Quantitative Macroeconomics

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Week 13

Willi Mutschler
willi@mutschler.eu

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Contents

1. DSGE Models: Definition, Key Challenges, Basic Structure

1. Briefly define the term and key challenges of **D**ynamic **S**tochastic **G**eneral **E**quilibrium (DSGE) models. What are DSGE models useful for?

2. Outline the common structure of a DSGE model. How do Neo-Classical, New-Classical and New-Keynesian models differ?

3. Comment whether or not the assumptions underlying DSGE models should be as realistic as possible. For example, a very common assumption is that all agents live forever.

Readings

- Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016, Ch. 1)
- Torres (2013, Ch. 1)

2. RBC model

Consider the basic Real Business Cycle (RBC) model with leisure. The representative household maximizes present as well as expected future utility

$$\max E_t \sum_{j=0}^{\infty} \beta^j U_{t+j}$$

with $\beta < 1$ denoting the discount factor and E_t is expectation given information at time t. The contemporaneous utility function

$$U_t = \gamma \log(c_t) + \psi \log(1 - l_t)$$

is additively separable and has two arguments: consumption c_t and normalized labor supply l_t. The marginal utility of consumption is positive, whereas more labor supply reduces utility. Accordingly, γ is the consumption weight in the utility function and ψ the weight on leisure. In each period the household takes the real wage w_t as given and supplies perfectly elastic labor service to the representative firm. In return, she receives real labor income in the amount of $w_t l_t$ and, additionally, real profits div_t from the firm as well as revenue from lending capital k_{t-1} at real rental rate $r_{k,t}$ to the firms, as it is assumed that the firm and capital stock are owned by the household. Income and wealth are used to finance consumption c_t and investment i_t. In total, this defines the (real) budget constraint of the household:

$$c_t + i_t = w_t l_t + r_{k,t} k_{t-1} + div_t$$

The law of motion for capital k_t at the end of period t is given by

$$k_t = (1 - \delta) k_{t-1} + i_t$$

where δ is the capital depreciation rate.[1] Assume that the transversality condition is full-filled.

Productivity a_t is the driving force of the economy and evolves according to

$$\log a_t = \rho_a \log a_{t-1} + \varepsilon_{a,t}$$

where ρ_a denotes the persistence parameter and $\varepsilon_{a,t}$ is assumed to be normally distributed with mean zero and variance σ_a^2.

Real profits div_t of the representative firm are revenues from selling output y_t minus costs from labor $w_t l_{d,t}$ and renting capital $r_{k,t} k_{d,t-1}$:

$$div_t = y_t - w_t l_{d,t} - r_{k,t} k_{d,t-1}$$

The representative firm maximizes expected profits

$$\max E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} div_{t+j}$$

subject to a Cobb-Douglas production function

$$f(k_{d,t-1}, l_{d,t}) = y_t = a_t k_{d,t-1}^{\alpha} l_{d,t}^{1-\alpha}$$

The stochastic discount factor $\Lambda_{t,t+j}$ takes into account that firms are owned by the household, i.e. it is the present value of a unit of consumption in period $t + j$ or, respectively, the marginal utility of an additional unit of profit; therefore

$$\Lambda_{t,t+j} = \beta^j \frac{\partial U_{t+j} / \partial c_{t+j}}{\partial U_t / \partial c_t}$$

Finally, we have non-negativity constraints $k_t \geq 0$, $c_t \geq 0$ and $0 \leq l_t \leq 1$.

[1] Note that we use the end-of-period timing convention for capital, i.e. k_t instead of k_{t+1}, because the investment decision is done in period t and hence capital is also determined in t. In older papers and books you will often find beginning-of-period timing convention for capital, so always think about when a variable is decided and determined.

1. Show that the first-order conditions of the representative household are given by

$$U_{c,t} = \beta E_t \left[U_{c,t+1} \left(1 - \delta + r_{k,t+1}\right)\right]$$
$$w_t = -\frac{U_{l,t}}{U_{c,t}}$$

   where $U_{c,t} = \gamma c_t^{-1}$ and $U_{l,t} = \frac{-\psi}{1-l_t}$. Interpret these equations in economic terms.

2. Show that the first-order conditions of the representative firm are given by

$$w_t = f_l$$
$$r_{k,t} = f_k$$

   where $f_l = (1-\alpha)a_t\left(\frac{k_{d,t-1}}{l_{d,t}}\right)^{\alpha}$ and $f_k = \alpha a_t\left(\frac{k_{d,t-1}}{l_{d,t}}\right)^{1-\alpha}$. Interpret these equations in economic terms.

3. Show that combining the optimal decisions with clearing of both the labor market, i.e. $l_t^s = l_t$, and the capital market, $k_t^d = k_t$ implies clearing of the goods market:

$$y_t = c_t + i_t$$

4. Derive the steady-state of the model, in the sense that there is a set of values for the endogenous variables that in equilibrium remain constant over time.

5. Discuss how to calibrate the following parameters α, β, δ, γ, ψ, ρ_a and σ_a.

6. Briefly provide intuition behind the transversality condition.

7. Write a script for this RBC model with a feasible calibration for an OECD country that computes the steady-state of the model.

8. Write a DYNARE mod file for this RBC model with a feasible calibration for an OECD country and compute the steady-state of the model by using a `steady_state_model` block. Compare this to the steady-state computed above.

9. Now assume a contemporaneous utility function of the CRRA (constant Relative Risk Aversion) type:[2]

$$U_t = \gamma \frac{c_t^{1-\eta_c} - 1}{1-\eta_c} + \psi \frac{(1-l_t)^{1-\eta_l} - 1}{1-\eta_l}$$

   a) Derive the model equations and steady-state analytically.
   b) Write a script to compute the steady-state for this model.
   c) Write a DYNARE mod file and compute the steady-state for this model by using a helper function in the `steady_state_model` block.

Readings

- McCandless (2008, Ch. 3, Ch. 6)
- Torres (2013, Ch. 1, Ch. 2)

[2]Note that due to L'Hopital's rule $\eta_c = \eta_l = 1$ implies the original specification, $U_t = \gamma \log c_t + \psi \log(1 - l_t)$.

3. The Algebra of New Keynesian Models

Consider the basic New Keynesian (NK) model without capital, a linear production function and Calvo (1983) price frictions.

Households The economy is assumed to be inhabited by a large number of identical households. The representative household maximizes present as well as expected future utility

$$\max E_t \sum_{j=0}^{\infty} \beta^j U(c_{t+j}, l_{t+j}, z_{t+j})$$

with $\beta < 1$ denoting the discount factor and E_t is the expectation operator conditional on information at time t. The contemporaneous utility function

$$U(c_t, l_t, z_t) = z_t \cdot \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{l_t^{1+\varphi}}{1+\varphi} \right)$$

has three arguments: a consumption index c_t, a labor supply index l_t which corresponds to either hours worked or employed household members, and an exogenous preference shifter z_t. Note that the marginal utility of consumption is positive, whereas more labor reduces utility. The inverse of σ is the intertemporal elasticity of substitution, whereas the inverse of φ is the Frisch elasticity of labor. Note that the exogenous preference shifter z_t influences only intertemporal decisions, but not intratemporal ones. The consumption index is formed by aggregating a continuum of goods represented on the interval $h \in [0, 1]$ into a single consumption good using a Dixit and Stiglitz (1977) aggregation technology:

$$c_t = \left(\int_0^1 c_t(h)^{\frac{\epsilon-1}{\epsilon}} dh \right)^{\frac{\epsilon}{\epsilon-1}}$$

That is, $c_t(h)$ denotes the quantity of good h consumed by the household in period t. $\epsilon > 1$ is an elasticity parameter measuring the *love-of-variety*. The household decides how to allocate its consumption expenditures among the different goods by taking the price $P_t(h)$ of good h as given and maximizing the consumption index c_t for any given level of expenditures. Similarly, in each period the household takes the nominal wage W_t as given and supplies perfectly elastic labor service to the firm sector. In return she receives nominal labor income $W_t l_t$ and, additionally, nominal profits and dividends $P_t \int_0^1 div_t(f)df$ from each firm $f \in [0, 1]$ in the intermediate goods sector, because the firms are owned by the household. Moreover, the household purchases a quantity of one-period nominally riskless bonds B_t at price Q_t. The bond matures the following period and pays one unit of money at maturity. Income and wealth are used to finance consumption expenditures. In total this defines the (nominal) budget constraint of the household

$$\int_0^1 P_t(h)c_t(h)dh + Q_t B_t \leq B_{t-1} + W_t l_t + P_t \int_0^1 div_t(f)df$$

In addition, it is assumed that the household is subject to a solvency constraint that prevents it from engaging in Ponzi-type schemes:

$$\lim_{T \to \infty} E_t \left\{ \Lambda_{t,T} \frac{B_T}{P_T} \right\} \geq 0$$

for all periods t, where

$$\Lambda_{t,T} = \beta^{T-t} \frac{\partial U(c_T, l_T, z_T)/\partial c_T}{\partial U(c_t, l_t, z_T)/\partial c_t} \tag{1}$$

denotes the stochastic discount factor.

Furthermore, let $\Pi_t = P_t/P_{t-1}$ denote the gross inflation rate, then the following relationships for the nominal interest rate R_t and the real interest rate r_t hold:

$$Q_t = \frac{1}{R_t} \tag{2}$$

$$R_t = r_t E_t \Pi_{t+1} \tag{3}$$

1. Explain the economic intuition behind equation (2) that determines the nominal interest rate and equation (3) that determines the real interest rate.

2. Is there debt in this model? In other words, what is the optimal path for B_t in this model?

3. Explain the difference between the solvency constraint $\lim_{T\to\infty} E_t\left\{\Lambda_{t,T}\frac{B_T}{P_T}\right\} \geq 0$ and the transversality condition $\lim_{T\to\infty} E_t\left\{\Lambda_{t,T}\frac{B_T}{P_T}\right\} = 0$ which holds in the optimum allocation.

4. Show that cost minimization of consumption expenditures implies

$$c_t(h) = \left(\frac{P_t(h)}{P_t}\right)^{-\epsilon} c_t$$

$$P_t = \left(\int_0^1 P_t(h)^{1-\epsilon} dh\right)^{\frac{1}{1-\epsilon}}$$

   Interpret these equations. What does this imply for the budget constraint?

5. Derive the intratemporal and intertemporal optimality conditions:

$$w_t := \frac{W_t}{P_t} = -\frac{\frac{\partial U(c_t,l_t,z_t)}{\partial l_t}}{\frac{\partial U(c_t,l_t,z_t)}{\partial c_t}}$$

$$\frac{\partial U(c_t,l_t,z_t)}{\partial c_t} = \beta E_t\left[\frac{\partial U(c_{t+1},l_{t+1},z_{t+1})}{\partial c_{t+1}} r_t\right]$$

   where w_t denotes the real wage and $\Pi_{t+1} = P_{t+1}/P_t$ the gross inflation rate. Interpret these equations.

Firms: final good The economy is populated by a continuum of firms indexed by $f \in [0,1]$ that produce differentiated goods $y_t(f)$. The technology for transforming these intermediate goods into the final output good y_t has the Dixit and Stiglitz (1977) form:

$$y_t = \left[\int_0^1 (y_t(f))^{\frac{\epsilon-1}{\epsilon}} df\right]^{\frac{\epsilon}{\epsilon-1}} \tag{4}$$

where $\epsilon > 1$ is the substitution elasticity between inputs, the so-called *love-of-variety*.

6. Show that profit maximization in the final goods sector implies:

$$y_t(f) = \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} y_t$$

$$P_t = \left[\int_0^1 P_t(f)^{1-\epsilon} df\right]^{\frac{1}{1-\epsilon}}$$

   Interpret these equations. What does this imply for profits in the final goods sector?

Firms: intermediate goods Intermediate firm f uses the following linear production function to produce their differentiated good

$$y_t(f) = a_t l_{d,t}(f) \tag{5}$$

where a_t denotes the common technology level available to all firms. Firms face perfectly competitive factor markets for hiring labor $l_{d,t}(f)$. Real profits of firm f are equal to revenues from selling its differentiated good at price $P_t(f)$ minus costs from hiring labor at wage w_t:

$$div_t(f) = \frac{P_t(f)}{P_t} y_t(f) - w_t l_{d,t}(f) \tag{6}$$

The objective of the firm is to choose contingent plans for $P_t(f)$ and $l_{d,t}(f)$ so as to maximize the present discounted value of nominal dividend payments given by

$$E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} P_{t+j} div_{t+j}(f)$$

where household's stochastic discount factor $\Lambda_{t,t+j}$ takes into account that firms are owned by the household.

Prices of intermediate goods are determined by nominal contracts as in Calvo (1983) and Yun (1996). In each period firm f faces a constant probability $1-\theta, 0 \leq \theta \leq 1$, of being able to re-optimize the price $P_t(f)$ of its good $y_t(f)$. The probability is independent of the time it last reset its price. Formally:

$$P_t(f) = \begin{cases} \widetilde{P}_t(f) & \text{with probability } 1-\theta \\ P_{t-1}(f) & \text{with probability } \theta \end{cases} \tag{7}$$

where $\widetilde{P}_t(f)$ is the re-optimized price in period t. Accordingly, when a firm cannot re-set its price for j periods, its price in period $t+j$ is given by $\widetilde{P}_t(f)$ and stays there until the firm can optimize it again. Hence, the firm's objective in t is to set $\widetilde{P}_t(f)$ to maximize expected profits until it can re-optimize the price again in some future period $t+j$. The probability to be stuck at the same price for j periods is given by θ^j.

7. Derive the following expression for the stochastic discount factor:

$$\Lambda_{t,t+1+j} = \beta \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{-1} \Lambda_{t+1,t+1+j}$$

8. Show that the optimal labor demand schedule of intermediate good firm f is given by:

$$w_t = mc_t(f) a_t = mc_t(f) \frac{y_t(f)}{l_{d,t}(f)}$$

where $mc_t(f)$ are real marginal costs of firm f. What does this imply for aggregate real marginal costs $mc_t = \int_0^1 mc_t(f) df$?

9. Denote $\widetilde{p}_t := \frac{\widetilde{P}_t(f)}{P_t}$ and show that optimal price setting of intermediate firms must satisfy:

$$\begin{aligned} \widetilde{p}_t \cdot s_{1,t} &= \frac{\epsilon}{\epsilon-1} \cdot s_{2,t} \\ s_{1,t} &= y_t \frac{\partial U(c_t, n_t^s, z_t)}{\partial c_t} + \beta\theta E_t \Pi_{t+1}^{\epsilon-1} s_{1,t+1} \\ s_{2,t} &= mc_t y_t \frac{\partial U(c_t, n_t^s, z_t)}{\partial c_t} + \beta\theta E_t \Pi_{t+1}^{\epsilon} s_{2,t+1} \end{aligned}$$

Explain why firms that reset prices set the same price, i.e. $\widetilde{P}_t(f) = \widetilde{P}_t$ or in other words we can drop the f.

10. Show that the law of motion for the optimal re-set price $\widetilde{p}_t = \frac{\widetilde{P}_t(f)}{P_t}$ is given by:

$$1 = \theta \Pi_t^{\epsilon-1} + (1-\theta) \widetilde{p}_t^{1-\epsilon}$$

Monetary Policy The central bank adjusts the nominal interest rate R_t according to an interest rate rule in response to deviations of (i) gross inflation Π_t from a target Π^* and (ii) output y_t from steady-state output y:

$$R_t = R\left(\frac{\pi_t}{\pi^*}\right)^{\phi_\pi}\left(\frac{y_t}{y}\right)^{\phi_y} e^{\nu_t} \tag{8}$$

where R denotes the nominal interest rate in steady state, ϕ_π the sensitivity parameter to inflation deviations, ϕ_y the feedback parameter of the output gap and ν_t an exogenous deviation to the rule.

Exogenous variables and stochastic shocks The exogenous preference shifter z_t, the level of technology a_t and the exogenous deviations ν_t from the monetary rule evolve according to

$$\log z_t = \rho_z \log z_{t-1} + \varepsilon_{z,t} \tag{9}$$

$$\log a_t = \rho_a \log a_{t-1} + \varepsilon_{a,t} \tag{10}$$

$$\nu_t = \rho_\nu \nu_{t-1} + \varepsilon_{\nu,t} \tag{11}$$

with persistence parameters ρ_z, ρ_a and ρ_ν. The preference shock $\varepsilon_{z,t}$, the productivity shock $\varepsilon_{a,t}$ and the monetary policy shock $\varepsilon_{\nu,t}$ are iid Gaussian:

$$\begin{pmatrix}\varepsilon_{z,t}\\ \varepsilon_{a,t}\\ \varepsilon_{\nu,t}\end{pmatrix} \sim N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}\sigma_z^2 & 0 & 0\\ 0 & \sigma_a^2 & 0\\ 0 & 0 & \sigma_\nu^2\end{pmatrix}\right)$$

Market clearing

11. What does market clearing imply for private bonds B_t?
12. Explain why labor market clearing implies:

$$l_t = \int_0^1 l_{d,t}(f)df$$

13. Show that aggregate real profits of the intermediate firms are given by

$$div_t \equiv \int_0^1 div_t(f)df = y_t - w_t l_t$$

14. Show that aggregate demand is given by

$$y_t = c_t \tag{12}$$

15. Denote $p_t^* = \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df$. Show that aggregate supply is given by

$$p_t^* y_t = a_t l_t$$

Explain why p_t^* is called the price efficiency distortion.

16. Derive the law of motion for the price efficiency distortion p_t^*:

$$p_t^* = (1-\theta)\,\widetilde{p_t}^{-\epsilon} + \theta \pi_t^\epsilon p_{t-1}^* \tag{13}$$

Readings

- Galí (2015, Ch. 3)
- Heijdra (2017, Ch. 19)
- Romer (2019, Ch. 7)
- Woodford (2003, Ch. 3)
- Walsh (2017, Ch. 8)

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A. Solutions

1 Solution to DSGE Models: Definition, Key Challenges, Basic Structure

1. DSGE models use modern macroeconomic theory to explain and predict co-movements of aggregate time series. DSGE models start from what we call the micro-foundations of macroeconomics (i.e. to be consistent with the underlying behavior of economic agents), with a heart based on the rational expectation forward-looking economic behavior of agents. In reality all macro variables are related to each other, either directly or indirectly, so there is no "cetribus paribus", but a dynamic stochastic general equilibrium system.

   - General Equilibrium (GE): equations must always hold.
     Short-run: decisions, quantities and prices adjust such that equations are full-filled.
     Long-run: steady-state, i.e. a condition or situation where variables do not change their value (e.g. balanced-growth path where the rate of growth is constant).
   - Stochastic (S): disturbances (or shocks) make the system deviate from its steady-state, we get business cycles or, more general, a data-generating process
   - Dynamic (D): Agents are forward-looking and solve intertemporal optimization problems. When a disturbance hits the economy, macroeconomic variables do not return to equilibrium instantaneously, but change gradually over time, producing complex reactions. Furthermore, some decisions like investment or saving only make sense in a dynamic context. We can analyze and quantify the effects after (i) a temporary shock: how does the economy return to its steady-state, or (ii) a permanent shock: how does the economy transition to a new steady-state.

   Basic model structure:

   $$E_t\left[f(y_{t+1}, y_t, y_{t-1}, u_t)\right] = 0$$

   where E_t is the expectation operator with information conditional up to and including period t, y_t is a vector of endogenous variables at time t, u_t a vector of exogenous shocks or random disturbances with proper density functions. $f(\cdot)$ is what we call economic theory.
   First key challenge: values of endogenous variables in a given period of time depend on future expected values. We need dynamic programming techniques to find the optimality conditions which define the economic behavior of the agents. The solution to this system is called a decision or **policy function**:

   $$y_t = g(y_{t-1}, u_t)$$

   describing optimal behavior of all agents given the current state of the world y_{t-1} and after observing current shocks u_t.
   Second key challenge: DSGE models cannot be solved analytically, except for some very simple and unrealistic examples. We have to resort to numerical methods and a computer to find an approximated solution.
   third key challenge: Once the theoretical model and solution is at hands, the next step is the application to the data. A common procedure called calibration is assigning values to the parameters of the model by using previous information or matching some key ratios or moments provided by the data. More recently, researchers are commonly applying formal statistical methods to estimate the parameters using maximum likelihood, Bayesian techniques, indirect inference, or a method of moments.

2. The dynamic equilibrium is the result from the combination of economic decisions taken by all economic agents. For example, the following agents or sectors are commonly included:

   - Households: benefit from private consumption, leisure and possibly other things like money holdings or state services; subject to a budget constraint in which they finance their expenditures via (utility-reducing) work, renting capital and buying (government) bonds $\hookrightarrow$ maximization of utility

- Firms produce a variety of products with the help of rented equipment (capital) and labor. They (possibly) have market power over their product and are responsible for the design, manufacture and price of their products. ↪ cost minimization or profit maximization
- Monetary policy follows a feedback rule for either interest rates or money supply (growth). For instance: nominal interest rate reacts to deviations of the current (or lagged) inflation rate from its target and of current output from potential output.
- Fiscal policy (the government) collects taxes from households and companies in order to finance government expenditures (possibly utility-enhancing) and government investment (possibly productivity-enhancing). In addition, the government can issue debt securities.

There is no limitation, i.e. you can also add other agents and sectors like financial intermediaries (banks), international trade, research & development, climate, etc.

3. Neoclassical or New-Classical models are basically the same terminology (unless you study economic history or really want to dive into the different school of thoughts). Basically, both approaches focus on so-called **micro-foundations**, the one more in a classical sense (focus on real rigidities) and the other more in a Keynesian sense (focus on nominal rigidities). In principle this is already evident in the baseline RBC model and the baseline New-Keynesian model:

   - RBC model is the canonical neoclassical model: reduce economy to the interaction of just one (representative) consumer/household and one (representative) firm. Representative household takes decisions in terms of how much to consume (save) and how much time is devoted to work (leisure). Representative firm decides how much it will produce. Equilibrium of the economy will be defined by a situation in which all decisions taken by all economic agents are compatible and feasible. One can show that business cycles can be generated by one special disturbance: total factor productivity or neutral technological shock; hence, the model generates so-called real business cycles without nominal frictions. Moreover, there is monetary neutrality in the model.
   - New-Keynesian models have the same foundations as New-Classical general equilibrium models, but incorporate different types of rigidities in the economy. Whereas new classical DSGE models are constructed on the basis of a perfect competition environment, New-Keynesian models include additional elements to the basic model such as imperfect competitions, existence of adjustment costs in investment process, liquidity constraints or rigidities in the determination of prices and wages. Due to these nominal rigidities there is no monetary neutrality in the short run. Moreover, New-Keynesian models have become the leading macroeconomic paradigm.

   Noth that the scale of DSGE models has grown over time with incorporation of a large number of features. To name a few: consumption habit formation, nominal and real rigidities, non-Ricardian agents, investment adjustment costs, investment-specific technological change, taxes, public spending, public capital, human capital, household production, imperfect competition, monetary union, steady-state unemployment, green vs. brown production sector etc.

4. The degree of realism offered by an economic model is not a goal per se to be pursued by macroeconomists; typically we are focused on the model's **usefulness** in explaining macroeconomic reality. General strategy is the construction of formal structures through equations that reflect the interrelationships between the different economic variables. These simplified structures is what we call a model. The essential question is not that these theoretical constructions are realistic descriptions of the economy, but that they are able to explain the dynamics observed in the economy. Therefore, it is not possible to reject a model ex-ante because it is based on assumptions that we believe are not realistic. Rather, the validations must be based on the usefulness of these models to explain reality, and whether they are more useful than other models. Of course, most of the times unrealistic assumptions will yield non-useful models; often, however, simplified assumptions that are a very rough approximation of reality yield quite

useful models. Either way, the DSGE model paradigm is up-front with our assumptions and provide the EXACT model dynamics in terms of mathematical correct formulations that can be challenged, adapted and, ideally, improved.

Regarding the assumption that the lifetime of economic agents is assumed to be infinite: We know that the lifetime of consumers, firms and governments is in fact finite. Nevertheless, in most models this is a valid approximation of reality, because for solving and simulating these models is is not important that agents actually live forever, but that they use the infinite time horizon as **their reference period for taking economic decisions**. Framed this way, the assumption becomes highly realistic. Viewing at the economy from a macroeconomic point of view: No government thinks it will cease to exist at some point in the future and no entrepreneur takes decisions based on the idea that the firm will go bankrupt sometime in the future. Granted, for consumers this is rather weak; however„ we may think about families, dynasties or households rather than individual consumers. Again, the infinite time planning horizon assumption is a feasible one. On the other hand, if you want to study the finite life cycle of an agent (school-work-retirement) or pension schemes, the so-called Overlapping-Generations (OLG) framework is probably more adequate. Either way, we need the same methods and techniques to deal with OLG models as we do with New-Keynesian models or RBC models, because all these models belong to the same class, i.e. are all DSGE models.

2 Solution to RBC model

1. Due to the transversality condition, we will not have corner solutions and can neglect the non-negativity constraints. Moreover, due to the concave optimization problem, we only need to focus on the first-order conditions.

   The Lagrangian for the household problem is

$$
\begin{aligned}
£^H = E_t \sum_{j=0}^{\infty} & \beta^j \left\{\gamma \log(c_{t+j}) + \psi \log(1 - l_{t+j})\right\} \\
& + \beta^j \lambda_{t+j} \left\{(w_{t+j} l_{t+j} + r_{k,t+j} k_{t-1+j} - c_{t+j} - i_{t+j})\right\} \\
& + \beta^j \mu_{t+j} \left\{((1-\delta) k_{t-1+j} + i_{t+j} - k_{t+j})\right\}
\end{aligned}
$$

   Note that the problem is not to choose $\{c_t, i_t, l_t, k_t\}_{t=0}^{\infty}$ all at once in an open-loop policy, but to choose these variables sequentially given the information at time t in a closed-loop policy, i.e. at period t decision rules for $\{c_t, i_t, l_t, k_t\}$ given the information set at period t; at period $t+1$ decision rules for $\{c_{t+1}, i_{t+1}, l_{t+1}, k_{t+1}\}$ given the information set at period $t+1$, and so on.

   The first-order condition w.r.t. c_t is given by

$$
\begin{aligned}
& \frac{\partial £^H}{\partial c_t} = E_t \left(\gamma c_t^{-1} - \lambda_t\right) = 0 \\
& \Leftrightarrow \lambda_t = \gamma c_t^{-1} \qquad (I)
\end{aligned}
$$

   The first-order condition w.r.t. l_t is given by

$$
\begin{aligned}
& \frac{\partial £^H}{\partial l_t} = E_t \left(\frac{-\psi}{1 - l_t} + \lambda_t w_t\right) = 0 \\
& \Leftrightarrow \lambda_t w_t = \frac{\psi}{1 - l_t} \qquad (II)
\end{aligned}
$$

   The first-order condition w.r.t. i_t is given by

$$
\begin{aligned}
& \frac{\partial £^H}{\partial i_t} = E_t \left(-\lambda_t + \mu_t\right) = 0 \\
& \Leftrightarrow \lambda_t = \mu_t \qquad (III)
\end{aligned}
$$

   The first-order condition w.r.t. k_t is given by

$$
\begin{aligned}
& \frac{\partial £^H}{\partial k_t} = E_t(-\mu_t) + E_t \beta \left(\lambda_{t+1} r_{k,t+1} + \mu_{t+1}(1-\delta)\right) = 0 \\
& \Leftrightarrow \mu_t = E_t \beta(\mu_{t+1}(1-\delta) + \lambda_{t+1} r_{k,t+1}) \qquad (IV)
\end{aligned}
$$

   (I) and (III) in (IV) yields

$$
\underbrace{\gamma c_t^{-1}}_{U_{c,t}} = \beta E_t \underbrace{\gamma c_{t+1}^{-1}}_{U_{c,t+1}} (1 - \delta + r_{k,t+1})
$$

   This is the Euler equation of **intertemporal optimality**. It reflects the trade-off between consumption and savings. If the household saves a (marginal) unit of consumption, i.e. invest this into the capital stock, she can consume $(1 - \delta + r_{k,t+1})$ units in the following period. The marginal utility of consuming a unit today is equal to $U_{c,t}$, whereas consuming tomorrow has expected utility equal to $E_t(U_{c,t+1})$. Discounting expected marginal utility with β, an optimum is characterized by a situation in which the household must be indifferent between both choices.

(I) in (II) yields:

$$w_t = -\frac{\frac{-\psi}{1-l_t}}{\gamma c_t^{-1}} \equiv -\frac{U_{l,t}}{U_{c,t}}$$

This equation reflects **intratemporal optimality**; in other words, the labor supply function. According to the equation, the real wage must be equal to the marginal rate of substitution between consumption and leisure.

2. First, note that even though firms maximize expected profits it is actually a static problem as there are no forward-looking terms. That is, the objective is to maximize profits

$$div_t = a_t k_{d,t-1}^{\alpha} l_{d,t}^{1-\alpha} - w_t l_{d,t} - r_{k,t} k_{d,t-1}$$

The first-order condition w.r.t. $l_{d,t}$ is given by

$$\frac{\partial div_t}{\partial l_{d,t}} = (1-\alpha) a_t k_{d,t-1}^{\alpha} l_{d,t}^{-\alpha} - w_t = 0$$

$$\Leftrightarrow w_t = (1-\alpha) a_t k_{d,t-1}^{\alpha} l_{d,t}^{-\alpha} = f_l = (1-\alpha)\frac{y_t}{l_{d,t}}$$

The real wage must be equal to the marginal product of labor. Due to the Cobb-Douglas production function it is a constant proportion $(1-\alpha)$ of the ratio of total output to labor. Simply put, this is the labor demand function.

The first-order condition w.r.t. $k_{d,t-1}$ is given by

$$\frac{\partial div_t}{\partial k_{d,t-1}} = \alpha a_t k_{d,t-1}^{\alpha-1} l_{d,t}^{1-\alpha} - r_{k,t} = 0$$

$$\Leftrightarrow r_{k,t} = \alpha a_t k_{d,t-1}^{\alpha-1} l_{d,t}^{1-\alpha} = f_k = \alpha \frac{y_t}{k_{d,t-1}}$$

The real rental rate for capital must be equal to the marginal product of capital. Due to the Cobb-Douglas production function it is a constant proportion α of the ratio of total output to capital. Simply put, this is the capital demand function.

3. Making use of clearing of labor and capital markets implies that the firms profits in the optimum are given by:

$$div_t = y_t - w_t l_t - r_{k,t} k_{t-1} = y_t - (1-\alpha) y_t - \alpha y_t = 0$$

$$\Leftrightarrow y_t = w_t l_t + r_{k,t} k_{t-1}$$

Insert into the budget restriction of the households yields:

$$c_t + i_t = w_t l_t + r_{k,t} k_{t-1} + div_t = y_t$$

This is a manifestation of Walras law: if 2 out of 3 markets are cleared, the last market must clear as well.

4. First, consider the steady-state value of technology:

$$\log a = \rho_a \log a + 0 \Leftrightarrow \log a = 0$$

$$\Leftrightarrow a = 1$$

The Euler equation in steady-state becomes:

$$U_c = \beta U_c (1 - \delta + r_k)$$

$$\Leftrightarrow r_k = \frac{1}{\beta} + \delta - 1$$

Next we will provide recursively closed-form expressions for all variables in terms of steady-state labor. That is the right-hand sides of the following equations are given in terms of parameters or previously computed expressions.

- The firms demand for capital in steady-state becomes

$$
\begin{aligned}
r_k &= \alpha a k^{\alpha-1} l^{1-\alpha} \\
\Leftrightarrow \frac{k}{l} &= \left(\frac{\alpha a}{r_k}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

- The firms demand for labor in steady-state becomes:

$$
w = (1-\alpha) A k^{\alpha} l^{-\alpha} = (1-\alpha) a \left(\frac{k}{l}\right)^{\alpha}
$$

- The law of motion for capital in steady-state implies

$$
\frac{i}{l} = \delta \frac{k}{l}
$$

- The production function in steady-state becomes

$$
\frac{y}{l} = a \left(\frac{k}{l}\right)^{\alpha}
$$

- The clearing of the goods market in steady-state implies

$$
\frac{c}{l} = \frac{y}{l} - \frac{i}{l}
$$

Now, we have expressions for all variables as a ratio to steady-state labor. Hence, once we compute l, we can revisit the above expressions to compute all values in closed-form. Due to the log-utility function, we can actually derive a closed-form expression for l. To this end, set labor demand equal to labor supply and express the right hand side in terms of previously computed expressions.

$$
\begin{aligned}
\psi \frac{1}{1-l} &= \gamma c^{-1} w \\
\Leftrightarrow \psi \frac{l}{1-l} &= \gamma \left(\frac{c}{l}\right)^{-1} w \\
\Leftrightarrow l &= (1-l) \frac{\gamma}{\psi} \left(\frac{c}{l}\right)^{-1} w \\
\Leftrightarrow l &= \frac{\frac{\gamma}{\psi} \left(\frac{c}{l}\right)^{-1} w}{1 + \frac{\gamma}{\psi} \left(\frac{c}{l}\right)^{-1} w}
\end{aligned}
$$

Lastly, it is straightforward to compute the remaining steady-state values, i.e.

$$
c = \frac{c}{l} l, \qquad i = \frac{i}{l} l, \qquad k = \frac{k}{l} l, \qquad y = \frac{y}{l} l
$$

5. The transversality condition for an infinite horizon dynamic optimization problem is the boundary condition determining a solution to the problem's first-order conditions together with the initial condition. The transversality condition requires the present value of the state variables (here k_t and a_t) to converge to zero as the planning horizon recedes towards infinity. The first-order and transversality conditions are sufficient to identify an optimum in a concave optimization problem. Given an optimal path, the necessity of the transversality condition reflects the impossibility of finding an alternative feasible path for which each state variable deviates from the optimum at each time and increases discounted utility. These conditions are implicit only, we don't enter them in a computer program. But implicitly we do consider them when we focus on unique and stable solutions or when we pick certain steady-state values.

6. General hints: construct and parameterize the model such, that it corresponds to certain properties of the true economy. One often uses steady-state characteristics for choosing the parameters in accordance with observed data. For instance, long-run averages (wages, hours worked, interest rates, inflation, consumption-shares, government-spending-ratios, etc.) are used to target certain steady-state values of the endogenous variables, which implies certain values for some parameters. You can also use micro-studies, however, one has to be careful about the aggregation.

   We will focus on OECD countries and discuss one "possible" way to calibrate the model parameters (there are many other ways):

   α productivity parameter of capital. Due to the Cobb Douglas production function thus should be equal to the proportion of capital income to total income of economy. So, one looks inside the national accounts for OECD countries and sets α to 1 minus the share of labor income over total income. For most OECD countries this implies a range of 0.25 to 0.40.

   β subjective intertemporal preference rate of households: it is the value of future utility in relation to present utility. Usually this parameter takes a value slightly less than unity, indicating that agents discount the future. For quarterly data, we typically set it around 0.99 and for yearly data 0.96. These values imply a certain steady-state real rental rate. To see this, re-consider the Euler equation in steady-state: $\beta = \frac{1}{\bar{r}_k+1-\delta}$ where $r_k = \alpha\frac{y}{k}$. Looking at OECD data one usually finds that the average capital productivity k/y is in the range of 9 to 11.

   δ depreciation rate of capital stock. For quarterly data the literature uses values in the range of 0.02 to 0.03, for yearly data you often find 0.10. Again let's use a steady-state relationship to get a reasonable value. That is, $\delta = \frac{\bar{I}}{\bar{K}} = \frac{\bar{I}/\bar{Y}}{\bar{K}/\bar{Y}}$. For OECD data one usually finds an average ratio of investment to output, $\bar{I}/\bar{Y}$, around 0.25.

   γ and ψ households's preference regarding consumption and leisure. Often a certain interpretation in terms of elasticities of substitutions is possible. In the RBC mode, we can make use of the First-Order-Conditions in steady-state, i.e.

$$\frac{\psi}{\gamma} = \bar{W}\frac{(1-\bar{L})}{\bar{C}} = (1-\alpha)\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}\frac{(1-\bar{L})}{\bar{C}} = (1-\alpha)\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha}\frac{\frac{1}{\bar{L}}(1-\bar{L})}{\frac{\bar{C}}{\bar{L}}}$$

   Note that $\bar{C}/\bar{L}$ as well as $\bar{K}/\bar{L}$ are given in terms of already calibrated parameters (see steady-state computations). Therefore, one possible way is to normalize one of the parameters to unity (e.g. $\gamma = 1$) and calibrate the other one in terms of steady-state ratios for which we would only require to set a value for steady-state hours worked $\bar{L}$. In the specification of the utility function, we see that labor is normalized to be between 0 and 1. So, targeting 8 hours a day implies $l = 8/24 = 1/3$.

   ρ_A and σ_A parameters of process for total factor productivity. These do not influence the steady-state values, but the dynamics of the model. Often you can calibrate these by e.g. estimating the Cobb-Douglas production function with OLS and then compute the Solow residuals. Then look at the persistence and standard error of the residuals. Typically we find that ρ_A is mostly set above 0.9 to reflect persistence of the technological process and σ_A around 0.6 in the simple RBC model.

7. The function might look like this:

progs/matlab/rbcLogutilSS.m

```
function [SS,PARAMS,error_indicator] = rbcLogutilSS(SS,PARAMS)
% [SS,PARAMS,error_indicator] = rbcLogutilSS(SS,PARAMS)
% ----------------------------------------------------------------------
```

```
% computes the steady-state of the RBC model with log utility analytically
% -----------------------------------------------------------------------
% INPUTS
%   - SS     : structure with initial steady state values, fieldnames are variable names (usually empty, but might be useful for initial values)
%       - params : structure with values for the parameters, fieldnames are parameter names
% -----------------------------------------------------------------------
% OUTPUTS
%   - SS     : structure with computed steady state values, fieldnames are variable names
%       - params : structure with updated values for the parameters, fieldnames are parameter names
%   - error_indicator: 0 if no error when computing the steady-state
% -----------------------------------------------------------------------
% Willi Mutschler (willi@mutschler.eu)
% Version: January 26, 2023
% -----------------------------------------------------------------------
error_indicator = 0; % initialize no error

% read-out parameters
ALPHA = PARAMS.ALPHA;
BETA  = PARAMS.BETA;
DELTA = PARAMS.DELTA;
GAMMA = PARAMS.GAMMA;
PSI   = PARAMS.PSI;
RHOA  = PARAMS.RHOA;

% compute steady-state
a = 1;
rk = 1/BETA+DELTA-1;
k_l = ((ALPHA*a)/rk)^(1/(1-ALPHA));
if k_l <= 0
    error_indicator = 1;
end
w = (1-ALPHA)*a*k_l^ALPHA;
iv_l = DELTA*k_l;
y_l = a*k_l^ALPHA;
c_l = y_l - iv_l;
if c_l <= 0
    error_indicator = 1;
end
l = GAMMA/PSI*c_l^(-1)*w/(1+GAMMA/PSI*c_l^(-1)*w); % closed-form expression for l

c = c_l*l;
iv = iv_l*l;
k = k_l*l;
y = y_l*l;
uc  = GAMMA*c^(-1);
ul  = -PSI/(1-l);
fl  = (1-ALPHA)*a*(k/l)^ALPHA;
fk  = ALPHA*a*(k/l)^(ALPHA-1);
```

```

% write to output structure
SS.y  = y;
SS.c  = c;
SS.k  = k;
SS.l  = l;
SS.a  = a;
SS.rk = rk;
SS.w  = w;
SS.iv = iv;
SS.uc = uc;
SS.ul = ul;
SS.fl = fl;
SS.fk = fk;

end
```

You can try it out with the following parametrization (same as in the Dynare mod file):

progs/matlab/rbcLogutilSSTest.m

```
% computes the steady-state of the RBC model with log utility
% -----------------------------------------------------------------------
% Willi Mutschler (willi@mutschler.eu)
% Version: January 26, 2023
% -----------------------------------------------------------------------

% calibration
PARAMS.ALPHA = 0.35;
PARAMS.BETA = 0.9901;
PARAMS.DELTA = 0.025;
PARAMS.GAMMA = 1;
PARAMS.PSI = 1.7333;
PARAMS.RHOA = 0.9;
SS = []; % no need for initial values

% compute steady-state
[SS,PARAMS,error_indicator] = rbcLogutilSS(SS,PARAMS);
if ~error_indicator
    disp(SS);
else
    error('steady-state could not be computed')
end
```

8. The mod file might look like this:

progs/dynare/rbcLogutil.mod

```
%% Declare Variables and Parameters
var y c k l a rk w iv uc ul fl fk;
varexo eps_a;
parameters ALPHA BETA DELTA GAMMA PSI RHOA;

%% Calibration of parameters (simple)
% ALPHA = 0.35;
```

```
% BETA = 0.9901;
% DELTA = 0.025;
% GAMMA = 1;
% PSI = 1.7333;
% RHOA = 0.9;

%% Calibration of parameters (advanced) for OECD countries

% target values
kss_yss    = 10;    % average capital productivity found in long-run averages of data
ivss_yss   = 0.25;  % average investment to ouput ratio found in long-run averages of data
wsslss_yss = 0.65;  % average share of labor income to total income
lss        = 1/3;   % 8h/24h working time

% flip steady-state relationships to get parameters in terms of the target values
ALPHA = 1-wsslss_yss;          % labor demand in steady-state combined with Cobb-Douglas production function
DELTA = ivss_yss / kss_yss;    % capital accumulation in steady-state
rkss  = ALPHA/kss_yss;         % capital demand in steady-state combined with Cobb-Douglas production function
BETA  = 1/(rkss - DELTA + 1); % Euler equation in steady-state
% normalize GAMMA and calibrate PSI to get targeted lss
GAMMA = 1; % normalize
ass = 1; % tfp in steady-state
kss_lss = ((ALPHA*ass)/rkss)^(1/(1-ALPHA));
kss = kss_lss*lss;
yss = kss/kss_yss;
ivss = DELTA*kss;
wss = (1-ALPHA)*ass*kss_lss^ALPHA;
css = yss-ivss;
PSI = GAMMA*(css/lss)^(-1)*wss*(lss/(1-lss))^(-1); % flipped steady-state labor equation

RHOA  = 0.9; % does not affect the steady-state

%% Model Equations
model;
uc = GAMMA*c^(-1);
ul = -PSI/(1-l);
fl = (1-ALPHA)*a*(k(-1)/l)^ALPHA;
fk = ALPHA*a*(k(-1)/l)^(ALPHA-1);

uc = BETA*uc(+1)*(1-DELTA+rk(+1));
w = - ul/uc;
w = fl;
rk = fk;
y = a*k(-1)^ALPHA*l^(1-ALPHA);
k = (1-DELTA)*k(-1) + iv;
y = c + iv;
log(a) = RHOA*log(a(-1)) + eps_a;
```

```
end;

%% Steady State
steady_state_model;
a = 1;
rk = 1/BETA+DELTA-1;
K_L = ((ALPHA*a)/rk)^(1/(1-ALPHA));
w = (1-ALPHA)*a*K_L^ALPHA;
I_L = DELTA*K_L;
Y_L = a*K_L^ALPHA;
C_L = Y_L - I_L;
l = GAMMA/PSI*C_L^(-1)*w/(1+GAMMA/PSI*C_L^(-1)*w);
c = C_L*l;
iv = I_L*l;
k = K_L*l;
y = Y_L*l;
uc  = GAMMA*c^(-1);
ul  = -PSI/(1-l);
fl  = (1-ALPHA)*a*(k/l)^ALPHA;
fk  = ALPHA*a*(k/l)^(ALPHA-1);
end;

steady;

shocks;
var eps_a = 1;
end;

stoch_simul(order=1,irf=30,periods=400) y c k l rk w iv a;

figure('name','Simulated Data')
subplot(3,3,1); plot(oo_.endo_simul(ismember(M_.endo_names,'a'),300:end)); title('productivity');
subplot(3,3,2); plot(oo_.endo_simul(ismember(M_.endo_names,'y'),300:end)); title('output');
subplot(3,3,3); plot(oo_.endo_simul(ismember(M_.endo_names,'c'),300:end)); title('consumption');
subplot(3,3,4); plot(oo_.endo_simul(ismember(M_.endo_names,'k'),300:end)); title('capital');
subplot(3,3,5); plot(oo_.endo_simul(ismember(M_.endo_names,'iv'),300:end)); title('investment');
subplot(3,3,6); plot(oo_.endo_simul(ismember(M_.endo_names,'rk'),300:end)); title('rental rate');
subplot(3,3,7); plot(oo_.endo_simul(ismember(M_.endo_names,'l'),300:end)); title('labor');
subplot(3,3,8); plot(oo_.endo_simul(ismember(M_.endo_names,'w'),300:end)); title('wage');
```

Obviously, the results are the same.

9. a) For the first-order conditions of the household we know use

$$U_{c,t} = \gamma c_t^{-\eta_c}$$
$$U_{l,t} = -\psi(1-l_t)^{-\eta_l}$$

The steady-state for labor changes to

$$w\gamma\left(\frac{c}{l}\right)^{-\eta_c} = \psi(1-l)^{-\eta_l}L^{\eta_c}$$

This cannot be solved for l in closed-form. Rather, we need to condition on the values of the parameters and use a numerical optimizer to solve for l.

b) The function might look like this:

progs/matlab/rbcSS.m

```matlab
function [SS,PARAMS,error_indicator] = rbcSS(SS,PARAMS)
% [SS,PARAMS,error_indicator] = rbcSS(SS,PARAMS)
% -------------------------------------------------------------------------
% computes the steady-state of the RBC model with CES utility using a
% numerical optimizer to compute steady-state labor
% -------------------------------------------------------------------------
% INPUTS
%   - SS      : structure with initial steady state values, fieldnames are variable names
%       - params : structure with values for the parameters, fieldnames are parameter names
% -------------------------------------------------------------------------
% OUTPUTS
%   - SS      : structure with computed steady state values, fieldnames are variable names
%       - params : structure with updated values for the parameters, fieldnames are parameter names
%   - error_indicator: 0 if no error when computing the steady-state
% -------------------------------------------------------------------------
% Willi Mutschler (willi@mutschler.eu)
% Version: January 26, 2023
% -------------------------------------------------------------------------

error_indicator = 0; % initialize no error

% read-out parameters
ALPHA = PARAMS.ALPHA;
BETA  = PARAMS.BETA;
DELTA = PARAMS.DELTA;
GAMMA = PARAMS.GAMMA;
PSI   = PARAMS.PSI;
RHOA  = PARAMS.RHOA;
ETAC  = PARAMS.ETAC;
ETAL  = PARAMS.ETAL;

% compute steady-state
a = 1;
rk = 1/BETA+DELTA-1;
k_l = ((ALPHA*a)/rk)^(1/(1-ALPHA));
```

```
if k_l <= 0
    error_indicator = 1;
end
w = (1-ALPHA)*a*k_l^ALPHA;
iv_l = DELTA*k_l;
y_l = a*k_l^ALPHA;
c_l = y_l - iv_l;
if c_l <= 0
    error_indicator = 1;
end
if (ETAC == 1 && ETAL == 1)
    % closed-form expression for l
    l = GAMMA/PSI*c_l^(-1)*w/(1+GAMMA/PSI*c_l^(-1)*w);
else
    % no closed-form solution and we therefore use a fixed-point algorithm
    if error_indicator == 0
        l0 = SS.l;
        [l,~,exitflag] = fsolve(@findL,l0,optimset('Display','off','TolX',1e-12,'TolFun',1e-12));
        if exitflag <= 0
            error_indicator = 1;
        end
    else
        l = NaN;
    end
end
c  = c_l*l;
iv = iv_l*l;
k  = k_l*l;
y  = y_l*l;
uc = GAMMA*c^(-1);
ul = -PSI/(1-l);
fl = (1-ALPHA)*a*(k/l)^ALPHA;
fk = ALPHA*a*(k/l)^(ALPHA-1);

% write to output structure
SS.y = y;
SS.c = c;
SS.k = k;
SS.l = l;
SS.a = a;
SS.rk = rk;
SS.w = w;
SS.iv = iv;
SS.uc = uc;
SS.ul = ul;
SS.fl = fl;
SS.fk = fk;

%% Auxiliary function used in optimization
% note that some variables are not explicitly declared as input arguments but get their value from above,
```

```
% i.e. the scope of some variables spans multiple functions
function error = findL(L)
    error = w*GAMMA*c_l^(-ETAC) - PSI*(1-L)^(-ETAL)*L^ETAC;
end

end
```

You can try it out with the following parametrization (same as in the Dynare mod file):

progs/matlab/rbcSSTest.m

```
% computes the steady-state of the RBC model with CES utility
% -----------------------------------------------------------------------
% Willi Mutschler (willi@mutschler.eu)
% Version: January 26, 2023
% -----------------------------------------------------------------------

% calibration
PARAMS.ALPHA = 0.35;
PARAMS.BETA  = 0.9901;
PARAMS.DELTA = 0.025;
PARAMS.GAMMA = 1;
PARAMS.PSI   = 1.7333;
PARAMS.RHOA  = 0.9;
PARAMS.ETAC  = 2;
PARAMS.ETAL  = 1.5;
SS.l = 1/3; % initial guess for labor

% compute steady-state
[SS,PARAMS,error_indicator] = rbcSS(SS,PARAMS);
if ~error_indicator
    disp(SS);
else
    error('steady-state could not be computed')
end
```

c) In Dynare we could use the following mod file:

progs/dynare/rbcCES.mod

```
%% Declare Variables and Parameters
var y c k l a rk w iv uc ul fl fk;
varexo eps_a;
parameters ALPHA BETA DELTA GAMMA PSI RHOA ETAC ETAL;

%% Calibration of parameters (simple)
ALPHA = 0.35;
BETA  = 0.9901;
DELTA = 0.025;
GAMMA = 1;
PSI   = 1.7333;
RHOA  = 0.9;
ETAC  = 2;
ETAL  = 1.5;

```

```
%% Model Equations
model;
uc = GAMMA*c^(-ETAC);
ul = -PSI*(1-l)^(-ETAL);
fl = (1-ALPHA)*a*(k(-1)/l)^ALPHA;
fk = ALPHA*a*(k(-1)/l)^(ALPHA-1);

uc = BETA*uc(+1)*(1-DELTA+rk(+1));
w = - ul/uc;
w = fl;
rk = fk;
y = a*k(-1)^ALPHA*l^(1-ALPHA);
k = (1-DELTA)*k(-1) + iv;
y = c + iv;
log(a) = RHOA*log(a(-1)) + eps_a;
end;

%% Steady State
steady_state_model;
a = 1;
rk = 1/BETA+DELTA-1;
k_l = ((ALPHA*a)/rk)^(1/(1-ALPHA));
w = (1-ALPHA)*a*k_l^ALPHA;
iv_l = DELTA*k_l;
y_l = a*k_l^ALPHA;
c_l = y_l - iv_l;
l0 = 1/3; % initial guess
l = rbcCEShelper(l0,PSI,ETAL,ETAC,GAMMA,c_l,w);
c = c_l*l;
iv = iv_l*l;
k = k_l*l;
y = y_l*l;
uc = GAMMA*c^(-ETAC);
ul = -PSI*(1-l)^(-ETAL);
fl = (1-ALPHA)*a*(k_l)^ALPHA;
fk = ALPHA*a*(k_l)^(ALPHA-1);
end;

steady;
```

and the following helper function:

progs/dynare/rbcCEShelper.m

```
function l = rbcCEShelper(l0,PSI,ETAL,ETAC,GAMMA,c_l,w)
    if ETAC == 1 && ETAL == 1
        % close-form expression
        l = GAMMA/PSI*c_l^(-1)*w/(1+GAMMA/PSI*c_l^(-1)*w);
    else
        % use numerical optimizer
        l = fsolve(@(L) w*GAMMA*c_l^(-ETAC) - PSI*(1-L)^(-ETAL)*L^ETAC ,...
                      l0, optimset('Display','Final','TolX',1e-12,'TolFun',1e
                        -12));
```

```
    end
end
```

Obviously, the results are the same.

3 Solution to The Algebra of New Keynesian Models

1. Equation (2) captures that bond prices are inversely related to interest rates. When the interest rate goes up, the price of bonds falls. Intuitively, this makes sense because if you are paying less for a fixed nominal return (at par), your expected return should be higher. More specifically to our model, we consider so-called zero-coupon bonds or discount bonds. These bonds don't pay any interest but derive their value from the difference between the purchase price and the par value (or the face value) paid at maturity. On maturity the bondholder receives the face value of his investment. So instead of interest payments, you get a large discount on the face value of the bond; that is the price is lower than the face value. In other words, investors profit from the difference between the buying price and the face value, contrary to the usual interest income. In our model, we consider zero-coupon bonds with a face value of 1. So suppose that you buy such a bond at a price of 0.8, then although the bond pays no interest, your compensation is the difference between the initial price and the face value. Let R_t denote the *gross yield to maturity* of a zero-coupon bond, that is the discount rate that sets the present value of the promised bond payments equal to the current market price of the bond. So the price of a Zero-Coupon bond is equal to

$$Q_t = \frac{1}{R_t}$$

   In our example this would imply $R_t = 1.25$. As there are no other investment opportunities in this model R_t is also equal to the nominal interest rate in the economy.

   Equation (3) is the so-called Fisherian equation which states that the gross real return on a bond r_t is equivalent to the gross nominal interest rate divided by the expected gross inflation rate. Inflation expectations are responsible for the difference between nominal and real interest rates, showing that future expectations matter for the economy.

2. In equilibrium, bond-holding is always zero in all periods: $B_t = 0$. This is due to the fact that in this model we have a representative agent and only private bonds. If all agents were borrowing, there would be nobody they could be borrowing from. If all were lenders, nobody would like to borrow from them. In sum the price of bonds (or more specifically the nominal interest rate) adjusts such that bonds across all agents are in zero net supply as markets need to clear in equilibrium. Note, though, that this bond market clearing condition is imposed *after* you derive the households optimality conditions as household savings behavior in equilibrium still needs to be consistent with the bond market clearing.

3. The *No-Ponzi-Game* or *solvency* condition is an external constraint imposed on the individual by the market or other participants. You forbid your agent from acquiring infinite debt that is never repaid, a so-called Ponzi-scheme. That is, the individual is restricted from financing consumption by raising debt and then raising debt again to repay the previous debt and finance again consumption and so on. The individual would very much like to violate it, though, so we need to impose this constraint. In short: the *solvency* condition prevents that households consume more than they earn and refinance their additional consumption with excessive borrowing.

   The *transversality condition* is an optimality condition that states that it is not optimal to start accumulating assets and never consume them, i.e. $\lim_{T\to\infty} E_t \left\{\Lambda_{t,T}\frac{B_T}{P_T}\right\} \leq 0$. But with respect to optimality you would still want to run a Ponzi-scheme if allowed one. $\lim_{T\to\infty} E_t \left\{\Lambda_{t,T}\frac{B_T}{P_T}\right\} \leq 0$ combined with $\lim_{T\to\infty} E_t \left\{\Lambda_{t,T}\frac{B_T}{P_T}\right\} \geq 0$ yields $\lim_{T\to\infty} E_t \left\{\Lambda_{t,T}\frac{B_T}{P_T}\right\} = 0$. This condition must be satisfied in order for the individual to maximize intertemporal utility implying that at the limit wealth should be zero. In other words, if at the limit wealth is positive it means that the household could have increased its consumption without necessarily needing to work more hours; thus implying that consumption was not maximized and therefore contradicting the fact that the household behaves optimally. In short: transversality conditions make sure that households do no have any leftover savings (in terms of bonds or capital) as this does not correspond to an optimal path of utility-enhancing consumption.

We never need to actually include this condition into our codes, but implicitly we use it to pick a certain steady state or trajectory. For instance in the RBC model we have three possible steady states $(k_t = 0, c_t = 0)$, $k_t > 0, c_t = 0$ or $k_t > 0, c_t > 0$. We do not consider the first one because in this case the economy does not exist. All the trajectories leading to the second one violate the transversality condition, so finally we select the third steady state as the *good one* and this is exactly the one that is most interesting from an economic point of view.

Coming back to our model, both the *solvency* and *transversality condition* are actually full-filled already as bond-holding is always zero in all periods including the hypothetical asymptotic end of life: $B_t = 0$ for all t. So these conditions are rather trivial in this model setting, but are important in more sophisticated models.

4. The household minimizes consumption expenditures $\int_0^1 P_t(h)c_t(h)dh$ by choosing $c_t(h)$ and taking the aggregation technology into account. That is, the Lagrangian is given by:

$$£^c = \int_0^1 P_t(h)c_t(h)dh + P_t\left(c_t - \left[\int_0^1 (c_t(h))^{\frac{\epsilon-1}{\epsilon}}dh\right]^{\frac{\epsilon}{\epsilon-1}}\right)$$

where P_t denotes the Lagrange multiplier, i.e. the cost of an additional unit in the index c_t. Setting the derivative with respect to $c_t(h)$ equal to zero yields:

$$\frac{\partial £^c}{\partial c_t(h)} = P_t(h) - P_t\left(\frac{\epsilon}{\epsilon-1}\right)\underbrace{\left[\int_0^1 (c_t(h))^{\frac{\epsilon-1}{\epsilon}}dh\right]^{\frac{\epsilon}{\epsilon-1}-1}}_{c_t^{1/\epsilon}}\left(\frac{\epsilon-1}{\epsilon}\right)(c_t(h))^{\frac{\epsilon-1}{\epsilon}-1} = 0$$

which can be simplified to:

$$c_t(h) = \left(\frac{P_t(h)}{P_t}\right)^{-\epsilon}c_t$$

Note that this is the demand function for each consumption good $c_t(h)$. Accordingly, ϵ is the (constant) demand elasticity.

Plugging this expression into the aggregation technology yields:

$$c_t^{\frac{\epsilon-1}{\epsilon}} = \int_0^1 (c_t(h))^{\frac{\epsilon-1}{\epsilon}}dh = \int_0^1\left(\left(\frac{P_t(h)}{P_t}\right)^{-\epsilon}c_t\right)^{\frac{\epsilon-1}{\epsilon}}dh = c_t^{\frac{\epsilon-1}{\epsilon}}P_t^{\epsilon-1}\int_0^1 (P_t(h))^{1-\epsilon}dh$$

$$\Leftrightarrow P_t = \left[\int_0^1 (P_t(h))^{1-\epsilon}dh\right]^{\frac{1}{1-\epsilon}}$$

$$\Leftrightarrow 1 = \int_0^1\left(\frac{P_t(h)}{P_t}\right)^{1-\epsilon}dh \tag{14}$$

Similar to the aggregation technology for the consumption index c_t, P_t can be interpreted as the aggregation technology for the different prices $P_t(h)$.

In the budget constraint, we can now get rid of one integral $\int_0^1 c_t(h)P_t(h)dh = P_tc_t$, because:

$$\int_0^1 c_t(h)P_t(h)dh = \int_0^1\left(\frac{P_t(h)}{P_t}\right)^{-\epsilon}c_tP_t(h)dh = P_tc_t\underbrace{\int_0^1\left(\frac{P_t(h)}{P_t}\right)^{1-\epsilon}dh}_{\overset{(14)}{=}1} = P_tc_t$$

That is, conditional on optimal behavior of households, total consumption expenditures can be rewritten as the product of the aggregate price index times the aggregate consumption quantity index.

5. Due to our assumptions, the solvency and transversality conditions as well as the concave optimization problem, we can rule out corner solutions and neglect the non-negativity constraints in the variables and the budget constraint; hence, we only need to focus on the first-order conditions. The Lagrangian for the household's problem is:

$$
\begin{aligned}
\pounds^{HH} = & E_t \sum_{j=0}^{\infty} \beta^j \left\{ U\left(c_{t+j}, l_{t+j}, z_{t+j}\right) \right\} \\
& + \beta^j \lambda_{t+j} \left\{ \int_0^1 div_{t+j}(f)df + w_{t+j} l_{t+j} + \underbrace{\frac{B_{t-1+j}}{P_{t-1+j}}}_{b_{t-1+j}} \underbrace{\frac{P_{t-1+j}}{P_{t+j}}}_{\Pi_{t+j}^{-1}} - Q_{t+j} \underbrace{\frac{B_{t+j}}{P_{t+j}}}_{b_{t+j}} - c_{t+j} \right\}
\end{aligned}
$$

where $\beta^j \lambda_{t+j}$ are the Lagrange multipliers corresponding to period $t+j$'s **real** budget constraint (be aware of the difference between nominal and real variables and constraints; for instance, $b_t = B_t/P_t$ is real debt). The problem is not to choose $\{c_t, l_t, b_t\}_{t=0}^{\infty}$ all at once in an open-loop policy, but to choose these variables sequentially given the information at time t in a closed-loop policy, i.e. at period t decision rules for $\{c_t, l_t, b_t\}$ given the information set at period t; at period $t+1$ decision rules for $\{c_{t+1}, l_{t+1}, b_{t+1}\}$ given the information set at period $t+1$, etc.

First-order condition with respect to c_t

$$\lambda_t = \frac{\partial U(c_t, l_t, z_t)}{\partial c_t} = z_t c_t^{-\sigma} \tag{15}$$

This is the marginal consumption utility function, i.e. the benefit (shadow price) of an additional unit of revenue (e.g. dividends or labor income) in the budget constraint.

First-order condition with respect to l_t

$$w_t = -\frac{\partial U(c_t, l_t, z_t)/\partial l_t}{\lambda_t} = -\frac{\partial U(c_t, l_t, z_t)/\partial l_t}{\partial U(c_t, l_t, z_t)/\partial c_t} = l_t^{\varphi} c_t^{\sigma} \tag{16}$$

This is the **intratemporal** optimality condition or, in other words, the labor supply curve of the household. Note that the preference shifter z_t has no effect on this intratemporal decision.

First-order condition with respect to b_t

$$\lambda_t Q_t = \beta E_t \left[\lambda_{t+1} \Pi_{t+1}^{-1} \right] \tag{17}$$

Combined with (2) and (3) this yields the so-called Euler equation, i.e. the **intratemporal** choice between consumption and saving:

$$\frac{\partial U(c_t, l_t, z_t)}{\partial c_t} = \beta E_t \left[\frac{\partial U(c_{t+1}, l_{t+1}, z_{t+1})}{\partial c_{t+1}} R_t \Pi_{t+1}^{-1} \right] \tag{18}$$

$$z_t c_t^{-\sigma} = \beta E_t \left[z_{t+1} c_{t+1}^{-\sigma} \right] r_t \tag{19}$$

In words, intertemporal optimality is characterized by an indifference condition: An additional unit of consumption yields either marginal utility today in the amount of $\frac{\partial U(c_t, l_t, z_t)}{\partial c_t}$ (left-hand side). Or, alternatively, this unit of consumption can be saved given the real interest rate r_t. This saved consumption unit has a present marginal utility value of $\beta E_t \left[z_{t+1} c_{t+1}^{-\sigma} \right] r_t$ (right-hand side). An optimal allocation equates these two choices.

6. The output packers maximize profits $P_t y_t - \int_0^1 P_t(f) y_t(f) df$ subject to (4). The Lagrangian is

$$\pounds^p = P_t y_t - \int_0^1 P_t(f) y_t(f) df + \Lambda_t^p \left\{ \left[\int\limits_0^1 (y_t(f))^{\frac{\epsilon-1}{\epsilon}} df \right]^{\frac{\epsilon}{\epsilon-1}} - y_t \right\}$$

where Λ_t^p is the Lagrange multiplier corresponding to the aggregation technology. The first-order condition w.r.t y_t is

$$P_t = \Lambda_t^p \tag{20}$$

Λ_t^p is the gain of an additional output unit; hence, equal to the aggregate price index P_t.

The first-order condition w.r.t $y_t(f)$ yields:

$$\frac{\partial \pounds^p}{\partial y_t(f)} = -P_t(f) + \Lambda_t^p \frac{\epsilon}{\epsilon - 1} \left[\int_0^1 y_t(f)^{\frac{\epsilon-1}{\epsilon}} df \right]^{\frac{\epsilon}{\epsilon-1}-1} \frac{\epsilon - 1}{\epsilon} (y_t(f))^{\frac{\epsilon-1}{\epsilon}-1} = 0 \tag{21}$$

Note that $\left[\int_0^1 (y_t(f))^{\frac{\epsilon-1}{\epsilon}} df \right] = y_t^{\frac{\epsilon-1}{\epsilon}}$ and $\Lambda_t^p = P_t$. Therefore:

$$P_t(f) = P_t \left[y_t^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon-\epsilon+1}{\epsilon-1}} (y_t(f))^{\frac{\epsilon-1-\epsilon}{\epsilon}} = P_t \left(\frac{y_t(f)}{y_t} \right)^{\frac{-1}{\epsilon}} \tag{22}$$

Reordering yields

$$y_t(f) = \left(\frac{P_t(f)}{P_t} \right)^{-\epsilon} y_t \tag{23}$$

This is the demand curve for intermediate good $y_t(f)$. Again we see that ϵ is the constant demand elasticity.

The aggregate price index is implicitly determined by inserting the demand curve (23) into the aggregator (4)

$$y_t = \left[\int\limits_0^1 \left(\left(\frac{P_t(f)}{P_t} \right)^{-\epsilon} y_t \right)^{\frac{\epsilon-1}{\epsilon}} df \right]^{\frac{\epsilon}{\epsilon-1}} \tag{24}$$

$$\Leftrightarrow P_t = \left[\int_0^1 (P_t(f))^{1-\epsilon} df \right]^{\frac{1}{1-\epsilon}} \tag{25}$$

7. As the firms are owned by the households, the nominal stochastic discount factor, $\Lambda_{t,t+j}$, between t and $t + j$ is derived from the Euler equation (19) of the households $\lambda_t = \beta E_t \left[\lambda_{t+1} R_t \Pi_{t+1}^{-1} \right]$ which implies for the stochastic discount factor:

$$E_t \Lambda_{t,t+j} = E_t 1/R_{t+j} = E_t \beta^j \frac{\lambda_{t+j}}{\lambda_t} \frac{P_t}{P_{t+j}}$$

From here, we can establish the following relationships:

$$\Lambda_{t,t} = 1 \tag{26}$$

$$\Lambda_{t+1,t+1+j} = \beta^j \frac{\lambda_{t+1+j}}{\lambda_{t+1}} \frac{P_{t+1}}{P_{t+1+j}} \tag{27}$$

$$\Lambda_{t,t+1+j} = \beta^{j+1} \frac{\lambda_{t+1+j}}{\lambda_t} \frac{P_t}{P_{t+1+j}} = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{P_t}{P_{t+1}} \beta^j \frac{\lambda_{t+1+j}}{\lambda_{t+1}} \frac{P_{t+1}}{P_{t+1+j}} = \beta \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{-1} \Lambda_{t+1,t+1+j} \tag{28}$$

We will need this later to derive the recursive nonlinear price setting equations.

8. The Lagrangian of the intermediate firm is

$$\pounds^f = E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} P_{t+j} \left[\frac{P_{t+j}(f)}{P_{t+j}} y_{t+j}(f) - w_{t+j} l_{d,t+j}(f) + mc_{t+j}(f) \left(a_{t+j} l_{d,t+j}(f) - y_{t+j}(f) \right) \right]$$

$mc_t(f)$ denotes the Lagrange multiplier which is the shadow price of producing an additional output unit in the optimum; obviously, this is our understanding of real marginal costs. Taking the derivative wrt $l_{d,t}(f)$ actually boils down to a static problem (as we only need to evaluate for $j = 0$) and yields:

$$w_t = mc_t(f) a_t = mc_t(f) \frac{y_t(f)}{l_{d,t}(f)} \tag{29}$$

where we substituted the production function (5) for a_t. This is the labor demand function, which implies that the labor-to-output ratio is the same across firms and equal to a_t. Note that all firms face the same factor prices and all have access to the same production technology a_t; hence, from the above equation it is evident that marginal costs are identical across firms

$$mc_t(f) = \frac{w_t}{a_t} \tag{30}$$

This means that aggregate marginal costs are also equal to the ratio between the real wage and technology:

$$mc_t = \int_0^1 mc_t(f) df = \frac{w_t}{a_t}$$

9. The Lagrangian of the intermediate firm is

$$\pounds^f = E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} P_{t+j} \left[\left(\frac{P_{t+j}(f)}{P_{t+j}} \right)^{1-\epsilon} y_{t+j} - w_{t+j} l_{d,t+j}(f) + mc_{t+j}(f) \left(a_{t+j} l_{d,t+j}(f) - \left(\frac{P_{t+j}(f)}{P_{t+j}} \right)^{-\epsilon} y_{t+j} \right) \right] \tag{31}$$

where (compared to above) we used the demand curve (23) to substitute for $y_t(f)$. When firms decide how to set their price they need to take into account that due to the Calvo mechanism they might get stuck at $\widetilde{P}_t(f)$ for a number of periods $j = 1, 2, \dots$ before they can re-optimize again. The probability of such a situation is θ^j. Therefore, when firms are able to change prices in period t, they take this into account and the above Lagrangian of the expected discounted sum of nominal profits becomes:

$$\pounds^{f^c} = E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} P_{t+j} \left[\left(\frac{\widetilde{P}_t(f)}{P_{t+j}} \right)^{1-\epsilon} y_{t+j} - w_{t+j} l_{d,t+j}(f) + mc_{t+j} \left(a_{t+j} l_{d,t+j}(f) - \left(\frac{\widetilde{P}_t(f)}{P_{t+j}} \right)^{-\epsilon} y_{t+j} \right) \right] \tag{32}$$

$$= E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} P_{t+j}^{\epsilon} y_{t+j} \left[\widetilde{P}_t(f)^{1-\epsilon} - P_{t+j} \cdot mc_{t+j} \cdot \widetilde{P}_t(f)^{-\epsilon} \right] + \dots \tag{33}$$

where in the second line we focus only on relevant parts for the optimization wrt to $\widetilde{P}_t(f)$. Moreover, we took into account that $mc_t(f) = mc_t$.

The first-order condition of maximizing $\pounds^{f^c}$ wrt to $\widetilde{P}_t(f)$ is

$$0 = E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} P_{t+j}^{\epsilon} y_{t+j} \left[(1-\epsilon) \cdot \widetilde{P}_t(f)^{-\epsilon} + \epsilon \cdot P_{t+j} \cdot mc_{t+j} \widetilde{P}_t(f)^{-\epsilon-1} \right] \tag{34}$$

As $\widetilde{P}_t(f) > 0$ does not depend on j, we multiply by $\widetilde{P}_t(f)^{\epsilon+1}$:

$$0 = E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} P_{t+j}^{\epsilon} y_{t+j} \left[(1-\epsilon) \cdot \widetilde{P}_t(f) + \epsilon \cdot P_{t+j} \cdot mc_{t+j} \right] \tag{35}$$

Rearranging

$$\widetilde{P}_t(f)\cdot E_t\sum_{j=0}^{\infty}\theta^j\Lambda_{t,t+j}P_{t+j}^{\epsilon}y_{t+j}=\frac{\epsilon}{\epsilon-1}\cdot E_t\sum_{j=0}^{\infty}\theta^j\Lambda_{t,t+j}P_{t+j}^{\epsilon+1}y_{t+j}mc_{t+j} \tag{36}$$

Dividing both sides by $P_t^{\epsilon+1}$

$$\underbrace{\frac{\widetilde{P}_t(f)}{P_t}}_{\widetilde{p}_t}\cdot\underbrace{E_t\sum_{j=0}^{\infty}\theta^j\Lambda_{t,t+j}\left(\frac{P_{t+j}}{P_t}\right)^{\epsilon}y_{t+j}}_{S_{1,t}}=\frac{\epsilon}{\epsilon-1}\cdot\underbrace{E_t\sum_{j=0}^{\infty}\theta^j\Lambda_{t,t+j}\left(\frac{P_{t+j}}{P_t}\right)^{\epsilon+1}y_{t+j}mc_{t+j}}_{S_{2,t}} \tag{37}$$

Note that all firms that reset prices face the same problem and therefore set the same price, $\widetilde{P}_t(f)=\widetilde{P}_t$. This is also evident by looking at the infinite sums, $S_{1,t}$ and $S_{2,t}$, because these do not depend on f. Therefore, we can drop the f in $\widetilde{P}_t(f)$ and define $\widetilde{p}_t:=\frac{\widetilde{P}_t}{P_t}$. The first-order condition can thus be written compactly:

$$\widetilde{p}_t\cdot S_{1,t}=\frac{\epsilon}{\epsilon-1}\cdot S_{2,t} \tag{38}$$

Moreover, the two infinite sums can be written recursively. For this we make use of the relationships for the stochastic discount factor (26) and (28). The first recursive sum can be written as:

$$\begin{aligned}
S_{1,t}&=E_t\sum_{j=0}^{\infty}\theta^j\Lambda_{t,t+j}\left(\frac{P_{t+j}}{P_t}\right)^{\epsilon}y_{t+j}\\
&=y_t+E_t\sum_{j=1}^{\infty}\theta^j\Lambda_{t,t+j}\left(\frac{P_{t+j}}{P_t}\right)^{\epsilon}y_{t+j}\\
&=y_t+E_t\sum_{j=0}^{\infty}\theta^{j+1}\Lambda_{t,t+j+1}\left(\frac{P_{t+j+1}}{P_t}\right)^{\epsilon}y_{t+j+1}\\
&=y_t+E_t\sum_{j=0}^{\infty}\theta^{j+1}\Lambda_{t,t+j+1}\left(\frac{P_{t+j+1}}{P_{t+1}}\frac{P_{t+1}}{P_t}\right)^{\epsilon}y_{t+j+1}\\
&=y_t+E_t\sum_{j=0}^{\infty}\theta^{j+1}\beta\frac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{-1}\Lambda_{t+1,t+1+j}\left(\frac{P_{t+j+1}}{P_{t+1}}\Pi_{t+1}\right)^{\epsilon}y_{t+j+1}\\
&=y_t+\theta\beta E_t\frac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{\epsilon-1}\underbrace{E_t\sum_{j=0}^{\infty}\theta^j\Lambda_{t+1,t+1+j}\left(\frac{P_{t+j+1}}{P_{t+1}}\right)^{\epsilon}y_{t+j+1}}_{=S_{1,t+1}}
\end{aligned}$$

The second recursive sum can be written as

$$\begin{aligned}
S_{2,t} &= E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t,t+j} \left(\frac{P_{t+j}}{P_t}\right)^{\epsilon+1} y_{t+j} mc_{t+j} \\
&= y_t mc_t + E_t \sum_{j=1}^{\infty} \theta^j \Lambda_{t,t+j} \left(\frac{P_{t+j}}{P_t}\right)^{\epsilon+1} y_{t+j} mc_{t+j} \\
&= y_t mc_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \Lambda_{t,t+j+1} \left(\frac{P_{t+j+1}}{P_t}\right)^{\epsilon+1} y_{t+j+1} mc_{t+j+1} \\
&= y_t mc_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \Lambda_{t,t+j+1} \left(\frac{P_{t+j+1}}{P_{t+1}} \frac{P_{t+1}}{P_t}\right)^{\epsilon+1} y_{t+j+1} mc_{t+j+1} \\
&= y_t mc_t + E_t \sum_{j=0}^{\infty} \theta^{j+1} \beta \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{-1} \Lambda_{t+1,t+1+j} \left(\frac{P_{t+j+1}}{P_{t+1}} \Pi_{t+1}\right)^{\epsilon+1} y_{t+j+1} mc_{t+j+1} \\
&= y_t mc_t + \theta \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{\epsilon} \underbrace{E_t \sum_{j=0}^{\infty} \theta^j \Lambda_{t+1,t+1+j} \left(\frac{P_{t+j+1}}{P_{t+1}}\right)^{\epsilon+1} y_{t+j+1} mc_{t+j+1}}_{=S_{2,t+1}}
\end{aligned}$$

To sum up:

$$S_{1,t} = y_t + \theta \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{\epsilon-1} S_{1,t+1} \tag{39}$$

$$S_{2,t} = y_t mc_t + \theta \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \Pi_{t+1}^{\epsilon} S_{2,t+1} \tag{40}$$

10. The law of motion for $\widetilde{p}_t = \frac{\widetilde{P}_t}{P_t}$ is given by the aggregate price index (25) which can be re-arranged to

$$1 = \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon} df \tag{41}$$

Due to the Calvo mechanism we get that $(1-\theta)$ firms can re-set their price to $\widetilde{P}_t$, whereas the remaining θ firms cannot and set their price equal to P_{t-1}. Therefore:

$$1 = \int_{optimizers} \left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon} df + \int_{non-optimizers} \left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon} df \tag{42}$$

$$1 = (1-\theta) \left(\frac{\widetilde{P}_t}{P_t}\right)^{1-\epsilon} + \theta \int_0^1 \left(\frac{P_{t-1}(f)}{P_t} \frac{P_{t-1}}{P_{t-1}}\right)^{1-\epsilon} df \tag{43}$$

$$1 = (1-\theta) \widetilde{p}_t^{1-\epsilon} + \theta \left(\frac{P_{t-1}}{P_t}\right)^{1-\epsilon} \int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{1-\epsilon} df \tag{44}$$

$$1 = (1-\theta) \widetilde{p}_t^{1-\epsilon} + \theta \Pi_t^{1-\epsilon} \underbrace{\int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{1-\epsilon} df}_{\overset{(41)}{=} 1} \tag{45}$$

$$1 = (1-\theta) \widetilde{p}_t^{1-\epsilon} + \theta \Pi_t^{\epsilon-1} \tag{46}$$

11. Private bonds B_t are in zero net supply on the budget constraint. Note that this condition can only be imposed after taking first order conditions. It would be invalid to eliminate bonds already in the budget constraint of the household. Even if bonds are in zero net supply, households savings behavior in equilibrium still needs to be consistent with the bond market clearing.

12. In an equilibrium, labor demand from the intermediate firms needs to be equal to the labor supply of the households; hence:

$$\int_0^1 l_{d,t}(f)df = l_t \tag{47}$$

so l_t denotes equilibrium hours worked (both supplied and demanded).

13. Given the demand for good $y_t(f)$ and the Dixit-Stiglitz aggregation technology, we get:

$$\int_0^1 y_t(f)P_t(f)df = \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} y_t P_t(f)df = P_t y_t \underbrace{\int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon} df}_{\overset{(14)}{=}1} = P_t y_t$$

Moreover, from the labor market we have $l_t = \int_0^1 l_{d,t}(f)df$. Plugging both expressions into aggregate real profits:

$$div_t = \int_0^1 div_t(f)df = \int_0^1 \frac{P_t(f)}{P_t} y_t(f)df - \int_0^1 w_t l_{d,t}(f)df = y_t - w_t l_t$$

14. Revisit the budget constraint in real terms:

$$\int_0^1 \frac{P_t(h)}{P_t} c_t(h)dh + Q_t b_t \leq b_{t-1}\Pi_t^{-1} + w_t l_t + \int_0^1 div_t(f)df$$

which becomes

$$c_t = w_t l_t + (y_t - w_t l_t) = y_t$$

in an optimal allocation with cleared markets. This is the aggregate demand equation.

15. Define $y_t^{sum} = \int_0^1 y_t(f)df$. Using the production function (5) and labor market clearing we get:

$$y_t^{sum} = \int_0^1 a_t l_{d,t}(f)df = a_t l_t \tag{48}$$

Furthermore, due to the demand for intermediate good $y_t(f)$ in (23) we get:

$$y_t^{sum} = y_t \underbrace{\int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df}_{=p_t^*} \tag{49}$$

Equating both yields:

$$p_t^* y_t = a_t l_t \tag{50}$$

This is the aggregate supply equation. Price frictions, however, imply that resources will not be efficiently allocated as prices are too high because not all firms can re-optimize their price in every period. This inefficiency is measured by $p_t^* < 1$.

16. The law of motion for the efficiency distortion p_t^* is given due to the Calvo price mechanism, i.e.:

$$
\begin{aligned}
p_t^* &= \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df \\
p_t^* &= \int_{optimizers} \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df + \int_{non-optimizers} \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df \\
p_t^* &= (1-\theta)\widetilde{p_t}^{-\epsilon} + \theta \int_0^1 \left(\frac{P_{t-1}(f)}{P_t}\right)^{-\epsilon} df \\
p_t^* &= (1-\theta)\widetilde{p_t}^{-\epsilon} + \theta \int_0^1 \left(\frac{P_{t-1}(f)}{P_t}\frac{P_{t-1}}{P_{t-1}}\right)^{-\epsilon} df \\
p_t^* &= (1-\theta)\widetilde{p_t}^{-\epsilon} + \theta \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} \int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{-\epsilon} df \\
p_t^* &= (1-\theta)\widetilde{p_t}^{-\epsilon} + \theta \Pi_t^{\epsilon} \underbrace{\int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{-\epsilon} df}_{=p_{t-1}^*} \\
p_t^* &= (1-\theta)\widetilde{p_t}^{-\epsilon} + \theta \Pi_t^{\epsilon} p_{t-1}^*
\end{aligned}
$$