1st Canadian Mathematical Olympiad 1969

1. If $a_1/b_1 = a_2/b_2 = a_3/b_3$ and p_1, p_2, p_3 are not all zero, show that for all $n \in \mathbb{N}$,
$$\left(\frac{a_1}{b_1}\right)^n = \frac{p_1a_1^n + p_2a_2^n + p_3a_3^n}{p_1b_1^n + p_2b_2^n + p_3b_3^n}$$
2. Let $c \geq 1$. Determine which of $\sqrt{c+1} - \sqrt{c}$ and $\sqrt{c} - \sqrt{c-1}$ is larger.
3. Let a hypothenuse of a right-angled triangle has length c and the other two sides have lengths a, b. Prove that $a + b \leq c\sqrt{2}$. When does the equality hold?
4. Let P be an arbitrary point within an equilateral triangle ABC, and PD, PE, PF be the perpendiculars drawn to the sides of the triangle. Show that
$$\frac{PD+PE+PF}{AB+BC+CA} = \frac{1}{2\sqrt{3}}.$$
5. Let a triangle ABC have side lengths a, b and c. The bisector of the angle C intersects AB at D. Prove that $CD = \dfrac{2ab\cos(C/2)}{a+b}$.
6. Find the sum $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!$.
7. Show that there are no integers a, b, c satisfying $a^2 + b^2 = 8c + 6$.
8. Let $f : \mathbb{N} \to \mathbb{Z}$ be a function such that $f(2) = 2$, $f(mn) = f(m)f(n)$ for all m, n, and $f(m) > f(n)$ whenever $m > n$. Prove that $f(n) = n$.
9. If a quadrilateral is inscribed in a circle of radius 1, show that the length of its shortest side does not exceed $\sqrt{2}$.
10. Let ABC be a triangle with $AC = BC = 1$ and $\angle C = 90^\circ$. Let P be an arbitrary point on side AB and Q, R be the feet of perpendiculars from P to AC, BC, respectively. Consider the areas of APQ, PBR and $QCRP$. Prove that the largest of these three areas is at least $2/9$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com