Explanation of the problem with splitting the wave action equation for ice and no ice sources

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This document uses simple simulations to illustrate where the *time-splitting* solution is inaccurate. Consider the wave equation written:

$$\frac{dN}{dt} = \beta N$$

with exact solution for one global time step dt

$$N(t+dt) = N(t) \times \exp(\beta dt)$$

We assume beta includes both hypothetical attenuation by sea ice and hypothetical positive wave energy input sources. In reality, the input source is more complicated, but for illustrative purposes, we can assume that the ice attenuation and input are both linear and can be combined into a single beta coefficient.

In this case,

$$\beta = -\alpha c_g + \gamma,$$

where alpha is the attenuation rate in ice, c_g is the group velocity, and gamma is an arbitrary number representing input. If gamma is very large, waves should grow despite attenuation by ice (i.e. beta becomes positive).

To inspect the time-splitting error, we look at three solutions:

1) **Exact solution for hypothetical linear example**
   $N(t+dt) = N(t) \times \exp(\beta dt)$
2) **Semi-implicit approximate solution for combination of attenuation and input without time splitting**

$$N(t+dt) = N(t)\left(\frac{1+\beta dt}{max(1, 1-\beta dt)}\right)$$

3) **Time-split solution – splitting input and attenuation, with input solved with semi-implicit approximation and attenuation solved exactly (roughly as in ww3)**

$$N^* = N(t)\left(\frac{1+\gamma dt}{max(1, 1-\gamma dt)}\right)$$
$$N(t+dt) = N^* \times \exp(-\alpha c_g dt)$$

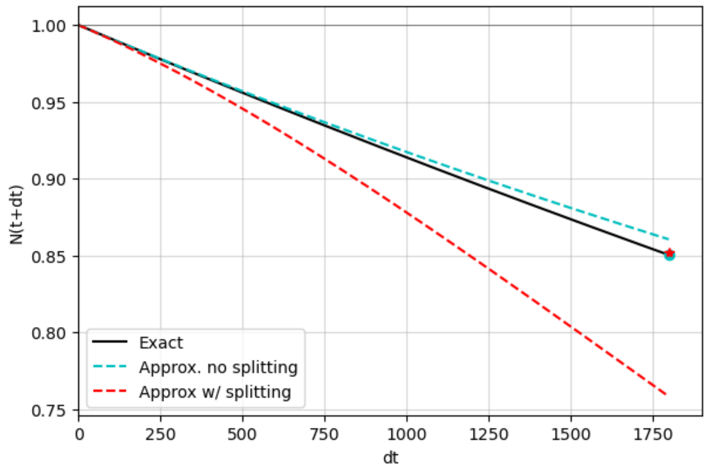
We show these solutions for **3 cases:**

a) **Small input (net effect is decay)**
b) **Large input (net effect is growth)**

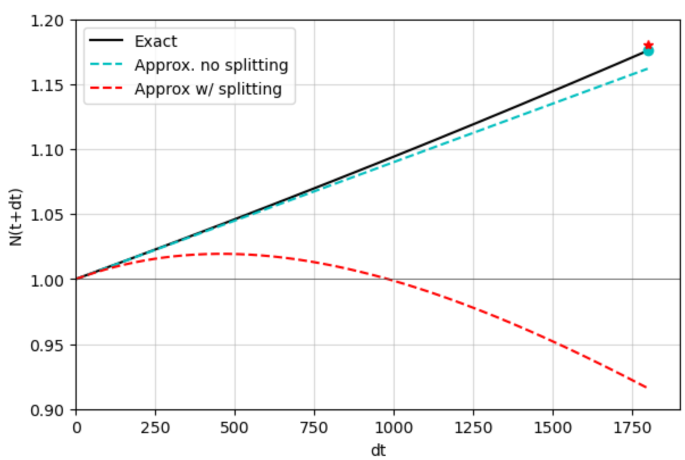
c) **No input; only exponential decay from ice**

In all cases, we show illustrative values for a 5-second wave (frequency = 0.2 Hz). The attenuation rate is alpha = 10^(-4), which is the attenuation rate for IC4M1 on a 5-second wave. The group velocity is approximately 0.8 * T where T is the period. The timestep dt is sensitized from 0 to 1800 seconds. Initial wave action N(t) = 1 (in normalized units).

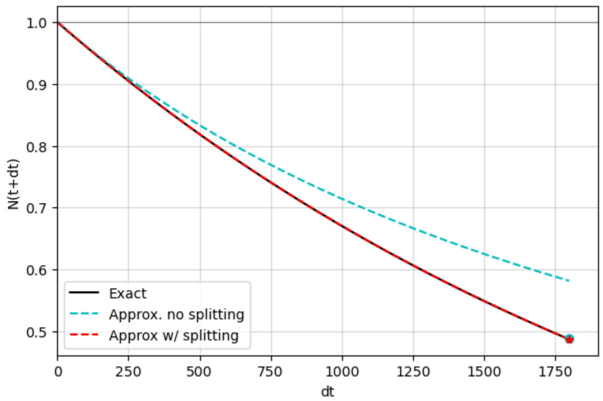
Case (a): Small input (net effect is decay).
The no time-splitting solution is more accurate than the time-splitting solution, especially for the large timesteps used in climate-scale models (e.g. dt>500 s).
*Beta = -9 * 10^(-5)*



Case (b): Large input (net effect should be growth).
The no time-splitting solution is more accurate than the time-splitting solution, especially for the large timesteps used in climate-scale models (e.g. dt>500 s).
*Beta = +9 * 10^(-5)*



Case (c): No input; only exponential decay from ice.
Time-splitting solution is exact when there is no input (and superior to the no time-splitting solution in this case), though the waves are generally heavily damped and therefore not very important. *Beta = -alpha * cg*



Subcycling
When gamma is nonzero and the global timestep is large, subcycling can be helpful.

In all figures at dt=1800 s, we show the approximate solutions with 90 subcycles (dtmin=20 s) with a

- Cyan dot for the "no time splitting" solution with no timesplitting, the equation is solved repeatedly with timestep dtmin until reaching the global timestep
- Red star for the "time splitting" solution, only the approximate portion (with the gamma coefficient) is solved repeatedly using dtmin; then ice attenuation (-alpha * cg) is applied at the end (upon reaching the global timestep)

Generally both solutions very good, so this is another option.

In our research we eliminated the time splitting and forced subcycling whenever the sea ice fraction is between something like 5 and 95%. We found the wave energy was substantially higher, since the time splitting solution for large dt overdamps the solution.