Bounding distances to unsafe sets

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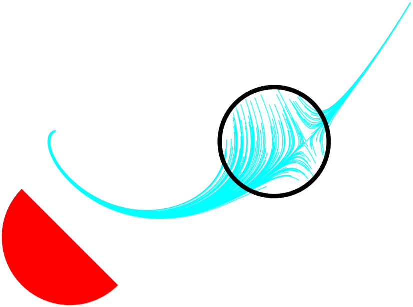
Main Ideas

Quantify safety of trajectories by distance to unsafe set

Relax distance using optimal transport theory

Develop occupation measure programs to bound distance

Flow System Setting



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3] \qquad \forall t \in [0, 5]$$

$$X_0 = \{x \mid (x_1 - 1.5)^2 + x_2 \leq 0.4^2\}$$
$$X_u = \{x \mid x_1^2 + (x_2 + 0.7)^2 \leq 0.5^2, \quad \sqrt{2}/2(x_1 + x_2 - 0.7) \leq 0\}$$

Distance Function

Metric space (X, c) satisfying $\forall x,\ y \in X$:

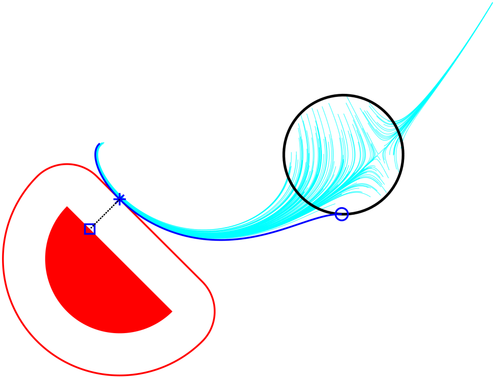
$$
\begin{aligned}
c(x,y) &> 0 && x \neq y \\
c(x,x) &= 0 \\
c(x,y) &= c(y,x) \\
c(x,y) &\leq c(x,z) + c(z,y) && \forall z \in X
\end{aligned}
$$

Point-Unsafe Set distance: $c(x; X_u) = \min_{y \in X_u} c(x,y)$

Distance Estimation Problem

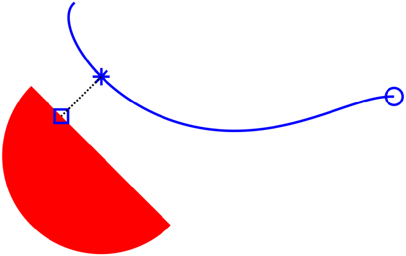
$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$

$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Optimal Trajectories (Distance)



Optimal trajectories described by $(x_p^*,\ y^*\ x_0^*,\ t_p^*)$:

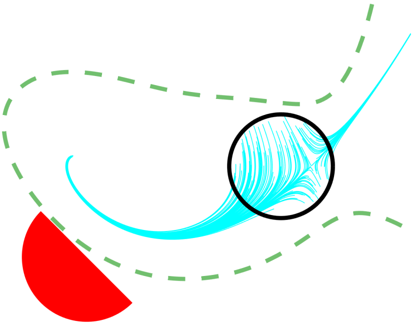
x_p^*	location on trajectory of closest approach
y^*	location on unsafe set of closest approach
x_0^*	initial condition to produce x_p^*
t_p^*	time to reach x_p^* from x_0^*

Safety Background

Barrier Program

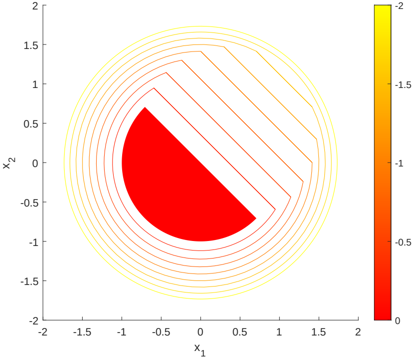
Barrier function $B : X \to \mathbb{R}$ indicates safety

$$\begin{aligned} B(x) &\leq 0 && \forall x \in X_u \\ B(x) &> 0 && \forall x \in X_0 \\ \mathcal{L}_f B(x) &\geq 0 && \forall x \in X \end{aligned}$$



Half-circle Contours

Unsafe set $X_u = \{x \mid 1 - x_1^2 - x_2^2 \geq 0, -x_1 - x_2 \geq 0\}$



$q \leq 1 - x_1^2 - x_2^2$ $\qquad$ $q \leq -x_1 - x_2$

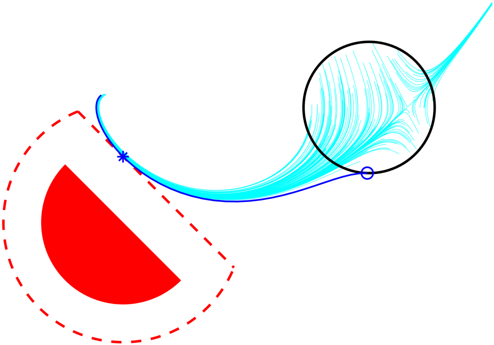
$$q = -0.25, -0.5, \ldots, -2$$

Safety Margin

Unsafe set $X_u = \{x \mid p_i(x) \geq 0 \; \forall i = 1 \ldots N_u\}$

Safety margin $p^* = \max \min_i p_i(x)$ along trajectories

If $p^* < 0$, no trajectories enter X_u (safe)

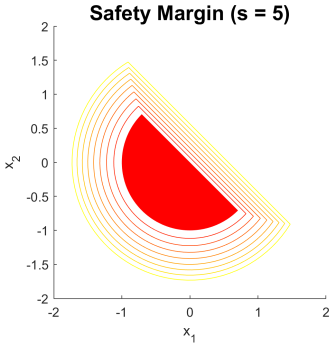


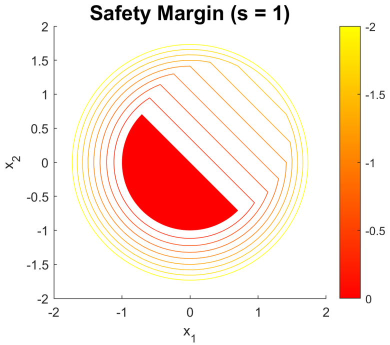
safe: $p^* \leq -0.2831$

Safety Margin Scaling

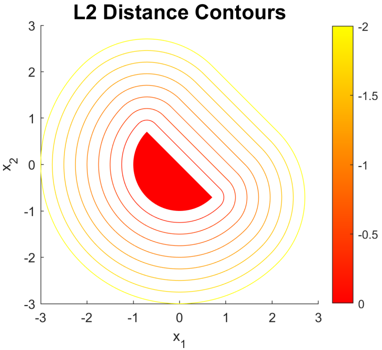
Scale factor in constraints

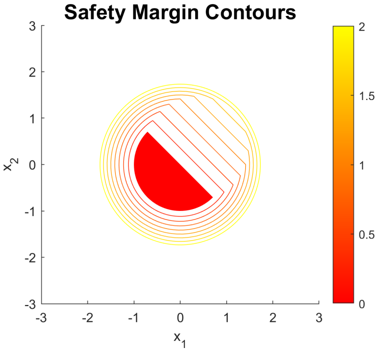
$$q \leq 1 - x_1^2 - x_2^2 \qquad\qquad q \leq s(-x_1 - x_2)$$





Distance vs. Safety Margin





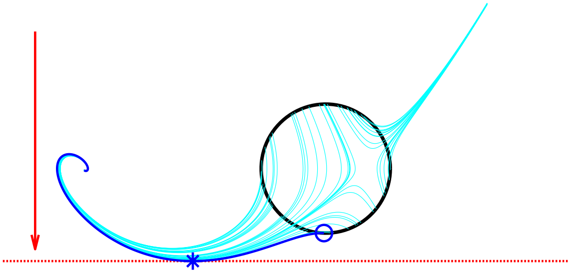
Peak Estimation

Peak Estimation Background

Find maximum value of $p(x)$ along trajectories

$$P^* = \max_{t,\ x_0 \in X_0} p(x(t \mid x_0))$$

$$\dot{x}(t) = f(t, x(t)) \qquad\qquad t \in [0, T]$$



$$\dot{x} = [x_2, -x_1 - x_2 + x_1^3/3]$$

Peak Estimation Program (Measure)

Infinite dimensional linear program (Cho, Stockbridge, 2002)

$$
\begin{aligned}
p^* = \ \max \quad & \langle p(x), \mu_p \rangle \\
& \mu_p = \delta_{t=0} \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \langle 1, \mu_0 \rangle = 1 \\
& \mu, \mu_p \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Peak measure μ_p: free terminal time

Maximin Program

Solve $P^* = \max_x \min_i p_i(x)$ (M., Henrion, Sznaier 2020)

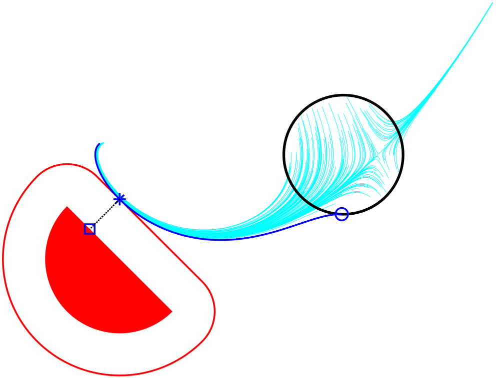
$$
\begin{aligned}
p^* = \ \max \quad & q \\
& q \leq \langle p_i(x), \mu_p \rangle & \forall i \\
& \mu_p = \delta_{t=0} \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \langle 1, \mu_0 \rangle = 1 \\
& \mu, \mu_p \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Used for safety margins, $p^* \leq p_d^* < 0$

Distance Program

Distance Estimation Problem (reprise)

$$P^* = \min_{t,\, x_0 \in X_0} c(x(t \mid x_0); X_u)$$
$$\dot{x}(t) = f(t, x), \quad \forall t \in [0, T].$$



L_2 bound of 0.2831

Distance Relaxation

Distance in points $\rightarrow$ Earth-Mover distance

$$\begin{array}{lcl} c(x, y) & & \langle c(x, y), \eta \rangle \\ x \in X & \rightarrow & \langle 1, \eta \rangle = 1 \\ y \in X_u & & \eta \in \mathcal{M}_+(X \times X_u) \end{array}$$

Joint (Wasserstein) probability measure η

Measures from Optimal Trajectories

Form measures from each $(x_p^*,\ x_0^*,\ t_p^*,\ y^*)$

Atomic Measures (rank-1)

$$\mu_0^* : \qquad \delta_{x=x_0^*}$$

$$\mu_p^* : \qquad \delta_{t=t_p^*} \otimes \delta_{x=x_p^*}$$

$$\eta^* : \qquad \delta_{x=x_p^*} \otimes \delta_{y=y^*}$$

Occupation Measure $\qquad \forall v(t, x) \in C([0, T] \times X)$

$$\mu^* : \qquad \langle v(t, x), \mu \rangle = \int_0^{t_p^*} v(t, x^*(t \mid x_0^*))dt$$

Distance Program (Measures)

Infinite Dimensional Linear Program

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x,y), \eta \rangle \\
& \pi^x_\# \eta = \pi^x_\# \mu_p \\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \langle 1, \mu_0 \rangle = 1 \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Prob. Measures: $\langle 1, \mu_0 \rangle = \langle 1, \mu_p \rangle = \langle 1, \eta \rangle = 1$

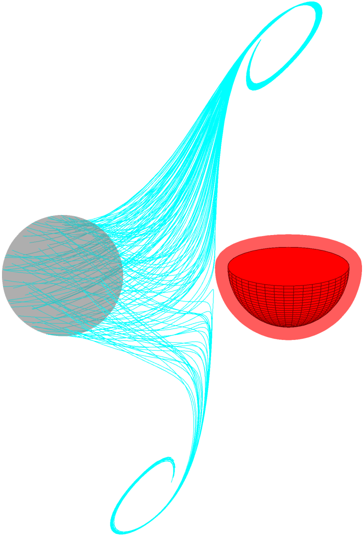
Distance Example (Twist)

'Twist' System, $T = 5$

$$\dot{x}_i = A_{ij}x_j - B_{ij}(4x_j^3 - 3x_j)/2$$

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

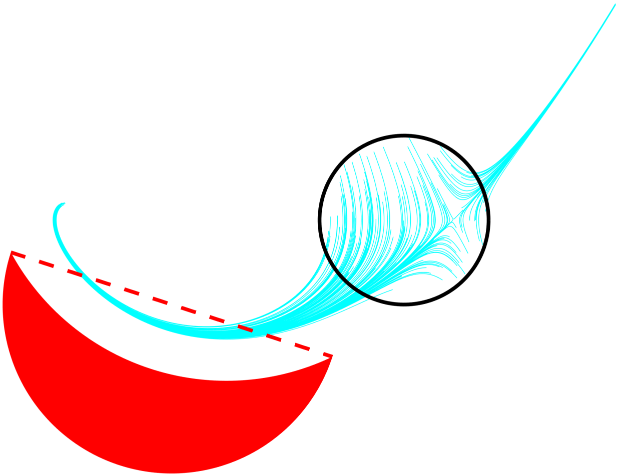


L_2 bound of 0.0425

Moon L2 Contours

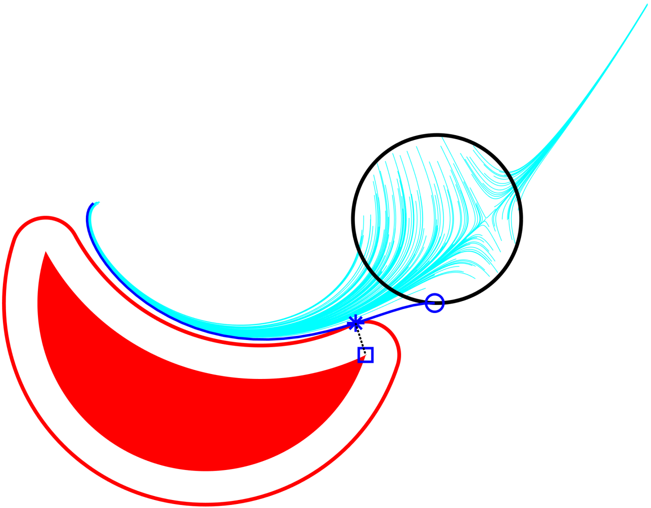
Inside one circle, outside another

Distance Example (Flow Moon)



Collision if X_u was a half-circle

Distance Example (Flow Moon)

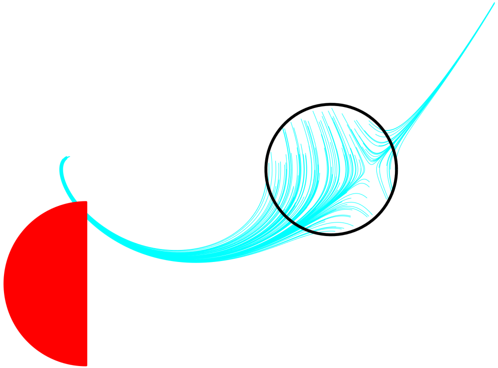


L_2 bound of 0.1592

Spurious Distance Bound

Numerical error may yield $p^* > 0$ even when unsafe

Ensure trajectories are safe before computing distance bound



Invalid L_2 bound of 4.371×10^{-4}

Distance Variations

Distance Variations

Uncertainty in dynamics

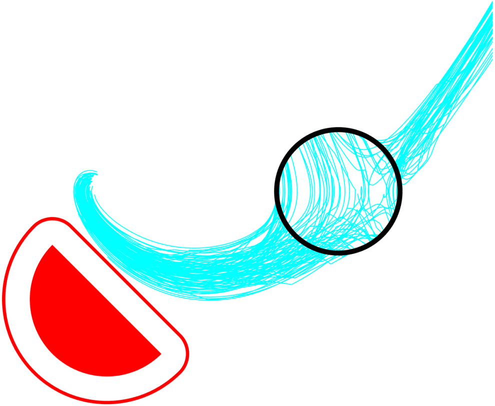
Lifted distances (with absolute values)

Set-Set distances for shape safety

Distance Uncertainty

Time dependent uncertainty $w(t) \in W \ \forall t \in [0, T]$

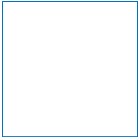
Young measure μ, Liouville $\mu_p = \delta_0 \otimes \mu_0 + \pi^{tx}_{\#} \mathcal{L}_f^{\dagger} \mu$



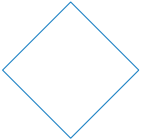
L_2 bound of 0.1691

Lifted Distance

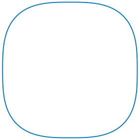
LP lifts to deal with absolute values



$$\|x-y\|_\infty \qquad \min \quad q$$
$$-q \le \langle x_i - y_i, \eta \rangle \le q \qquad \forall i$$

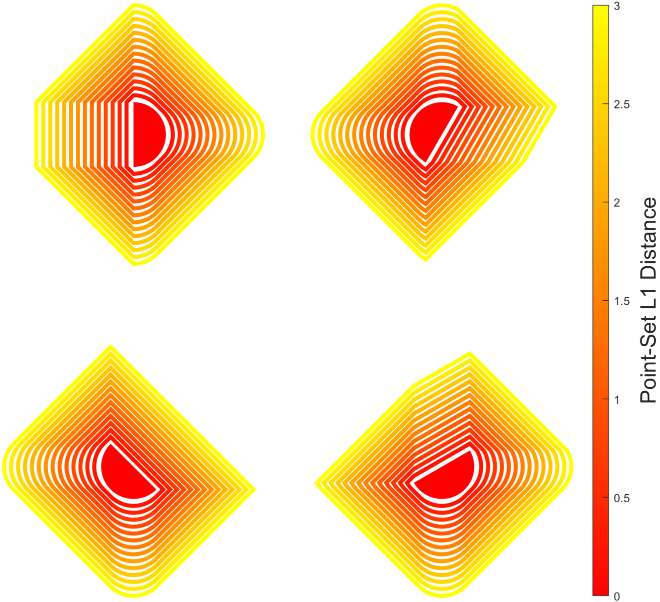


$$\|x-y\|_1 \qquad \min \quad \textstyle\sum_i q_i$$
$$-q_i \le \langle x_i - y_i, \eta \rangle \le q_i \qquad \forall i$$

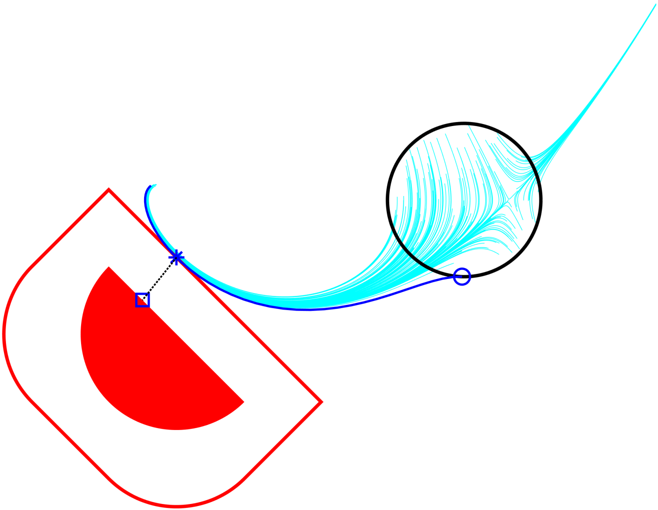


$$\|x-y\|_3^3 \qquad \min \quad \textstyle\sum_i q_i$$
$$-q_i \le \langle (x_i - y_i)^3, \eta \rangle \le q_i \quad \forall i$$

Half-Circle L1 Contours



Lifted Distance (L1) Example



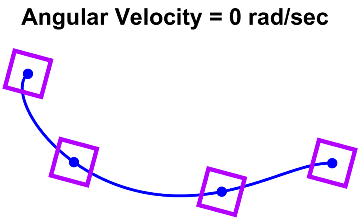
L_1 bound of 0.4003

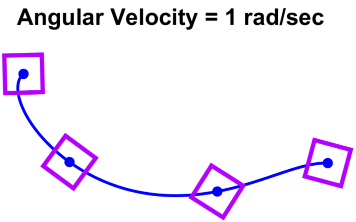
Shapes along Trajectories

Orientation $\omega(t) \in \Omega$, shape S

Body to global coordinate transformation A:

$$A : S \times \Omega \to X \qquad (s, \omega) \mapsto A(s; \omega)$$





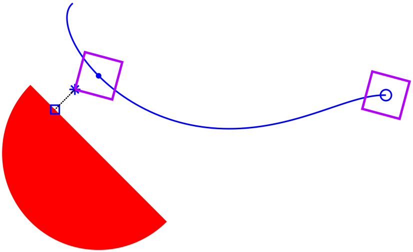
Set-Set Distance Problem

Set-Set distance between $A_\omega \circ S$ and X_u given ω

$$P^* = \min_{t,\, \omega_0 \in \Omega_0,\, s \in S} c(x(t); X_u)$$

$$x(t) = A(s;\, \omega(t \mid \omega_0)) \quad \forall t \in [0, T]$$

$$\dot{\omega}(t) = f(t, \omega) \quad \forall t \in [0, T]$$



L_2 bound of 0.1465

Set-Set Program (Measures)

Add new 'shape' measure μ_s

$$
\begin{aligned}
p^* = \ \min \quad & \langle c(x,y), \eta \rangle \\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \pi^\omega_\# \mu_p = \pi^\omega_\# \mu_s \\
& \pi^x_\# \eta = A(s;\omega)_\# \mu_s \\
& \langle 1, \mu_0 \rangle = 1 \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_s \in \mathcal{M}_+(\Omega \times S) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times \Omega) \\
& \mu_0 \in \mathcal{M}_+(\Omega_0)
\end{aligned}
$$

Take-aways

Conclusion

Distance Estimation with occupation measures

Approximate recovery if moment matrices are low-rank

Extend to uncertain, lifted, set-set scenarios

Future Work

- No relaxation gap
- Sparsity
- Time Delays
- Implementation
- Paper

Acknowledgements

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Thank you

Thank you for your attention

Bonus Material and Ideas

Distance Program (Functions)

Auxiliary $v(t, x)$, point-set proxy $w(x) \leq c(x; X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0, x) \geq \gamma & & \forall x \in X_0 \\
& w(x) \geq v(t, x) & & \forall (t, x) \in [0, T] \times X \\
& c(x, y) \geq w(x) & & \forall (x, y) \in X \times X_u \\
& \mathcal{L}_f v(t, x) \geq 0 & & \forall (t, x) \in [0, T] \times X \\
& v \in C^1([0, T] \times X) \\
& w \in C(X)
\end{aligned}
$$

Lifted Distance Program (Measure)

New terms for lifted distance

$$
\begin{aligned}
p^* = \ \min \quad & \sum_i q_i \\
& \mu_p = \delta_0 \otimes \mu_0 + \mathcal{L}_f^\dagger \mu \\
& \pi^x_\# \eta = \pi^x_\# \mu_p \\
& \langle 1, \mu_0 \rangle = 1 \\
& -q_i \leq \langle c_{ij}(x, y), \eta \rangle \leq q_i & \forall i, j \\
& \eta \in \mathcal{M}_+(X \times X_u) \\
& \mu_p, \ \mu \in \mathcal{M}_+([0, T] \times X) \\
& \mu_0 \in \mathcal{M}_+(X_0)
\end{aligned}
$$

Same process as maximin peak

Lifted Distance Program (Function)

New terms $\beta_i^{\pm}$ on costs

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0, x) \geq \gamma & & \forall x \in X_0 \\
& w(x) \geq v(t, x) & & \forall (t, x) \in [0, T] \times X \\
& \textstyle\sum_{i,j} (\beta_{ij}^{+} - \beta_{ij}) c_{ij}(x, y) \geq w(x) & & \forall (x, y) \in X \times X_u \\
& \mathcal{L}_f v(t, x) \geq 0 & & \forall (t, x) \in [0, T] \times X \\
& \mathbf{1}^T (\beta_i^{+} + \beta_i^{-}) = 1, \ \beta_i^{\pm} \in \mathbb{R}_+^{n_i} & & \forall i \\
& v \in C^1([0, T] \times X) \\
& w \in C(X)
\end{aligned}
$$

Set-Set Barrier Program

Original unsafe set $X_u \subset X$

Lifted unsafe set $\tilde{X}_u$:

$$\tilde{X}_u = \{(s, \omega) \in S \times \Omega \mid A(s; \omega) \in X_u\}$$

Barrier Program for $B : \Omega \to \mathbb{R}$

$$\begin{aligned}
&B(\omega) > 0 && \forall \omega \in \Omega \\
&\mathcal{L}_f B(\omega) \geq 0 && \forall \omega \in \Omega \\
&B(\omega) \leq 0 && \forall (s, \omega) \in \tilde{X}_u
\end{aligned}$$

Set-Set Program (Function)

Set-Set distance proxy $z(\omega) \leq \max_{s \in S} c(A(s;\omega); X_u)$:

$$
\begin{aligned}
d^* = \max_{\gamma \in \mathbb{R}} \quad & \gamma \\
& v(0, \omega) \geq \gamma & & \forall x \in \Omega_0 \\
& c(x, y) \geq w(x) & & \forall (x, y) \in X \times X_u \\
& w(A(s;\omega)) \geq z(\omega) & & \forall (s, \omega) \in S \times \Omega \\
& z(\omega) \geq v(t, \omega) & & \forall (t, \omega) \in [0, T] \times \Omega \\
& \mathcal{L}_f v(t, \omega) \geq 0 & & \forall (t, \omega) \in [0, T] \times \Omega \\
& v \in C^1([0, T] \times X) \\
& w \in C(X), \ z \in C(\Omega)
\end{aligned}
$$