The rules of type theory

Overview

- following[1] consider MLTT with
  - Σ-types (dependent sums)
  - Π-types (dependent products)
  - Id-types (identity types), and
  - a base type $\boldsymbol{N}$ of natural numbers (no universes and (higher) inductive types)
- use a very verbose syntax with lots of *type annotations*
  - good for theoretical reasoning, but not practical
  - omitting type decorations can be justified a posteriori since in most cases they can be uniquely reconstructed
- basic syntactic entities are **types** $A, B, C, \ldots$, **terms** $s, t, u, \ldots$, **contexts**

  $$\Gamma, \Delta, \ldots \;=\; (x_1{:}A_1, x_2{:}A_2, \ldots, x_n{:}A_n)$$

  and **judgments**

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[1] M. Hofmann. “Syntax and semantics of dependent types”. In: *Extensional Constructs in Intensional Type Theory*. Springer, 1997, pp. 13–54.

6 forms of judgments

$\Gamma \vdash \ \mathrm{cxt}$	'Γ is a context'
$\Gamma \equiv \Delta \vdash \ \mathrm{cxt}$	'Γ and Δ are equal contexts'
$\Gamma \vdash A \ \mathrm{type}$	A is a type in context Γ
$\Gamma \vdash A \equiv B \ \mathrm{type}$	A and B are equal types in context Γ
$\Gamma \vdash t : A$	t is a term of type A in context Γ
$\Gamma \vdash t \equiv u : A$	t and u are equal terms of type A in context Γ

Types of rules

For each type former – i.e. Σ, Π, Id, $\boldsymbol{N}$ – there are

- formation
- introduction
- elimination
- computation
- congruence

rules.
Furthermore, there are the following basic rules.

- empty context and context extension
- variable, weakening, substitution
- reflexivity, transitivity, symmetry for definitional equality
- conversion

Context rules

$$\frac{}{\Diamond \vdash \ \text{cxt}}$$ empty context

$$\frac{\Gamma \vdash A \ \text{type}}{\Gamma, x{:}A \vdash \ \text{cxt}}$$ context extension

$$\frac{\Gamma \equiv \Delta \vdash \ \text{cxt} \qquad \Gamma \vdash A \equiv B \ \text{type}}{(\Gamma, x{:}A) \equiv (\Delta, x{:}B) \vdash \ \text{cxt}}$$ congruence for context extension

Variable, weakening and substitution

$$\frac{\Gamma, x{:}A, \Delta \vdash \ \mathrm{cxt}}{\Gamma, x{:}A, \Delta \vdash x : A}$$

variable

$$\frac{\Gamma, \Delta \vdash \mathcal{J} \qquad \Gamma \vdash A \ \mathrm{type}}{\Gamma, x{:}A, \Delta \vdash \mathcal{J}}$$

weakening

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J} \qquad \Gamma \vdash t : A}{\Gamma, \Delta[t/x] \vdash \mathcal{J}[t/x]}$$

substitution

Equivalence rules for definitional equality

$$\frac{\Gamma \vdash \ \ \mathrm{cxt}}{\Gamma \equiv \Gamma \vdash \ \ \mathrm{cxt}} \qquad \frac{\Gamma \vdash A \ \ \mathrm{type}}{\Gamma \vdash A \equiv A \ \ \mathrm{type}} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash t \equiv t : A}$$

$$\frac{\Gamma \equiv \Delta \vdash \ \ \mathrm{cxt}}{\Delta \equiv \Gamma \vdash \ \ \mathrm{cxt}} \qquad \frac{\Gamma \vdash A \equiv B \ \ \mathrm{type}}{\Gamma \vdash B \equiv A \ \ \mathrm{type}} \qquad \frac{\Gamma \vdash t \equiv u : A}{\Gamma \vdash u \equiv t : A}$$

$$\frac{\Gamma \equiv \Delta \vdash \ \ \mathrm{cxt} \qquad \Delta \equiv \Theta \vdash \ \ \mathrm{cxt}}{\Gamma \equiv \Theta \vdash \ \ \mathrm{cxt}} \qquad \frac{\Gamma \vdash A \equiv B \ \ \mathrm{type} \qquad \Gamma \vdash B \equiv C \ \ \mathrm{type}}{\Gamma \vdash A \equiv C \ \ \mathrm{type}} \qquad \frac{\Gamma \vdash t \equiv u : A \qquad \Gamma \vdash u \equiv v : A}{\Gamma \vdash t \equiv v : A}$$

Conversion rules

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \equiv \Delta \vdash \text{ cxt}}{\Delta \vdash A \text{ type}}$$

type conversion

$$\frac{\Gamma \vdash t : A \qquad \Gamma \equiv \Delta \vdash \text{ cxt} \qquad \Gamma \vdash A \equiv B \text{ type}}{\Delta \vdash t : B}$$

term conversion

Π *types*

$$\frac{\Gamma \vdash A \ \text{type} \qquad \Gamma, x{:}A \vdash B \ \text{type}}{\Gamma \vdash \Pi x{:}A \,.\, B \ \text{type}}$$ formation

$$\frac{\Gamma, x{:}A \vdash t : B}{\Gamma \vdash (\lambda x{:}A \,.\, t^B) : \Pi x{:}A \,.\, B}$$ introduction

$$\frac{\Gamma \vdash t : \Pi x{:}A \,.\, B \qquad \Gamma \vdash u : A}{\Gamma \vdash \text{app}_{[x:A]B}(t, u) : B[u/x]}$$ elimination

$$\Gamma \vdash \text{app}_{[x:A]B}(\lambda x \,.\, t^B, u) \equiv t[u/x] : B[u/x]$$ computation (β-rule)

$$\Gamma \vdash t \equiv (\lambda x{:}A \,.\, \text{app}_{[x:A]B}(t, x)^B) : \Pi x{:}A \,.\, B$$ computation (η-rule)

$$\frac{\Gamma \vdash A \equiv A' \ \text{type} \qquad \Gamma, x{:}A \vdash B \equiv B' \ \text{type}}{\Gamma \vdash \Pi x{:}A \,.\, B \equiv \Pi x{:}A' \,.\, B'}$$ formation congruence

$$\frac{\Gamma, x{:}A \vdash t \equiv u : B}{\Gamma \vdash (\lambda x{:}A \,.\, t^B) \equiv (\lambda x{:}A \,.\, u^B) : \Pi x{:}A \,.\, B}$$ introduction congruence

$$\frac{\Gamma \vdash t \equiv t' : \Pi x{:}A \,.\, B \qquad \Gamma \vdash u \equiv u' : A}{\Gamma \vdash \text{app}_{[x:A]B}(t, u) \equiv \text{app}_{[x:A]B}(t', u') : B[u/x]}$$ elimination congruence

Σ *types*

$$\frac{\Gamma \vdash A \ \text{type} \qquad \Gamma, x{:}A \vdash B \ \text{type}}{\Gamma \vdash \Sigma x{:}A\,.\,B \ \text{type}}$$

formation

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B[t/x]}{\text{pair}_{[x:A]B}(t, u) : \Sigma x : A\,.\,B}$$

introduction

$$\frac{\begin{array}{l}\Gamma, z : \Sigma x : A\,.\,B \vdash C \ \text{type} \\ \Gamma, x{:}A, y{:}B \vdash t : C[\text{pair}_{[x:A]B}(x, y)/z] \\ \Gamma \vdash u : \Sigma x : A\,.\,B\end{array}}{\Gamma \vdash \text{rec}_{[z]C}^{\Sigma x:A\,.\,B}([x, y]t, u) : C[u/z]}$$

elimination

$$\begin{array}{l}\Gamma \ \vdash \text{rec}_{[z]C}^{\Sigma x:A\,.\,B}([x, y]u, \text{pair}_{[x:A]B}(s, t)) \\ \quad \equiv u[s/x, t/y] \qquad : \quad C[\text{pair}_{[x:A]B}(s, t)/z]\end{array}$$

computation

$\ldots$

congruence rules

The type $\mathbb{N}$ of natural numbers

$$\frac{\Gamma \vdash \ \text{cxt}}{\Gamma \vdash \boldsymbol{N} \ \text{type}}$$

formation

$$\frac{\Gamma \vdash \ \text{cxt}}{\Gamma \vdash 0 : \boldsymbol{N}} \qquad \frac{\Gamma \vdash t : \boldsymbol{N}}{\Gamma \vdash \text{succ}(t) : \boldsymbol{N}}$$

introduction

$$\frac{\begin{array}{l}\Gamma, x : \boldsymbol{N} \vdash A \ \text{type} \\ \Gamma \vdash s : A[0/x] \\ \Gamma, x : \boldsymbol{N}, y : A \vdash t : A[\text{succ}(x)/x] \\ \Gamma \vdash u \in \boldsymbol{N}\end{array}}{\Gamma \vdash \text{rec}^{\boldsymbol{N}}_{[x]A}(s, [x, y]t, u) : A[u/x]}$$

elimination

$$\Gamma \vdash \text{rec}^{\boldsymbol{N}}_{[x]A}(s, [x, y]t, 0) \equiv s : A[u/x]$$

computation

$$\Gamma \vdash \text{rec}^{\boldsymbol{N}}_{[x]A}(s, [x, y]t, \text{succ}(u))$$
$$\equiv t[u/x, \text{rec}^{\boldsymbol{N}}_{[x]A}(s, [x, y]t, u)/y] : A[u/x]$$

computation

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2 congruence rules

Identity types

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathrm{Id}_A(t,u) \ \ \text{type}}$$

formation

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathrm{refl}_A(t) : \mathrm{Id}_A(t,t)}$$

introduction

$$\frac{\begin{array}{l}\Gamma, x{:}A, y{:}A, p{:}\mathrm{Id}_A(x,y) \vdash B \ \ \text{type} \\ \Gamma, z{:}A \vdash s : B[z/x, z/y, \mathrm{refl}_A(z)/p] \\ \Gamma \vdash t : A \qquad \Gamma \vdash u : A \qquad \Gamma \vdash v : \mathrm{Id}_A(t,u)\end{array}}{\Gamma \vdash \mathrm{rec}^{\mathrm{Id_A}}_{[x,y,p]B}([z]s, t, u, v) : B[t/x, u/y, v/p]}$$

elimination

$$\begin{array}{l}\Gamma \vdash \mathrm{rec}^{\mathrm{Id_A}}_{[x,y,p]B}([z]s, t, t, \mathrm{refl}_A(t)) \\ \quad \equiv s[t/z] : B[t/x, t/y, \mathrm{refl}_A(t)/p]\end{array}$$

computation

$\ldots$

3 congruence rules

Pre-syntax

The syntactic classes of **pre-contexts**, **pre-types**, and **pre-terms** are defined by the following grammar.

$$\Gamma \quad ::= \quad \diamondsuit \mid \Gamma, x{:}A$$

$$A, B \quad ::= \quad \Pi x{:}A\,.\,B \mid \Sigma x \mid B \mid \mathrm{Id}_A(t, u) \mid \boldsymbol{N}$$

$$\begin{aligned} s, t, u, v \quad ::= \quad & x \mid \lambda x{:}A\,.\,t^B \mid \mathrm{app}_{[x:A]B}(t, u) \\ & \mid \mathrm{pair}_{[x:A]B}(t, u) \mid \mathrm{rec}^{\Sigma x:A\,.\,B}_{[z]C}([x, y]t, u) \\ & \mid 0 \mid \mathrm{succ}(t) \mid \mathrm{rec}^{\boldsymbol{N}}_{[z]C}(s, [x, y]t, u) \\ & \mid \mathrm{refl}_A(t) \mid \mathrm{rec}^{\mathrm{Id}_A}_{[x,y,p]C}([z]s, t, u, v) \end{aligned}$$