Model Predictive Control

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Kawa-lab seminar, 4/13

Motivation

Model Predictive Control is attracting researchers in many field.

Space X’s autonomous landing

Autonomous driving Car

Boston Dynamics’s robots

Let's start with a simple optimal control problem

A single dimensional car-like dynamics

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A quadratic stage cost function

$$\Sigma_{\mathrm{k}=0}^{\infty} \left(\boldsymbol{x}_k^T Q \boldsymbol{x}_k + Ru_k^2\right) \qquad Q = \mathrm{diag}(1, 1), R = 0.01$$

Finding a sequence of actions $\{u_k\}_{k=0}^{\infty}$ that will minimize $\Sigma_{\mathrm{k}=0}^{\infty} \left(\boldsymbol{x}_k^T Q \boldsymbol{x}_k + Ru_k^2\right)$ subject to a set of constraints $x_{k+1} = Ax_k + Bu_k$ is called discrete LQR problem.

The optimal set of actions are solved as $u_k^* = K\boldsymbol{x}_k$

where

$$K = -(R + B^T P B)^{-1} B^T P A$$
$$P = C^T Q C + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

The simple problem with some constraints

A single dimensional car-like dynamics

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k$$

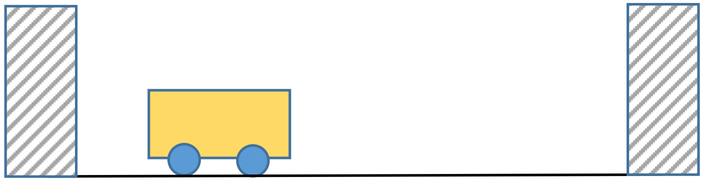
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with additional constraints

$-1 < x_1 < 0.1$

$-0.5 < x_2 < 0.5$ ⟵ acc. limit

$-1 < u < 1$ ⟵ velocity limit



wall ($x_1 = -1$)   wall ($x_1 = 0.1$)

Of course, the input that LQR gives will sometime violate the above constraints.

A trick

Intuitively, an actions c_k to the LQR policy as the following way would works well:

$$u_k = Kx_k + c_k$$

because, although Kx_k returns infeasible input regarding the constraints,
the c_k forcibly make it feasible.

MPC using Dual mode-prediction paradigm

In that case, the optimization problem is formulated as:

Find $\{c_k\}_{k=0}^{\infty}$ that minimizes $\Sigma_{k=0}^{\infty}\left(\boldsymbol{x}_k^T Q \boldsymbol{x}_k + R u_k^2\right)$

subject to
$$\begin{cases} \boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k \\ u_k = K\boldsymbol{x}_k + c_k \\ F\boldsymbol{x}_k + Gu_k \leq \mathbf{1} \end{cases}$$

⇓

$$(F + GK)\boldsymbol{x}_k + Gc_k \leq \mathbf{1}$$
$$\boldsymbol{x}_{k+1} = (A + BK)\boldsymbol{x}_k + Gc_k$$

(example of $F\boldsymbol{x} + Gu \leq \mathbf{1}$)

$$F = \begin{bmatrix} -1 & 0 \\ 10 & 0 \\ -2 & 0 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$-1 < x_1 < 0.1$

$-0.5 < x_2 < 0.5$

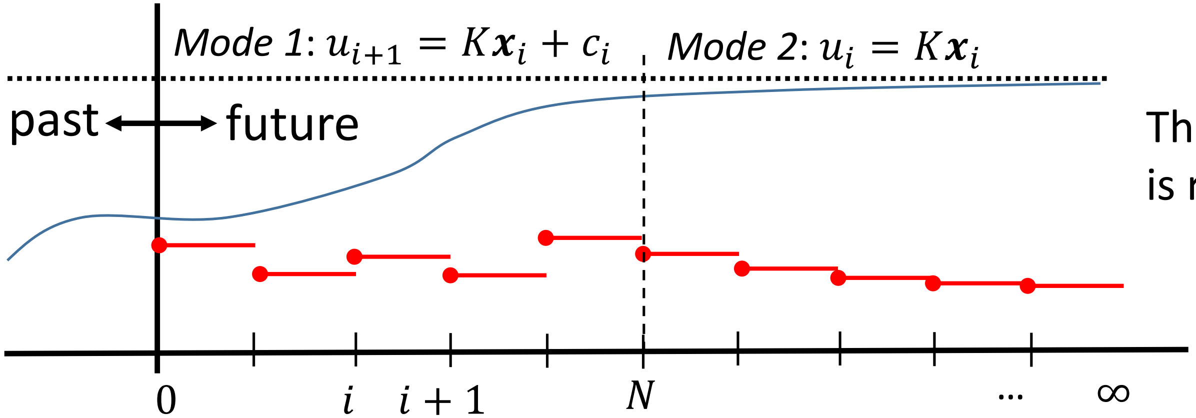
$-1 < u < 1$

※ Note that $\{\boldsymbol{x} | F\boldsymbol{x} + Gu \leq \mathbf{1}\}$ is a convex set.

MPC using Dual mode-prediction paradigm

Q. But how we can avoid that “infinite dimensional optimization” problem?

A. Dual mode-prediction paradigm



The number of optimization variables is reduced.

$\{c_i\}_{i=0}^{\infty} \rightarrow \{c_i\}_{i=0}^{N-1}$

infinite → N

Model Predictive Control with Dual mode-prediction paradigm

*Hereinafter we refer to c_i, u_i and $\boldsymbol{x}_{i|k}$ solved at time step k as $c_{i|k}$, $x_{i|k}$ and $u_{i|k}$, respectively.

1. Solve above-mentioned optimization problem at every time-step and obtain set of $\{c_{i|k}\}_{i=0}^{N-1}$.
2. Use $u_{0|\mathrm{k}} = Kx_{0|k} + c_{0|k}$ for single time step.

By iterating the above process, we can deal with additive disturbance and modelling error. Of course, without these error, the iteration does not make sense.

Model Predictive Control

※ Hereinafter Let me refer to (A+BK) as ϕ

Optimization problem can be written as follows.

Find $\{c_{i|k}\}_{i=0}^{N-1}$ that minimizes $\Sigma_{\mathrm{i=0}}^{N-1}\left(\boldsymbol{x}_{i|k}^T Q \boldsymbol{x}_{i|k} + R u_{i|k}^2\right) + \boldsymbol{x}_{N|k}^T P \boldsymbol{x}_{N|k}$

subject to $\begin{cases} \boldsymbol{x}_{i+1|k} = \phi \boldsymbol{x}_{i|k} + G c_{i|k} \\ (F + GK)\boldsymbol{x}_{i|k} + G c_{i|k} \leq \mathbf{1} \end{cases}$

LQR cost at step N
which is analytically available

Q. In this formulation, it is obvious that when $i \leq N-1$,
constraint $F\boldsymbol{x}_{i|k} + G u_{i|k} \leq \mathbf{1}$ holds. However, it is not always the case when $k > N$.
How can we deal with this problem?

A. Add a terminal constraint

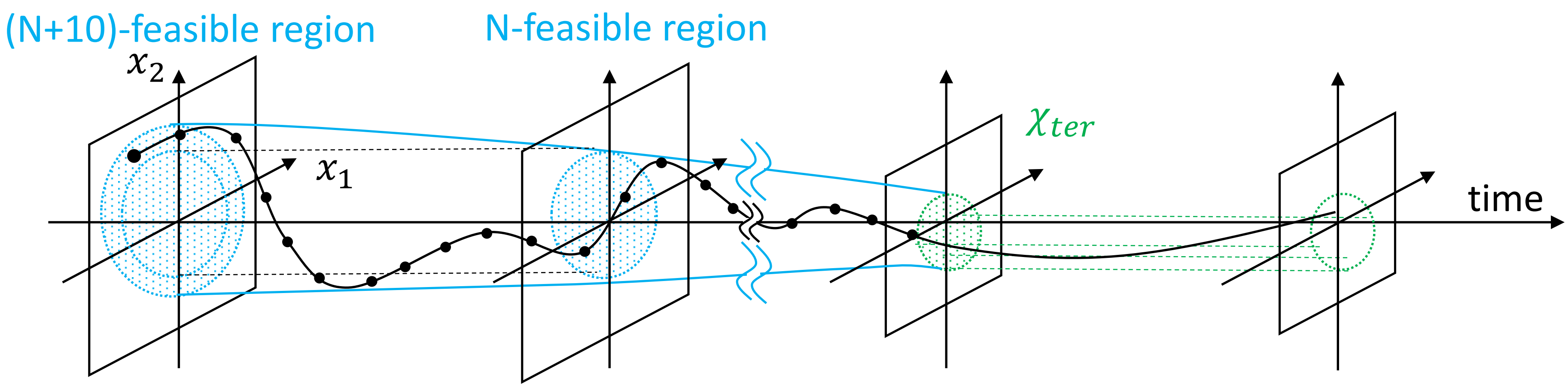
Suppose we can find a set χ_{ter} in which any $\boldsymbol{x} \in \chi_{ter}$ always will be drives into the set χ following the derived dynamics of LQR (i.e. $\boldsymbol{x}_{i+1|k} = \phi \boldsymbol{x}_{i|k}$) satisfying the constraint $F\boldsymbol{x}_{i|k} + G u_{i|k} \leq \mathbf{1}$.

If we add a constraint $\boldsymbol{x}_{i|k} \in \chi_{ter}$ to the above optimization problem,
$F\boldsymbol{x}_{i|k} + G u_{i|k} \leq \mathbf{1}$will be always satisfied even when $i > N$.

Maximum Positive Invariant (MPI) Set

The existence of set like χ_{ter} is obvious if we think about a set $S \coloneqq \{\mathrm{x}: |\boldsymbol{x}| < \epsilon\}$.
In that region, all $\boldsymbol{x} \to \mathbf{0}$ without violating the constraints.

But if we take a smaller χ_{ter}, of course constraints becomes more strict, which requires many prediction step and makes optimization computationally-heavy.

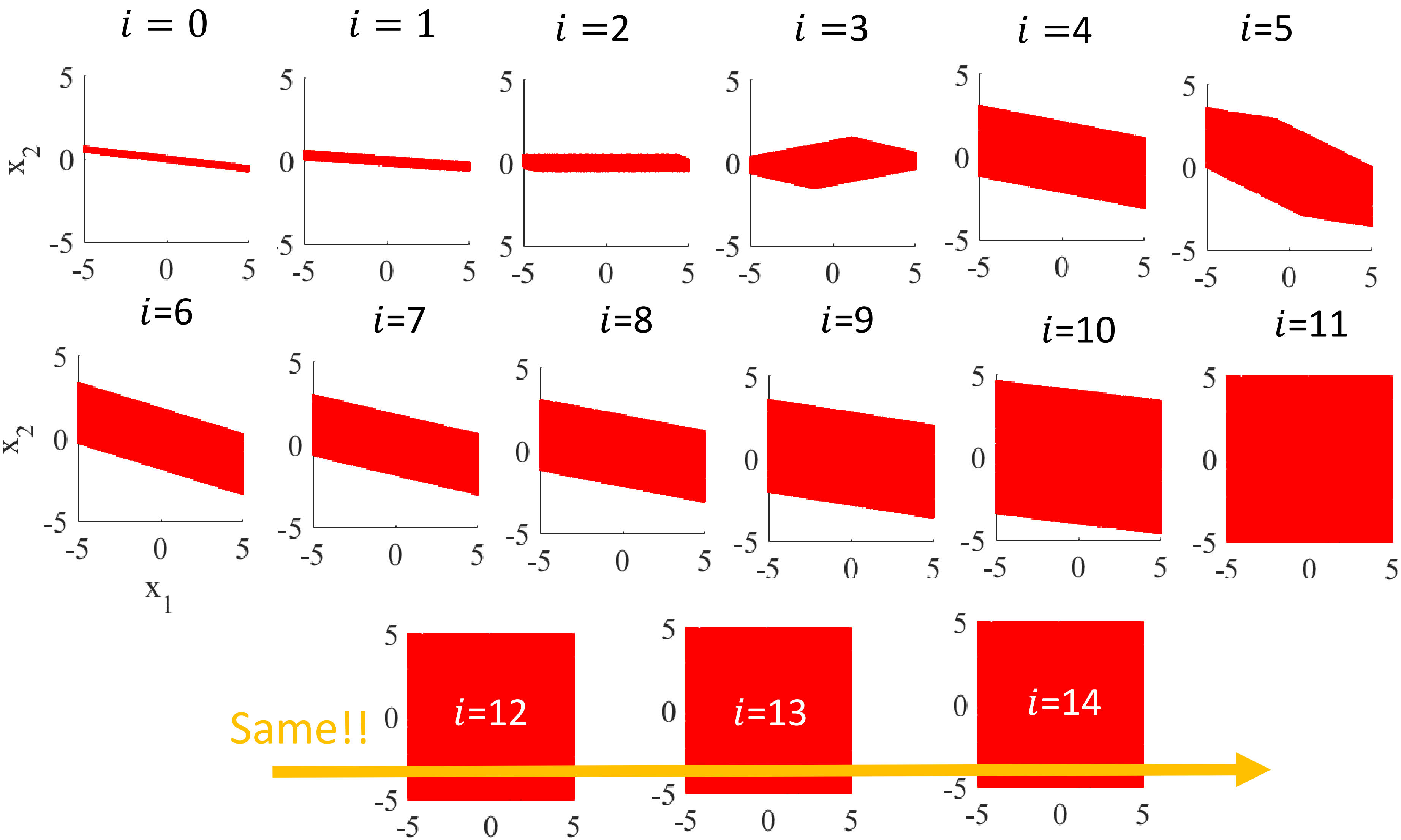


So what we want to have is the "largest χ_{ter}",
which we call Maximum Positive Invariant (MPI) Set χ_{MPI}.

Find MPI Set by Monte Carlo

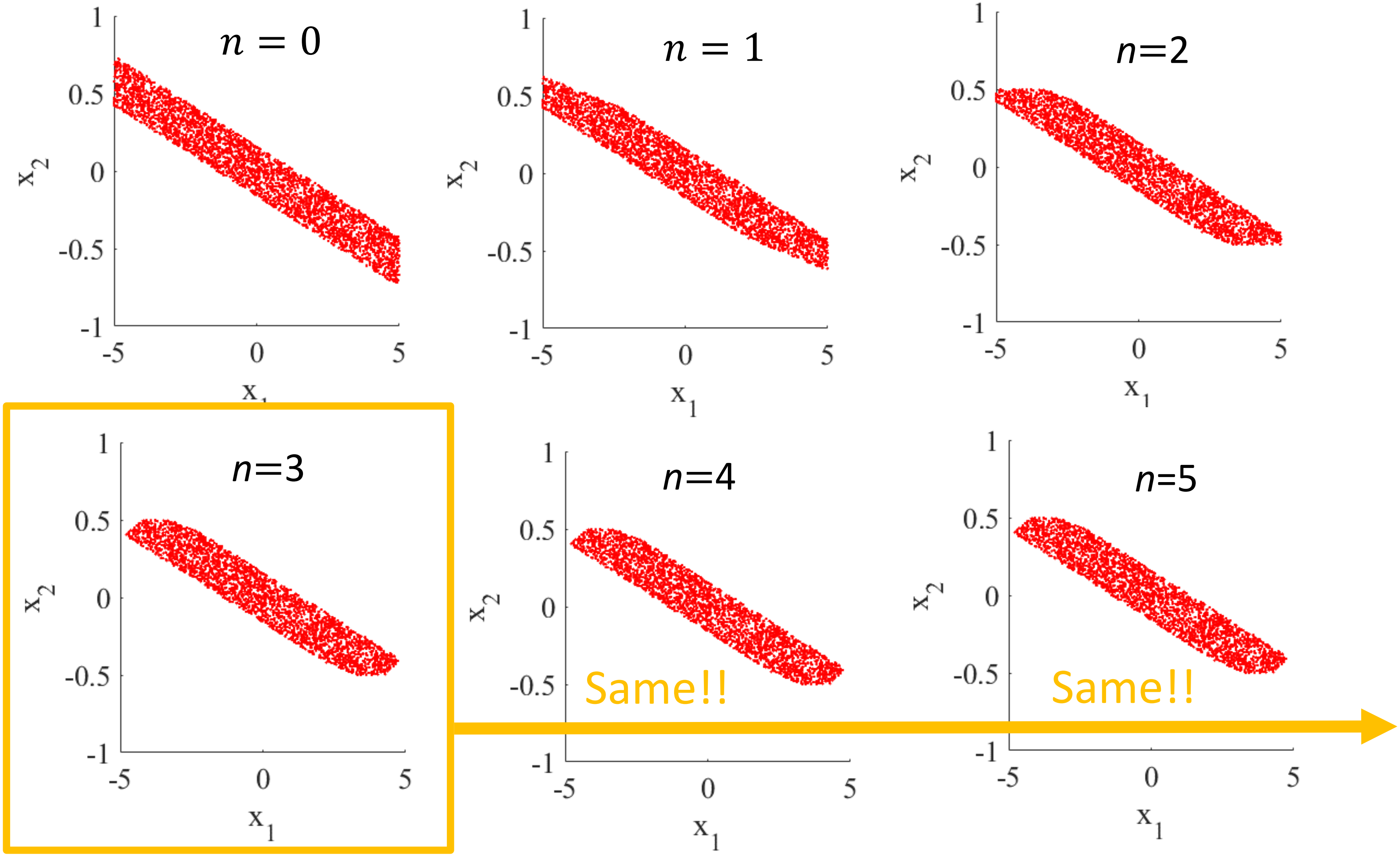
Let's find χ_{MPI} for the *simple car* case using Monte Carlo !!

Scatter $\chi^{(i)} := \{\boldsymbol{x} | (F + GK)\phi^i \boldsymbol{x} \leq \mathbf{1}\}$



Find MPI Set by Monte Carlo

When we calculate the intersection $\cap_{i=0}^{n} \chi^{i}$...



We can say that $\chi_{MPI} = \cap_{i=0}^{3} \chi^{i}$

Find MPI Set Semi-analytically

What we did actually using Monte Carlo is obtaining the intersection set

$$\chi_{MPI} = \cap_{n=0}^{\nu}, \quad \chi^{(n)} := \{\boldsymbol{x} | (F + GK)\phi^i \boldsymbol{x} \le \mathbf{1}\}.$$

where ν is smallest positive integer such that $(F + GK)\phi^{\nu+1}\boldsymbol{x} \le \mathbf{1}$. (*)

The terminal constraint $\boldsymbol{x}_{i|k} \in \chi_{ter}$ for the optimization can be expressed as
The following inequality:

$$V_T \boldsymbol{x}_{i|k} \le \mathbf{1}, \quad V_T = \begin{bmatrix} F + GK \\ (F + GK)\phi \\ \vdots \\ (F + GK)\phi^{\nu} \end{bmatrix}$$

Now that we can have the following.

Model Predictive Control using the Dual-Mode Prediction

Find $\{c_k\}_{i=0}^{N-1}$ that minimizes $\Sigma_{\mathrm{i}=0}^{N-1} \left(\boldsymbol{x}_{i|k}^T Q \boldsymbol{x}_{i|k} + R u_{i|k}^2 \right) + \boldsymbol{x}_{N|k}^T P \boldsymbol{x}_{N|k}$

subject to
$$\begin{cases} \boldsymbol{x}_{i+1|k} = \phi \boldsymbol{x}_{i|k} + G c_i \\ (F + GK)\boldsymbol{x}_{i|k} + G c_k \le \mathbf{1} \\ V_T \boldsymbol{x}_{i|k} \le \mathbf{1} \end{cases}$$

(* Please see Ref. 2, pp. 22—23 for the exact proof)

Find MPI Set Semi-analytically

ν can be found by iteration of the solve-and-check procedure of the following linear programming (LP):

While$(\mathrm{m} \neq \nu)${

For j=0 : n_c linear programming

$$\boldsymbol{x}_j^{\max} \leftarrow \mathrm{argmax}_{\boldsymbol{x}} \, (F+GK)_j \phi^{m+1} \boldsymbol{x}$$

$$\text{subject to } (F+GK)\phi^{\nu+1}\boldsymbol{x} \leq \mathbf{1}, i = 0 \dots m$$

end for

if $(F+GK)\phi^{n+1}\boldsymbol{x}_j^{\mathbf{max}} \leq 1$ **then** $\mathrm{m} = \nu$ is proven

end while

Return m

where n_c is the number of constraints (i.e. number of rows of $F+GK$) and $(F+GK)_j$ is the j-th row vector of $(F+GK)$.

※Don’ t worry. This procedure is performed offline.

Model Predictive Control with Uncertainty

Consider a dynamics with additive disturbance (can be caused by modelling-error)

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + \boldsymbol{w}_k$$

where $w_k \in W$ and W is a convex set.

For this problem, disturbance invariant set Z plays a key role.

Def. (disturbance invariant set)

A set Z is a disturbance invariant set if $(A + BK)\boldsymbol{x} + \boldsymbol{w} \in Z$ is satisfied for all $\boldsymbol{x} \in Z$, and for all $\boldsymbol{w} \in W$.

The following contents may be explained on the white board:

1. Minkovski sum
2. Minimum disturbance invariant set (1dim)
3. Minimum disturbance invariant set (n-dim)
4. Robust Model Predictive Control

Reference

[1] Kouvaritakis, Basil, and Mark Cannon. *Model predictive control*. Springer, Switzerland, 2016. (*pdf is freely available online)

[2] Langson, W., Chryssochoos, I., Raković, S. V., & Mayne, D. Q. (2004). Robust model predictive control using tubes. *Automatica*, *40*(1), 125-133.