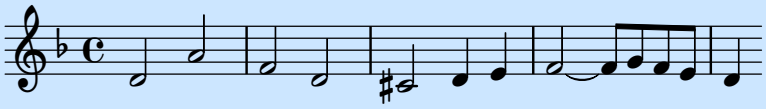
Johann Sebastian
Bach

Die Kunst der Fuga

BWV 1080

For Piano or Harpsichord



common subject in « The Art of Fugue »

LAVENDER
BLUE
Open Scores

Engraved by Davide Madrisan
https://github.com/madrisan/open-scores/

Based on a previous Mutopia work of Pierre-Luc Gauthier & Chris Sawer

Mutopia Project
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INDEX

List of Subjects 2

— **CONTRAPUNCTI**

Contrapunctus I 4
four-voice fugue on principal subject

Contrapunctus II 7
four-voice fugue on principal subject, accompanied by a "French" style dotted rhythm

Contrapunctus III 12
four-voice fugue on principal subject in inversion, employing intense chromaticism

Contrapunctus IV 15
four-voice fugue on principal subject in inversion, employing counter-subjects

Contrapunctus V 21
four-voice fugue with many stretto entries

Contrapunctus VI [per Diminutionem] in Stylo Francese 25
four-voice fugue with many stretto entries a 4 in Stylo Francese

Contrapunctus VII per per Augmentationem et Diminutionem 32
fugue that uses augmented (doubling all note lengths) and diminished versions of the main subject and its inversion

Contrapunctus VIII 38
triple fugue with three subjects, having independent expositions

Contrapunctus IX alla Duodecima 46
double fugue, with two subjects occurring dependently and in invertible counterpoint at the twelfth

Contrapunctus X alla Decima 51
double fugue, with two subjects occurring dependently and in invertible counterpoint at the tenth

Contrapunctus XI 56
triple fugue, employing the three subjects of Contrapunctus VIII in inversion

Contrapunctus XII [rectus] 64
Contrapunctus XII inversus 67
mirror fugues, in which a piece is notated once and then with voices and counterpoint completely inverted

Contrapunctus XIII [rectus] 70
Contrapunctus XIII inversus 74
mirror fugues, in which a piece is notated once and then with voices and counterpoint completely inverted

Contrapunctus XIV 78
four-voice triple fugue (not completed by Bach, but likely to have become a quadruple fugue: see below)

Brief notes on the completion of Contrapunctus XIV 87

— **CANONI**

Canon in Hypodiapason (Canon alla Ottava) 92
canon in imitation at the octave

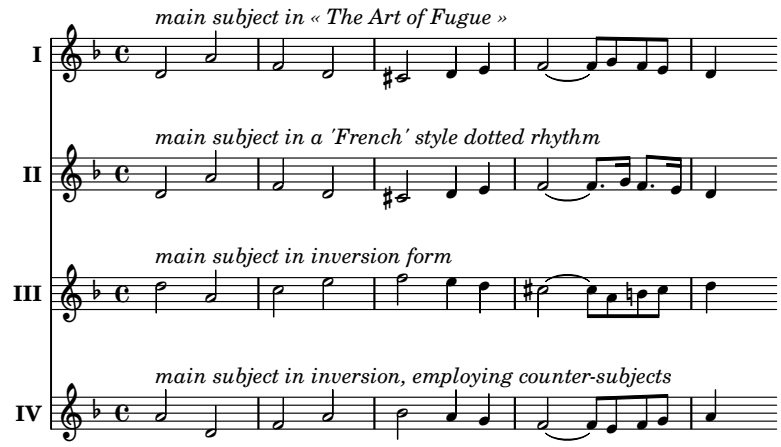
Canon alla Decima [in] Contrapunto alla Terza 96
canon in imitation at the tenth

Canon alla Duodecima in Contrapunto alla Quinta 101
canon in imitation at the twelfth

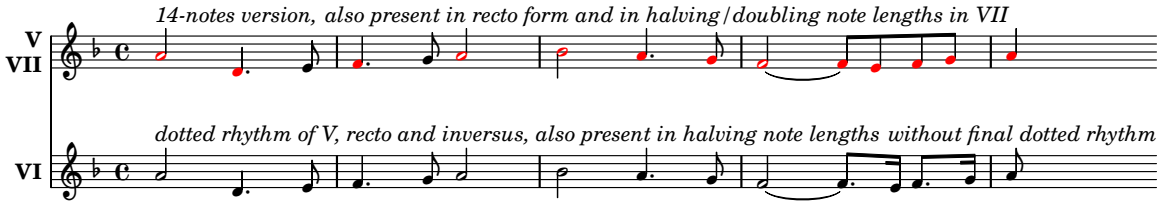
Canon [in Hypodiatesseron] per augmentationem in Contratio Motu 104
canon in which the following voice is both inverted and augmented

List of Subjects

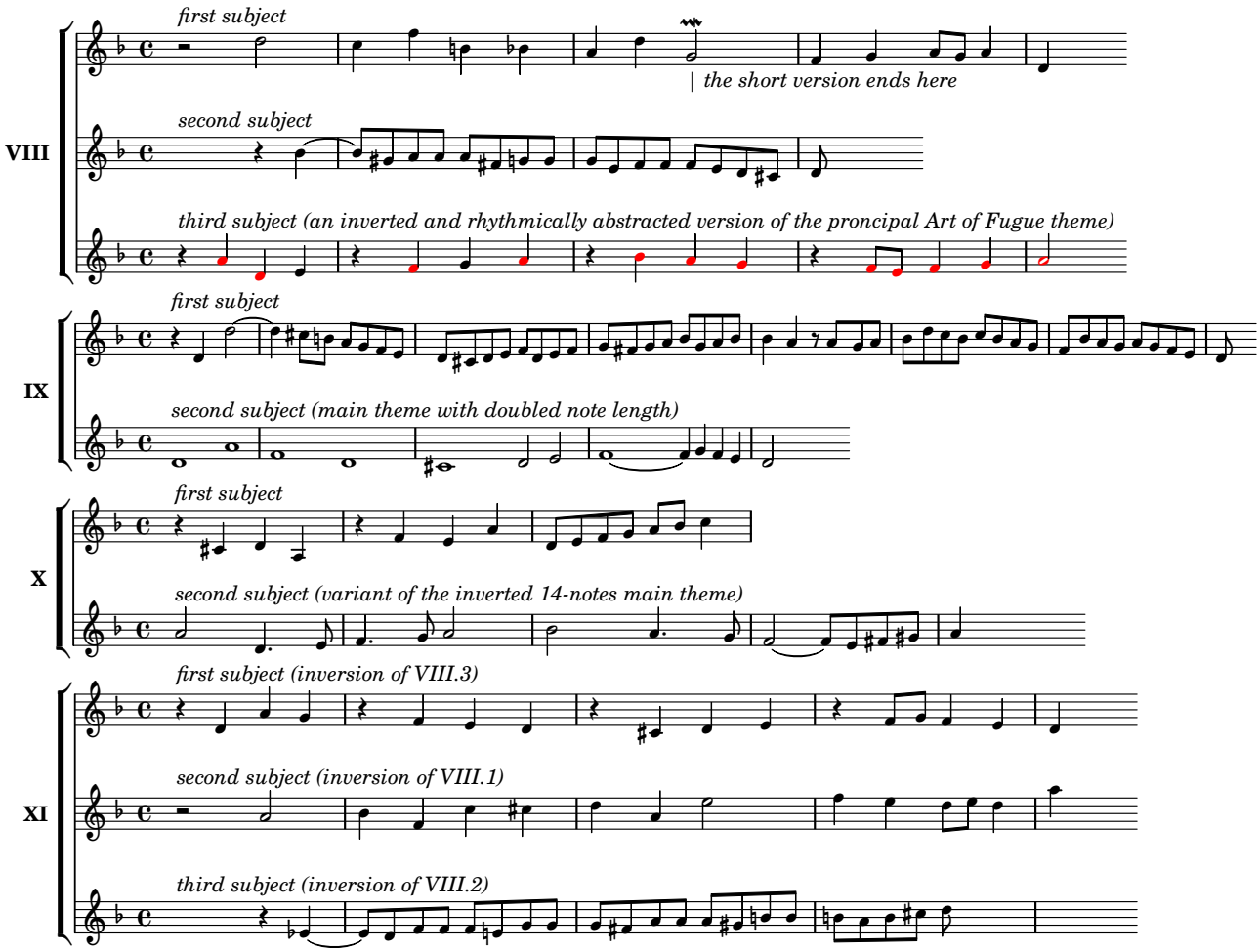
Simple fugues



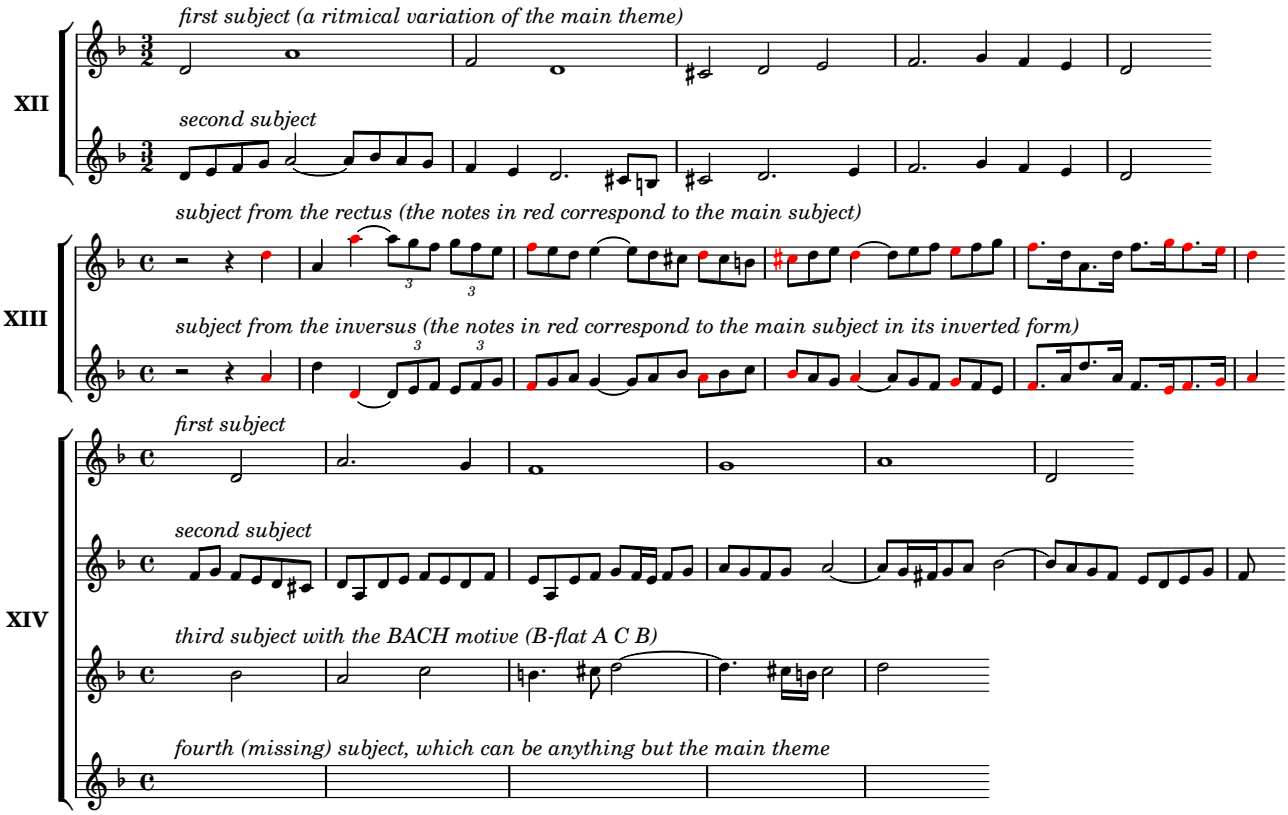
Stretto-Fugues (Counter-Fugues)



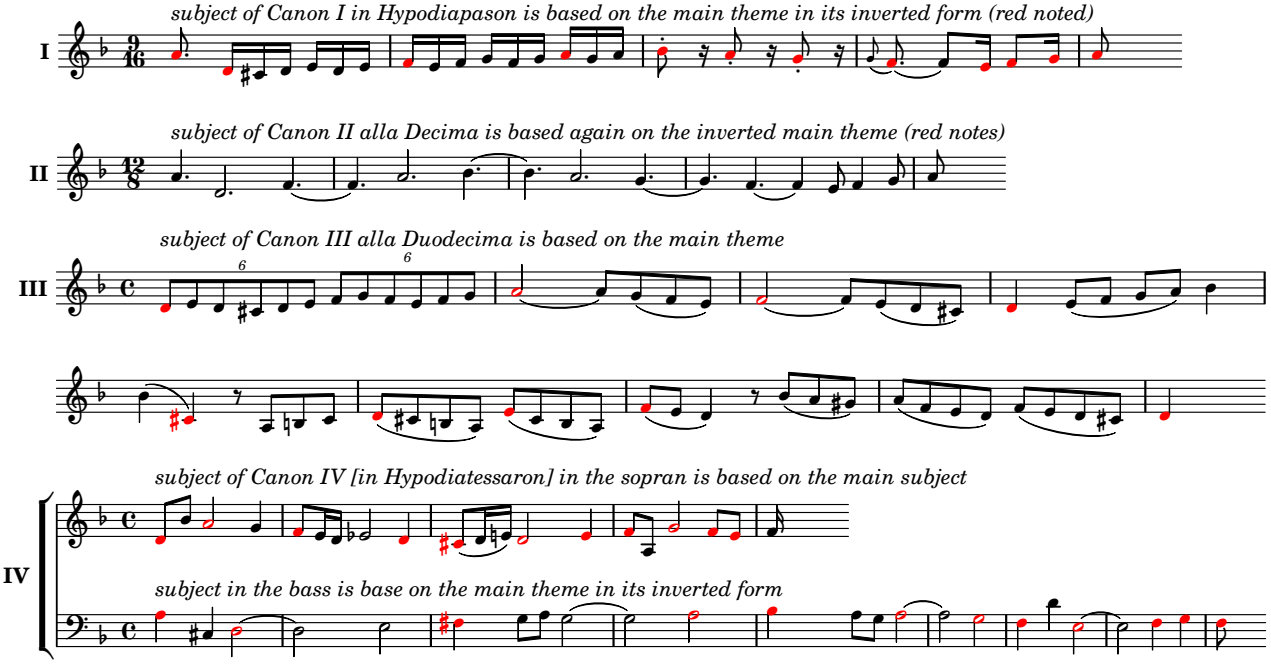
Double and Triple Fugues



Mirror-Fugues



Canons



Die Kunst der Fuga

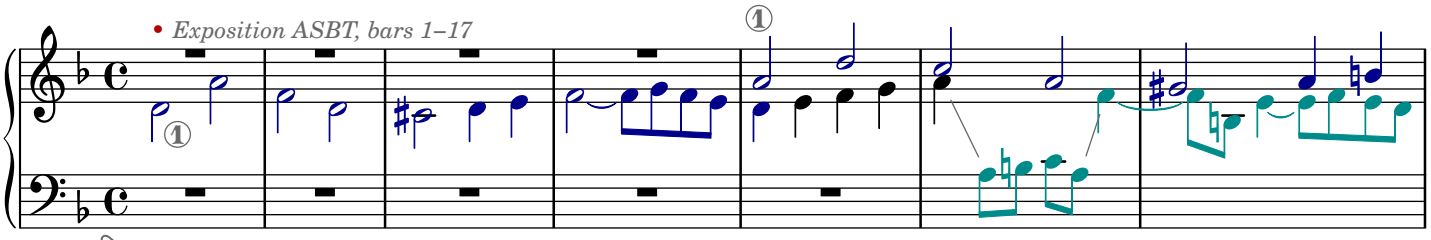
Contrapunctus I

Four-voice fugue on principal subject

Exposition ASBT | 1 | Second exposition ASBT | 2 | S | B | 3 | T

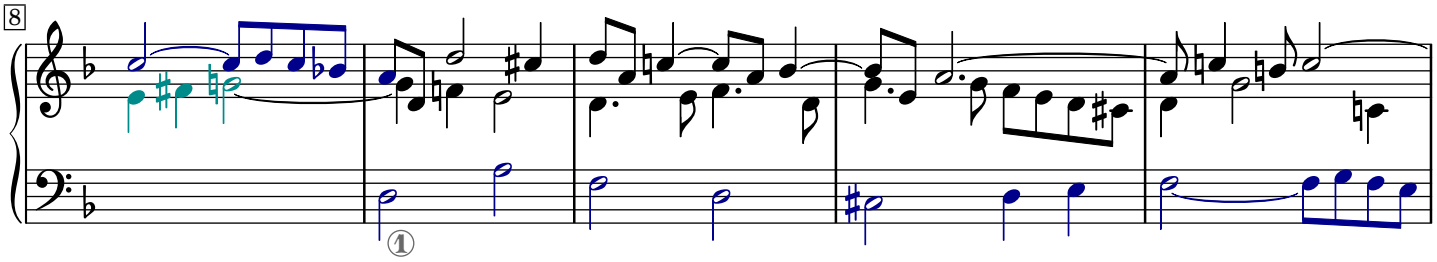
BWV 1080, 1

• *Exposition ASBT, bars 1–17*

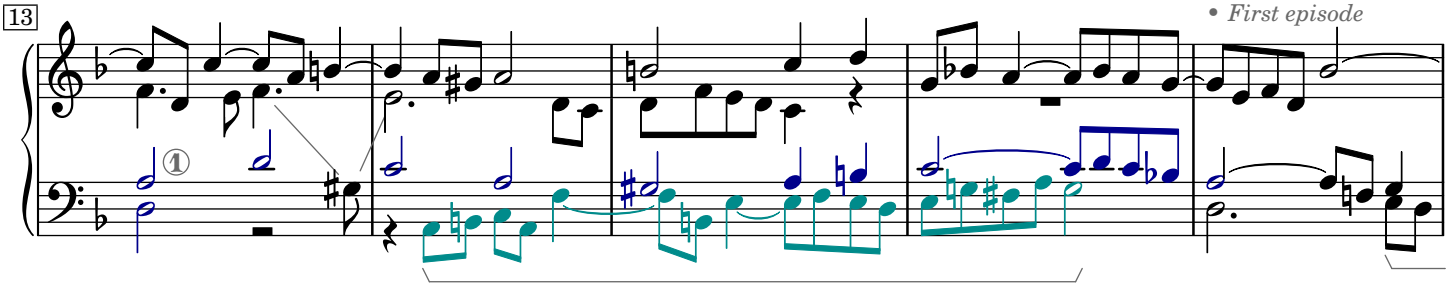


[H.A.Kellner] *158 = Die(18) + Kunst(80) + der(26) + Fuga(34) = Johann(58) + Sebastian(86) + Bach(14)*
[H.A.Kellner] *The first two notes of the entire work are D and A: 4 and 1, where 41 = J(9)+S(18)+B(2)+A(1)+C(3)+H(8)*

8



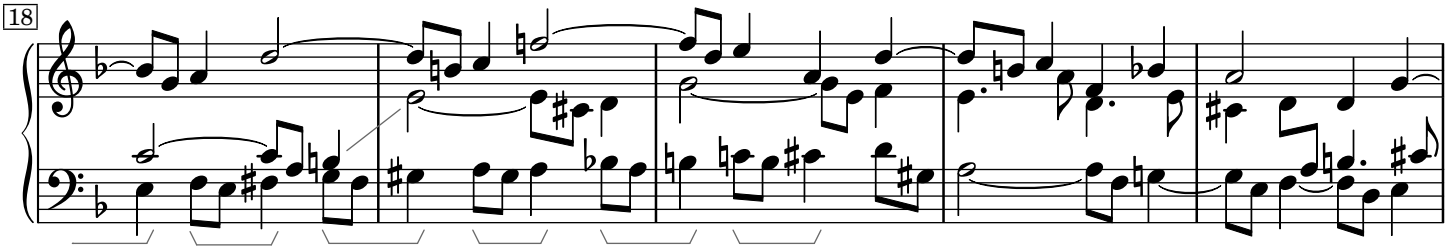
13



• *First episode*

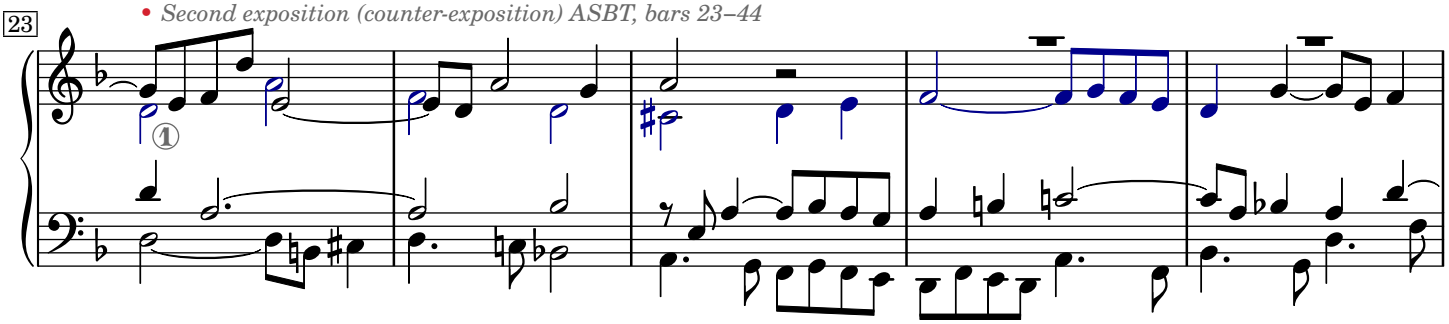
same counterpoint used with the first answer

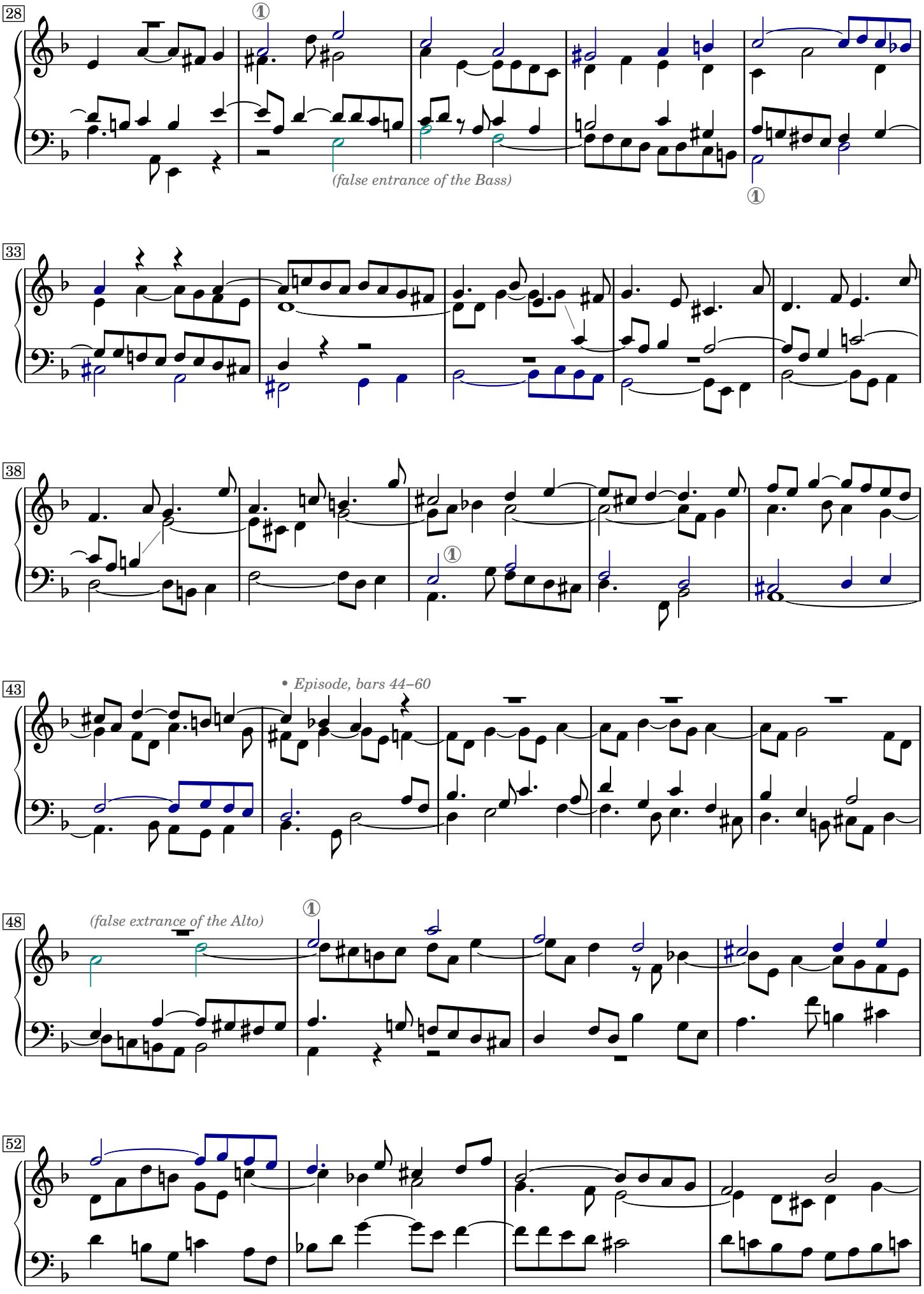
18

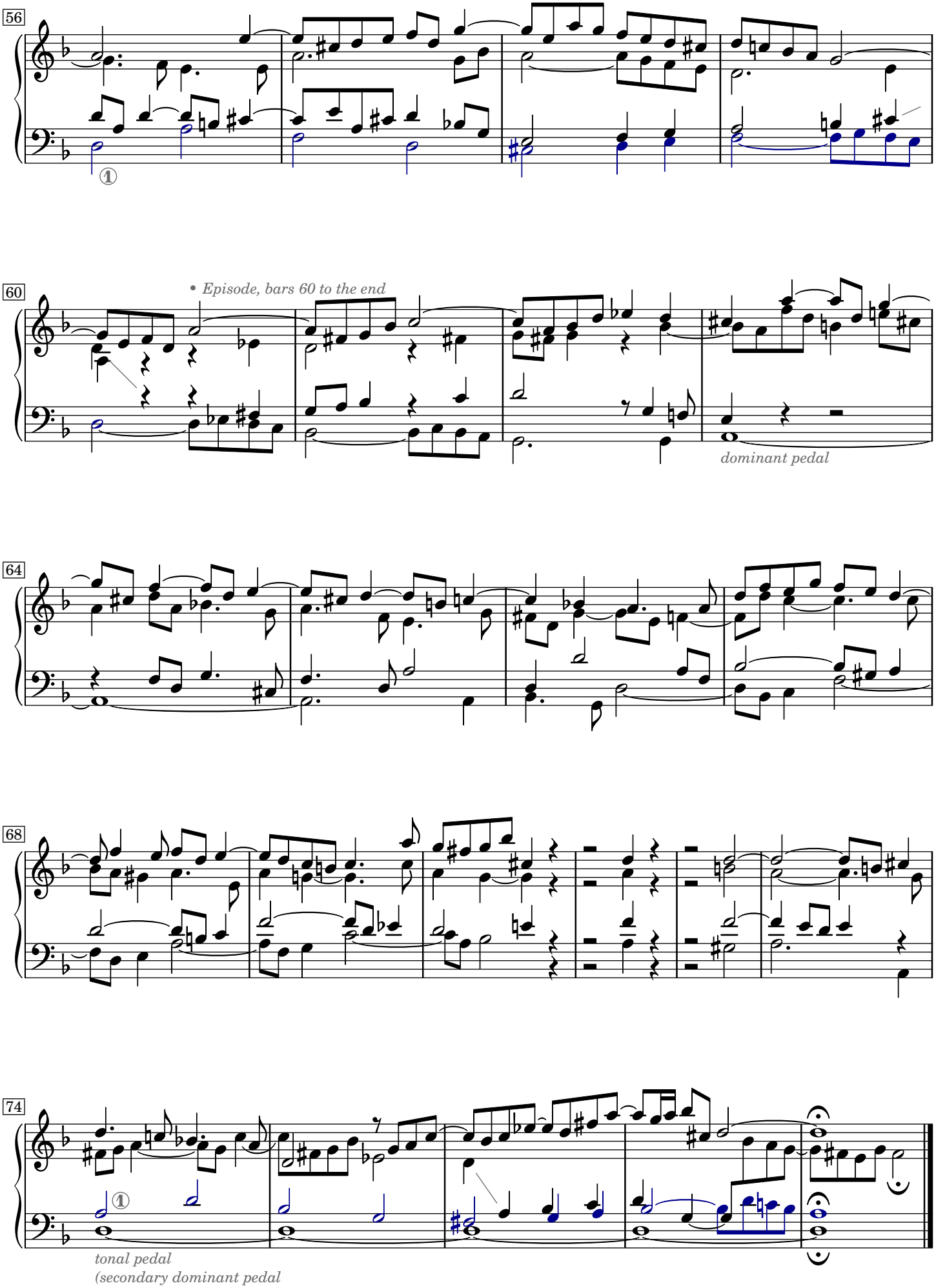


23

• *Second exposition (counter-exposition) ASBT, bars 23–44*




28
(false entrance of the Bass)
33
38
43
• Episode, bars 44–60
48
(false extrance of the Alto)
52

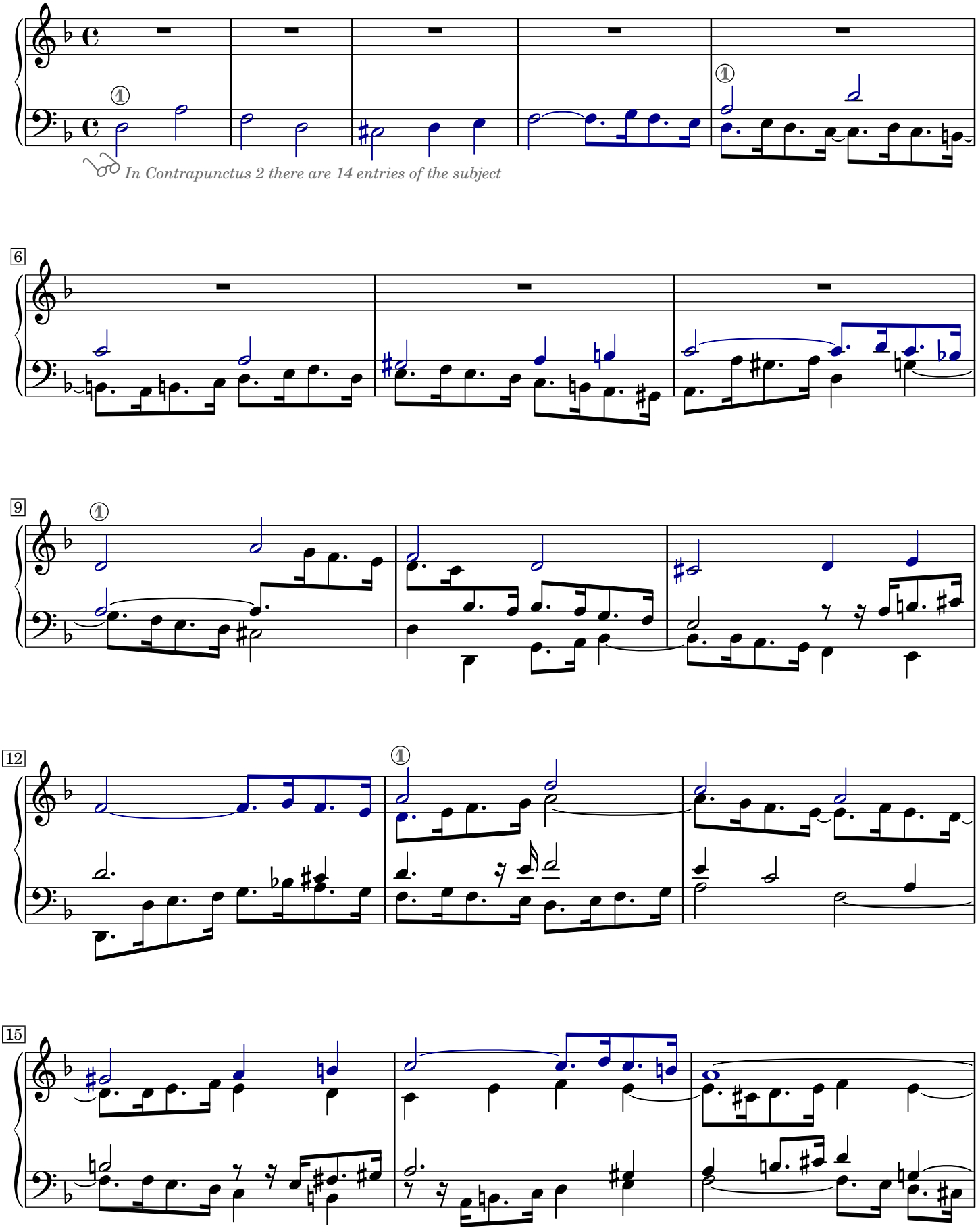

56
60
• Episode, bars 60 to the end
dominant pedal
64
68
74
tonal pedal
(secondary dominant pedal
in bars 74–76 in G minor key)

Contrapunctus II

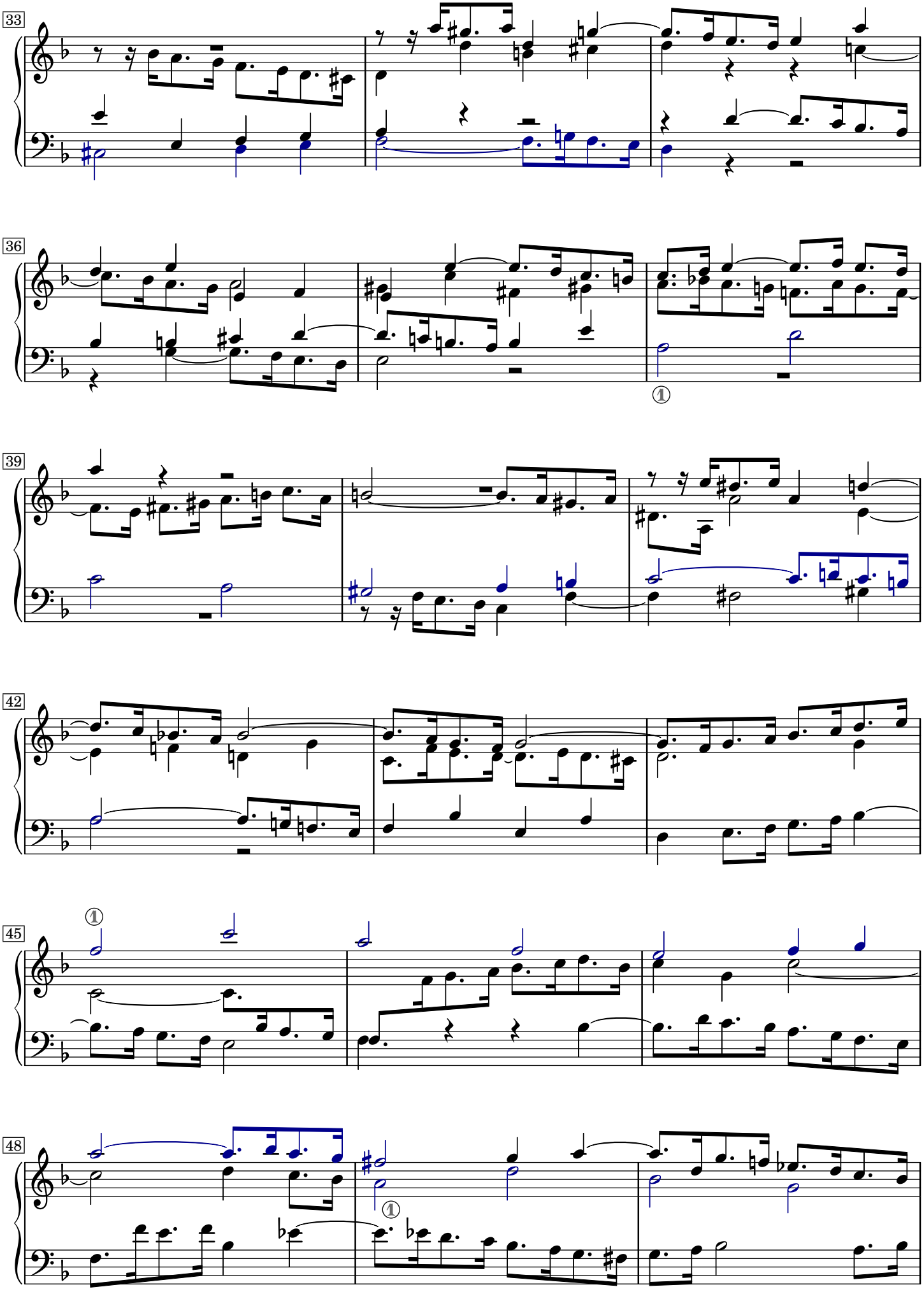
Four-voice fugue on principal subject, accompanied by a “French” style dotted rhythm

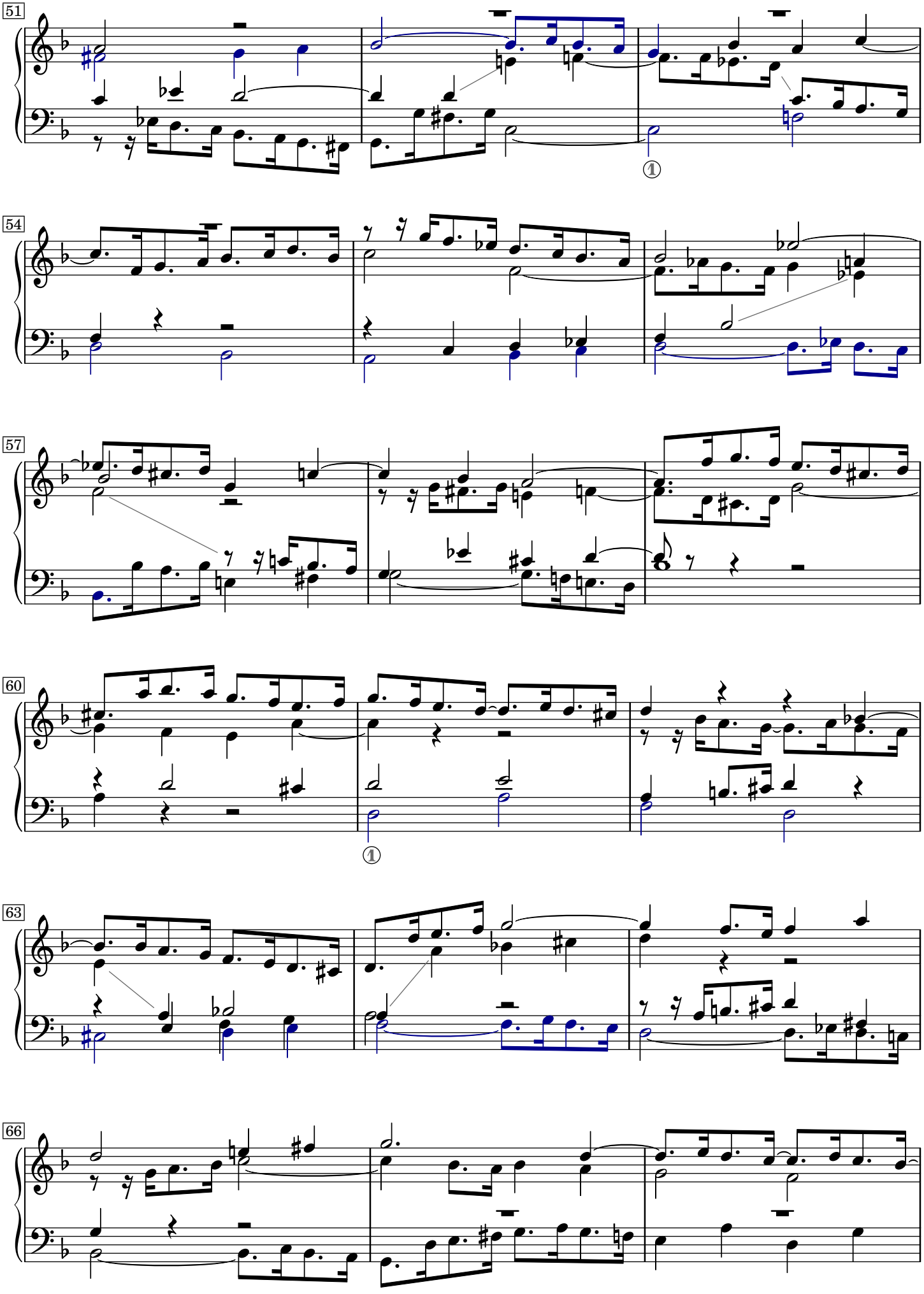
Exposition BTAS | 1 | Second exposition ASBT | 2 | SAB | 3 | B | 4 | T | 5 | S

BWV 1080, 2



18
21
24
27
30


33
36
39
42
45
48


51
54
57
60
63
66

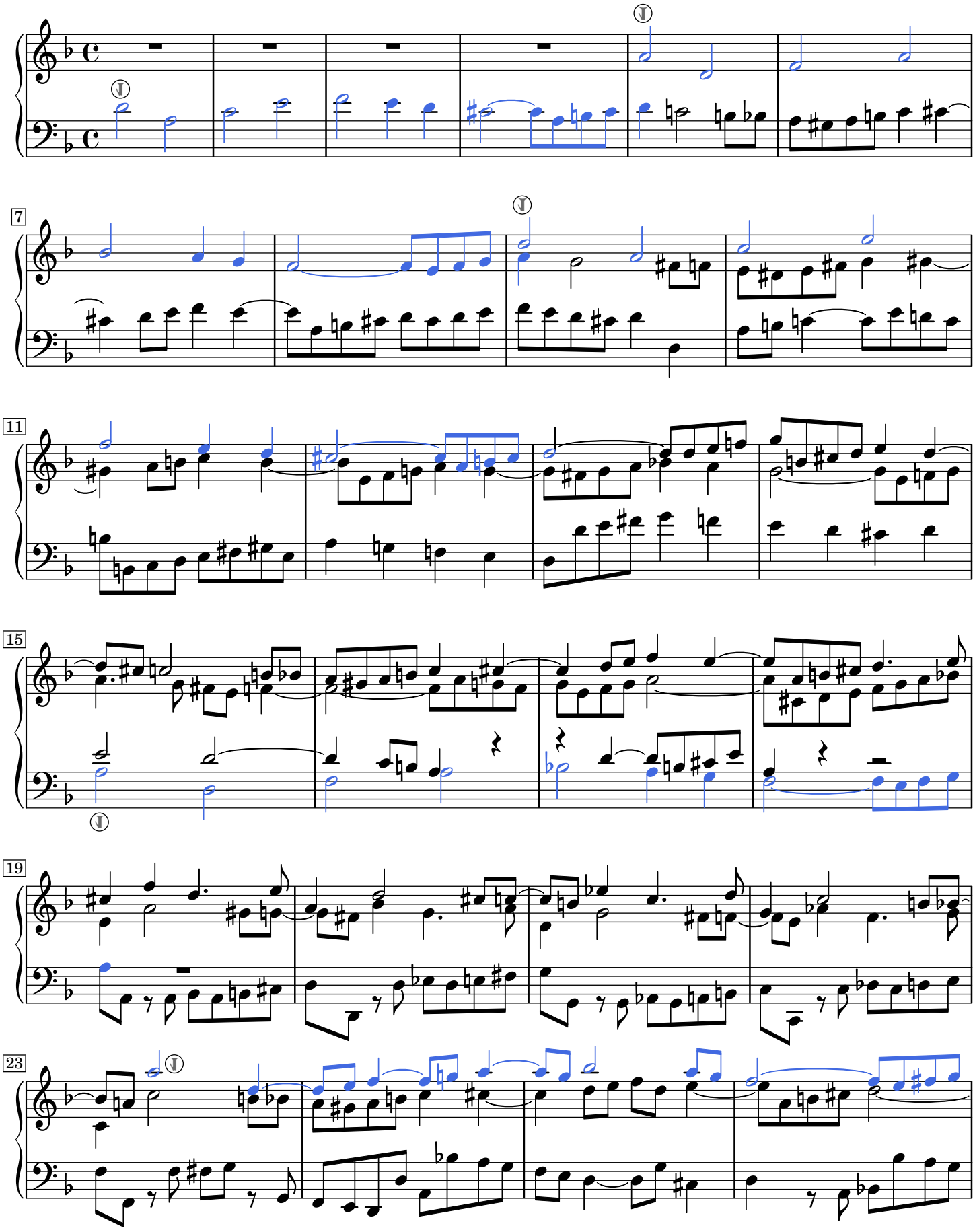
69
72
75
78
81

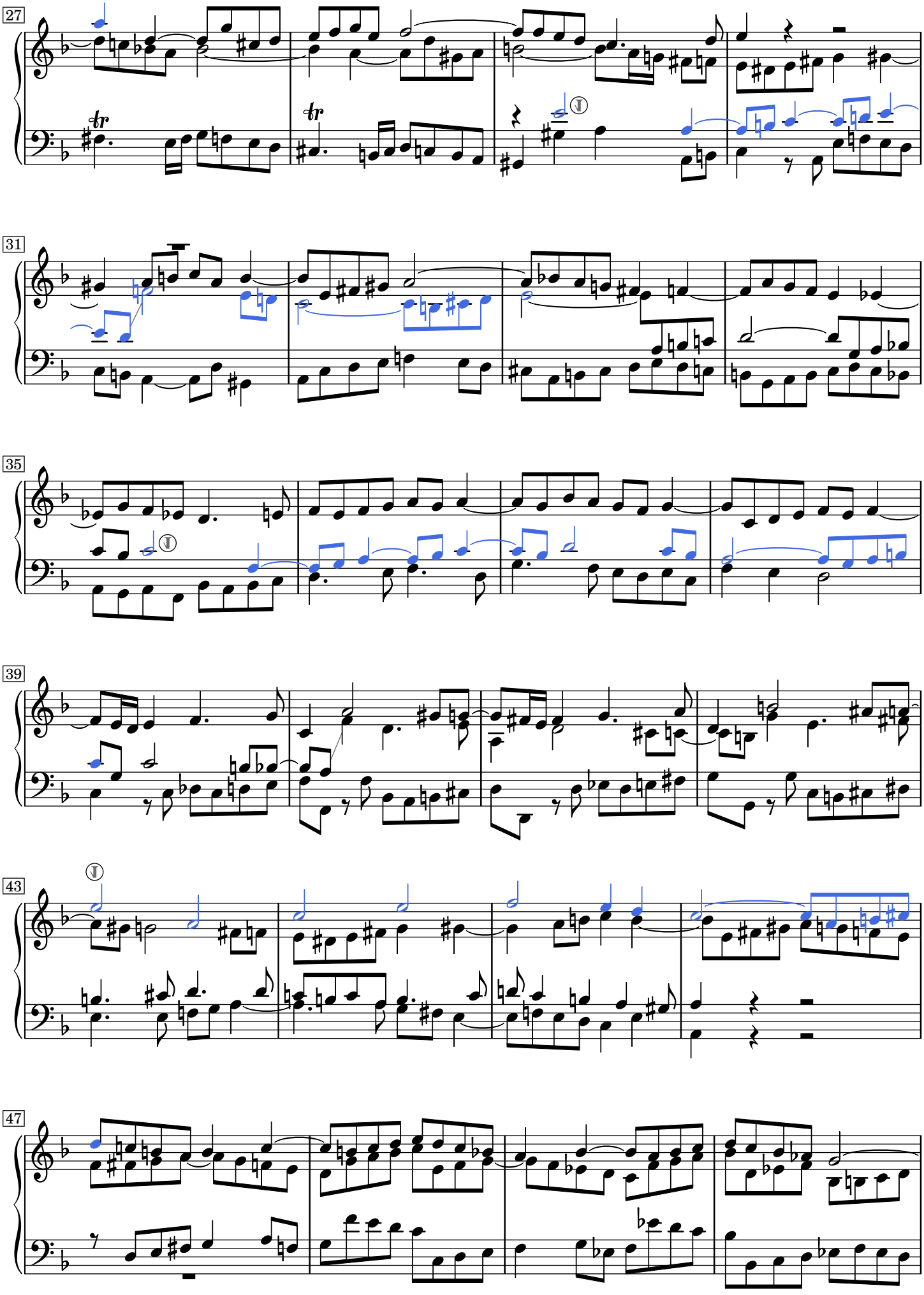
Contrapunctus III

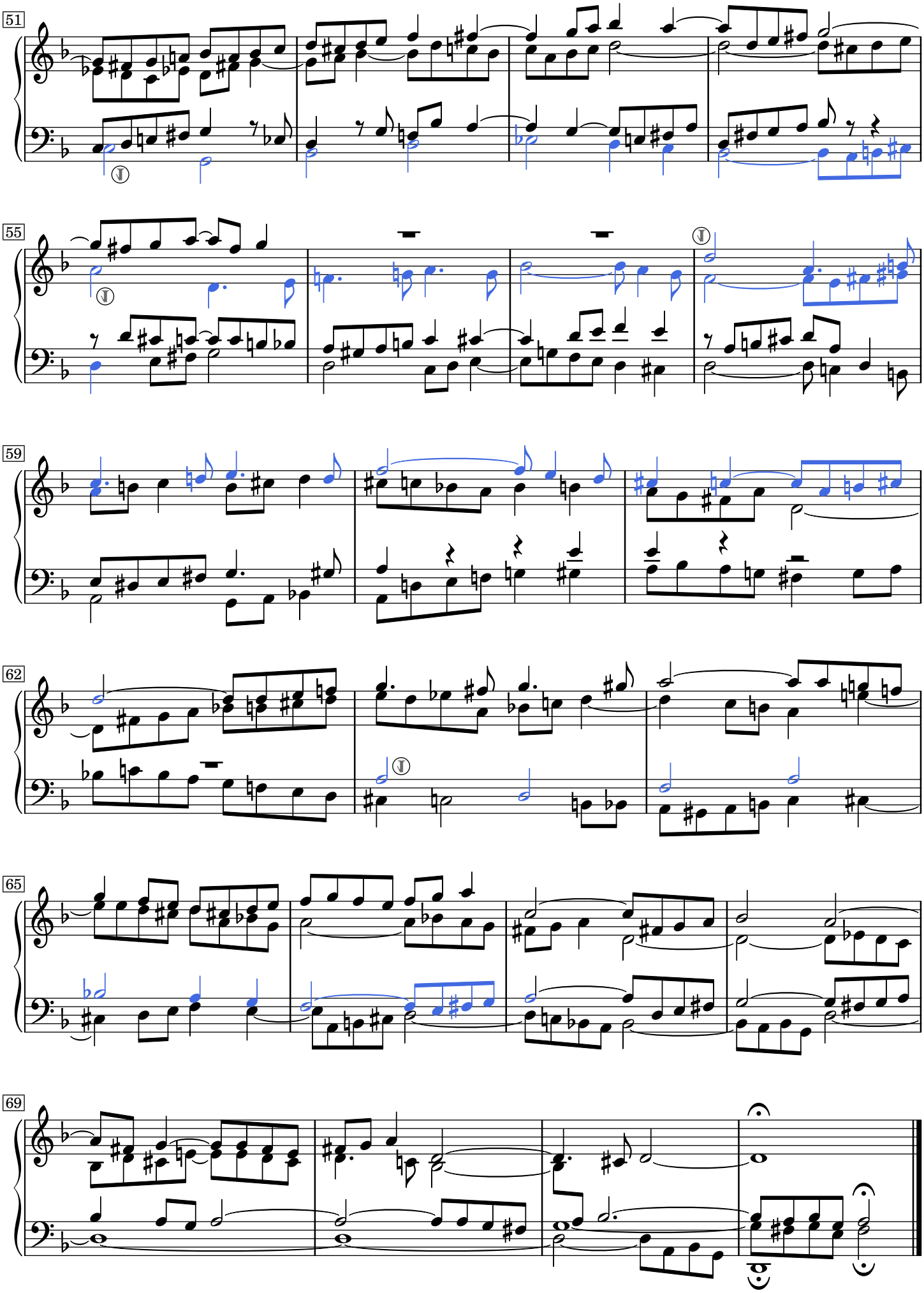
Four-voice fugue on principal subject in inversion, employing intense chromaticism

Exposition TASB | 1 | S | 2 | T | 3 | T | 4 | S | 5 | BAST | Coda

BWV 1080, 3




27
31
35
39
43
47

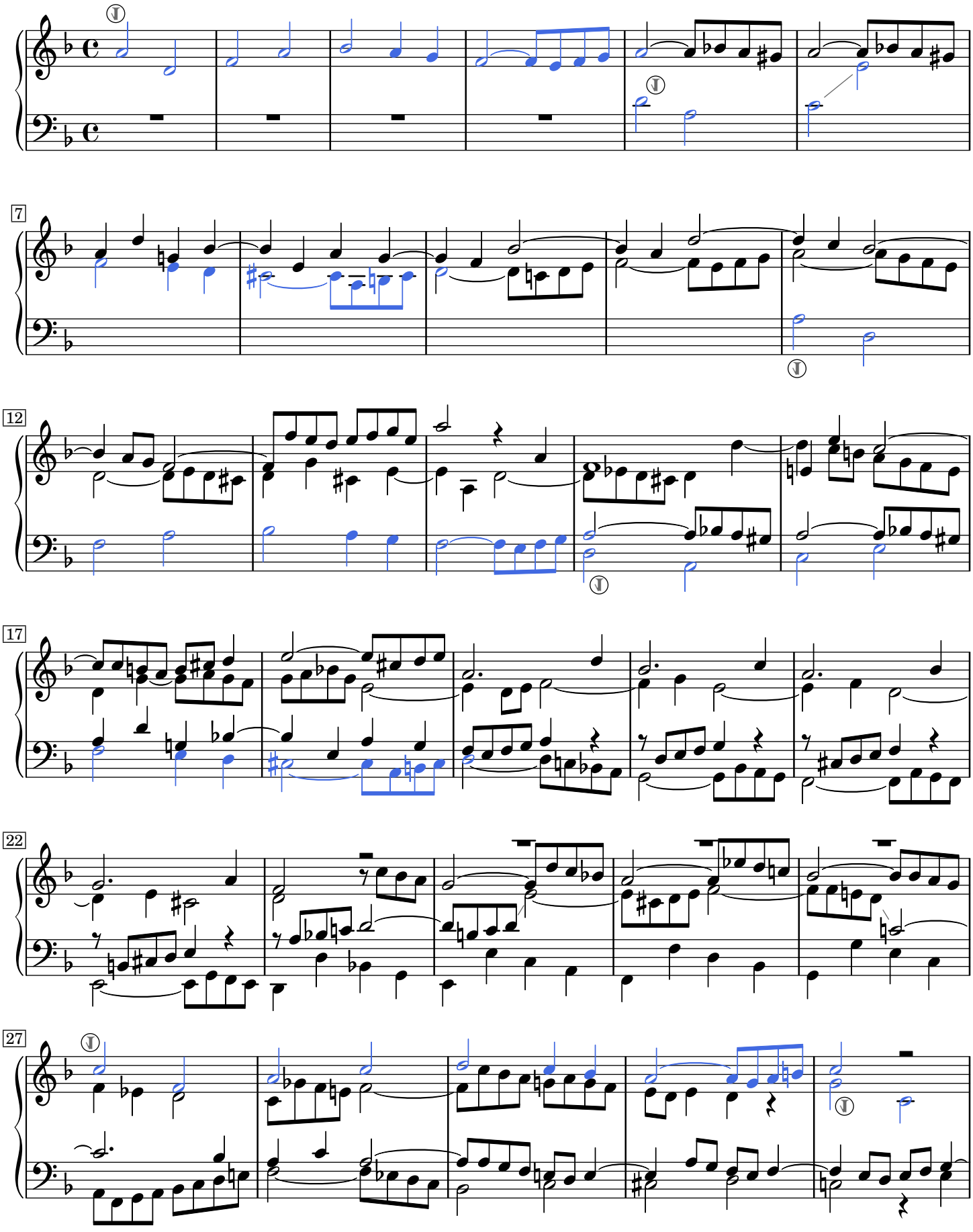

51
55
59
62
65
69

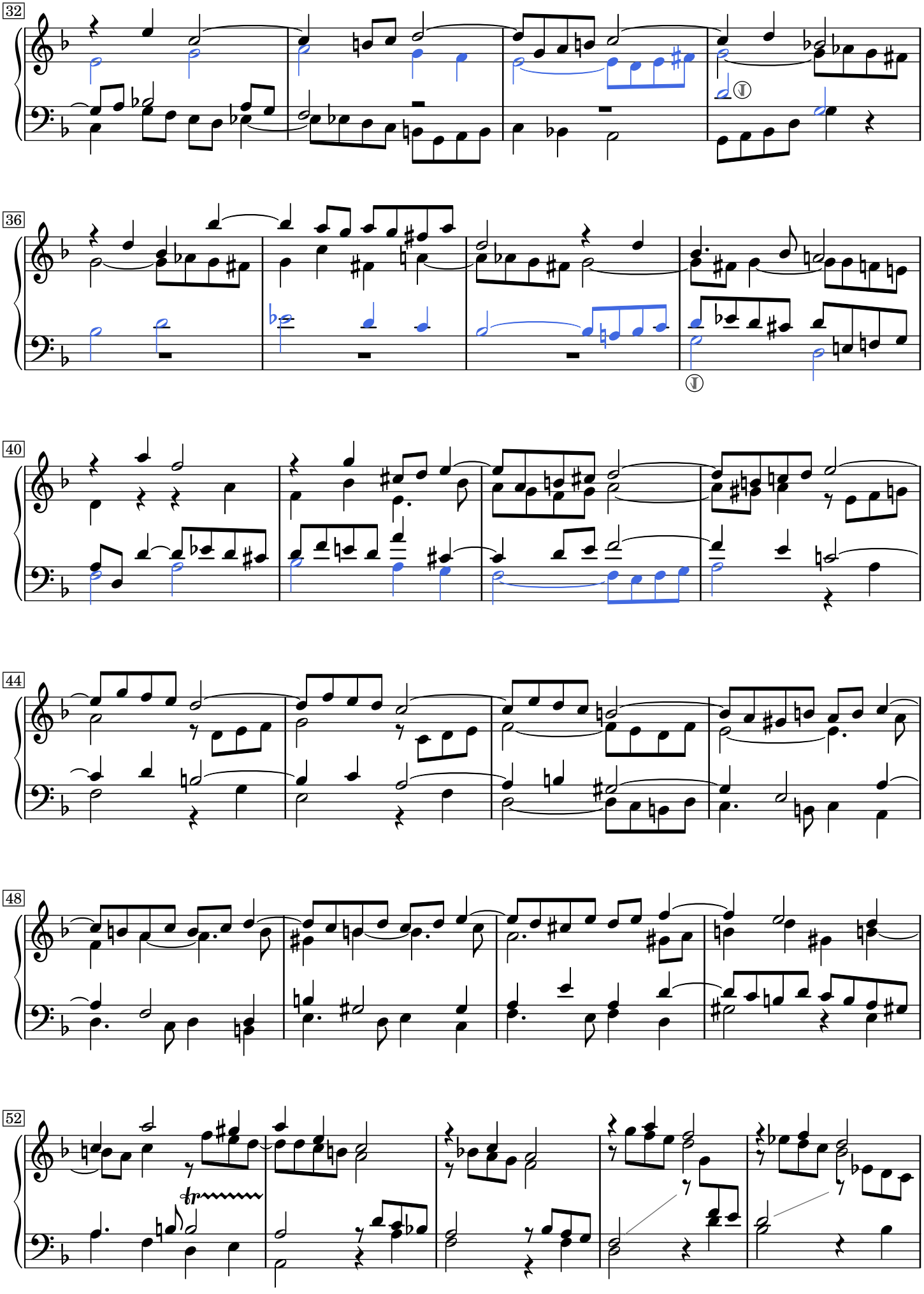
Contrapunctus IV

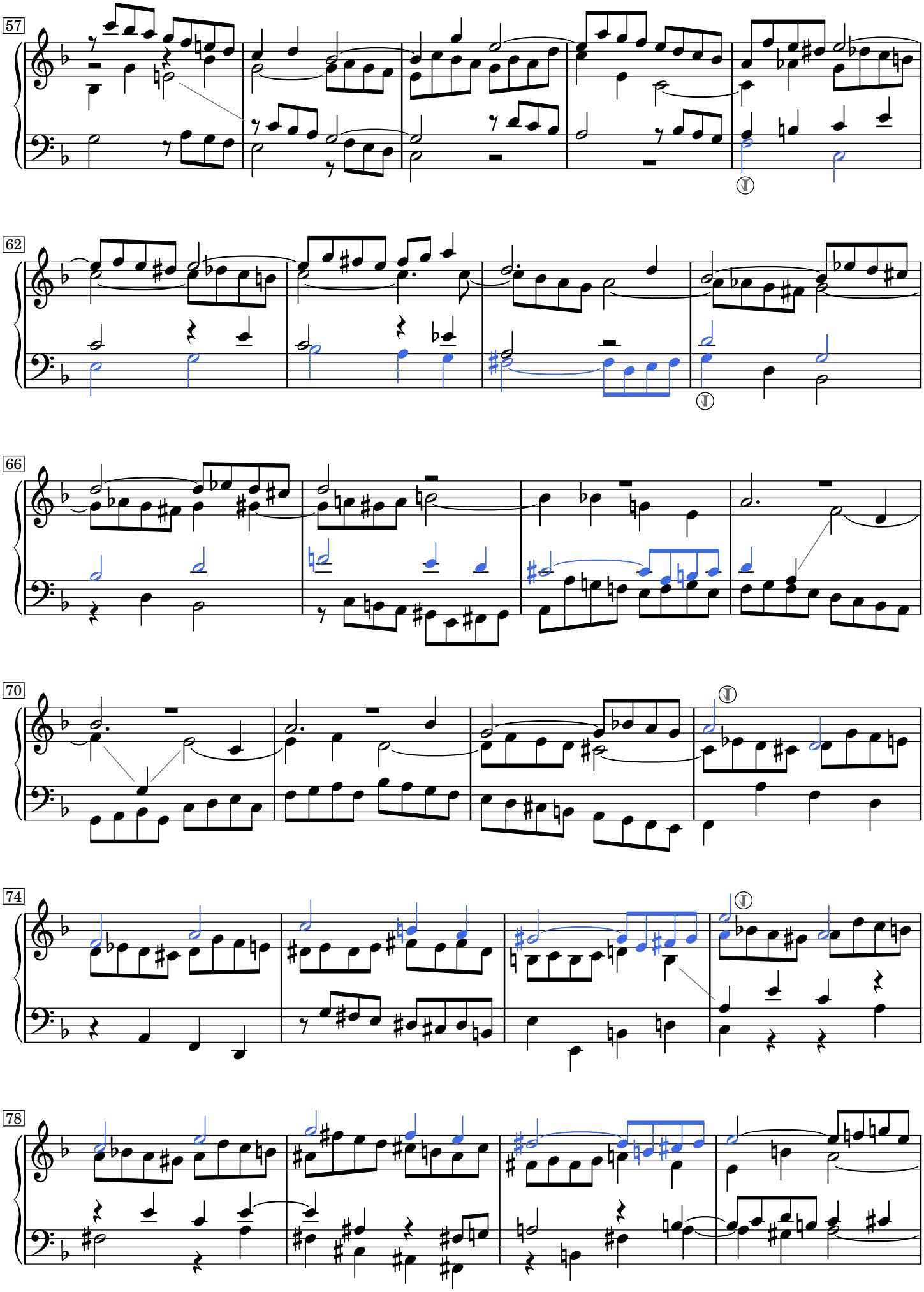
Four-voice fugue on principal subject in inversion, employing counter-subjects

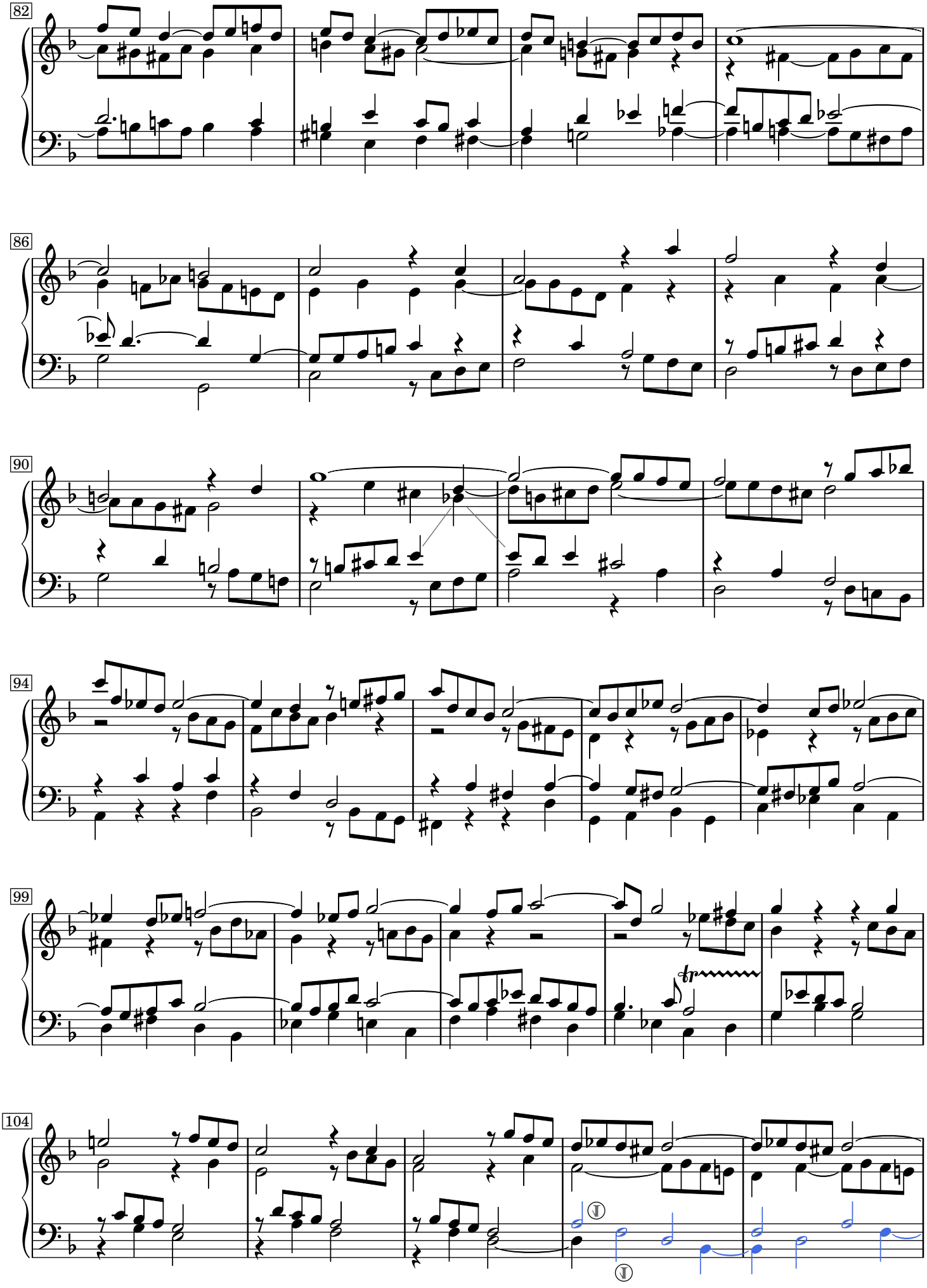
Exposition SATB | 1 | Second exposition SATB | 2 | BT | 3 | AS | 4 | T•B•SA | 5 | TA

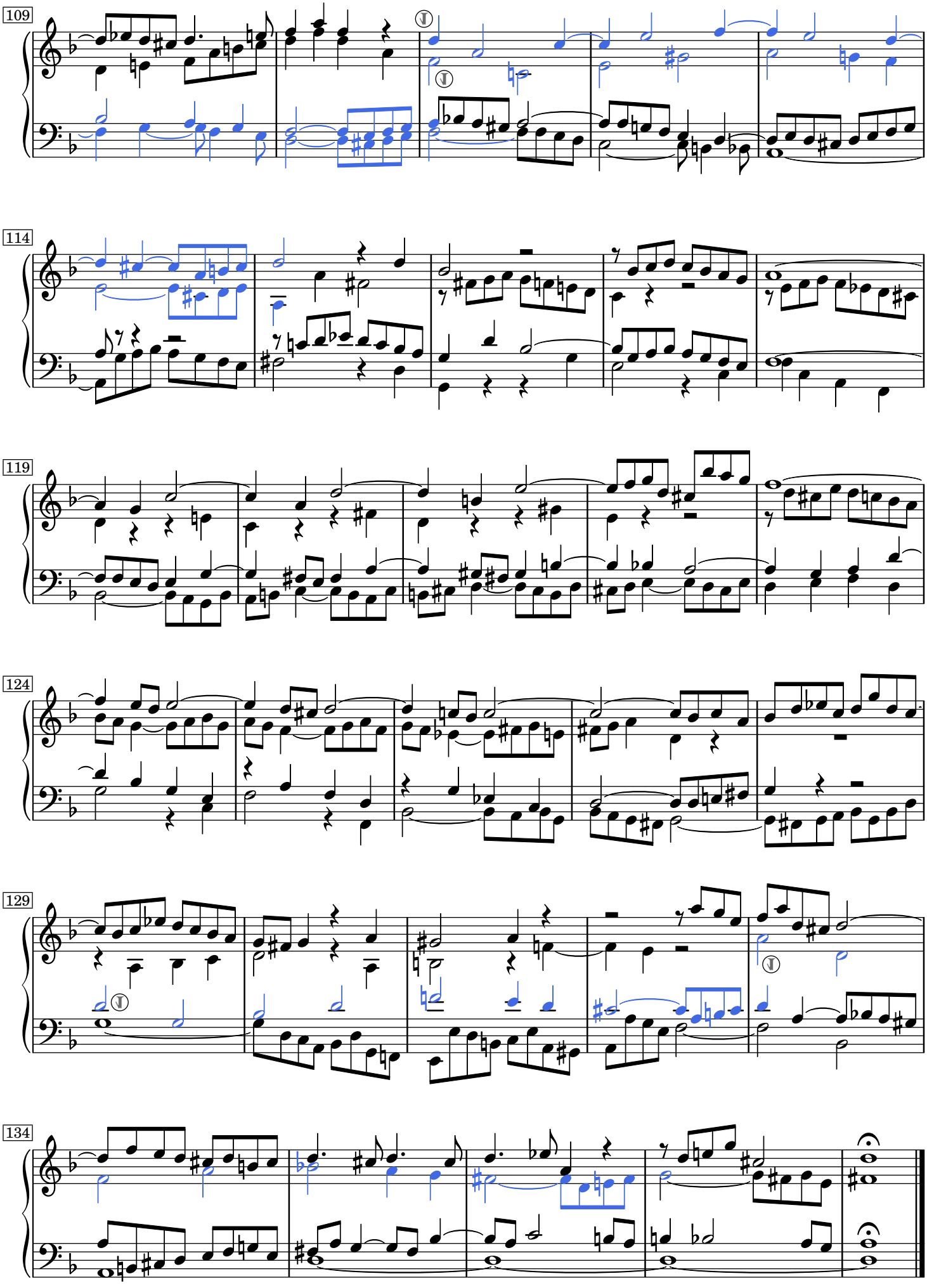
BWV 1080, 4




32
36
40
44
48
52


57
62
66
70
74
78


82
86
90
94
99
104

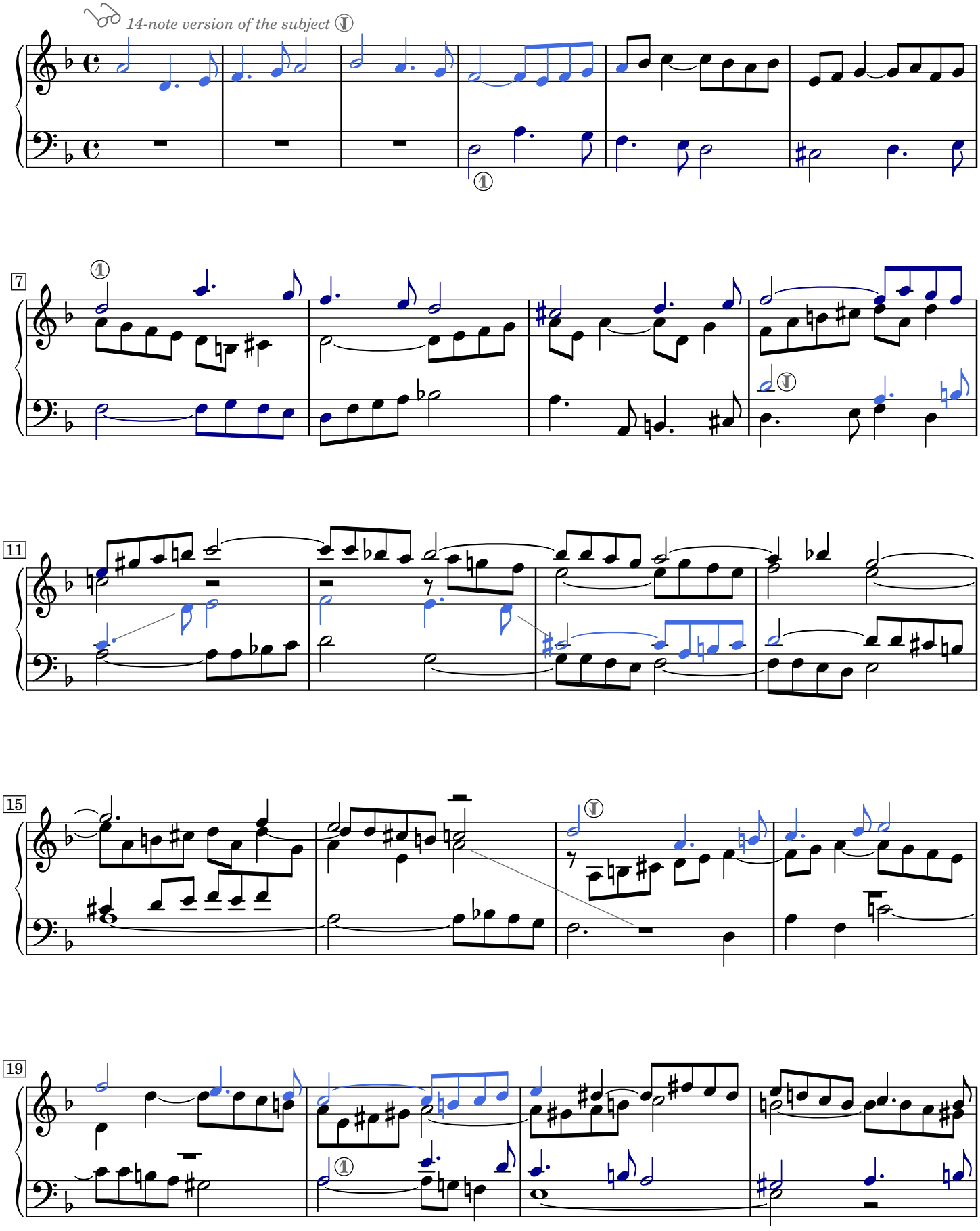

109
114
119
124
129
134

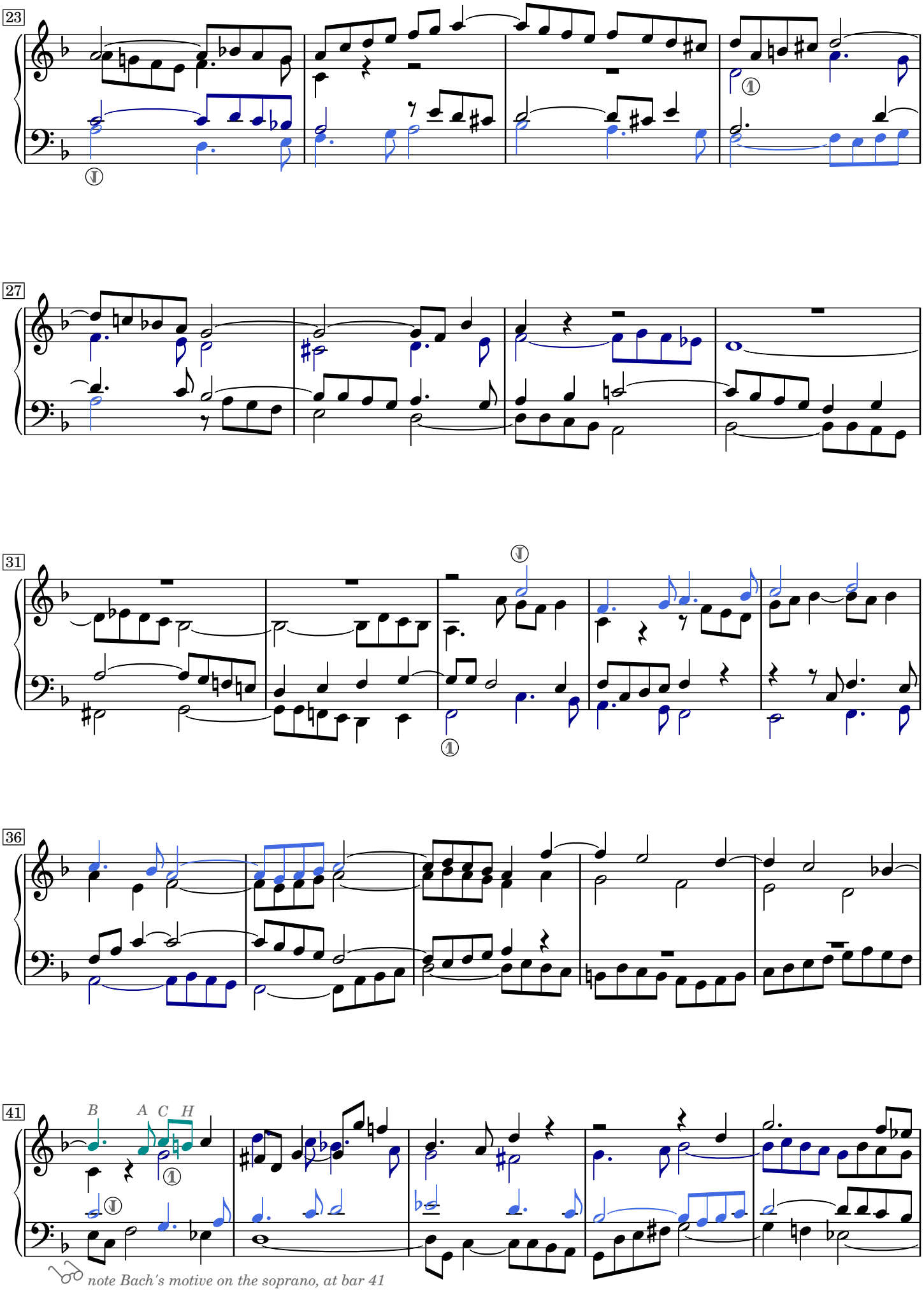
Contrapunctus V

Four-voice fugue with many stretto entries

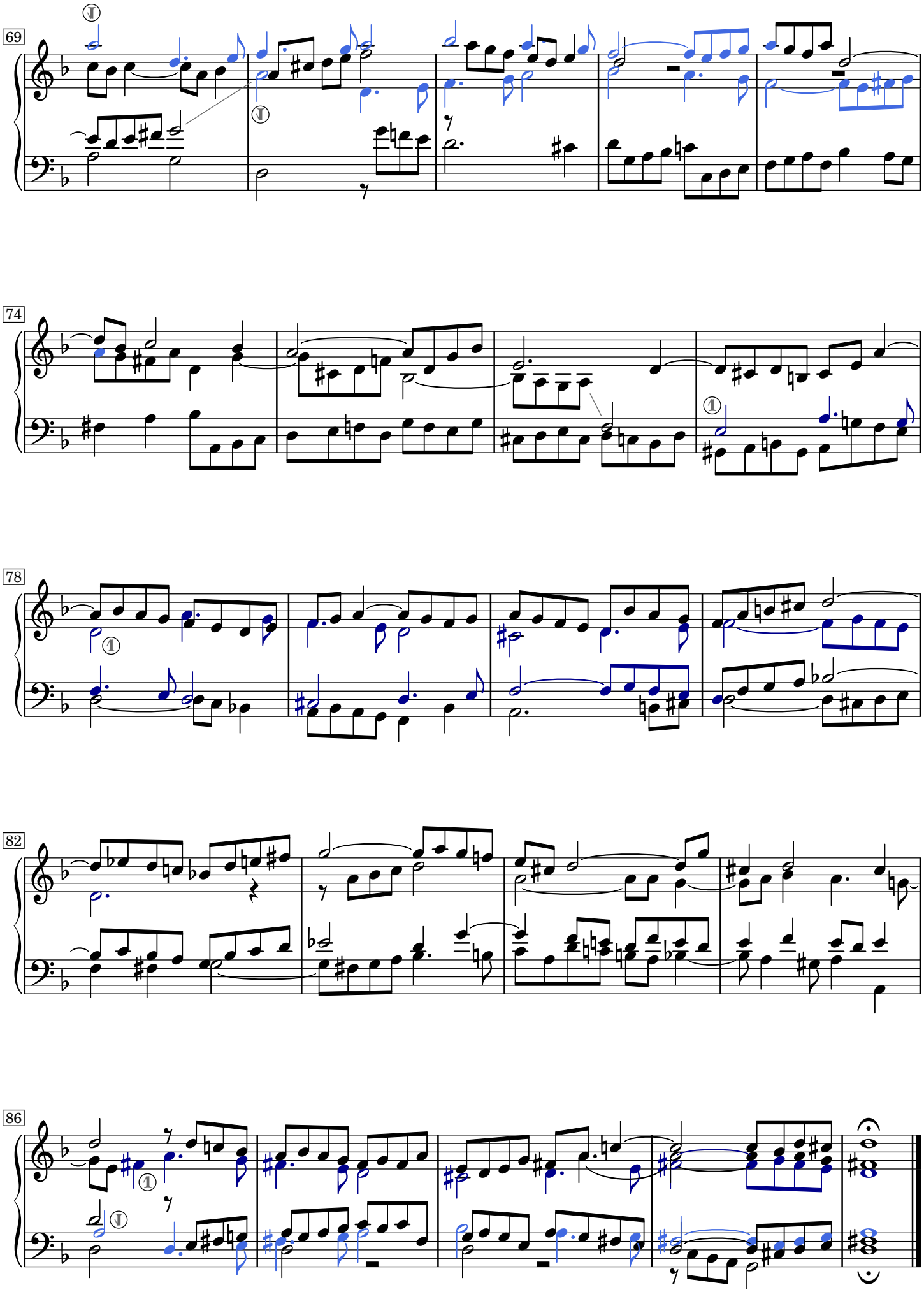
Exposition ABST | 1 | Second exposition STBA | 2 | Stretto BS | TA | BT | 3 | SA | 4 | ST | TA | BA

BWV 1080, 5




23
27
31
36
41
B
A
C
H
note Bach's motive on the soprano, at bar 41

46
51
56
60
64

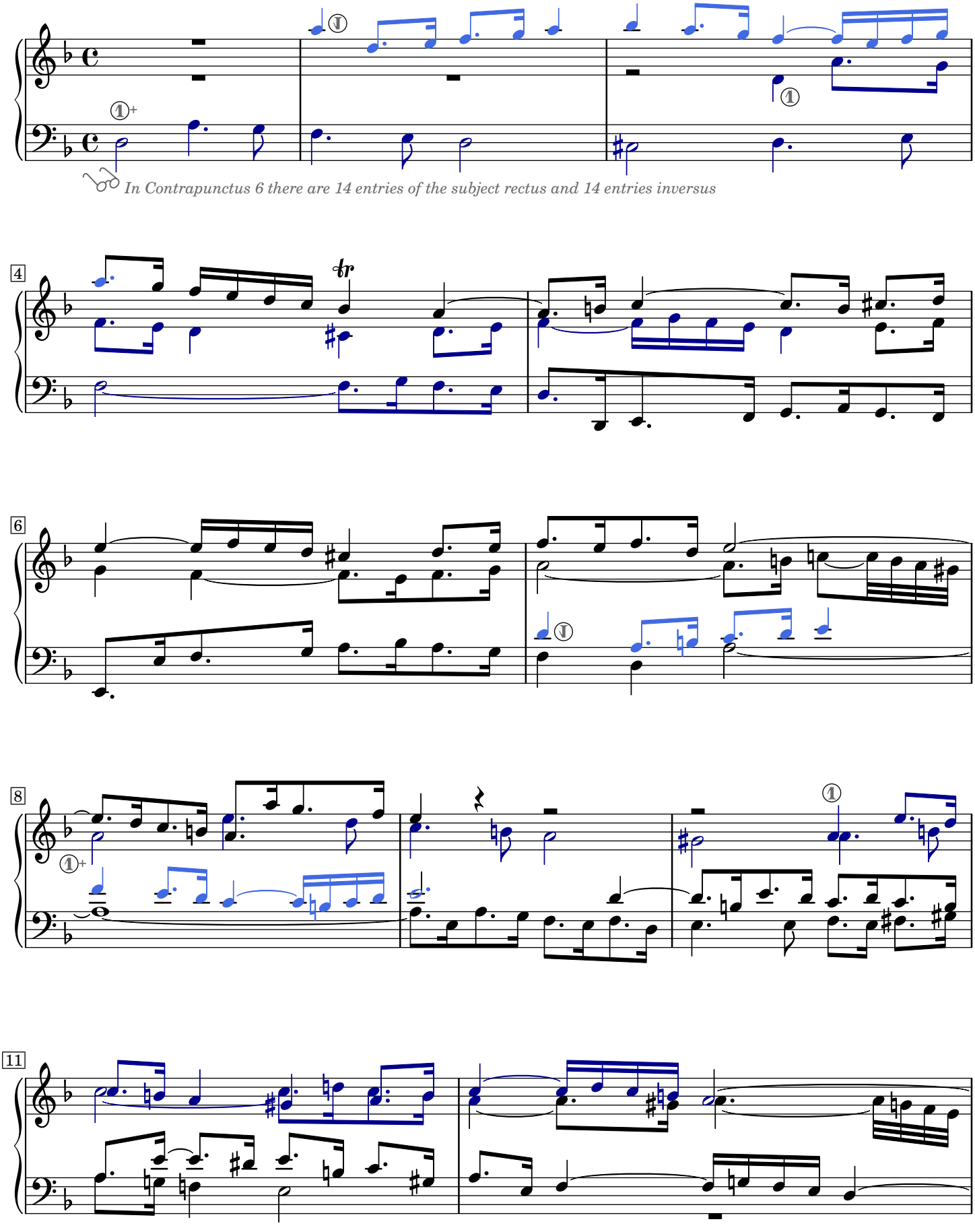

69
74
78
82
86

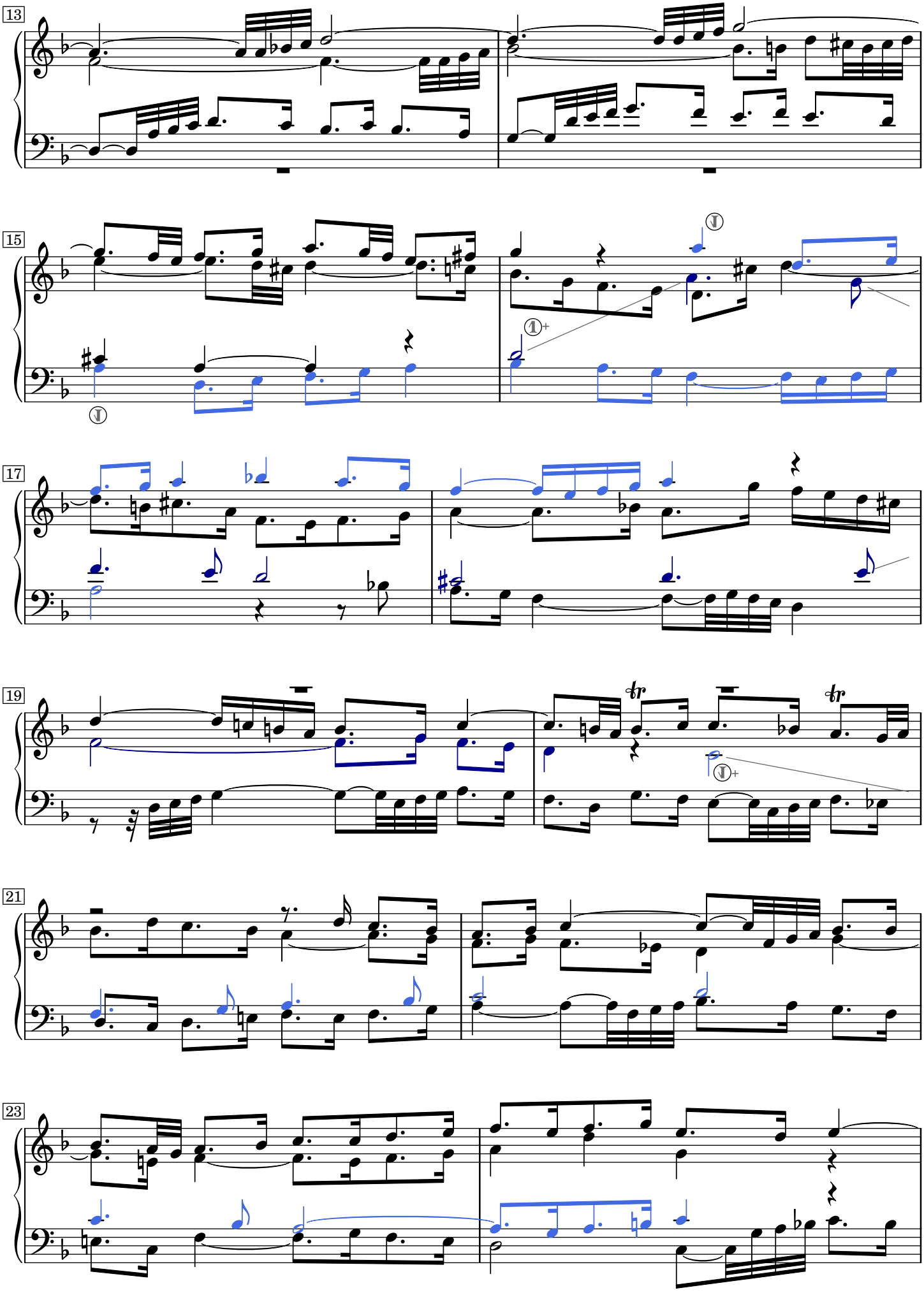
Contrapunctus VI

[per Diminutionem] in Stilo Francese

Stretto BSA | 1 | TAS | 2 | BTS | T | AT | 3 | TASBT | 4 | A | 5 | BT | 6 | SAT | 7 | TSA

BWV 1080, 6

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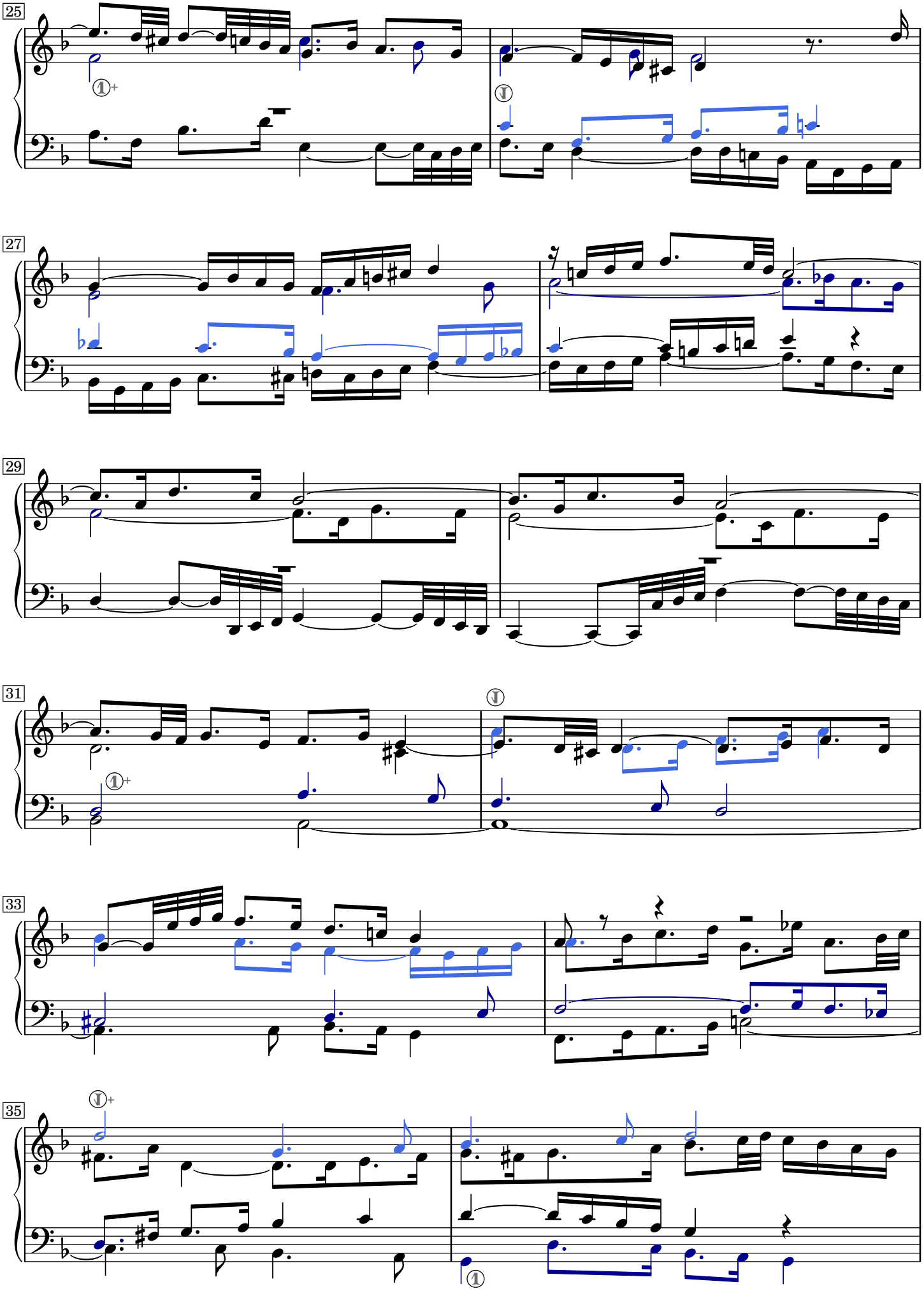
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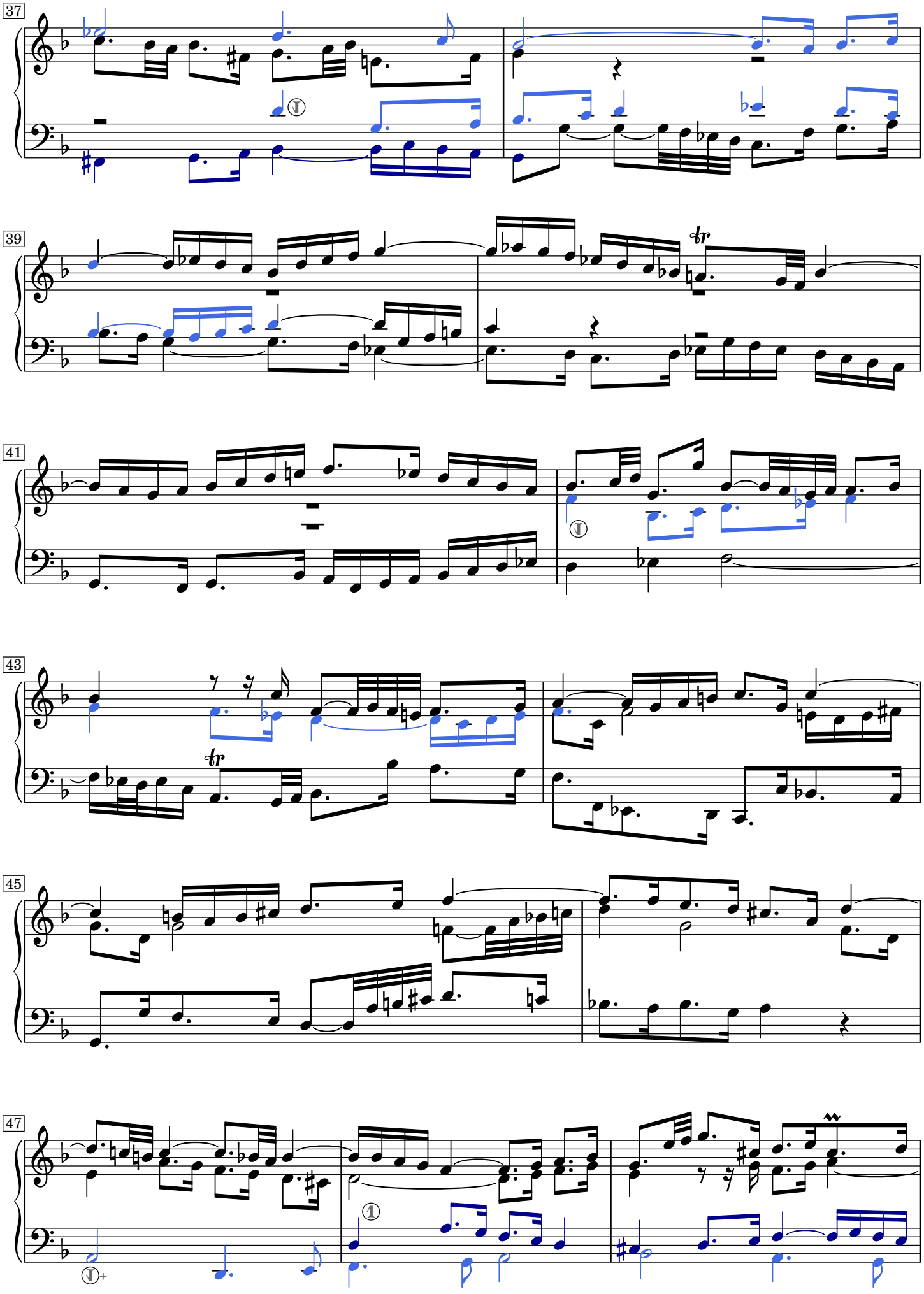
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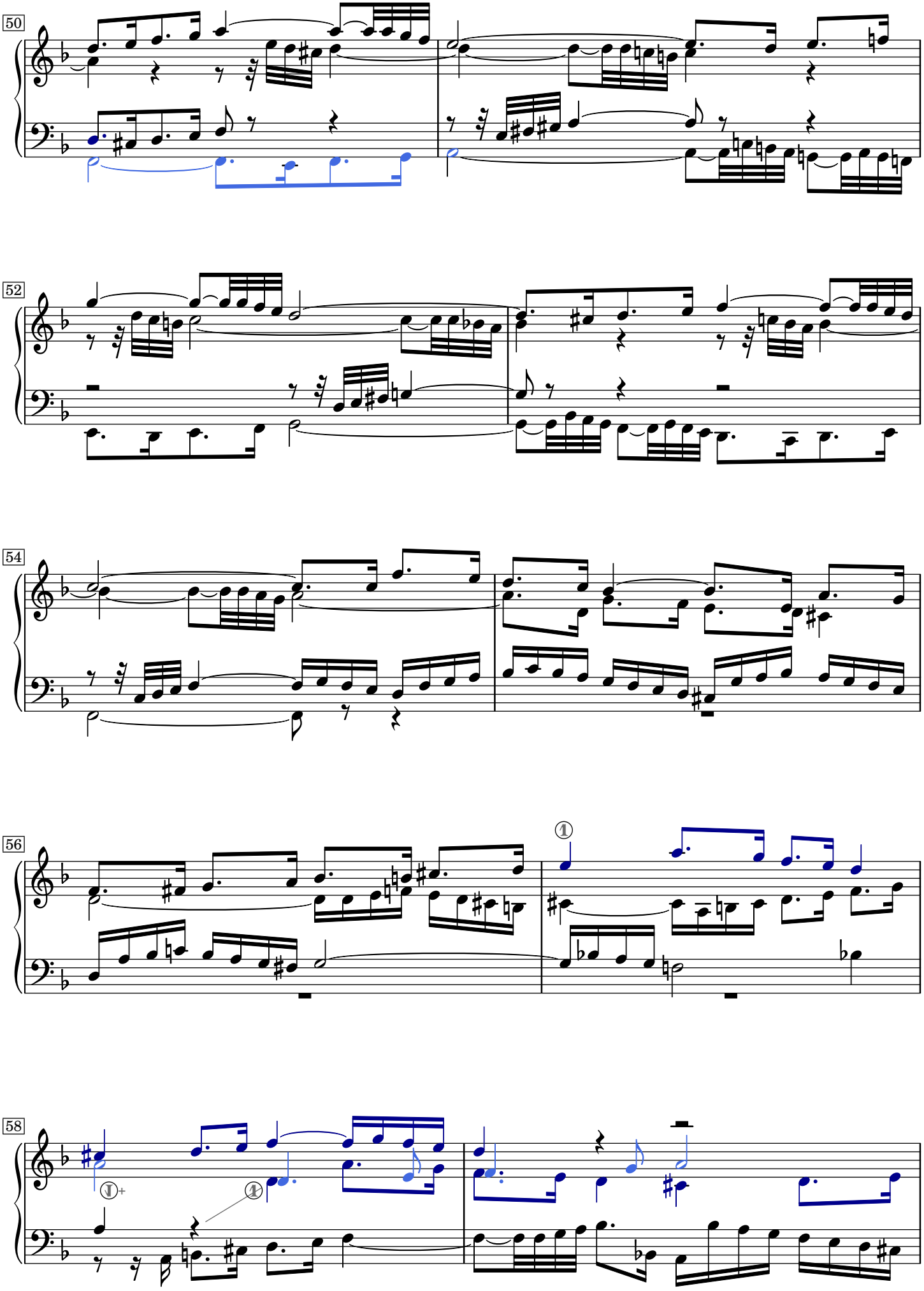
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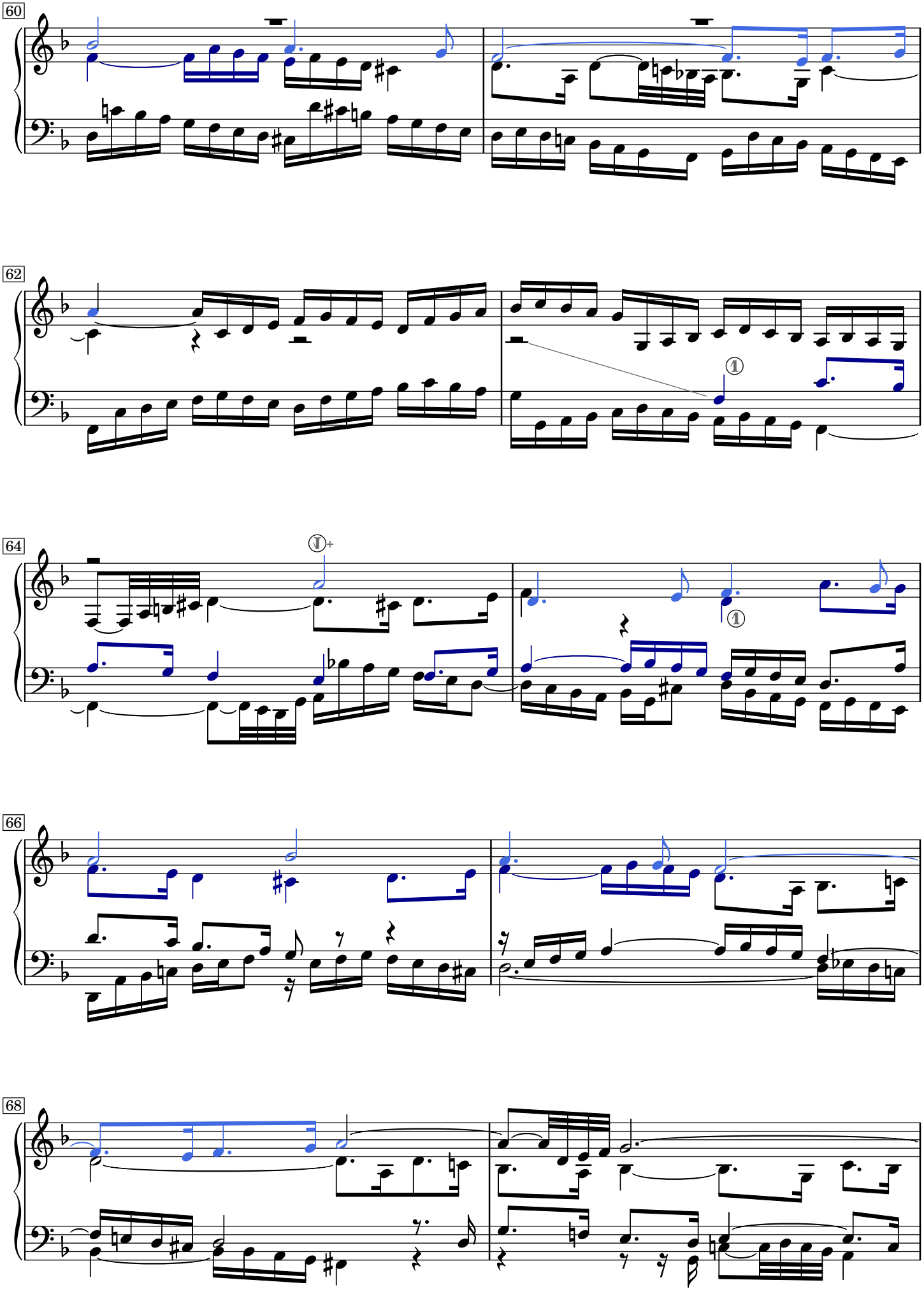
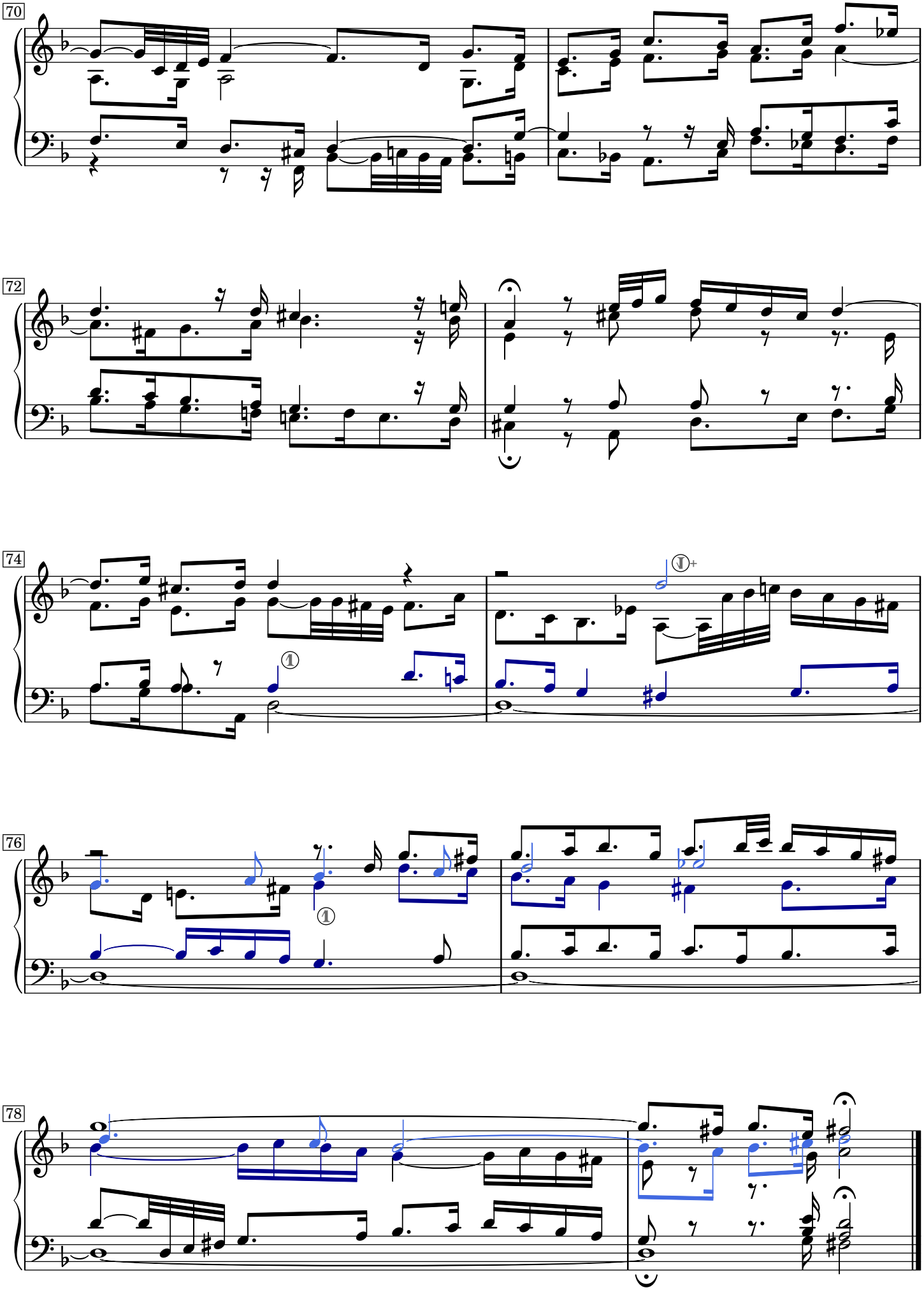

60
62
64
66
68

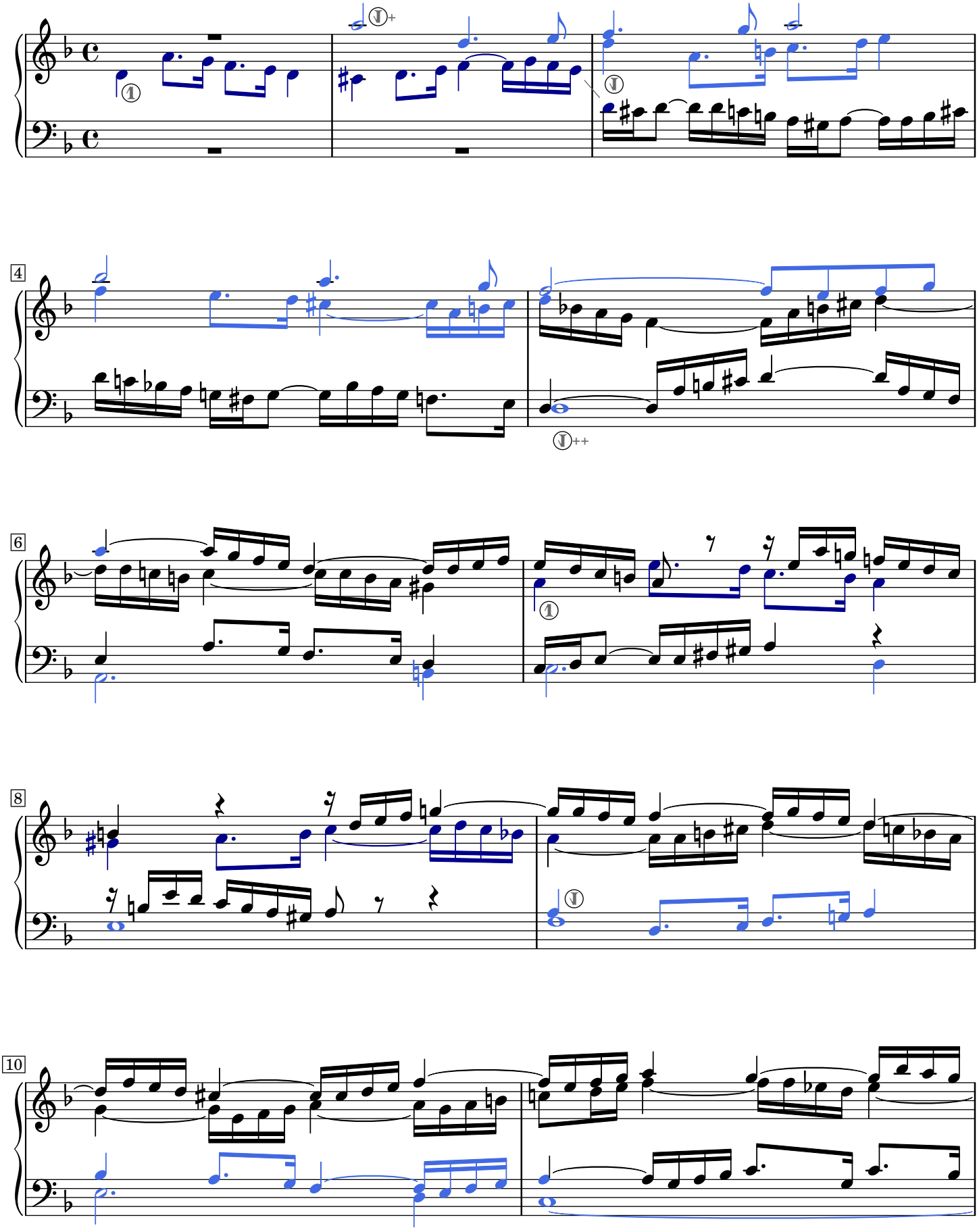

70
72
74
76
78

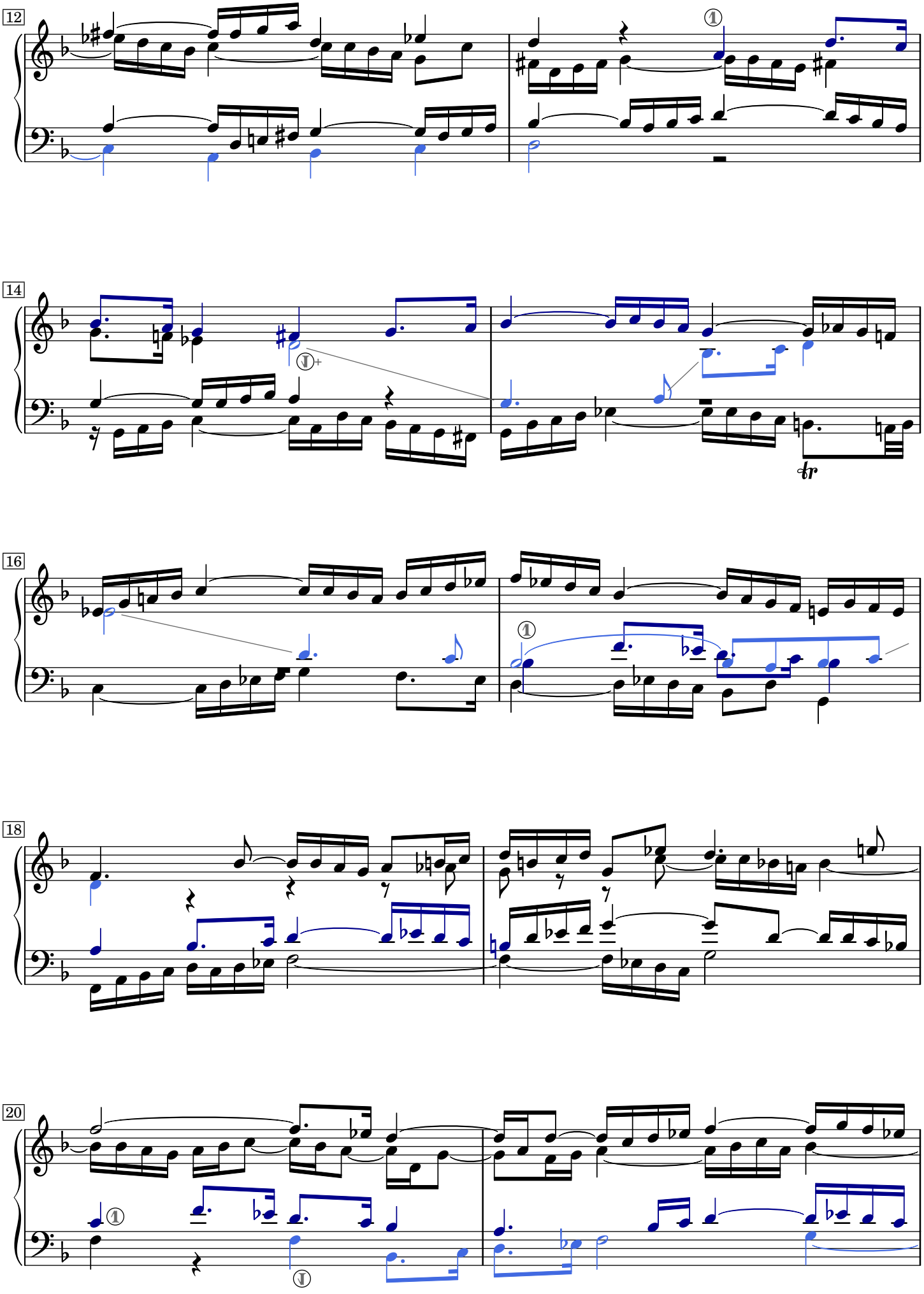
Contrapunctus VII

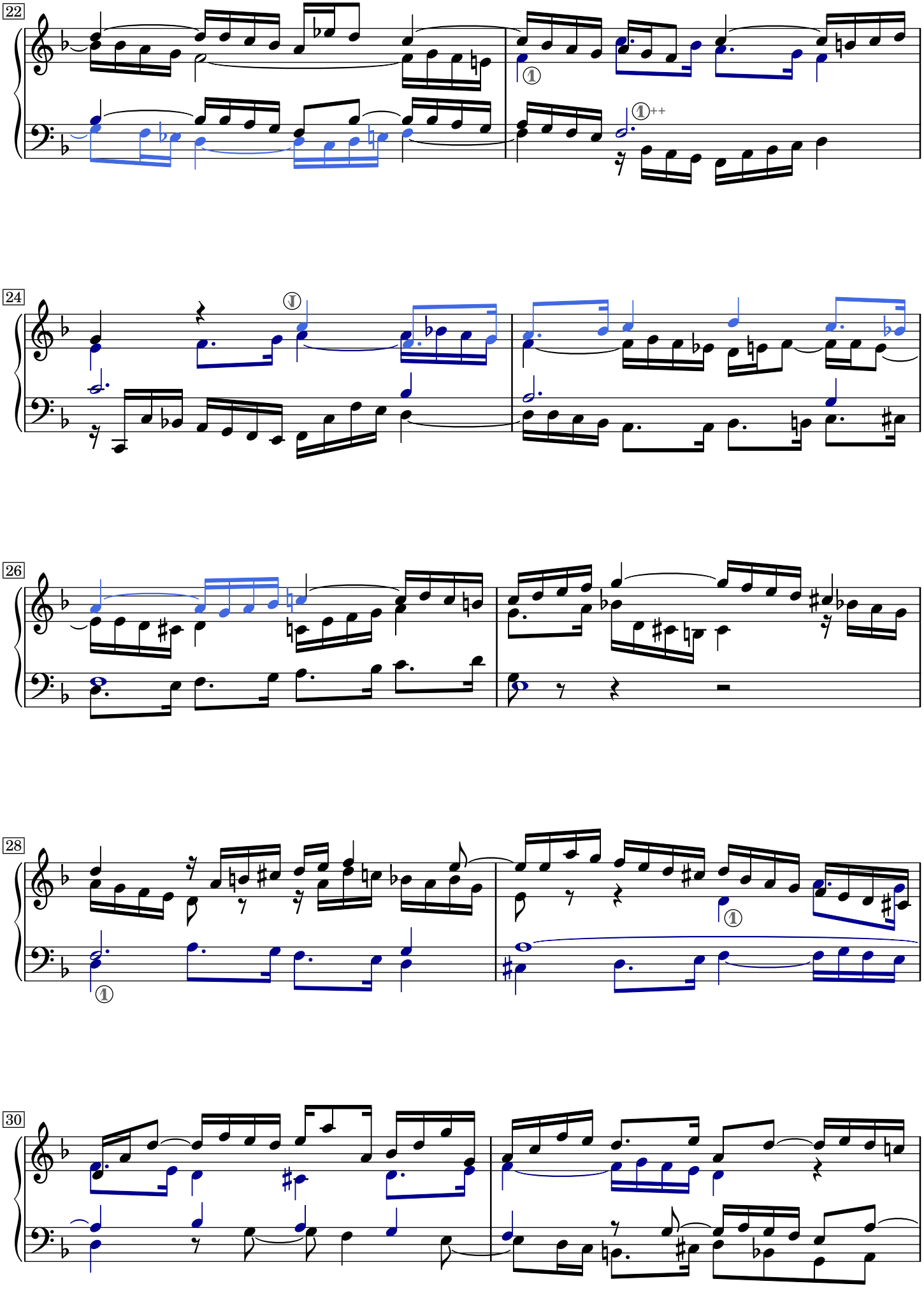
per Augmentationem et Diminutionem

Stretto TSABAT | SAT | TB | ATSBA | 1 | ATSSAATB | SAAT | Coda

BWV 1080, 7




12
14
16
18
20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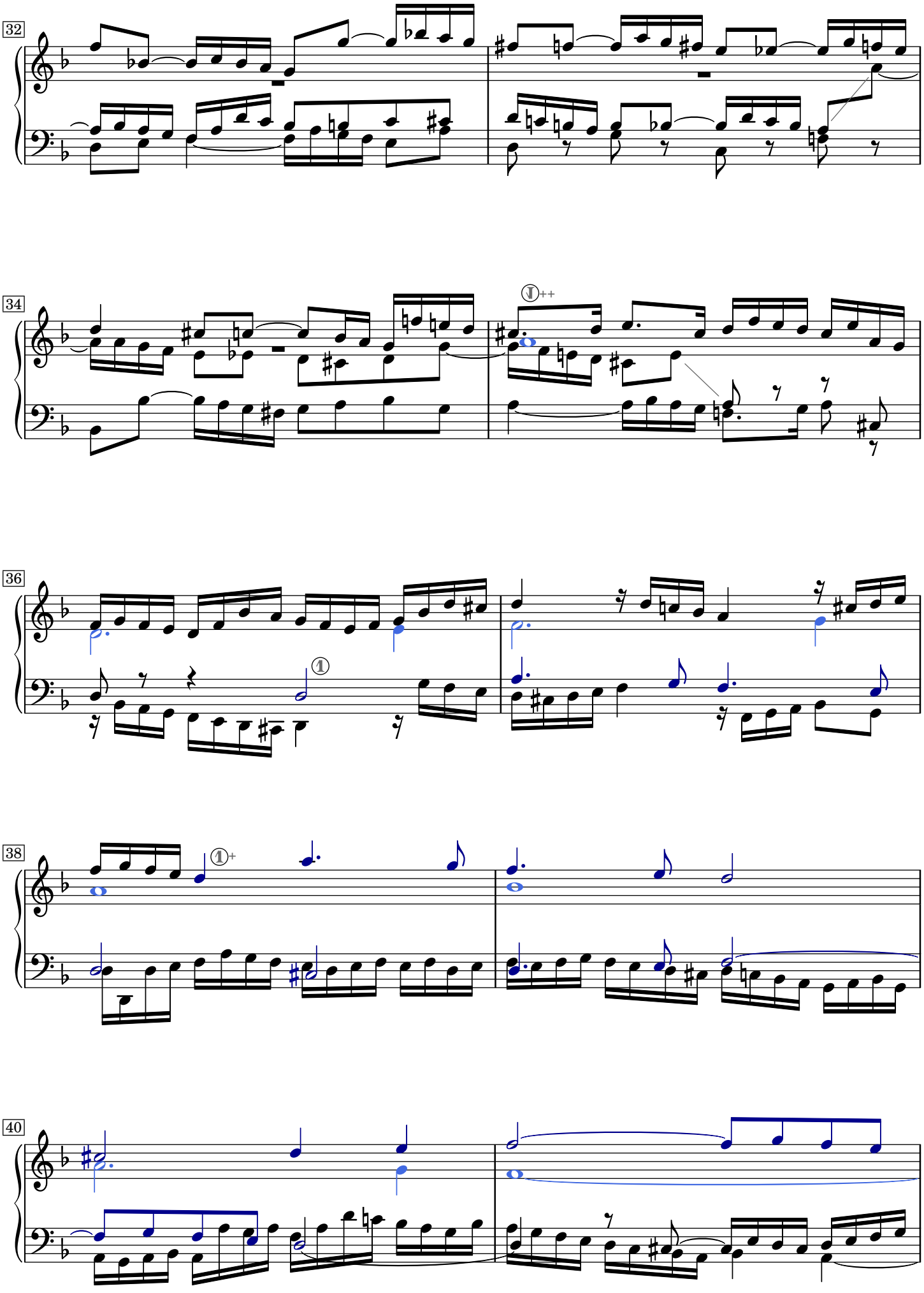
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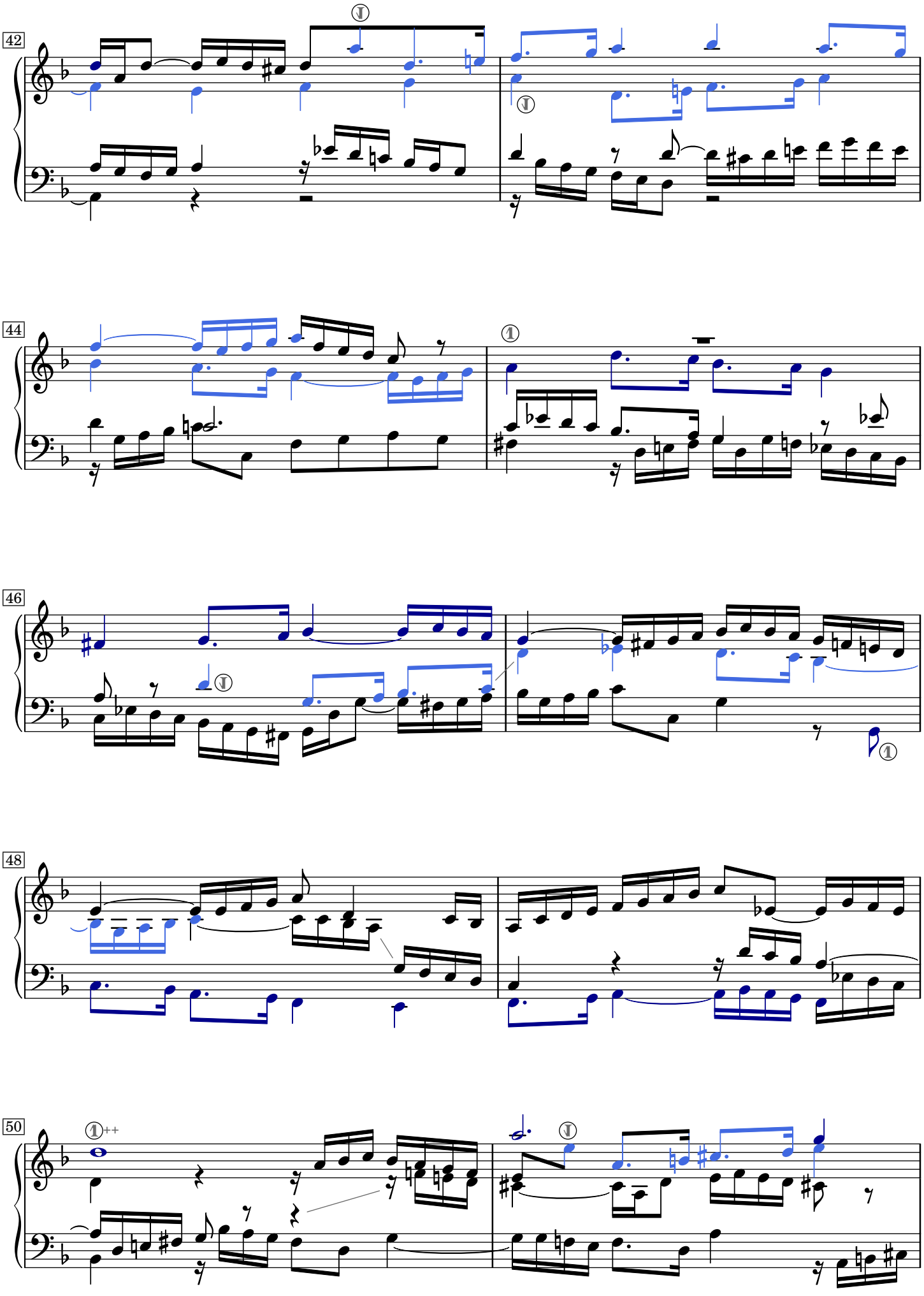
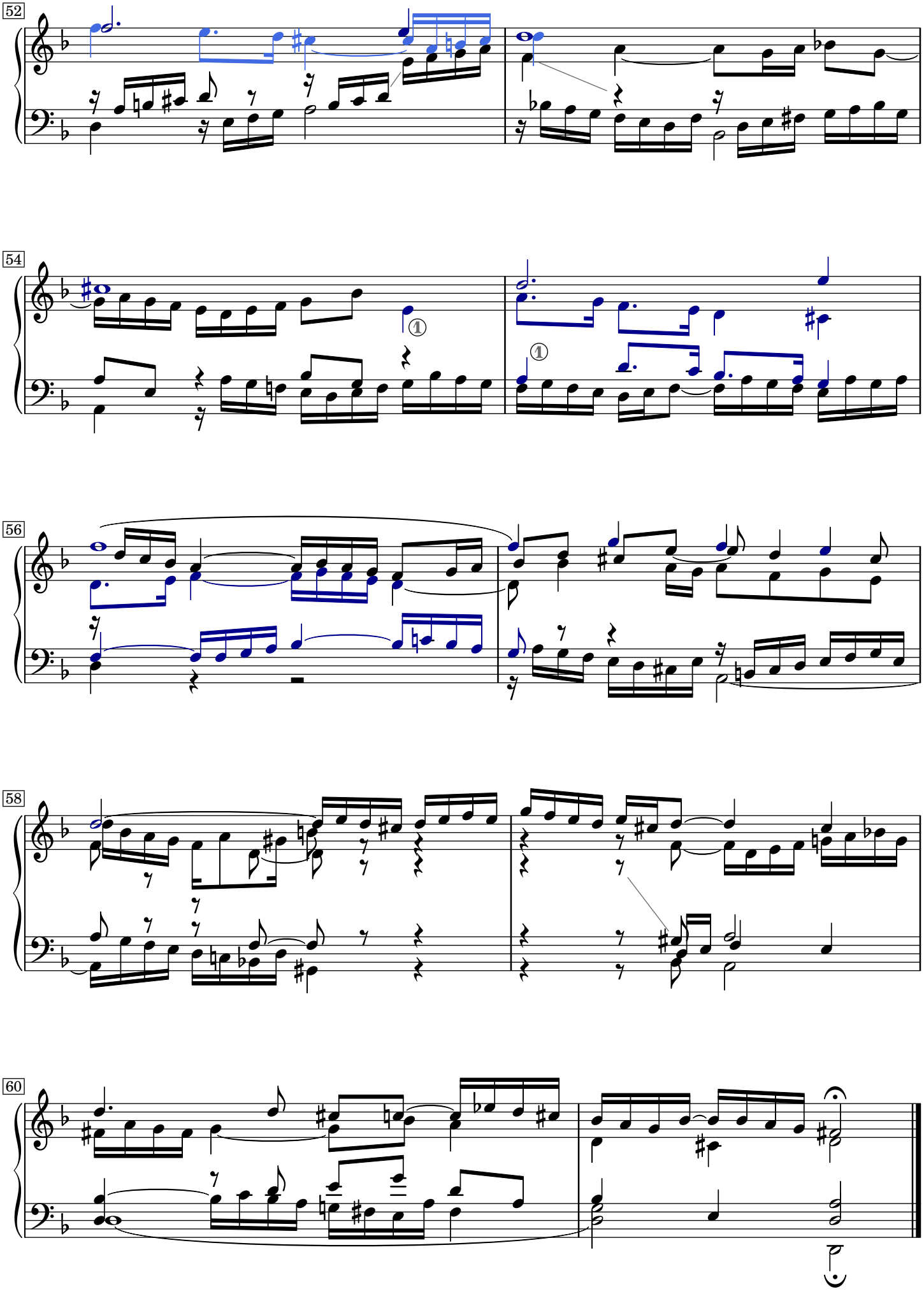

42
44
46
48
50


52
54
56
58
60

Contrapunctus VIII

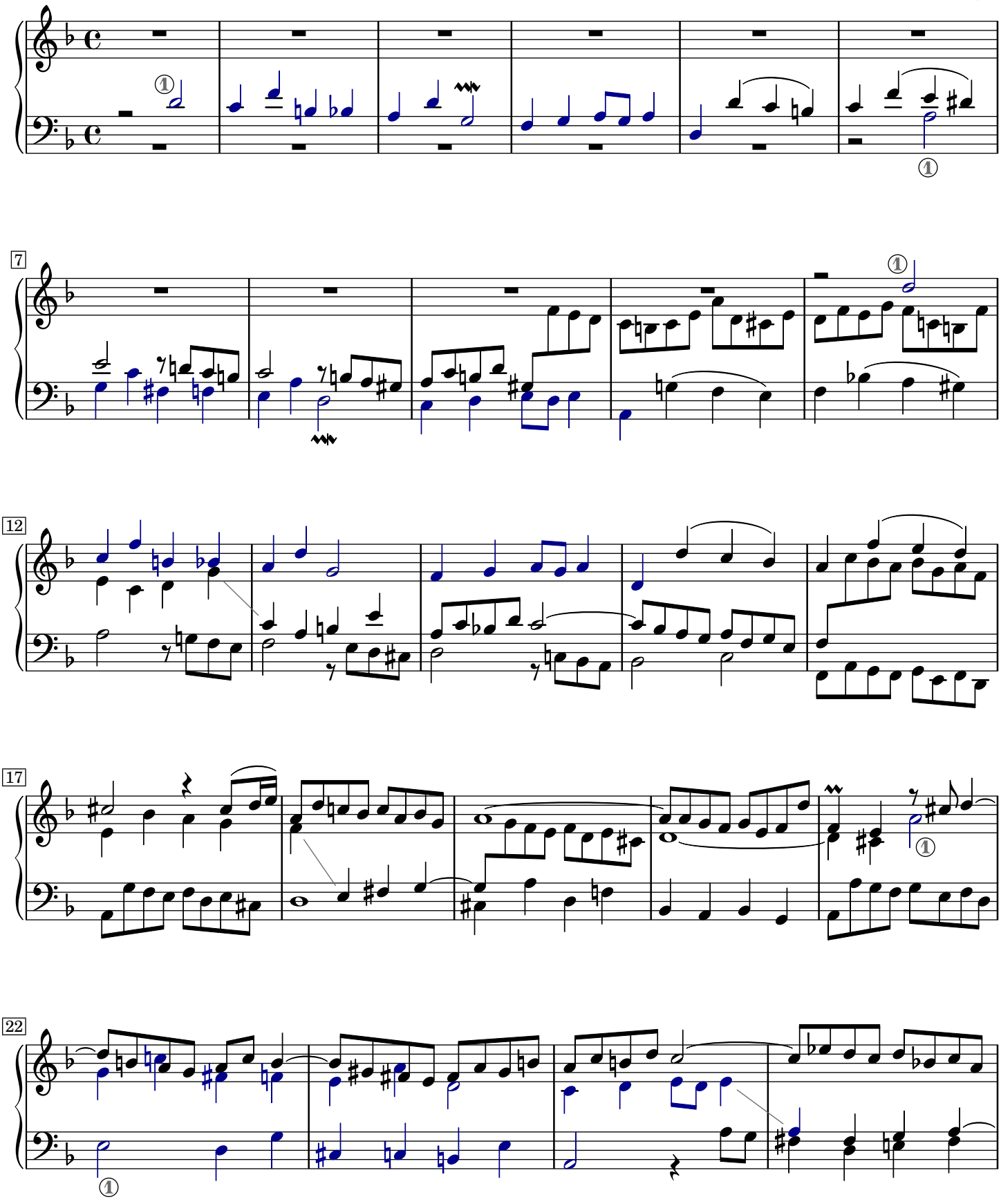
[Triple Fugue]

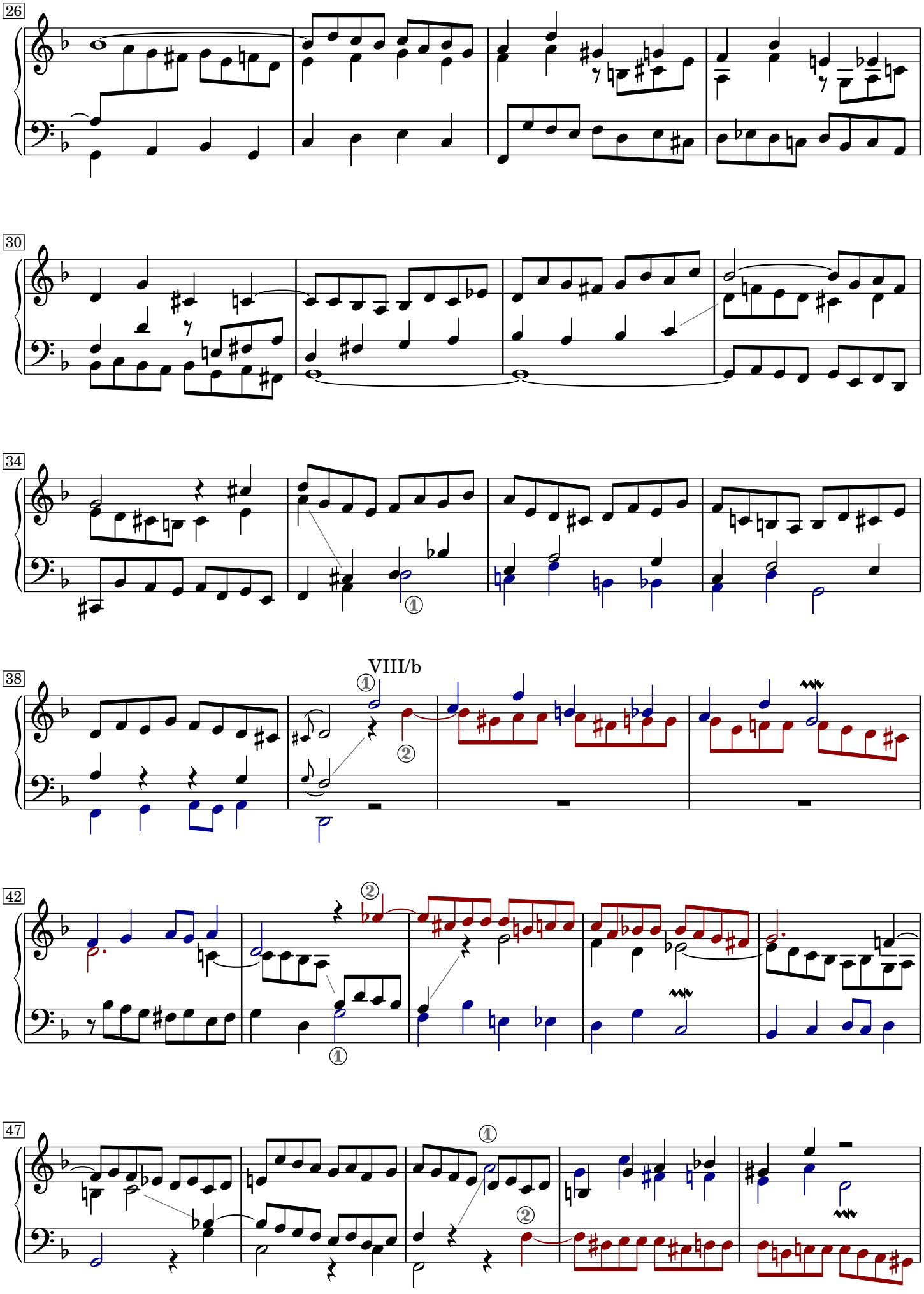
Exposition 1 A1 B1 S1 | 1 | Stretto A1 B1 | 2 | B1

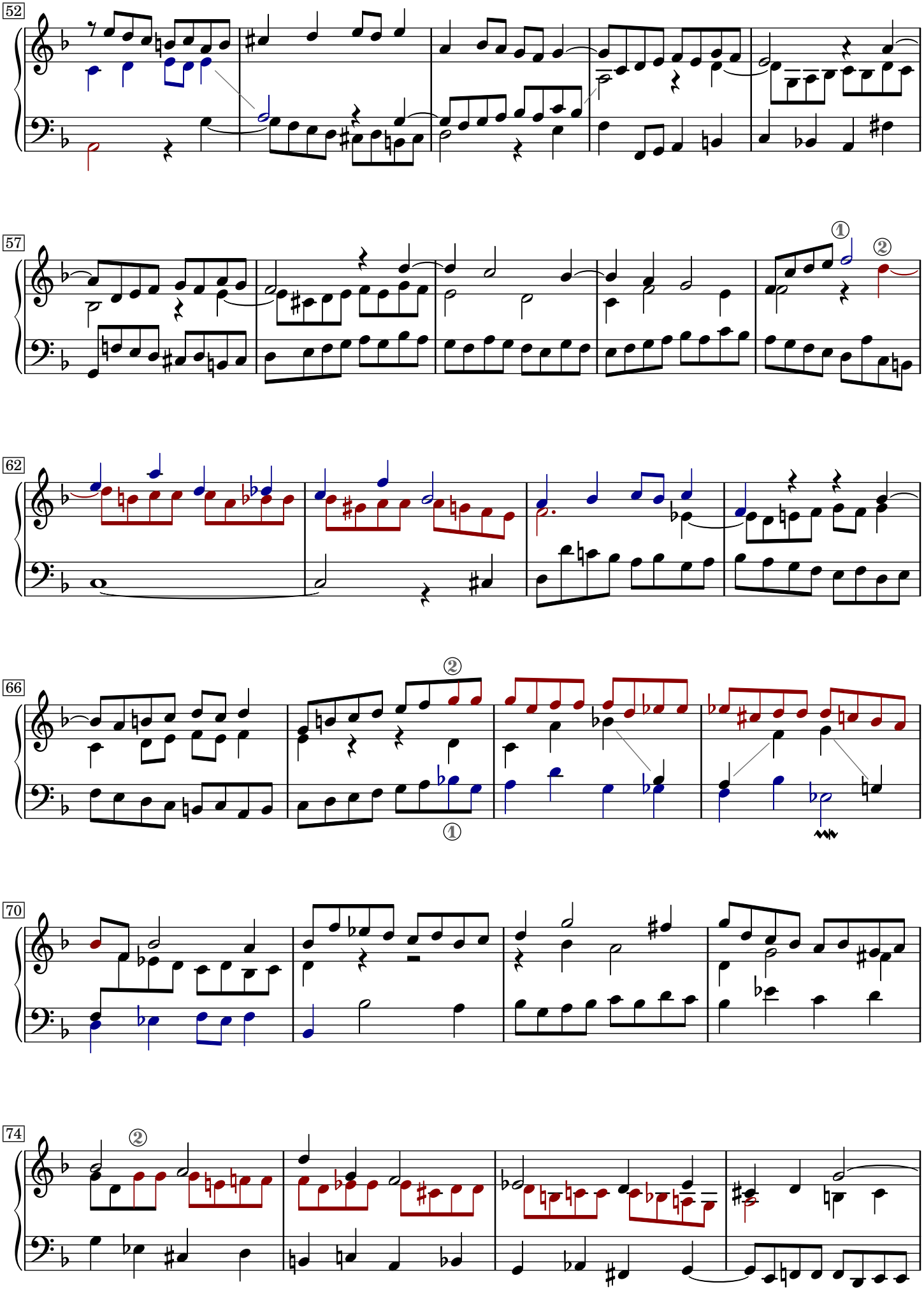
Exposition 2 S1 A2 B1 S2 A1 B2 | 3 | S1 A2 | B1 S2 | 4 | S1 A2 | S1 A2

Exposition 3 A3 B3 S3 | 5 | A1 | B1 | 6 | S1 A2 B3 | A1 B2 S3 | B1 S2 A3 | B1 S2 A3 | A2 S1 B3

BWV 1080, 8




26
30
34
38
VIII/b
42
47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57
62
66
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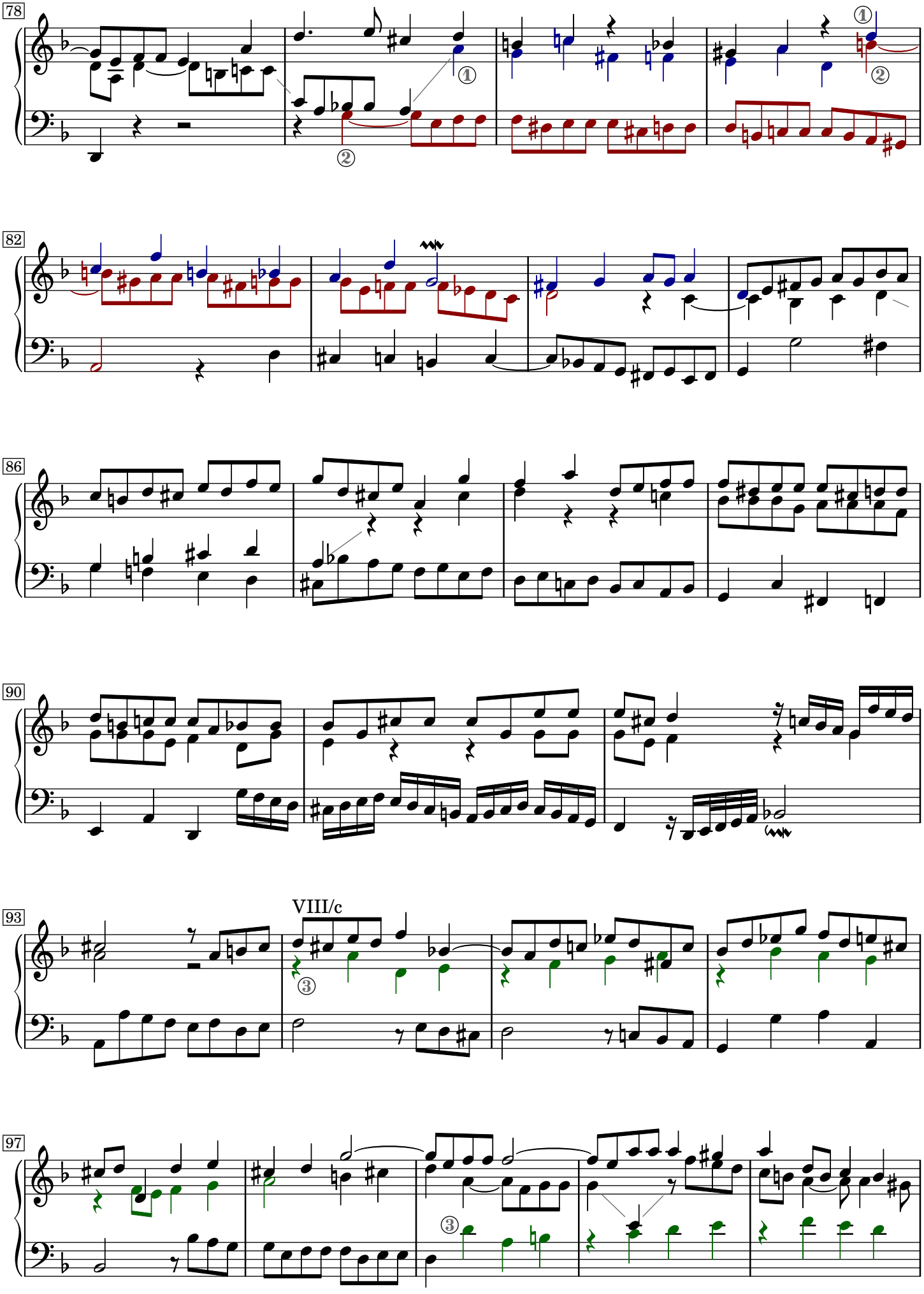
74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VIII/c
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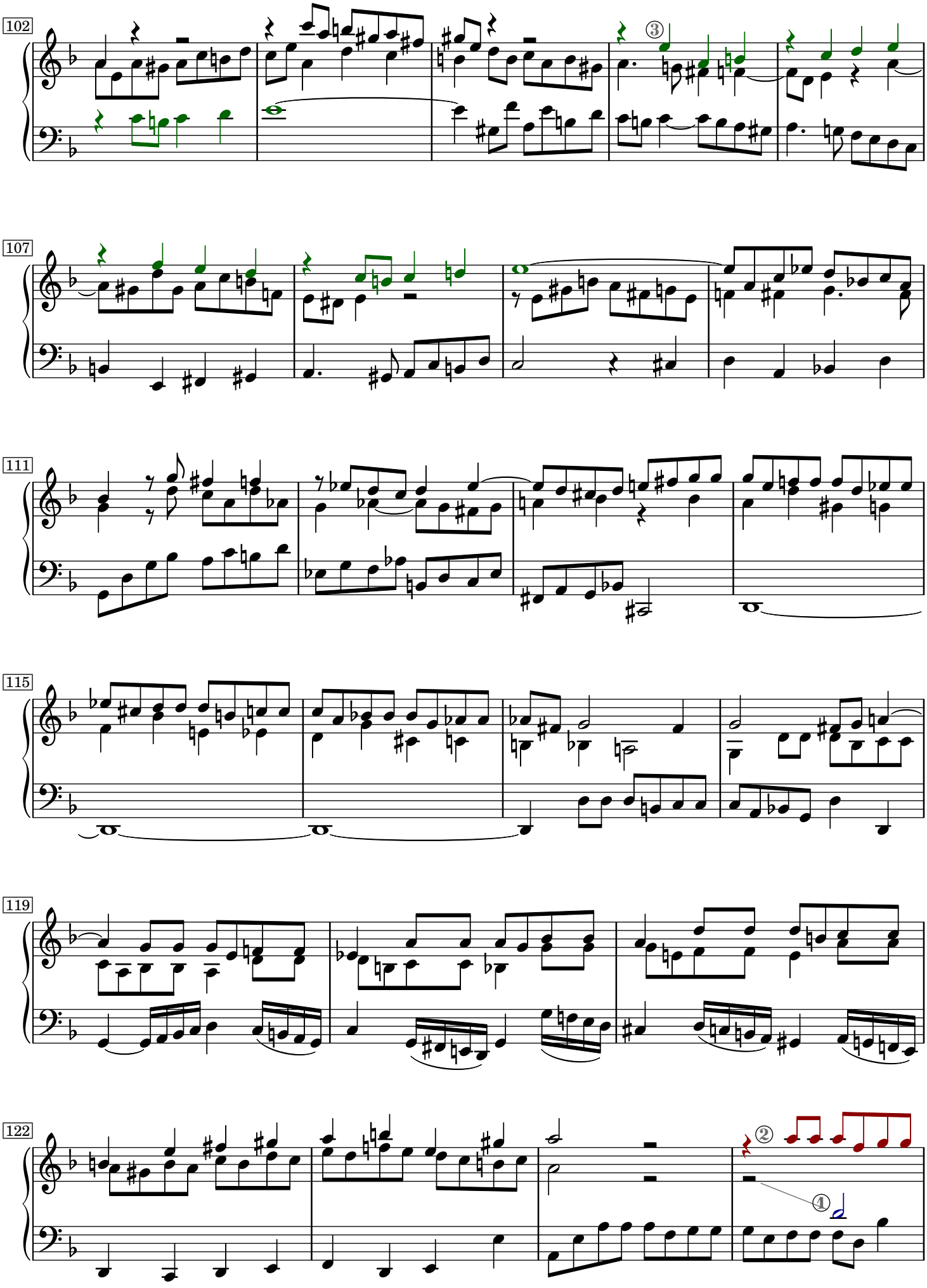
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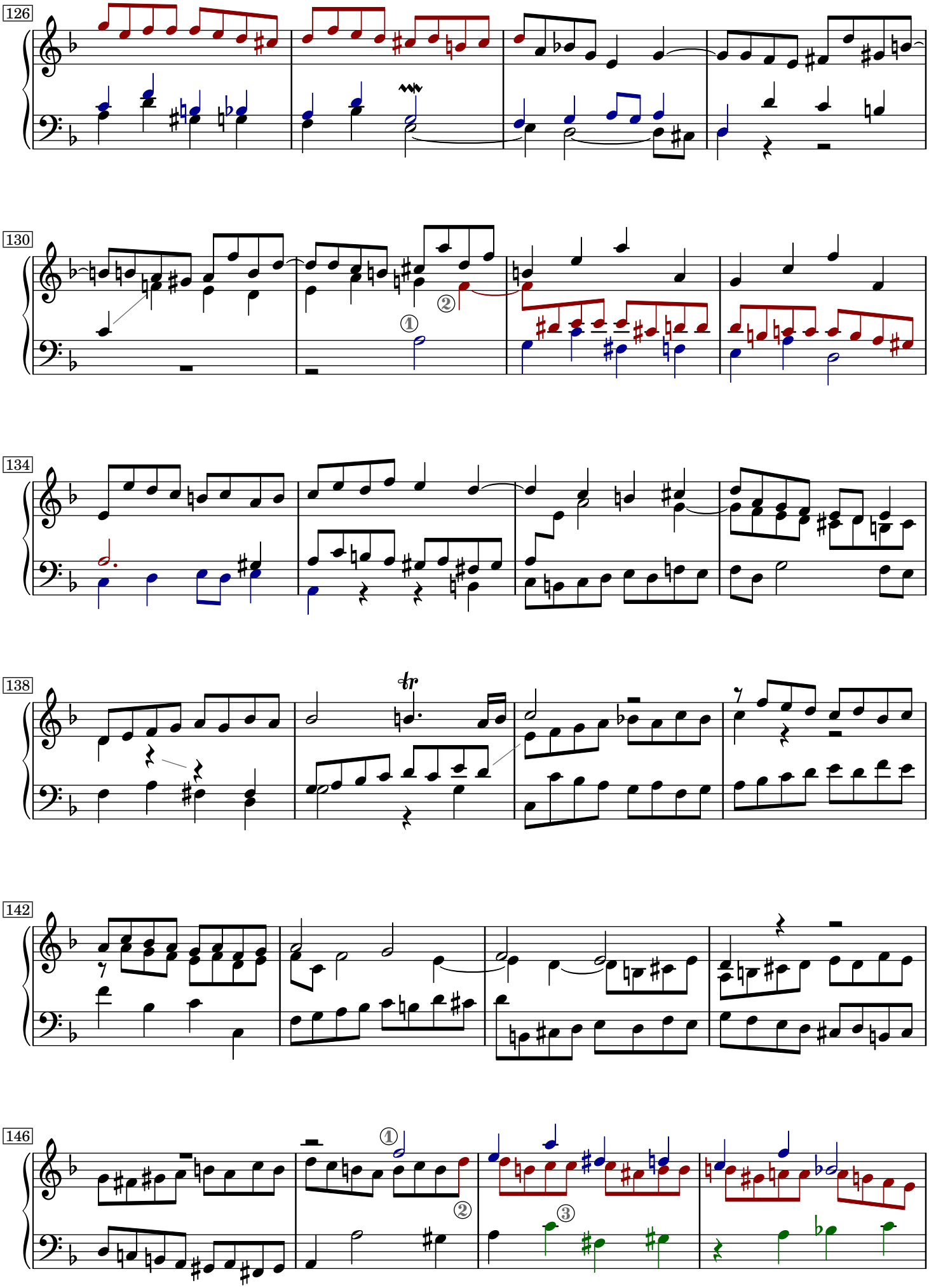
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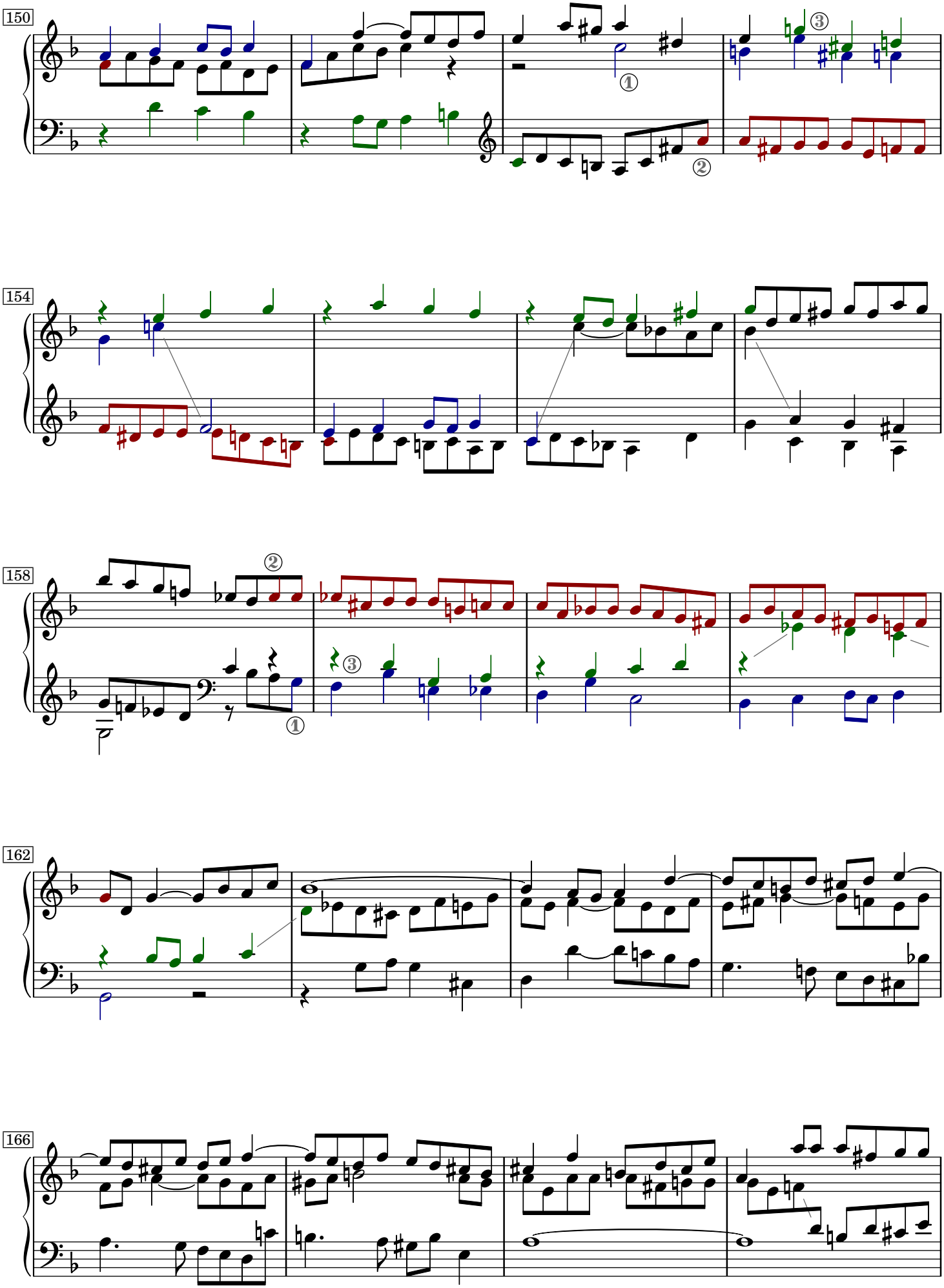
150
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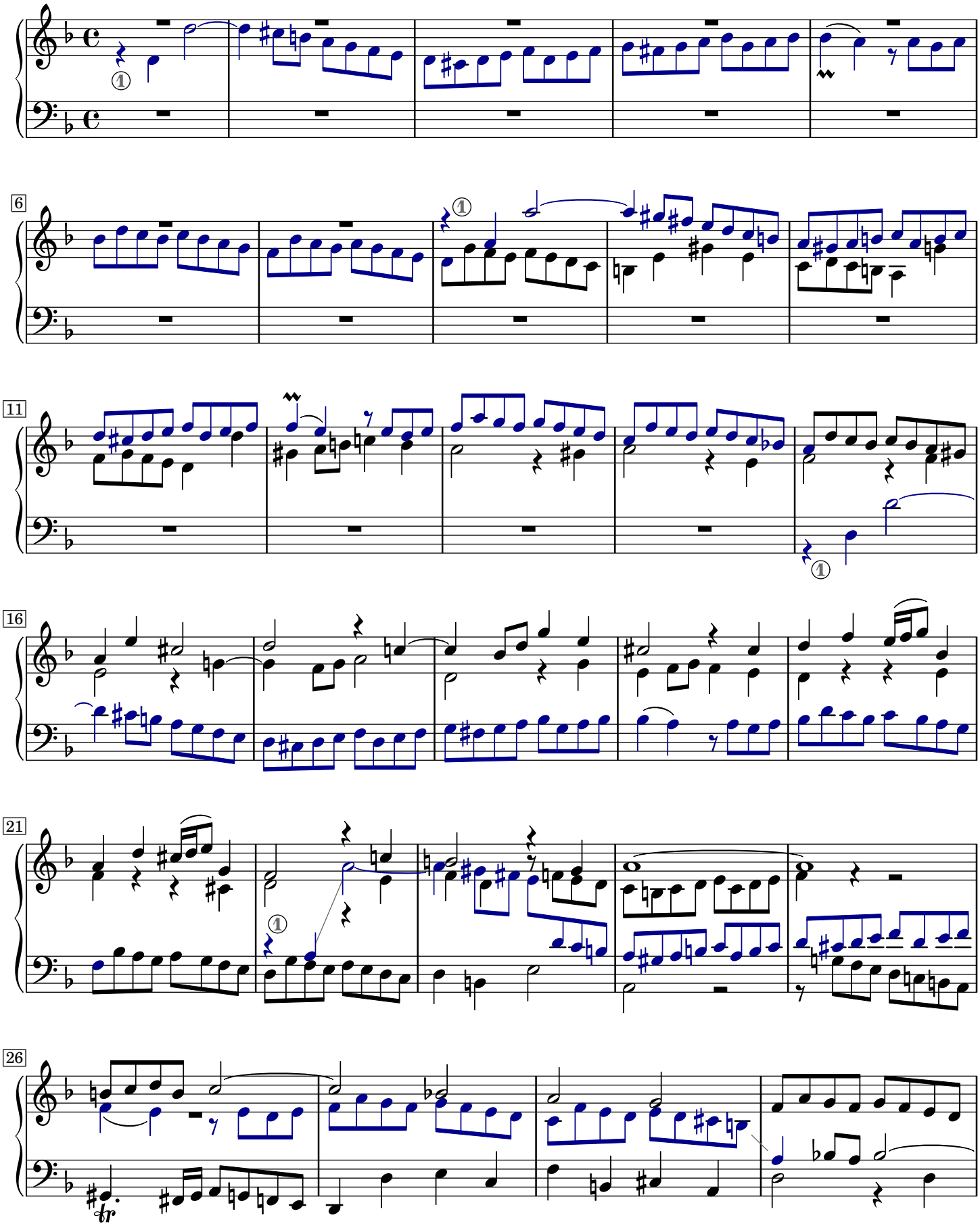
170
174
178
181
185
tr

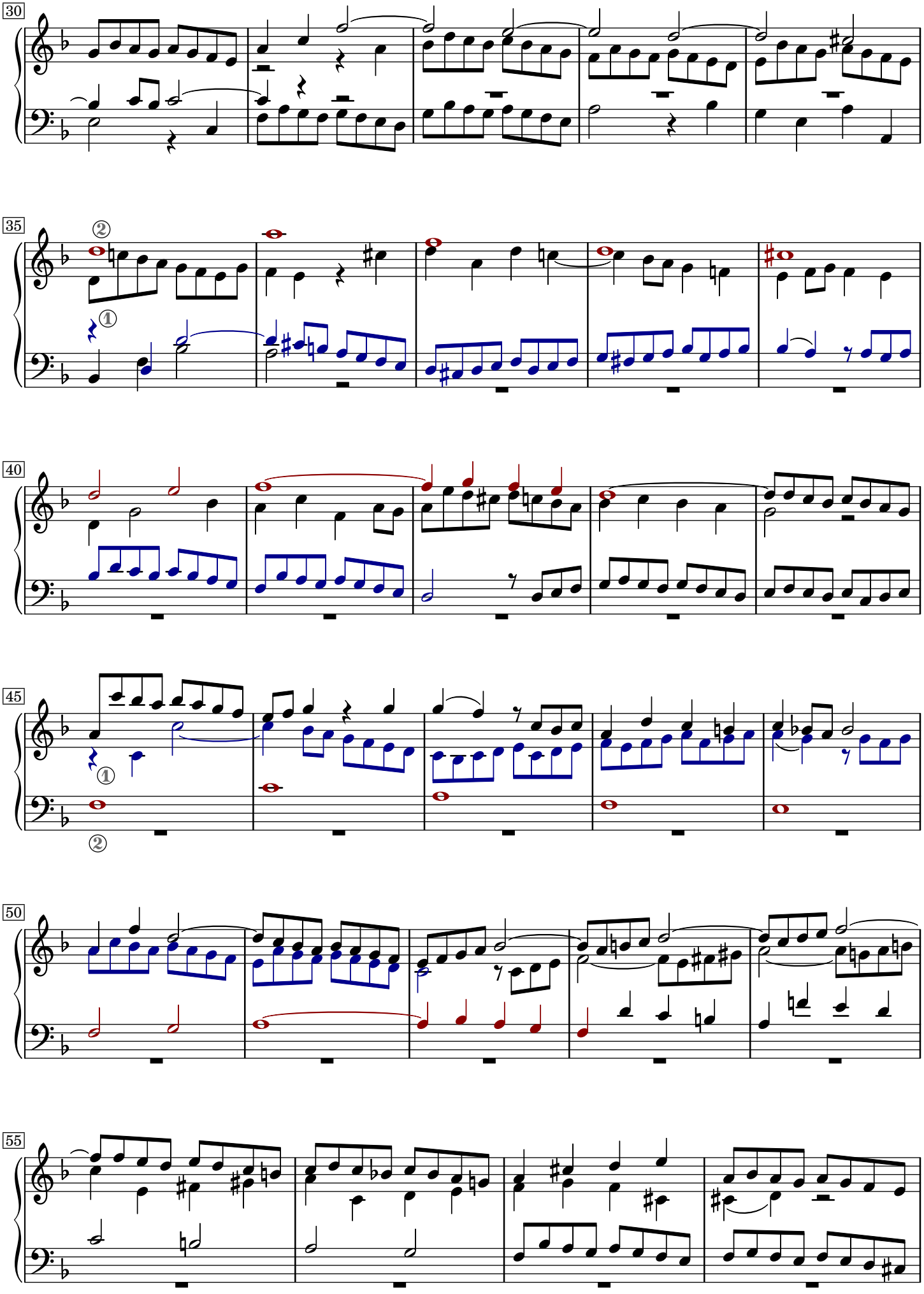
Contrapunctus IX

alla Duodecima

Exposition A1 S1 B1 T1 | 1 | S2 T1 | 2 | T2 A1 | 3 | A2 B1 | 4 | T2 A1 | 5 | B2 S1 | 6 | T2 A1 | 7 | A2 T1 | Coda

BWV 1080, 9

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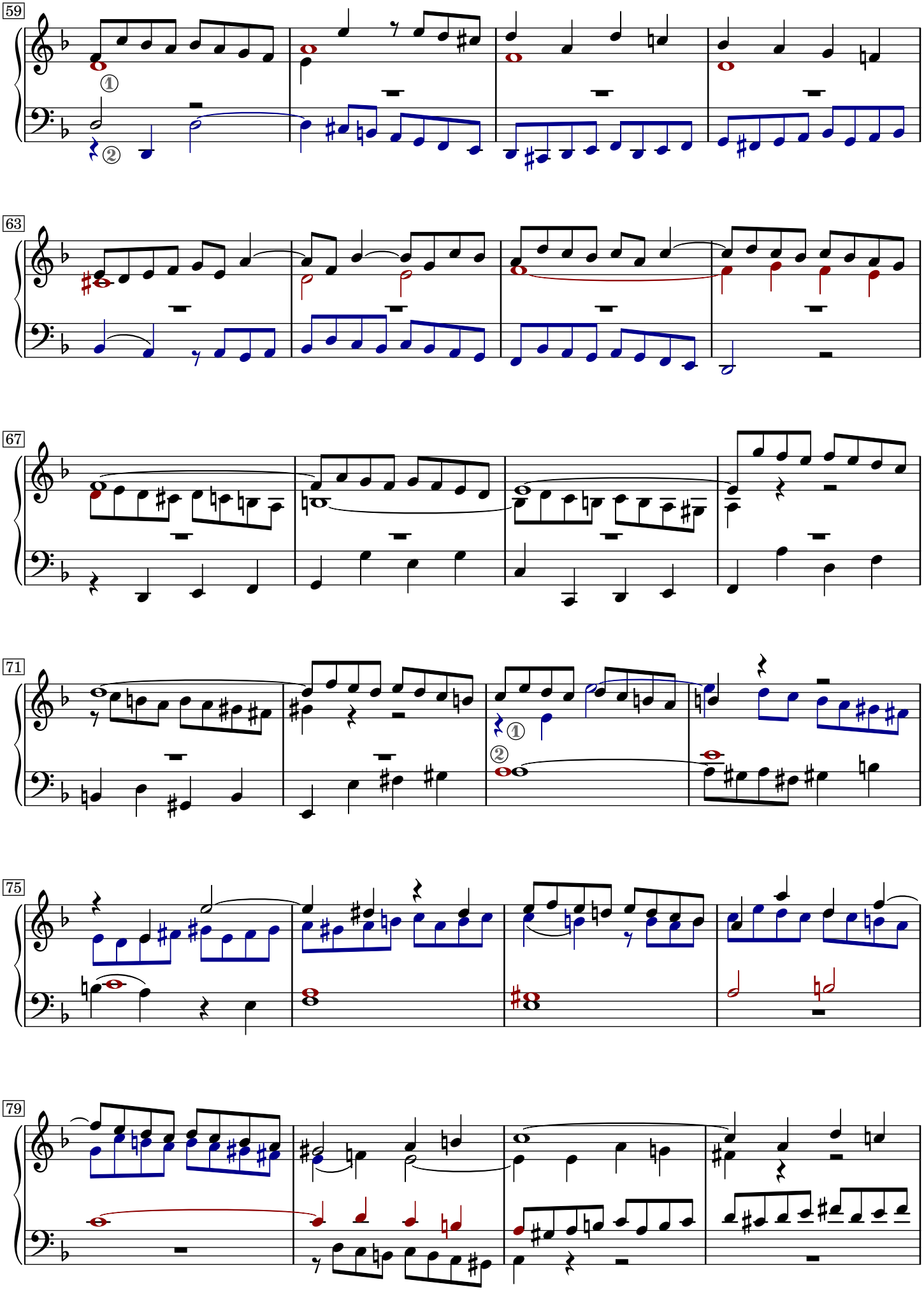
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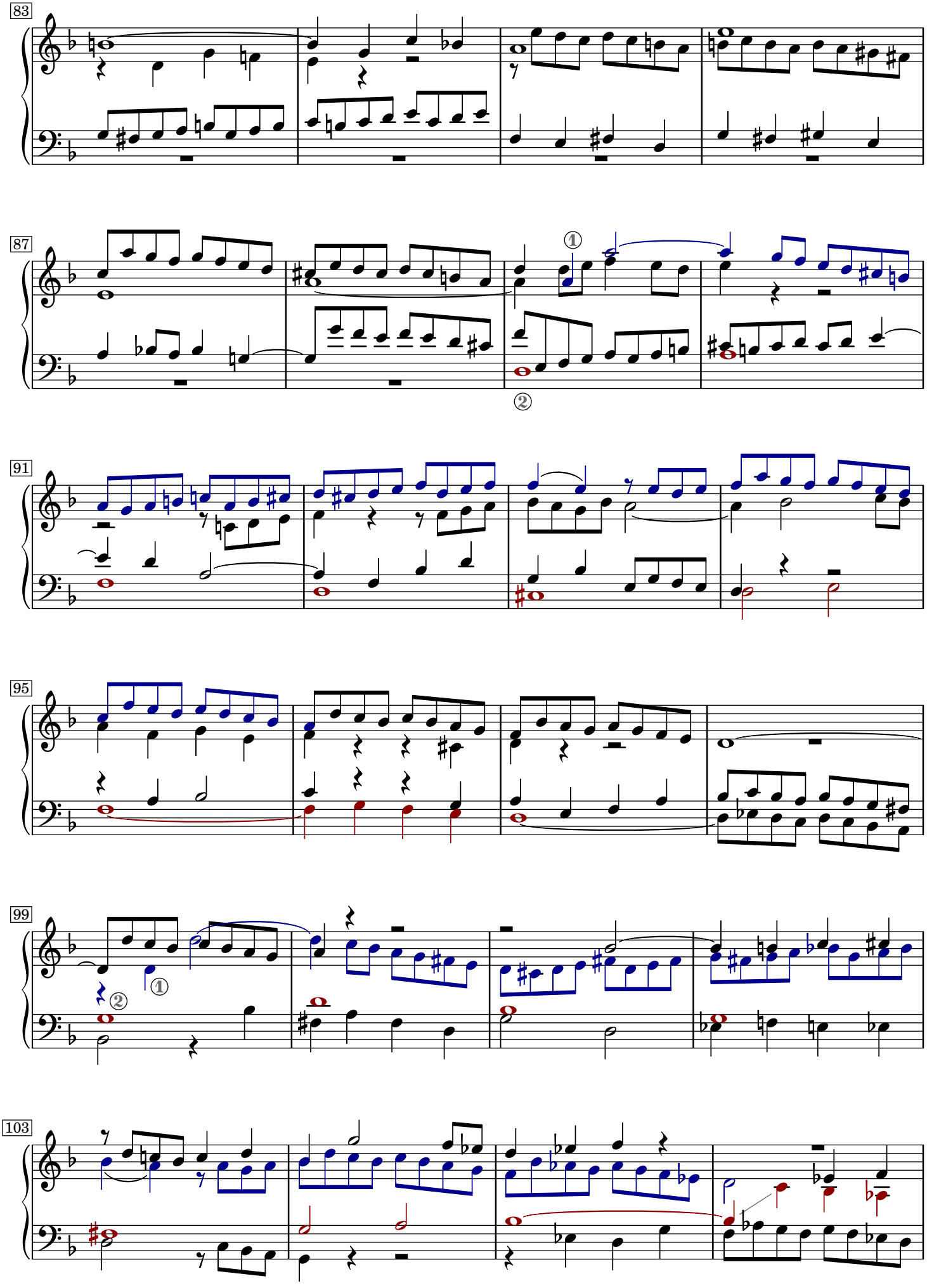
83
87
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103

107
111
115
120
125

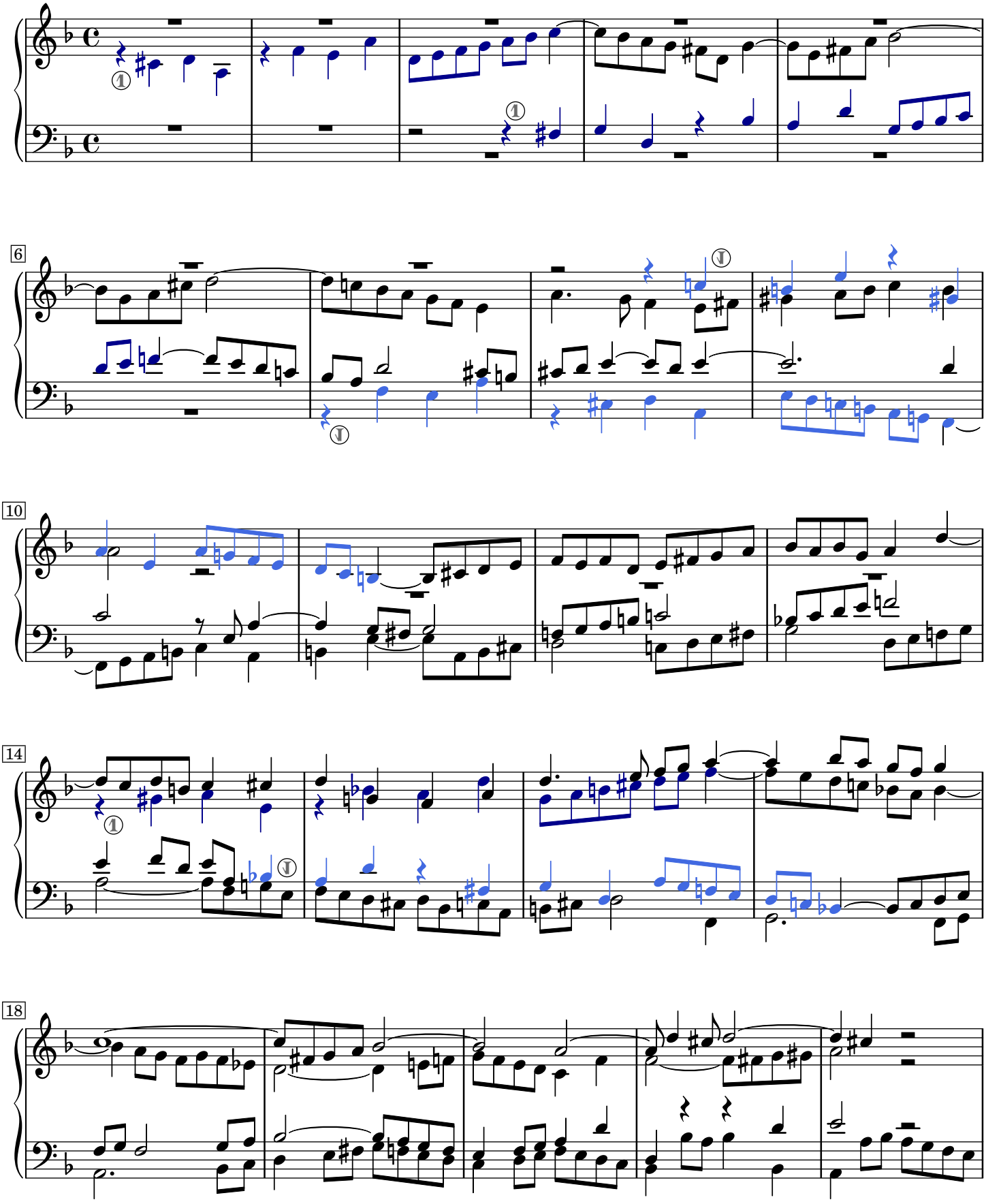
Contrapunctus X

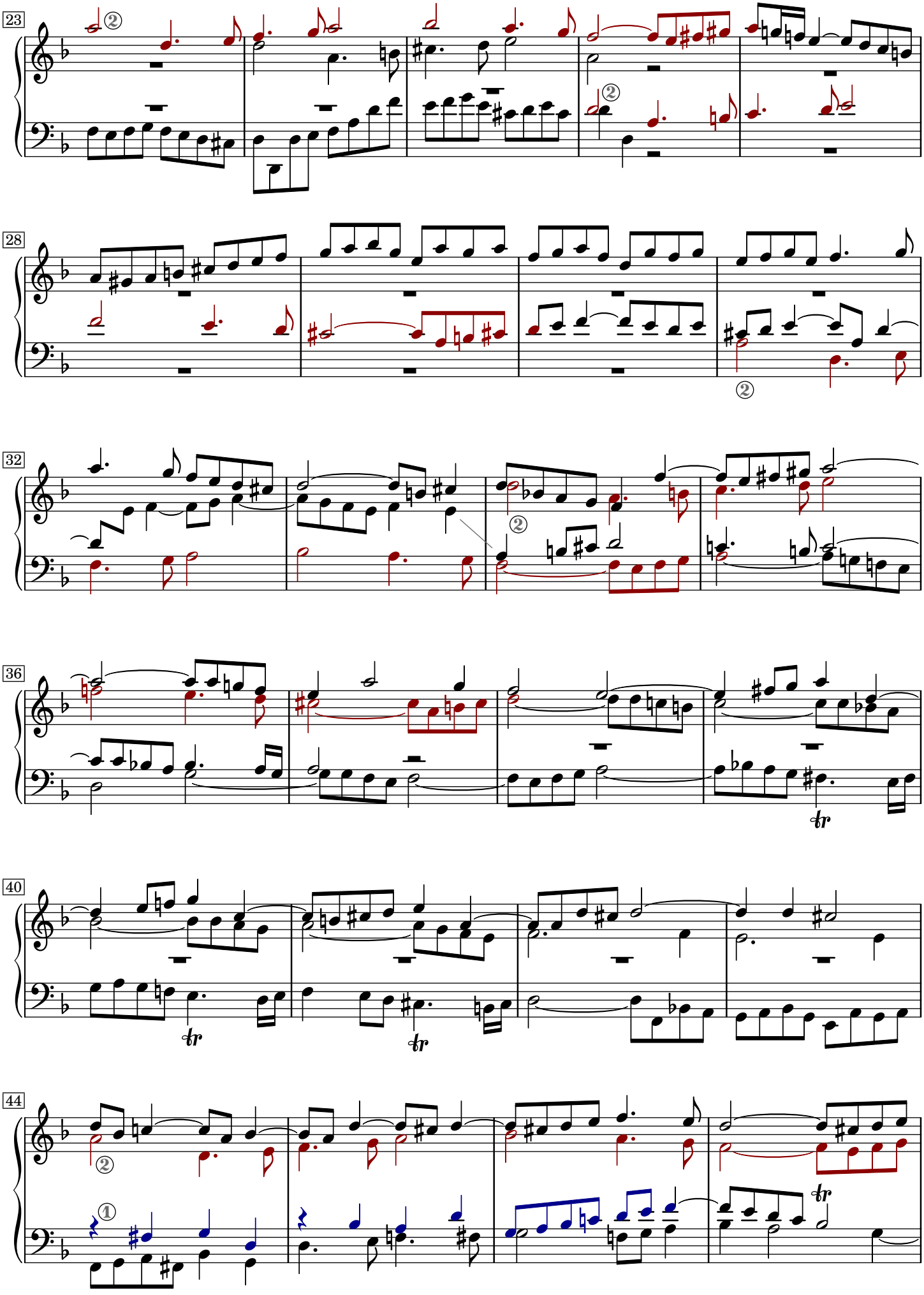
alla Decima

Exposition 1 A1 T1 B1 S1 | 1 | Stretto A1 T1 | Exposition 2 S2 T2 B3 A2 | 2

T1 A2 | 3 | A1 B2 | 4 | T2 S1 | 5 | S2 A1 B1 | 6 | B2 A1 S1 | 7 | B1 T1 S2 | 8 | T1 A1

BWV 1080, 10

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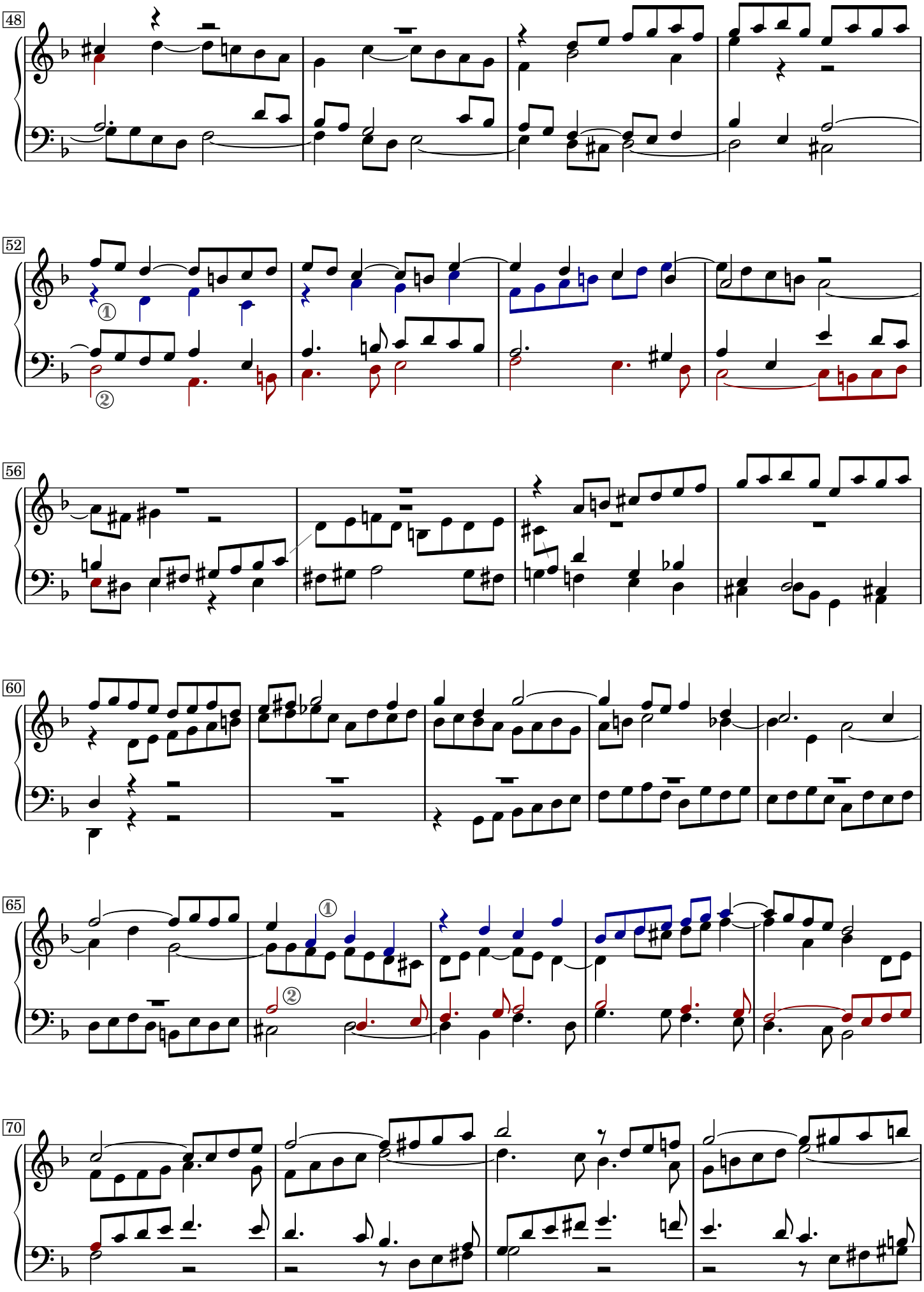
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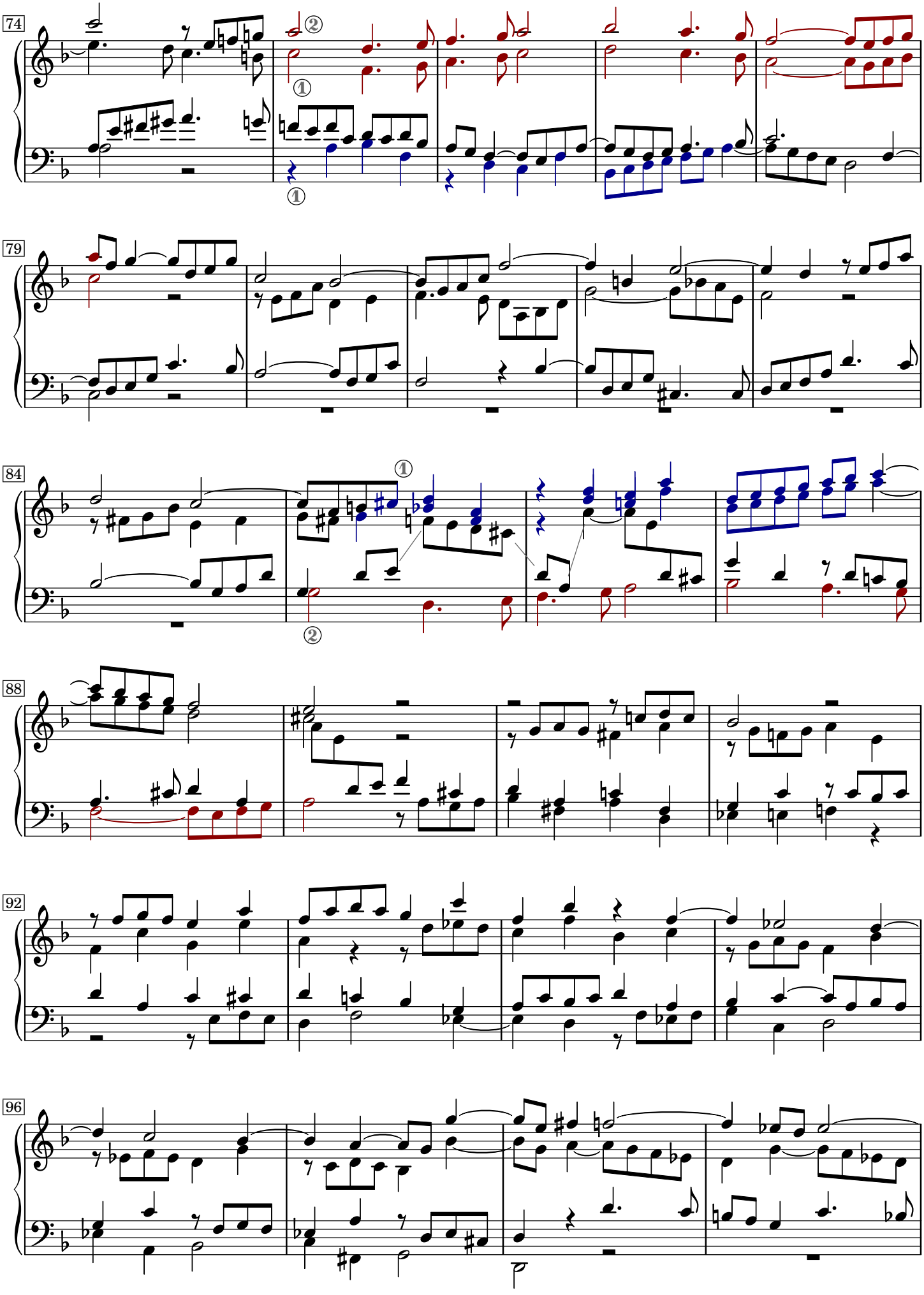
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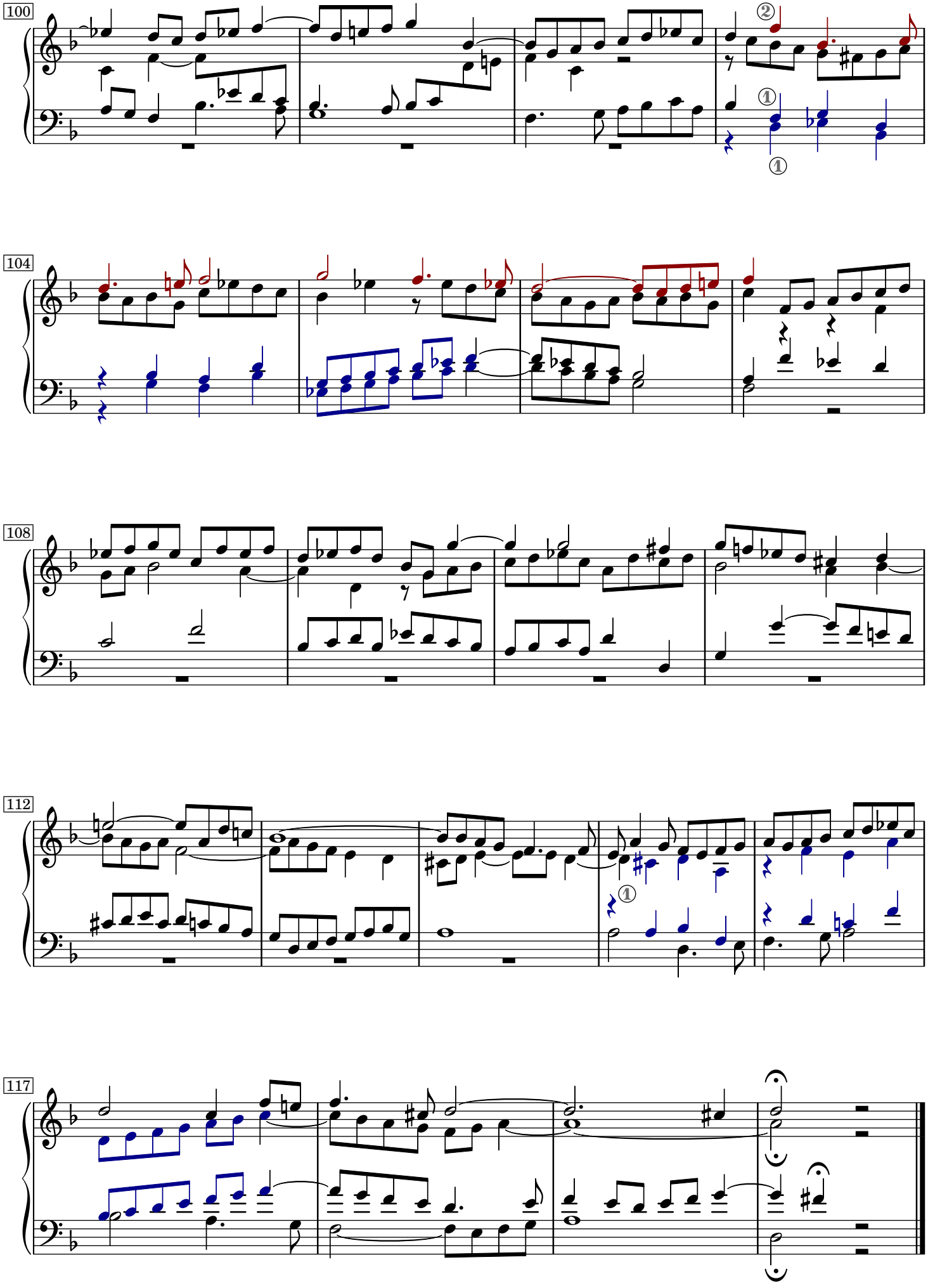
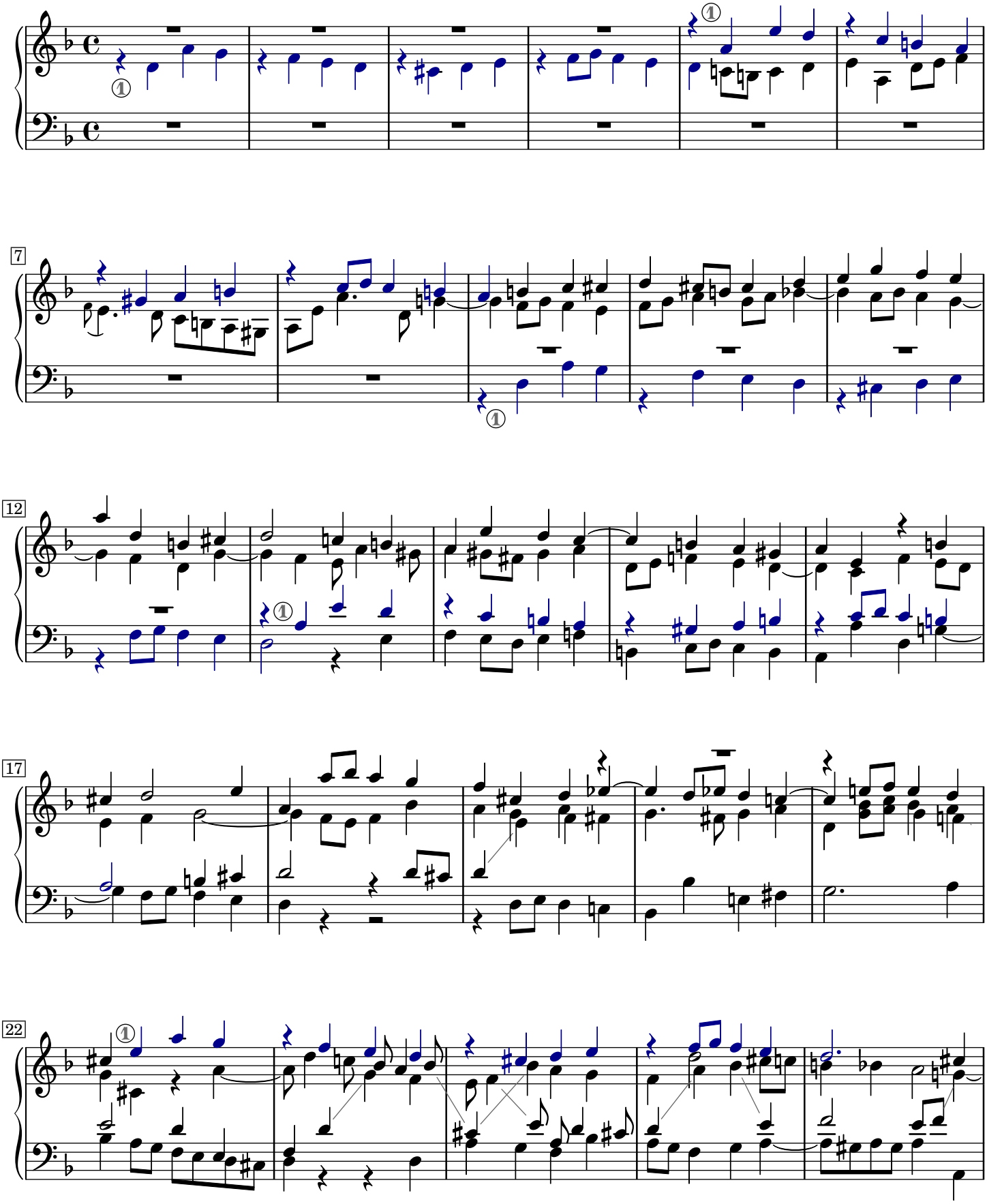
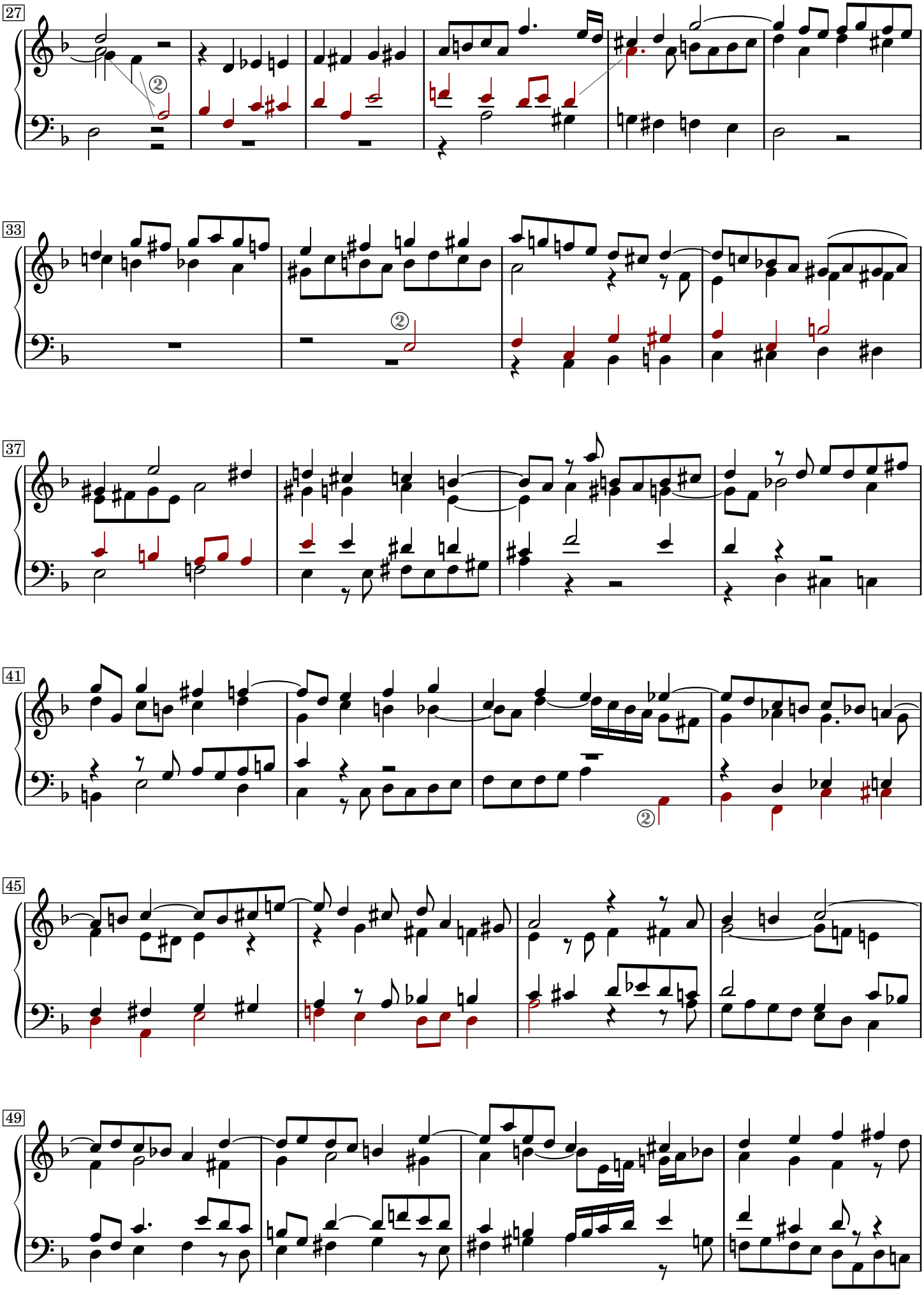

100
104
108
112
117

Contrapunctus XI

Exposition 1/a A1 S1 B1 T1 | 1 | S1 | Exposition 2 A2 T2 B2 S2 | 2 | B2
Exposition 1/b T1 S1 B1 A1 | B2 T3 | A2 S3 | A1 | A3 T2 | S2 A3 | 3 | B1 | T2 A3 | 4
A3 T1 S2 | 5 | A1 S1 | B1 T1 | 6 | B2 T3 A1 | B3 T2 S2

BWV 1080, 11

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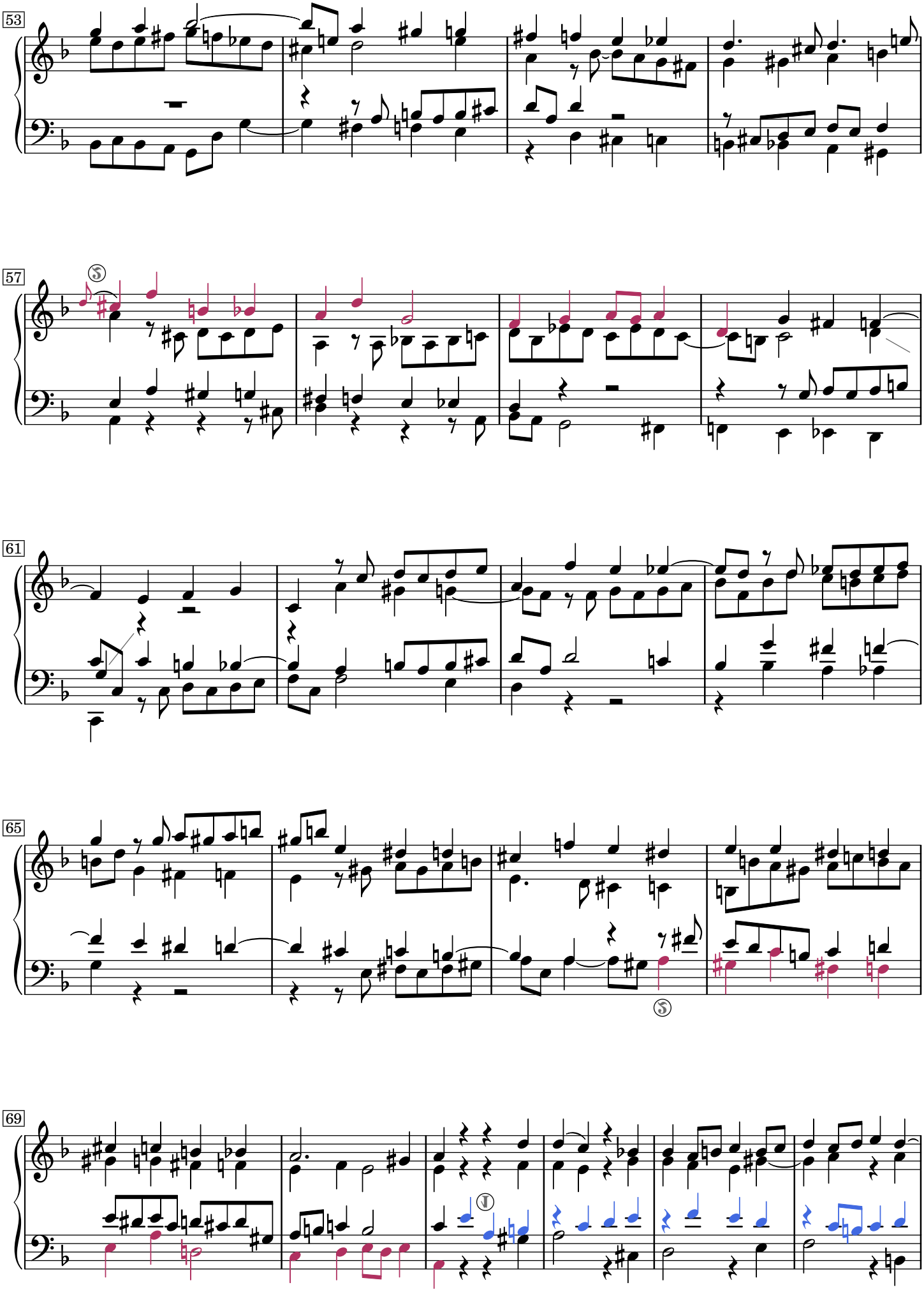
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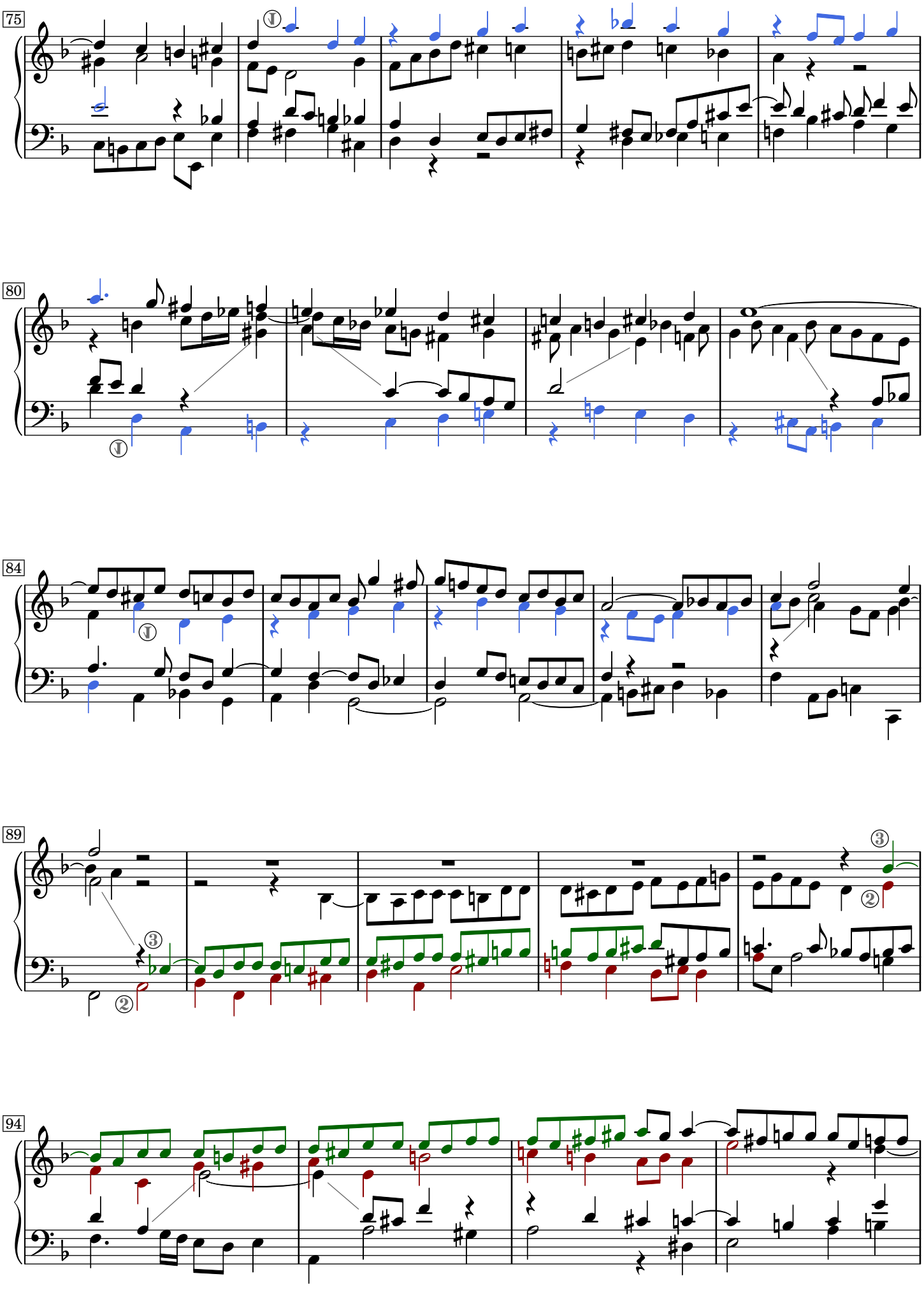
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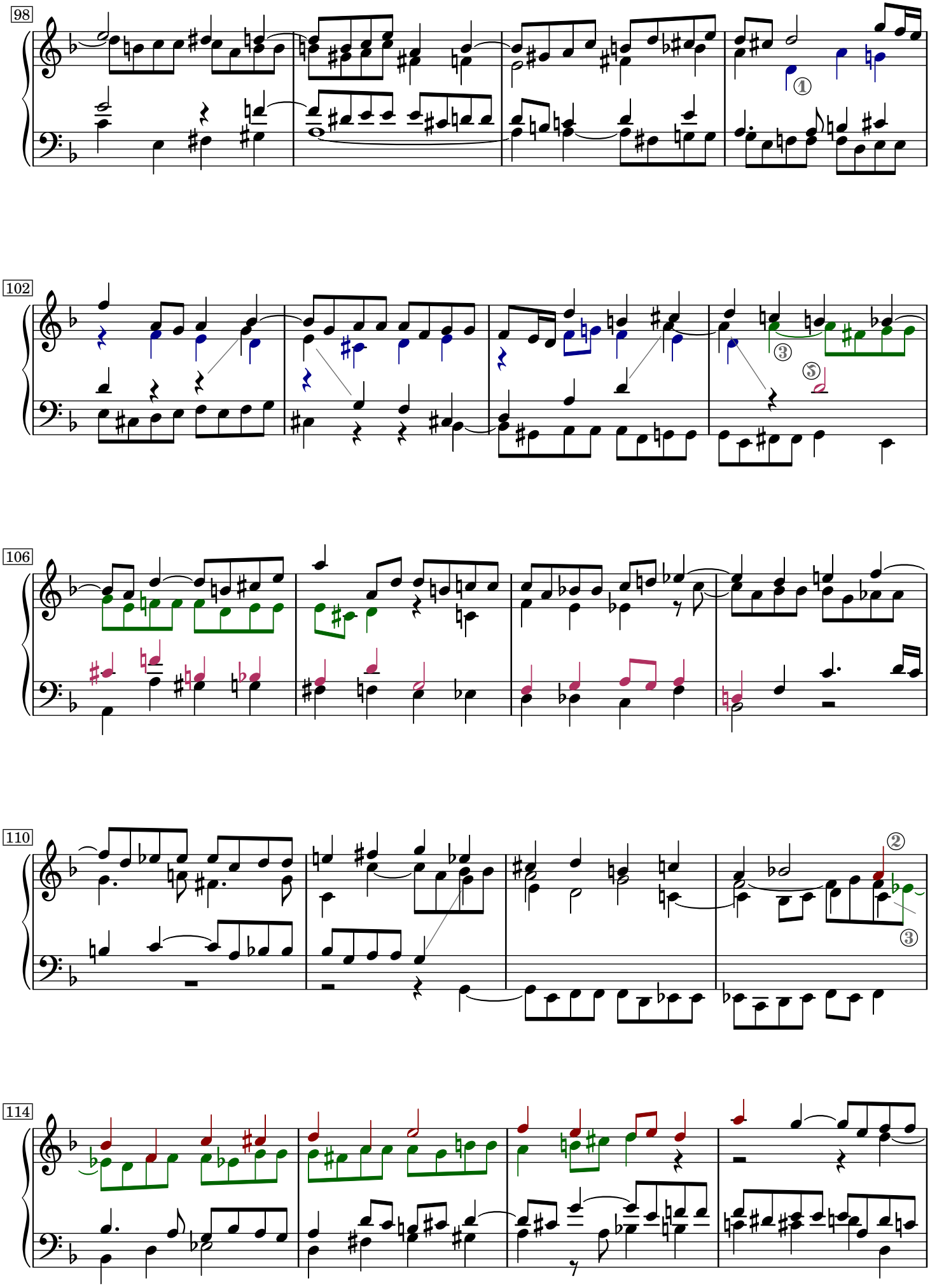
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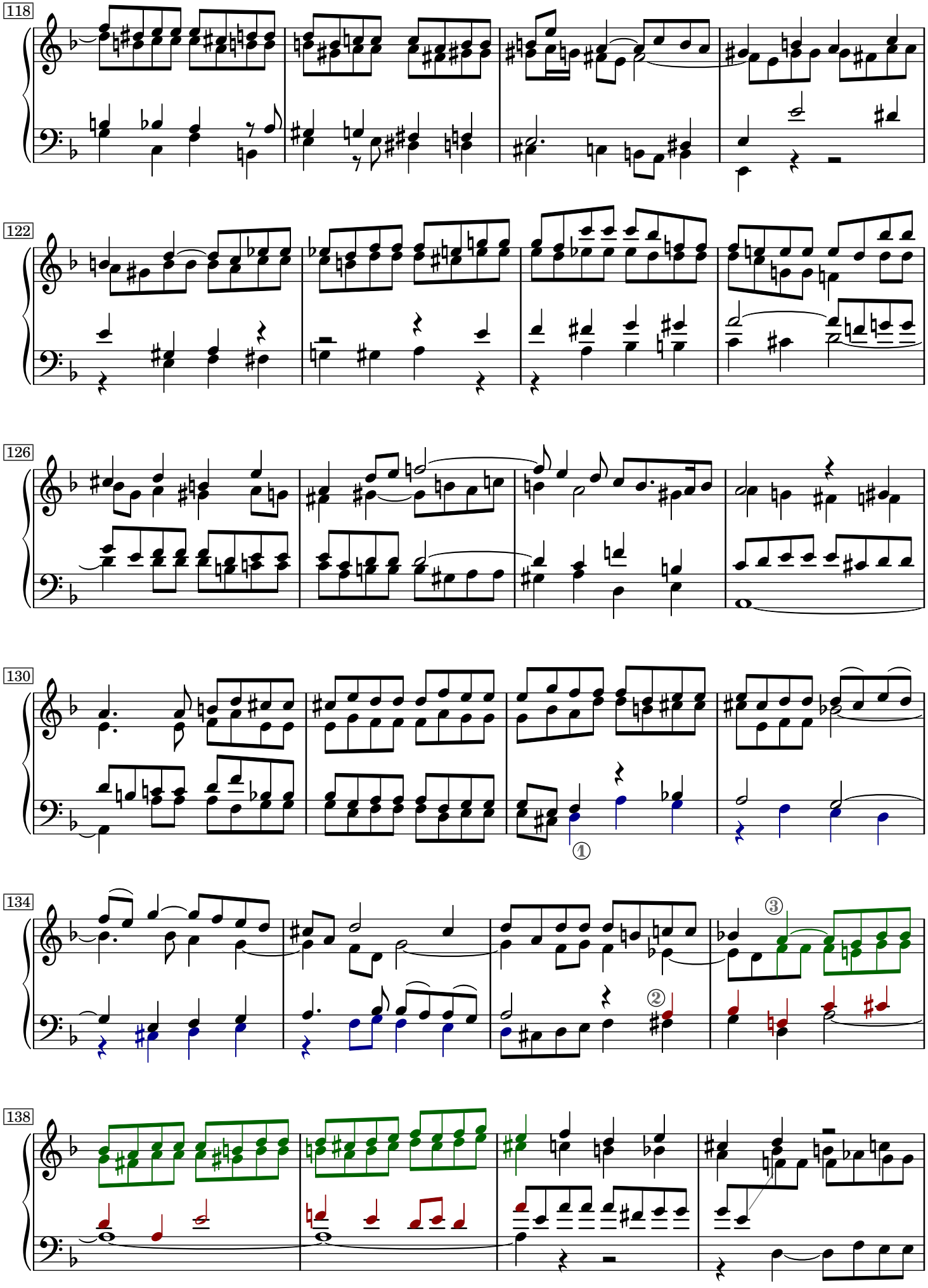
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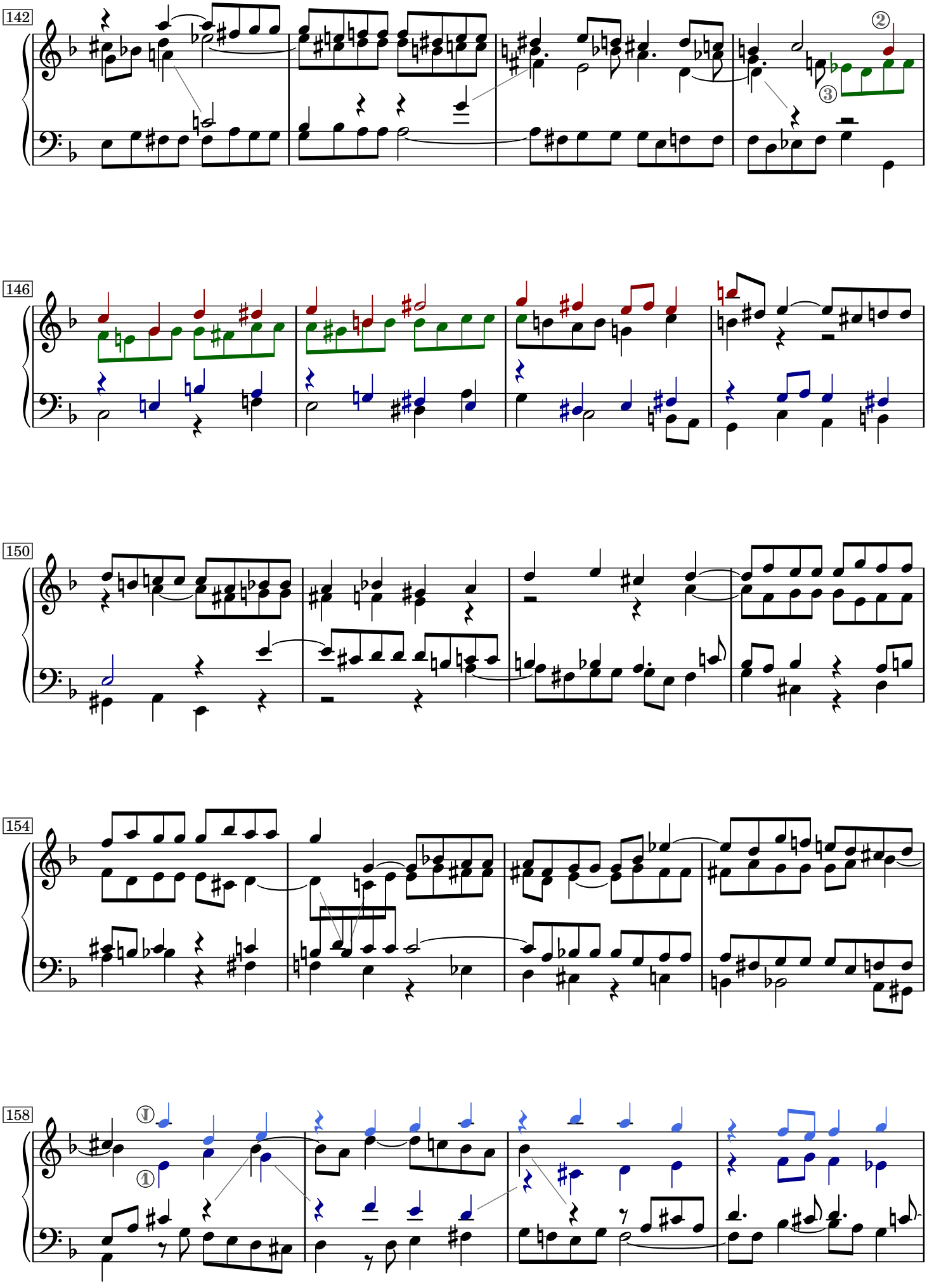

142
146
150
154
158

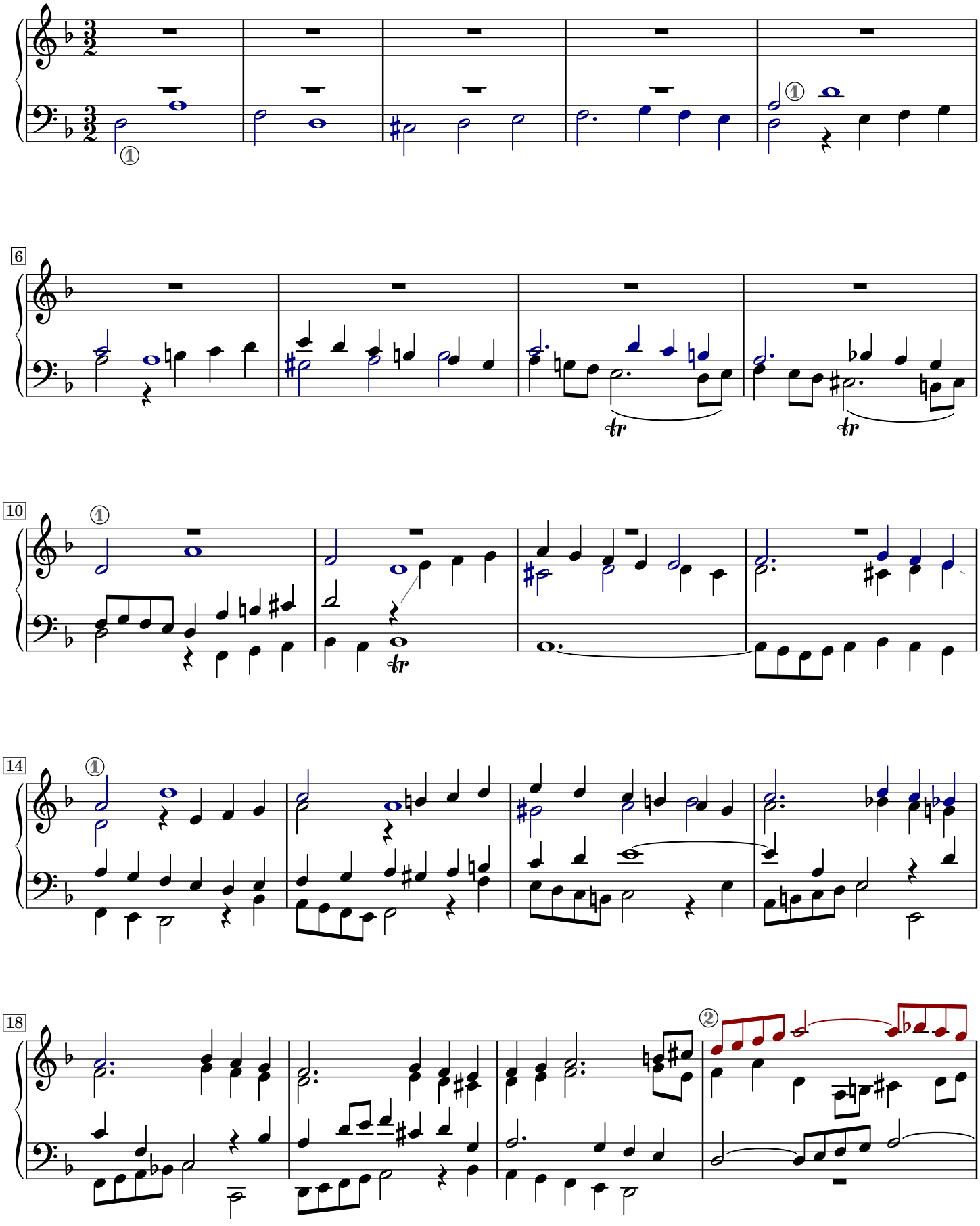
162
166
170
175
180

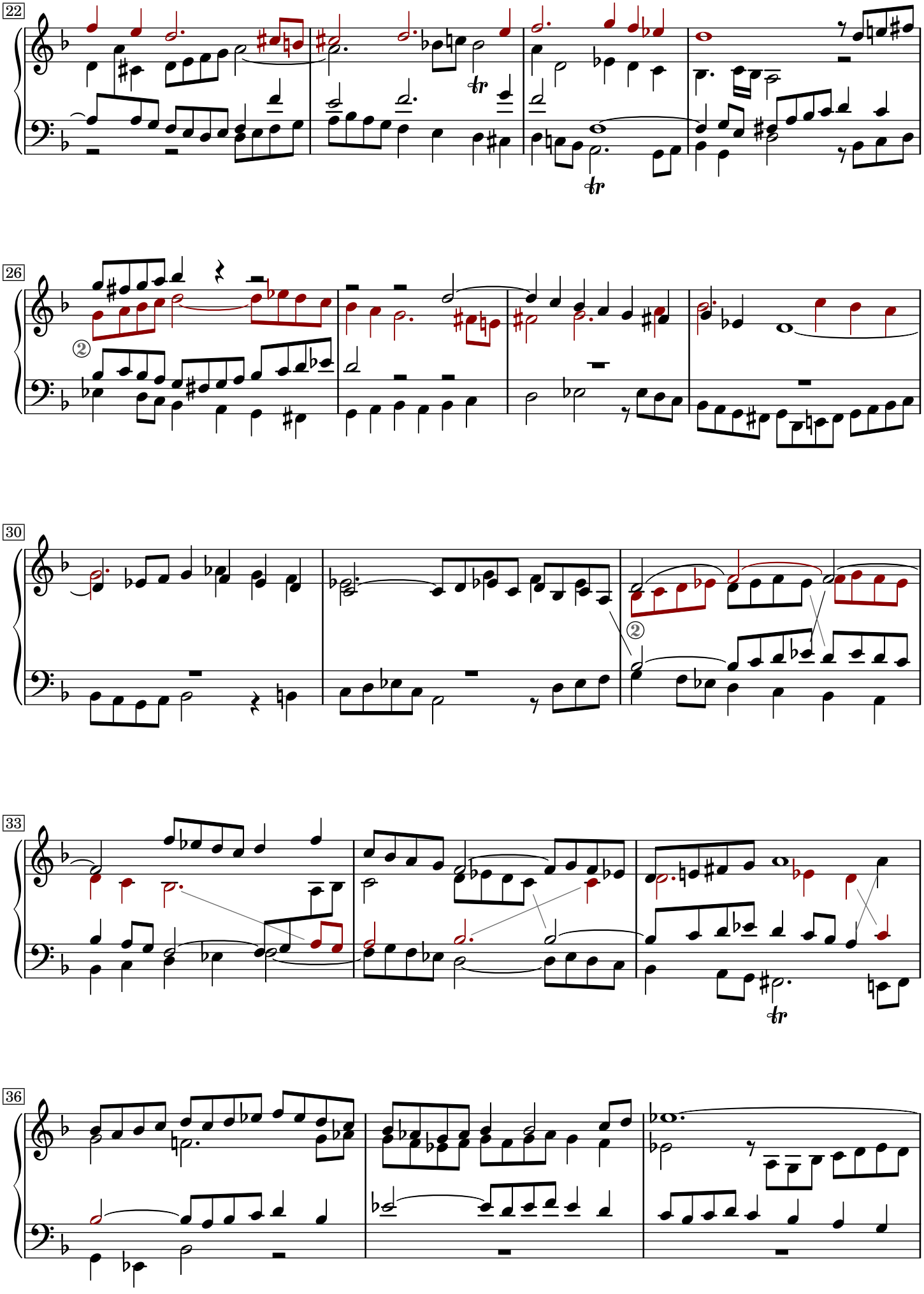
Contrapunctus [12]

[rectus]

Exposition 1 B1 T1 A1 S1 | 1 | Exposition 2 S2 A2 T2 B2 | 2 | A2 | S2 | Coda

BWV 1080, 12/1

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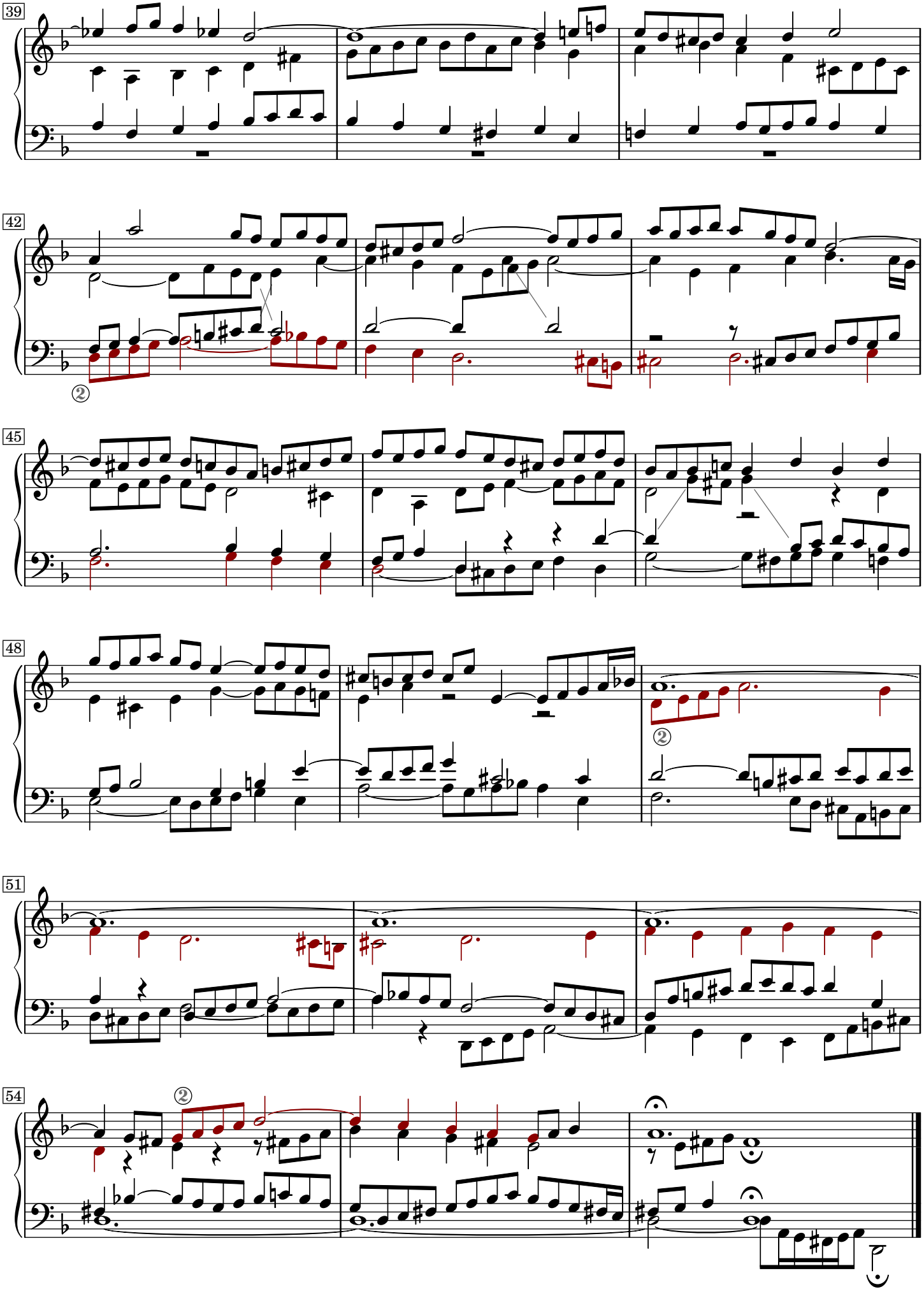
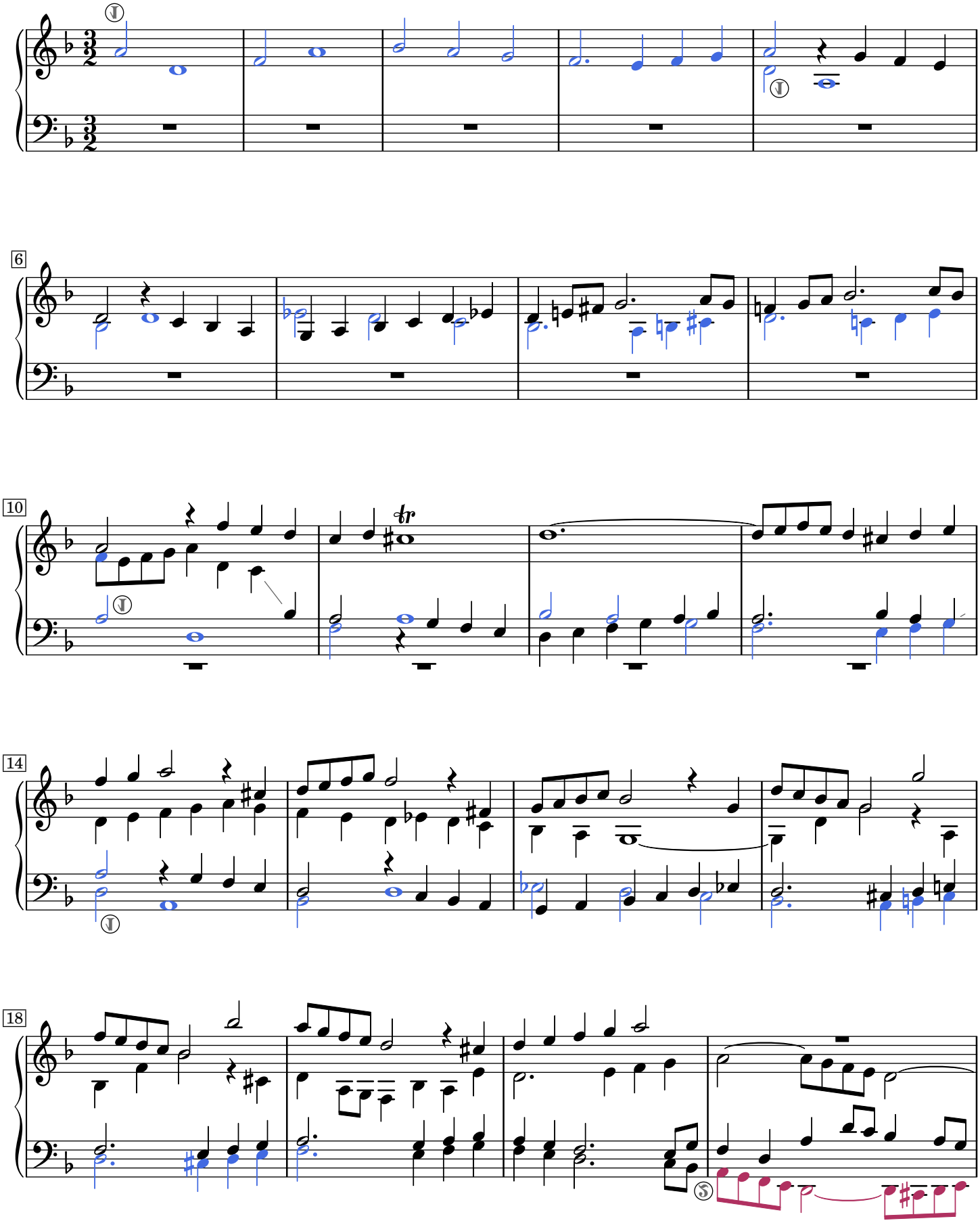

39
42
②
45
48
②
51
54
②

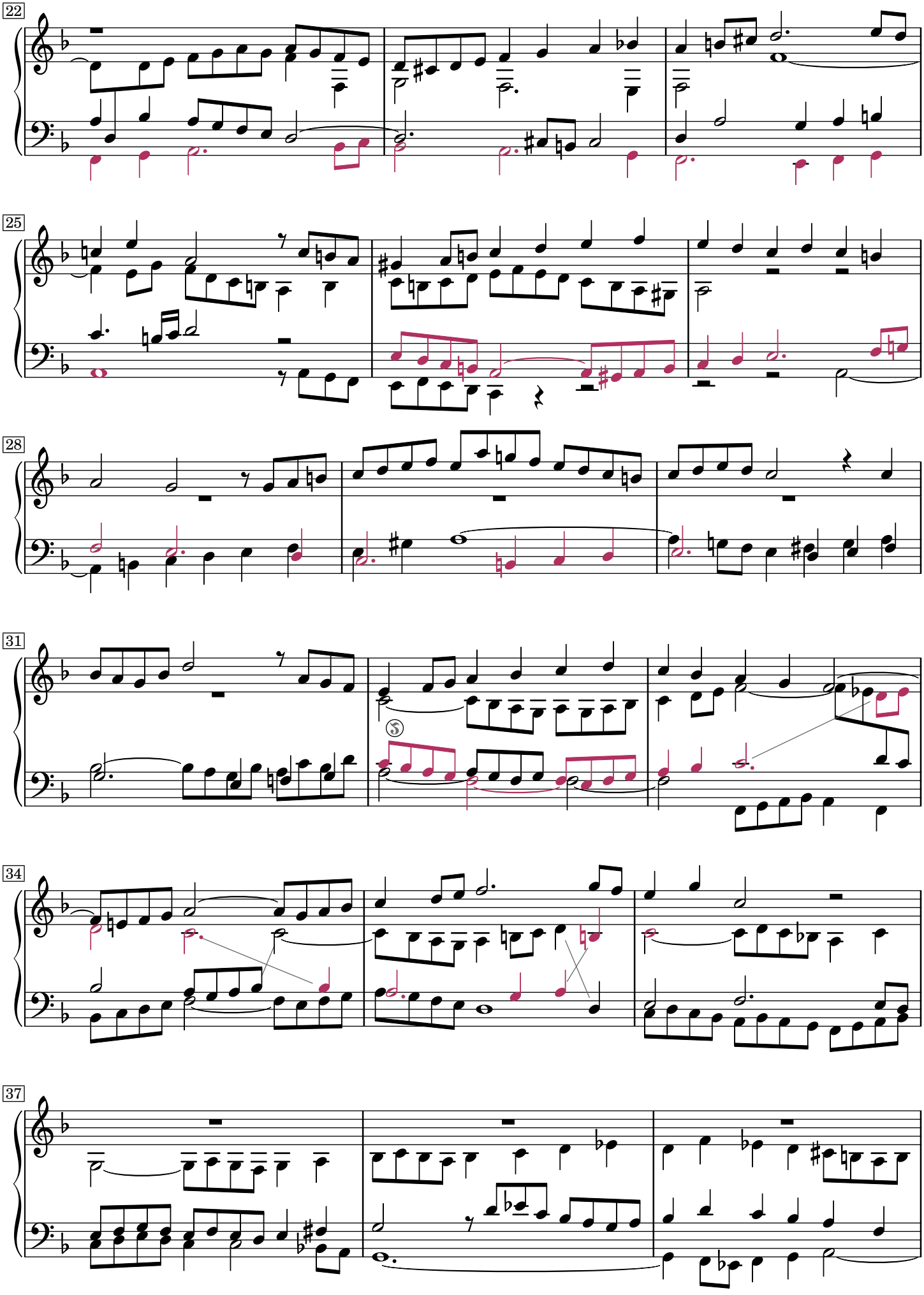
Contrapunctus XII

inversus

Exposition 1 S1 A1 T1 B1 | 1 | Exposition 2 B2 T2 A2 S2 | 2 | T2 | B2 | Coda

BWV 1080, 12/2

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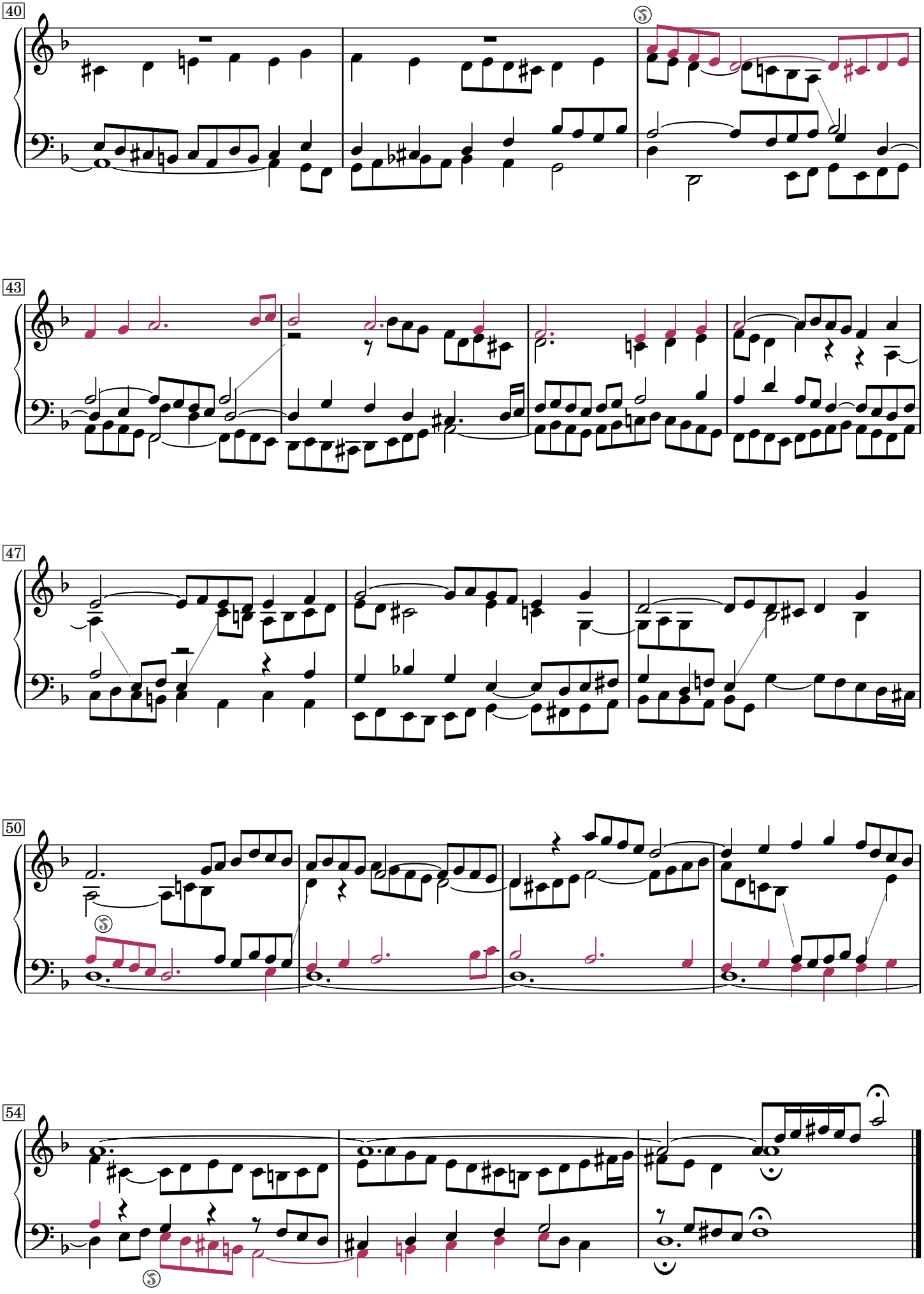
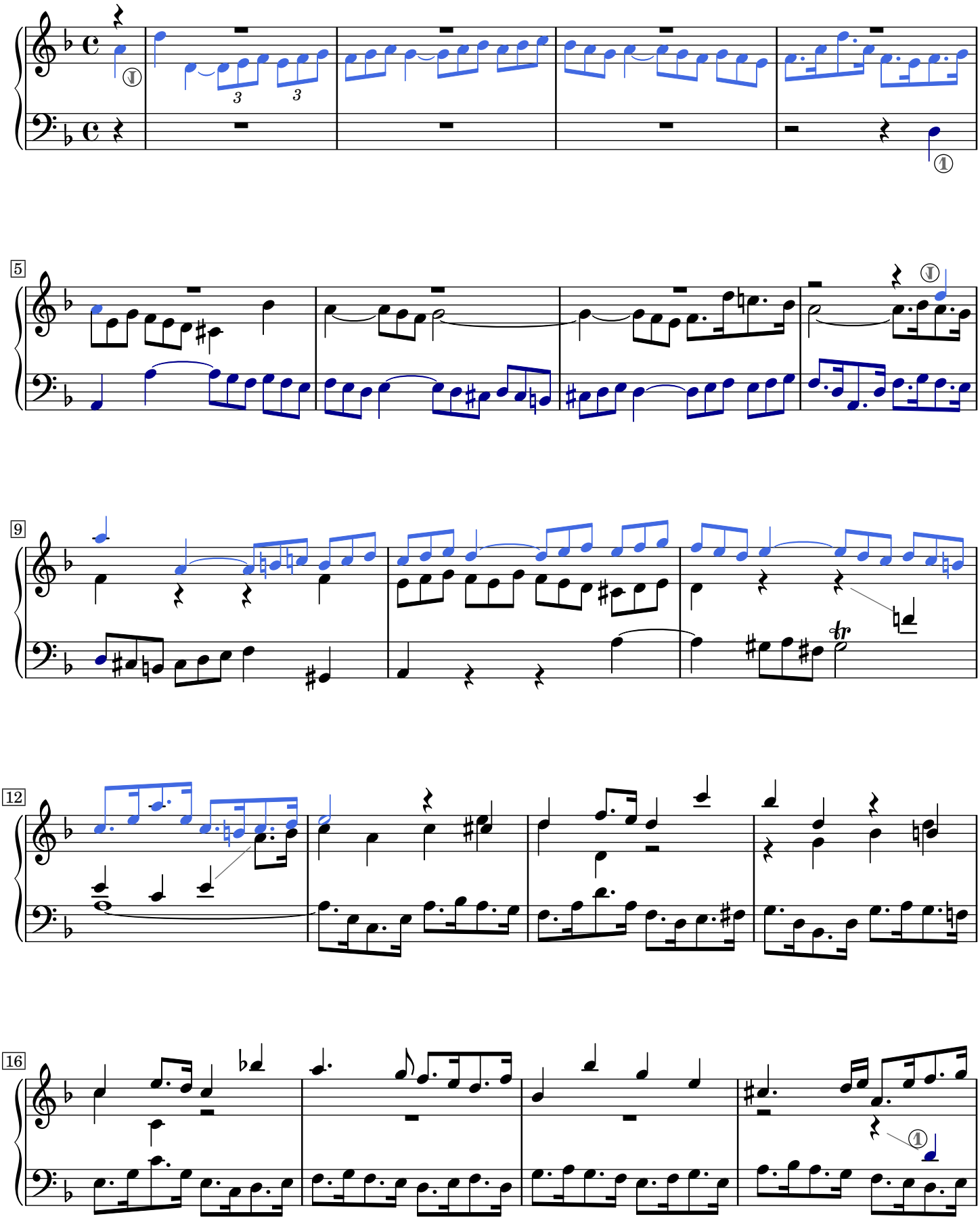

40
43
47
50
54

Contrapunctus [XIII]

[rectus]

Exposition 1 A B S | 1 | A | 2 | Exposition 2 B S A | 3 | A | 4 | Final presentation B S

BWV 1080, 13/1



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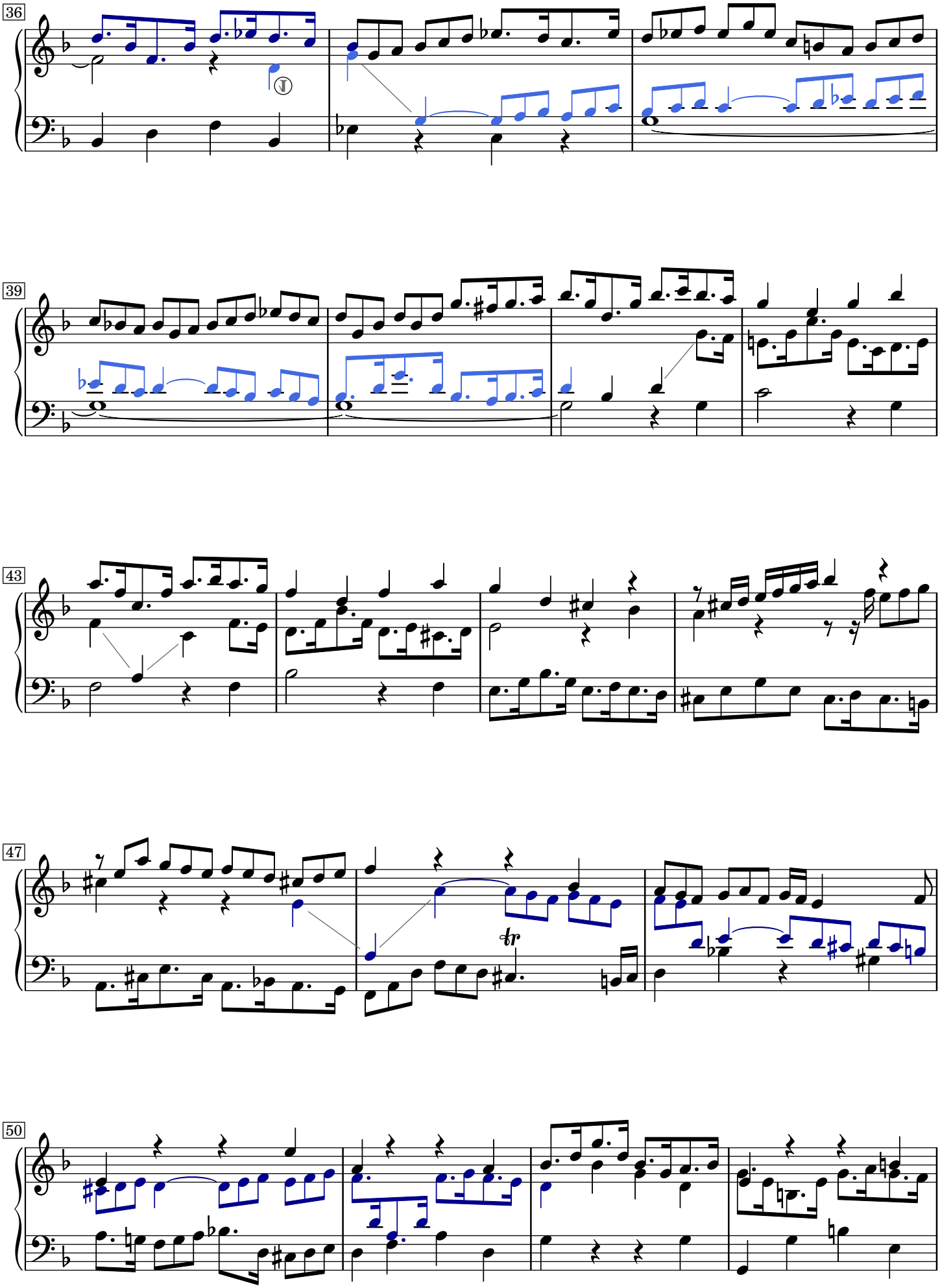
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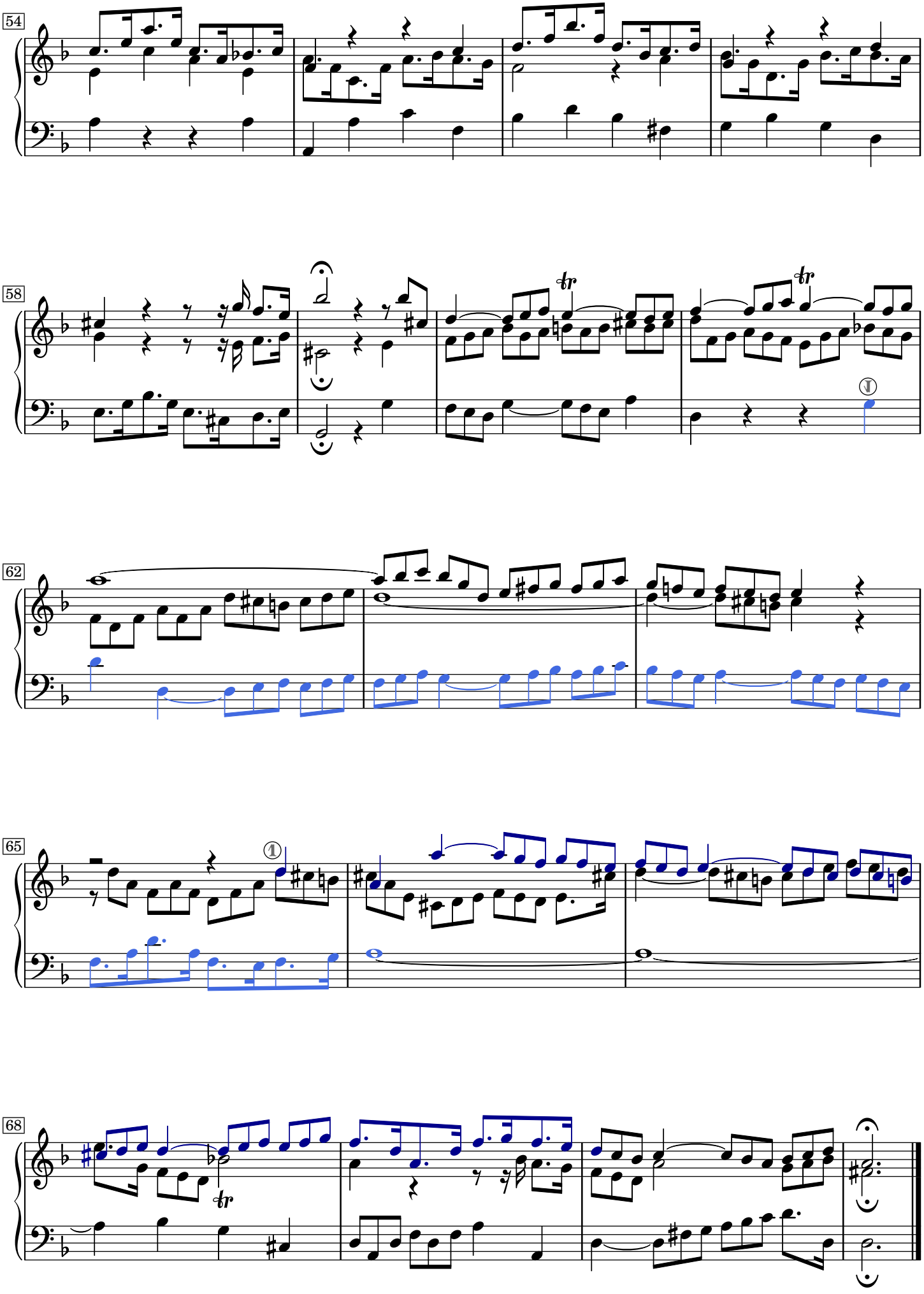
47
50

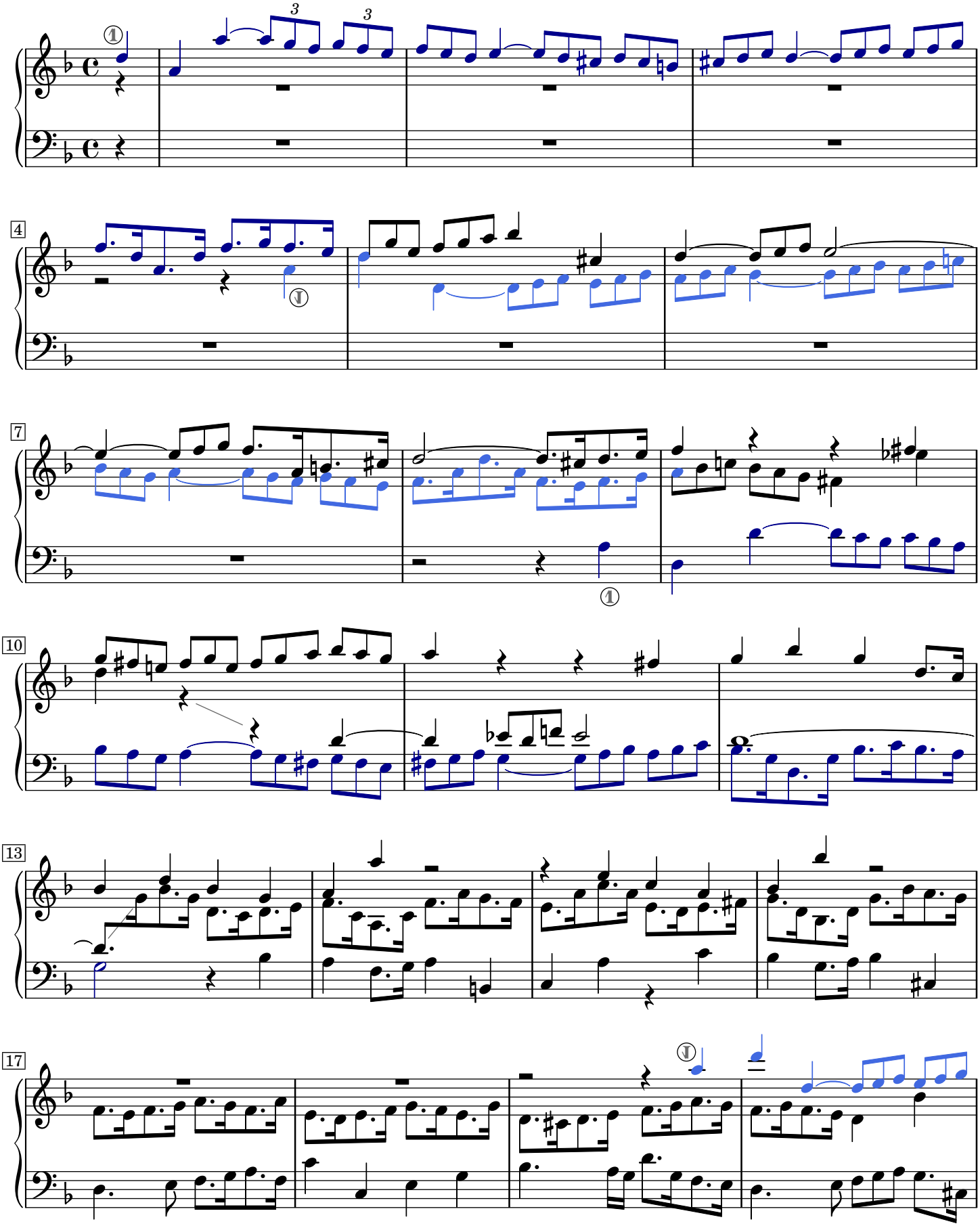

54
58
62
65
68

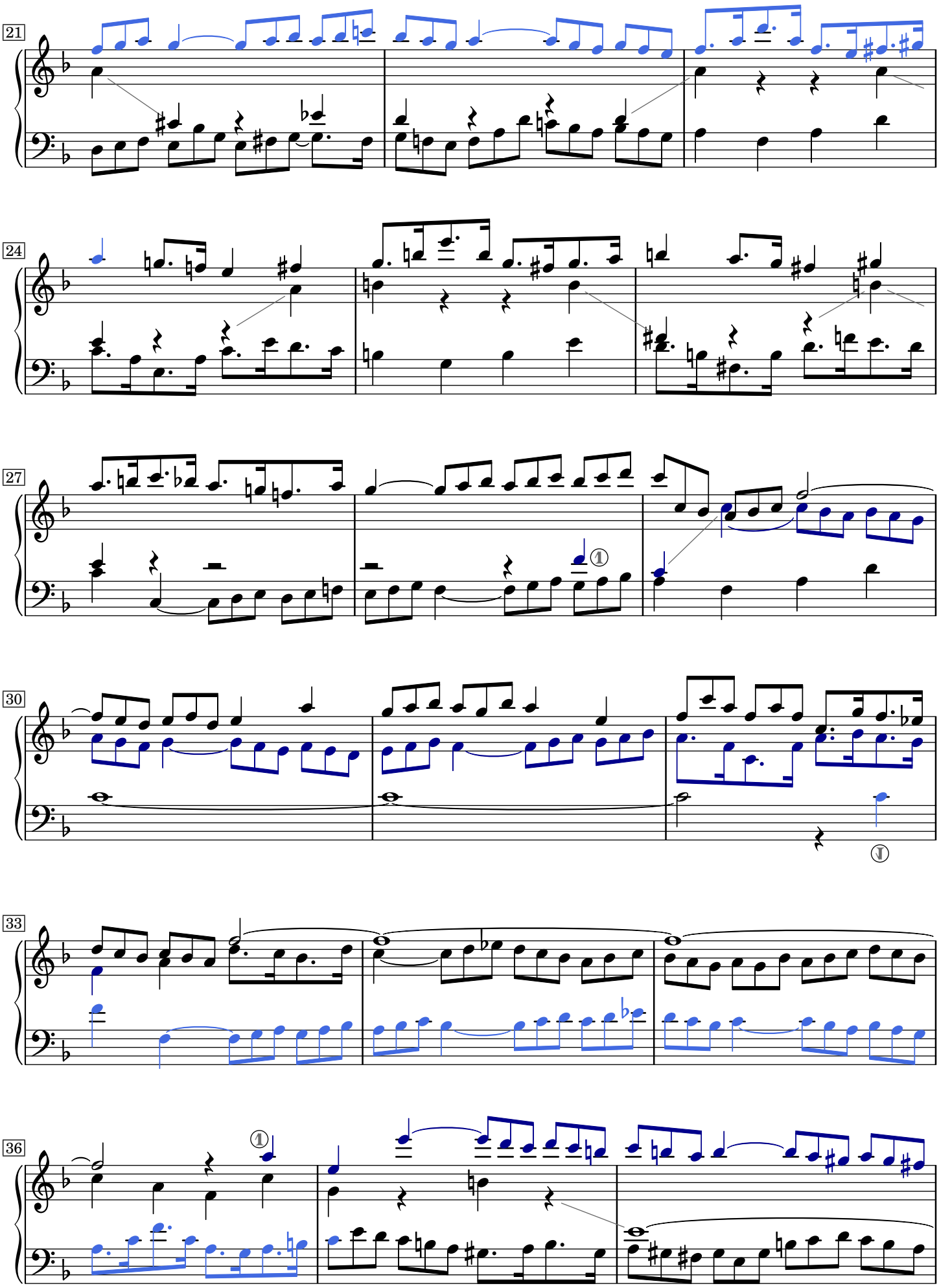
Contrapunctus [XIII]

inversus

Exposition 1 S A B | 1 | S | 2 | Exposition 2 A B S | 3 | S | 4 | Final presentation A B

BWV 1080, 13/2

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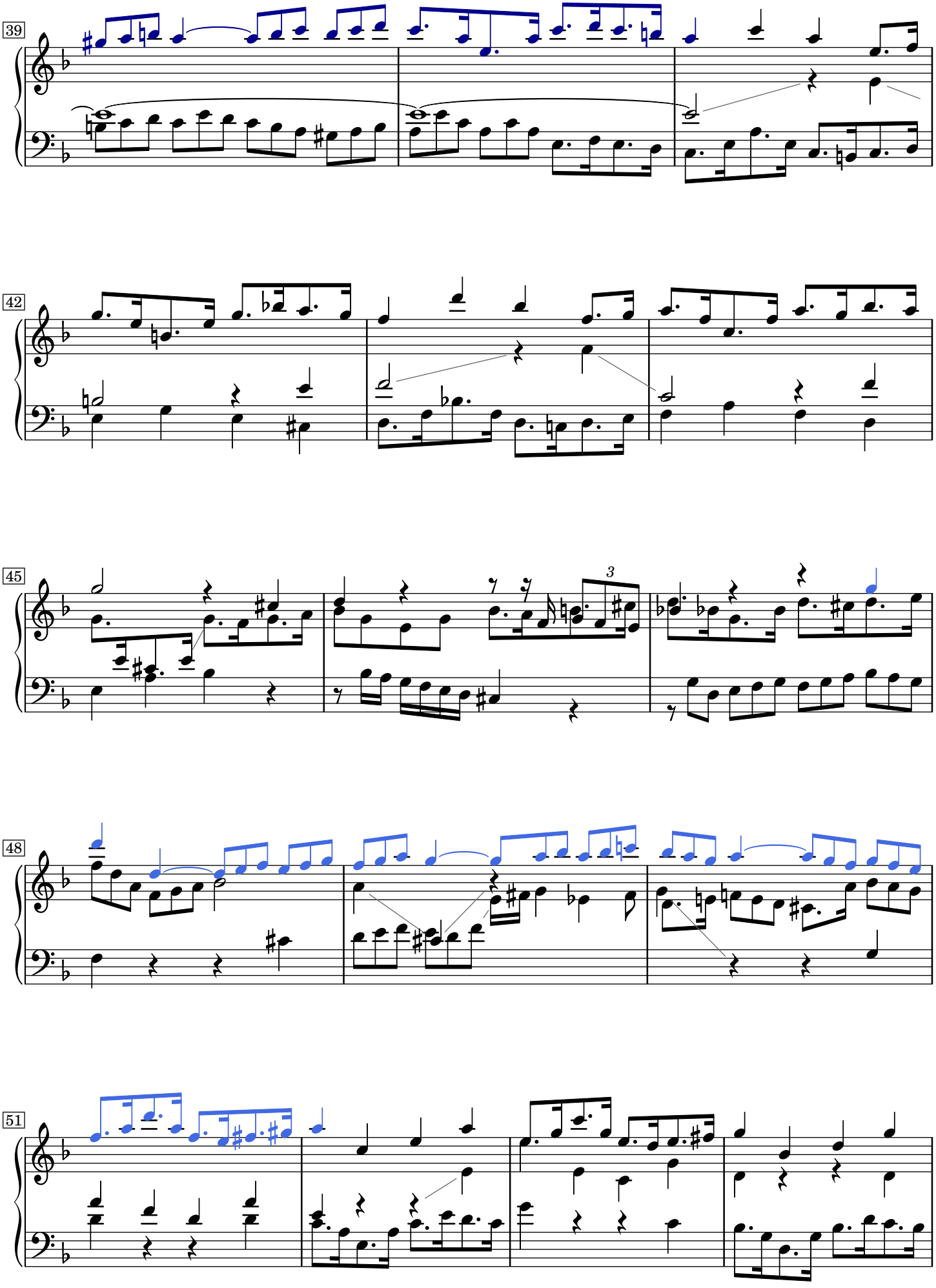
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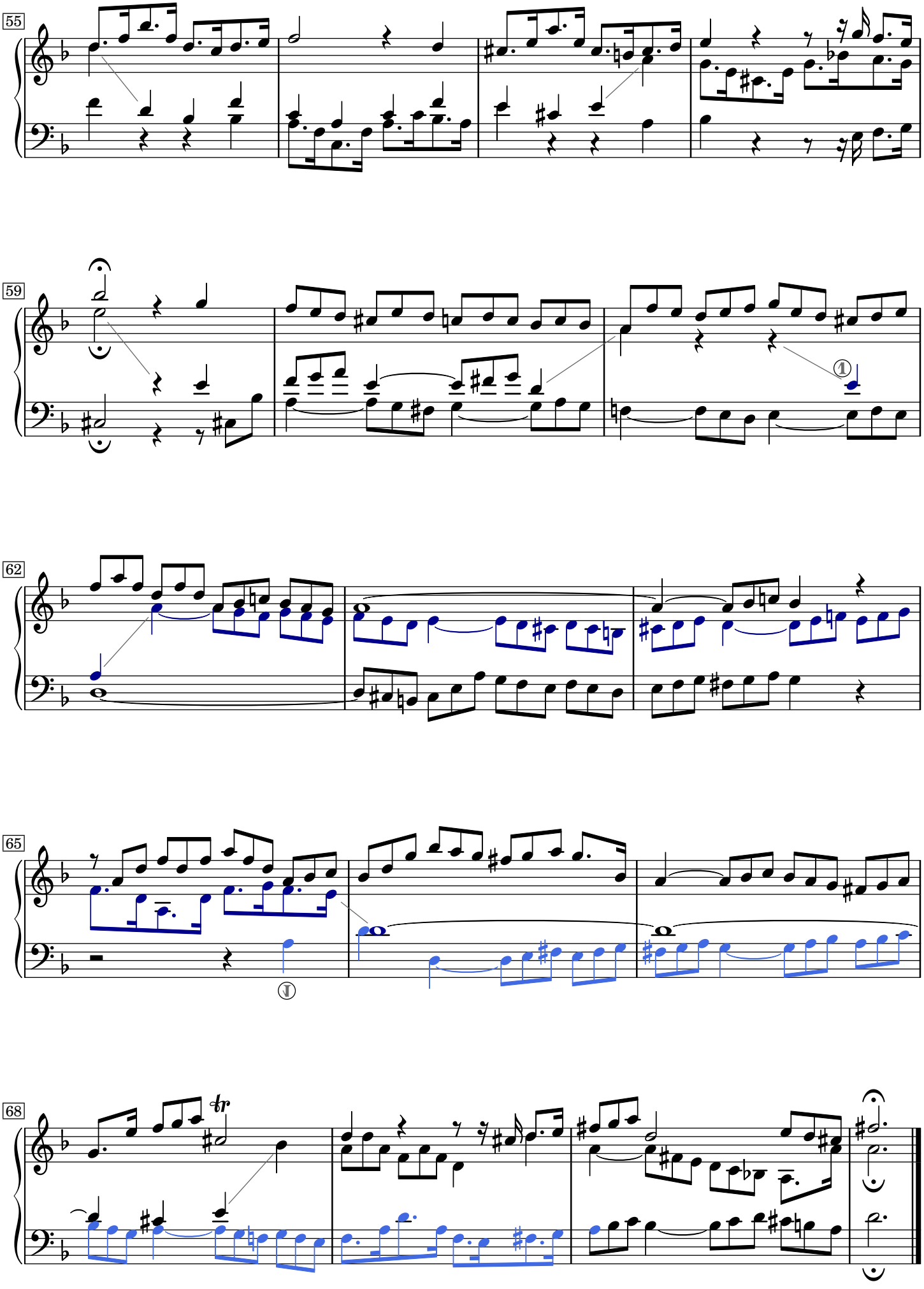
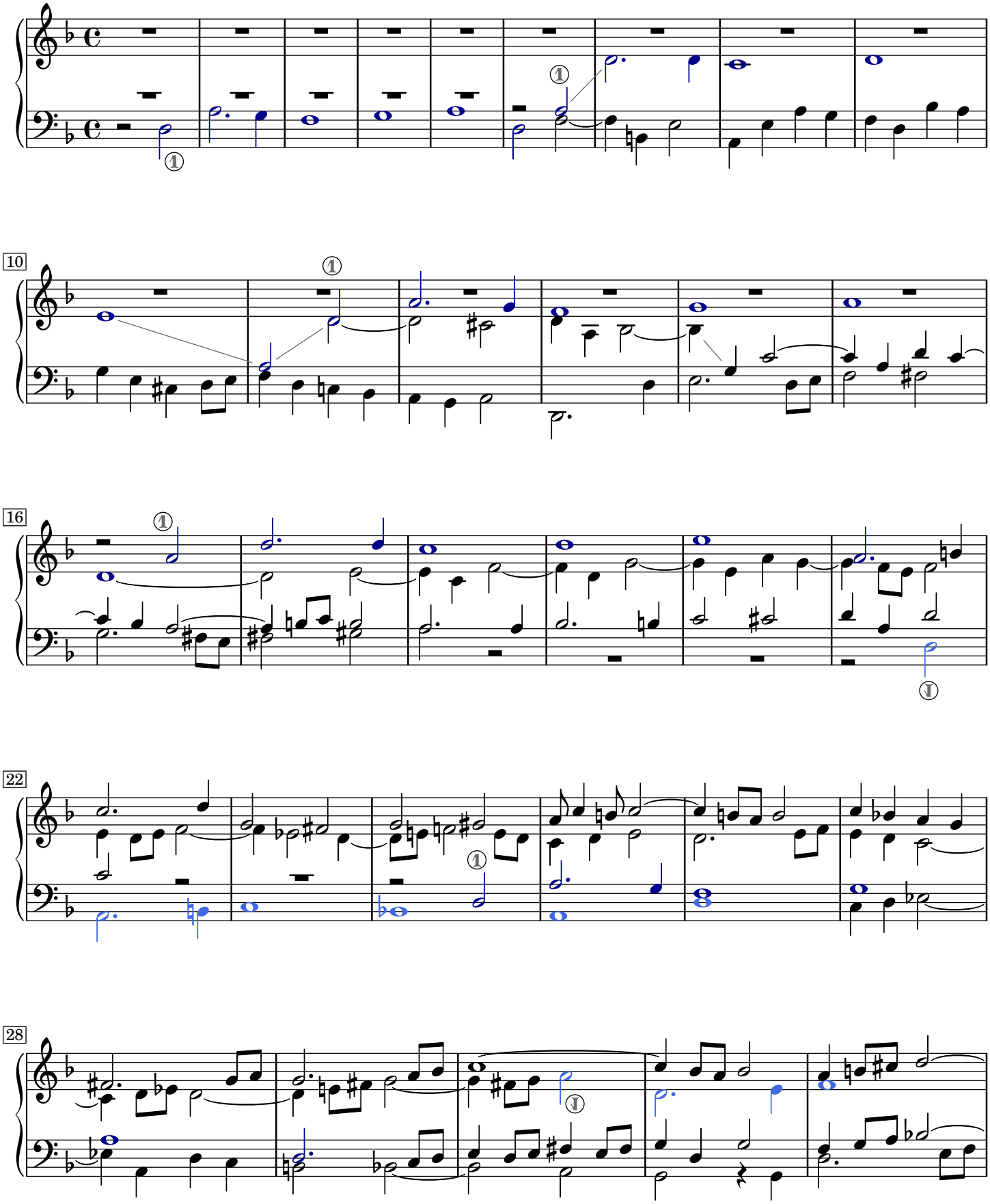
48
51

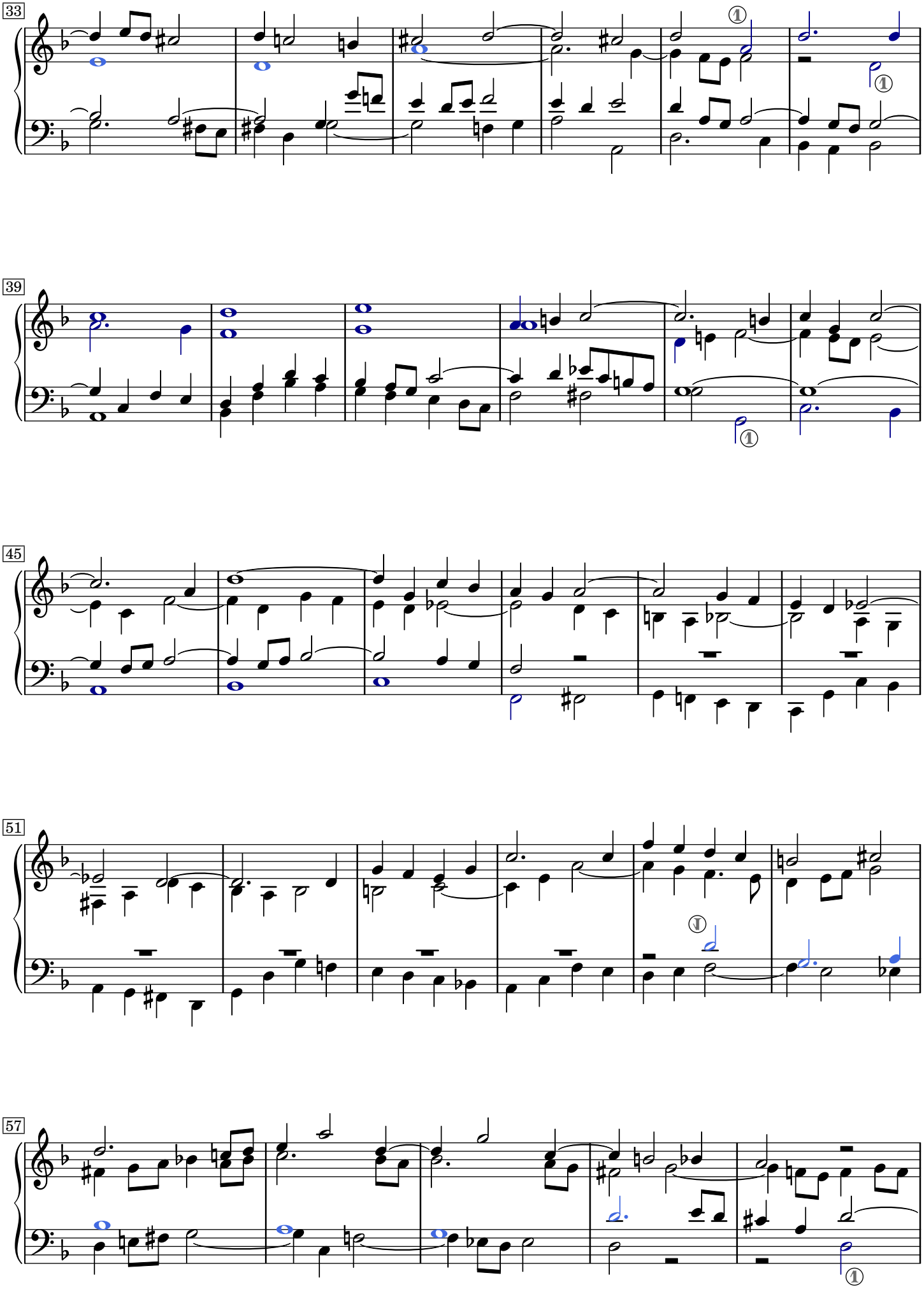

55
59
62
65
68

[Contrapunctus XIV]

Exposition 1 B1 A1 T1 S1 | Stretto B1 T1 | A1 | S1 A1 | B1 | 1 | T1 | B1 S1 | 2 | A1 T1 | S1 | 3 | B1 A1 T1 S1 A1 | B1 | 4
Exposition 2 A2 S2 B2 T2 | 5 | S2 B1 | 6 | A2 T1 | 7 | T2 S1 | 8 | B2 A1 S1 | 9
Exposition 3/a T3 A3 S3 B3 | 10 | Exposition 3/b T3 A3 S3 B3 | B3 T3 A3 | 11 | A2 B1 T3 | ?

BWV 1080, 19

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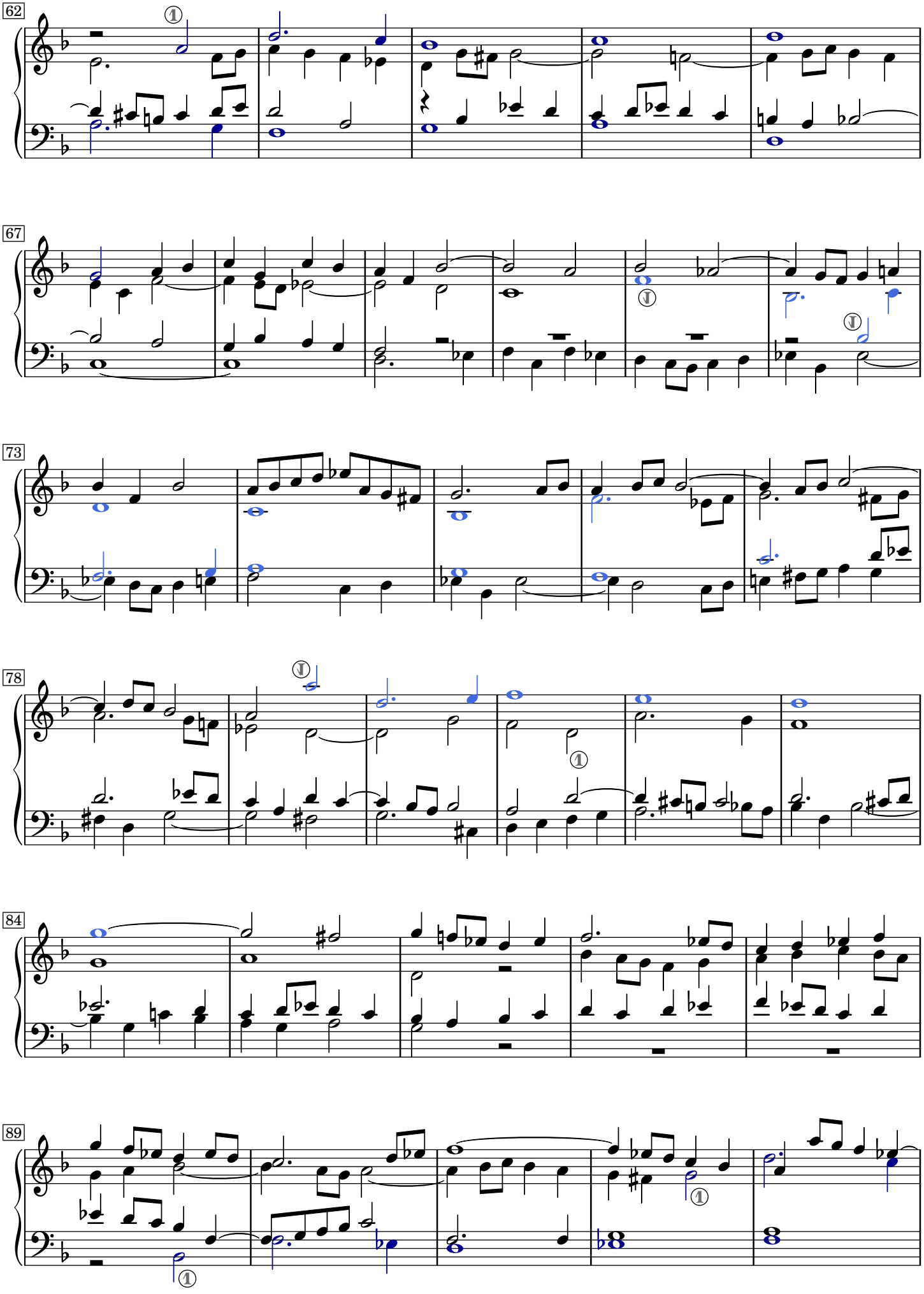
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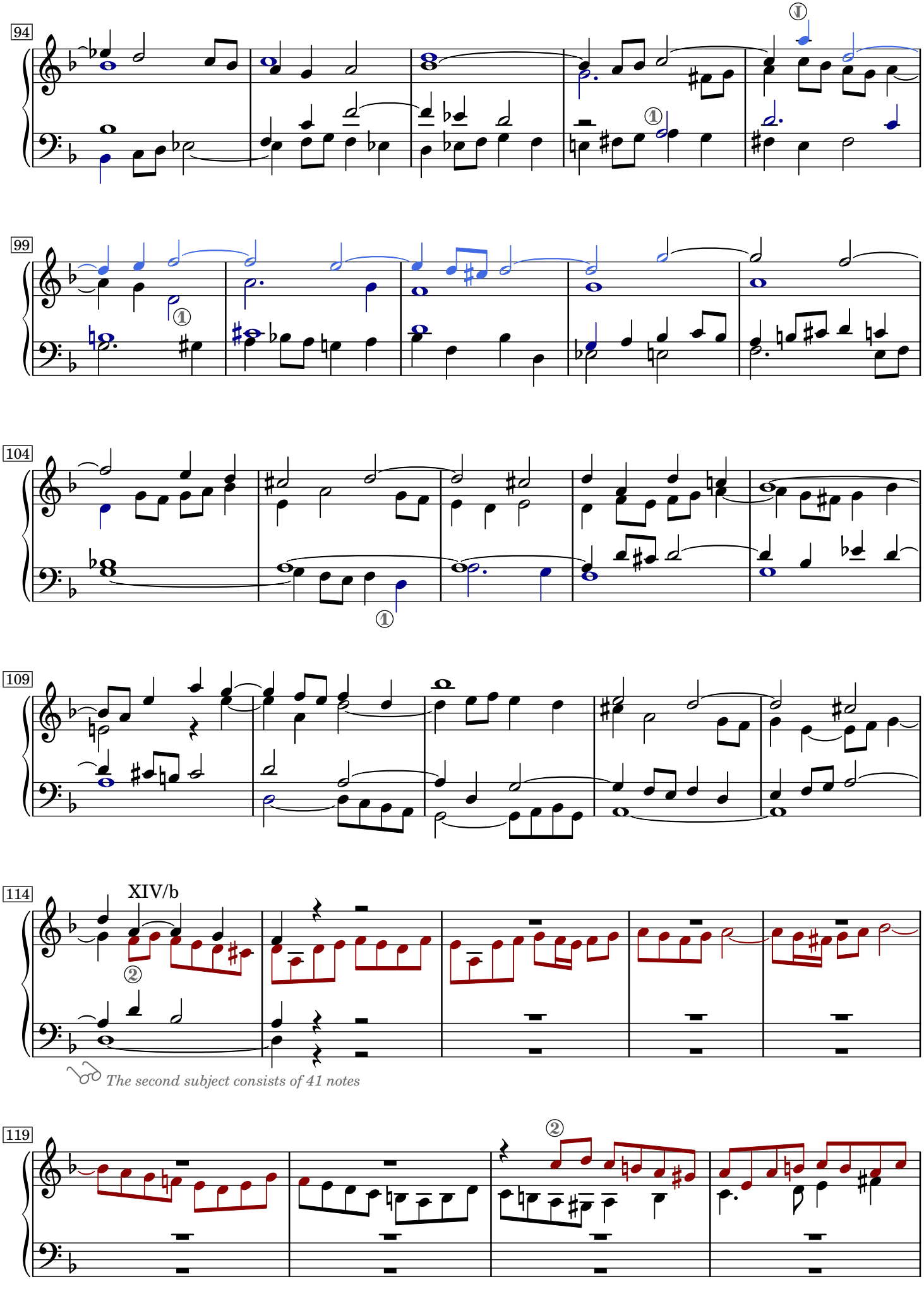
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99
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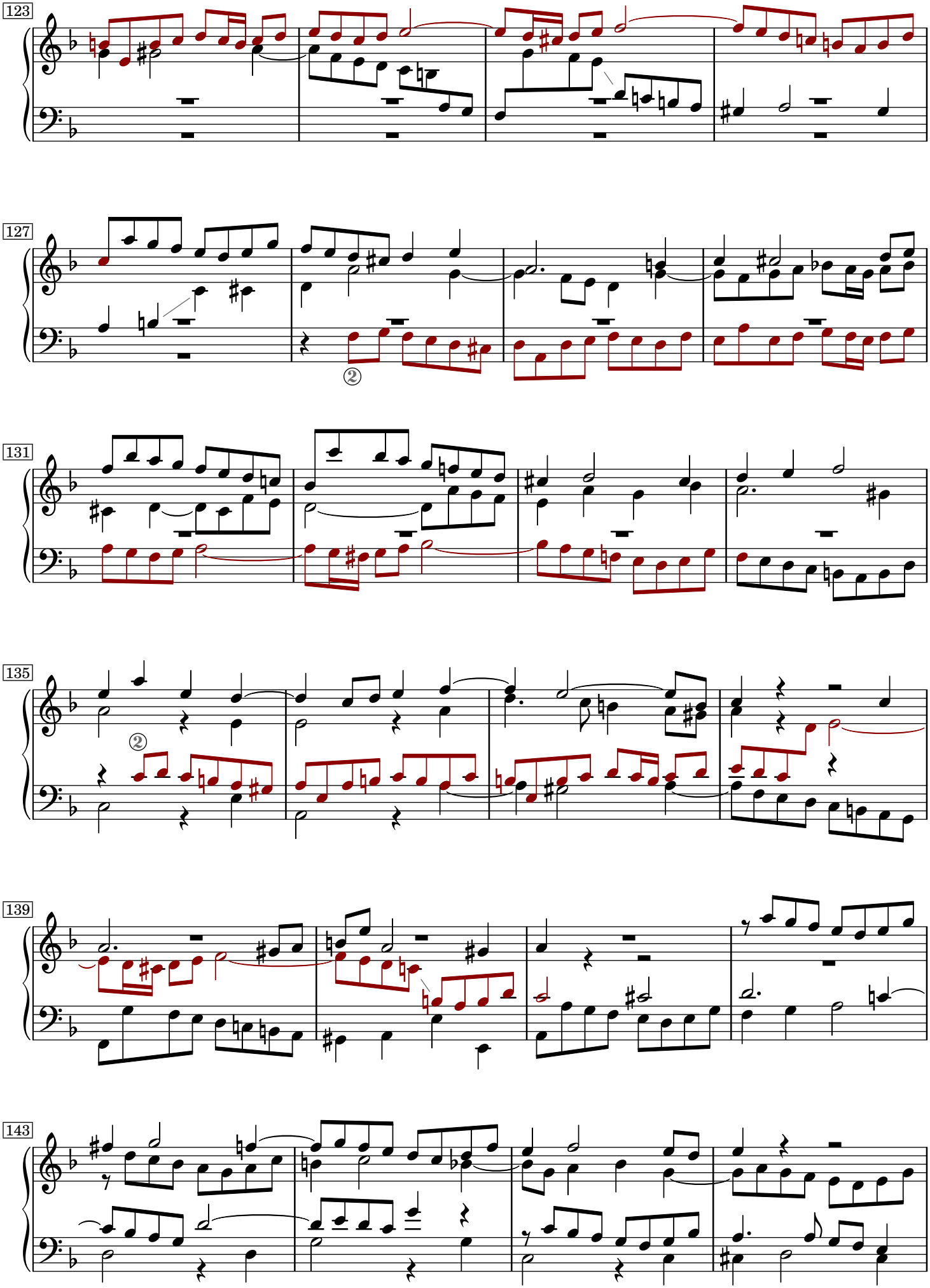
109
114
XIV/b
The second subject consists of 41 notes
119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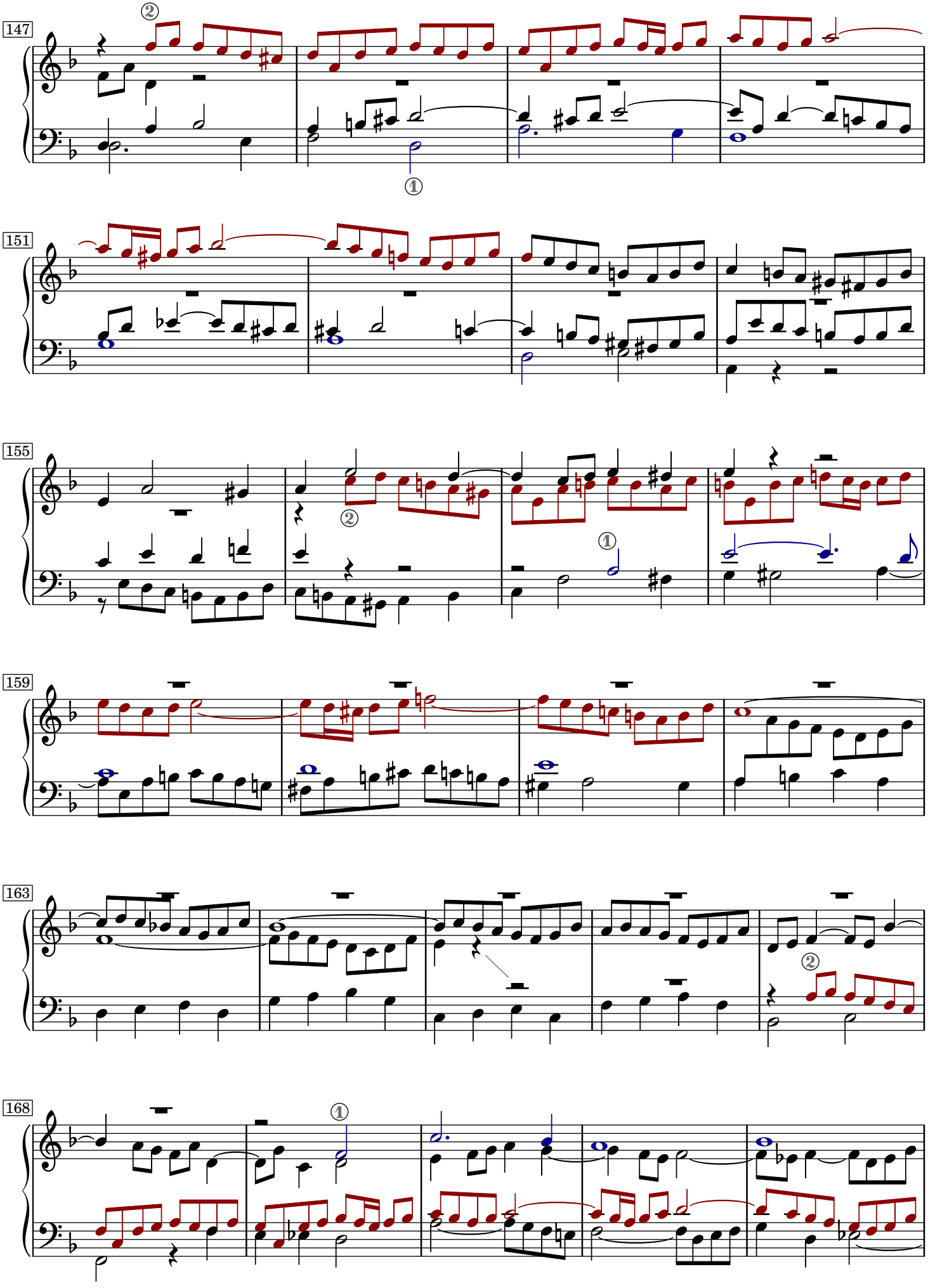
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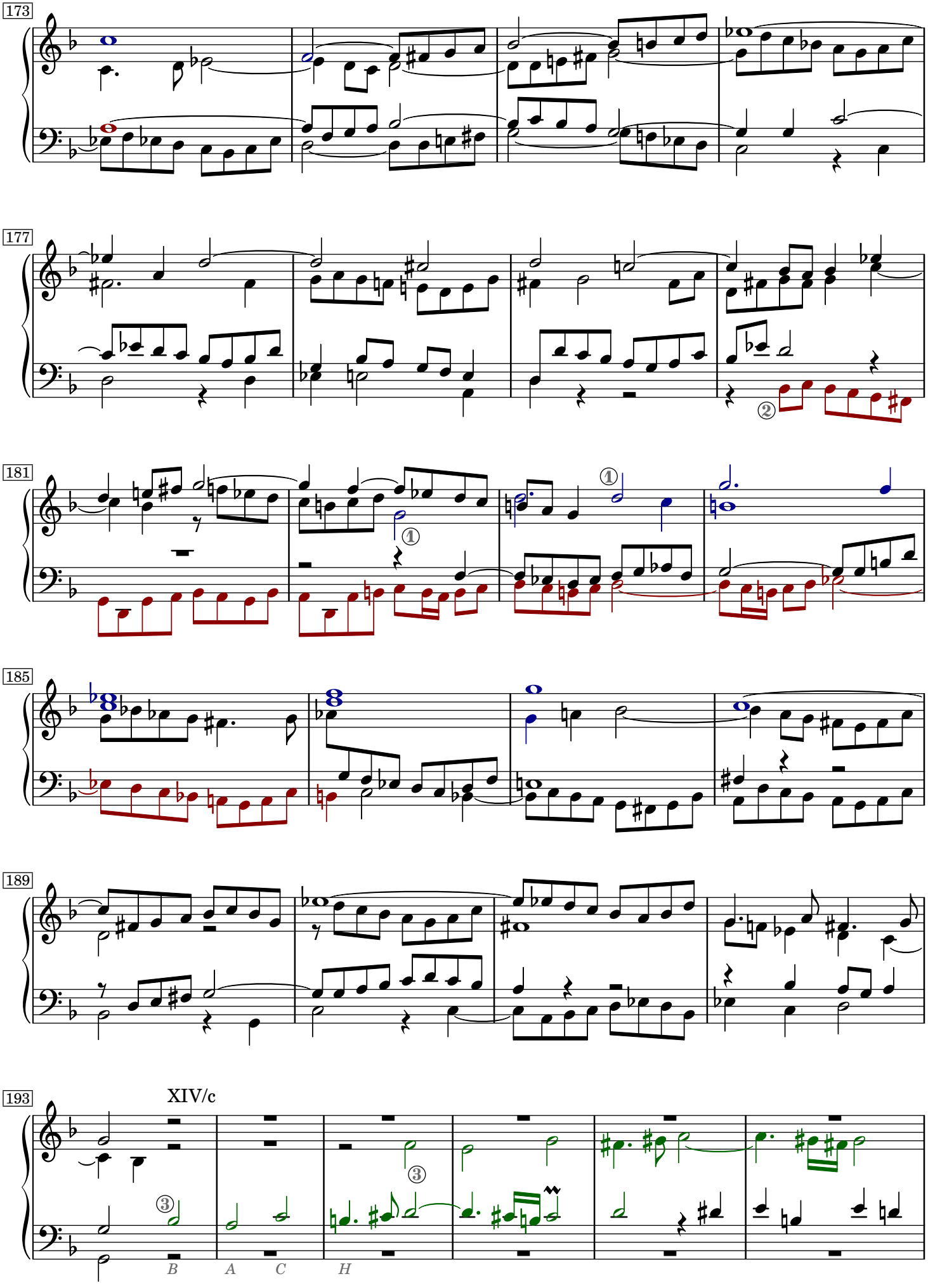
173
177
181
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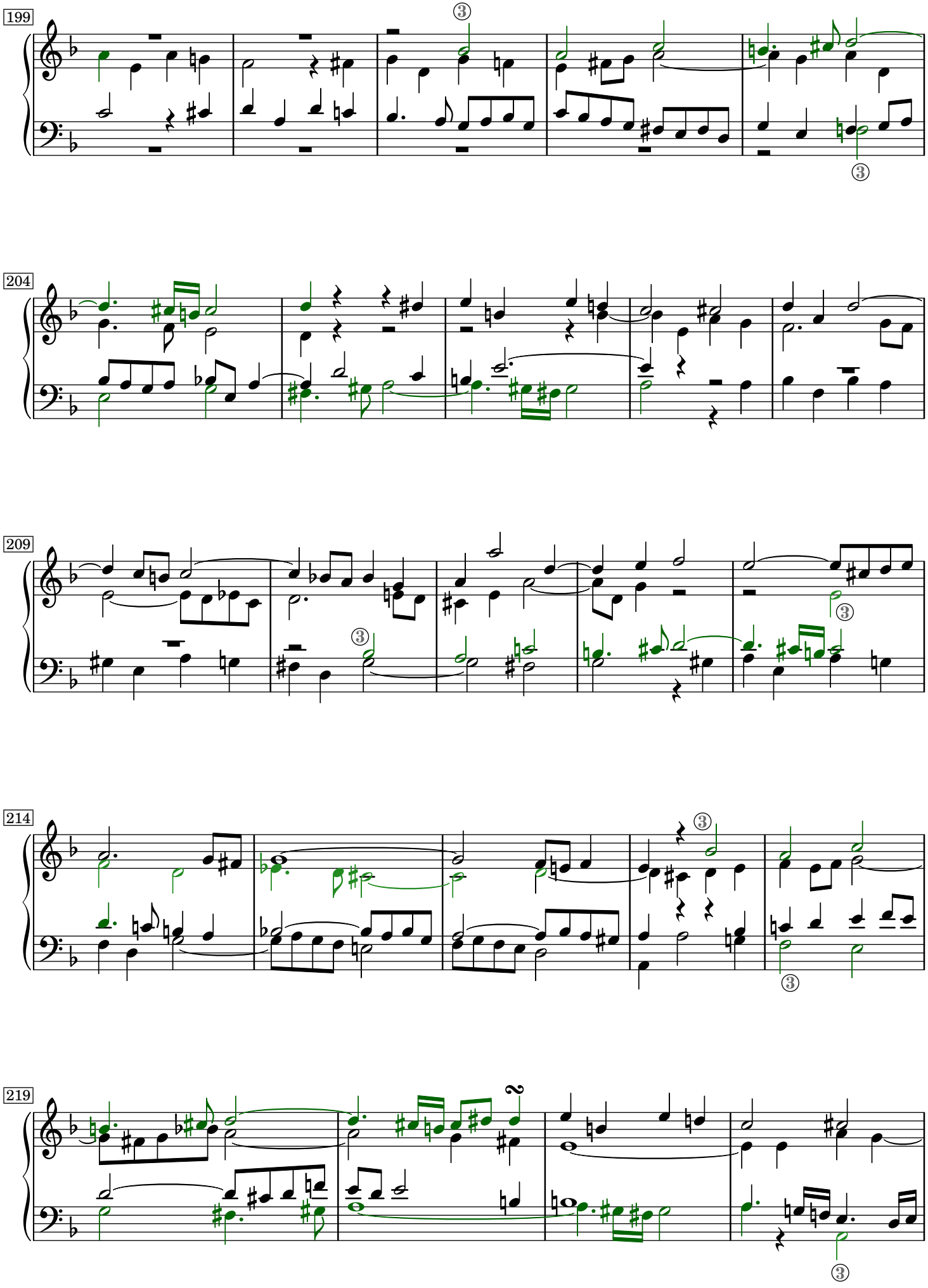
193
XIV/c
B
A
C
H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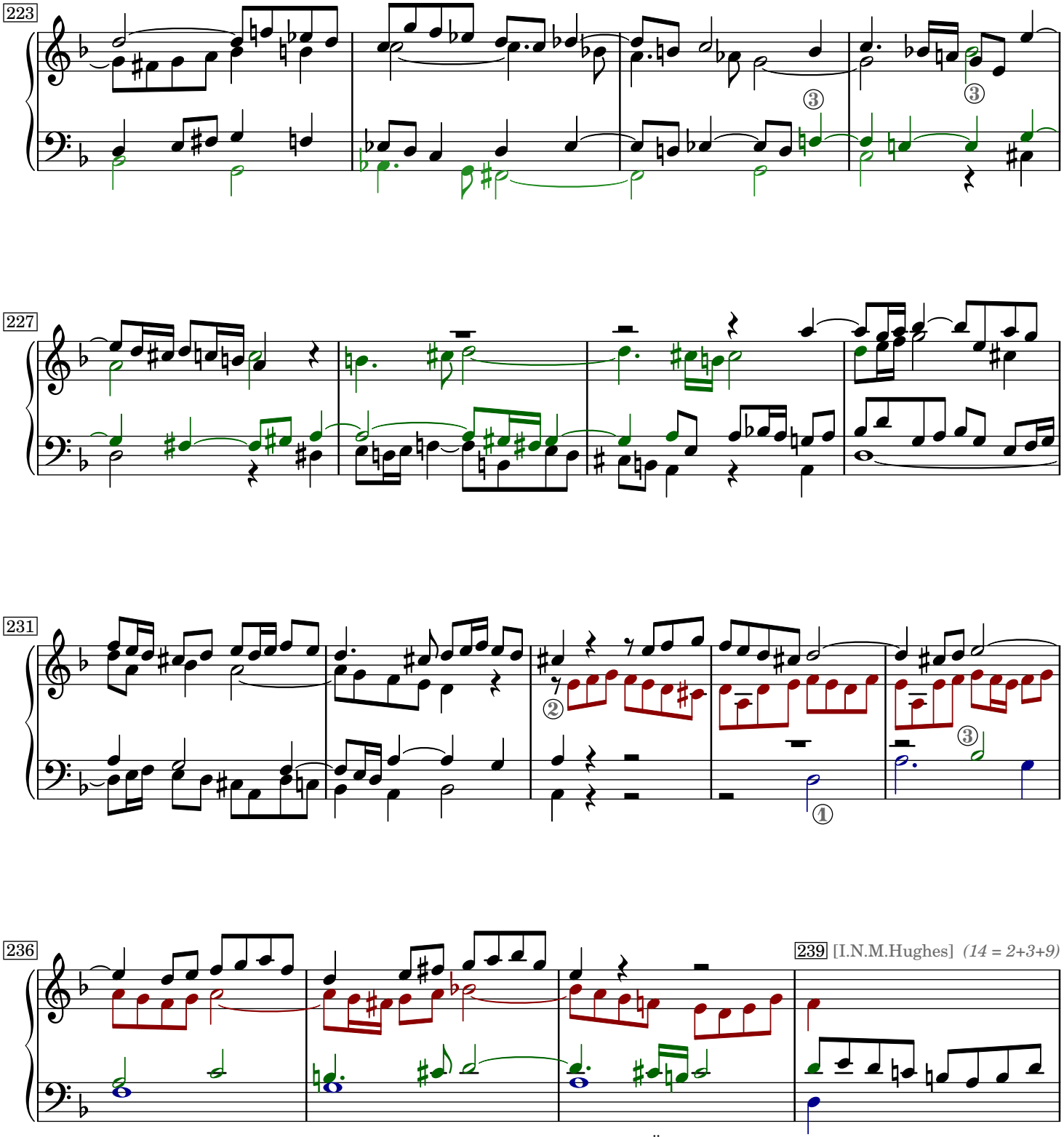


Über dieser Fuge, wo der Nahme B A C H
im Contrasubject angebracht worden ist
der Verfasser gestorben
— in the autograph, in the hand of (?) C.P.E. Bach (*)

(*) This line ("At the point where the composer introduces the name B A C H in the countersubject to this fugue, the composer died.") has generated romantic images of Bach dictating these last notes shortly before his death, but calligraphic research has shown that this is false. The handwriting on the last page of Contrapunctus XIV is definitely Bach's, and it is in a clear, steady hand as opposed to the erratic handwriting of Bach's final years. The final page of Contrapunctus XIV was definitely written several years before Bach's death.

Brief notes on the completion of Contrapunctus XIV

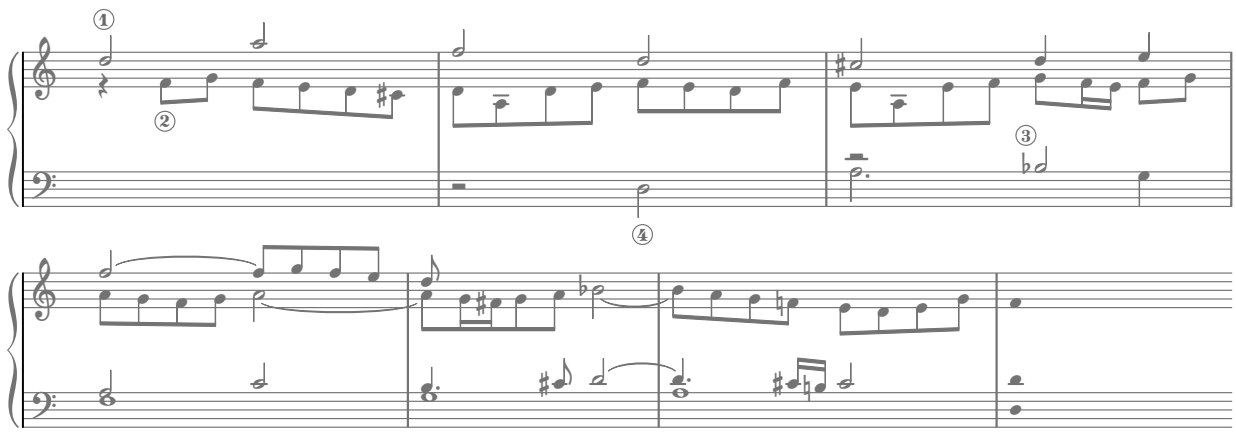
Ruth Tatlow states that the Art of Fugue is “the most developed thesis of Bach as a mathematician” [Bach and the riddle of the number alphabet – 1991, Cambridge University Press]. We will see some examples in these notes, where *“mathematician”* is to be understood primarily in the geometric-algebraic sense of the proportions of the composition and the relationships between musical voices. Most of the material that follows is based on Indra Nicholas Martindale Hughes' doctoral thesis *“Accident or Design? New Theories on the Unfinished Contrapunctus 14 in J. S. Bach’s The Art of Fugue BWV 1080”* — 2006, University of Auckland.

1
XIV and no further

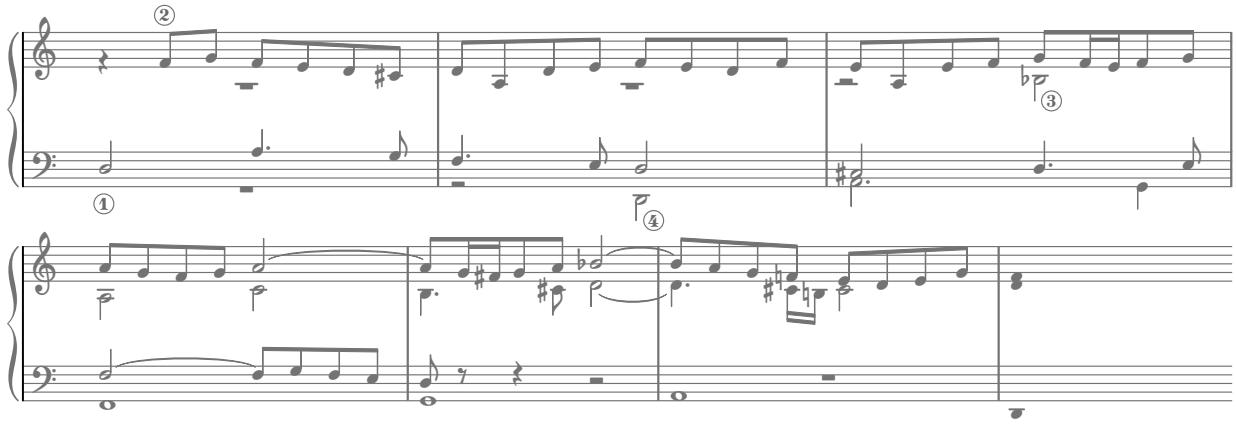
> *“Seine letzte Kranckheit, hat ihn verhindert, seinem Entwurfe nach, die vorletze Fuge völlig zu Ende zu bringen, und die letzte, welche 4 Themata enthalten, und nachgehends in allen 4 Stimmen Note für Note umgekehret werden sollte, auszuarbeiten.” — from Bach's Obituary, 1754*
>
> [His last illness prevented him from completing the penultimate fugue according to his draft, and from working out the last one, which was to contain 4 themes and subsequently be reversed note for note in all 4 voices. This work only came to light after the blessed author's death.]

The Bach's Obituary was written in the latter half of 1750, within months of Bach’s death, and published in 1754 in Lorenz Christoph Mizler's *Musikalische Bibliothek,* the organ of Mizler's Musical Society of which Bach had been a member from 1747 (the 14th member admitted to the society, see below). Although no author is indicated in the article, its authors are known to be Carl Philipp Emanuel, Bach's son, and Johann Friedrich Agricola, one of Bach's pupils.

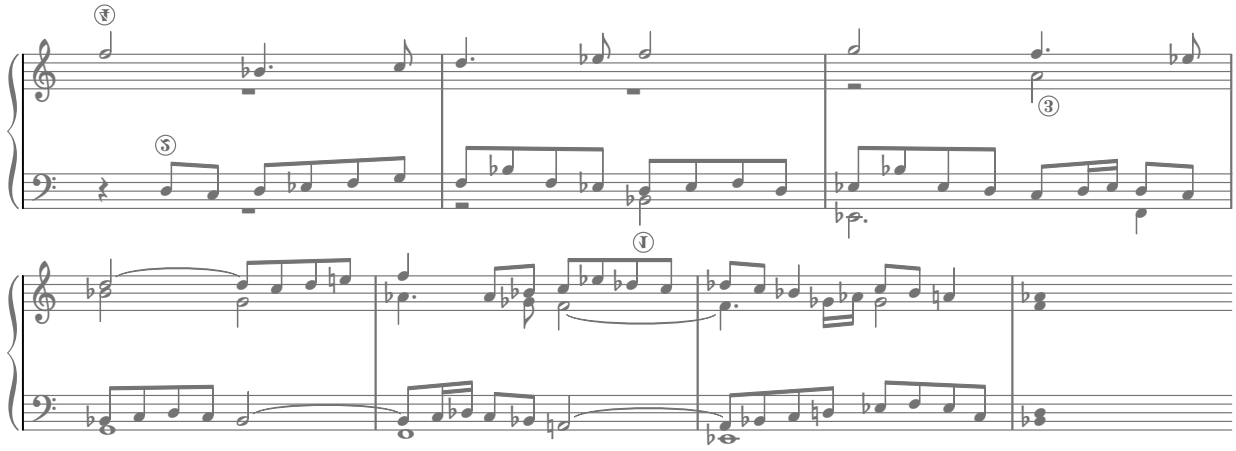
Huges believes that the words *“penultimate”* and *“last”* in the obituary excerpt above should be interpreted as the third (incomplete) and fourth (missing) parts of Contrapunctus XIV, and that a Contrapunctus XV was never planned by Bach. The Contrapunctus XIV was presumably meant by Bach as a *quadruple fugue* – a fugue with four subjects. Only three subjects are present in the extant portion of the fugue, but a fourth one is visibly missing. The Art of Fugue is in fact an in-depth exploration of the contrapuntal possibilities inherent in a single musical subject and this subject cannot fail to appear in every fugue. However, it was not until 1881 that Gustave Nottebohm discovered that this main Art of Fugue subject is able to be combined with the first three subjects, and that it must therefore have been the fourth subject of this quadruple fugue [Nottebohm, G. (1881) “J. S. Bach’s letzte Fuge” Music-Welt: 234]. The combination of the four subjects as given by Nottebohm and Christoph Wolff follows.



Thus, according to the obituary, the fourth and final section of Contrapunctus XIV is to include at least two episodes in which the four subjects appear together in a recto and inversus combination. Here is an alternative solution to the quadruple subjets combination, as proposed by Hughes.



And his solution to the arduous problem of the quadruple inversus combination, using the 14-note form of ④ .



2
Symbology of the number 14

The Art of Fugue is rich in symbolism related to the number 14, which is a transposition of the name Bach

B + A + C + H = 2 + 1 + 3 + 8 = 14

according to number alphabet designated by Tatlow as “Latin natural-order: variant 1” [op. cit.]

A: 1 B: 2 C: 3 D: 4 E: 5 F: 6 G: 7 H: 8 I/J: 9 K: 10 L: 11 M: 12 N: 13
O: 14 P: 15 Q: 16 R: 17 S: 18 T: 19 U/V: 20 W: 21 X: 22 Y: 23 Z: 24

The number 41, the retrograde of 14, is also present in the Art of Fugue, where:

J + S + B + A + C + H = 9 + 18 + 2 + 1 + 3 + 8 = 41

Occurrences of the number 14 and 41 in the Art of Fugue:

- In The Art of Fugue there are 14 Contrapuncti
- Contrapunctus I — the first two notes are D and A: 4 and 1 [Kellner, H. A. (May 1999) The Diapason]
- Contrapunctus II — there are 14 entries of the subject
- Contrapunctus V and onwards — the subject with and addition of passing notes arrives at a total of 14 notes
- Contrapunctus V — appearance of the BACH motive at bar 41
- Contrapunctus VI — there are 14 entries of the subject rectus and 14 entries inversus
- Contrapunctus XIV — the second subject consists of 41 notes
- Contrapunctus XIV — the BACH motive makes its most prominent appearance as the third subject
- Contrapunctus XIV — the last (incomplete) bar has number 239, and 2 + 3 + 9 = 14

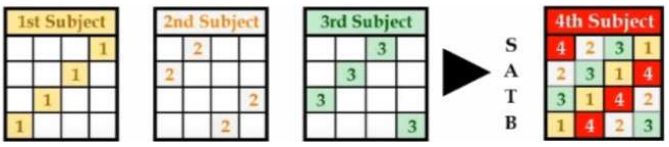
This should reinforce Huges' theory of the non-existence of an additional Contrapunctus XV designed by Bach at the close of the Art of Fugue cycle: ending the cycle with Fugue XIV would better adhere to this symbolism.

3
Subjects exposition and 4-chars permutations

The order of voices in which each subject enters at its exposition in Contrapunctus XIV is

Section 1	B T A S
Section 2	A S B T
Section 3	T A S B
Section 4	?

where “B” stands for bass, “T” for tenor, “A” for alto and “S” for soprano. One permutation is missing, and given Bach's love of algebraic games (or simply to vary the order of items in the expositions as much as possible) the likely order of appearance of the subjects in the fourth section should be reasonably: S B T A. This leads to what is known as *Zoltán Göncz's permutation matrix.*



4
Ratio of section lengths in Contrapunctus XIV

Some of Bach's compositions show a constant proportional ratio in the number of bars of the different sections that make them up. This principle also seems to be present in Contrapunctus XIV according to Gregory Butler.

> *“If we examine the relative proportions of the three extant sections of this fugue, we notice a consistent diminution in the lengths of successive sections. Moreover, section 2 (78 measures) is almost exactly two-thirds the length of section 1 (115 measures), and section 3, not quite complete, occupies forty-six measures and conceivably in its complete state would have occupied approximately two-thirds the length of section 2 (52 measures). Adhering to the same proportions, section 4 may well have occupied approximately two-thirds the length of section 3, that is, approximately thirty-four measures.”* [Butler, G.: Ordering Problems in J S Bach's 'Art of Fugue' Resolved]

This theory was refined by Hughes as follows.

Section 1	bars 1 – 114.25		
Section 2	bars 114.25 – 192.50	→	79.25 bars (1.4(4) = ratio 2·1)
Section 3	bars 192.50 – 248.50	←	**55 bars** to achieve a ratio 3·2 of 1.4, **9 bars** missing (up to bar 248)
Section 4	bars 248.50 – 286.50	←	**38 bars** to achieve a ratio 4·3 of 1.4 (up to bar 286)

This extrapolation may seem really far-fetched, but it becomes less so when one takes into account that the ratio 1.4 also appears among the Kyrie sections in the *Mass in B minor,* where the three movements of the Kyrie (Kyrie I – Christe – Kyrie II) sit in a diminishing proportional relationship to one another, a ratio of 1.4 existing between the relative lengths of each movement 122 : 84.75 : 59.

Finally, note that 55 = 14 + 41. Another number frequently seen in Bach's music.

5
Multi-level proportions

The Art of Fugue shows remarkable symmetries on multiple levels.

At the macroscopic level, as Yasuhiko Nakajima points out on his website The Art of Fugue Laboratory one can see a symmetry by total number of bars per section.

Style	*G. Butler's reconstruction*	*Bar number*	*Tot. bars*
Simple	Contrapunctus I	78	
	Contrapunctus II	84	
	Contrapunctus III	72	372
	Contrapunctus IV	138	
Inversion	Contrapunctus V	90	
	Contrapunctus VI	79	230
	Contrapunctus VII	61	
Three themes	Contrapunctus VIII	188	
Two themes	Contrapunctus IX	130	438
	Contrapunctus X	120	
Three themes	Contrapunctus XI	184	
Mirror	Contrapunctus XII rectus/inversus	112	438
	Contrapunctus XIII recus/inversus	142	
Four themes	Contrapunctus XIV	239	**239**
Canons	Canon I	103	
	Canon II	82	
	Canon III	78	372
	Canon IV	109	

Counterpoint XIV breaks the symmetry, by not having 230 bars.

A deeper mathematical analysis of The Art of Fugue's structure made by Loïc Sylvestre and Marco Costa [The Mathematical Architecture of Bach's The Art of Fugue - Il Saggiatore Musicale, 2010] shows an amazing adherence to the *Fibonacci sequence*

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, ...

Fibonacci numbers are intimately linked to the irrational number called the *golden ratio*

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

because if F(n) describes the nth Fibonacci number, the quotient of the adjacent terms F(n+1) / F(n) will approach the limit ϕ . The golden ratio, in its geometric manifestation, was already known to ancient Greek mathematicians and was also very popular among artists even in times before Bach's. Euclid's Elements provides the first known written definition: *"A straight line is said to have been cut into the extreme and golden ratio when, as the whole line stands to the greater segment, so the greater stands to the lesser."*

We will focus on the ratios in the number of bars of the Art of Fugue, using the following rationale. Since in the manuscript the *rectus* and *inversus* of Contrapuncti XII and XIII are not written separately, but aligned, one below the other as a unit, we consider here their number of bars only once instead of summing rectus and inversus. Some examples of golden ratios follows. (Here ∑ means 'sum of bars'.)

(1) ∑ Contrapuncti 1,2,3,4,5,6,7 / ∑ Contrapuncti 1,2,3,4 = 602 / 372 = 1.618 ≈ ϕ
(2) ∑ Contrapuncti 1,2,3,4 / ∑ Contrapuncti 5,6,7 = 372 / 230 = 1.617 ≈ ϕ
(3) ∑ Contrapuncti 1,2,3,4,...,12,13,14 / ∑ Contrapuncti 8,9,10,11,12,13,14 = 1590 / 988 = 1.609 ≈ ϕ
(4) ∑ Contrapuncti 8,9,10,11,12,13,14 / ∑ Contrapuncti 1,2,3,4,5,6,7 = 988 / 602 = 1.641 ≈ ϕ
(5) ∑ Contrapuncti 1,2,3,4,5,6,7 / ∑ Canons 1,2,3,4 = 602 / 230 = 1.618 ≈ ϕ

These examples also show a beautiful fractal phenomenon of *self-similarity*. It can also be observed that the ratio (3) would be 1.614 with a 230 bar long Contrapunctus XIV. Likewise the ratio (4) would be 1.626. Both better approximations of the ϕ value.

Other symmetries are present within the Contrapuncti.

Contrapunctus I / Contrapunctus III = 78 / 72 = 1.083
Contrapunctus IX / Contrapunctus X = 130 / 120 = 1.083

And also within the four Canons, which can be divided into two pairs that share nearly the same ratio.

Canon alla ottava / Canon alla duodecima = 103 / 78 = 1.320
Canon per augmentationem in contrario motu / Canon alla decima = 109 / 82 = 1.329

≻ ≺

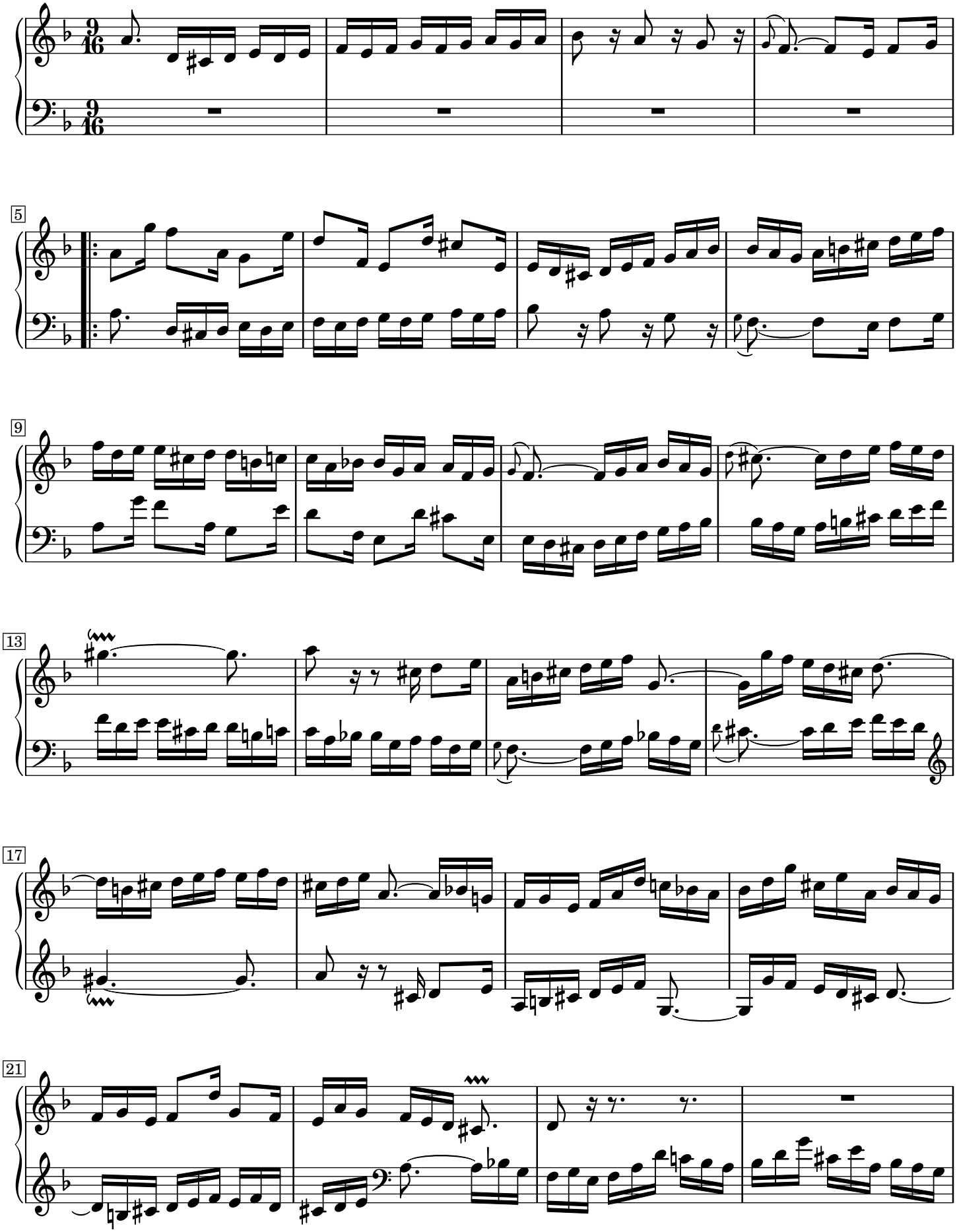
The completion of the Contrapunctus XIV remains an interesting and challenging problem in itself, but in light of the above considerations, Nakajima's words seem agreeable.

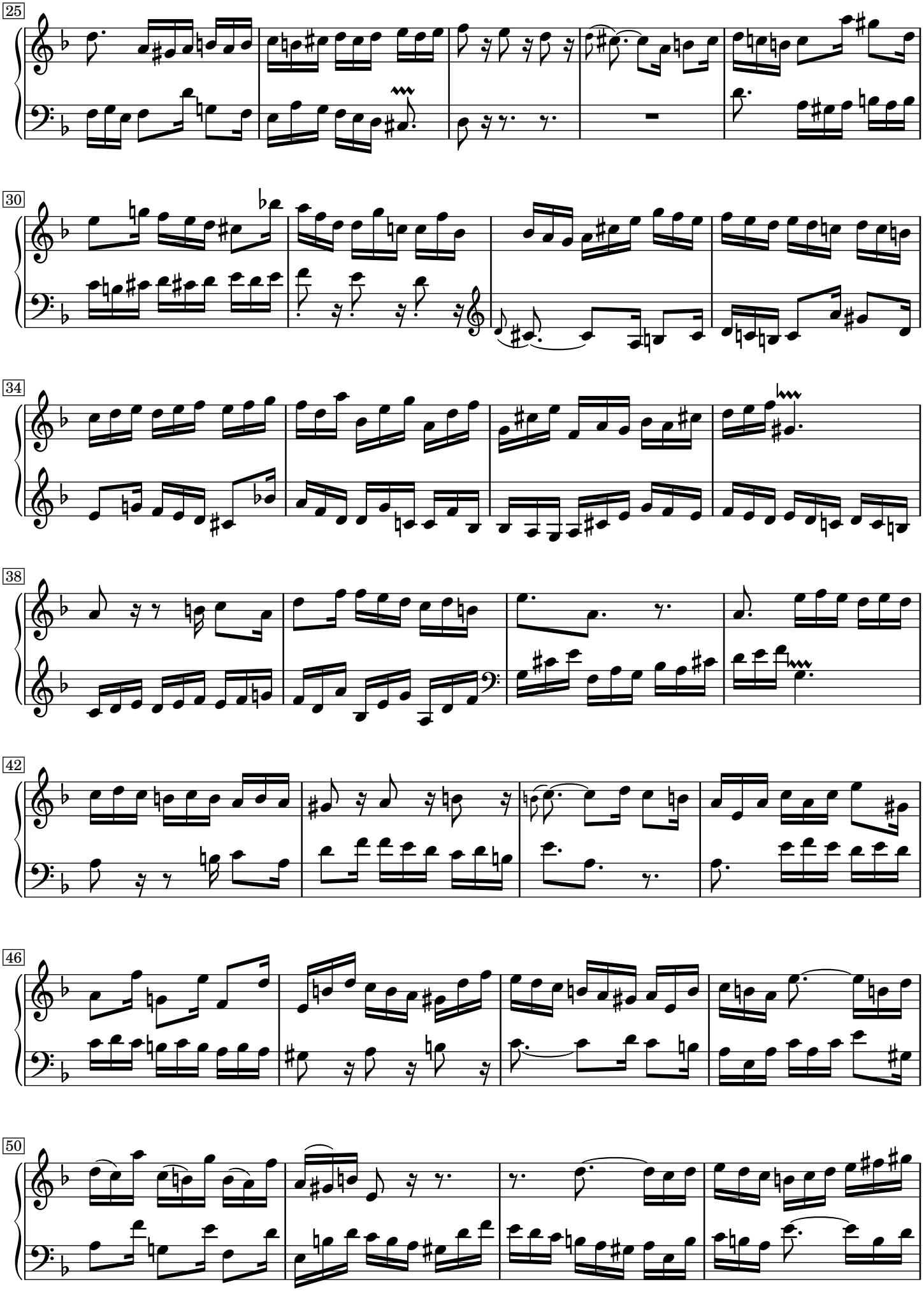
> *Considering its structure without the common theme and with this asymmetry for the cycle of the art of fugue, the unfinished fugue might not be the member of the cycle. This is just my presumption but Bach was discouraged from completing it because of this excessive bar number. It is believed that composing, publication, and extension of pieces and addition of new pieces, all these were simultaneously ongoing at that time, then halfway built unfinished fugue could not match with the finally created symmetric structure.*

Canon in Hypodiapason

(Canon alla Ottava)

BWV 1080, 15

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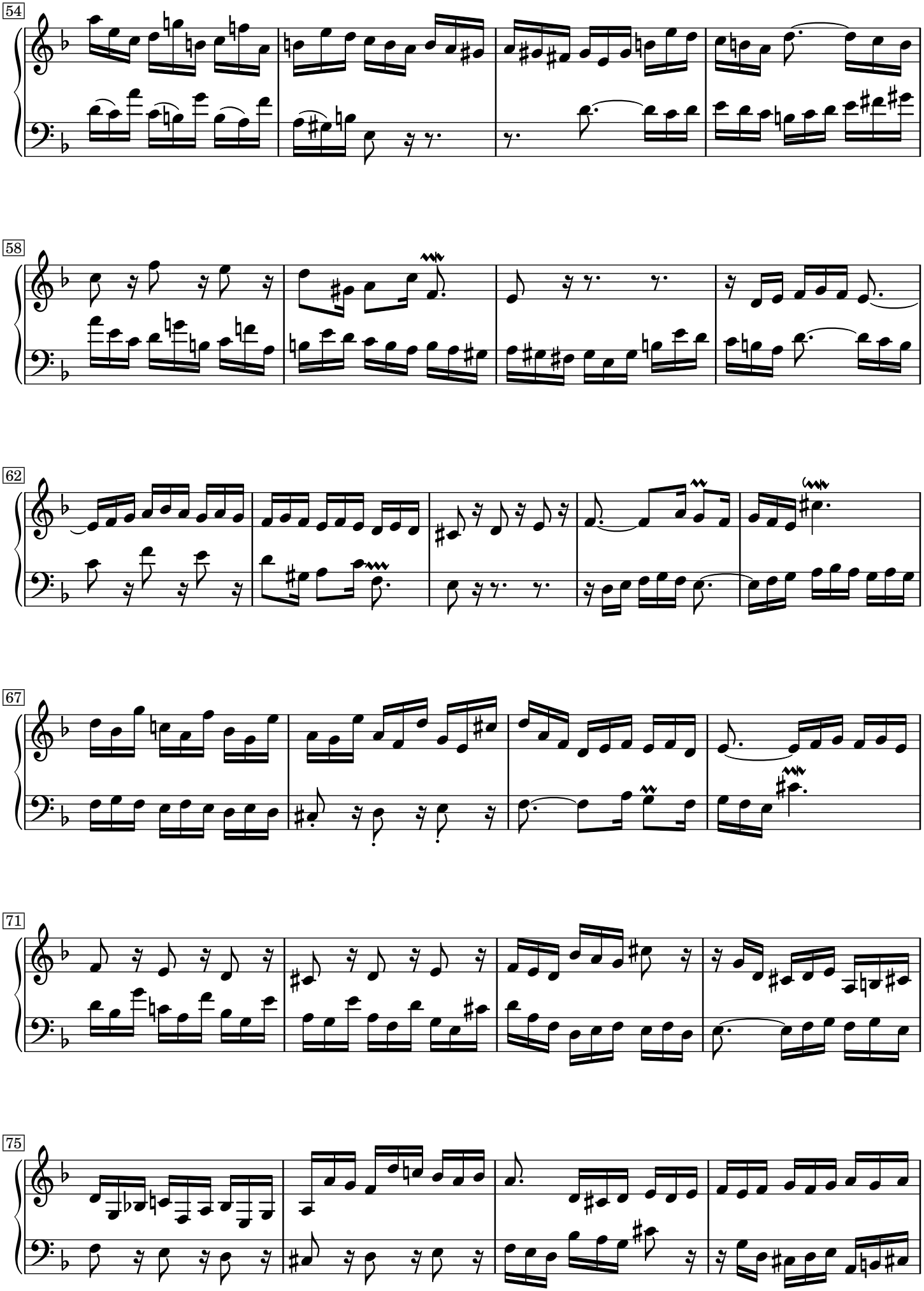
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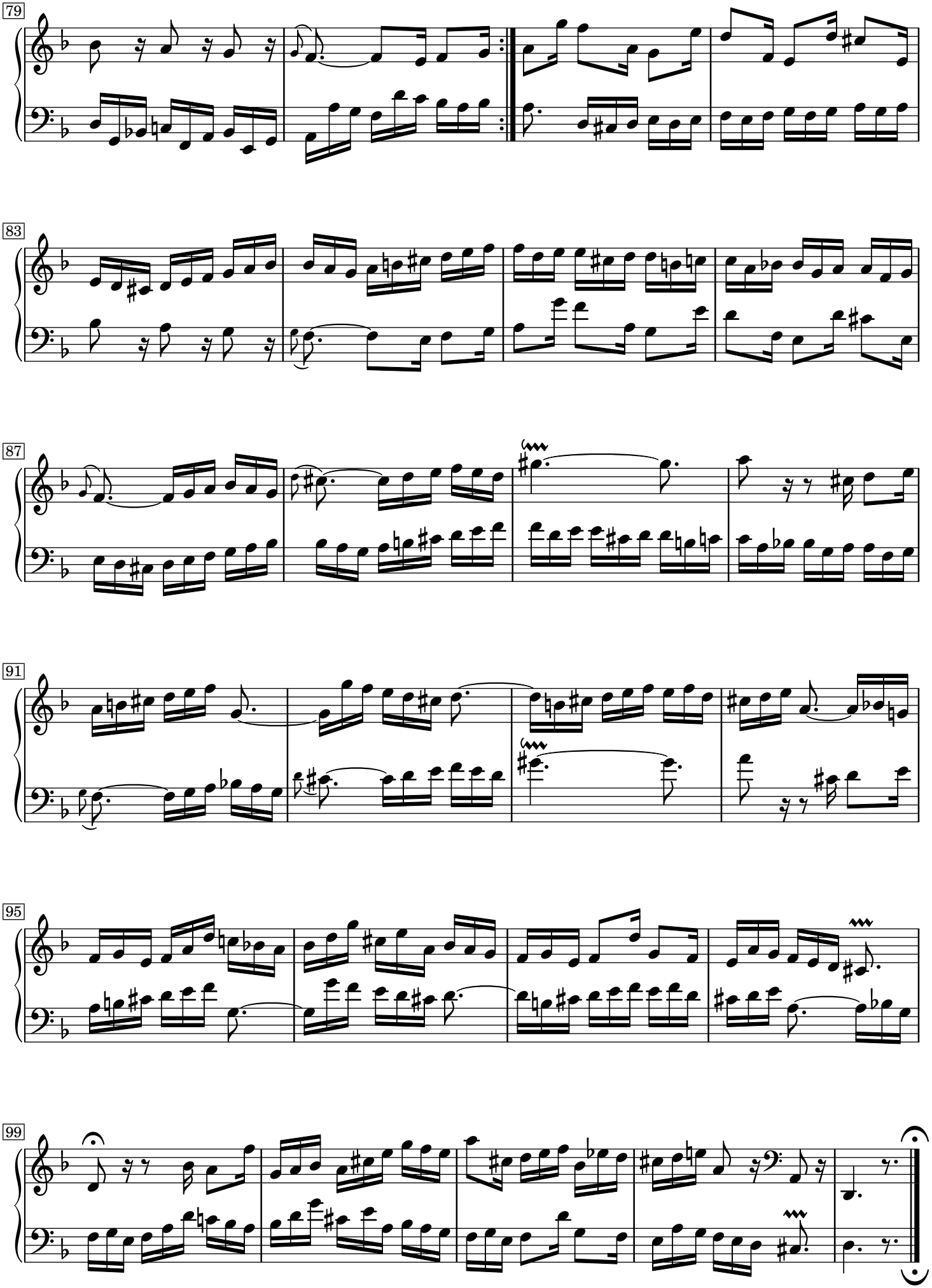
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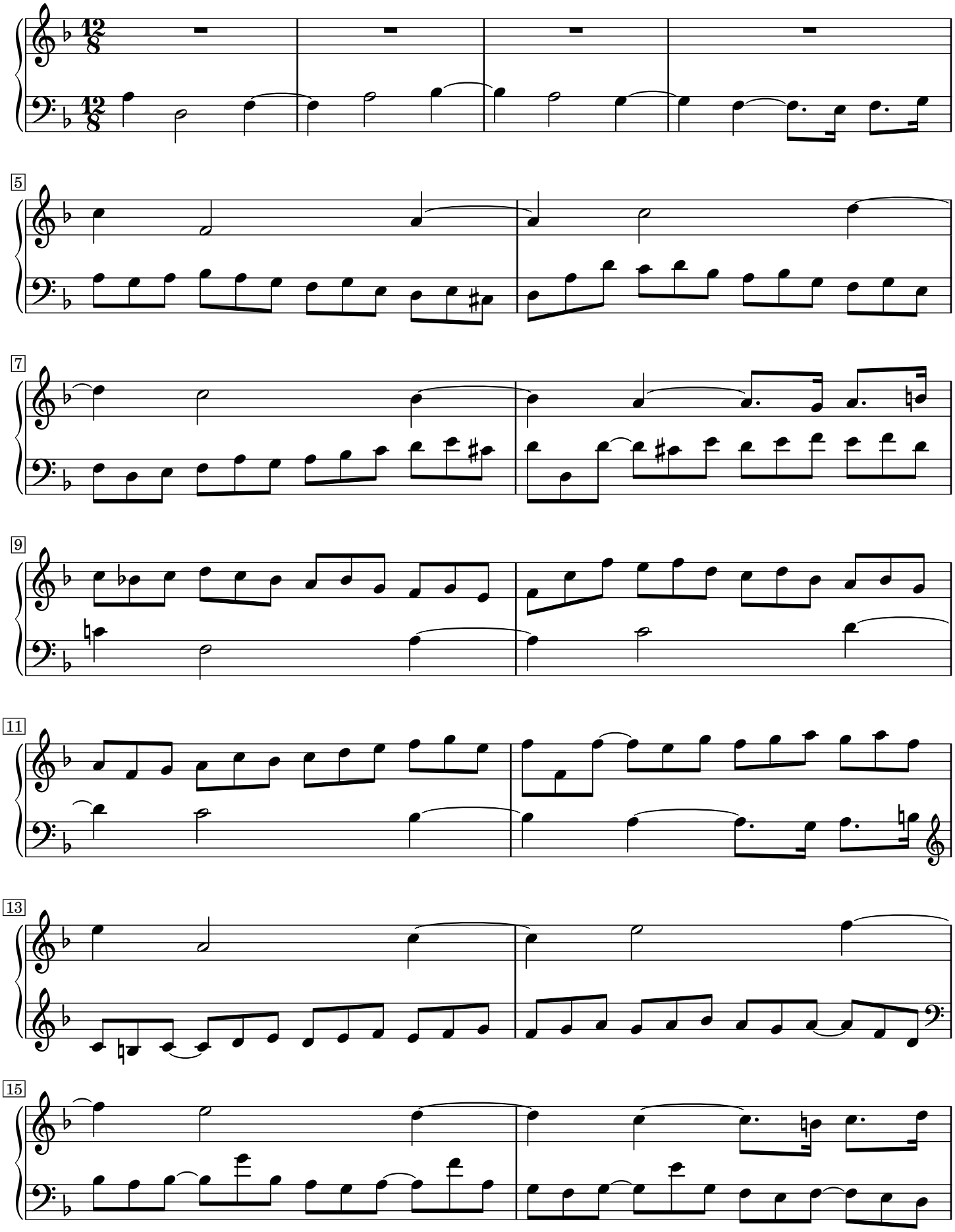
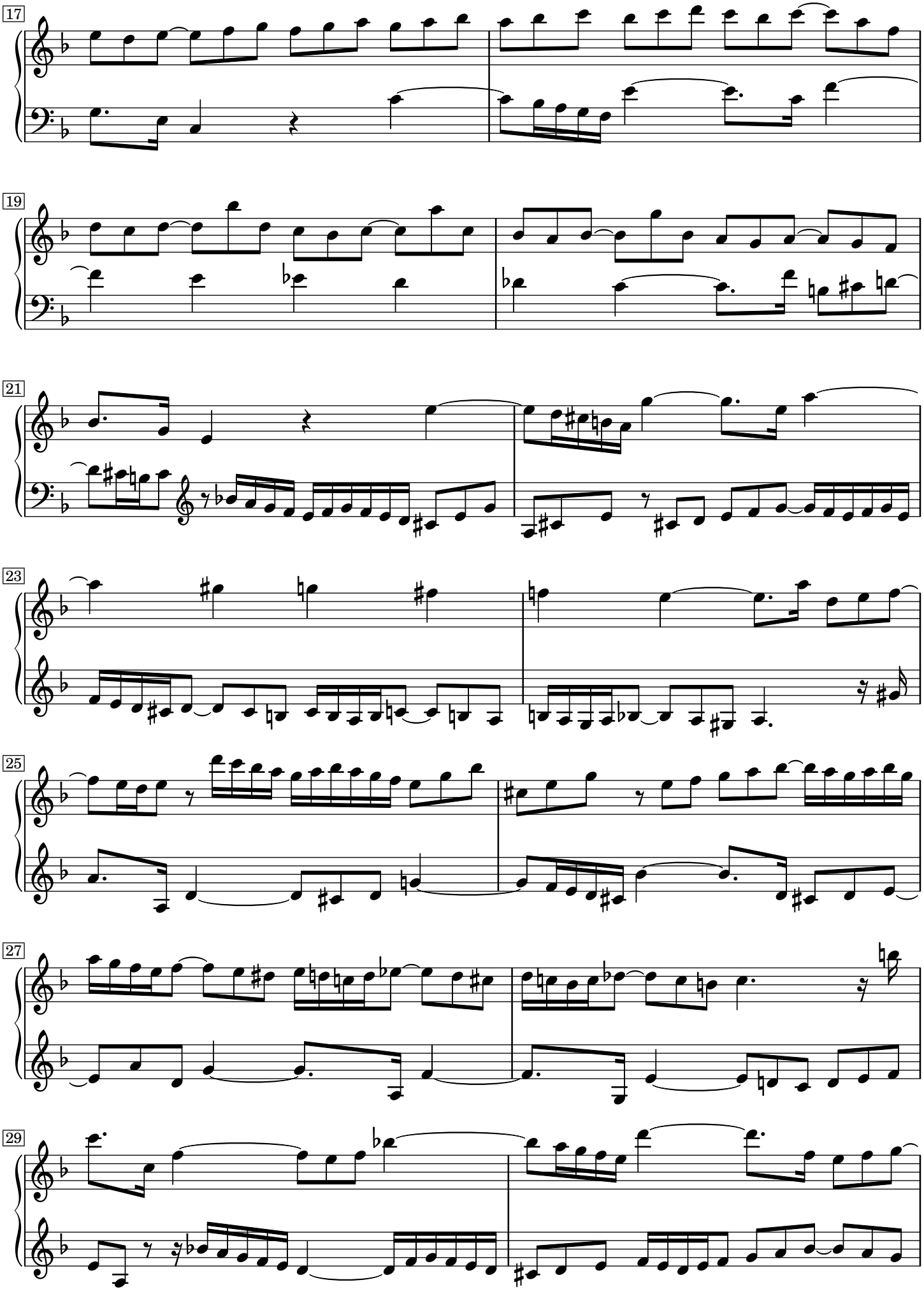
Canon alla Decima

Contrapunto alla Terza

BWV 1080, 16

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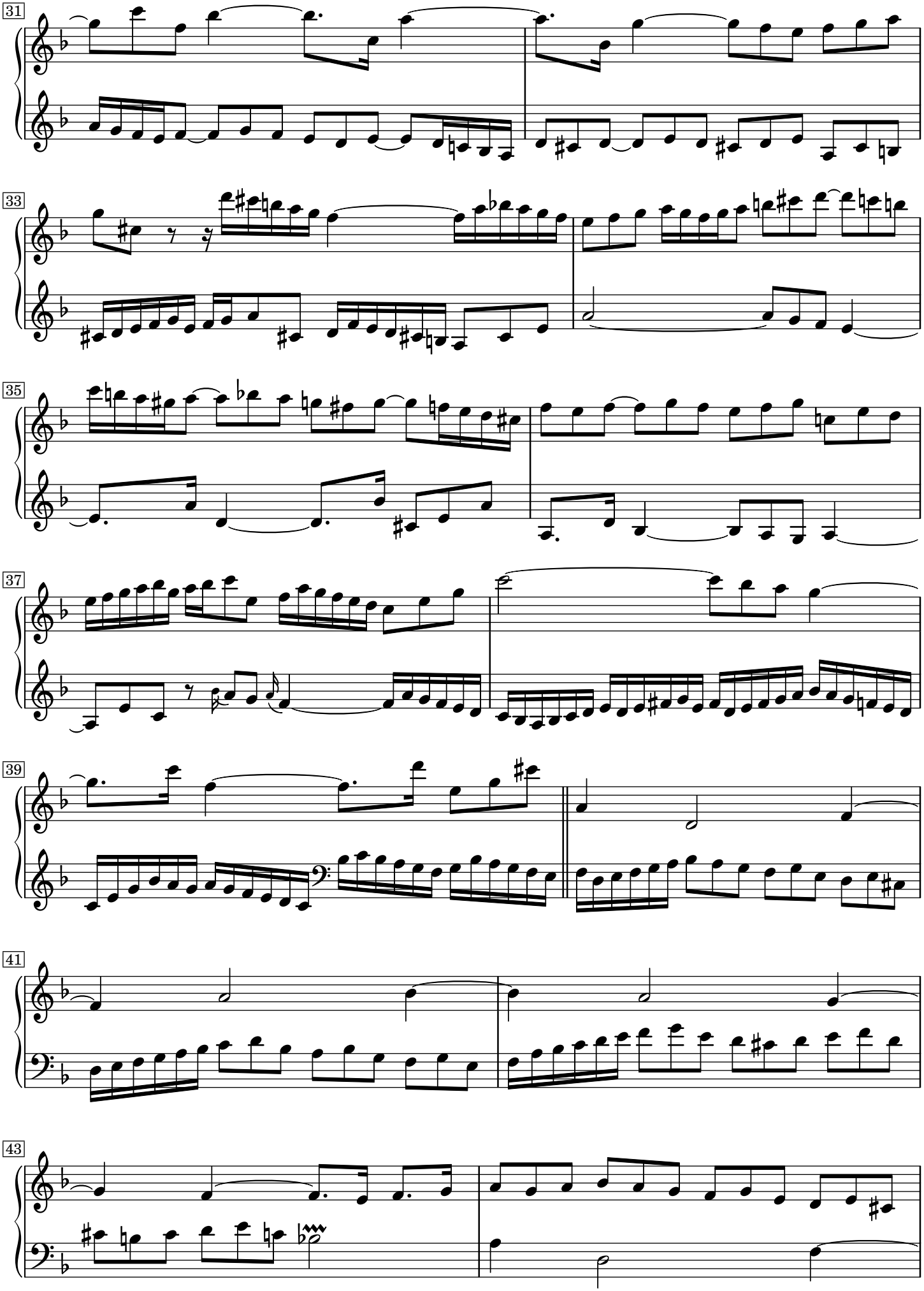
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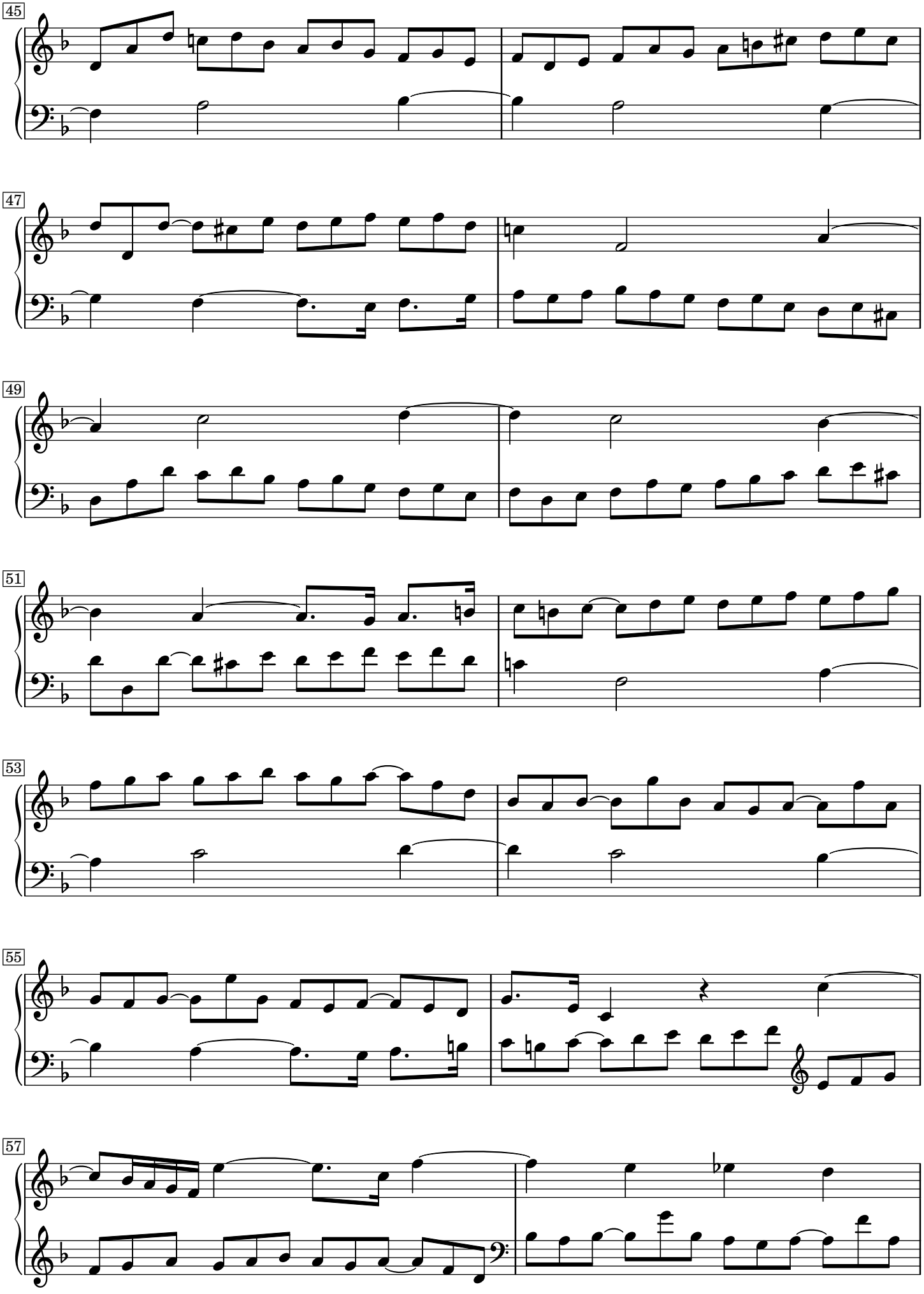
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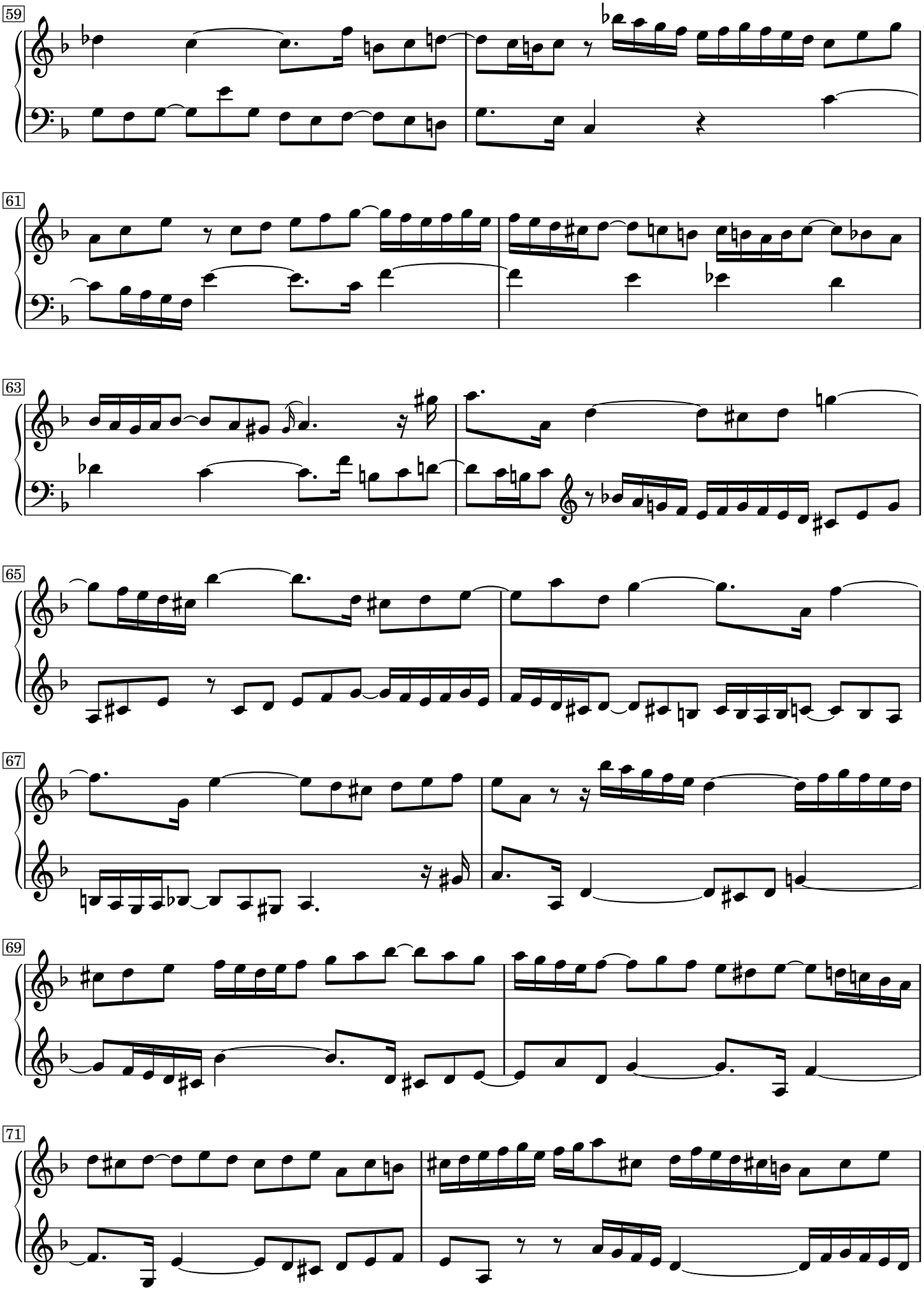
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Cadenza

Canon alla Duodecima

in Contrapunto alla Quinta

BWV 1080, 17

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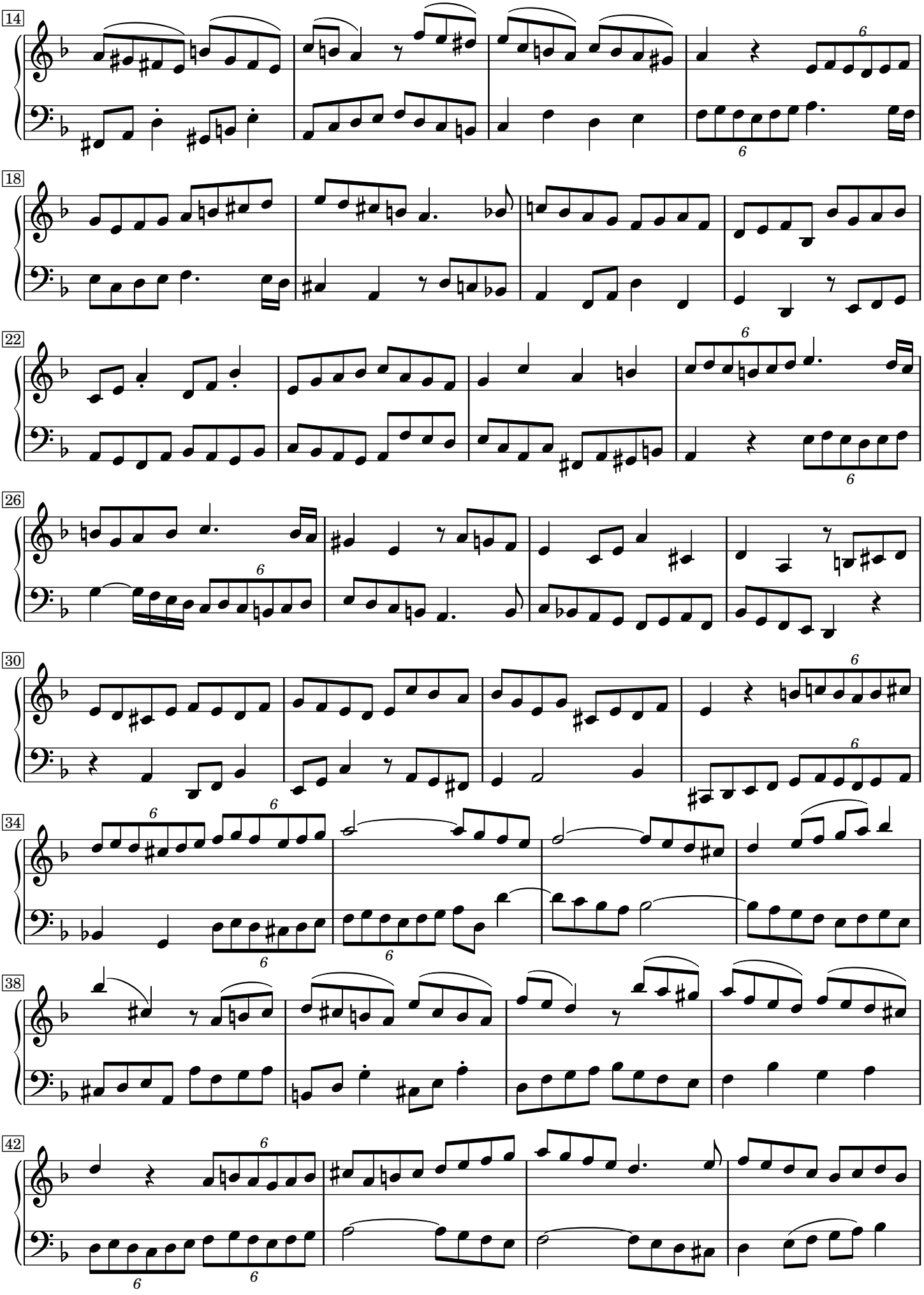
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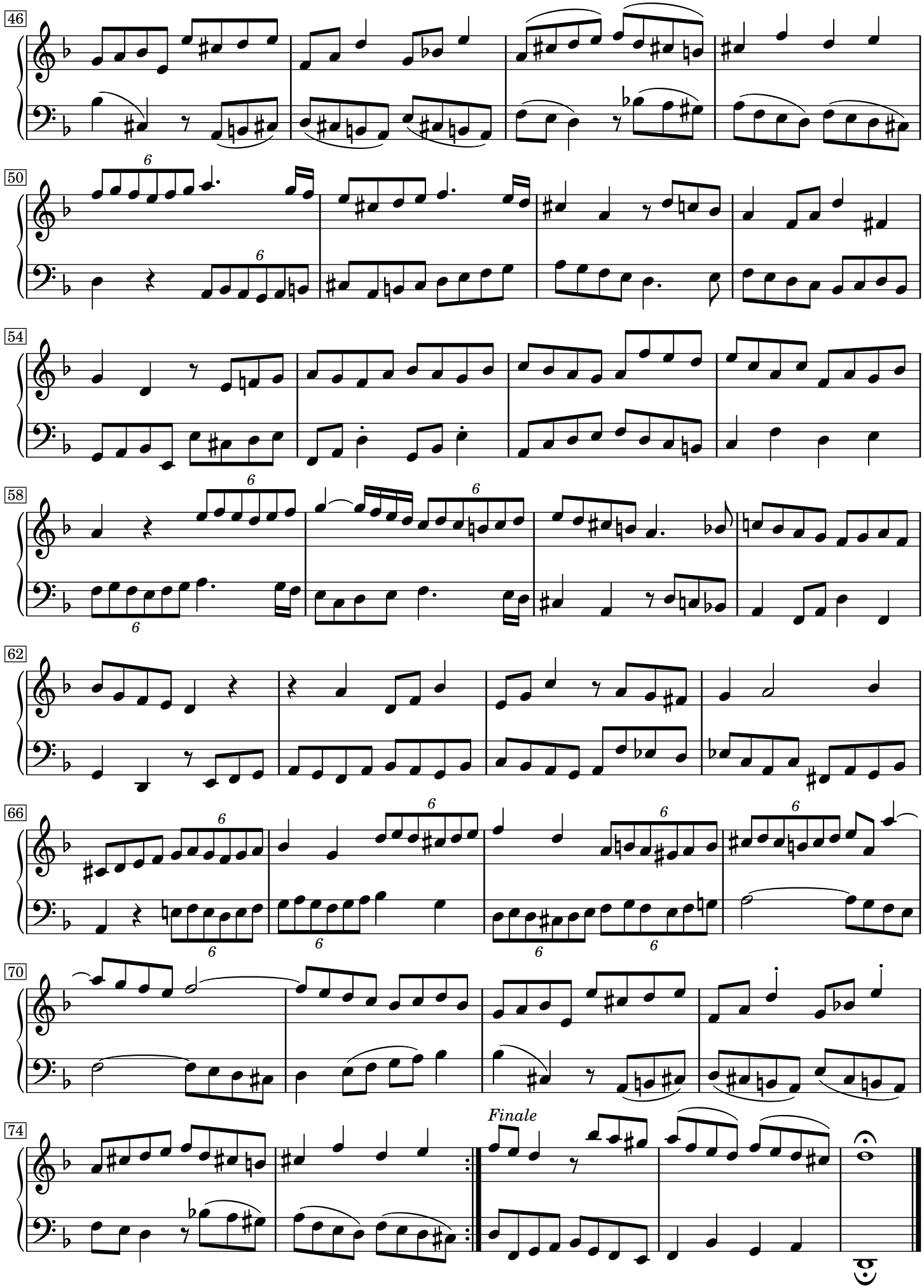
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Canon [in Hypodiatessaron]

per Augmentationem in Contrario Motu

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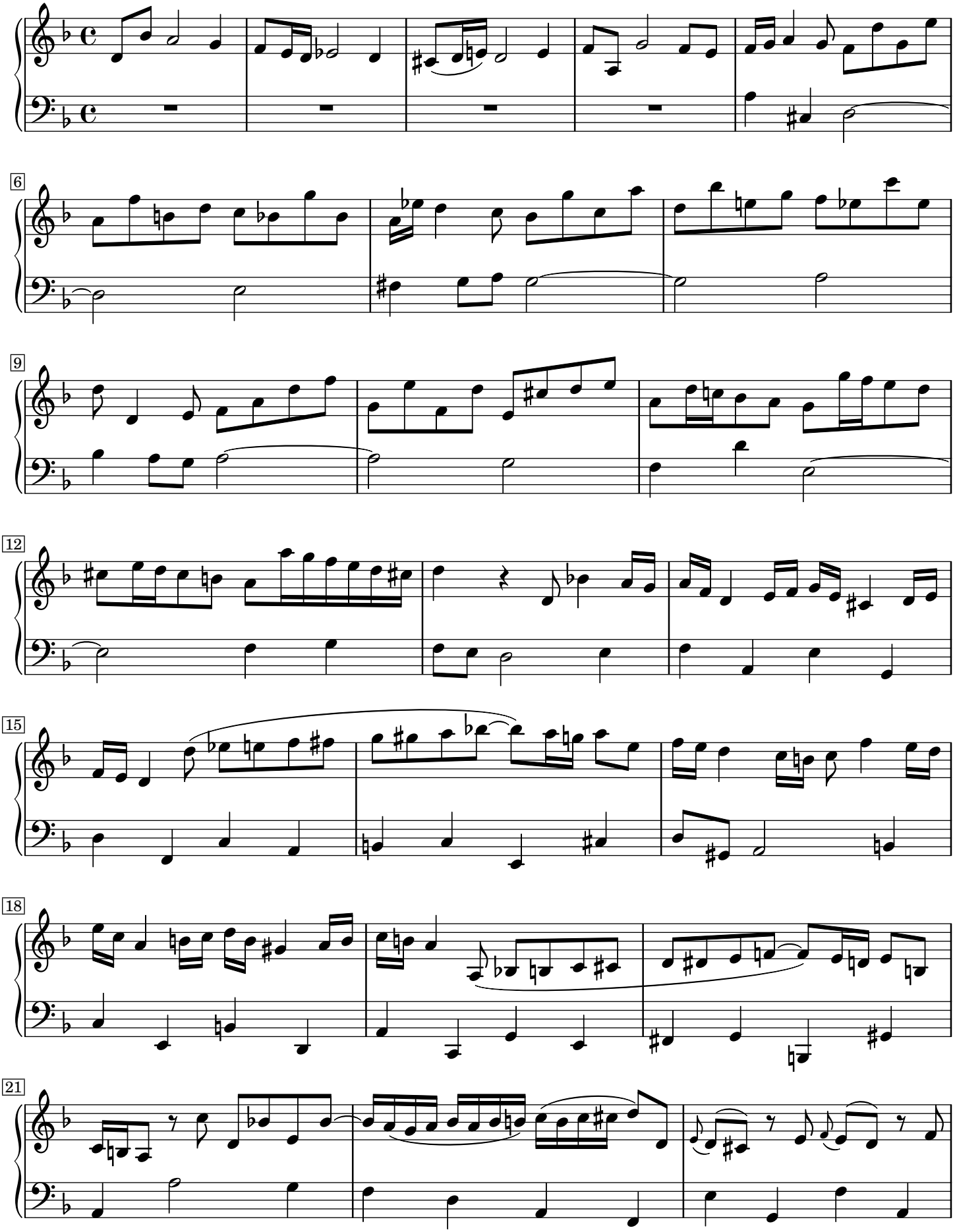

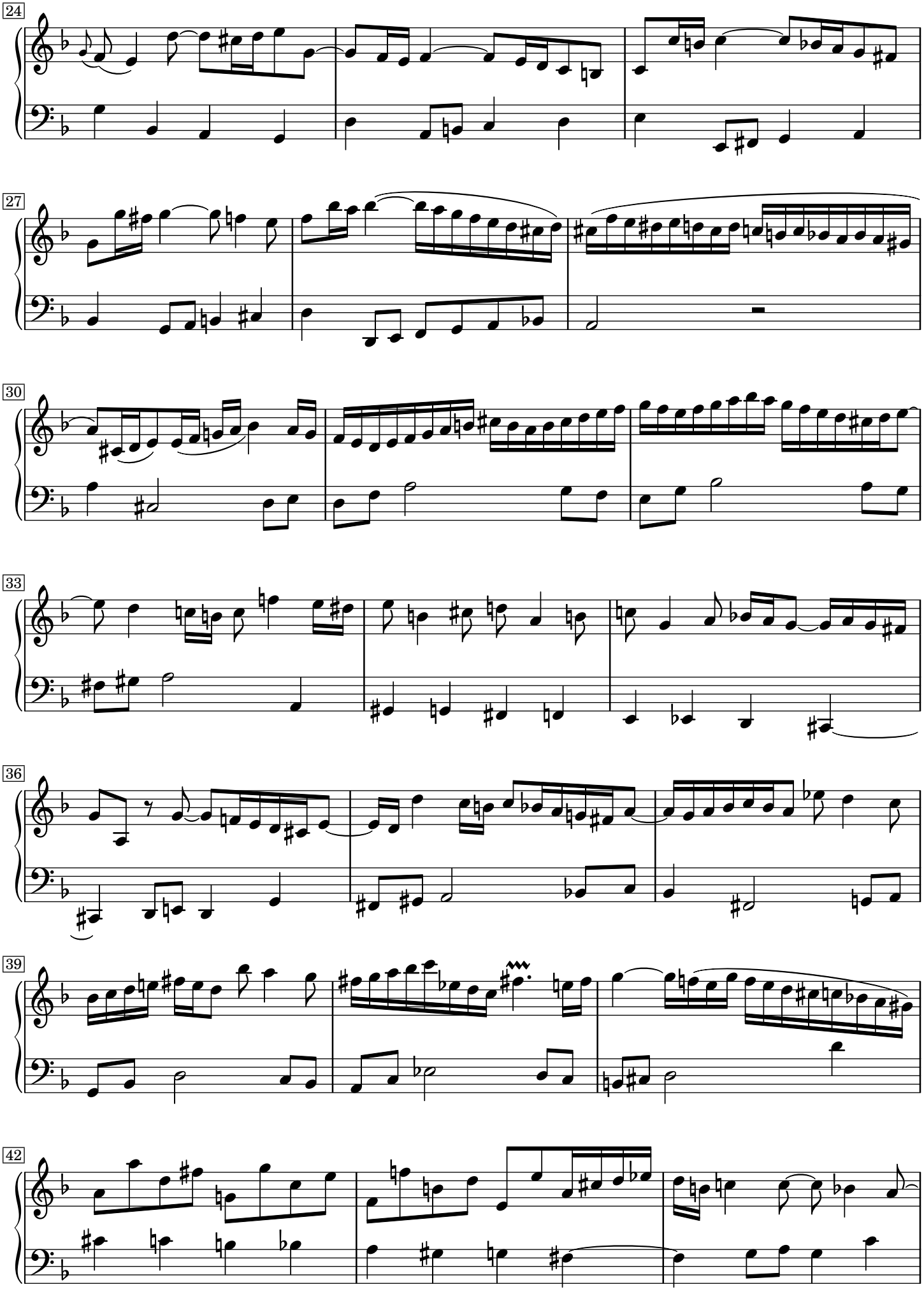
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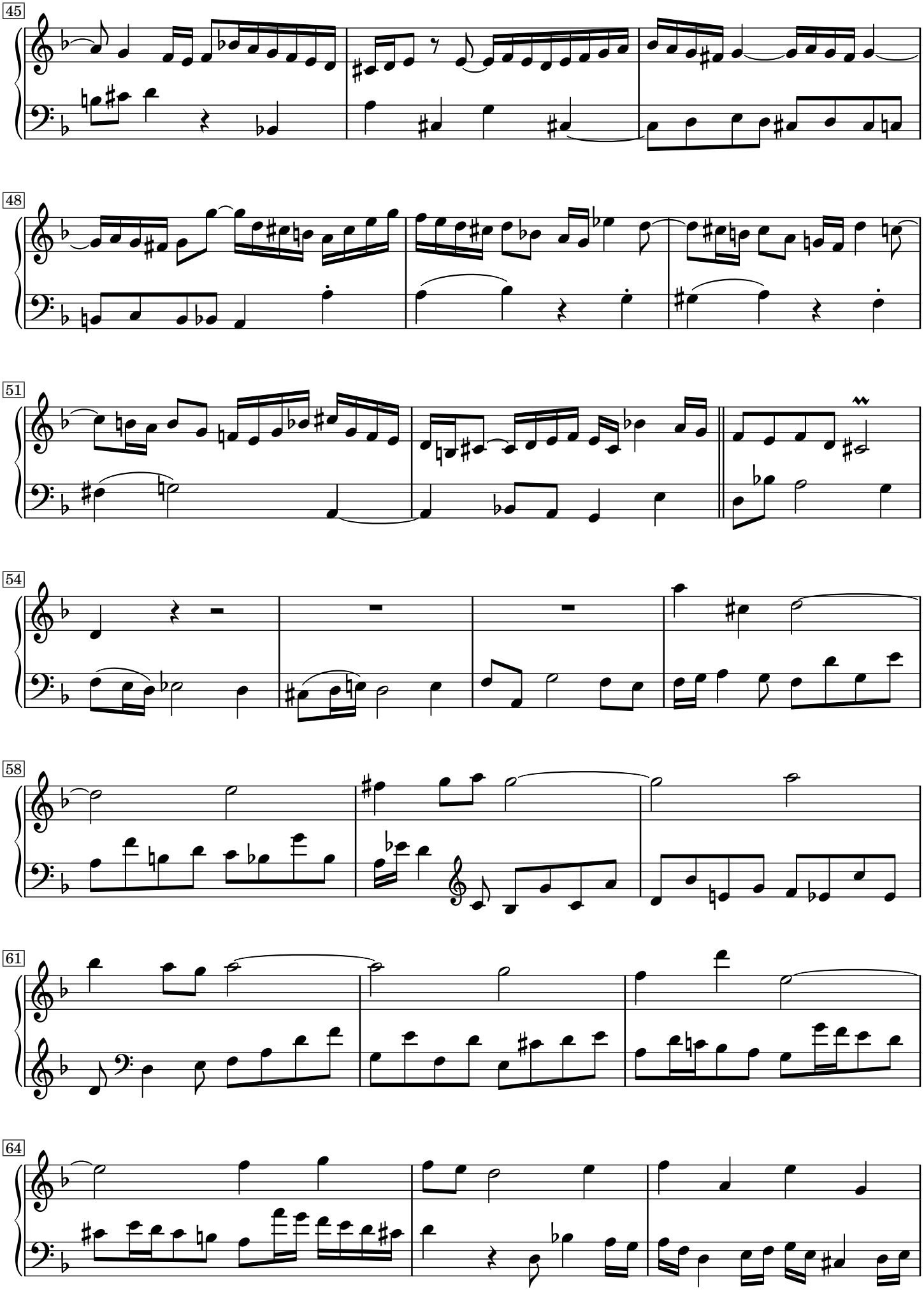
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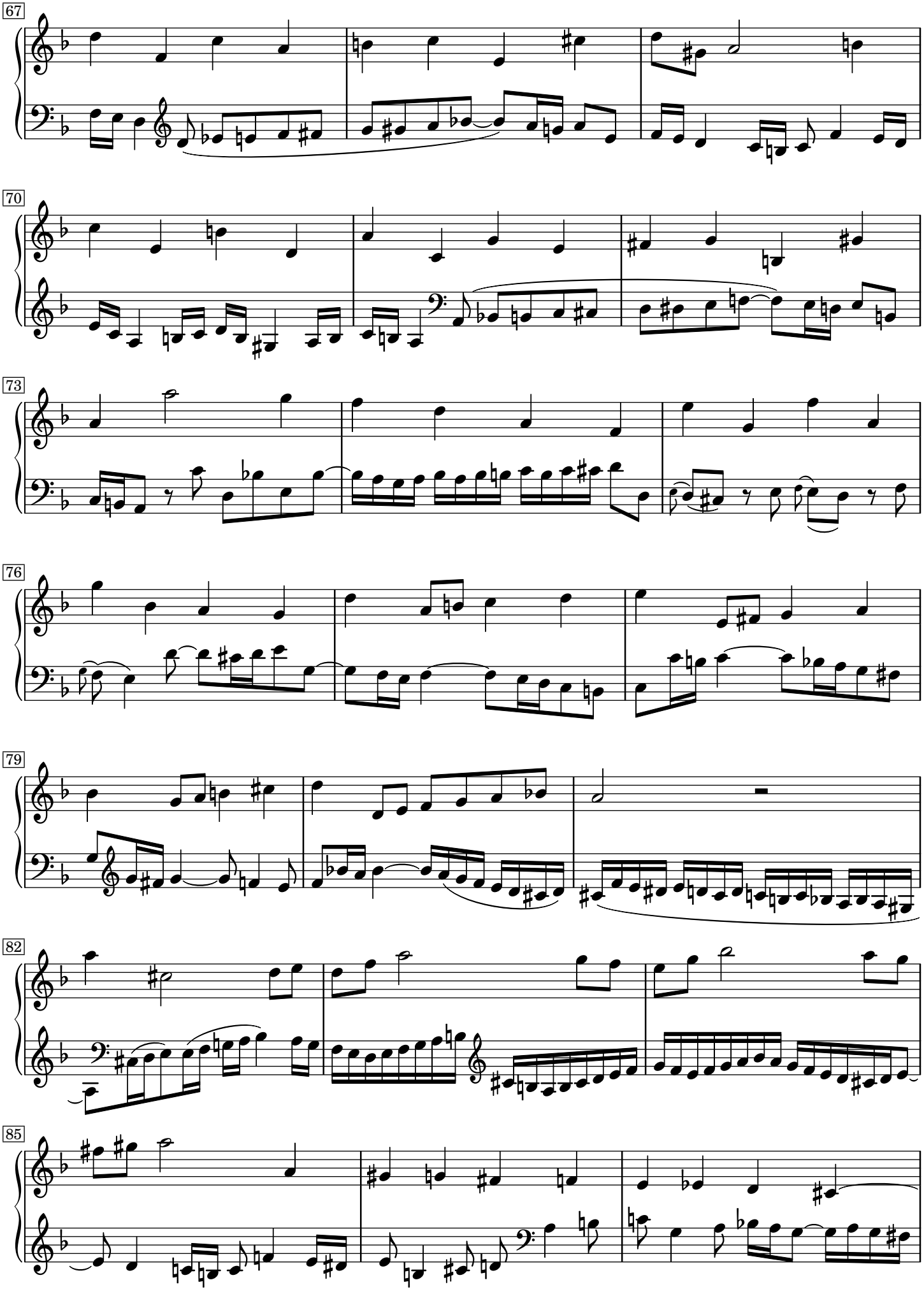
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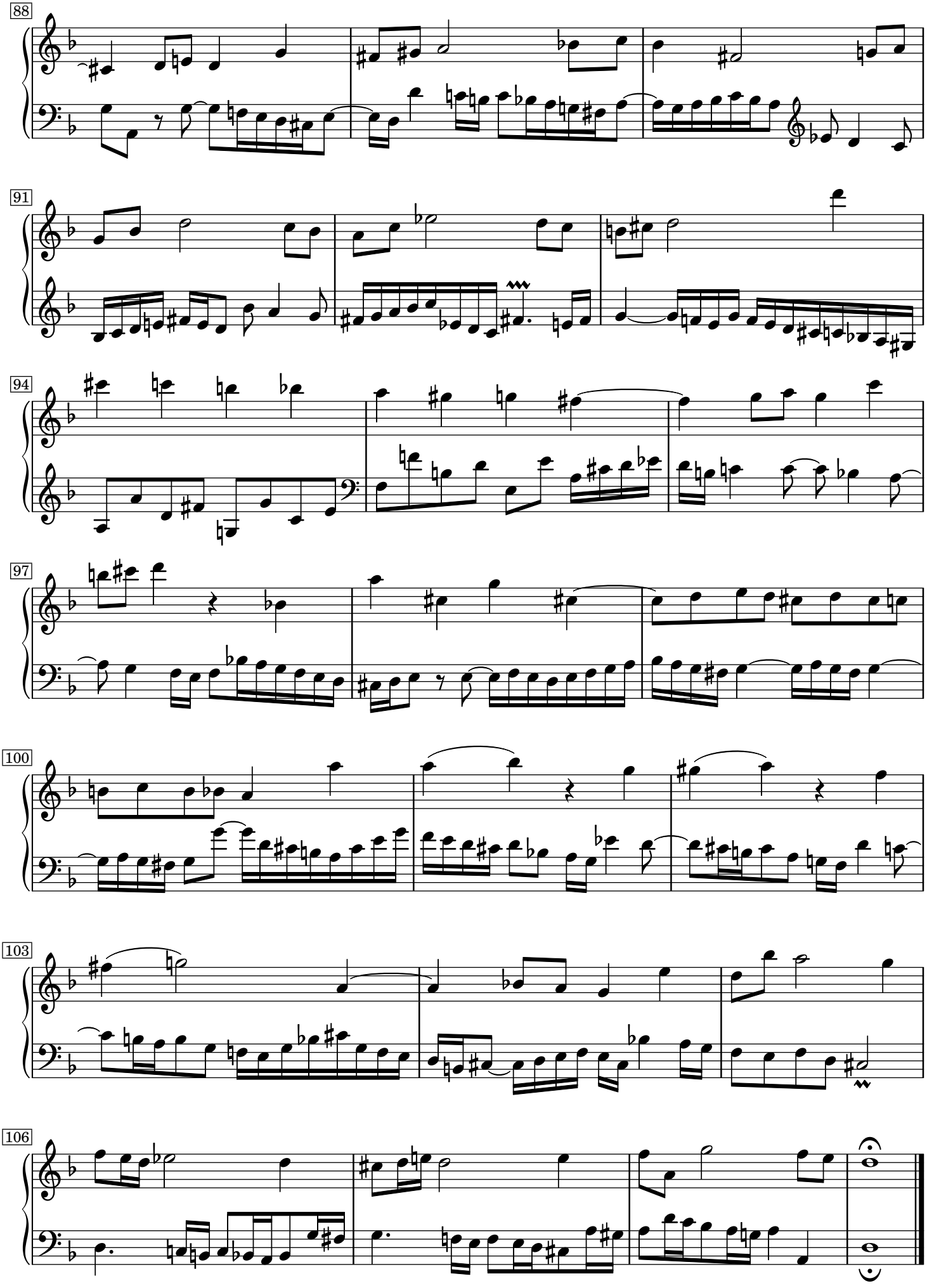
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