure of the third equation, does not detract from the inviolable truth of the law of the conservation of the same force in the world. For that which is absorbed by the minute parts is not

[34] *Ibid.*, p. 484.

[35] Leibniz, "Essay de dynamique" (see n. 5).

[36] *Ibid.*, Langley, p. 668.

[37] Jean Bernoulli, "Discours sur les loix de la communication du mouvement," *Recueil des pieces qui a remporté les prix de l'Académie royale des sciences*, 1727, *2*: 1–108; see p. 29.

[38] Leibniz, "Essay de dynamique," Langley, pp. 657, 658.

[39] *Ibid.*, p. 658.

[40] *Ibid.*, p. 668.

absolutely lost for the universe, although it is lost for the total force of the concurrent bodies."[41] Although Leibniz argues on philosophical grounds that the dissipated *vis viva* is conserved for the universe, he gives no empirical proof and does not recognize the heat changes which accompany this phenomenon.[42]

In "Specimen dynamicum" (1695), Leibniz presents a mature synthesis of his concept of force, drawing together the observations and opinions expressed since 1686 in his paper on dynamics and incorporating philosophical views developed concurrently with his work in physics.[43] It summarizes his attack on the foundation of Descartes' explanation of the universe as extended matter in motion. He gives an interpretation of force as the very foundation for an understanding of both the physical and spiritual universe. What is real in the universe is activity; the essence of substance is action, not extension as Descartes had insisted. This activity is constituted as a primitive force or a striving toward change; it is the innermost nature of a body. The basic indivisible substances whose essence is a continual tendency toward action were later, in 1714, called monads by Leibniz.[44] These units of primitive force can neither be created nor destroyed naturally, and all must begin simultaneously and be annihilated at once.

[41] *Ibid.*, p. 670. The complete argument reads: Now when the parts of the bodies absorb the force of the impact as a whole as when two pieces of rich earth or clay come into collision, or in part as when two wooden balls meet, which are much less elastic than two globes of jasper or tempered steel; when I say some force is absorbed in the parts, it is as good as lost for the absolute force and for the respective velocity, that is to say for the third and the first equation which do not succeed, since that which remains after the impact has become less than what it was before the impact, by reason of a part of the force being turned elsewhere. But the quantity of progress or rather the second equation is not concerned therein.... But in the semi-elastics, as two wooden balls, it happens still further that the bodies mutually depart after the impact, although with a weakening of the first equation, following this force of the impact which has not been absorbed.... But this loss of the total force, or this failure of the third equation, does not detract from the inviolable truth of the law of the conservation of the same force in the world. For that which is absorbed by the minute parts is not absolutely lost for the universe, although it is lost for the total force of the concurrent bodies.

[42] For an evaluation of Leibniz's statement see Hiebert, *Historical Roots*, pp. 88–90:

In these passages Leibniz apparently postulated an inner *force* of motion for the invisible smallest parts of bodies. These smallest parts were thought to acquire the kinetic force lost by bodies for inelastic deformable collisions. Leibniz also assumed this inner force to be equivalent to the external force of motion, since he stated that the total *force* remains unchanged for the universe even for inelastic collisions. There is I believe no statement in Leibniz which would lead one to credit him with either observation or knowledge of the fact that this phenomenon is accompanied by heat changes. Nevertheless by this time it was common belief especially among philosophers that heat was due to or synonymous with the motion of the smallest parts of matter.

[43] Gottfried Wilhelm Leibniz, "Specimen dynamicum," Loemker, Vol. II, pp. 711–738. In "Specimen dynamicum" Leibniz again attempts refutation of the Cartesian principle of "force." In an argument based on the $m|v|$ and mv^2 acquired by two pendula of equal length but different mass, Leibniz argues that perpetual motion could arise if Descartes' measure of "force" is accepted. To achieve a mechanical perpetual motion machine, an interacting mechanical system would be necessary. Again no such system is specified by Leibniz. The argument is based on the mental substitution of an equipollent body. See pp. 724–727.

[44] Gottfried Wilhelm Leibniz, "The Monadology" and "The Principles of Nature and of Grace," Loemker, Vol. II, pp. 1044–1046 and 1033–1034. In his discussions on physics Leibniz conformed the language of his philosophical system to that of ordinary speaking. Thus all these points on the level of physics have a counterpart in Leibniz's system of monads, or souls, in which there is no real space or motion and in which there is no real communication of motion. For each case of impact in the world of phenomena there is a counterpart in the real world of monads which consists in the heightening and diminution of the states of perception of infinite numbers of monads. All of this takes place in accordance with the system of pre-established

Thus the conservation of substance and force form the basis of Leibniz's philosophical viewpoint. Since time and space are neither realities nor substances, but merely relations, motion which is the continuous change in both space and time is likewise only a relation.[45] What is real in motion is force, a momentary state which carries with it a striving toward a future state. It is therefore clear why motion and extension cannot be the essence of reality for Leibniz, as they were for Descartes.

We are thus able to view Leibniz's attack on the Cartesian measure of force as primarily an attempt to establish his own philosophical system based on the conservation of "force" and to place less emphasis on a simple attempt to substitute the mathematical formula mv^2 for the formula $m|v|$. This latter aim is encompassed in the more general purpose of the former. Perhaps his insight into a universe which was fundamentally energistic led him to make assumptions about the possibility of transferring that energy and to identify implicitly the conservation and the measure of force, the establishment of both being an integral part of his ultimate aim.

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harmony. For Leibniz's discussion of this problem see *Correspondence with Arnauld.* (*Discourse on Metaphysics* and *Correspondence with Arnauld,* La Salle: Open Court, 1957):

> Thus the souls change nothing in the ordering of the body nor do the bodies effect changes in the ordering of souls (and it is for this reason that forms should not be employed to explain the phenomena of nature). One soul changes nothing in the sequence of thought of another soul, and in general one particular substance has no physical influence upon another; such influence would besides be useless since each substance is a complete being which suffices of itself to determine by virtue of its own nature all that must happen to it. [P. 153.]

> . . . all the phenomena of the body can be explained mechanically or by the corpuscular philosophy in accordance with certain assumed mechanical principles without troubling oneself whether there are souls or not. In the ultimate analysis of the principles of physics and mechanics, however, it is found that these assumed principles cannot be explained solely by the modifications of extension, and the very nature of force calls for something else. [P. 163.]

> Nevertheless, we have the right to say that one body pushes another; that is to say, that one body never begins to have a certain tendency excepting when another which touches it loses proportionally, according to the constant laws which we observe in phenomena; and since movements are rather real phenomena than beings, a movement as a phenomenon is in my mind the immediate consequence of effect of another phenomenon, and the same is true in the mind of others. The condition of one *substance*, however, is not the immediate consequence of the condition of another particular substance. [P. 183.]

[45] Gottfried Wilhelm Leibniz, "Clarification of Bayles' Difficulties," Loemker, Vol. II, p. 806.