Indian Journal of History of Science, 48.2 (2013) 291-313

Summation of Convergent Geometric Series and the Concept of Approachable *Śūnya*

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Derivation of the formula for summation of convergent geometric series of rational numbers assumes summing of $\propto$ terms. However, cardinality of rational numbers can only be aleph$_0$ ($\aleph_0$) and nothing higher. Application of this fact in the derivation of the formula leads to the emergence of the concept of 'Approachable *Śūnya*' which is $1/2^{\aleph_0}$. This affords an analysis of the expression 1= 0.999…., which has been problematic for students everywhere to accept. Further, all divergent geometric series of rational numbers 'converge' to $2^{\aleph_0}$. Arithmetic of first Approachable *Śūnya* shows that zero is more like a transfinite cardinal than its finite neighbors on the linear number line.

Key words: Approachable *Śūnya*, Geometric series, Summation, Cardinals

1. Introduction

I took out few 25 paise coins to pay the fare
The bus conductor sized me from head to toe -
25 paise is zero, the gaze implied…

I dropped a 10 paise coin into an outstretched hand
It threw it away in cold disgust -
10 paise is zero, the murmur meant…

I gave a 5 paise coin to credit my account
The bank clerk viewed me with a blank look -
5 paise is zero, the wide eyes conveyed…

This gives a vague idea of 25, 10, 5 $\rightarrow$ 0 approaching zero. A few other examples in this context will be of interest.

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Summation of geometric series was well known to the ancient mathematicians of both Orient and Occident.

Proposition 35 of Book 9 of Euclid's *Elements* (Heath, 1908, p. 420), affords the formula for the summation of the series. His insight was that when a_1 to a_n are in geometric progression, $a_2–a_1: a_1 = a_n–a_1:(a_1+a_2+\ldots+a_n)$. From this, the modern formula for the partial sum of n terms can easily be deduced.

Archimedes (Heath, 1953) used geometrical construction to prove that $\frac{1}{4} + \frac{1}{4^2} + \ldots = \frac{1}{3}$. His reasoning is based on the idea that $(\frac{3}{4} + \frac{3}{4^2} + \ldots) = 1$, which is equivalent to the series $3\,(\frac{1}{4} + \frac{1}{4^2} + \ldots) = 1$.

Ācārya Bhadrabāhu (circa 433 BC - circa 355 BC) in his *Kalpasūtra* gives the sum of a geometric series. Mathematician Mahāvīra (Singh, 1936) in 9th century gives the formula for the summation of the convergent geometric series in his *Gaṇita Sārasaṃgraha.*

Thus this accepted formula for summation of geometric series has had a passage through two millennia. For the sake of historical interest, and to develop the concept of 'Approachable *Śūnya*', it is possible to derive this formula from first principles.

Let the partial sum of n terms of the convergent geometric series be S_n i.e. $S_n = a + ar + ar^2 + ar^3 + ar^4 +\ldots+ ar^{n-1}$, where 'a' is the first term and 'r' is the common ratio such that r < 1, then

$$rS_n = ar + ar^2 + ar^3 + ar^4 +\ldots+ ar^n$$

$$S_n - rS_n = (a + ar + ar^2 + ar^3 + ar^4 +\ldots+ ar^{n-1}) - (ar + ar^2 + ar^3 + ar^4 +\ldots+ ar^n)$$

$$= a - ar^n$$

$$S_n (1 - r) = a (1- ar^n)$$

$$\text{Hence } S_n = \frac{a(1-r^n)}{1-r} \qquad \ldots(1)$$

When the number of terms tend to infinity (∝), $\frac{a}{1-r}$ r^n tends to zero and the formula reduces to S = $\frac{a}{1-r}$ …(2)

While there is near-universal acceptance of Equation (2), the summation of at least one geometric series evokes persistent skepticism among students. This series is

1 = $\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots$, which is more often written as 1 = 0.999…

One paper (Tall, 1978) states that first year university students, fresh from school, when asked whether 0.999… was equal to 1, replied by a majority that 0.999… was *less* than 1. Katz (2010) states the persistent report of teachers that students' 'naïve initial intuition' is that 0.999… is less than 1. It argues that "the students' hunch that 0.999… falls infinitesimally short of 1 can be justified in a rigorous fashion, in the framework of Abraham Robinson's (1996) non-standard analysis."

In yet another paper (Sierpinska, 1994) the arguments gone through by a group of 17-year-old Humanity students while initially rejecting the equality 1 = 0.999… and how finally just one student came around to accepting it, is reported. Byers (2007) recalls asking students in a class on real analysis the question, "does 1 = .999…?" "Something about this expression made them nervous. They were not prepared to say that .999… is equal to 1, but they all agreed that it was 'very close' to 1."

Besides this perplexity of students everywhere, there is another reason for a second look at Equation (2). The limit concept that is used to derive Equation (2) assumes the summation of infinite terms. This assumption, obviously, has been made based on the idea that there is only one infinity ∝. However, Cantor (Byers, 2007) has shown the inevitability of infinity of infinities or transfinite cardinals. Further, he has shown by the famous zigzag argument that the cardinality of rational numbers is only, the cardinality of natural numbers. Hence, as the geometric series is formed of rational numbers, the cardinality of terms of these series must be considered as $\aleph_0$ instead of ∝. These factors call for a closer look at Equation (2).

2. Summation of $\aleph_0$ instead of $\propto$ terms

First of all, let us sum the series to $\aleph_0$ instead of $\propto$ terms. Take the following convergent infinite geometric series as an example

$$1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \qquad \ldots(3)$$

Here the common ratio is $\frac{1}{2}$ and the terms are infinite. The Right Hand Side (RHS) of Equation (3) contains terms that are a subset of rational numbers. Thus, by Cantor's zigzag argument, the cardinality or the number of terms on the (RHS) can only be $\aleph_0$ and nothing higher.

Let us construct the following table to view the sum of the series after n terms $(\sum_1^n \frac{1}{2^n})$ and the difference from Left Hand Side (LHS) at that point.

Table 1

Number of terms (n)	**Sum of RHS after n terms** $\sum_1^n \frac{1}{2^n}$	**Difference LHS - RHS** $(1 - \sum_1^n \frac{1}{2^n})$
1	$\frac{1}{2} = 1 - \frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{2^2}$	$\frac{1}{2^2}$
3	$\frac{7}{8} = 1 - \frac{1}{2^3}$	$\frac{1}{2^3}$
4	$\frac{15}{16} = 1 - \frac{1}{2^4}$	$\frac{1}{2^4}$
	$1 - \frac{1}{2^{\aleph_0}}$	$\frac{1}{2^{\aleph_0}}$

So we are left with the result that after all the terms of the RHS of Equation (3) have been exhausted, the series is still short of the LHS by $\frac{1}{2^{\aleph_0}}$.

Let us take another convergent geometrical series

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots \qquad \ldots(4)$$

As in the case of Equation (3), let us prepare a table

Table 2

Number of terms (n)	**Sum of RHS after n terms** $\sum_1^n \frac{1}{3^n}$	**Difference LHS - RHS** $(1/2 - \sum_1^n \frac{1}{3^n})$
1	$\frac{1}{3} = \frac{1}{2} \times (1 - \frac{1}{3})$	$\frac{1}{2} - \frac{1}{2} \times (1 - \frac{1}{3}) = \frac{1}{2} \times \frac{1}{3}$
2	$\frac{1}{3} + \frac{1}{9} = \frac{4}{9} = \frac{1}{2} \times (1 - \frac{1}{3^2})$	$\frac{1}{2} - \frac{1}{2} \times (1 - \frac{1}{3^2}) = \frac{1}{2} \times \frac{1}{3^2}$
3	$\frac{4}{9} + \frac{1}{27} = \frac{13}{27} = \frac{1}{2} \times (1 - \frac{1}{3^3})$	$\frac{1}{2} - \frac{1}{2} \times (1 - \frac{1}{3^3}) = \frac{1}{2} \times \frac{1}{3^3}$
4	$\frac{13}{27} + \frac{1}{81} = \frac{40}{81} = \frac{1}{2} \times (1 - \frac{1}{3^4})$	$\frac{1}{2} - \frac{1}{2} \times (1 - \frac{1}{3^4}) = \frac{1}{2} \times \frac{1}{3^4}$
$\aleph_0$	$\frac{1}{2} \times (1 - \frac{1}{3^{\aleph_0}})$	$\frac{1}{2} \times \frac{1}{3^{\aleph_0}}$

Here RHS of Equation (4) is short of the LHS by $\frac{1}{2} \times \frac{1}{3^{\aleph_0}}$.

Let us find out the value of $\frac{1}{2} \times \frac{1}{3^{\aleph_0}}$.

Let k be any finite number such that $2 \leq k \leq 2^{\aleph_0}$. Raising all three terms to $\aleph_0$, we have

$$2^{\aleph_0} \leq k^{\aleph_0} \leq \left(2^{\aleph_0}\right)^{\aleph_0}, \text{ but } = \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0 x \aleph_0}$$

$\aleph_0 \times \aleph_0 = \aleph_0$ by the rules of transfinite arithmetic.

Thus $2^{\aleph_0} \leq k^{\aleph_0} \leq 2^{\aleph_0}$. Since the first and last terms are equal, the middle term must be equal to the other two and hence

$2^{\aleph_0} = k^{\aleph_0}$. Therefore $2^{\aleph_0} = 3^{\aleph_0}$ and hence $\frac{1}{2} \times \frac{1}{3^{\aleph_0}} = \frac{1}{2} \times \frac{1}{2^{\aleph_0}}$.

Again by the rules of cardinal arithmetic, if n is any finite number, then $n \times \aleph_0 = \aleph_0$. Therefore, $\frac{1}{2} \times \frac{1}{2^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$, which was the value of LHS - RHS in the case of Equation (3) also.

Similarly it is possible to show that for the series $\frac{1}{3} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$, the reminder (LHS - RHS) is $\frac{1}{3} \times \frac{1}{4^{\aleph_0}}$ which reduces to $\frac{1}{2^{\aleph_0}}$. Also, for the series $\frac{1}{4} = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots$, the reminder is $\frac{1}{4} \times \frac{1}{5^{\aleph_0}}$ which also reduces to $\frac{1}{2^{\aleph_0}}$.

Putting these results into a table

Table 3

Sum on the LHS	First term	Common ratio (r)	Reminder (LHS-RHS)	Equivalent reminder
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2^{\aleph_0}}$	$\frac{1}{2^{\aleph_0}}$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2} \times \frac{1}{3^{\aleph_0}}$	$\frac{1}{2^{\aleph_0}}$
$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3} \times \frac{1}{4^{\aleph_0}}$	$\frac{1}{2^{\aleph_0}}$
$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{4} \times \frac{1}{5^{\aleph_0}}$	$\frac{1}{2^{\aleph_0}}$

Thus the equivalent reminder in the case of all these summations is $\frac{1}{2^{\aleph_0}}$. Does it mean that the results for summation of convergent geometric series, accepted for long, must be treated as not rigorous enough?

3. Explanation

Let us look at formula (1) which is the Actual Sum (AS) of *n* terms and the formula (2) which is the Ideal Sum (IS) of infinite terms. The term in the AS that tends to zero as the number of terms tend to infinity is $\frac{ar^n}{1-r}$. Also, since $\frac{ar^n}{1-r} = \frac{a}{1-r}$ r^n, we can say that

$$\text{LHS} - \text{RHS} = \text{IS} - \text{AS} = \text{IS } r^n \qquad \ldots(5)$$

This can be seen in Column 4 of Table 3. For example, take the second row. Here the IS is $\frac{1}{2}$ and $r = \frac{1}{3}$. Hence the value of $\text{LHS} - \text{RHS} = \frac{1}{2} \times \frac{1}{3^{\aleph_0}}$. Similarly, in the third row, IS is $\frac{1}{3}$ and $r = \frac{1}{4}$. Here the value of $\text{LHS} - \text{RHS} = \frac{1}{3} \times \frac{1}{4^{\aleph_0}}$ and so on.

Truly speaking, the part that tends to zero is r^n, since both a and r are non-zero. As such the equivalent reminder, as shown in Column 5 of Table 3 is only $r^{\aleph_0} = \frac{1}{2^{\aleph_0}}$.

If we examine the argument for the derivation of formula (2), we are left with a clearer perspective. We had argued that as $n \to \infty$, $r^n \to 0$. *We then went on to refine* $\propto$ *and replace it with more exact* $\aleph_0$. *This has resulted in a more refined concept of 0, which in this case is* $\frac{1}{2^{\aleph_0}}$.

To Droupadī who scrubbed the vessel clean
The *Akṣayapātra* contained nothing;
To Sri Krishna with Cosmic Vision
The rim contained a fraction of a leaf…

This result can be seen even more clearly on a logarithmic number line.

4. Logarithmic Scale

The linear number line, shown in Fig.1, is addition-based and has equal intervals for each additional step. It stretches from – ∝ on the left to + ∝ on the right with 0 in the middle.

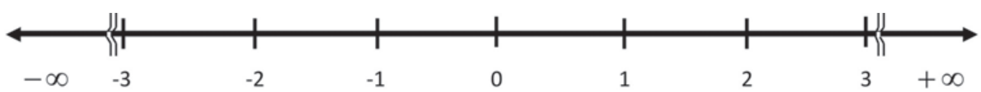

Fig. 1. Linear number line

In contrast to the above is the logarithmic scale, which is ratio-based. Here the ratio between adjacent terms is constant. Shown in Fig. 2 is a logarithmic scale to base 2. Here the common ratio between adjacent terms is 2. As can be seen, this numberline stretches from Unattainable Zero on the left to Unattainable Infinity on the right with 2^0 or 1 in the middle.

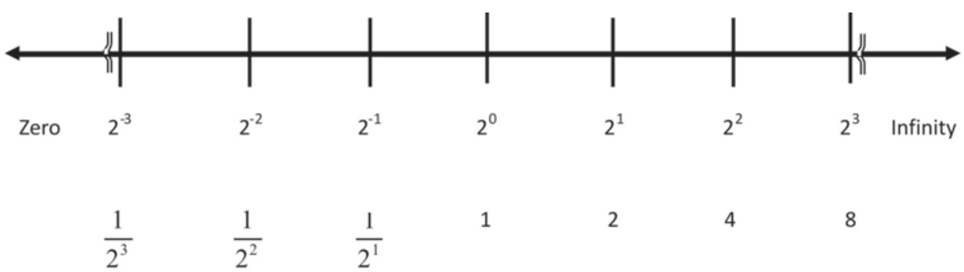

Fig. 2. Logarithmic number line

Experiments with school children of US indicate that kindergarten students, who are yet to be exposed to the rigour of formal education, mark numbers on a line in a logarithmic manner (Seigler, 2004). However, with increasing age and experience of linear numberline, this logarithmic sense starts declining (Seigler, Opfer, 2003). Similar studies were conducted among Munduruku, an Amazonian tribe with hardly any education and very few number words in their lexicon. In fact, their repertoire of numbers (Pica, Lemer, Izard & Dehaene, 2004) does not extend beyond 5. In experiments involving the mapping of numbers on a scale of 0 to 10 (or 10 and 100), they invariably placed numbers in a logarithmic proportion rather than a linear one (Dehane, Pica, Spelke, Izard & Dehaene,2008).

The authors of the study (Dehane, Pica, Spelke, Izard & Dehaene,2008) have concluded that "the mapping of numbers onto space is a universal intuition, and this initial intuition of number is logarithmic. The concept of a linear number line appears to be a cultural invention that fails to develop in the absence of formal education". Even studies among animals have revealed that they too have a sense of number (Dehaene, 1997) and this follows the logarithmic rather than the linear scale (Dehaene, 2003). Thus there is scale-tipping evidence that logarithmic or ratio-based sense of numbers is more innate and primary than addition-based linear numberline and that this ratio-based sense of numbers is hardwired into Nature.

Each of the positions to the left of 2^0 of Fig. 2 is, in fact, the reminder as shown in column 3 of Table 1. Thus the reminder (LHS – RHS) after one term is $\frac{1}{2^1}$, after two terms is $\frac{1}{2^2}$ and so on. After how many such steps can zero on the extreme left be reached? Obviously, one can reach only $\frac{1}{2^{\aleph_0}}$ after $\aleph_0$ steps.

Since attaining Absolute Zero (or reminder of RHS - LHS = 0) is not possible on this numberline even after $\aleph_0$ steps, it is self evident that LHS and RHS of Equation (3) cannot be equal even after all rational numbers have been exhausted.

At the same time we have seen that $\frac{1}{2^{\aleph_0}}$ *is the zero of convergent geometric series of rational numbers*. And in that way, the LHS and RHS of Equation (3) are equal since the term that is the reminder of LHS – RHS ($\frac{1}{2^{\aleph_0}}$) is *defined as zero*. It is possible to denote this term $\frac{1}{2^{\aleph_0}}$ as 'First

Approachable *Śūnya*' (S_1) which we encounter on the way to Unapproachable Absolute Zero.

Equation (3) can therefore be restated as

$$1= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...+ S_1 \qquad ...(6)$$

And Equation (4) as

$$1= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...+ S_1 \qquad ...(7)$$

To Hiraṇyakaśipu who relied on the sinews
The pillar was just a material support;
To Prahlāda who relied on the Spirit
The same pillar contained Subtle Infinite…

It can be easily seen that Equations (6) and (7) are like the statement 5 = (5 – x) + x, where x = 0, whereas Equations (3) and (4) are like the statement 5 = (5 – x), where also x = 0.

Theorem 1: The actual sum of any convergent geometric series of rational numbers (where r < 1; $r = \frac{1}{n}$ where $n \in N; n > 1$) *is less than its ideal sum by* $\frac{1}{2^{N_0}}$, *which is the First Approachable Śūnya (S_1).*

5. Do divergent geometric series 'converge'?

The formula for the summation of a divergent geometric series is $S = \frac{a(r^n - 1)}{r - 1}$ where 'a' is the first term, 'r' is the common ratio between terms such that |r| >1

Therefore for the series $S= 2 + 2^2 + 2^3 + ...$ ...(8)

the sum is $\frac{2(2^{\aleph_0} - 1)}{2 - 1} = 2 \times 2^{\aleph_0} = 2^{\aleph_0}$.

Similarly take the series $S= 3 + 3^2 + ...$ ...(9)

Here the sum is $\frac{3(3^{\aleph_0}-1)}{3-1}=\frac{3(3^{\aleph_0})}{2}=\frac{3(2^{\aleph_0})}{2}=2^{\aleph_0}$.

If n is any rational number such that $2 \le n \le 2^{\aleph_0}$, then the sum of the series

S= n + n^2 +n^3 + … …(10)

$$= \frac{n(n^{\aleph_0}-1)}{n-1} = \frac{n(2^{\aleph_0})}{n-1} = \frac{n}{n-1}(2^{\aleph_0}) = 2^{\aleph_0}$$

Putting them all in a table,

Table 4

First term (a)	**Common Ratio (r)**	**Sum of the series (S)**	**Equivalent Sum**
2	2	$2^{\aleph_0}$	$2^{\aleph_0}$
3	3	$\frac{3(2^{\aleph_0})}{2}$	$2^{\aleph_0}$
4	4	$\frac{4(2^{\aleph_0})}{3}$	$2^{\aleph_0}$
n	N	$\frac{n(2^{\aleph_0})}{n-1}$	$2^{\aleph_0}$

Thus all divergent geometric series with rational numbers 'converge' to 2^{N_0}, which is their infinity.

To a boy standing on the beach
A sounding rocket may ascend to limitless heights
To someone looking from above
The blazing arrow cannot cross the atmosphere…

Theorem 2: All divergent geometric series of rational numbers (where r >1; r ∈ N) converge to 2^{N_0}.

6. Is 1 = 0.999…?

We now come to the statement that has persistently aroused perplexity among students everywhere. Their intuitional hunch is that in the statement 1 = 0.999…, the RHS is somehow just short of LHS. It has been asked pertinently (Tall, 1981): "…why do we so persistently obtain these early intuitions of infinity as a direct product of school experience?...If everyone seems to get such wild ideas, in what sense is the accepted mathematical definition so much the better?"

Various proofs have been advanced to convince doubting students about the veracity of the statement. Let us have a closer look at some of them:

a) In the expression 1 = 0.999…, the RHS could be written as a convergent geometric series

$$1 = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots. \qquad \ldots(11)$$

So by Equation (2), the IS of the series is $\dfrac{\frac{9}{10}}{1-\frac{1}{10}} = \dfrac{\frac{9}{10}}{\frac{9}{10}} = 1$

Using Equation (5), $\text{LHS} - \text{RHS} = \text{IS} \times r^n = 1 \times \frac{1}{10^{\aleph_0}} = \frac{1}{10^{\aleph_0}}$. But $10^{\aleph_0} = 2^{\aleph_0}$ and so RHS is less than LHS by $\frac{1}{2^{N_0}}$. Thus Equation (11) can be written as

$$1 = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots + \frac{1}{2^{\aleph_0}}$$

Or $1 = 0.999\ldots + \dfrac{1}{2^{\aleph_0}}$ $\qquad \ldots(12)$

It can be easily seen that Equation (12) is similar to Equations (6) and (7) in that they all obey Theorem 1.

The wave expands as 0.999…
The particle coalesces into 1
Śūnya wedges itself in between
And dances reciprocally with Aleph-One…

b) Another argument for the identity 1= 0.999… is $\frac{1}{9} = 0.111\ldots$ hence $9 \times \frac{1}{9} = 0.999\ldots$ or 1= 0.999…

Take $\frac{1}{9} = 0.111\ldots$

We can rewrite the above as

$$\frac{1}{9} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots. \qquad \ldots(13)$$

Using Equation (5), $\text{LHS} - \text{RHS} = \text{IS} \times r^n = \frac{1}{9 \times 10^{\aleph_0}}$

Now we can write $\frac{1}{9} = 0.111\ldots + \frac{1}{9 \times 10^{\aleph_0}}$

Multiplying both sides by 9,

$$9 \times \frac{1}{9} = (9 \times 0.111...) + 9 \times \frac{1}{9 \times 10^{\aleph_0}}$$

Or $1 = 0.999\ldots + \frac{1}{10^{\aleph_0}}$ or

$$1 = 0.999\ldots + \frac{1}{2^{\aleph_0}} \qquad \ldots(14)$$

which is the same as Equation (12)

The argument $\frac{1}{3} = 0.333\ldots$, hence $3 \times \frac{1}{3} = 0.999\ldots$, and so 1= 0.999… is similar to the derivation of Equation (14) and can be explained in a similar manner to yield $1 = 0.999\ldots + \frac{1}{2^{N_0}}$

The *Viśvarūpa* is revealed in Unending Glory
The reassuring form of Śrī Kṛṣṇa reappears;
An incidental cause, Arjuna stands astounded
With ten fingers held together as one…

c) An algebraic argument for the identity 1 = 0.999… goes like this.

x = 0.999…
hence 10x = 9.999…
so 10x – x = 9.999… – 0.999…
or 9x = 9 and hence x = 1

This proof hinges on the assumption that multiplication of an infinite string of decimals by 10 merely shifts the decimal point by one place to the right but has no effect on the possible last digit. This is implied by the line 10x = 9.999… (The question can always be asked, what about the value of 2x ?)

Further, one must accept the 'consistency of arithmetical operations'. What is meant by this is that if a number *x* is subjected to many arithmetic operations like multiplication, subtraction and division, and if we get *x* itself as the final answer, then the value of *x* should be the same at the end of the operation as at the beginning. For example, take x = 2, multiply by 10 (10 × 2 = 20), subtract 2 (20 – 2 = 18) and divide by 9 ($\frac{18}{9} = 2$), then we should get 2 itself as the final answer. It can neither be 1.9999 or even 2.0001. This rigour is not maintained in the argument of the above proof. We start with x = 0.999… but end with x = 1 by mere arithmetical operations. This is vaguely dissatisfying.

It has been insightfully commented (Thurston, 1994) that "On the most fundamental level, the foundations of mathematics are much shakier than the mathematics that we do. Most mathematicians adhere to foundational principles that are known to be polite fictions… There is considerable evidence (but no proof) that we can get away with these polite fictions without being caught out, but that doesn't make them right."

There is another view of multiplication by 10. Here multiplying by 10 involves adding a 0 at the extreme right end and shifting the number to the left of this 0 so that the number of decimal places before and after the operation remains the same. The trailing 0 may not contribute any value but may only be a placeholder. To understand the concept let us first look at the following table

Table 5

Decimal string (x)	**No.of decimal places**	**10x**
0.9	1	9.0
0.99	2	9.90
0.999	3	9.990
0.999…	1 million	9.999…90 (1 million decimal places)
0.999…	1 billion	9.999…90 (1 billion decimal places)

With this pattern in mind, let us look at the following metaphors to guide our understanding

Countless bogies in a shunting yard,
And a guard's compartment is attached to the rear;
Now more bogies can be added
Only to the front or in between…

Exuberant Bhāgīrathī gushes along
Only to find her feet chained by a dam;
What once was an endless flow
Is now a stagnant pool about to overflow…

Let us see what will happen if we follow this logic: when the end is restrained, the middle starts bulging. In the operations shown below, all the decimals strings are assumed to be digits long.

$x = 0.999...$
so $10x = 9.999….90$
Hence $10x – x = 9.999….90 – 0.999…$
or $9x = 8.999…1$

and hence $x = 0.999...$ the number we started with. It is obvious that this approach to 10x is more 'consistent' arithmetically.

Let us check the 'consistency' of this approach

Here $5x + 3x = 4.99…95 + 2.99…97 = 7.99…92 = 8x$.

Table 6

x	0.999…
2x	1.99…98
3x	2.99…97
4x	3.99…96
5x	4.99…95
6x	5.99…94
7x	6.99…93
8x	7.99…92
9x	8.99…91
10x	9.99…90

Also, $\frac{8x}{2} = \frac{7.99...92}{2} = 3.99\ldots96 = 4x$.

Thus it can be seen that this view of multiplication of an infinite string is the more 'consistent' approach.

But the argument that 10x = 9.999… is the basis for the conversion of recurring decimals to fractions. So, does it mean that all these well accepted proofs are invalid?

Let us once again look at the 'consistent' approach and see what happens.

x = 0.999…

10x = 9.99…90

10x – x = 9.99…90 – 0.999...

Hence 9x = 8.99…91

But 8.99…91 can be expressed as 9 – 0.00…09

Thus $9x = 9 - 0.00\ldots09 = 9 - \frac{9}{10^{\aleph_0}}$

Or $x = 1 - \frac{1}{10^{\aleph_0}} = 1 - \frac{1}{2^{\aleph_0}}$

Rearranging terms, $1 = x + \frac{1}{2^{\aleph_0}}$

Or $1 = 0.999\ldots + \frac{1}{2^{\aleph_0}}$ …(15)

which is the same as Equations (12) and (14).

This can be seen even more clearly if the decimal string x is expressed as a geometric series

$$x = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots + \frac{9}{10^{\aleph_0}}$$

$$10x = 9 + \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots + \frac{0}{10^{\aleph_0}}$$

$$10x - x = 9 - \frac{9}{10^{\aleph_0}}$$

$$9x = 9 - \frac{9}{10^{\aleph_0}}$$

Dividing by 9, $x = 1 - \frac{1}{10^{\aleph_0}} = 1 - \frac{1}{2^{\aleph_0}}$ and therefore $1 = x + \frac{1}{2^{\aleph_0}}$

Or $1 = 0.999\ldots + \frac{1}{2^{\aleph_0}}$ as in Equation (15)

From Equation (12), (14) and (15), it can be seen that $1 = 0.999\ldots + \frac{1}{2^{\aleph_0}}$. *However, as* $\frac{1}{2^{\aleph_0}}$ *is defined as zero or First Approachable Śūnya, the identity 1 = 0.999… is also valid. But perhaps for better understanding in classrooms, it could initially be explained as* $1 = 0.999\ldots + \frac{1}{2^{\aleph_0}}$.

Add a spoonful of curd
And the milk turns into a potful of curd;
Add a tiny fraction of *Śūnya*
And the recurring decimal curdles into a fraction…

7. Cantor-Gulmohur

Take the convergent geometric series $1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ If the different terms of this series could be compared to the segments of a stem, then the zero of the series (or First Approachable *Śūnya*) could be compared to its flower.

Branch is similar to stem
Twig is similar to branch
But the flaming flower born at the end
Is dazzlingly different from them all…

Further, we know that $1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ and in general $\frac{1}{n} = \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \cdots$ where n is any positive integer. So the terms of the series $1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ like $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ etc could again be expanded as convergent geometric series as shown below

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots \text{ and } \frac{1}{8} = \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \ldots \text{ etc.}$$

The terms of these series could also be expanded similarly and so on. Thus the single stem of 1 would consist of $\aleph_0$ segments, each of which could branch into $\aleph_0$ sub-branches and so on without end. And each such diverse series will have a flower of $\frac{1}{2^{\aleph_0}}$ at its end. This tree of convergent geometric series of rational numbers, fully covered with flowers, could be designated Cantor-Gulmohar in honour of the discoverer of transfinite cardinals.

Like Prometheus Cantor stole the fire of infinity
And was punished daily by the gnawing eagles of doubt;
Like a portion of the liver growing back to fullness
His intellectual rigour would then shoo away the demon...

Cantor was the Abhimanyu among mathematicians
And entered the arena like a glad gladiator;

Surrounded by enemies within and outside
He fell, but not before blazing a new trail…

Across the starlit heights of empyrean
Cantor built the transfinite pathway;
A staircase fit for Divine Descend –
It now tests the endurance of seekers…

As the bitter turns nectar inside the jackfruit
The rind becomes thicker and thornier;
As the core of his Cardinals fructified into brilliance
His discourse was enveloped by Shakespeare-Bacon…

Hubble trained the telescope on the skies
And lo, there were flaming galaxies with increasing distances in between;
Cantor trained the power set on numberline
And lo, there were blazing cardinals with increasing numbers in between…

8. First Approachable *Śūnya* as a probability

Library of Babel, as conceived by Jorge Luis Borges (1979), is an interminably mammoth structure that hosts all possible books of 410 pages each, formed by every possible combination of 25 characters. Borges concludes: "The Library is unlimited and cyclical. If an eternal traveler were to cross it in any direction, after centuries he would see that the same volumes were repeated in the same disorder (which, thus repeated, would be an order: the Order)." Perhaps a fitting embellishment to the gate of this infinite library would be a combination lock with $\aleph_0$ dials, each of which would consist of the 10 digits from 0 to 9. The probability that a blind librarian can open it in a single try is $\frac{1}{10^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$, the First Approachable *Śūnya*. (If each dial consisted of the two choices 'True / False', 'Yes / No' etc., then also this probability is $\frac{1}{2^{\aleph_0}}$, straightaway.)

And what I assume you shall assume…

Do I contradict myself?
Very well then I contradict myself,
(I am large, I contain multitudes.)(Whitman, 1855)

9. Arithmetic of First Approachable *Śūnya*

It is easier to understand Transfinite Cardinals and First Approachable *Śūnya* (which, after all, is only a reciprocal of Transfinite Cardinal) by using the metaphor of Fire. A very large number of bamboos in a forest give birth to spark by rubbing against each other. This grows into a huge conflagration engulfing the entire forest. Now the cardinality or 'numerosity' of the trees is different from the 'numerosity' of the conflagration that is born of them. Let us consider the trees as finite numbers and the conflagration or Agni as an infinite cardinal.

> Despite consuming innumerable trees
> The immense wildfire roars in hunger;
> Few logs of wood added on subtracted
> Mean nothing to the menacing apparition…

Now for their arithmetic

tree + agni = agni
agni – tree = agni
tree × agni = agni
tree ÷ agni = ember

More specifically

1 trees + agni = 2 trees + agni
But 1 ≠ 2
Agni – 2 trees = agni – 3 trees
But 2 ≠ 3
3 trees × agni = 4 trees × agni
But 3 ≠ 4
4 tree ÷ agni = 5 trees ÷ agni
But 4 ≠ 5

The mere contact with agni modifies / destroys the limited cardinality of trees. Hence it is better to put parenthesis and show the relationship in clearer light.

(1 tree + agni) = (2 trees + agni) = agni
(agni – 2 trees) = (agni – 3 trees) = agni
(3 trees × agni) = (4 trees × agni) = agni
(4 tree ÷ agni) = (5 trees ÷ agni) = ember

If n is any finite number and a Transfinite Cardinal, then

$$n + 2^{\aleph_0} = 2^{\aleph_0}$$

$$2^{\aleph_0} - n = 2^{\aleph_0}$$

$$n \times 2^{\aleph_0} = 2^{\aleph_0}$$

$\dfrac{n}{2^{\aleph_0}} = n \times \dfrac{1}{2^{\aleph_0}} = \dfrac{1}{2^{\aleph_0}} = S_1$, where S_1 is the First Approachable *Śūnya*.

Similarly, for any finite number n and the First Approachable *Śūnya* S_1

$$n + S_1 = n$$

$$n - S_1 = n$$

$$n \times S_1 = n \times \frac{1}{2^{\aleph_0}} = \frac{1}{2^{\aleph_0}} = S_1$$

$\dfrac{n}{S_1} = n \times \dfrac{2^{\aleph_0}}{1} = 2^{\aleph_0}$, the corresponding Transfinite Cardinal.

Arithmetic of zero closely follows the arithmetic of First Approachable *Śūnya*, except for division. And arithmetic of First Approachable *Śūnya* shows the same pattern as the arithmetic of transfinite cardinals. Does it give a clue that zero on the linear numberline — though it appears to be like its finite neighbours — is more of a transfinite cardinal than a mere nothing?

> Masquerading as a man among men
> The Infinite acts as charioteer to Arjuna;
> On both sides are arrayed mighty warriors
> While He sits with a mere whip in hand…

Table 7

Arithmetic of *Śūnya*	**Arithmetic of Zero**
$n + S_1 = n$	$n + 0 = n$
$n - S_1 = n$	$n - 0 = n$
$n \times S_1 = S_1$	n x 0 = 0
$\dfrac{n}{S_1} = 2^{\aleph_0}$	$\dfrac{n}{0}$ = Undefined

10. Concluding Remarks

Since the cardinality of rational numbers is limited to aleph$_0$ ($\aleph_0$), summation of convergent geometric series of rational numbers leads inevitably to the concept of Approachable *Śūnya*. Findings of neuroscience point to the pre-eminence of logarithmic number line over linear one. And on such a number line, the First Approachable *Śūnya* is practically another zero that one encounters on the way to Unattainable Absolute Zero. Could there be other Approachable *Śūnyas*? What is their relationship with number lines, both linear and logarithmic? These could be possible leads for future work.

Does the concept of Approachable *Śūnya* have any bearing on the paradoxes of Zeno?

I kept two ice cubes on the table
After sometime, the cubes were zero
But the water was not…

I kept the water in a vessel
After a long time the water was zero
But the vapour was not…

I kept the vapour in a dragon's cauldron
After Bhāgīratha-like effort the vapour was zero
But the plasma was not…

When will I reach zero, I asked myself
But no answer was forthcoming.
Later an inner prompting rose on its own:
Kalpa must pass as per His *Saṅkalpa*…

Acknowledgement

To the One who shines as the Many by a fraction of His Splendour.

Bibliography

Byers, W., *How Mathematicians Think: Using Ambiguity, Contradiction, And Paradox To Create Mathematics,* Princeton University Press, 2007, pp. 40–41 & 269-270.

Borges, J.L., *Labyrinths: Selected stories and other writings*, Penguin Books, 1979, pp.78–86.

Dehaene, S., Izard, V., Spelke, E., Pica, P., "Log or linear? distinct intuitions of the number scale in Western and Amazonian indigene cultures", *Science*, 320(5880)(2008), 1217–1220.

Dehaene, S., *The Number Sense: How The Mind Creates Mathematics*, Oxford University Press, 1997.

Dehaene, S., "The neural basis of the Weber–Fechner law: a logarithmic mental number line", *Trends in Cognitive Sciences*, 7.4 (2003) 145-147.

Katz, K., Katz, M., "When is .999... less than 1?", *The Montana Mathematics Enthusiast*, 7.1 (2010) 3-30.

Heath, T.L., *The Thirteen Books of Euclid's Elements*, Cambridge University Press, 1908

Heath, T.L., *The Works Of Archimedes*, Cambridge University Press, 1953.

Pica, P., Lemer, C., Izard, V., Dehaene, S., "Exact and Approximate Arithmetic in an Amazonian Indigene Group", *Science*, 306(5695) (2004) 499-503.

Robinson, A., *Non-standard analysis* (Revised edition), Princeton University Press, 1996.

Sierpinska, A., *Understanding in mathematics, (Studies in Mathematics Education Series,* 2*)*, Falmer Press, Bristol PA, 1994.

Siegler, R.S., Booth, J. L., "Development of numerical estimation in young children", *Child Development*, 75.2 (2004) 428-444.

Siegler, R. S., Opfer, J. E., "The development of numerical estimation: evidence for multiple representations of numerical quantity", *Psychological Science*, 14.3 (2003) 237-43.

Singh, A. N., "On the use of series in Hindu mathematics", *Osiris*, 1(1936), 606-–628.

Tall, D., Schwarzenberger, R.L.E., "Conflicts in the learning of real numbers and limits", *Mathematics Teaching*, 82 (1978) 44-49.

Tall, D., "Intuitions of Infinity", *Mathematics in School*, 10.3 (1981) 30-33.

Thurston, W.P., "On proof and progress in mathematics", *Bulletin of the American Mathematical Society*, 30.2 (1994) 161-177.

Whitman, W., *Song of Myself*, Dover Publications, 1855. (Republished 2001), lines 2 & 1324-1326.