Quantitative Macroeconomics

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Week 9

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Contents

1. Long-Run Restrictions

Blanchard and Quah (1989) consider a bivariate model of the U.S. economy, where ur_t denotes the U.S. unemployment rate and gdp_t the log of U.S. real GDP. There is some evidence that ur_t is covariance stationary, whereas gdp_t exhibits a unit root; that is, GDP growth, $\Delta gdp_t = gdp_t - gdp_{t-1}$, is covariance-stationary. Blanchard and Quah (1989) set up a SVAR model for $y_t = (\Delta gdp_t, ur_t)'$ and analyze the effects of two structural shocks, an aggregate supply shock ε_t^{AS} and an aggregate demand shock ε_t^{AD}.

1. Why are short-run restrictions sometimes (or even often) *problematic*? What about long-run restrictions?

2. Assume for simplicity a VAR(1) model for y_t. Derive the effect of the structural shocks on the behavior of ur_{t+h}, Δgdp_{t+h} and gdp_{t+h} for $h = 0, 1, 2 \cdots$. What happens in the long-run, i.e. for $h \to \infty$?

3. Discuss the implications on the structural impulse responses of requiring gdp_t to return to its initial level in the long-run in response to an aggregate demand shock.

4. Given knowledge of the reduced-form VAR model parameters, show how to recover the short-run impact matrix B_0^{-1} from the long-run structural impulse response matrix

$$\Theta(1) = (I - A_1 - \cdots - A_p)^{-1} B_0^{-1} = A(1)^{-1} B_0^{-1}$$

   where $A(1)$ denotes the lag polynomial evaluated at $L = 1$.

5. Consider the data given in `BlanchardQuah1989.csv`. Estimate a SVAR(8) model with a constant term. The structural shocks are identified by imposing that ε_t^{AD} has no long-run effect on the level of real GDP. Estimate the impact matrix B_0^{-1} using

   a) the Cholesky decomposition on $\hat{A}(1)^{-1} \hat{\Sigma}_u \hat{A}(1)^{-1'} = \Theta(1)\Theta(1)'$

   b) a nonlinear equation solver that minimizes

$$F(B_0^{-1}) = \begin{bmatrix} vech(B_0^{-1} B_0^{-1'} - \hat{\Sigma}_u) \\ \text{restrictions on } \Theta(1) \end{bmatrix}$$

   where $\Theta(1) = (I - A_1 - \cdots - A_p)^{-1} B_0^{-1} = A(1)^{-1} B_0^{-1}$. Assume that $E(\varepsilon_t \varepsilon_t') = I_2$ and the diagonal elements of B_0^{-1} are positive.

6. Plot the structural impulse response functions using `irfPlots.m` for the level of GDP and the unemployment rate. Interpret your results in economic terms.

Readings

- Kilian and Lütkepohl (2017, Ch. 10.1, 10.3, 11.1, 11.2)

2. Combining Short-Run And Long-Run Restrictions

Consider a stylized VAR(4) model of U.S. monetary policy with only three quarterly variables. Let $y_t = (\Delta gnp_t, i_t, \Delta p_t)'$ be stationary variables, where gnp_t denotes the log of U.S. real GNP, p_t the corresponding GNP deflator in logs, and i_t the federal funds rate, averaged by quarter. The estimation period is restricted to 1954q4-2007q4 in order to exclude the period of unconventional monetary policy measures. Defining $\varepsilon_t = (\varepsilon_t^{policy}, \varepsilon_t^{AD}, \varepsilon_t^{AS})'$, the identifying restrictions can be summarized as

$$B_0^{-1} = \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \text{ and } \Theta(1) = A(1)^{-1}B_0^{-1} = \begin{bmatrix} 0 & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

The long-run restrictions are imposed on the cumulated impulse responses.

1. Provide intuition given the above identifying restrictions.
2. Consider the data given in `RWZ2010.csv`. Estimate a VAR(4) model with a constant term.
3. Estimate the structural impact matrix using a nonlinear equation solver, i.e. the objective is to find the unknown elements of B_0^{-1} such that

   $$\begin{bmatrix} vech(B_0^{-1}B_0^{-1\prime} - \hat{\Sigma}_u) \\ \text{short-run restrictions on } B_0^{-1} \\ \text{long-run restrictions on } \Theta(1) \end{bmatrix}$$

   is minimized where the normalization $\Sigma_\varepsilon = I_3$ is imposed. Furthermore, use the following insight to normalize the signs of the columns of B_0^{-1}:

   - a monetary policy shock (first column) raises the interest rate (second row) (monetary tightening)
   - a positive aggregate demand shock (second column) does not lower real GNP (first row) and inflation (third row)
   - a positive aggregate supply shock (3rd column) does not lower real GNP (first row) and does not raise inflation (third row)

4. Use the implied estimate of the structural impact matrix to plot the structural impulse response functions for the level of real GNP, the Federal Funds rate and the Deflator Inflation with response to a tightening in monetary policy. Interpret your results economically.

Readings

- Kilian and Lütkepohl (2017, Ch. 10.4, 10.5, 11.3)

3. Inference In SVARs Identified By Exclusion Restrictions

Consider an exactly-identified structural VAR model subject to short- and/or long-run restrictions, where the structural impulse response of variable j to structural shock k at horizon h is denoted as $\theta_{jk,h}$, which we simply denote as θ. We are interested in the distribution of θ, in particular deriving $(1-\gamma)\%$ point-wise confidence intervals given a consistent estimate $\hat{\theta}$ of θ.

1. Consider the asymptotic confidence intervals which are derived using the delta method:

$$\hat{\theta} \pm z_{\gamma/2} \widehat{std}(\hat{\theta})$$

   where $z_{\gamma/2}$ denotes the $\gamma/2$ percentile of the standard normal distribution and $\widehat{std}(\hat{\theta})$ a consistent estimate of the standard deviation of θ. Name the assumptions and shortcomings of this approach.

2. Outline the idea and algorithm of the Standard Residual-Based Recursive-Design Bootstrap approach.

3. Name the central idea underlying the Residual-Based Wild Bootstrap.

4. Discuss the choice of significance level γ.

5. Discuss how to draw initial conditions for a resampling method.

6. Given a bootstrap approximation to the distribution of the structural impulse-response function, discuss how to construct bootstrap confidence intervals from this distribution. Particularly, explain
   a) intervals based on bootstrap standard errors
   b) Efron's percentile interval
   c) equal-tailed percentile-t intervals

Readings

- Kilian and Lütkepohl (2017, Ch. 12.1-12.5, 12.9)

References

Blanchard, Olivier J. and Danny Quah (Sept. 1989). "The Dynamic Effects of Aggregate Demand and Supply Disturbances". In: *American Economic Review* 79.4, pp. 655–673.

Kilian, Lutz and Helmut Lütkepohl (2017). *Structural Vector Autoregressive Analysis*. Themes in Modern Econometrics. Cambridge: Cambridge University Press. ISBN: 978-1-107-19657-5. URL: https://doi.org/10.1017/9781108164818.

A. Solutions

1 Solution to Long-Run Restrictions

1. Short-run restrictions are often quite restrictive as we require knowledge of how certain variables react instantaneously to certain shocks. Economic theory does not really give us much guidance in the short-run, so we usually argue that certain variables are *sluggish* or information of variables is only available with a time lag. If we can agree on such behavior, we can be pretty confident about these restrictions. But most of the times, economists do not agree about the behavior of the variables in the short-run or have competing models about it. On the other hand, there is much more agreement what happens in the long-run. For instance, there is a vast literature on the effect of monetary policy shocks on output and inflation in the short-run with disagreeing results. On the other hand, most economists would agree that the effect of demand shocks such as monetary policy shocks have no effects on output and a positive effect on the price level. This suggests an alternative approach, i.e use theoretically-inspired long-run restrictions to identify shocks and impulse responses not only for the long-run behavior but also for the short-run behavior of all variables.

2. We have already shown that in a VAR(1) model we get the following structural impulse response function for horizons $h = 0, 1, 2, \cdots$

$$\frac{\partial y_t}{\partial \varepsilon_t'} = B_0^{-1}, \qquad \frac{\partial y_{t+1}}{\partial \varepsilon_t'} = A_1 B_0^{-1}, \qquad \frac{\partial y_{t+2}}{\partial \varepsilon_t'} = A_1^2 B_0^{-1}, \qquad \frac{\partial y_{t+h}}{\partial \varepsilon_t'} = A_1^h B_0^{-1}$$

The effects of Δgdp_t are given by the first row and the second row contains the effects on ur_t. So the effect of the structural shocks on the covariance stationary variables ur_t and Δgdp_t is given by

- B_0^{-1} on impact
- $A_1 B_0^{-1}$ after one period
- $A_1^2 B_0^{-1}$ after two periods
- $A_1^h B_0^{-1}$ after h periods

Now, because ur_t and Δgdp_t are covariance stationary, the Eigenvalues of A_1 are inside the unit circle. So, in the long-run, $h \to \infty$, we have that $A^h \to 0$. That is, the effect of the structural shocks on the covariance-stationary variables ur_t and Δgdp_t vanishes over time and they return to their expected mean value. On the other hand, given the IRFs of $\frac{\partial \Delta gdp_{t+h}}{\partial \varepsilon_t'}$ we can derive the IRF of gdp_{t+h} by the cumulative sum

$$\frac{\partial gdp_{t+h}}{\partial \varepsilon_t'} = \frac{\partial \Delta gdp_t}{\partial \varepsilon_t'} + \frac{\partial \Delta gdp_{t+1}}{\partial \varepsilon_t'} + \cdots + \frac{\partial \Delta gdp_{t+h}}{\partial \varepsilon_t'}$$

That is, the effect on the level of GDP to an increase in the structural shocks is equal to the **first row** in

- $(I)B_0^{-1}$ on impact
- $(I + A_1)B_0^{-1}$ after one period
- $(I + A_1 + A_1^2)B_0^{-1}$ after two periods
- $(I + A_1 + A_1^2 + A_1^3)B_0^{-1}$ after three periods
- $(I + A_1 + A_1^2 + \cdots + A_1^h)B_0^{-1}$ after h periods

In the long-run, $h \to \infty$, the effect of the structural shocks on the level of gdp_t is given by the **first row** of

$$(I + A_1 + A_1^2 + A_1^3 + \cdots)B_0^{-1} = (I - A)^{-1} B_0^{-1}$$

Again, we make use of the fact that ur_t and Δgdp_t are covariance-stationary, which implies that the Eigenvalues of A_1 are inside the unit circle and we can make use of the formula for the geometric sum.

More generally, for VAR(p) models with variables in first-differences, we get the long-run effect matrix for the corresponding level variables

$$\Theta(1) = A(1)^{-1} B_0^{-1}$$

where $A(1) = (I - A_1 - \cdots - A_p)$ is the lag polynomial evaluated at $L = 1$.

3. Re-consider the long-run multiplier matrix:

$$\Theta(1) = A(1)^{-1} B_0^{-1} \equiv \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

where a and c are the long-run effects of ε_t^{AS} and ε_t^{AD} on the level of gdp_t. Economic theory tells us that in the long-run gdp_t is not affected by an aggregate demand shock and it returns to its initial value. In other words, economic theory requires that $c = 0$:

$$\Theta(1) = A(1)^{-1} B_0^{-1} \equiv \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$$

Note that we don't have anything to say about a (the effect of an AS shock on the level of GDP) and leave it unrestricted. Similarly, as ur_t is covariance-stationary and does not enter y_t as first difference, we can't really identify or economically interpret b and d and leave it unrestricted as well.

4. There are two ways to infer B_0^{-1} from the restricted $\Theta(1) = A(1)^{-1} B_0^{-1} \equiv \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ matrix:

1. Method: Cholesky decomposition
Note that $\Theta(1)$ is lower triangular and looks like a Cholesky factor. Indeed:

$$\Theta(1)\Theta(1)' = A(1)^{-1} B_0^{-1} B_0^{-1'} A(1)^{-1'} = A(1)^{-1} \Sigma_u A(1)^{-1'}$$

This right-hand side can be readily computed as both $A(1)$ as well as Σ_u are given by the reduced-form estimation. That is computing the lower triangular Cholesky factor of $A(1)^{-1} \Sigma_u A(1)^{-1'}$ provides $\Theta(1)$. From $\Theta(1)$ we can compute B_0^{-1}:

$$B_0^{-1} = A(1)\Theta(1)$$

Once we have B_0^{-1} we can proceed as usual and compute the impulse response function.

2. Method: Numerical optimization
Use a numerical optimizer to find a B_0^{-1} matrix that fulfills the following restrictions:

- covariance restrictions: $vech(B_0^{-1} B_0^{-1'}) = vech(\Sigma_u)$
- long-run restriction: $\Theta(1)_{1,2} = 0$

where $\Theta(1) = A(1)^{-1} B_0^{-1}$. Note that both Σ_u as well as $A(1)^{-1}$ are fixed parameters and given by the reduced-form estimation.

5. The code might look like this:

progs/matlab/BlanchardQuahLR.m

```
% -------------------------------------------------------------------------
% Replicates the Blanchard and Quah (1989) model using long-run restrictions
% to identify the structural shocks either via a Cholesky decomposition or
```

```matlab
% using a numerical solver for expository purposes
% ----------------------------------------------------------------------
% Willi Mutschler, December 14, 2022
% willi@mutschler.eu
% ----------------------------------------------------------------------

clearvars; clc;close all;

%% settings and options
nlag = 8;                                    % number of lags
nsteps = 40;                                 % horizon of IRFs
IRFcumsum = [1 0];                           % cumulate (1) or not (0) IRFs for each variable
varnames = ["GDP level", "Unemployment"];    % variable names in IRF plots
epsnames = ["Supply Shock", "Demand Shock"]; % shock names in IRF plots

% file where identification restrictions are set up
f = str2func('BlanchardQuahLR_f');
StartValueMethod = 1;   % 0: Use identity matrix, 1: use square root, 2: use cholesky as starting value
% options for fsolve
TolX = 1e-4;          % termination tolerance on the current point
TolFun = 1e-9;        % termination tolerance on the function value
MaxFunEvals = 50000; % maximum number of function evaluations allowed
MaxIter = 1000;       % maximum numberof iterations allowed
OptimAlgorithm = 'trust-region-dogleg'; % algorithm used in fsolve
options=optimset('TolX',TolX,'TolFun',TolFun,'MaxFunEvals',MaxFunEvals,'MaxIter',MaxIter,'Algorithm',OptimAlgorithm);

%% data handling
BlanchardQuah1989 = importdata('../../data/BlanchardQuah1989.csv'); % load data
ENDO = BlanchardQuah1989.data;
[obs_nbr,var_nbr] = size(ENDO);

%% reduced-form estimation
opt.const = 1; % 0: no constant, 1: constant, 2: constant and linear trend
VAR = VARReducedForm(ENDO,nlag,opt); % Estimate reduced form
A1inv_big = inv(eye(size(VAR.Acomp,1))-VAR.Acomp); % Long-run matrix using companion form
J = [eye(var_nbr),zeros(var_nbr,var_nbr*(nlag-1))];
LRMat = J*A1inv_big*J';  % total impact matrix inv(eye(nvars)-A1hat-A2hat-...-Aphat)

%% structural Estimation
% identification using Cholesky
THETA_cholesky = chol(LRMat*VAR.SigmaOLS*LRMat','lower');
B0inv_cholesky = inv(LRMat)*THETA_cholesky;

% identification using numerical optimization
if StartValueMethod == 0
    B0inv_opt = eye(var_nbr); % use identity matrix as starting value
elseif StartValueMethod == 1
```

```
    B0inv_opt = VAR.SigmaOLS^.5; % use square root of vcov of reduced form as starting value
elseif StartValueMethod == 2
    B0inv_opt = chol(VAR.SigmaOLS,'lower'); % use Cholesky decomposition of vcov of reduced form
end
f(B0inv_opt,VAR.SigmaOLS,LRMat)' % test whether function works at initial value (should give you no error)

% Call optimization routine fsolve to minimize f
[B0inv_opt,fval,exitflag,output] = fsolve(f,B0inv_opt,options,VAR.SigmaOLS,LRMat);

% normalization rule on impact matrix: diagonal elements of B0inv are supposed to be positive (only needed for optimized values, cholesky is always positive on diagonal)
if any(diag(B0inv_opt)<0)
    x = diag(B0inv_opt)<0;
    B0inv_opt(:,find(x==1)) = -1*B0inv_opt(:,find(x==1));
end

% compare
table(B0inv_cholesky, B0inv_opt)
impact = B0inv_opt;

% check that B0inv solution is correct (result should be (close to a) K x K zero matrix)
impact*impact'-VAR.SigmaOLS
% check that structural innovations are orthogonal to one another (result should be identity matrix for correlations)
E=inv(impact)*VAR.residuals;
corrcoef(E(2,:),E(1,:))

%% compute and plot structural impulse response function
IRFpoint = irfPlots(VAR.Acomp,impact,nsteps,IRFcumsum,varnames,epsnames);
```

Here is the helper function to impose the restrictions:

progs/matlab/BlanchardQuahLR_f.m

```
function f = BlanchardQuahLR_f(B0inv,SIGMAUHAT,LRMat)
% f = BlanchardQuahLR_f(B0inv,SIGMAUHAT,LRMat)
% -----------------------------------------------------------------------
% Evaluates the system of nonlinear equations
% vech(SIGMAUHAT) = vech(B0inv*B0inv')
% subject to the long-run restrictions
% [* 0;
%  * *];
% -----------------------------------------------------------------------
% INPUTS
%   - B0inv     : candidate for short-run impact matrix. [var_nbr x var_nbr]
%   - SIGMAUHAT : covariance matrix of reduced-form residuals. [var_nbr x var_nbr]
%       - LRMat     : total long-run impact matrix. [var_nbr x var_nbr]
```

```
%                 - for VAR model A(1) = inv(eye(nvars)-A1hat-A2hat-...-Aphat)
% -----------------------------------------------------------------------
% OUTPUTS
%   - f : function value of system of nonlinear equations
% -----------------------------------------------------------------------
% Willi Mutschler, November 2017
% willi@mutschler.eu
% -----------------------------------------------------------------------

THETA = LRMat*B0inv; % cumulated (long-run) impulse response function
f=[vech(B0inv*B0inv'-SIGMAUHAT);
    THETA(1,2) - 0;
    ];

end
```

The supply shock has on impact a positive effect on the log-level of GDP, it then drops for one quarter, but increases again afterwards. The long-run effect of a supply shock on the log-level of GDP is positive. On the other hand, the demand shock obviously has no long-run effect on the log-level of GDP (due to our identifying restriction), whereas in the short-run the effect is negative. The effects on the unemployment rate are basically flipped with the exception that in the long-run the effects vanish as ur_t is covariance-stationary.

2 Solution to Combining Short-Run And Long-Run Restrictions

1. The Federal Reserve Board is assumed to control the interest rate by setting the policy innovation after observing the forecast errors for deflator inflation and real GNP growth. The model is fully identified and includes an aggregate demand shock and an aggregate supply shock in addition to the monetary policy shock. The monetary policy shock does not affect real GNP either within the current quarter or in the long run. The only shock to affect the log-level of real GNP in the long run is the aggregate supply shock.

2/3/4 The code might look like this:

progs/matlab/RWZSRLR.m

```matlab
% -------------------------------------------------------------------------
% Replicates the Rubio-Ramirez, Waggoner, Zha (2010)'s model using both
% short-run and long-run restrictions to identify the structural shocks
% using a numerical solver (instead of the more efficient algorithm
% proposed by the authors)
% -------------------------------------------------------------------------
% Willi Mutschler, December 14, 2022
% willi@mutschler.eu
% -------------------------------------------------------------------------

clearvars; clc; close all;

% data handling
RWZ2010 = importdata('../../data/RWZ2010.csv');
ENDO = RWZ2010.data;

% settings and options
varnames = ["GDP level", "Federal Funds Rate", "Deflator Inflation"];
epsnames = ["Policy", "Aggregate Demand", "Aggregate Supply"] + "Shock"; % note that structural shocks are not ordered in the same row as the corresponding variable!!!
IRFcumsum = [1 0 0];
nsteps = 40;
nlag = 4;
[obs_nbr,var_nbr] = size(ENDO);

% estimate reduced-form
opt.const = 1;
VAR = VARReducedForm(ENDO,nlag,opt);
A1inv_big = inv(eye(size(VAR.Acomp,1))-VAR.Acomp); % from the companion form
LRMat = A1inv_big(1:var_nbr,1:var_nbr); % total impact matrix inv(eye(nvars)-A1hat-A2hat-...-Aphat)
% one can also use the J matrix
% J = [eye(var_nbr),zeros(var_nbr,var_nbr*(nlag-1))];
% LRMat = J*A1inv_big*J';

% options for fsolve
TolX = 1e-4;          % termination tolerance on the current point
TolFun = 1e-9;        % termination tolerance on the function value
MaxFunEvals = 50000; % maximum number of function evaluations allowed
MaxIter = 1000;       % maximum numberof iterations allowed
OptimAlgorithm = 'trust-region-dogleg'; % algorithm used in fsolve
```

```matlab
optim_options = optimset('TolX',TolX,'TolFun',TolFun,'MaxFunEvals',MaxFunEvals,'MaxIter',MaxIter,'Algorithm',OptimAlgorithm);

% initital guess
StartValueMethod = 2; % 0: Use identity matrix, 1: use square root, 2: use cholesky as starting value
if StartValueMethod == 0
    B0inv = eye(var_nbr); % Use identity matrix as starting value
elseif StartValueMethod == 1
    B0inv = VAR.SigmaOLS^.5; % Use square root of vcov of reduced form as starting value
elseif StartValueMethod == 2
    B0inv = chol(VAR.SigmaOLS,'lower'); % Use Cholesky decomposition of vcov of reduced form
end

% structural identification
f = str2func('RWZSRLR_f');   % identification restrictions are set up in RWZSRLR_f
f(B0inv,VAR.SigmaOLS,LRMat)' % test whether function works at initial value (should give you no error)

% call optimization routine fsolve
[B0inv,fval,exitflag,output] = fsolve(f,B0inv,optim_options,VAR.SigmaOLS,LRMat);
disp(B0inv);

% normalization rules
if sign(B0inv(2,1)) == -1
    B0inv(:,1)=-B0inv(:,1); % normalize sign of first column such that a monetary policy shock (first column) raises the interest rate (second row) (monetary tightening)
end
if sign(B0inv(1,2)) == -1 && sign(B0inv(3,2)) == -1
    B0inv(:,2)=-B0inv(:,2); % normalize sign of second column such that a positive aggregate demand shock (2nd column) does not lower real GNP (first row) and inflation (third row)
end
if sign(B0inv(1,3)) == -1 && sign(B0inv(3,3)) == 1
    B0inv(:,3)=-B0inv(:,3); % normalize sign of third column such that a positive aggregate supply shock (3rd column) does not lower real GNP (first row) and does not raise inflation (third row)
end

impact = B0inv;

% check that B0inv solution is correct (result should be (close to a) K x K zero matrix)
impact*impact'-VAR.SigmaOLS
% check that structural innovations are orthogonal to one another (result should be identity matrix for correlations)
E = inv(impact)*VAR.residuals;
corrcoef(E(1,:),E(2,:))
```

```
corrcoef(E(1,:),E(3,:))
corrcoef(E(2,:),E(3,:))

% compute and plot structural impulse response function
IRFpoint = irfPlots(VAR.Acomp,impact,nsteps,IRFcumsum,varnames,epsnames);
```

Here is the helper function to impose the restrictions:

progs/matlab/RWZSRLR_f.m

```
function f = RWZSRLR_f(B0inv,SIGMAUHAT,LRMat)
% f = RWZSRLR_f(B0inv,SIGMAUHAT,LRMat)
% -------------------------------------------------------------------------
% Evaluates the system of nonlinear equations vech(SIGMAUHAT) = vech(B0inv*B0inv')
% subject to the short-run and long-run restrictions.
% -------------------------------------------------------------------------
% INPUTS
%   - B0inv     : candidate for short-run impact matrix. [var_nbr x var_nbr]
%   - SIGMAUHAT : covariance matrix of reduced-form residuals. [var_nbr x var_nbr]
%       - LRMat     : total long-run impact matrix. [var_nbr x var_nbr]
%                   - for VAR model A(1) = inv(eye(nvars)-A1hat-A2hat-...-Aphat)
% -------------------------------------------------------------------------
% OUTPUTS
%   - f : function value, see below
% -------------------------------------------------------------------------
% Willi Mutschler, December 14, 2022
% willi@mutschler.eu
% -------------------------------------------------------------------------

THETA = LRMat*B0inv;

f = [vech(SIGMAUHAT-B0inv*B0inv');
         B0inv(1,1) - 0;
         THETA(1,1) - 0;
         THETA(1,2) - 0;
    ];

end
```

An unexpected monetary policy tightening is associated with a persistent decline in real GNP and a rather short-lived response in inflation. This is an interesting finding considering the dual mandate of the Federal Reserve.

3 Solution to Inference In SVARs Identified By Exclusion Restrictions First, some notation. The underlying VAR model is given by:

$$y_t = \nu + A_1 y_{t-1} + \dots + A_p y_{t-p} + u_t$$

where the structural shocks are related to the residuals by the impact matrix, $u_t = B_0^{-1}\varepsilon_t$, such that $\Sigma_u = B_0^{-1}B_0^{-1\prime}$. Now let's collect the reduced-form coefficients in a vector α:

$$\alpha = vec\left(\begin{bmatrix}\nu & A_1 & \cdots & A_p\end{bmatrix}\right)$$

and similarly, the unique coefficients of Σ_u in a vector σ:

$$\sigma = vech\left(\Sigma_u\right)$$

Key insight: the impulse response function for variable j with respect to shock k at horizon h is a function of α and σ:

$$\theta \equiv \theta_{jkh} = g(\alpha, \sigma)$$

1. Under some suitable conditions (u_t is Gaussian iid or u_t is iid with finite 4 moments), one can use a central limit theorem to derive the asymptotic distribution of α and σ given OLS (or ML) estimates $\hat{\alpha}$ and $\hat{\sigma}$:

$$\sqrt{T}\begin{pmatrix}\hat{\alpha}\\ \hat{\sigma}\end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix}\alpha\\ \sigma\end{pmatrix}, \begin{pmatrix}\Sigma_{\hat{\alpha}} & 0\\ 0 & \Sigma_{\hat{\sigma}}\end{pmatrix}\right)$$

where the block diagonal structure of the covariance matrix is without loss of generality to simplify the exposition and also since it appears under Gaussianity. Particularly, assuming Gaussian u_t consistent estimates can be derived:

$$\hat{\Sigma}_{\hat{\alpha}} = \left(\frac{1}{T}ZZ'\right)^{-1} \otimes \hat{\Sigma}_u$$
$$\hat{\Sigma}_{\hat{\sigma}} = 2D_K^{+}\left(\hat{\Sigma}_u \otimes \hat{\Sigma}_u\right)D_K^{+\prime}$$

where D_K is a duplication matrix such that $D_K vech(\Sigma_u) = vec(\Sigma_u)$ and $D_K^{+} = (D_K' D_K)^{-1}D_K'$ its Moore-Penrose inverse. In other words, both D_K or D_K^{+} are sparse matrices with some entries being equal to either 1/2, 1 or 2, so it is a fixed matrix.

DELTA METHOD: The delta method is an analytical way to derive the asymptotic distribution of a function of asymptotically normally distributed variables with known (estimated) mean and variance. More precisely, if asymptotically $X \sim N(\mu_X, \Sigma_X)$ we are able to derive the asymptotic distribution of a continuous function $\theta = g(X)$ based on a first-order Taylor expansion of g at $x = \mu_X$:

$$\theta = g(X) \approx g(\mu_X) + \frac{\partial g(\mu_X)}{\partial X'}(X - \mu_X)$$

Making use of the Gaussian distribution we can easily compute the expectation and variance to derive the asymptotic distribution of θ:

$$\theta = g(X) \sim N\left(g(\mu_X), \frac{\partial g(\mu_X)}{\partial X'}\Sigma_X \frac{\partial g(\mu_X)}{\partial X}\right)$$

The delta method requires consistent estimates and is an important tool to derive asymptotic distributions.[1] For us, if u_t is a normally distributed white noise process, we know the asymptotic distributions of α and σ_u, and hence, we may rely on asymptotics based on the normal

[1] For example, the delta method is often used to derive the asymptotic distribution of standard errors when estimated with Maximum Likelihood methods. That is, a variance σ_x^2 is a positive number; however, many numerical optimizers are not designed to work well with a bounded domain of parameters. One often employed approach is to do a parameter transformation, i.e. $\theta = \log(\sigma_x^2)$ to get the ML estimates $\hat{\theta}$ and corresponding variance $\hat{V}(\hat{\theta})$. As ML estimators are asymptotically normally distributed, one uses the delta method to report the point estimate for $\hat{\sigma}_X = e^{0.5\hat{\theta}}$ and corresponding standard error $\hat{std}(\hat{\sigma_X}) = \sqrt{0.5e^{0.5\hat{\theta}}\hat{V}(\hat{\theta})0.5e^{0.5\hat{\theta}}}$.

distribution to get the stated confidence interval

$$\hat{\theta} \pm z_{\gamma/2}\widehat{std}(\hat{\theta})$$

However, this works only under very restrictive assumptions. A more general approach is to rely on simulation-based methods (sampling techniques), i.e. to bootstrap the distribution of θ based on sample analogues. There are many different bootstrap approaches, we will cover only the main ones (and not the most recent ones), to understand the intuition and general approach.

One more thing to consider, even when using this asymptotic confidence interval, is that we rely on a consistent estimate for $\widehat{std}(\hat{\theta})$, for which we either may use closed-form expressions (valid under Gaussianity) or, better, which we can alternatively base on a bootstrap approximation.

2. u_t is iid white noise with distribution F, $u_t \overset{iid}{\sim} F$. Idea: approximate the unknown stationary VAR(p) data generating process (DGP) of known order p:

$$y_t = v + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t$$

by the bootstrap DGP

$$y_t^* = \hat{\nu} + \hat{A}_1 y_{t-1}^* + \cdots + \hat{A}_p y_{t-p}^* + u_t^*$$

where $u_t^* \overset{iid}{\sim} \hat{F}_T$ and $\hat{F}_T$ is the implied estimate of the error distribution F. $*$ marks values corresponding to the bootstrap DGP. Usually we use a nonparametric approach, i.e. we draw u_t^* with replacement from the set of residuals $\{\hat{u}\}_{t=1}^T$ of the consistent reduced-form estimation. The key insight is that u_t^* has the same distribution as u_t.
Side note: it is advisable to use a nonparametric approach instead of a wrong parametric one (e.g. normal distribution for u_t^*)!
Algorithm:
Given random draws for $u_t^{*r}, t = 1, \ldots, T$ and initial conditions $[y_{-p+1}^{*r}, \ldots, y_0^{*r}]$ recursively generate for each bootstrap replication $r = 1, \ldots, R$ a sequence of bootstrap realizations $\{y_t^{*r}\}_{t=-p+1}^T$ as

$$\begin{aligned}
y_1^{*r} &= \hat{v} + \hat{A}_1 y_0^{*r} + \cdots + \hat{A}_p y_{-p+1}^{*r} + u_1^{*r} \\
y_2^{*r} &= \hat{v} + \hat{A}_1 y_1^{*r} + \cdots + \hat{A}_p y_{-p+2}^{*r} + u_2^{*r} \\
&\vdots \\
y_T^{*r} &= \hat{v} + \hat{A}_1 y_{T-1}^{*r} + \cdots + \hat{A}_p y_{-p+T}^{*r} + u_T^{*r}
\end{aligned}$$

Then proceed as usual: estimate reduced-form (if you estimated the lag length you should estimate lag length again as well), use identification restrictions to compute bootstrapped impulse response function in each replication r. Given this approach we get a bootstrap approximation to the distribution of the IRFs, which we can use for inference.

3. The Standard Residual-Based Bootstrap approach requires iid regression errors, this is a quite strong and restrictive assumption. An alternative is to use the so-called *Wild Bootstrap*, i.e. instead of drawing u_t^{*r} with replacement, we multiply each element of the residual vector by a scalar draw η_t from an auxiliary distribution that has mean zero and unit variance:

$$u_t^{*r} = \hat{u}_t \eta_t, \eta_t \overset{iid}{\sim} (0, 1)$$

Possible distributions are usually

a) standard normal distribution

b) $\eta_t = 1$ with probability 0.5 and $\eta_t = -1$ with probability 0.5

c) $\eta_t = -(\sqrt{5}-1)/2$ with probability p and $\eta_t = (\sqrt{5}+1)/2$ with probability $1-p$, where $p = (\sqrt{5}+1)/(2\sqrt{5})$.

Note 1: Usually there is not much difference which distribution is chosen.
Note 2: Any t-statistic based on the wild bootstrap will have to be computed based on heteroskedasticity-robust standard errors.

4. We are trained to rely on 5%, i.e. use 5% for the p-value or compute $100\% - 5\% = 95\%$ confidence intervals. However, there is no statistical foundation for this, but it is simply common practice![2] Nevertheless, due to the case that typically in SVARs we have rather short samples, applied researcher prefer to use ± 1 standard error bands, i.e. 68% confidence intervals, instead of 2 standard error bands which correspond to 95% confidence intervals. Moreover, for a bootstrap approximation, the number of draws to accurately estimate the 2.5th and 97.5th percentiles tends to be much larger than required for the 16th and 84th percentiles.

5. The usual approach is to draw the initial conditions at random with replacement as a block of p consecutive vector valued observations. For each bootstrap replication r a new block is selected. Another approach would be to always use e.g. the mean of y_t as initial conditions. Or simulate a burnin-phase, e.g. of 1000 observations, and discard these.

6. Note that there is much development and ongoing research within this field to overcome the shortcomings of the following traditional approaches. Nevertheless, these approaches are still most widely used in applied work, and it is important to know them:

   a) Intervals based on bootstrap standard errors: Take the asymptotic CI but estimate the standard deviation of the bootstrap draws of $\hat{\theta}^*$ numerically, i.e.

   $$\hat{\theta} \pm z_{\gamma/2}\widehat{std}(\hat{\theta}^*)$$

   This allows us to relax the assumption of Gaussian iid innovations underlying the conventional interval computed with the delta method.

   b) Efron's percentile interval: Let $\hat{\theta}^*_{\gamma/2}$ and $\hat{\theta}^*_{1-\gamma/2}$ be the critical points defined by the $\gamma/2$ and $1-\gamma/2$ quantiles of the distribution of $\hat{\theta}^*$. Then Efron's percentile interval is:

   $$[\hat{\theta}^*_{\gamma/2}, \hat{\theta}^*_{1-\gamma/2}]$$

   However, this approach is based on the assumption of an unbiased estimator of θ; in SVAR models we typically need to correct for the inherent small-sample bias.

   c) equal-tailed percentile-t intervals is based on the idea that instead of using the critical points based on the standard normal distribution, we create our own table with t-statistics by a bootstrap approximation. That is, we approximate the distribution of the asymptotically pivotal (i.e. independent of other parameters) t-statistic

   $$\hat{t} = \frac{\hat{\theta} - \theta}{\widehat{std}(\hat{\theta})}$$

   by

   $$\hat{t}^* = \frac{\hat{\theta}^* - \hat{\theta}}{\widehat{std}(\hat{\theta}^*)},$$

   where $\hat{\theta}$ is treated as a fixed parameter in the bootstrap DGP. Let $\hat{t}^*_{\gamma/2}$ and $\hat{t}^*_{1-\gamma/2}$ be the critical points defined by the $\gamma/2$ and $1-\gamma/2$ quantiles of the distribution of $\hat{t}^*$, then the CI is given by

   $$[\hat{\theta} - \hat{t}^*_{1-\gamma/2}\widehat{std}(\hat{\theta}); \hat{\theta} - \hat{t}^*_{\gamma/2}\widehat{std}(\hat{\theta})]$$

[2] Actually, in the last 5 years there has been an increased debate in statistics and many social sciences about NOT relying solely on p-values for assessing statistical significance!

The bootstrapped t-values allow for possible asymmetry in the distribution and implicitly correct for bias. Note that again we need an estimate (either analytically or via bootstrap) for $\widehat{std}(\hat{\theta})$.