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Logical Geometry of the Rhombic Dodecahedron of Oppositions

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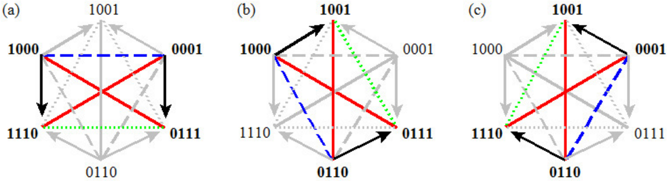
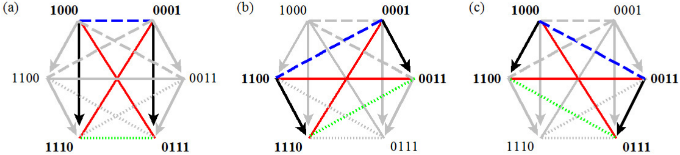
Introduction: Aristotelian subdiagrams

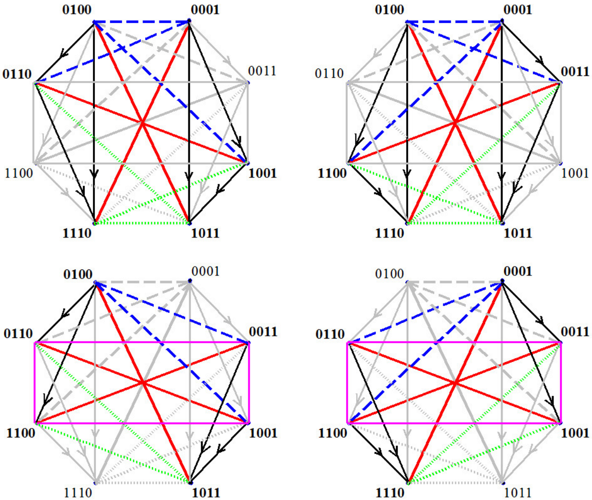
3 squares embedded in (strong) Jacoby-Sesmat-Blanché hexagon (JSB)



3 squares embedded in Sherwood-Czezowski hexagon (SC)



4 hexagons embedded in Buridan octagon



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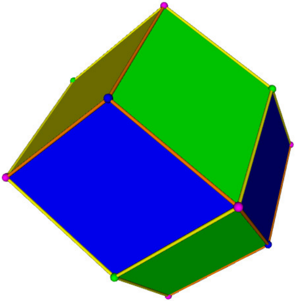
Internal structure of bigger/3D Aristotelian diagrams ? Some initial results:

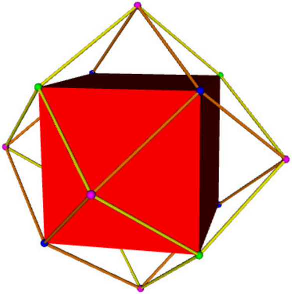
- 4 **weak** JSB-hexagons in logical cube (Moretti-Pellissier)
- 6 **strong** JSB hexagons in bigger 3D structure with 14 formulas/vertices
  - tetra-hexahedron (Sauriol)
  - tetra-icosahedron (Moretti-Pellissier)
  - nested tetrahedron (Lewis, Dubois-Prade)
  - **rhombic dodecahedron = RDH** (Smessaert-Demey) $\rightsquigarrow$ joint work

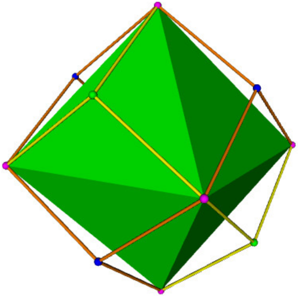
Greater complexity of RDH $\rightsquigarrow$ exhaustive analysis of internal structure ??
Main aim of this talk $\rightsquigarrow$ tools and techniques for such an analysis

- examine larger substructures (octagon, decagon, dodecagon, ...)
- distinguish families of substructures (strong JSB, weak JSB, ...)
- establish the exhaustiveness of the typology

cube	+	octahedron	=	cuboctahedron	$\overset{dual}{\Longrightarrow}$	**rhombic dodecahedron**
Platonic		Platonic		Archimedean		Catalan
6 faces		8 faces		14 faces		**12 faces**
8 vertices		6 vertices		12 vertices		**14 vertices**







14 vertices of RDH decorated with 14 bitstrings of length 4

Modal Logic S5	Propositional Logic	bitstrings level 1	bitstrings level 3	Propositional Logic	Modal Logic S5
$\Box p$	$p \wedge q$	1000	0111	$\neg(p \wedge q)$	$\neg\Box p$
$\neg\Box p \wedge p$	$\neg(p \rightarrow q)$	0100	1011	$p \rightarrow q$	$\Box p \vee \neg p$
$\Diamond p \wedge \neg p$	$\neg(p \leftarrow q)$	0010	1101	$p \leftarrow q$	$\neg\Diamond p \vee p$
$\neg\Diamond p$	$\neg(p \vee q)$	0001	1110	$p \vee q$	$\Diamond p$

Modal Logic S5	Propositional Logic	bitstrings level 2/0	bitstrings level 2/4	Propositional Logic	Modal Logic S5
p	p	1100	0011	$\neg p$	$\neg p$
$\Box p \vee (\Diamond p \wedge \neg p)$	q	1010	0101	$\neg q$	$\neg\Diamond p \vee (\neg\Box p \wedge p)$
$\Box p \vee \neg\Diamond p$	$p \leftrightarrow q$	1001	0110	$\neg(p \leftrightarrow q)$	$\neg\Box p \wedge \Diamond p$
$\Box p \wedge \neg\Box p$	$p \wedge \neg p$	0000	1111	$p \vee \neg p$	$\Box p \vee \neg\Box p$

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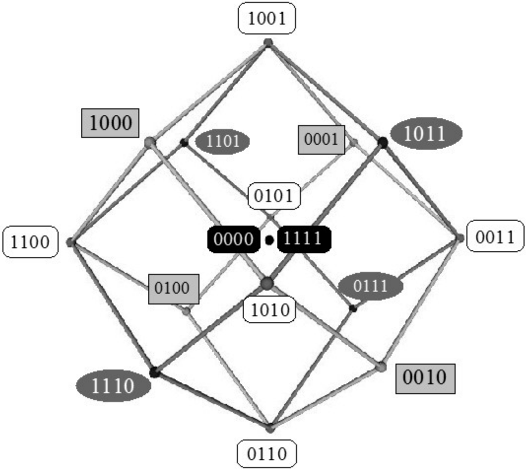
cube = 4 × L1 + 4 × L3 / **octahedron** = 6 × L2 / **center** = L0 + L4



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Bitstrings have been used to encode

- **logical systems**: e.g. classical propositional logic, first-order logic, modal logic and public announcement logic
- **lexical fields**: e.g. comparative quantification, subjective quantification, color terms and set inclusion relations

Contradiction relation is visualized using the **central symmetry** of RDH:

- contradictory bitstrings (e.g. 1100 and 0011) occupy diametrically opposed vertices
- the negation of a bitstring is located at a maximal (Euclidean) distance from that bitstring.
- nearly all Aristotelian diagrams discussed in the literature observe central symmetry ("contradictories are diagonals")

Bitstrings/formulas come in **pairs of contradictories (PCD)**
Key notion in describing RDH is σ_n-**structure**.

- A σ_n-structure consists of n PCDs
- A σ_n-structure is visualized by means of a centrally symmetrical diagram
- Examples

a square has 2 PCDs	$\Rightarrow$	σ_2-structure
a hexagon has 3 PCDs	$\Rightarrow$	σ_3-structure
an octagon has 4 PCDs	$\Rightarrow$	σ_4-structure
a cube has 4 PCDs	$\Rightarrow$	σ_4-structure

Remarks

- 1 σ-structure may correspond to different σ-diagrams:
  - alternative 2D visualisations
  - 2D versus 3D representations
- All σ-structures have an even number of formulas/bitstrings
- Nearly all Aristotelian diagrams in the literature are σ-structures

σ_n-structures

Original question of Aristotelian subdiagrams ("How many smaller diagrams inside bigger diagram?") can now be reformulated in terms of σ-structures.

- For n $\leq$ k, the nummer of σ_n-structures embedded in a σ_k-structure can be calculated as the number of combinations of n PCDs out of k by means of the simple combinatorial formula: $\binom{k}{n} = \frac{k!}{n!(k-n)!}$
- This combinatorial technique $\rightsquigarrow$ recover well-known results:
  - #squares (σ_2) inside a hexagon (σ_3) is $\binom{3}{2}$: $\frac{3!}{2!(1)!} = \frac{6}{2} = 3$
  - #hexagons (σ_3) inside octagon (σ_4) is $\binom{4}{3}$: $\frac{4!}{3!(1)!} = \frac{24}{6} = 4$
- This combinatorial technique $\rightsquigarrow$ obtain new results for RDH:
  - RDH contains 14 vertices, hence 7 PCDs $\rightsquigarrow$ RDH = σ_7-structure
  - Calculate the number of σ_n-structures inside a σ_7-structure as the number of combinations of n PCDs out of 7: $\binom{7}{n} = \frac{7!}{n!(7-n)!}$

σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
$\binom{7}{0}$	$\binom{7}{1}$	$\binom{7}{2}$	$\binom{7}{3}$	$\binom{7}{4}$	$\binom{7}{5}$	$\binom{7}{6}$	$\binom{7}{7}$
$\frac{7!}{0!(7)!}$	$\frac{7!}{1!(6)!}$	$\frac{7!}{2!(5)!}$	$\frac{7!}{3!(4)!}$	$\frac{7!}{4!(3)!}$	$\frac{7!}{5!(2)!}$	$\frac{7!}{6!(1)!}$	$\frac{7!}{7!(0)!}$
$\frac{5040}{1\times5040}$	$\frac{5040}{1\times720}$	$\frac{5040}{2\times120}$	$\frac{5040}{6\times24}$	$\frac{5040}{24\times6}$	$\frac{5040}{120\times2}$	$\frac{5040}{720\times1}$	$\frac{5040}{5040\times1}$
1	7	21	35	35	21	7	1

- 3 squares in 1 JSB $\times$ 6 JSB in RDH $=$ 18 squares in RDH. Remaining 3 ?? Unconnected/degenerate squares
- 6 strong JSB $+$ 4 weak JSB $=$ 10 hexagons in RDH. Remaining 25 ?? Sherwood-Czezowski. Others ? Unconnected4/12.
- symmetry/mirror image ? Complementarity: $\#\sigma_0 = \#\sigma_7$, $\#\sigma_1 = \#\sigma_6$, $\#\sigma_2 = \#\sigma_5$, $\#\sigma_3 = \#\sigma_4$

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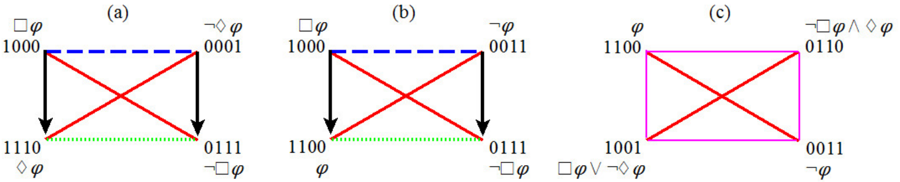
rhombic dodecahedron (RDH)	=	cube (C)	+	octahedron (O)
σ_7	=	σ_4	+	σ_3
7 PCDs	=	4 PCDs L1-L3	+	3 PCDs L2-L2

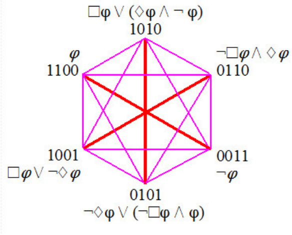
Construct a principled typology of families of σ-structures inside RDH.

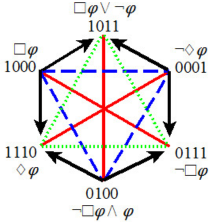
- $\sigma_n = n$ out of the 7 PCDs of RDH
- $\sigma_n = [k$ out of the 4 PCDs of $C] + [\ell$ out of the 3 PCDs of $O]$
- **CO-perspective**: every class of σ_n-structures can be subdivided into families of the form $C_k O_\ell$, for $0 \leq k \leq 4$; $0 \leq \ell \leq 3$ and $k + \ell = n$.
- For example, the cube C is $C_4 O_0$, and the octahedron O is $C_0 O_3$.
- The number of $C_k O_\ell$-structures inside RDH ($C_4 O_3$) can be calculated as $\binom{4}{k}\binom{3}{\ell}$.

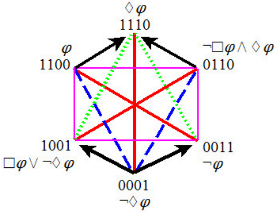
σ_2	$=$	C_2O_0	$+$	C_1O_1	$+$	C_0O_2
$\binom{7}{2}$		$\binom{4}{2}\binom{3}{0}$		$\binom{4}{1}\binom{3}{1}$		$\binom{4}{0}\binom{3}{2}$
21	$=$	6	$+$	12	$+$	3
squares		classical balanced 2×L1/2×L3		classical unbalanced 1×L1/2×L2/1×L3		degenerated (balanced) 4×L2



σ_3	$=$	C_0O_3	$+$	C_3O_0	$+$	C_1O_2	$+$	C_2O_1
$\binom{7}{3}$		$\binom{4}{0}\binom{3}{3}$		$\binom{4}{3}\binom{3}{0}$		$\binom{4}{1}\binom{3}{2}$		$\binom{4}{2}\binom{3}{1}$
35	$=$	1	$+$	4	$+$	12	$+$	18
hexagons		degener. U12		weak JSB		degener. U4		strong JSB Sher-Czez

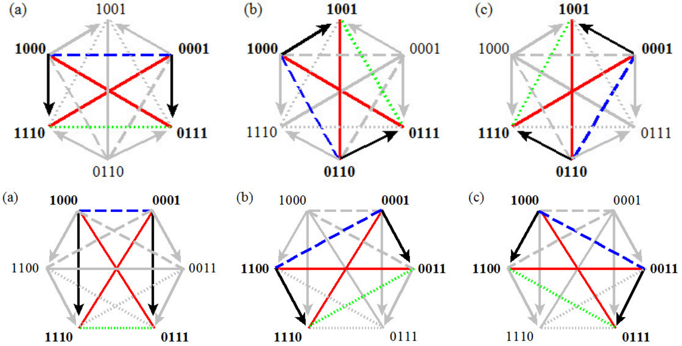






$18 \times C_2O_1 \quad = \quad 6 \times C_2O_1a \quad + \quad 12 \times C_2O_1b$

strong JSB — Sherwood-Czezowski



- CO-perspective: no distinction strong JSB *vs* Sherwood-Czezowski
- isomorphism perspective: no distinction strong JSB *vs* weak JSB

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Families of σ_n-structures: overview

σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
			C_0O_3	C_4O_0			
			1	1			
	C_1O_0	C_0O_2	C_3O_0	C_1O_3	C_4O_1	C_3O_3	
	4	3	4	4	3	4	
C_0O_0	C_0O_1	C_2O_0	C_2O_1a	C_2O_2a	C_2O_3	C_4O_2	C_4O_3
1	3	6	6	6	6	3	1
		C_1O_1	C_2O_1b	C_2O_2b	C_3O_2		
		12	12	12	12		
			C_1O_2	C_3O_1			
			12	12			
1	7	21	35	35	21	7	1

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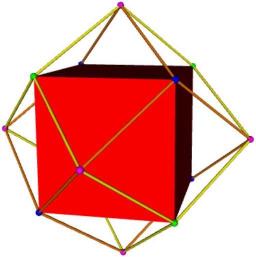
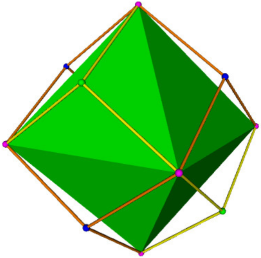
Fundamental complementarity between σ-structures inside RDH

- $|\sigma_n| = |\sigma_{7-n}|$
- $|C_k O_\ell| = |C_{4-k} O_{3-\ell}|$

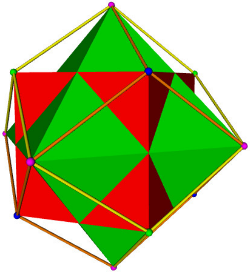
$C_4 O_0$



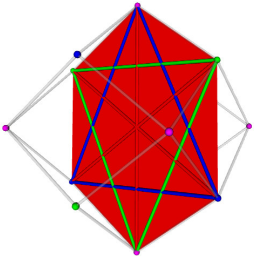
$C_0 O_3$

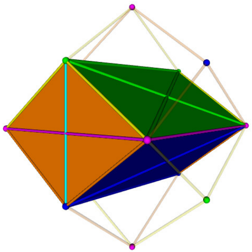


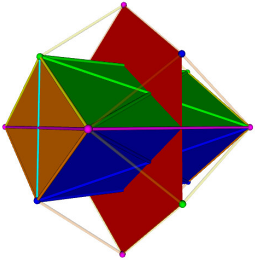
$C_4 O_3$



C_2O_1a	C_2O_2a	C_4O_3
strong JSB hexagon	Buridan octagon	rhombic dodecahedron







rhombicube

structure	subtype	N	subtype	structure
σ_0	C_0O_0	1	C_4O_3	σ_7
σ_1	C_1O_0	4	C_3O_3	σ_6
	C_0O_1	3	C_4O_2	
	C_0O_2	3	C_4O_1	
σ_2	C_2O_0	6	C_2O_3	σ_5
	C_1O_1	12	C_3O_2	
	C_0O_3	1	C_4O_0	
	C_3O_0	4	C_1O_3	
σ_3	C_2O_1a	6	C_2O_2a	σ_4
	C_2O_1b	12	C_2O_2b	
	C_1O_2	12	C_3O_1	

$\rightsquigarrow$ The logical geometry of rhombic dodecahedron RDH

$\rightsquigarrow$ Typology of Aristotelian subdiagrams of RDH

$\rightsquigarrow$ Tools/techniques for exhaustive analysis of internal structure of RDH

- define σ_n-structure $= n$ out of the 7 PCDs of RDH
- distinguish families of substructures $= C_k O_\ell$-perspective:
  $\sigma_n = [k$ out of the 4 PCDs of $C] + [\ell$ out of the 3 PCDs of $O]$
- establish the exhaustiveness of the typology $\rightsquigarrow$ complementarity

$\rightsquigarrow$ Frame of reference for classifying Aristotelian diagrams in the literature

σ_1	C_1O_0	Brown 1984
	C_0O_1	Demey 2012
σ_2	C_0O_2	Brown 1984, Béziau 2012
	C_2O_0	Fitting & Mendelsohn 1998, McNamara 2010, Lenzen 2012
	C_1O_1	Luzeaux, Sallantin & Dartnell 2008, Moretti 2009
σ_3	C_0O_3	Moretti 2009
	C_2O_1a	Sesmat 1951, Blanché 1966, Béziau 2012, Dubois & Prade 2013
	C_2O_1b	Czezowski 1955, Khomskii 2012, Chatti & Schang 2013
	C_1O_2	Seuren 2013, Seuren & Jaspers 2014, Smessaert & Demey 2014
	C_3O_0	Pellissier 2008, Moretti 2009, Moretti 2012
σ_4	C_1O_3	
	C_3O_1	
	C_2O_2b	Béziau 2003, Smessaert & Demey 2014
	C_2O_2a	Hughes 1987, Read 2012, Seuren 2012
	C_4O_0	Moretti 2009, Chatti & Schang 2013, Dubois & Prade 2013
σ_5	C_3O_2	Seuren & Jaspers 2014
	C_2O_3	
	C_4O_1	Blanché 1966, Joerden & Hruschka 1987, Wessels 2002
σ_6	C_4O_2	Béziau 2003, Moretti 2009, Moretti 2010
	C_3O_3	
σ_7	C_4O_3	Sauriol 1968, Moretti 2009, Smessaert 2009, Dubois & Prade 2013

Thank you!

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