Ragax: Ragalur Expressions

Using derivatives to validate Indian Classical Music

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Key Takeaways

Derivatives are **Intuitive** and **Extendable**

- implement matcher
- invent operators
- listen to music

Regular Expressions

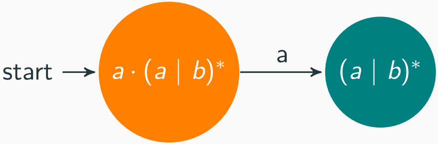
$$a(a|b)*$$

ab	✓
aabbba	✓
ac	×
ba	×

$$a \cdot (a \mid b)^*$$

What is a Brzozowski Derivative

The Brzozowski derivative (1964) [1] of an expression is the expression that is left to match after the given character has been matched.



$$
\begin{aligned}
\partial_a a \cdot b \cdot c &= b \cdot c \\
\partial_a (a \cdot b \mid a \cdot c) &= (b \mid c) \\
\partial_c c &= \epsilon \\
\partial_a a^* &= a^*
\end{aligned}
$$

Basic Operators

empty set	$\emptyset$
empty string	ε
character	a
concatenation	$r \cdot s$
zero or more	r^*
logical or	$r \mid s$

Basic Operators

```
data Regex = EmptySet
  | EmptyString
  | Character Char
  | Concat Regex Regex
  | ZeroOrMore Regex
  | Or Regex Regex
```

Nullable

Does the expression match the empty string.

$$
\begin{aligned}
\nu(\emptyset) &= \text{false} \\
\nu(\varepsilon) &= \text{true} \\
\nu(a) &= \text{false} \\
\nu(r \cdot s) &= \nu(r) \textit{ and } \nu(s) \\
\nu(r^*) &= \text{true} \\
\nu(r \mid s) &= \nu(r) \textit{ or } \nu(s)
\end{aligned}
$$

Nullable

Does the expression match the empty string.

```
nullable :: Regex -> Bool
nullable EmptySet = False
nullable EmptyString = True
nullable Character{} = False
nullable (Concat a b) = nullable a && nullable b
nullable ZeroOrMore{} = True
nullable (Or a b) = nullable a || nullable b
```

Nullable Examples

$$
\begin{aligned}
\nu(a \cdot b \cdot c) &= \times \\
\nu(\varepsilon) &= \checkmark \\
\nu(a \mid b) &= \times \\
\nu(\varepsilon \mid a) &= \checkmark \\
\nu(a \cdot \varepsilon) &= \times \\
\nu((a \cdot b)^*) &= \checkmark \\
\nu(c \cdot (a \cdot b)^*) &= \times
\end{aligned}
$$

Derivative Rules

$$
\begin{aligned}
\partial_a \emptyset &= \emptyset \\
\partial_a \epsilon &= \emptyset \\
\partial_a a &= \epsilon \\
\partial_a b &= \emptyset && \text{for } b \neq a \\
\partial_a (r \cdot s) &= \partial_a r \cdot s && \mathit{not}(\nu(r)) \\
\partial_a (r \cdot s) &= \partial_a r \cdot s \mid \partial_a s && \nu(r) \\
\partial_a (r^*) &= \partial_a r \cdot r^* \\
\partial_a (r \mid s) &= \partial_a r \mid \partial_a s
\end{aligned}
$$

Derivative Rules

```haskell
deriv :: Regex -> Char -> Regex
deriv EmptyString _ = EmptySet
deriv EmptySet _ = EmptySet
deriv (Character a) c = if a == c
  then EmptyString else EmptySet
deriv (Concat r s) c = if nullable r
  then (deriv r c `Concat` s) `Or` deriv s c
  else deriv r c `Concat` s
deriv (ZeroOrMore r) c =
  deriv r c `Concat` ZeroOrMore r
deriv (Or r s) c =
  deriv r c `Or` deriv s c
```

Our regular expression matcher

$$
\begin{aligned}
&\nu(foldl(\partial, r, str)) && && \text{where} \\
&foldl(\partial, r, str) &=& \; r && \text{if } str == "" \\
& &=& \; foldl(\partial, \partial_{s[0]}(r), s[1:]) && \text{otherwise}
\end{aligned}
$$

```haskell
match :: Regex -> String -> Bool
match r str = nullable (foldl deriv r str)
```

```go
func matches(r *expr, str string) bool {
    for _, c := range str {
        r = deriv(r, c)
    }
    return nullable(r)
}
```

Simplification

$$
\begin{aligned}
\emptyset \cdot r &\approx \emptyset \\
r \cdot \emptyset &\approx \emptyset \\
\varepsilon \cdot r &\approx r \\
r \cdot \varepsilon &\approx r \\
\\
r \mid r &\approx r \\
\emptyset \mid r &\approx r \\
\\
(r^*)^* &\approx r^* \\
\varepsilon^* &\approx \varepsilon \\
\emptyset^* &\approx \varepsilon
\end{aligned}
$$

Example: Matching a sequence of notes

Using a regex we can validate the C Major Pentatonic Scale.

$$c \cdot (c|d|e|g|a)^*$$

ceg ✓

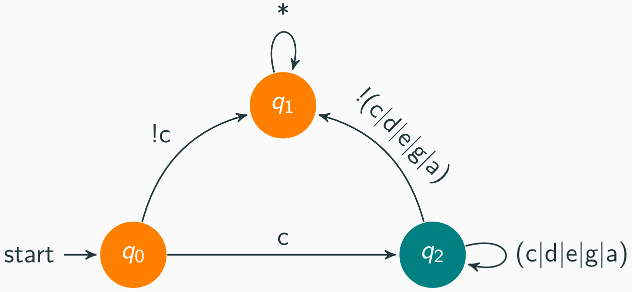
$$
\begin{aligned}
\partial_c c \cdot (c|d|e|g|a)^* &= \varepsilon \cdot (c|d|e|g|a)^* \\
\partial_e \varepsilon \cdot (c|d|e|g|a)^* &= (\emptyset \cdot (c|d|e|g|a)^*) \mid (\emptyset|\emptyset|\varepsilon|\emptyset|\emptyset) \cdot (c|d|e|g|a)^* \\
&= \emptyset \mid (\emptyset|\emptyset|\varepsilon|\emptyset|\emptyset) \cdot (c|d|e|g|a)^* \\
&= (\emptyset|\emptyset|\varepsilon|\emptyset|\emptyset) \cdot (c|d|e|g|a)^* \\
&= \varepsilon \cdot (c|d|e|g|a)^* \\
&= (c|d|e|g|a)^* \\
\partial_g (c|d|e|g|a)^* &= (\emptyset|\emptyset|\emptyset|\varepsilon|\emptyset) \cdot (c|d|e|g|a)^* \\
&= (c|d|e|g|a)^*
\end{aligned}
$$

$$\nu((c|d|e|g|a)^*) = \checkmark$$

Questions?

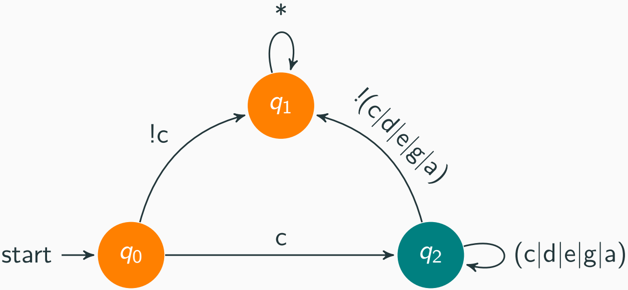
Deterministic Finite Automata

$c \cdot (c|d|e|g|a)^*$



Memoization and Simplification

$$
\begin{aligned}
q_0 &= c \cdot (c|d|e|g|a)^* \\
q_1 &= \emptyset \\
q_2 &= (c|d|e|g|a)^*
\end{aligned}
$$



- Memoizing deriv = transition function
- Memoizing nullable = accept function
- Simplification = minimization [4]

Recursive Regular Expressions

$$((a \cdot b)^* \mid c)$$

Two new operations:

- Define a reference: $\#myref = (a \cdot b)^*$
- Use a reference: $(@myref \mid c)$

$$\begin{aligned} \partial_a @q &= \partial_a \# q \\ \nu(@q) &= \nu(\# q) \end{aligned}$$

Ragas - Indian Classical Music

https://youtu.be/iElMWziZ62A?t=136

Ragas

Ragas are indian version of western scales [5]:

- Stricter
- Next note depends on current note.
- Notes named differently and relative to root note.

Raga	S	r	R	g	G	m	M	P	d	D	n	N
Western	c	c♯	d	d♯	e	f	f♯	g	g♯	a	a♯	b

Example Raga

- Raag Bhupali (a type of Pentatonic scale)
- Ascent: S R G P D S'
- Descent: S' D P G R S

- Western Labeling Relative to c
- Ascent: c d e g a c^1
- Descent: c^1 a g e d c

http://raag-hindustani.com/22_files/ArohBhupali.mp3

Questions?

A Grammar for a Raga

- Raag Bhupali (a type of Pentatonic scale)
- Ascent: S R G P D S'
- Descent: S' D P G R S

$$
\begin{aligned}
\#S &= (S \cdot (@R \mid @D))^* \\
\#R &= R \cdot (@G \mid \varepsilon) \\
\#G &= G \cdot (@P \mid @R) \\
\#P &= P \cdot (@D \mid @G) \\
\#D &= D \cdot (\varepsilon \mid @P)
\end{aligned}
$$

Demo

Context Free Grammars

Left Recursive Raga

$$
\begin{aligned}
\#S &= (S \cdot (@R \mid @D))^* = @S \cdot (S \cdot (@R \mid @D)) \mid \varepsilon \\
\#R &= R \cdot (@G \mid \varepsilon) \\
\#G &= G \cdot (@P \mid @R) \\
\#P &= P \cdot (@D \mid @G) \\
\#D &= D \cdot (\varepsilon \mid @P)
\end{aligned}
$$

nullable and derivative each have infinite recursion.

$$\nu(\#S) = (\nu(@S) \ \textit{and} \ \nu(S \cdot (@R \mid @D))) \ \textit{or} \ \nu(\varepsilon)$$

Parsing with Derivatives

This has been solved using [3] functional concepts:

- Laziness: Infinite Loop $\rightarrow$ Infinite Tree
- Memoization: Infinite Tree $\rightarrow$ Graph
- Least Fixed Point: Graph $\rightarrow$ Value

Laziness

Strict:

```go
func strictPlus(a, b int) int {
    return a + b
}
```

Lazy:

```go
func lazyPlus(a, b int) func() int {
    return func() int {
        return a + b
    }
}
```

Laziness - Infinite Tree

$$\lambda \implies \text{laziness}$$

$$\begin{aligned}
\partial_a(r|s) &= \partial_a r \mid \partial_a s &= \lambda(\partial_a r) \mid \lambda(\partial_a s) \\
\partial_a(r^*) &= \partial_a r \cdot r^* &= \lambda(\partial_a r) \cdot r^* \\
\partial_a(r \cdot s) &= \partial_a r \cdot s \mid \jmath(r) \cdot \partial_a s &= \lambda(\lambda(\partial_a r) \cdot s) \mid \lambda(\lambda(\jmath(r)) \cdot \lambda(\partial_a s))
\end{aligned}$$

$$\begin{aligned}
\jmath(r) &= \epsilon \quad \text{if } \nu(r) \\
&= \emptyset \quad \text{otherwise}
\end{aligned}$$

$$\partial_n \# S = \lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D)))) \mid \lambda(\partial_n \varepsilon)$$

Memoization - Graph

Nullable is called:

$$\begin{aligned} \nu(\partial_n \# S) &= \nu(\lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D)))) \mid \lambda(\partial_n \varepsilon)) \\ &= \nu(\lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D))))) \mid \nu(\lambda(\partial_n \varepsilon)) \end{aligned}$$

Lazy function is executed:

$$\begin{aligned} \lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D)))) &= \partial_n(@S \cdot (S \cdot (@R \mid @D))) \\ &= \lambda(\lambda(\partial_n @S) \cdot \lambda((S \cdot (@R \mid @D)))) \mid \\ &\quad\; \lambda(\lambda(\jmath(@S)) \cdot \lambda(\partial_n(S \cdot (@R \mid @D)))) \\ \lambda(\partial_n @S) &= \partial_n @S \end{aligned}$$

Infinite recursion:

$$\partial_n @S = \partial_n \# S$$

Memoizing closes the loop:

$$\partial_n @S = \lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D)))) \mid \lambda(\partial_n \varepsilon)$$

Memoization

```
func memoize(eval func(a) b) func(a) b {
    mem := make(map[a]b)
    return func(input a) b {
        if output, ok := mem[input]; ok {
            return output
        }
        output := eval(a)
        mem[input] = output
        return output
    }
}
```

Least Fixed Point

$$f(x) = x^2$$

$$f(0) = 0^2$$

$$f(1) = 1^2$$

$$\text{fixed points} = \{0, 1\}$$

$$\text{least fixed point} = 0$$

Least Fixed Point of Derivative

$$\partial_a r = r$$

$$\partial_a \emptyset = \emptyset$$

$$\partial_a a^* = a^*$$

$$\text{fixed points} = \{\emptyset, a^*\}$$

$$\text{least fixed point} = \emptyset$$

Least Fixed Point - Graph

Nullable is relentless:

$$
\begin{aligned}
\nu(\lambda(\partial_n \# S)) &= \ldots \\
\nu(\lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D))))) \mid \nu(\lambda(\partial_n \varepsilon)) &= \ldots \\
\nu(\lambda(\lambda(\partial_n @S) \cdot \lambda(\ldots))) &= \ldots \\
\nu(\lambda(\partial_n @S)) &= \nu(\text{fix}) \\
&= \nu(\emptyset) \\
&= \text{false} \\
\nu(\lambda(\lambda(\partial_n @S) \cdot \lambda(\ldots))) &= \text{false} \ \& \ \text{false} \\
\nu(\lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D))))) \mid \nu(\lambda(\partial_n \varepsilon)) &= \text{false} \mid \text{false} \\
\nu(\lambda(\partial_n \# S)) &= \text{false}
\end{aligned}
$$

http://awalterschulze.github.io/ragax/

Yacc is Dead

Yacc, Antlr, Flex, Bison, etc. perform better.

But derivatives:

- more intuitive than LR and LALR parsers;
- only use functional techniques;
- recognize generalized Context Free Grammars, not just a subset.

Trees

Relaxing

http://relaxng.org/ [2] - RELAX NG is a schema language for XML, like XSchema and DTD.

Derivatives used for Implementation and Specification.

Polymorphic Regular Expressions: Characters => XMLNodes.

New Operators:

$$\begin{aligned}
\partial_a(r \mathbin{\&\&} s) &= (\partial_a r \mathbin{\&\&} s) \mid (\partial_a s \mathbin{\&\&} r) \\
\nu(r \mathbin{\&\&} s) &= \nu(r) \text{ and } \nu(s) \\
\emptyset \mathbin{\&\&} r &\approx \emptyset \\
\varepsilon \mathbin{\&\&} r &\approx r \\
\\
\partial_a!(r) &= !(\partial_a r) \\
\nu(!(r)) &= \text{not}(\nu(r)) \\
\\
(r)? &\approx r \mid \varepsilon
\end{aligned}$$

TreeNode

```
data Expr = ...
    NodeExpr String Expr
    ...

deriv :: Expr -> Tree -> Expr
deriv (NodeExpr nameExpr childExpr) (Node name children) =
  if nameExpr == name &&
        nullable (foldl deriv childExpr children)
    then Empty
    else EmptySet

nullable NodeExpr{} = False
```

https://youtu.be/SvjSP2xYZm8

https://katydid.github.io

Katydid: Relapse

Relapse: Tree Validation Language.

JSON, Protobufs, Reflected Go Structures and XML

Go, Haskell + Cross language testsuite

New Operators:

$$
\begin{array}{lcl}
\partial_a(r \mathbin{\&} s) & = & (\partial_a r \mathbin{\&} \partial_a s) \\
\nu(r \mathbin{\&} s) & = & \nu(r) \text{ and } \nu(s) \\
\emptyset \mathbin{\&} r & \approx & \emptyset \\
r \mathbin{\&} r & \approx & r \\
\\
* & \approx & !(\emptyset) \\
\\
.r & \approx & * \cdot r \cdot *
\end{array}
$$

https://github.com/katydid/katydid-haskell `http://katydid.github.io/play/`

`http://katydid.github.io/tour/`

References

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