Fractal Representation of Images via the Discrete Wavelet Transform

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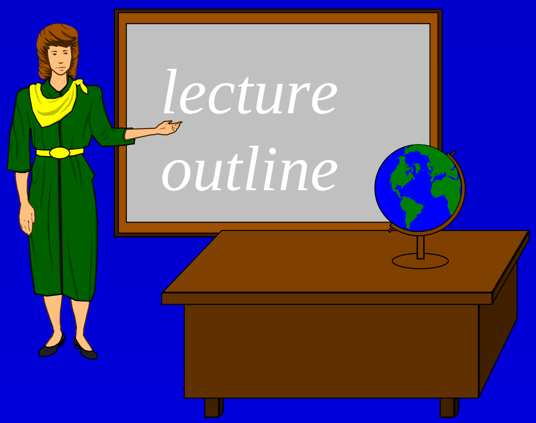
Lecture Outline :



- Fractal image representation - a review
- Image coding with *DWT* - a brief review
- Fractal representation via the *DWT*
- A Blockless Fractal Coder
- Image Super-resolution (if time permits)
- Summary and proposals for further research

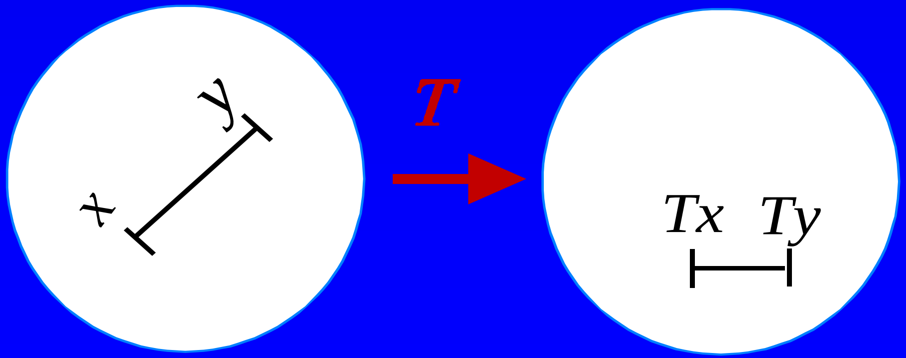
Fractal Image Coding

a Review

Contractive Transformations :

- <u>Definition</u> $T:(E,d)\rightarrow(E,d)$ is contractive iff $\forall x,y\in(E,d)$:

$$d(Tx,Ty)\leq s\cdot d(x,y)\qquad 0\leq s<1$$



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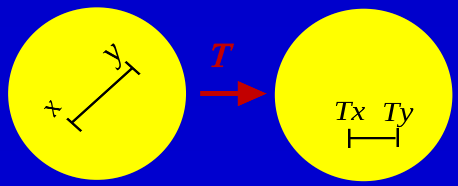
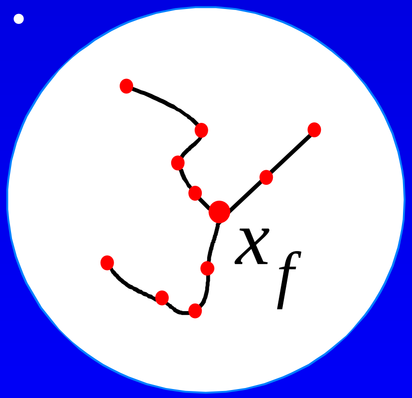


- There ***exists*** a ***unique*** Fixed-Point x^f :

  $$T\{x^f\} = x^f$$

- x^f can be iteratively obtained by :



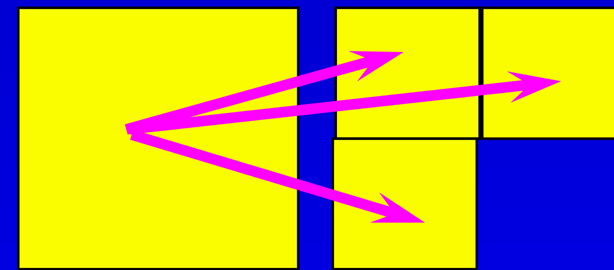
$$x^f = \lim_{n \to \infty} T^n\{x_0\} \qquad \forall x_0$$

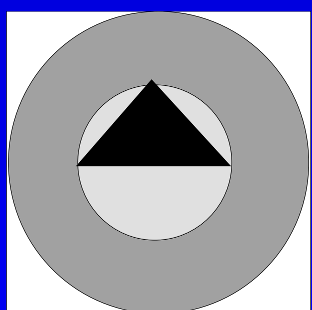
- Fractal representation for x :

  $$T : x^f\{T\} \approx x$$

Fractal Generation

Iterated Function systems (IFS)





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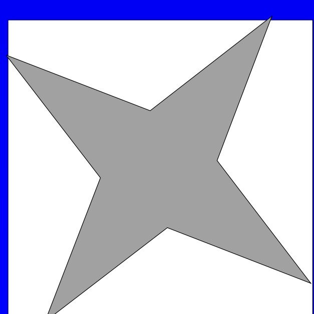

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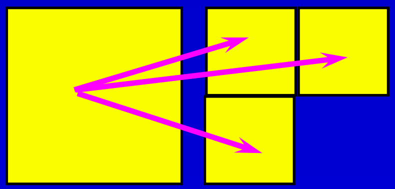
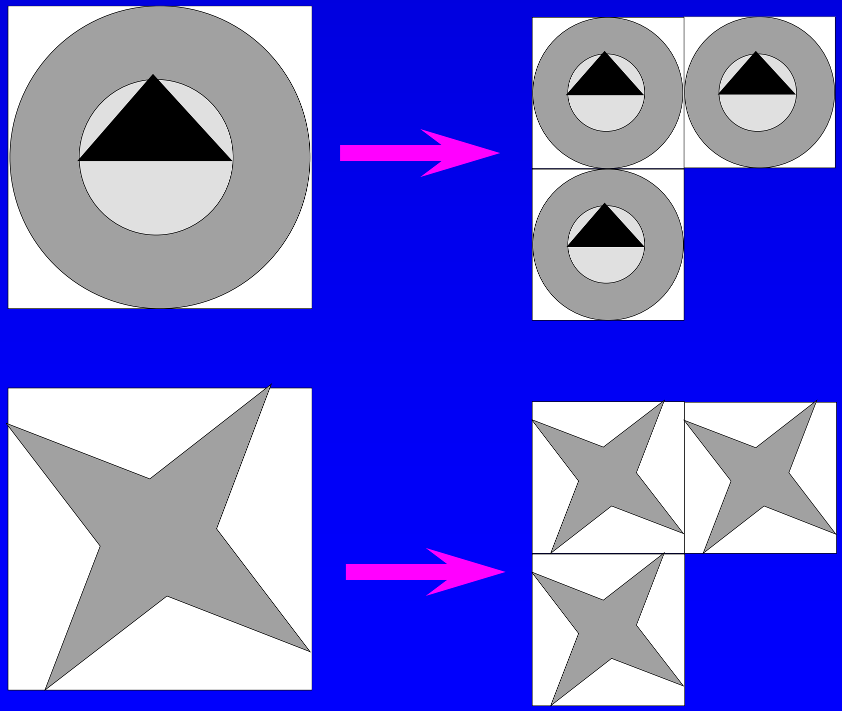
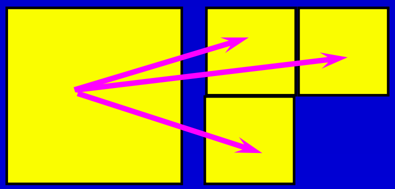
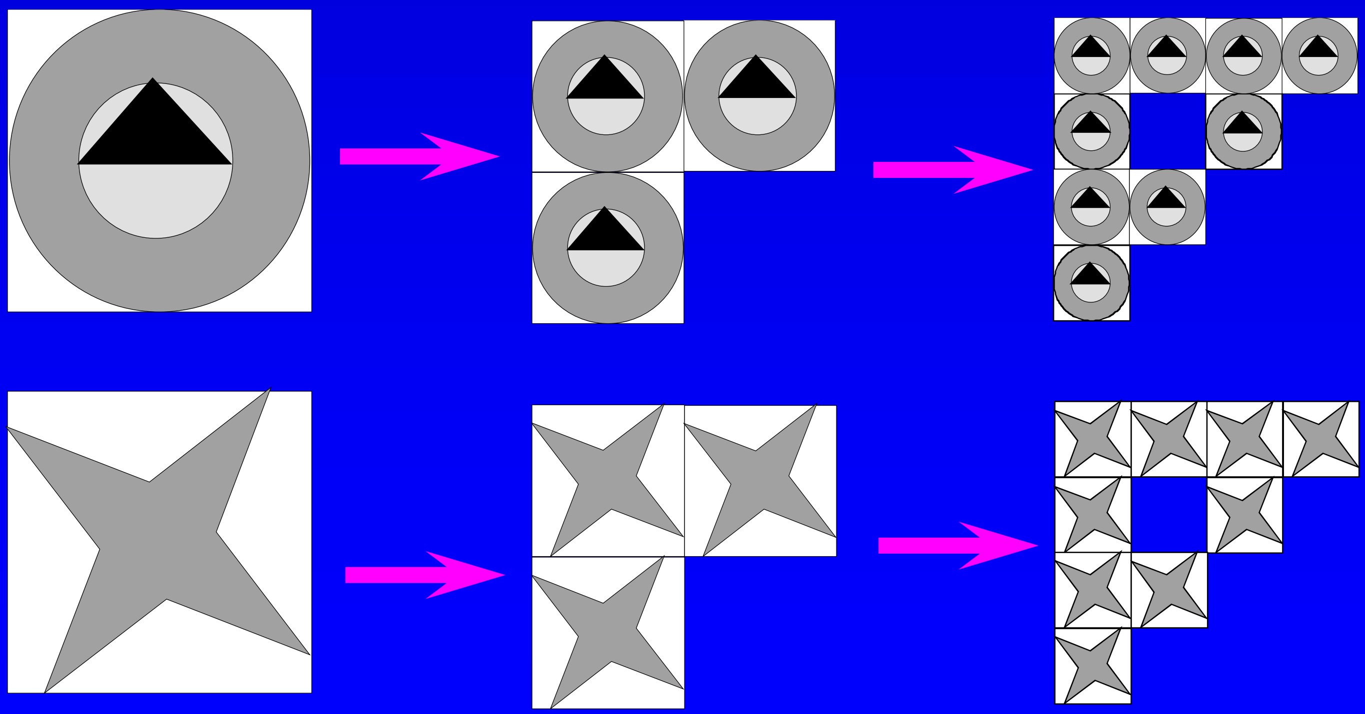
Fractal Generation ... Cont'd



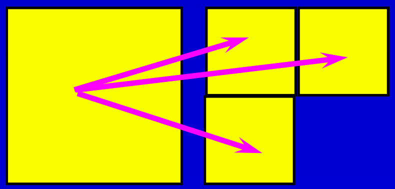


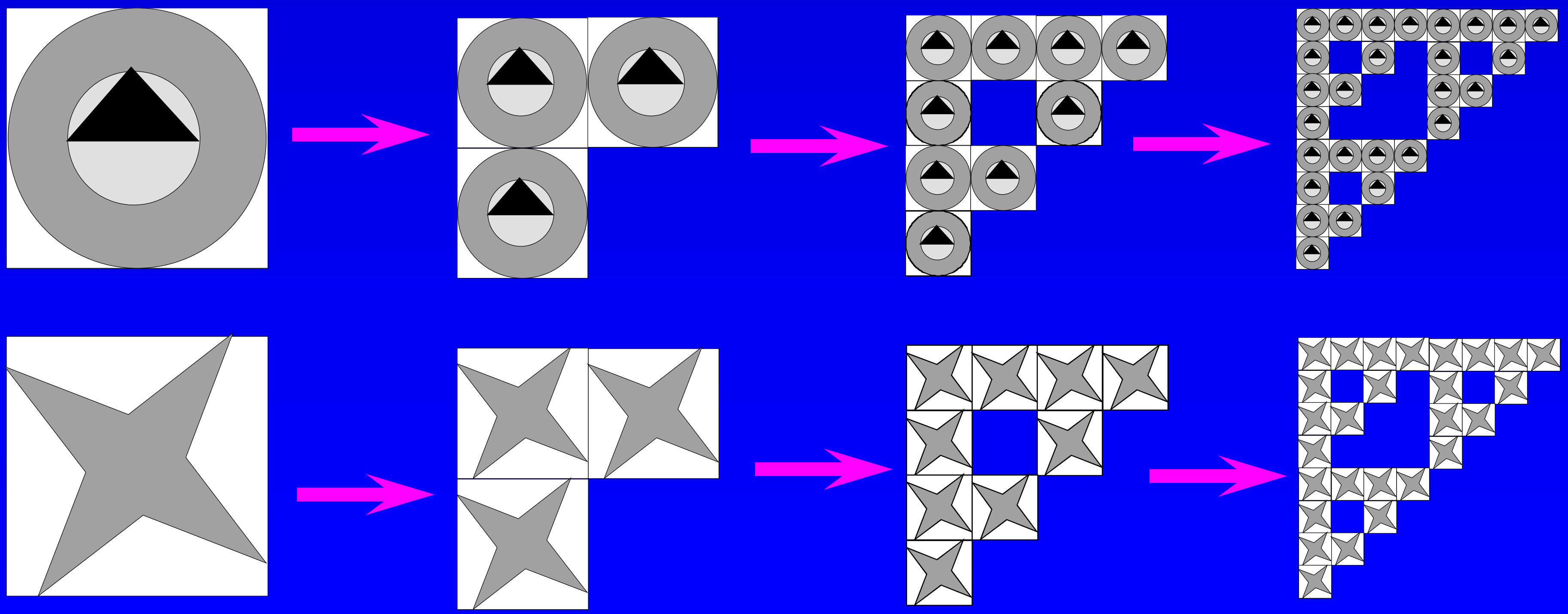
Fractal Generation ... Cont'd



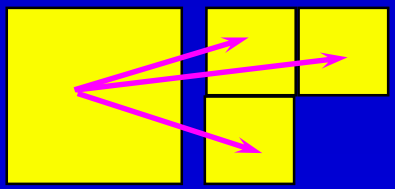


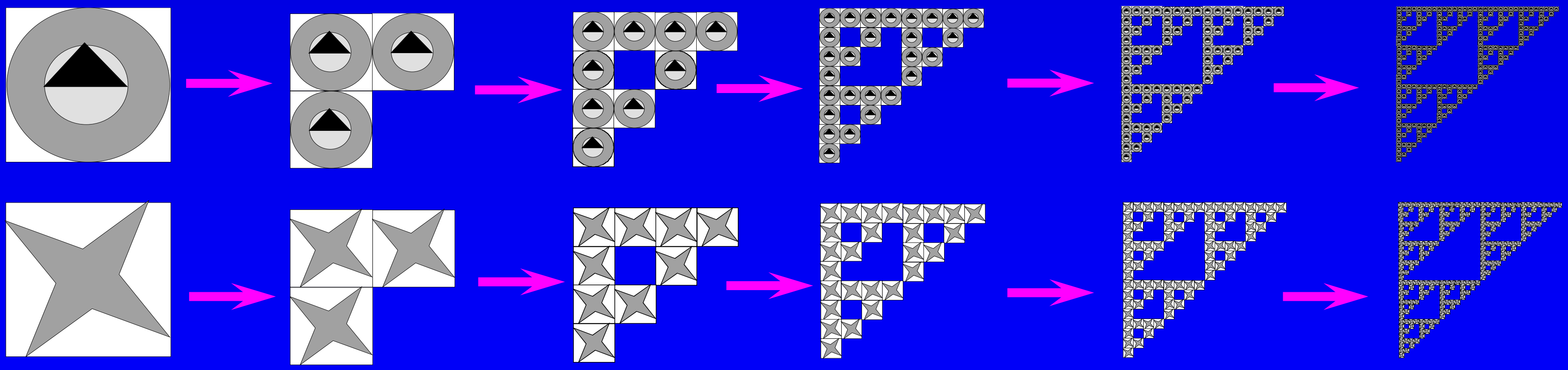
Fractal Generation ... Cont'd





Fractal Generation … Cont'd



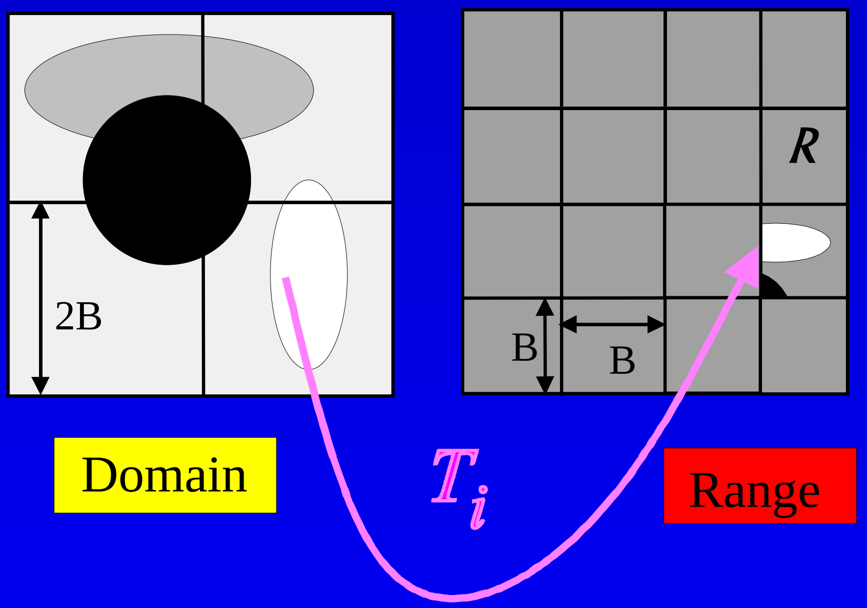


Fractal Image Coding [Jacquin 1989]

- Let

$$X = \bigcup_i R_i \; ; T = \bigcup_i T_i$$

- Find a Contractive *T* such that $x^f \approx x$

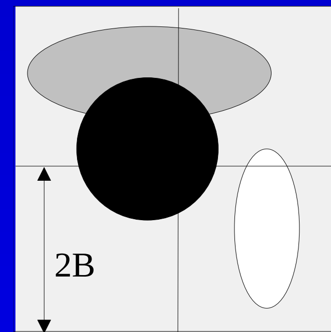


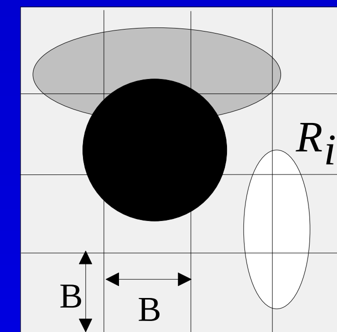
- <u>*Collage theorem*</u> : $d(x, x^f) < \frac{1}{1-s} d(x, Tx)$

- <u>Result</u> : find *T* that minimizes $d(x,Tx)$

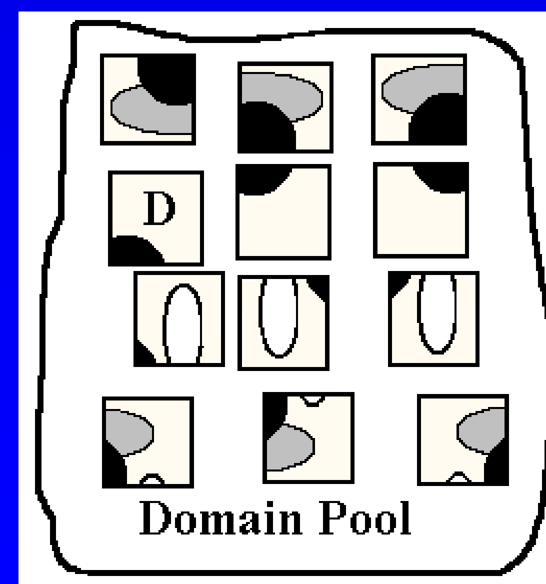
The Conventional Image Fractal Coder

- Extract *domain pool*
  - – I_p = Isometries
  - – φ = Scaling Function

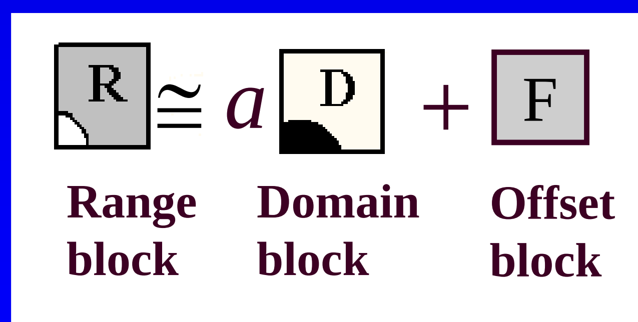




$$Dj = \varphi\{I_p\{\text{BLOCK}_{2\text{B}\times 2\text{B}}\}\}$$



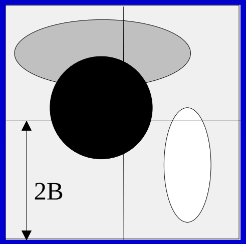
- For each R_i :

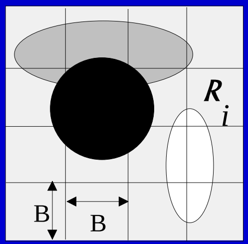


$$T_i = \{a_i, b_i, j_i\} \quad (b_i \cdot 1 = F)$$

$$= \arg\min \left\| R_i - (aD_j + F) \right\|_2^2$$

Encoding - ...Cont'd



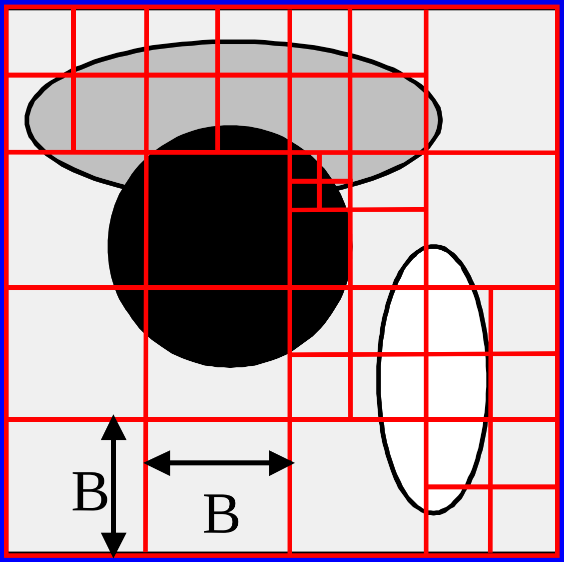


- $T = \bigcup_i T_i$ i.e. $T\{X\} = \bigcup_i R_i$
- T is contractive if $a_i < 1$

Quad-Tree approach

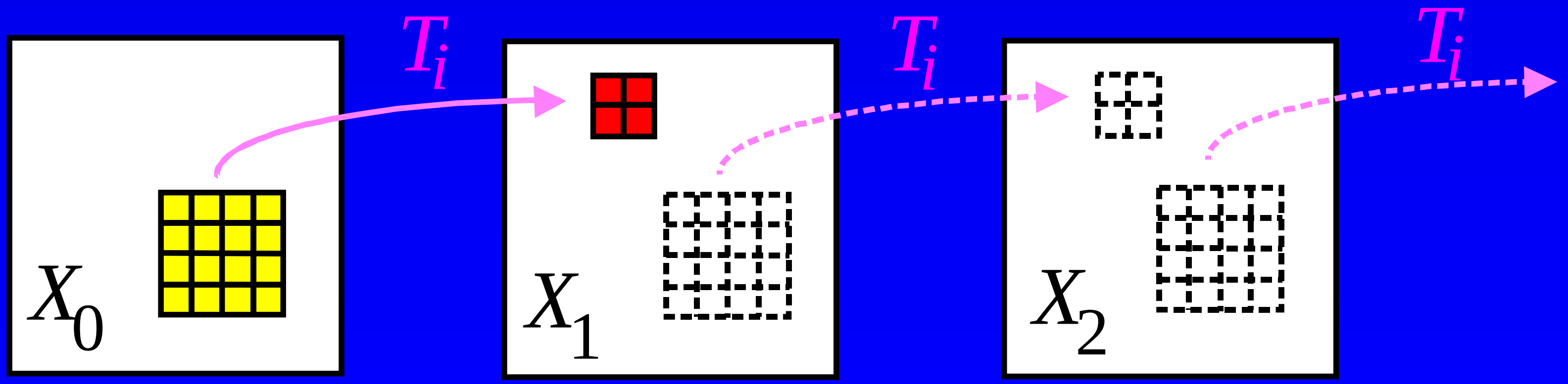
- *range blocks* of

different sizes [*Fisher* 92]



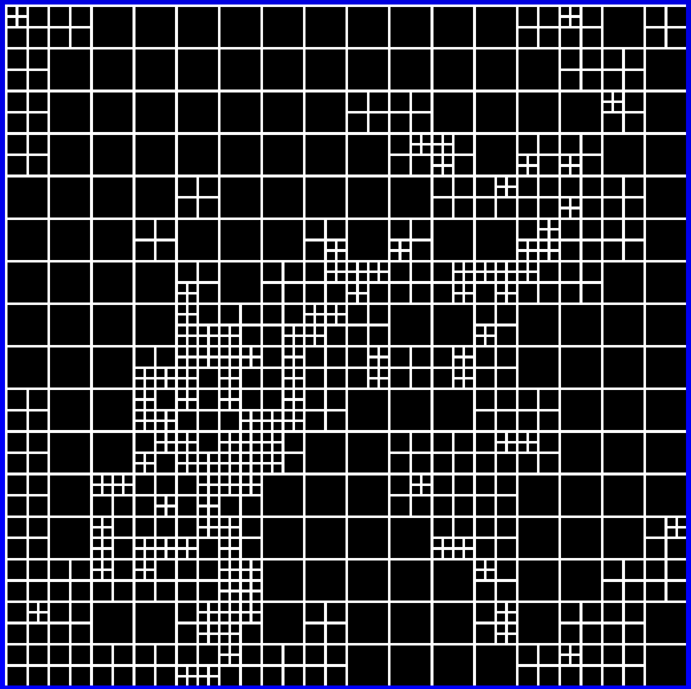
Finding the Fixed Point (*Decoding*)

- Start with any image X_0 .
- Apply T iteratively until the result converges .



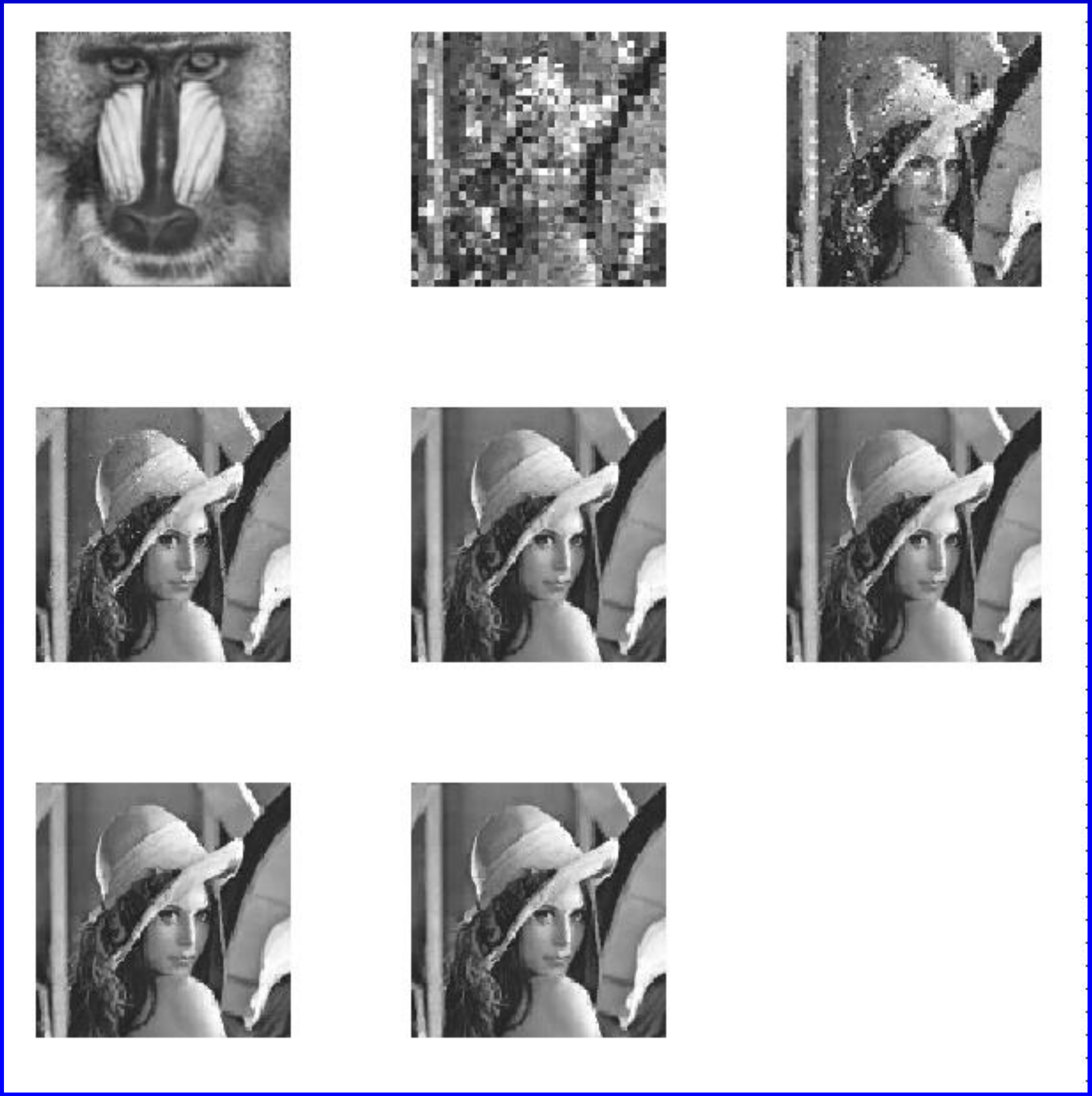
Example

T is found to represent "Lena"



Decoding - starting from "man"

Decoding - starting from "baboon"



Fixed Point Pyramid [Baharav et. al. 1993]

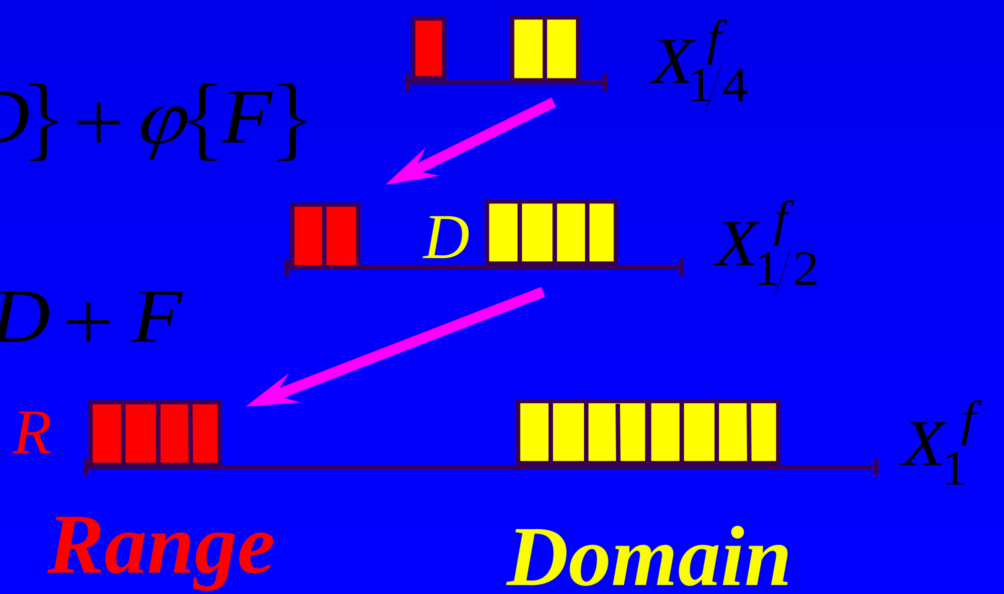
- $$X^f_{1/2^k} = \varphi\left\{X^f_{1/2^{k-1}}\right\} \; ; \quad k = 1, 2, \ldots, \log_2(\mathrm{B})$$

$$X^f_1 = T_1\left\{X^f_1\right\} \Rightarrow X^f_{1/2^k} = T_{1/2^k}\left\{X^f_{1/2^k}\right\}$$

- $$\mathrm{B}'\left\{T_{1/2^k}\right\} = 1/2^k \cdot \mathrm{B}$$

$$\varphi\{R\} = a\,\varphi\{D\} + \varphi\{F\}$$

$$R = aD + F$$



DC- Orthogonalization [Øien et. al. 1993]

A "conventional" T_i

$$R_i \approx a_i \cdot D_j + F_i \Rightarrow T_i = \{a_i, b_i, j\}$$

offset

T_i with *DC orthogonalization*

mean

$$R_i \approx a_i(D_j - \overline{D_j}) + \overline{R_i} \Rightarrow T_i = \{a_i, \overline{R_i}, j\}$$

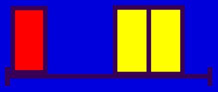
- Finite # of iterations
- Fast decoding (Combining Baharav & Øien) .../

...*Cont'd*

$$T_i = \{a_i, \overline{R}_i, j\}$$

- Fast Decoding

$$X^f_{1/4} = \overline{R}$$



...*Cont'd*

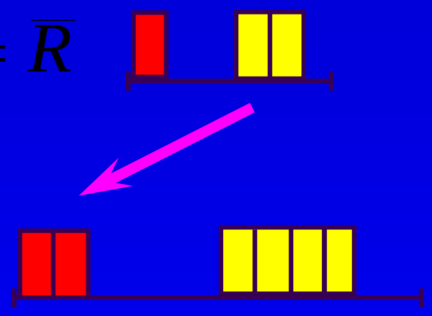
$$T_i = \{a_i, \overline{R}_i, j\}$$

- Fast Decoding

$$X^f_{1/4} = \overline{R}$$

$$B' = B/2 \ \ (T' = T_{1/2})$$

$$X^f_{1/2}$$



...Cont'd

$$T_i = \{a_i, \bar{R}_i, j\}$$

- Fast Decoding

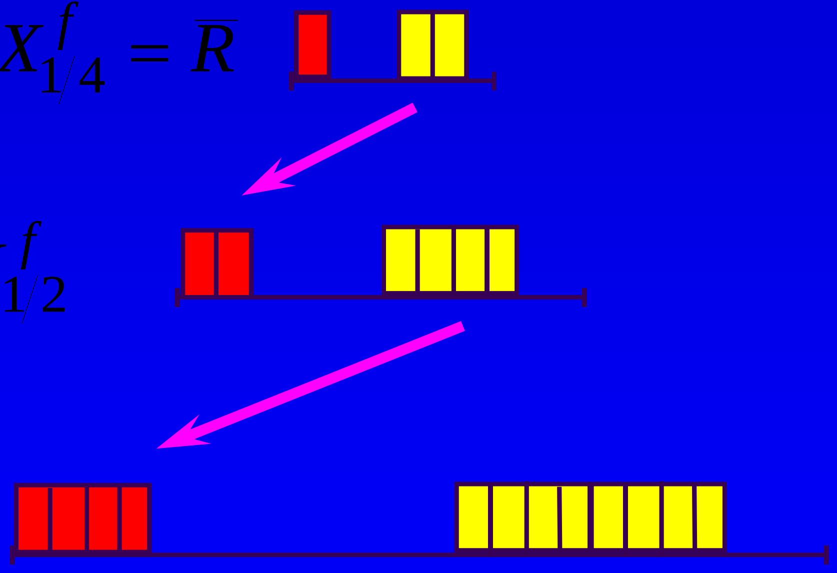
$$X^f_{1/4} = \bar{R}$$

$$B' = B/2 \quad (T' = T_{1/2})$$

$$X^f_{1/2}$$

$$B' = B \quad (T' = T)$$

$$X^f_1$$

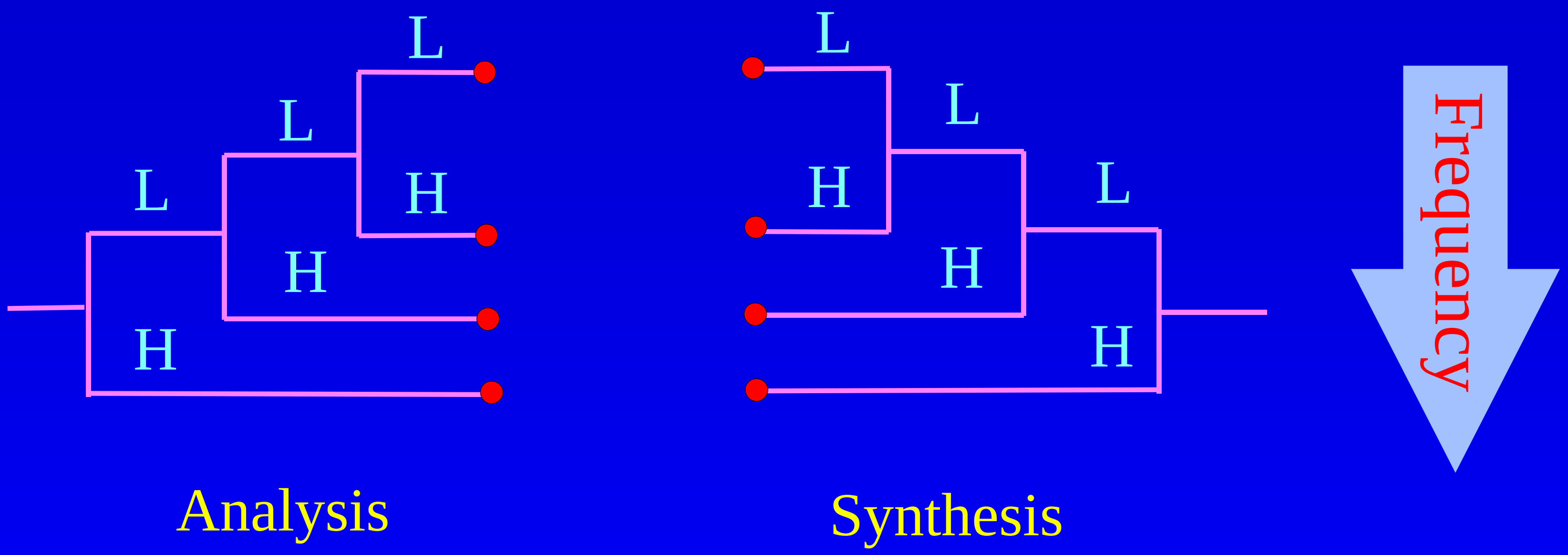


Discrete Wavelet Transforms (DWT)

Discrete Time

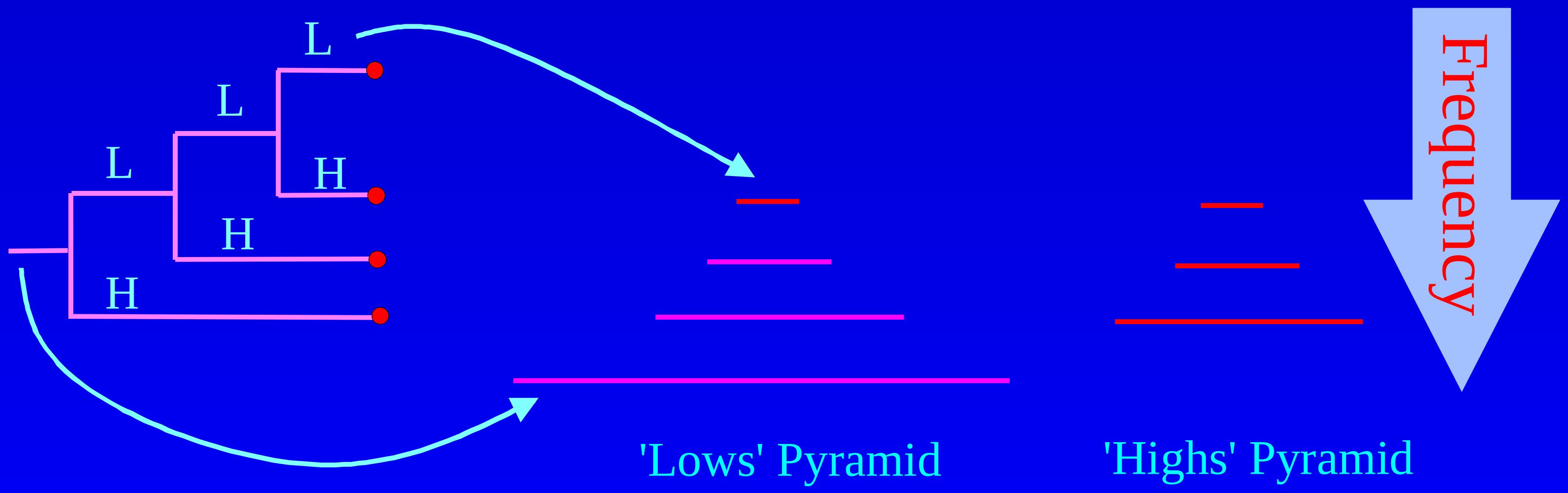
Octave-Band Subband Decomposition

Octave-Band Subband Decomposition

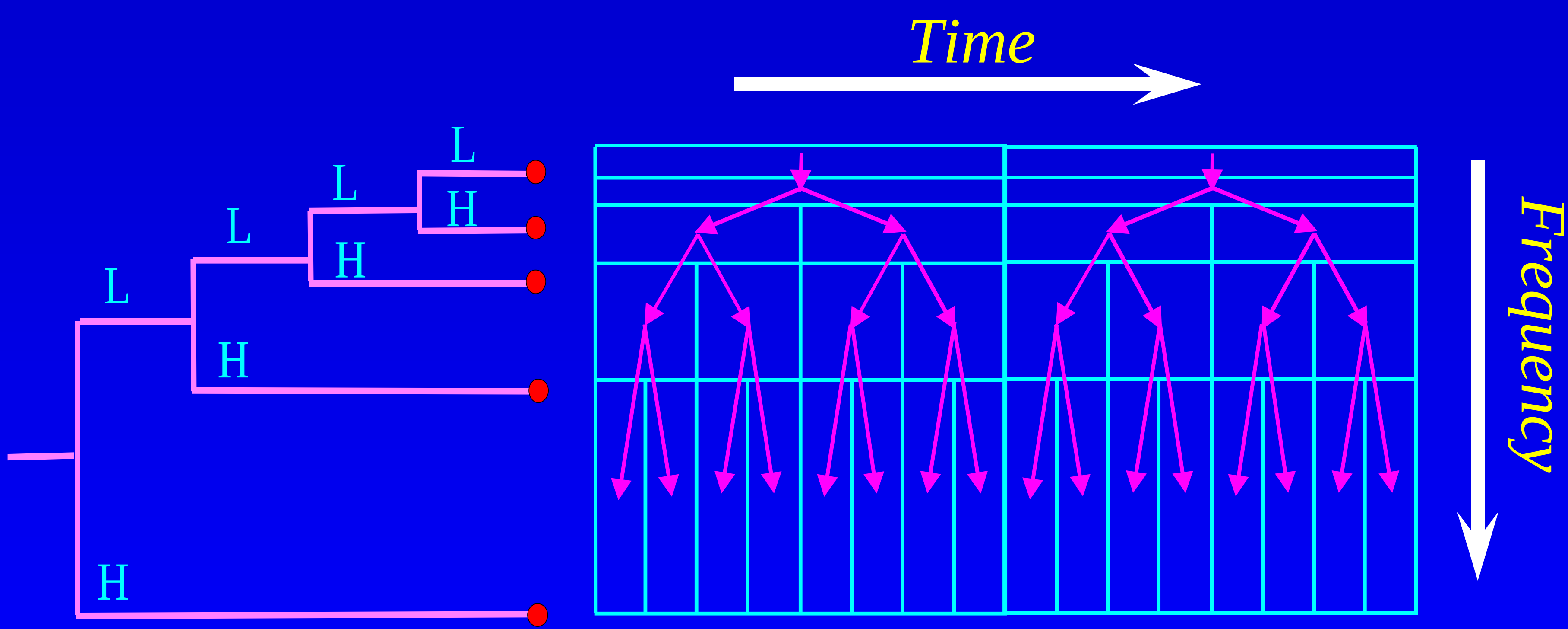


- Orthonormal Transformation
- Linear Phase (QMF) Vs. Perfect reconstruction

A Pyramidal Description



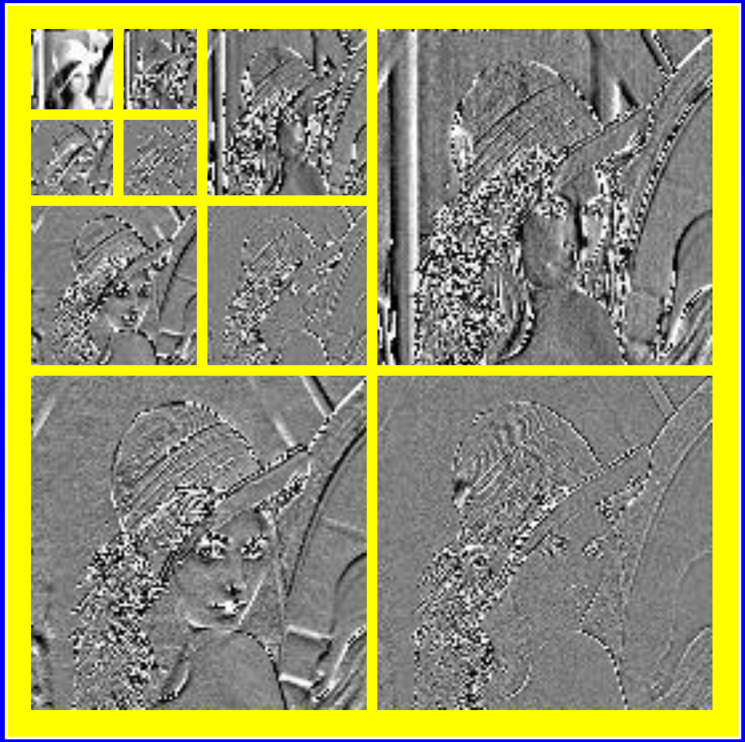
Wavelet Subtrees

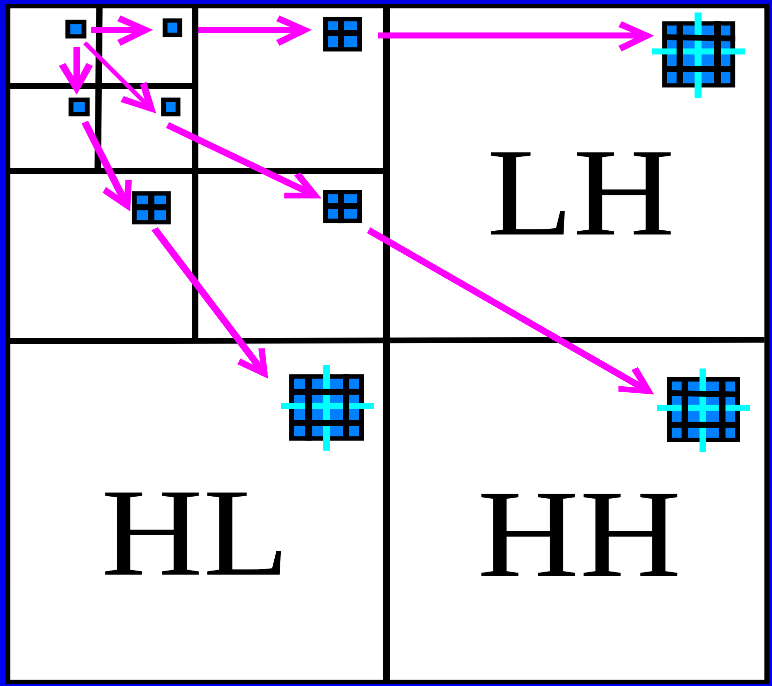


- The Sub-tree coefficients represent "a region".
- Every "*father*" has 2 "*sons*"(LP "*root*" has 1 "*son*") .

Wavelet Transforms in 2D

- Using A *separable QMF*.
- The LP "*root*" has three "*sons*" in three directions.
- Every other "*father*" has 4 "*sons*".





Wavelet Transforms in 2D

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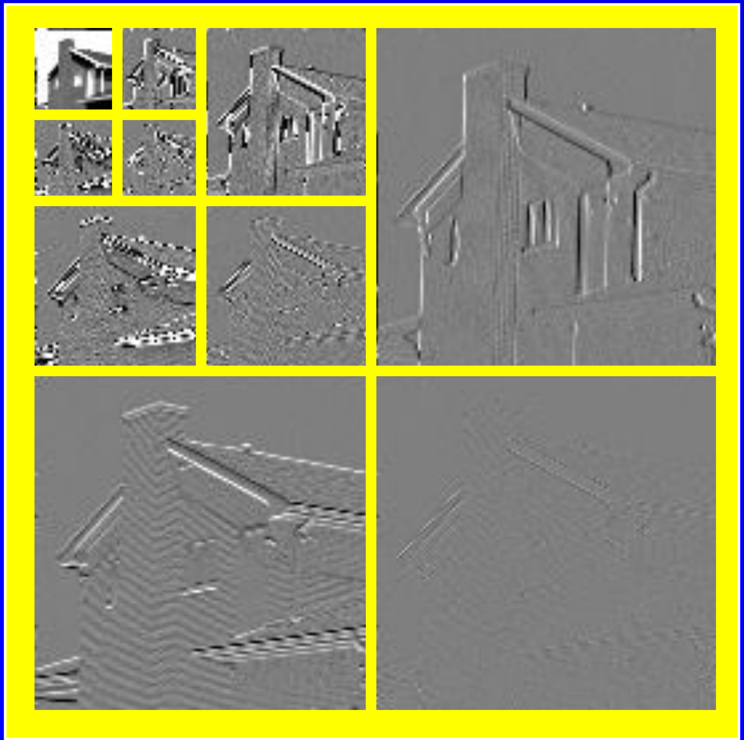


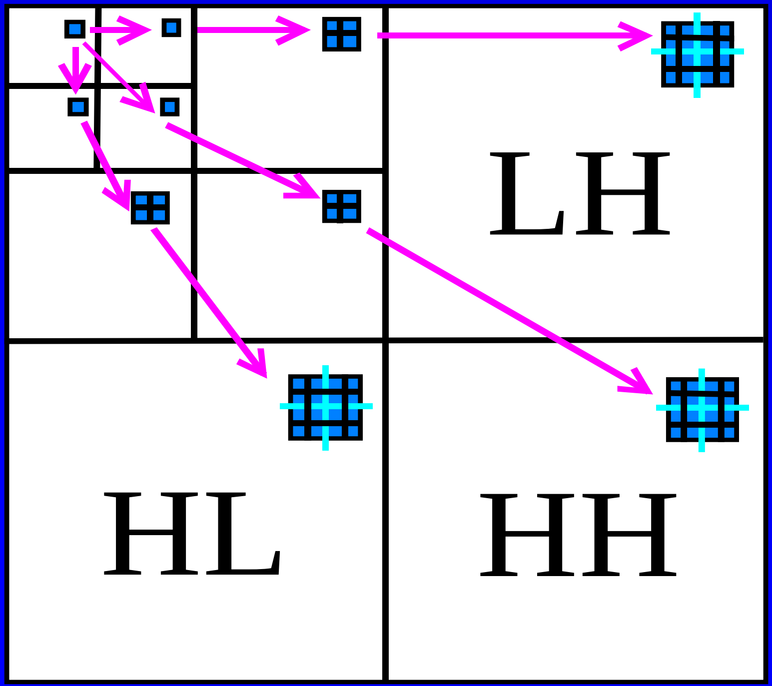


Image coding with wavelets

- Transformation (Filters, Wavelet Packets ...).
- Quantization (scalar, vector) and Entropy coding.
- Prediction of non-significant subtrees [*Shapiro..*].
- Prediction of coefficients

  [*Pentland, Rinaldo & Calvagno...*].

Fractal Transformations

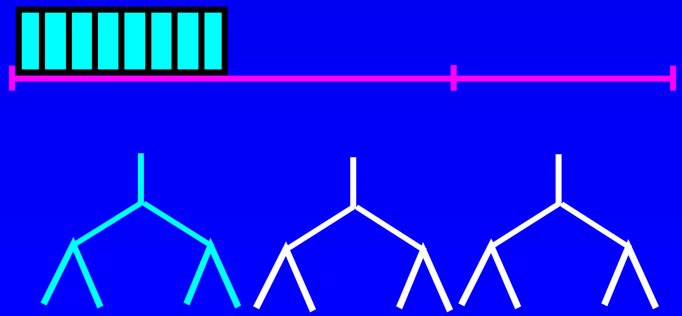
in the

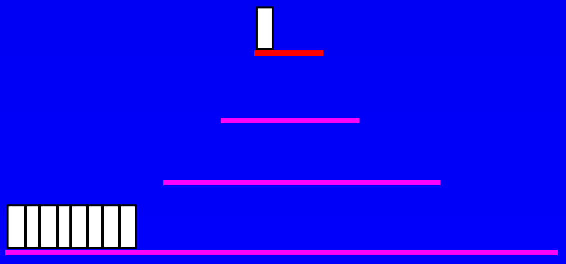
Haar-DWT Domain

The Haar-DWT

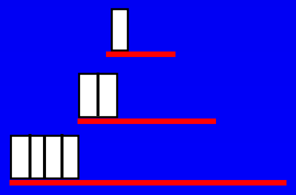
$$H_{lp} = \frac{1}{\sqrt{2}}[1\ 1]\ ;\ H_{hp} = \frac{1}{\sqrt{2}}[1\ -1]$$

- A "l" deep sub-tree is the *DWT* of a 2^l - samples input segment.
- The 'Lows' Pyramid upper level coefficients are the segments' means (up to a constant).





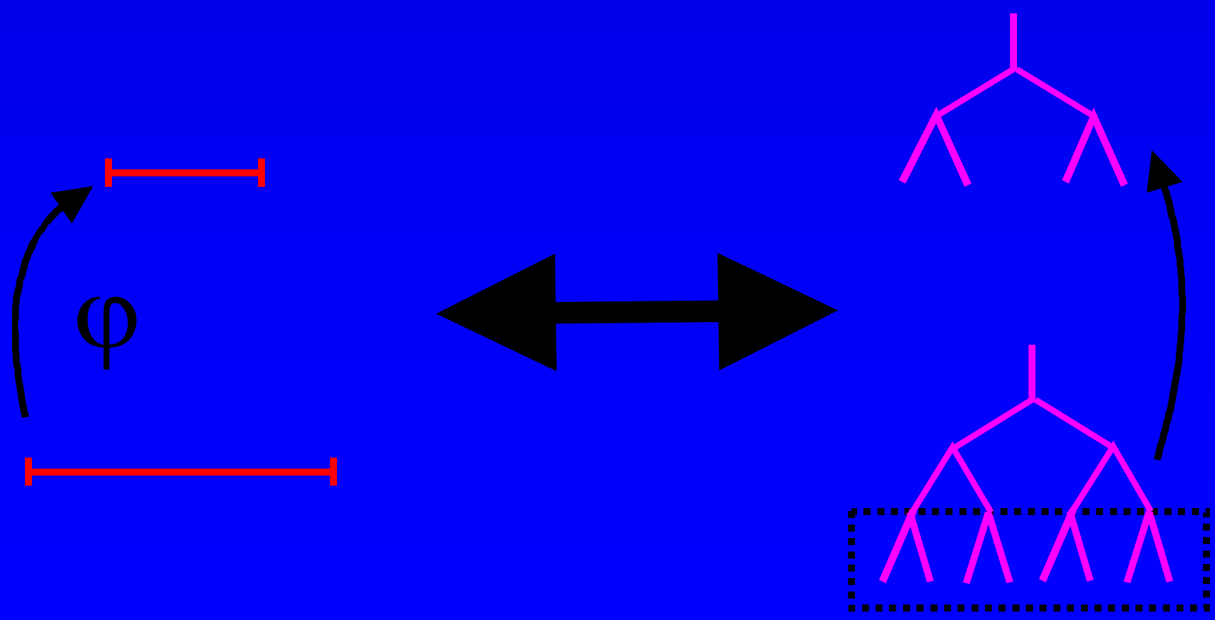
'Lows' Pyramid



'Highs' Pyramid

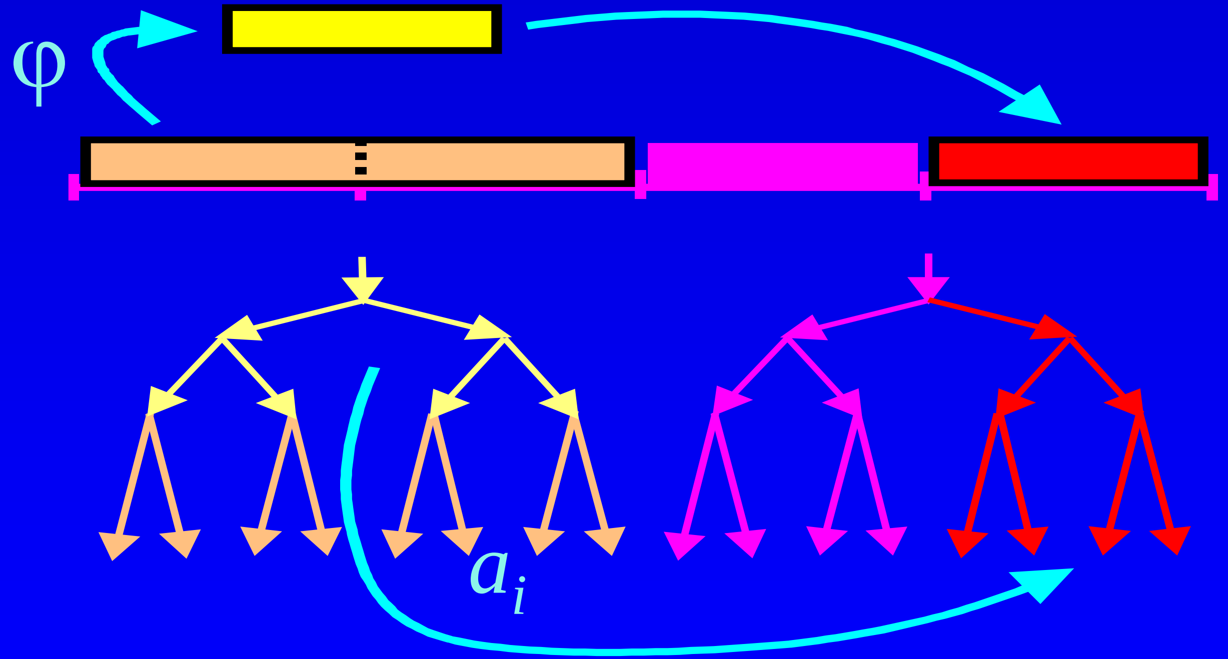
The Haar-DWT of a Fixed Point

Image Plane	*DWT* Domain
Down-scaling a block with φ $(=H_{lp}\downarrow 2)$	Pruning its subtree leafs



The Haar -DWT of a Fixed Point

$$a_i(D_j - \overline{D_j}) + \overline{R_i} = R_i$$



$a_i \cdot$ Domain Subtree = Range Subtree

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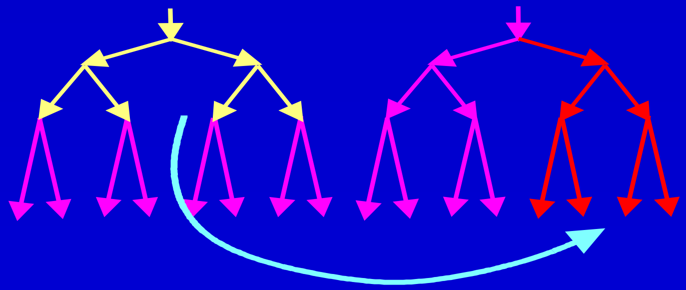
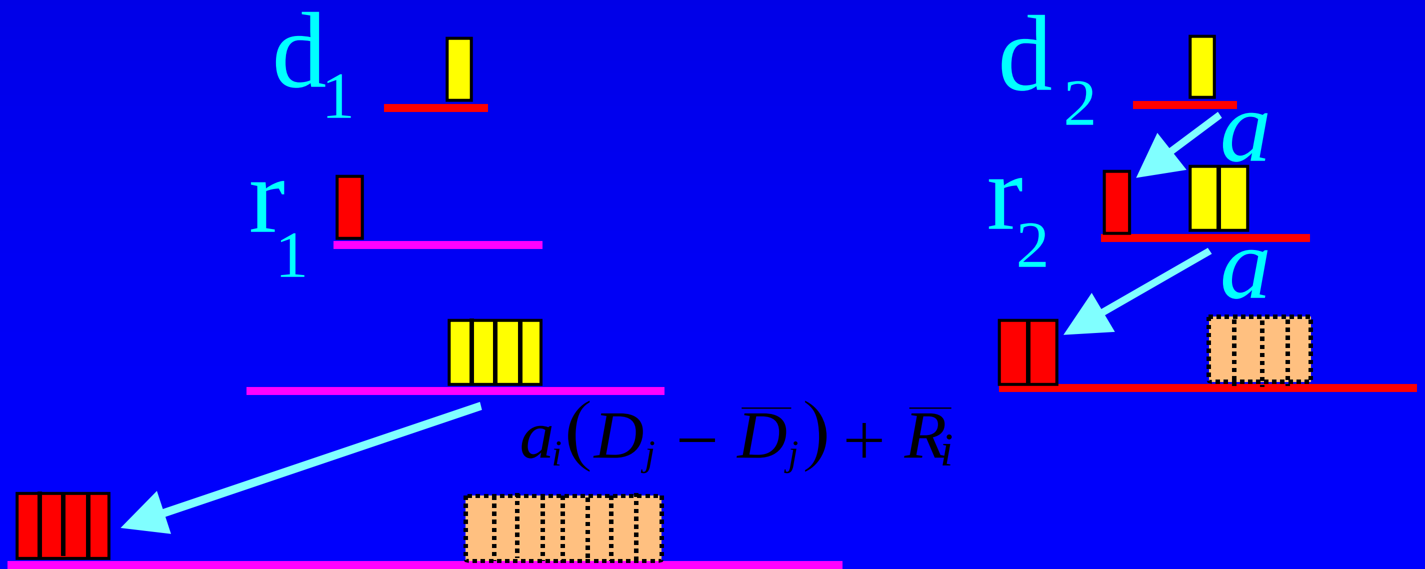


U = *Haar-DWT* unitary matrix (size BxB)

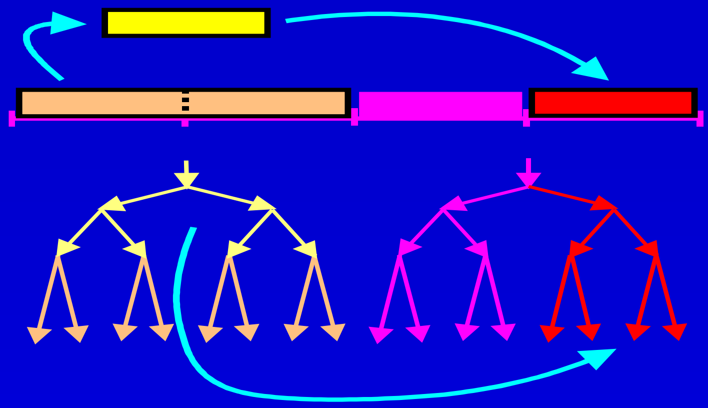
$$[\mathrm{r}_1,\ldots\mathrm{r}_B]^t = UR \; ; \; [\mathrm{d}_1,\ldots\mathrm{d}_B]^t = UD$$

$$R = a\left(D - \overline{D}\right) + \overline{R} \;\Rightarrow\; \mathrm{r}_k = a \cdot \mathrm{d}_k \quad 2 \le k \le \mathrm{B}$$

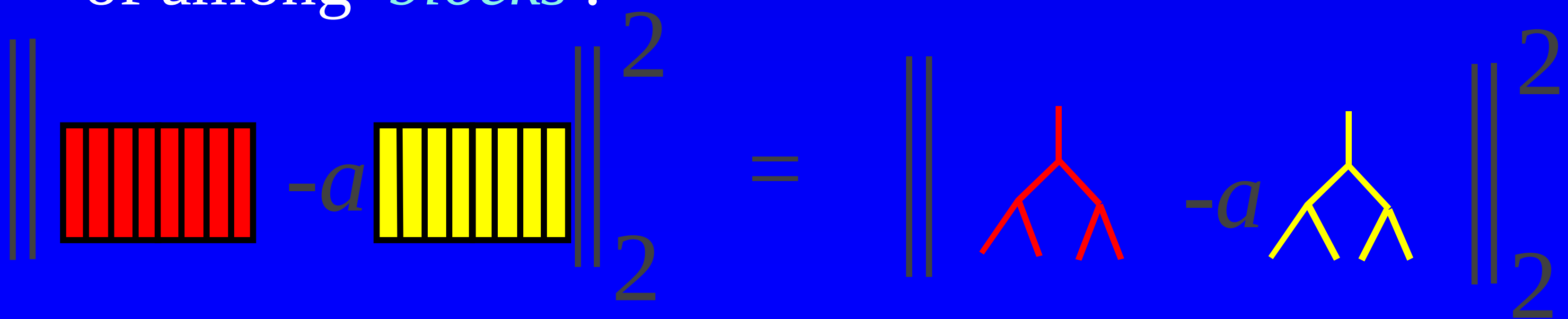


'Lows' Pyramid 'Highs' Pyramid

Domain-Pool Search

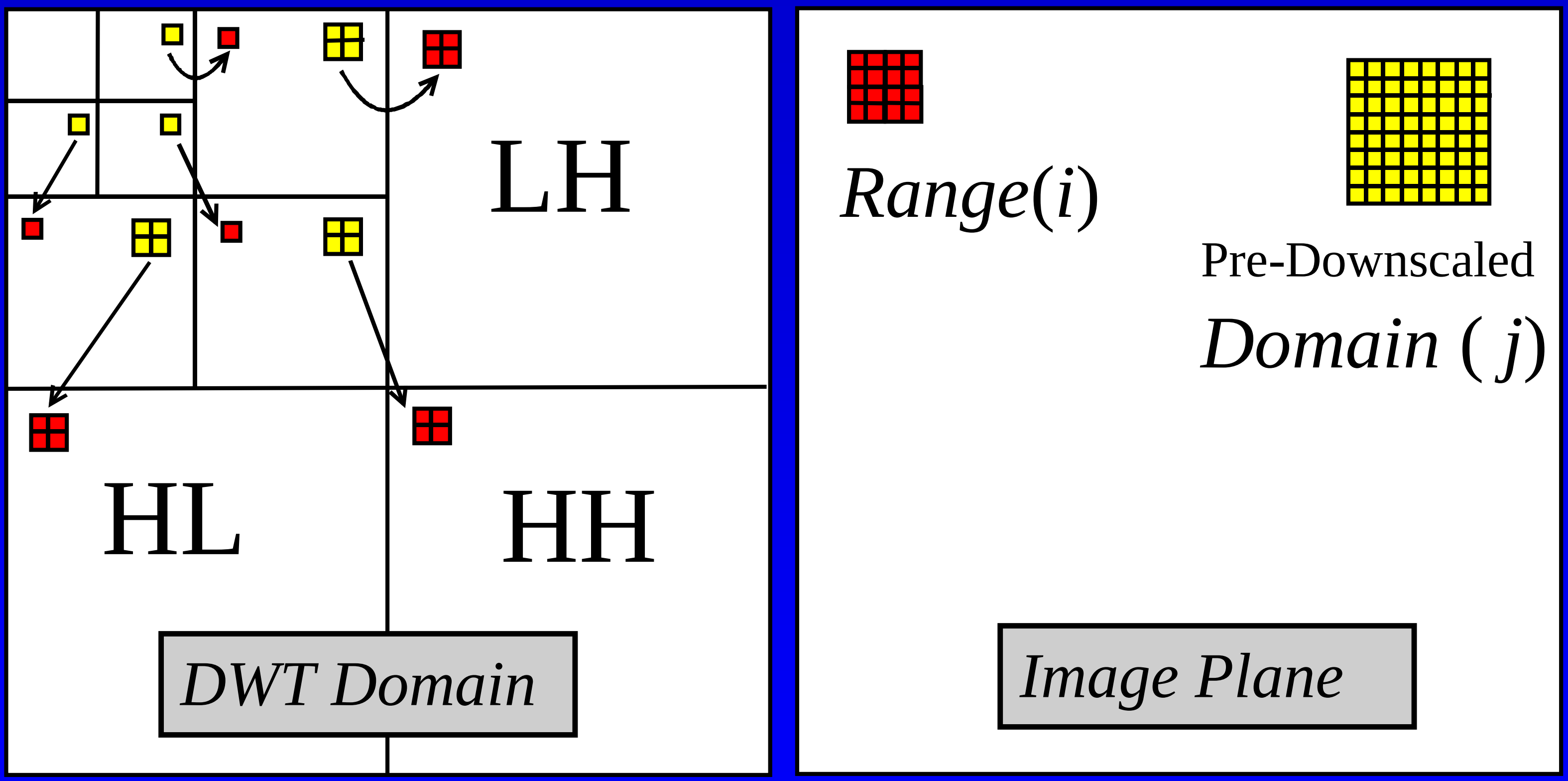


- Finding the best *Domain block* and *scaling factor ‘a’* :
  - Assume *l_2 norm*.
  - Recall that the *DWT* is orthonormal.
  - The search can be done among *subtrees* instead of among *blocks* .

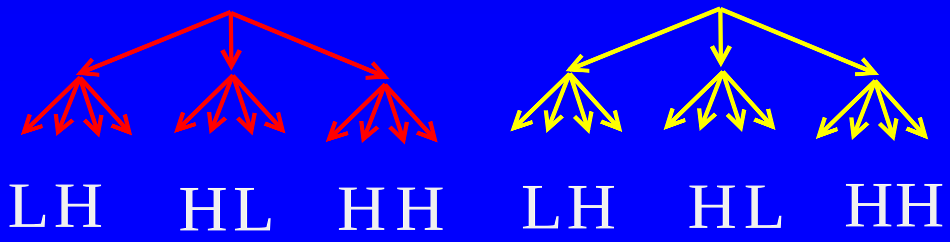


- Weighted Least Squares.

The 2D Case



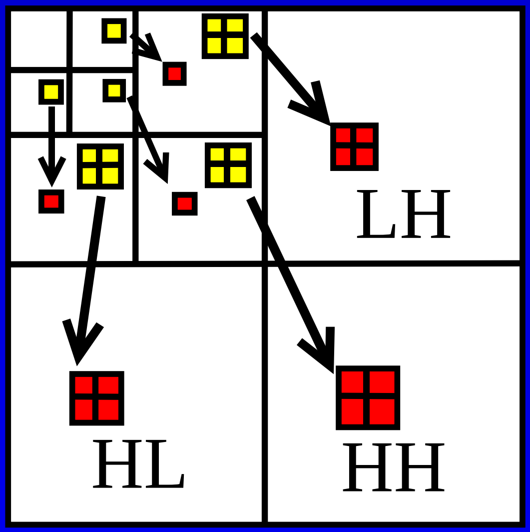
LH
HL
HH
DWT Domain
Range(i)
Pre-Downscaled
Domain (j)
Image Plane



LH
HL
HH
LH
HL
HH

An Equivalent Coder - Summary

- The Encoder
  - Calculate the *DWT* of the image
  - Construct *domain pool* of *Subtrees*
  - For each *range Subtree* Find the best match (*index* and *scaling factor*)



- The Decoder
  - Calculate the higher bands of coefficients recursively from the lower band
  - Calculate the *Inverse DWT*.

… *Cont'd*

Image Plane	Haar-*DWT* Domain
Flips and Rotates	Reordering of coeffs
Quadtree block splitting	Subdividing a subtree

A blockless Fractal Coder

Changing the Wavelet Filters

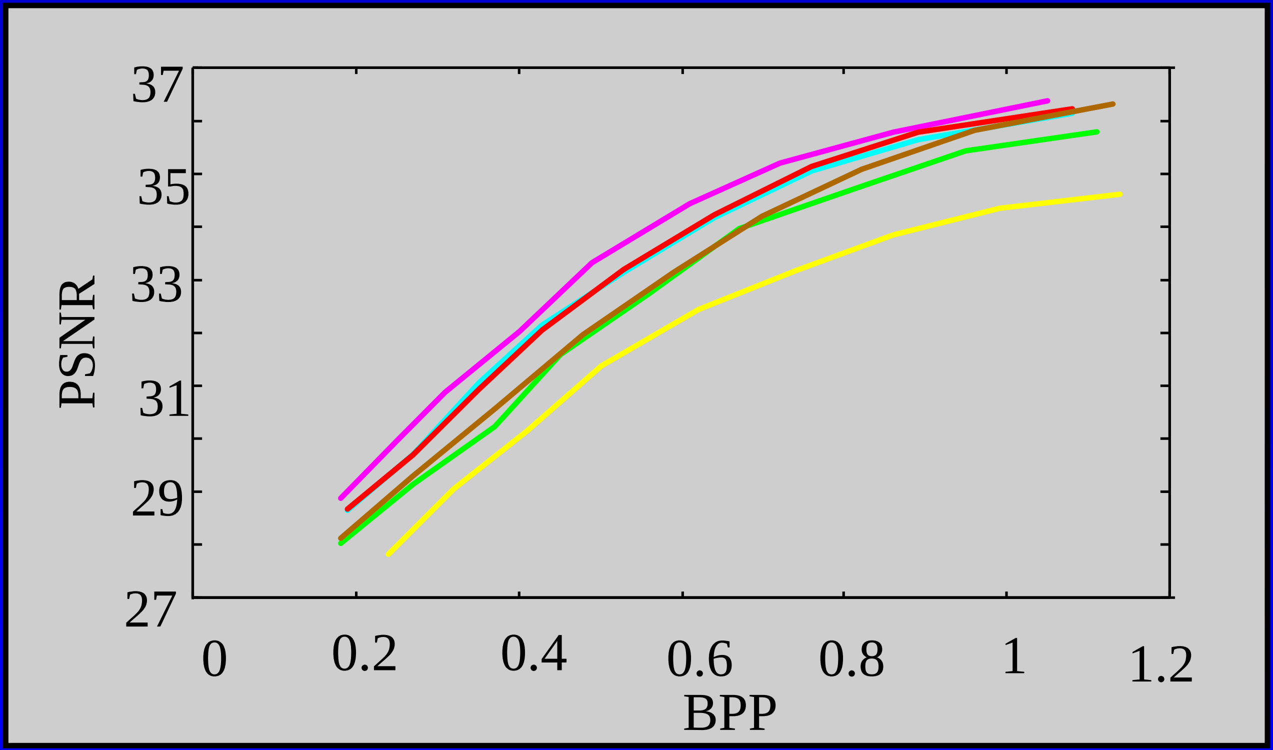
DWT-IFS with QMFs other then Haar

Haar	Others
Subtrees represent non overlapping blocks	Subtrees represent overlapping blocks
The IFS uses DC block orthogonalization	The IFS uses signal LP orthogonalization

The Choice of Wavelet Filters

- Zero Phase ⇒Keep the Subbands aligned
- Symmetric filters ⇒*Symmetric extensions* for finite duration signals
- Short and compact but with decaying ends
- Perfect reconstruction is not necessary
- Orthonormality

Results - Quadtree Approach



--- Haar (equivalent to the conventional fractal coder)

--- 8 Taps Least Asymmetric

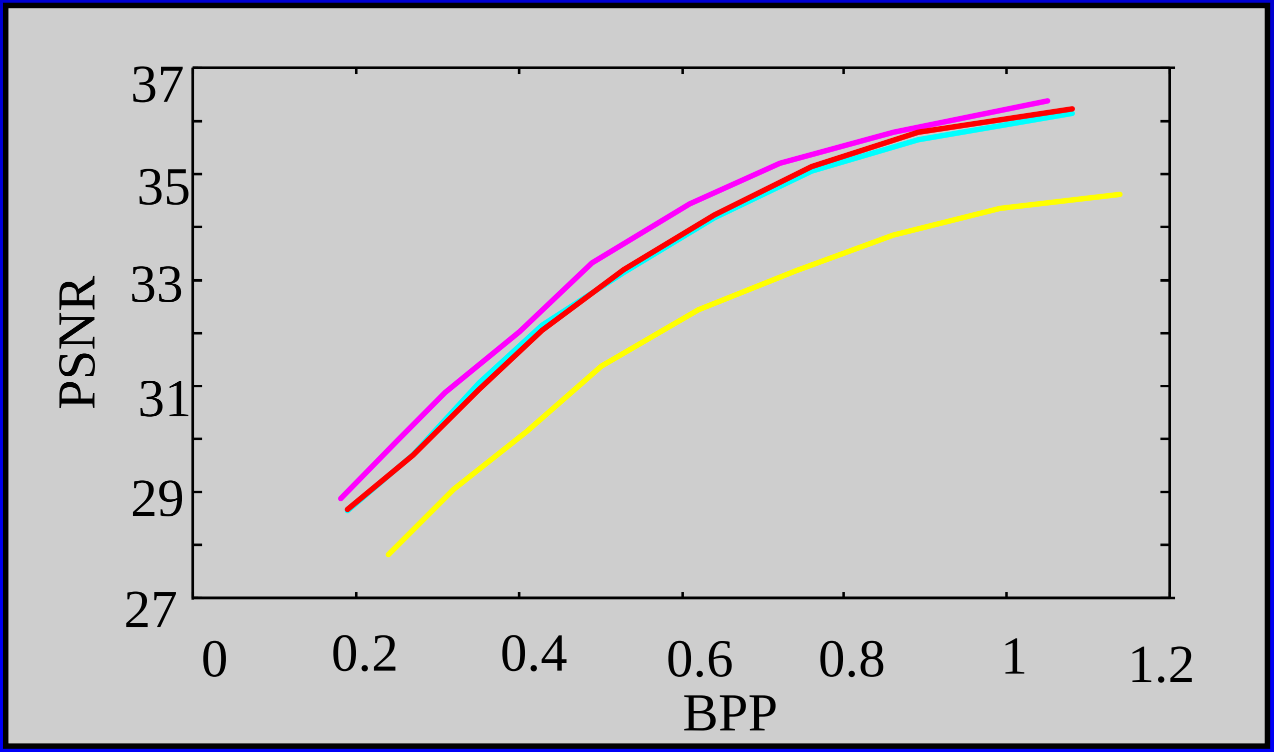
--- 8 Taps Min. Phase

--- 12 Taps Least Asymmetric

--- 12 Taps Min. Phase

--- Adelson et. al. 9 Taps QMF

Results - Quadtree Approach



--- Haar (equivalent to the conventional fractal coder)

--- 8 Taps Least Asymmetric

--- 12 Taps Least Asymmetric

--- Adelson et. al. 9 Taps QMF

The Visual Effect of the Error

~ 0.08 Bit/Pel ; "Lena"

Haar DWT IFS

Adelson DWT IFS

The Visual Effect of the Error

~ 0.08 Bit/Pel; Part of "Lena"

Haar DWT IFS

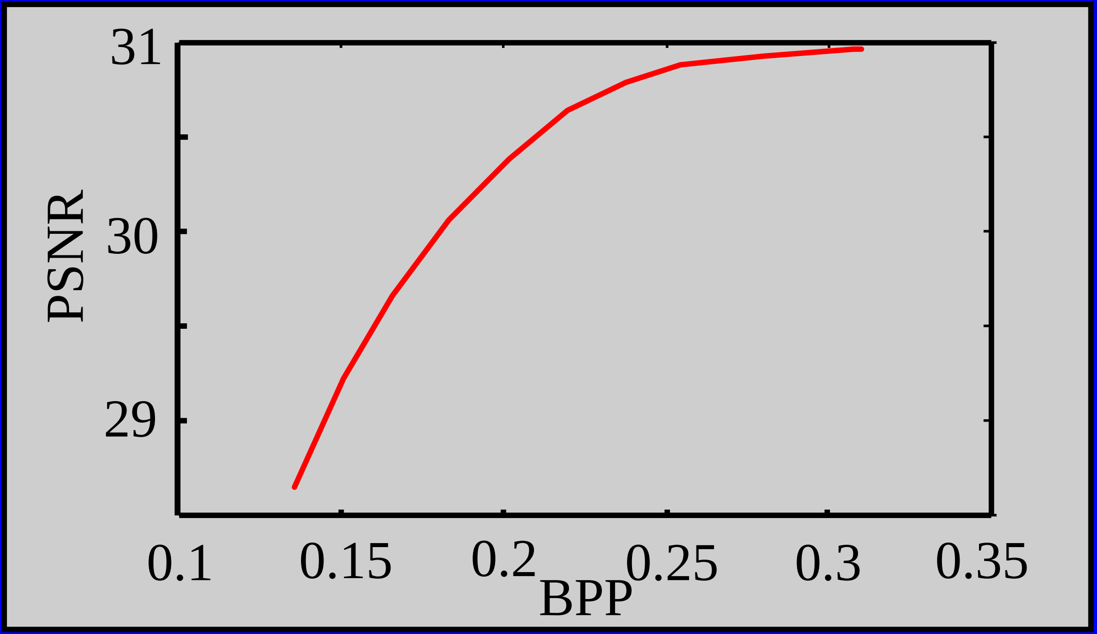
Adelson DWT IFS

Additional Improvements

Variable # of Domain Blocks

Low Energy Range-Subtrees

- Subtrees with low variance (smooth areas) can be zeroed .



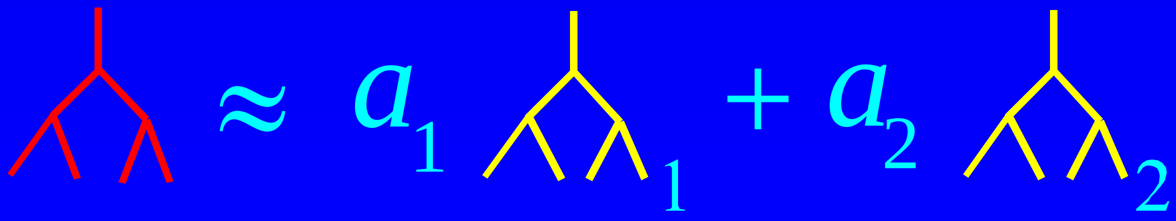
- This Causes disturbing blockiness with the *Haar-DWT* ***!!!***

Linear Combination of Domain blocks

- Use 2 Domain blocks :

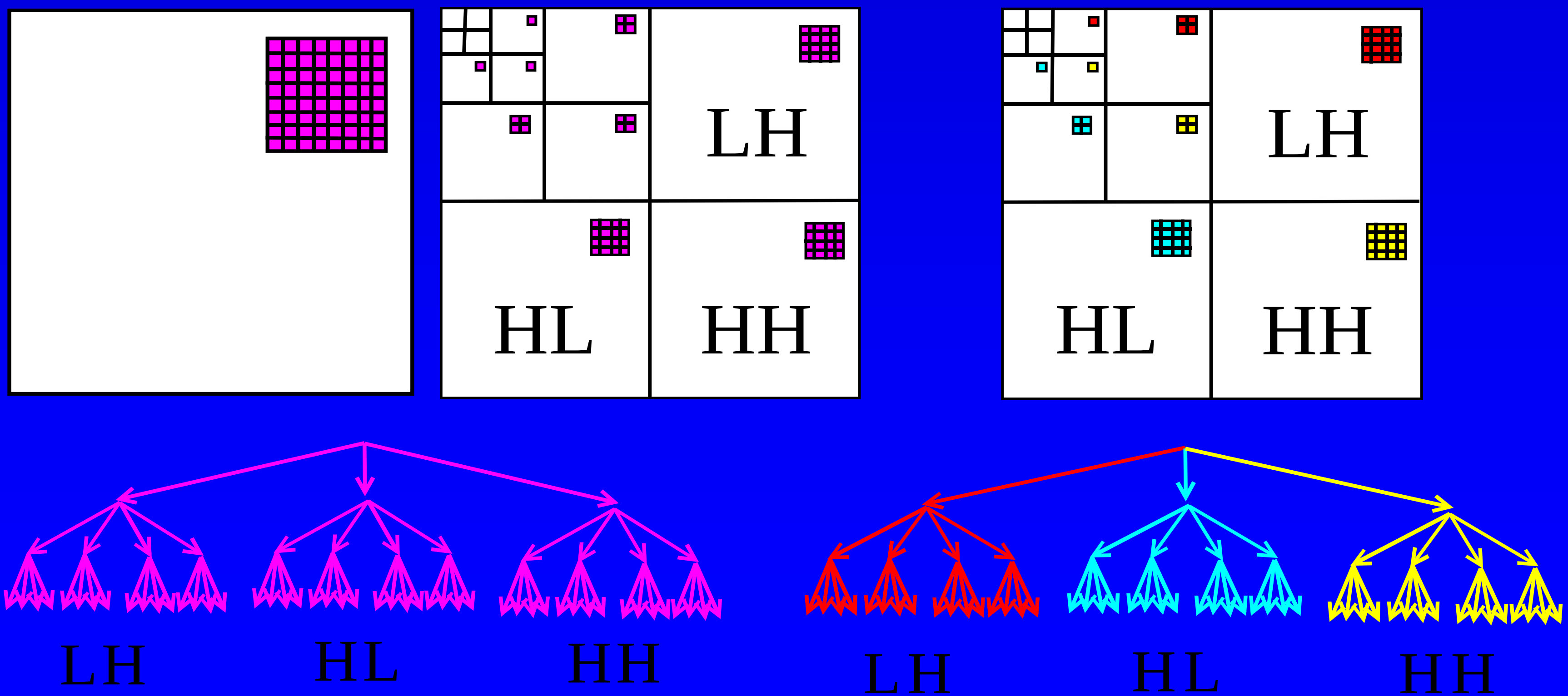
$$R \approx a_1(D_1 - \overline{D_1}) + a_2(D_2 - \overline{D_2}) + \overline{R}$$

- Apply *Matching Pursuits* to find a_1, D_1, a_2, D_2.
- In the *DWT Domain* :



Directional IFS

- If a ***Range-Subtree*** can't be estimated with 1 or 2 ***Domain Subtrees***, subdivide it into 3 ***Directional Subtrees*** and estimate each .

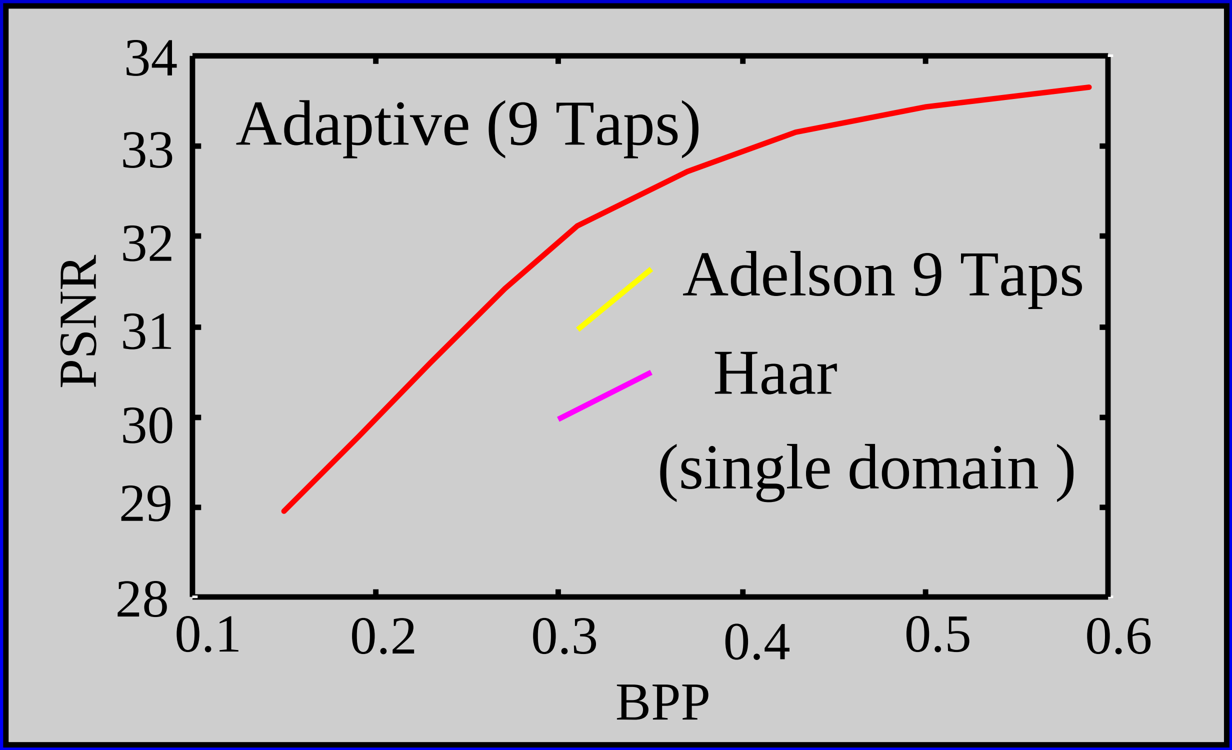


An Improved Coding Algorithm

- Encode the lower band
- For every range subtree - R :
  - Variance < Threshold ? T_i={Zero}
  - Single Domain ? T_i={Single, a_i, j_i}
  - Double Domain ? T_i={Double, a_i[1,2], j_i[1,2]}
  - Dir. Subtree ? T_i={Dir, a_i[LH,HL,HH], j_i[LH,HL,HH]}
  - If All above fail, subdivide the subtree (Quadtree)

Results - Adaptive # of domains

Image : “Lena”
Size : 512x512



Fixed Subtree size (B=8)

Uniform Quantization : 7bit - LP; 6bit - HP; 6bit-Scaling

Arithmetic coded

Example of coding "Lena"

Bit/Pel=0.31 PSNR=32.11dB

■ - None ■ - single ■ - Double ■ - Directional

Image "Super resolution"

- Decoding with a different Block size
- Subband extrapolation beyond the original
- Adding high bands without changing the original bands content
- The DWT-IFS embedded Function

Summary

and

Proposals for Further Research

Summary

- An equivalent *IFS* coder in the *Haar-DWT domain*
  - The IFS Predicts higher subbands coefficients from the lower ones
- A blockless fractal coder
  - Improving the coding results
- Variable # of domain blocks for the estimation
- An algorithm for “super resolution”

Proposals for Further Research

- "Fine Tuning" of the proposed coder
- Clustering the DWT-subtrees of the Domain-pool
- Combine with Wavelet coders
- Self-similar structured *Wavelet Packets*
- "Fractal Interpolation" - Theory and application

Questions ?