Quasigroups and Related Systems **8** (2001), 15 – 36

On some old and new problems in n-ary groups

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Abstract

In this paper some old unsolved problems connected with skew elements in n-ary groups are discussed.

1. Introduction

A nonempty set G together with one n-ary operation $f : G^n \longrightarrow G$ is called an *n-ary groupoid* and is denoted by $\langle G, f\rangle$. We say that this groupoid is *i-solvable* or *solvable at the place i* if for all $a_1, ..., a_n, b \in G$ there exists $x_i \in G$ such that

$$f(a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n) = b\,. \tag{1}$$

If this solution is unique, we say that this groupoid is *uniquely i-solvable*. An n-ary groupoid which is uniquely i-solvable for every $i = 1, 2, \ldots, n$ is called an *n-ary quasigroup* or *n-quasigroup* (cf. [3]).

We say that an n-ary groupoid $\langle G, f\rangle$ is *(i, j)-associative* if

$$f(a_1, ..., a_{i-1}, f(a_i, ..., a_{i+n-1}), a_{i+n}, ..., a_{2n-1}) \\ = f(a_1, ..., a_{j-1}, f(a_j, ..., a_{j+n-1}), a_{j+n}, ..., a_{2n-1}),$$

holds for all $a_1, \ldots, a_{2n-1} \in G$. If an n-ary operation is (i, j)-associative for every $i, j \in \{1, \ldots, n\}$, then it is called *associative.* An n-ary groupoid with an associative operation is called an *n-ary semigroup* or *n-semigroup.* An n-semigroup which is also an n-quasigroup is called an *n-ary group* (briefly: *n-group*) or a *polyadic group* (cf. [31]) .

2000 Mathematics Subject Classification: 20N15, 08N05
Keywords: n-ary group, skew element, variety
This paper is an extended version of my talk given at the First Conference of the Mathematical Society of the Republic of Moldova, Kishinev, August 16-18, 2001.

For $n = 2$ it is an ordinary group. For infinite n, where n is a countable infinite number, it is an *infinitary* group. Unfortunately all such groups are trivial (have only one element), but there are non-trivial infinitary quasigroups and semigroups (cf. [4]). In connection with this we assume throughout the whole text that $3 \leqslant n < \infty$.

The first idea of such generalization of groups was presented by E. Kasner in the lecture at the fifty-third annual meeting of the American Association for the Advancement of Science, reported by L. G. Weld in The Bulletin of the American Mathematical Society in 1904 (cf. [25]), but the first formal definition was given by W. Dörnte in the paper [6] based on his dissertation prepared under the inspiration of E. Noether.

Sets with one n-ary operation having different properties were investigated by many authors. For example, J. Certaine [5] and D. H. Lehmer [27] described some natural *ternary* (i.e. $n = 3$) operations defined on a group. Some ternary groupoids having interesting applications to projective and affine geometry were considered by R. Baer [2], H. Prüfer [32], A. K. Sushkevich [39] and V. V. Vagner [41]. Ternary quasigroups are used in [37] and [38] to the characterization of Mendelsohn and Steiner quadruple systems.

On the other hand, G. A. Miller [28] described sets of group elements involving only products of more than n elements. Some n-ary operations have interesting applications in physics. For example, Y. Nambu [29] proposed in 1973 the generalization of classical Hamiltonian mechanics based on the Poisson bracket to the case when the new bracket, now called the *Nambu bracket*, is an n-ary operation on classical observables. The author of [40] suspects that different n-ary structures such as n-Lie algebras, Lie ternary systems and linear spaces with additional internal n-ary operations, might clarify many important problems of modern mathematical physics (Yang-Baxter equation, Poisson-Lie groups, quantum groups). For example, ternary Z_3–graded algebras are important (cf. [26]) for their applications in physics of elementary interactions. Unfortunately, from the mathematical point of view all such structures are rather complicated, especially for $n > 3$.

The above definition of an n-ary group is a generalization of H. Weber's formulation of axioms of groups. Similar generalization of L. E. Dickson's axioms one leads to n-ary groups $\langle G, f\,\rangle$ *derived from a group* $\langle G, \cdot\,\rangle$, i.e. to n-ary groups with the operation

$$f(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

(cf. [1] and [33]). But for every $n \geqslant 3$ there are n-groups which are not derived from any group (cf. [6], [9], [10]).

E. L. Post observed in [31] that under the assumption of the n-ary associativity it suffices only to postulate the existence of the solution of (1) at the place $i = 1$ and $i = n$, or at one place i other than 1 and n. Then one can prove uniqueness of the solution of (1) for all $i = 1, 2, \dots, n$.

Also the assumption on the associativity can be given in the weaker form. For example, in [18] the following theorem is proved.

Theorem 1. (Dudek, Głazek, Gleichgewicht 1977) *An n-ary groupoid $\langle G, f \rangle$ is an n-ary group if and only if* (*at least*) *one of the following conditions is satisfied:*

a) *the $(i, i+1)$-associative law holds for some $i \in \{2, \dots, n-2\}$ and the equation* (1) *is uniquely solvable for i and some $k > i$,*

b) *the $(1, 2)$-associative law holds and the equation* (1) *is solvable for $i = n$ and uniquely solvable for $i = i$,*

c) *the $(n-1, n)$-associative law holds and the equation* (1) *is solvable for $i = 1$ and uniquely solvable for $i = n$.*

The class of n-ary groups can be characterized also as the class of n-ary semigroups with two binary operations satisfying two simple identities, or as the class of n-ary semigroups in which some two equations containing only two variables are solvable (cf. [13]).

2. Skew elements and endomorphisms

According to the definition of an n-ary group $\langle G, f \rangle$ for every $x \in G$ there exists only one $z \in G$ such that

$$f(x, ..., x, z) = x\,.$$

This element is called *skew* to x and is denoted by $\bar{x}$. Since for every $x \in G$ there exists only one $\overline{x}$, the above equation induces on G the new unary operation $^{-}: x \to \bar{x}$. This means that an n-ary group $\langle G, f \rangle$ can be considered as an algebra $\langle G, f, ^{-} \rangle$ of type $(n, 1)$ with two fundamental operations: an n-ary one f and an unary one $^{-}: x \to \bar{x}$, which gives some analogy with the binary case when a group is considered as an algebra $\langle G, \cdot, ^{-1} \rangle$ of type $(2, 1)$. In a binary group we have $xe = x$ for all x and some fixed e. For $n = 3$ this identity can be generalized to the form $f(x, e, e) = x$

or $f(x,x,e)=x$. The first form, for a ternary group derived from a binary group $\langle G,\cdot\rangle$, implies that e is the neutral element of $\langle G,\cdot\rangle$, the second – that e is the inverse of x (in $\langle G,\cdot\rangle$, obviously). Thus, in some sense, the skew element is a common generalization of the identity and the inverse element of a binary group.

In n-ary groups derived from binary groups we have $\bar{x}=x^{2-n}$ and

$$f(y,x,\ldots,\bar{x},\ldots,x)=f(x,\ldots,\bar{x},\ldots,x,y)=y \qquad (2)$$

for all x,y, where $\bar{x}$ can appear at any place under the sign of the n-ary operation. This shows that in an n-ary group derived from a group $\langle G,\cdot\rangle$ of the exponent $n-2$ the neutral element of $\langle G,\cdot\rangle$ is skew to every $x\in G$. In an n-ary group derived from a group $\langle G,\cdot\rangle$ of the exponent $n-3$ we have $\bar{x}=x^{-1}$ and $\bar{x}\neq\bar{y}$ for all $x\neq y$. If the exponent of $\langle G,\cdot\rangle$ is equal to $n-1$, then $\bar{x}=x$ for all $x\in G$.

An element $x=\bar{x}$ is called *idempotent*. It is also defined by the equation $f(x,\ldots,x)=x$. For every $n\geqslant 3$ there are n-ary groups without idempotents and n-ary groups in which only some elements are idempotent (cf. [10]). A group in which all elements are idempotent is called an *idempotent group*.

The operation $^{-}\colon x\to\bar{x}$ plays an important role in the theory of n-ary groups and in their applications to affine geometry (cf. [21] and [35]). This operation can be used also to the definition of n-ary groups (cf. [23] and [18]). The minimal axioms system defining of n-ary groups is given in the following theorem proved in [8].

Theorem 2. (Dudek 1980) *The class of n-ary groups $\langle G,f\rangle$ coincides with the variety of all $(1,2)$-associative n-ary groupoids $\langle G,f\rangle$ with an additional unary operation $^{-}\colon x\to\bar{x}$ satisfying the identity (2), where $\bar{x}$ appears at one fixed place.*

It is not difficult to see that in an n-ary group $\langle G,f\rangle$ derived from a commutative group the following identity holds:

$$\overline{f(x_1,x_2,\ldots,x_n)}=f(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)\,. \qquad (3)$$

It holds also in the non-commutative 8-group derived from the group S_3 and in every idempotent n-group. For $x_1=x_2=\ldots=x_n=x$ it is satisfied in any n-ary group.

From the proof of Theorem 3 in [22] it immediately follows that this identity holds in all *medial* (in the sense of Belousov [3]) n-ary groups, i.e.

in all n-ary groups in which the identity

$$f(f(x_{11},x_{12},\ldots,x_{1n}),f(x_{21},x_{22},\ldots,x_{2n}),\ldots,f(x_{n1},x_{n2},\ldots,x_{nn}))$$
$$= f(f(x_{11},x_{21},\ldots,x_{n1}),f(x_{12},x_{22},\ldots,x_{n2}),\ldots,f(x_{1n},x_{2n},\ldots,x_{nn}))$$

is satisfied. For $n=2$ it is the standard medial (entropic) law, which in the case of groups gives the commutativity. For $n\geqslant 3$ it not implies the commutativity of n-ary groups.

Since an n-ary group $\langle G,f\rangle$ is medial if and only if there exists $a\in G$ such that $f(x,a,\ldots,a,y)=f(y,a,\ldots,a,x)$ for all $x,y\in G$ (cf. [8]), the Hosszú theorem (cf. [24]) suggests the following result proved in [10].

Theorem 3. (Dudek 1988) *If for an n-ary group $\langle G,f\rangle$ there exists a commutative group $\langle G,+\rangle$, an element $b\in G$, and an automorphism φ of $\langle G,+\rangle$ such that $\varphi(b)=b$, $\varphi^{n-1}(x)=x$ for all $x\in G$ and*

$$f(x_1,x_2,\ldots,x_n)=x_1+\varphi(x_2)+\varphi^2(x_3)+\ldots+\varphi^{n-2}(x_{n-1})+x_n+b\,,$$

then (3) *is satisfied.*

Unfortunately the converse statement is not true.

In connection with this the following problem was posed in [10].

Problem 1. *Describe the class of all n-ary groups satisfying* (3)*, i.e. the class of n-ary groups for which $h(x)=\bar{x}$ is an endomorphism.*

For $n=3$ the answer is simple, because as proved W. Dörnte (cf. [6]) in all ternary groups we have $\overline{f(x,y,z)}=f(\bar{z},\bar{y},\bar{x})$. This means that a ternary group satisfies (3) if and only if it is medial.

For $n>3$ the problem is open. We know only the partial answer basing on the general connections between homomorphisms of n-ary groups and homomorphisms of their retracts (Theorem 2 from [20]).

Theorem 4. *A mapping $h:G\to G$ is an endomorphism of an n-ary group $\langle G,f\rangle$ if and only if there exists $a\in G$ such that*

(i) $h(f(x,a,\ldots,a,y))=f(h(x),b,\ldots,b,h(y))$,

(ii) $h(f(\bar{a},x,a,\ldots,a))=f(\bar{b},h(x),b,\ldots,b)$,

(iii) $h(f(\bar{a},\bar{a},\ldots,\bar{a}))=f(\bar{b},\bar{b},\ldots,\bar{b})$

for all $x, y \in G$ *and* $b = h(a)$.

Proof. Let $h : G \to G$ be an endomorphism of an n-ary group $\langle G, f\rangle$. If $h(a) = b$, then, according to the identity (2) and Theorem 2,

$$h(y) = h(f(y, a, \ldots, a, \bar{a}) = f(h(y), b, \ldots, b, h(\bar{a})),$$

which gives $h(\bar{a}) = \bar{b}$. Now, the conditions (i), (ii) and (iii) are obvious.

Conversely, assume that a mapping $h : G \to G$ satisfies the above three conditions for all $x, y \in G$, some fixed $a \in G$ and $b = h(a)$.

From the proof of Hosszú theorem given by E. I. Sokolov (cf. [36] or [19]) it immediately follows that $\langle G, +\rangle$, where $x + y = f(x, a, \ldots, a, y)$, is a binary group, $\varphi(x) = f(\bar{a}, x, a, \ldots, a)$ its automorphism such that for $c = f(\bar{a}, \bar{a}, \ldots, \bar{a})$ the following identity

$$f(x_1, x_2, \ldots, x_n) = x_1 + \varphi(x_2) + \varphi^2(x_3) + \ldots + \varphi^{n-1}(x_n) + c \qquad (5)$$

holds. Similarly, for $x \diamond y = f(x, b, \ldots, b, y)$, $\psi(x) = f(\bar{b}, x, b, \ldots, b)$ and $d = f(\bar{b}, \bar{b}, \ldots, \bar{b})$, we have

$$f(x_1, x_2, \ldots, x_n) = x_1 \diamond \psi(x_2) \diamond \psi^2(x_3) \diamond \ldots \diamond \psi^{n-1}(x_n) \diamond d\,.$$

Thus $h(x + y) = h(x) \diamond h(y)$ by (i), $h(\varphi(x)) = \psi(h(x))$ by (ii), and $h(c) = d$ by (iii). Therefore

$$\begin{aligned} h(f(x_1, x_2, \ldots, x_n)) &= h(x_1 + \varphi(x_2) + \varphi^2(x_3) + \ldots + \varphi^{n-1}(x_n) + c) \\ &= h(x_1) \diamond \psi(h(x_2)) \diamond \psi^2(h(x_3)) \diamond \ldots \diamond \psi^{n-1}(h(x_n)) \diamond d \\ &= f(h(x_1), h(x_2), \ldots, h(x_n))\,, \end{aligned}$$

which proves that h is an endomorphism. □

Putting in the above theorem $h(x) = \bar{x}$, we obtain

Corollary 1. *An n-ary group $\langle G, f\rangle$ satisfies* (3) *if and only if there exists $a \in G$ such that*

(i) $\overline{f(x, a, \ldots, a, y)} = f(\bar{x}, \bar{a}, \ldots, \bar{a}, \bar{y})$,

(ii) $\overline{f(\bar{a}, x, a, \ldots, a)} = f(\bar{\bar{a}}, \bar{x}, \bar{a}, \ldots, \bar{a})$,

(iii) $\overline{f(\bar{a}, \bar{a}, \ldots, \bar{a})} = f(\bar{\bar{a}}, \bar{\bar{a}}, \ldots, \bar{\bar{a}})$

for all $x, y \in G$, where $\bar{\bar{a}}$ is skew to $\bar{a}$.

Corollary 2. *An n-ary group $\langle G, f\rangle$ with an idempotent $a\in G$ satisfies* (3) *if and only if for all $x, y\in G$, we have*

$$(i)\quad \overline{f(x,a,\ldots,a,y)} = f(\bar{x}, a,\ldots,a,\bar{y}\,),$$

$$(ii)\quad \overline{f(a,x,a,\ldots,a)} = f(a,\bar{x},a,\ldots,a\,).$$

Proof. Indeed, if $a\in G$ is an idempotent, then $\bar{a}=a$ and, in the consequence, $\bar{\bar{a}}=a$, which together with $f(a,\ldots,a)=a$ gives the condition (*iii*) from Corollary 1. The rest is obvious. □

In the same manner as Theorem 4, putting $x+y=f(x,\bar{a},a,\ldots,a,y)$, $\varphi(x)=f(a,x,\bar{a},a,\ldots,a)$, $c=f(a,a,\ldots,a)$ and $x\diamond y=f(x,\bar{b},b,\ldots,b,y)$, $\psi(x)=f(b,x,\bar{b},b,\ldots,b)$, $d=f(b,b,\ldots,b)$, we can prove

Theorem 5. *A mapping $h: G\to G$ is an endomorphism of an n-ary group $\langle G, f\rangle$ if and only if there exists $a\in G$ such that*

$$(i)\quad h(f(x,\bar{a},a,\ldots,a,y)) = f(h(x),\bar{b},\ldots,b,h(y)),$$

$$(ii)\quad h(f(a,x,\bar{a},a,\ldots,a)) = f(b,h(x),\bar{b},b,\ldots,b),$$

$$(iii)\quad h(f(a,a,\ldots,a)) = f(b,b,\ldots,b)$$

for all $x,y\in G$ and $b=h(a)$.

Putting in this theorem $h(x)=\bar{x}$, we obtain

Corollary 3. *An n-ary group $\langle G, f\rangle$ satisfies* (3) *if and only if there exists $a\in G$ such that*

$$(i)\quad \overline{f(x,\bar{a},a,\ldots,a,y)} = f(\bar{x},\bar{\bar{a}},\bar{a},\ldots,\bar{a},\bar{y}\,),$$

$$(ii)\quad \overline{f(a,x,\bar{a},a,\ldots,a)} = f(\bar{a},\bar{x},\bar{\bar{a}},\bar{a},\ldots,\bar{a}\,),$$

$$(iii)\quad \overline{f(a,a,\ldots,a)} = f(\bar{a},\bar{a},\ldots,\bar{a}\,)$$

for all $x,y\in G$, where $\bar{\bar{a}}$ is skew to $\bar{a}$.

Corollary 4. *If an n-ary group $\langle G, f\rangle$ has an element $a\in G$ such that*

$$(i)\quad \overline{f(x,\bar{a},a,\ldots,a,y)} = f(\bar{x},\bar{a},a,\ldots,a,\bar{y}\,),$$

$$(ii)\quad \overline{f(a,x,\bar{a},a,\ldots,a)} = f(a,\bar{x},\bar{a},a,\ldots,a\,)$$

for all $x,y\in G$, then $h(x)=\bar{x}$ is an endomorphism of $\langle G, f\rangle$.

Proof. It is not difficult to verify (using (2) and Theorem 2) that for $x+y = f(x,\bar{a},a,\ldots,a,y)$, $\varphi(x) = f(a,x,\bar{a},a,\ldots,a)$ and $c = f(a,a,\ldots,a)$ the identity (5) holds. Obviously $\langle G,+\rangle$ is a group and a is its neutral element. Thus $a = a+a$ and, in the consequence, $\bar{a} = \overline{a+a} = \bar{a}+\bar{a}$ by (i). Hence $\bar{a} = a$ and $c = a$. Therefore, in our case, the identity (5) has the form

$$f(x_1,x_2,\ldots,x_n) = x_1+\varphi(x_2)+\varphi^2(x_3)+\ldots+\varphi^{n-1}(x_n)\,.$$

But, by (i) and (ii), for all $x,y\in G$ we have $\overline{x+y} = \bar{x}+\bar{y}$, $\overline{\varphi(x)} = \varphi(\bar{x})$, which gives

$$\begin{aligned}\overline{f(x_1,x_2,\ldots,x_n)} &= \bar{x}_1+\overline{\varphi(x_2)}+\overline{\varphi^2(x_3)}+\ldots+\overline{\varphi^{n-1}(x_n)}\\ &= \bar{x}_1+\varphi(\bar{x}_2)+\varphi^2(\bar{x}_3)+\ldots+\varphi^{n-1}(\bar{x}_n)\\ &= f(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)\,.\end{aligned}$$

Hence $h(x) = \bar{x}$ is an endomorphism of an n-ary group $\langle G,f\,\rangle$. □

The converse is not true. Indeed, in an n-ary group $\langle Z,f\,\rangle$, where Z is the set of integers, $f(x_1,\ldots,x_n) = x_1+\ldots+x_n+1$, $h(x) = \bar{x} = (2-n)x-1$ is an endomorphism, but (i) and (ii) are not satisfied. Moreover in this n-ary group $\bar{x}\neq\bar{y}$ for $x\neq y$. But there are n-groups in which $\bar{x}=\bar{y}$ for all x,y. In such n-groups one fixed element is skew to all others. Obviously this element is an idempotent. This suggest the following characterization given in [11].

Theorem 6. (Dudek 1990) *An n-ary group satisfies the identity $\bar{x}=\bar{y}$ if and only if it is derived from a binary group of the exponent $t\,|\,n-2$.*

If an element a is skew to all $x\in G$, then an n-group $\langle G,f\rangle$ is derived from a binary group $\langle G,\circ\,\rangle$, where $x\circ y = f(x,a,\ldots,a,y)$. Obviously a is the identity of $\langle G,\circ\,\rangle$. Moreover, by (2), for all $x\in G$ we have

$$a = f(a,x,\ldots,x,\bar{x}\,) = f(a,x,\ldots,x,a)\,,$$

which implies the identity

$$f(a,x,\ldots,x,a) = f(a,y,\ldots,y,a)\,. \tag{9}$$

Conversely, if there exists $a \in G$ such that (9) holds for all $x, y \in G$, then $f(a, x, \ldots, x, a) = f(a, \ldots, a)$. Therefore, applying (2), we obtain

$$\begin{aligned} f(a, \ldots, a, \bar{a}) = a = f(a, x, \ldots, x, \bar{x}) &= f(a, x, \ldots, x, f(a, \ldots, a, \bar{a}, \bar{x})) \\ &= f(f(a, x, \ldots, x, a), a, \ldots, a, \bar{a}, \bar{x}) \\ &= f(f(a, \ldots, a), a, \ldots, a, \bar{a}, \bar{x}) = f(a, \ldots, a, f(a, \ldots, a, \bar{a}, \bar{x})) \\ &= f(a, \ldots, a, \bar{x}), \end{aligned}$$

which implies $\bar{a} = \bar{x}$ for all $x \in G$.

Thus the following theorem is true.

Theorem 7. *An n-ary group satisfies the identity $\bar{x} = \bar{y}$ if and only if there exists $a \in G$ such that* (9) *holds for all $x, y \in G$.*

Problem 2. *Describe n-ary groups in which $\bar{x} \neq \bar{y}$ for all $x \neq y$.*

Problem 3. *When $h(x) = \bar{x}$ is an automorphism* ?

Let $\bar{x}^{(0)} = x$ and let $\bar{x}^{(k+1)}$ be the skew element to $\bar{x}^{(k)}$, where $k \geqslant 0$. In other words, $\bar{x}^{(0)} = x$, $\bar{x}^{(1)} = \bar{x}$, $\bar{x}^{(2)} = \bar{\bar{x}}$, $\bar{x}^{(3)} = \bar{\bar{\bar{x}}}$, etc.

For example, in a 4-group derived from the additive group Z_8, we have $\bar{x} \equiv 6x \,(mod\, 8)$, $\bar{\bar{x}} \equiv 4x \,(mod\, 8)$ and $\bar{x}^{(k)} \equiv 0 \,(mod\, 8)$ for $k \geqslant 3$. In the n-group derived from the additive group of integers: $\bar{x}^{(k)} \neq \bar{x}^{(t)}$ for all $x \neq 0$ and $k \neq t$. But in any ternary group $\bar{\bar{x}} = x$ for all x (cf. [6]).

If $\bar{x}^{(k)} = x$ and $\bar{y}^{(t)} = y$ for some $k, t > 1$, then $\bar{x} = \bar{y}$ if and only if $x = y$. If $h(x) = \bar{x}$ is an automorphism, then $h(x) = \bar{x}^{(k)}$ is an automorphism, too. The converse is not true, because $h(x) = \bar{\bar{x}}$ is an identity automorphism of any ternary group, but $h(x) = \bar{x}$ is an automorphism only in the case when this group is medial.

Problem 4. *Describe the class $\mathbf{W_k}$ of n-ary groups in which $h(x) = \bar{x}^{(k)}$ is an endomorphism* (*automorphism*)*.*

Obviously $\mathbf{W_1} \subset \mathbf{W_2} \subset \mathbf{W_3} \subset \ldots \subset \mathbf{W_0}$. When $\mathbf{W_k} = \mathbf{W_{k+1}}$?

As a simple consequence of Theorem 4, for $h(x) = \bar{x}^{(k)}$, we obtain

Corollary 5. *$h(x)=\bar{x}^{(k)}$ is an endomorphism of an n-ary group $\langle G,f\,\rangle$ if and only if there exists $a\in G$ such that*

$$(i)\quad \overline{f(x,a,\ldots,a,y)}^{\,(k)}=f(\bar{x}^{(k)},\bar{a}^{(k)},\ldots,\bar{a}^{(k)},\bar{y}^{(k)}\,),$$

$$(ii)\quad \overline{f(\bar{a},x,a,\ldots,a)}^{\,(k)}=f(\bar{a}^{(k+1)},\bar{x}^{(k)},\bar{a}^{(k)},\ldots,\bar{a}^{(k)}\,),$$

$$(iii)\quad \overline{f(\bar{a},\bar{a},\ldots,\bar{a})}^{\,(k)}=f(\bar{a}^{(k+1)},\bar{a}^{(k+1)},\ldots,\bar{a}^{(k+1)}\,)$$

for all $x,y\in G$.

Corollary 6. *If an n-ary group $\langle G,f\,\rangle$ contains an element a such that $a=\bar{a}^{(k)}$, then $h(x)=\bar{x}^{(k)}$ is an endomorphism of $\langle G,f\,\rangle$ if and only if*

$$(i)\quad \overline{f(x,a,\ldots,a,y)}^{\,(k)}=f(\bar{x}^{(k)},a,\ldots,a,\bar{y}^{(k)}\,),$$

$$(ii)\quad \overline{f(\bar{a},x,a,\ldots,a)}^{\,(k)}=f(\bar{a},\bar{x}^{(k)},a,\ldots,a\,),$$

$$(iii)\quad \overline{f(\bar{a},\bar{a},\ldots,\bar{a})}^{\,(k)}=f(\bar{a},\bar{a},\ldots,\bar{a}\,)$$

for all $x,y\in G$.

Corollary 7. *If an n-ary group $\langle G,f\,\rangle$ contains an idempotent a, then $h(x)=\bar{x}^{(k)}$ is an endomorphism if and only if*

$$(i)\quad \overline{f(x,a,\ldots,a,y)}^{\,(k)}=f(\bar{x}^{(k)},a,\ldots,a,\bar{y}^{(k)}\,),$$

$$(ii)\quad \overline{f(a,x,a,\ldots,a)}^{\,(k)}=f(a,\bar{x}^{(k)},a,\ldots,a\,)$$

for all $x,y\in G$.

We finish this section by the following problem.

Problem 5. *Describe the class $\mathbf{U_k}$ of n-ary groups in which $\bar{x}^{(k)}=\bar{y}^{(k)}$ for all elements x, y.*

The class $\mathbf{U_k}$ contains n-ary groups with only one k-*skew element*, i.e. n-ary groups in which there exists only one element a such that $a=\bar{x}^{(k)}$ for all x. Obviously $\mathbf{U_1}\subset\mathbf{U_2}\subset\mathbf{U_3}\subset\ldots$

It is not difficult to see that a ternary group belongs to $\mathbf{U_k}$ if and only if it is trivial (has only one element). The class $\mathbf{U_1}$ coincides with the class of all n-ary groups derived from binary groups of the exponent $t|n-2$ (Theorem 6). Generally, all n-ary groups derived from the binary group of the exponent $t\,|\,(n-2)^k$ belong to $\mathbf{U_k}$, but $\mathbf{U_k}$ contains also other groups.

3. Sequences

Now we consider the sequence

$$x,\ \bar{x}\,,\ \bar{x}^{(2)}\,,\ \bar{x}^{(3)}\,,\ \bar{x}^{(4)}\,,\ \ldots\,,\ \bar{x}^{(k)}\,,\ \ldots$$

If an n-ary group $\langle G, f\,\rangle$ is finite, then obviously $\bar{x}^{(k)} = \bar{x}^{(t)}$ for some $k \neq t$. (In a 6-ary group derived from the additive group Z_{12} for $x = 1$ we have: $1, 8, 4, 8, 4, 8, 4, \ldots$) But in some infinite n-ary groups (for example in an n-ary group derived from the additive group of integers) $\bar{x}^{(k)} \neq \bar{x}^{(t)}$ for all $k \neq t$.

In connection with this the following two problems were posed in [10].

Problem 6. *Describe infinite n-ary groups in which $\overline{x}^{(k)} \neq \overline{x}^{(m)}$ for all $k \neq m$ and all $x \in G$.*

Problem 7. *Describe n-ary groups in which there exists a natural number k such that $\overline{x}^{(k)} = \overline{x}^{(m)}$ for all $m \geqslant k$ and all $x \in G$.*

Following E. L. Post (cf. [31], p.282), we define the *n-ary power* putting

$$x^{<k>} = \begin{cases} f(x^{<k-1>}, x, \ldots, x) & \text{for} \quad k > 0, \\ x & \text{for} \quad k = 0, \\ y \,:\, f(y, x^{<-k-1>}, x, \ldots, x) = x & \text{for} \quad k < 0, \end{cases}$$

i.e. $x^{<0>} = x\,,$

$x^{<1>} = f(x, x, \ldots, x)\,,$

$x^{<2>} = f(x^{<1>}, x, x, \ldots, x)\,,$

$x^{<3>} = f(x^{<2>}, x, x, \ldots, x)\,,$

. . . .

A minimal natural number k (if it exists) such that $x^{<k>} = x$ is called an *n-ary order of* x and is denoted by $ord_n(x)$.

It is not difficult to verify that the following exponential laws hold

$$f(x^{<s_1>}, x^{<s_2>}, \ldots, x^{<s_n>}) = x^{<s_1+s_2+\ldots+s_n+1>},$$
$$(x^{<r>})^{<s>} = x^{<rs(n-1)+s+r>} = (x^{<s>})^{<r>}.$$

Using the above laws we can see that $\bar{x} = x^{<-1>}$ and, in the consequence

$$\bar{x}^{(2)} = (x^{<-1>})^{<-1>} = x^{<n-3>},$$
$$\bar{x}^{(3)} = ((x^{<-1>})^{<-1>})^{<-1>},$$

and so on. Generally: $\bar{x}^{(k)} = (\bar{x}^{(k-1)})^{<-1>}$ for all $k \geqslant 1$. This implies that $\bar{x}^{(k)} = x^{<S_k>}$ for

$$S_k = -\sum_{i=0}^{k-1}(2-n)^i = \frac{(2-n)^k - 1}{n-1}$$

Obviously $ord_n(\bar{x})$ is a divisor of $ord_n(x)$, and $ord_n(x)$ is a divisor of $Card(G)$. This last fact is a simple conclusion from Lagrange's theorem for finite n-ary groups (sf. [31], p.222). Hence

$$ord_n(x) \geqslant ord_n(\bar{x}) \geqslant ord_n(\bar{x}^{(2)}) \geqslant ord_n(\bar{x}^{(3)}) \geqslant \ldots$$

The first natural questions are:

1. When $ord_n(x) = ord_n(\bar{x})$?
2. When there exists k such that $ord_n(\bar{x}^{(k)}) = ord_n(\bar{x}^{(t)})$ for all $t \geqslant k$?
3. When $\lim\limits_{t\to\infty} ord_n(\overline{x}^{(t)}) = 1$?

From some results obtained by E. L. Post for a finite n-ary group generated by one element (cf. [31], p.283), we can deduce that

$$ord_n(x^{<s>}) = \frac{ord_n(x)}{gcd\{s(n-1)+1,\, ord_n(x)\}}$$

whenever $ord_n(x)$ is finite. Therefore for $k \geqslant 1$, we have

$$ord_n(\bar{x}^{(k)}) = ord_n(x^{<s_k>}) = \frac{ord_n(x)}{gcd\{n-2,\, ord_n(x)\}} \ .$$

Thus

$$ord_n(x) \geqslant ord_n(\bar{x}) = ord_n(\bar{x}^{(2)}) = ord_n(\bar{x}^{(3)}) = \ldots$$

Moreover, $ord_n(\bar{x}) = ord_n(x) < \infty$ if and only if $ord_n(x)$ and $n-2$ are relatively prime. Obviously $\lim\limits_{t\to\infty} ord_n(\overline{x}^{(t)}) = 1$ if and only if $ord_n(x)$ is a divisor of $n-2$.

This, together with Theorem 2 from [7], gives the following characterization of orders of skew elements.

Theorem 5. *If $ord_n(x) = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_m^{\alpha_m}$, where $p_1, p_2, \ldots, p_m$ are prime numbers, then for all $t \geqslant 1$ we have $ord_n(\overline{x}^{(t)}) = 1$ or $ord_n(\overline{x}^{(t)}) = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, where $k \leqslant m$ and $p_1 \nmid n-2,\, p_2 \nmid n-2,\, \ldots,\, p_k \nmid n-2$.*

Corollary 8. *If every prime divisor of $Card(G)$ is a divisor of $n-2$, then all skew elements of an n-ary group $\langle G, f\rangle$ are idempotent.*

A commutative n-ary group with this property is derived from some (commutative) binary group. All idempotents of such n-ary group are neutral elements in the sense of W. Dörnte (cf. [6]). The set of all neutral elements of a given n-ary group is empty or forms a commutative n-ary subgroup of this group (cf. [17]).

4. Special subgroups

An element x of an n-ary group $\langle G, f\rangle$ is called *potent* if for some natural $k \geqslant 1$ an element $x^{<k>}$ is idempotent. For any natural $n \geqslant 3$ there exist infinitely many pairwise non-isomorphic n-ary groups containing at least one potent element (cf. [17]). It is not difficult to see that x is potent if and only if $x^{<1>}$ is idempotent, or equivalently, if and only if $ord_n(x)$ is a divisor of n.

Problem 8. *When the set of all potents of a given n-ary group is an n-ary* (*normal*) *subgroup ?*

In [10] is considered the class $\mathbf{V_k}$ of n-ary groups in which $\overline{x}^{(k)} = x$ holds for all x. This class is a variety, $\mathbf{V_k} \cap \mathbf{V_{k+1}} = \mathbf{V_1}$ and $\mathbf{V_k} \subset \mathbf{V_{km}}$ for any natural k, m. Any $\mathbf{V_k}$ contains the variety of medial n-ary groups (and in the consequence – the variety of all commutative n-ary groups). But it contains also non-medial n-ary groups. $\mathbf{V_{2k}}$ contains the variety of ternary groups.

Problem 9. *Describe the variety* $\mathbf{V_k}$.

Note that if $h(x) = \bar{x}^{(k)}$ is an endomorphism of an n-ary group, then the relation
$$x \,\rho_k\, y \iff \overline{x}^{(k)} = \overline{y}^{(k)}$$
is a congruence on $\langle G, f\rangle$ and
$$G^{(k)} = \{\, \overline{x}^{(k)} \mid x \in G\}$$
is an n-ary subgroup of $\langle G, f\rangle$. Also
$$E^{(k)} = \{\, x \in G \mid \overline{x}^{(k)} = x\}$$

is an n-ary subgroup, if it is non-empty.

Generally $E^{(k)} \subset G^{(k)}$, but in some cases $E^{(k)} = G^{(k)}$. For example, in ternary groups we have $E^{(2k)} = G^{(2k)}$ for all natural k. Unfortunately, this not implies $E^{(2k+1)} = G^{(2k+1)}$. Nevertheless in ternary groups $G^{(k)} = G$ for all k.

Moreover, $E^{(k)} \in \mathbf{V_k}$, $E^{(1)} \subset E^{(k)}$, $E^{(s)} \subset E^{(sk)}$, $E^{(s)} \cap E^{(s+1)} = E^{(1)}$, $G^{(k+1)} = (G^{(k)})^{(1)}$ and

$$G \supset G^{(1)} \supset G^{(2)} \supset G^{(3)} \supset ...$$

In finite n-ary groups $G^{(k)} = G^{(k+1)} = ...$ for some $k \in N$, but in an n-ary group derived from the additive group of integers $G^{(k)} \neq G^{(m)}$ for all $k \neq m$.

Problem 10. *Describe the class of all n-ary groups (or only medial groups) satisfying the descending chain condition for $G^{(k)}$.*

If $G^{(k)} = G$ for some $k > 1$, then also $G^{(1)} = G$. Conversely, if $G^{(1)} = G$, then $G^{(2)} = (G^{(1)})^{(1)} = G$, and, in the consequence, $G^{(k)} = G$ for all $k > 1$. Thus the question on the equation $G^{(k)} = G$ can be reduced to the question on the equation $G^{(1)} = G$.

Problem 11. *Describe n-ary groups in which $G^{(1)} = G$.*

$G^{(k)}$ and $E^{(k)}$ are n-ary subgroups also in some n-ary groups in which $h(x) = \bar{x}^{(k)}$ is not an endomorphism. A simple illustration of such situation is a 4-group derived from the symmetric group S_3. In this 4-group we have $G^{(k)} = G^{(1)} = E^{(1)} = E^{(k)} = \{ z \in S_3 \mid z^3 = e \}$ and $\bar{x}^{(k)} = \bar{x}$ for all $x \in S_3$, but $\overline{f(a,z,a,z)} \neq f(\bar{a},\bar{z},\bar{a},\bar{z})$ for $a = (1\,2)$, $y = (1\,2\,3)$.

Problem 12. *Describe n-ary groups in which $G^{(1)}$ is an n-ary subgroup.*

Problem 13. *Describe n-ary groups in which $E^{(k)}$ is an n-ary subgroup.*

In a *distributive* n-ary group, i.e. in an n-ary group satisfying the identity

$$\overline{f(x_1,...,x_n)} = f(x_1,...,x_{i-1},\bar{x}_i,x_{i+1},...,x_n)\,, \qquad (10)$$

where $i = 1,2,...,n$, we have

$$\overline{x}^{(n-1)} = x = x^{<n-1>}$$

(cf. [14]). In such n-ary group all elements have the same finite n-ary order which is a divisor of $n-1$. Moreover, if $ord_n(x)=k$, then $x^{<t>}=\bar{x}^{(k-t)}$ and $\bar{x}^{(t)}=x^{<k-t>}$ for $t=0,1,\ldots,k$. Thus the smallest n-ary subgroup containing x has the form

$$C_x=\{x,x^{<1>},\ldots,x^{<k-1>}\}=\{x,\bar{x},\ldots,\overline{x}^{(k-1)}\},$$

where $k=ord_n(x)$. Obviously C_x is commutative and has no proper subgroups. This suggest the following theorem proved in [14].

Theorem 6. (Dudek 1995) *Any distributive n-ary group is a set-theoretic union of disjoint cyclic and isomorphic n-ary groups without proper subgroups.*

Theorem 7. (Dudek 1995) *Let $a\circ b=f(a,x,\ldots,x,b)$, where x is an arbitrary element of a distributive n-ary group $\langle G,f\rangle$. Then C_x is a normal subgroup of $\langle G,\circ\rangle$ and every coset of C_x in $\langle G,\circ\rangle$ is an n-ary subgroup of $\langle G,f\rangle$ isomorphic to $\langle C_x,f\rangle$.*

Problem 14. *Prove or disprove the converse of the above theorems.*

A distributive n-ary group is a set-theoretic union of commutative subgroups but it is not commutative in general. Indeed, if $t\geqslant 2$, $(t-1)|(n-1)$ and $p=t^{n-1}-1$, then $\varphi(x)\equiv tx(mod\,p)$ is an automorphism of the additive group Z_p such that $\varphi^{n-1}(x)\equiv x(mod\,p)$ for all $x\in Z_p$ and $\varphi(b)\equiv b$ for $b=1+t+t^2+\ldots+t^{n-2}$. It is not difficult to see that Z_p with the operation

$$f(x_1,x_2,\ldots,x_n)=(x_1+\varphi(x_2)+\ldots+\varphi^{n-2}(x_{n-1})+x_n+b)(mod\,p)$$

is a distributive n-ary group in which $\bar{x}^{(k)}\equiv(x-kb)(mod\,p)$. This n-ary group is a set-theoretic union of t disjoint commutative n-ary subgroups C_0, C_1, . . . , C_{t-1}, but it is *only* medial.

Any medial distributive n-ary group $\langle G,f\rangle$ is *autodistributive* (cf. [9]), i.e. the operation f is distributive with respect to itself. This means that for every $i=1,2,\ldots,n$ the following identity is satisfied

$$f(x_1,\ldots,x_{i-1},f(y_1,y_2,\ldots,y_n),x_{i+1},\ldots,x_n)=$$
$$f(f(x_1,\ldots,x_{i-1},y_1,x_{i+1},\ldots,x_n),\ldots,f(x_1,\ldots,x_{i-1},y_n,x_{i+1},\ldots,x_n)).$$

Any autodistributive n-ary group is distributive (cf. [9]), but for any $n>3$ there exists at least one idempotent distributive n-ary group which is not autodistributive. Such n-ary group can be induced by the group $(C^3, \bullet)$ and its automorphism $\varphi(x,y,z) = (\alpha x, \alpha^2 y, \alpha z)$, where C is the set of complex numbers,

$$(x,y,z) \bullet (a,b,c) = (x+a, b+xc+y, z+c)$$

and α is a primitive $(n-1)$-th root of unity (see [14], Theorem 6). For any $n \geqslant 7$ there are also non-idempotent distributive groups which are not autodistributive. Ternary distributive groups are autodistributive and vice versa. For $n = 4, 5, 6$ the problem is open.

In a distributive n-ary group $\langle G, f\rangle$ the operation $^{-}\colon x \to \bar{x}$ is an automorphism and induces the cyclic invariant subgroup $Aut_{\bar{s}}\langle G, f\rangle$ in the group of all automorphism $Aut\langle G, f\rangle$ and in the group $Aut_s\langle G, f\rangle$ of all *splitting-automorphism* in the sense of Płonka (cf. [30]).

Problem 15. *Describe the structure of groups:* $Aut\langle G, f\rangle / Aut_{\bar{s}}\langle G, f\rangle$, $Aut\langle G, f\rangle / Aut_s\langle G, f\rangle$ *and* $Aut_s\langle G, f\rangle / Aut_{\bar{s}}\langle G, f\rangle$.

If h is a splitting-automorphism of $\langle G, f\rangle$, then (as it is not difficult to see) $h(x) = h^n(x)$ for every $x \in G$.

Problem 16. *When* $Aut_s\langle G, f\rangle = Aut_{\bar{s}}\langle G, f\rangle$ *?*

Note by the way (cf. [14]), that if $\langle H, f\rangle$ is an n-ary subgroup of an autodistributive n-ary group $\langle G, f\rangle$, then for every $i = 1, \ldots, n$ and for all $a_1, a_2, \ldots, a_n \in G$ the coset

$$\{\, f(a_1, \ldots, a_{i-1}, h, a_{i+1}, \ldots, a_n) \mid h \in H\}$$

is an n-ary subgroup of $\langle G, f\rangle$ isomorphic to $\langle H, f\rangle$.

Moreover, in medial autodistributive n-ary groups

$$\{\, f(a_1, \ldots, a_{i-1}, h, a_{i+1}, \ldots, a_n) \mid h \in G^{(k)}\} = G^{(k)} = G$$

for all $k \geqslant 0$, and

$$\{\, f(a_1, \ldots, a_{i-1}, h, a_{i+1}, \ldots, a_n) \mid h \in E^{(t)}\} = E^{(t)} = G$$

for t such that $x^{<t>} = x$ for all $x \in G$. In this case we have also $G^{(k)} = G$ and $E^{(t)} = G$.

Unfortunately, this situation is not characteristic for medial autodistributive n-ary groups, because it takes place in some non-medial and non-autodistributive n-ary groups, too.

5. Fuzzy subgroups

By a *fuzzy set* μ in a set G we mean a function $\mu : G \to [0,1]$. The set

$$L(\mu,t) = \{x \in G : \mu(x) \geqslant t\},$$

where $t \in [0,1]$ is fixed, is called a *level subset of* μ.

A fuzzy set μ defined on a binary groupoid $\langle G, \cdot\rangle$ is called a *fuzzy subgroupoid* of G if $\mu(x \cdot y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$. A fuzzy set μ defined on a quasigroup $\langle G, \cdot, \backslash, /\rangle$ is called a *fuzzy subquasigroup* of G if $\mu(x * y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$ and $* \in \{\cdot, \backslash, /\}$. A fuzzy set μ defined on a group $\langle G, \cdot\rangle$ is called a *fuzzy subgroup* (or *Rosenfeld's fuzzy subgroup*) of G if it is a fuzzy subgroupoid such that $\mu(x^{-1}) \geqslant \mu(x)$ (or equivalently: $\mu(x^{-1}) = \mu(x)$) for all $x \in G$. (See the series of papers in *Fuzzy Sets and Systems.*)

The above concepts can be extended to n-ary systems in the way proposed in [16]. Namely, a fuzzy set μ defined on an n-ary groupoid $\langle G, f\rangle$ will be called an *n-ary fuzzy subgroupoid* of G if

$$\mu(f(x_1, x_2, \ldots, x_n)) \geqslant \min\{\mu(x_1), \ldots, \mu(x_n)\}$$

will be satisfied for all $x_1, \ldots, x_n \in G$.

This extension is good because for $n = 2$ it gives the standard definition. Moreover, all main results obtained for $n = 2$ can be proved also for $n > 2$ (cf. [16]).

Theorem 8. (Dudek 2000) *A fuzzy set μ of an n-ary groupoid $\langle G, f\rangle$ is an n-ary fuzzy subgroupoid of G if and only if for every $t \in [0,1]$, $L(\mu,t)$ is either empty or an n-ary subgroupoid of $\langle G, f\rangle$. Moreover, any n-ary subgroupoid of $\langle G, f\rangle$ can be realized as a level subgroupoid of some n-ary fuzzy subgroupoid.*

Theorem 9. (Dudek 2000) *If a fuzzy set μ of an n-ary groupoid $\langle G, f\rangle$ has the finite set of values $t_0 > t_1 > \ldots > t_m$ and $S_0 \subset S_1 \subset \ldots \subset S_m = G$ are n-ary subgroupoids of $\langle G, f\rangle$ such that $\mu(S_k \setminus S_{k-1}) = t_k$ for $0 \leqslant k \leqslant m$, where $S_{-1} = \emptyset$, then μ is an n-ary fuzzy subgroupoid.*

Theorem 10. (Dudek 2000) *If every n-ary fuzzy subgroupoid μ defined on $\langle G, f\rangle$ has the finite set of values, then every descending chain of n-ary subgroupoids of $\langle G, f\rangle$ terminates at finite step.*

A fuzzy set μ defined on G is said to be *normal* if there exists $x \in G$ such that $\mu(x) = 1$. A simple example of normal fuzzy sets are characteristic functions of subsets of G.

If an n-ary groupoid $\langle G, f\rangle$ is unipotent (cf. [12]), i.e. if there exists an element $\theta \in G$ such that $f(x, x, \dots, x) = \theta$ for all $x \in G$, then a fuzzy set μ defined on G is normal if and only if $\mu(\theta) = 1$.

The set $\mathcal{N}(G)$ of all normal n-ary fuzzy subgroupoids defined on an n-ary groupoid $\langle G, f\rangle$ is partially ordered by the relation

$$\mu \sqsubseteq \rho \Longleftrightarrow \mu(x) \leqslant \rho(x)$$

for all $x \in G$.

For any n-ary fuzzy subgroupoid μ of $\langle G, f\rangle$ there exists $\rho \in \mathcal{N}(G)$ such that $\mu \sqsubseteq \rho$. Moreover, if $\langle G, f\rangle$ is unipotent, then the maximal element of $(\mathcal{N}(G), \sqsubseteq)$ is either constant or characteristic function of some subset of G.

An *n-ary subquasigroup* of an n-ary quasigroup $\langle G, f\rangle$ must be defined as a non-empty subset S of G closed with respect to $n+1$ operations $f, f^{(1)}, \dots, f^{(n)}$, i.e. as a subset S of G such that $g(x_1, \dots, x_n) \in S$ for all $x_1, \dots, x_n \in S$ and all $g \in \mathcal{F} = \{f, f^{(1)}, f^{(2)}, \dots, f^{(n)}\}$, where $f^{(i)}$ is the i-th inverse operation of f (cf. [3] or [13]). This means that an *n-ary fuzzy quasigroup* must be defined as a fuzzy set such that

$$\mu(g(x_1, x_2, \dots, x_n)) \geqslant \min\{\mu(x_1), \dots, \mu(x_n)\}$$

for all $x_1, \dots, x_n \in G$ and $g \in \mathcal{F}$.

For such defined n-ary fuzzy quasigroups many of classical results are proved in (cf. [16]).

The problem is with the fuzzification on n-ary groups. As it is well known (cf. [6]), a non-empty subset S of an n-ary group $\langle G, f\rangle$ is an n-ary subgroup of $\langle G, f\rangle$ if it is closed with respect to f and $\bar{x} \in S$ for every $x \in S$. Thus, by the analogy to the binary case, an *n-ary fuzzy subgroup* can be defined as an n-ary fuzzy subgroupoid μ such that $\mu(\bar{x}) \geqslant \mu(x)$ for all $x \in G$ or as an n-ary fuzzy subgroupoid μ such that $\mu(\bar{x}) = \mu(x)$ for all $x \in G$.

Unfortunately these two concepts are not equivalent. Indeed, it is not difficult to see that in the unipotent 4-ary group derived from the additive group Z_4 the map μ defined by $\mu(0) = 1$ and $\mu(x) = 0.5$ for all $x \neq 0$ is an example of fuzzy subgroupoid in which $\mu(\bar{x}) \geqslant \mu(x)$ for all $x \in Z_4$. Thus μ is a fuzzy subgroup in the first sense. It is not a fuzzy subgroup in the second sense because for $x = 2$ we have $\mu(\bar{x}) > \mu(x)$.

These two concepts of an n-ary fuzzy group are equivalent for ternary groups and for all n-ary groups satisfying the identity $\bar{x}^{(k)} = x$, where $k > 0$ depends (or not) on x.

Problem 17. *Find the connection between n-ary fuzzy subgroups of a given n-ary group and fuzzy subgroups of its binary retracts* (*creating group*).

6. r-adic skew elements

r-adic skew elements were introduced by S. A. Rusakov (cf. [34]) as a generalization of skew elements and were used to the investigation of some properties of n-ary groups connected with their subgroups.

According to [34], an element $\tilde{a}$ of an n-ary group $\langle G, f\rangle$ is called *skew of type* k and is denoted by $\bar{a}^{(k,1)}$ if the equation

$$f(a^{<k-1>}, a, \ldots, a, \tilde{a}) = a$$

is satisfied. By the *r-adic skew element of type* k, where $k, r \in N$ and $\bar{a}^{(k,0)} = a$, we mean an element

$$\bar{a}^{(k,r)} = \overline{\bar{a}^{(k,r-1)}}^{(k,1)}.$$

It is easy to see that $\bar{a}^{(1,r)} = \bar{a}^{(r)}$, i.e. r-adic skew elements of type $k = 1$ are skew in the sense of Dörnte.

Moreover, r-adic skew elements of type k can be used to the definition of n-ary groups and have similar (but not identical) properties as elements skew in the sense of Dörnte. For example, $\bar{a}^{(k,r)} = a^{<S_{kr}>}$, where

$$S_{kr} = \frac{(1 - k(n-1))^r - 1}{n-1}$$

and

$$ord_n(\bar{a}^{(k,r)}) = \frac{ord_n(a)}{gcd\{(k(n-1)-1)^r, ord_n(a)\}} .$$

But on the other hand, in a ternary group derived from the additive group of integers we have $\bar{a} = -a$, $\bar{a}^{(2)} = a$ and $\bar{a}^{(k,r)} \neq \bar{a}^{(k,t)} = (1-2k)^t a$ for all $k > 1$ and $r \neq t$. In this group we have also $\bar{a}^{(14,t)} = \bar{a}^{(2,3t)}$ for all $t \in N$.

Problems for r-adic skew elements are similar to the problems posed for skew elements in the sense of Dörnte. For example, when $\bar{a}^{(k,r)} = a$ or when $h(x) = \bar{x}^{(k,r)}$ is an automorphism.

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Received August 16, 2001