Cobordism Categories

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enriched bicats p 68

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We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance.

Moore and Seiberg [37]

Acknowledgements

When Martin Hyland took over the task of supervising me, I was ready to leave Cambridge without attempting a Ph.D. That I have now submitted this dissertation is thanks to his constant concern and patience. I can only hope I was not too much of a trial.

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Sean Carmody

Cambridge

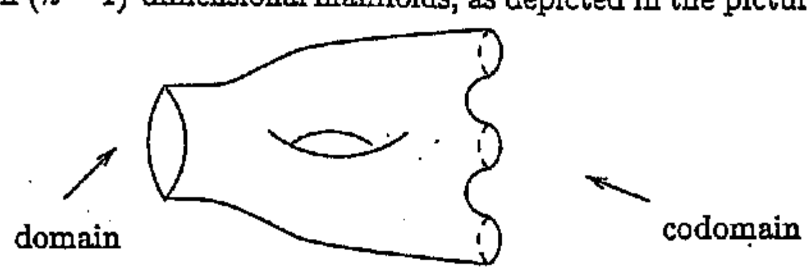
May 30, 1995

Contents

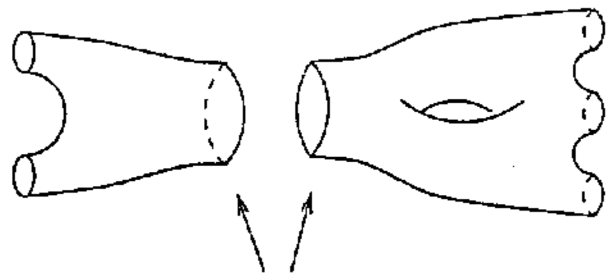
Introduction

The concept of a topological quantum field theory was introduced by Atiyah [1] as a step towards combining quantum field theory and general relativity. Needless to say, this is only one of a number of approaches to the elusive nirvana of a quantum theory of gravity. Topological quantum field theories associate algebraic structures to manifolds in a way which interacts well with the process of "gluing" manifolds along a common boundary. In their survey of conformal field theory, Moore and Seiberg [37] somewhat reluctantly introduce some category theory in an appendix, while other authors, including Blanchet, Habegger, Masbaum and Vogel [7], Freed [13, 14] and Reshetikhin and Turaev [40], have adopted the language of categories and their higher-dimensional analogues more enthusiastically. The single most important development in the application of category theory to topological quantum field theories is due to Segal [42], who realised that the "gluing" operation itself can be described in category theoretic terms.

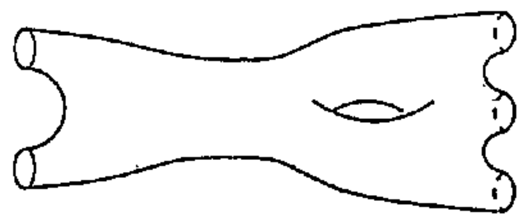
Segal's insight is to think of an n-dimensional manifold with boundary as an "arrow" between $(n-1)$-dimensional manifolds, as depicted in the picture below.



Cobordisms can then "glued" along common boundary components, as illustrated below. This operation of "gluing" behaves very much like a composition law for a category.



common boundary



After "gluing"

The resulting structure is the archetypal "cobordism category"[1] and we refer to it as *n-Cobord*. A topological field theory is then a representation of *n-Cobord* in an "algebraic" cobordism category. It is this point of view, also taken up by Walker [47], Baez and Dolan [2] and others, which is the focus of this dissertation. There is, however, an important complication. The gluing of manifolds is not strictly associative, so *n-Cobord* is not a category, but in fact a *bi*category. This is the first step into the realm of "higher-dimensional algebra".

The problem facing researchers investigating topological quantum field theories is that very little has been published on higher-dimensional algebra and indeed the subject is not well understood. Most authors have avoided handling higher-dimensional algebra rigourously, preferring to give a vague sketch of the subject and then refer to the recently published work of Kapranov and Voevodsky [22] for the detailed foundations of the subject. Kapranov and Voevodsky give definitions of braided monoidal categories, bicategories and braided monoidal 2-categories; however their treatment has a considerable number of errors, which is particularly disappointing in a work cited so often. We discuss some of their errors in an appendix.

[1]We are not using the term "cobordism category" in the same sense as Stong [43].

In this dissertation we give detailed descriptions of some of the categorical structures which arise in the study of topological quantum field theories. The aim of this treatment is to come closer to to a definition of cobordism category. In other words, we attempt to characterise the algebraic properties of *n-Cobord*. Our approach is close in spirit to the work of Ross Street. His perspective is exemplified by Street [46], an excellent, though sadly unpublished, survey of categorical structures as they apply to physics and other subjects. The measure of success of our work is that we are able to present an account of well-known simple topological quantum field theories with a rigour existence approaches have been unable to attain. Although these simple theories do not depend on the full structure of bicategories, we believe these theories are made clearer in the bicategorical setting.

The thesis begins with material that is largely introductory. In Chapter we review some elementary features of the theory of enriched categories. Our treatment of this subject follows Lawvere [32] in giving strong emphasis to enriched bimodules, partly because enriched bimodules provide a very natural example of a cobordism category. The discussion of enriched category theory concludes with the concept of Cauchy completeness, a key to understanding the notion of "dual objects" which arise as an algebraic analogue to reversing the orientation of manifolds in *n-Cobord*.

Bicategories are introduced in Chapter 1. Although it has become common to call bicategories "2-categories" and 2-categories "strict 2-categories", we have retained the traditional terminology. Defining bicategorical morphisms, transformations and modifications and discussing coherence theorems for bicategories concludes the introductory material and we proceed in Chapter 2 to the subject of monoidal bicategories. This material is essentially new, as no complete and correct definition of monoidal bicategories has yet been published, although this will change with the publication of the paper by Gordon, Power and Street [15] on "tricategories" since a monoidal bicategory can be thought of as a one-object tricategory. For our purposes the most important example of a monoidal bicategory is, of course, *n-Cobord*. Therefore a cobordism category should, at the very least, have the structure of a monoidal bicategory. Since monoidal bicategories are rather intricate algebraic structures, it comes as no surprise that only very simple topological field theories are well understood. Monoidal bicategories also allow us to formulate the notion of an enriched bicategory. Although new, this concept is a very natural generalization of enriched category and promises to be of significance in the future development of higher-dimensional algebra.

In Chapter 3 we return to the subject of "dual objects". Compact closed categories are categories with dual objects and have been the subject of much study. Several authors have implicitly acknowledged that there is an analogous notion of compact closed bicategory, however an explicit definition has never been published. We do define compact closed bicategories in detail and observe that *n-Cobord* is an example of such a structure.

As a preliminary definition of cobordism category, a compact closed bicategory has sufficient structure to allow a treatment of a simple 2-dimensional topological quantum field theory. A standard result says that this type of field theory amounts to a "Frobenius algebra". Although this result is well-known, it is rarely treated with rigour, so we present it at some length in Chapter 4.

Ultimately, a cobordism category should have more structure than a compact closed bicategory and in Chapter 5 we discuss "double structures" which provide one approach to a complete definition of cobordism category. The thesis concludes with an account of a two-dimensional topological field theory in terms of our notion of cobordism category.

We assume familiarity with the basic theory of categories and 2-categories. The standard reference on the theory of categories is Mac Lane [34], while 2-categories are defined by Mac Lane and discussed in greater depth by Kelly and Street [30]. Other less elementary notions are explicitly defined in detail. The one exception is the concept of "finite limit theory", which appears in Barr and

Wells [3] under the name "left exact theory". Although useful, the use of finite limit theories is not essential for the understanding of the theory of cobordism categories, and we make only occasional references to them. We adopt the following more or less conventional abbreviations: we use Set to denote the category of sets and functions, $\mathbf{Vect}_k$, the category of vector spaces over a field k, Ab, the category of abelian groups and group homomorphisms, and we use Cat to refer to both the category of categories and functors and the 2-category of categories, functors and natural transformations. chapterEnriched Categories Any category C comes equipped with a collection of *hom-sets*. The hom-set $\mathbf{C}(A, B)$ consists of all the arrows from A to B in C. In many cases, the sets $\mathbf{C}(A, B)$ have some additional structure: they may be abelian groups, or vector spaces for example. Furthermore, in these cases, the operation of composition respects the additional structure. A natural generalization of categories then suggests itself. An *enriched category* comes equipped with hom-*objects* $\mathbf{C}(A, B)$ rather than hom-*sets*, where these hom-objects come from some *base category*, such as Ab, the category of abelian groups. For ordinary categories, a composition law consists of functions

$$\mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$$

and so if a composition law for an enriched category is to be an arrow in the base category, we must have some analogue of the Cartesian product of sets. This analogue will be referred to as a *tensor product* and those categories with a tensor product are called *monoidal categories*. A monoidal category has all the necessary structure to be a base category for enriched categories. Generalized associativity laws and identity laws for enriched categories can be expressed as commuting diagrams in the base category. In this chapter we review the definition of monoidal categories, which are of great interest in their own right, and then give a treatment of enriched categories which is taken primarily from Kelly [28] and Lawvere [32]. The former is thorough but dense and the latter aesthetically pleasing but brief.

0.1 Monoidal Categories

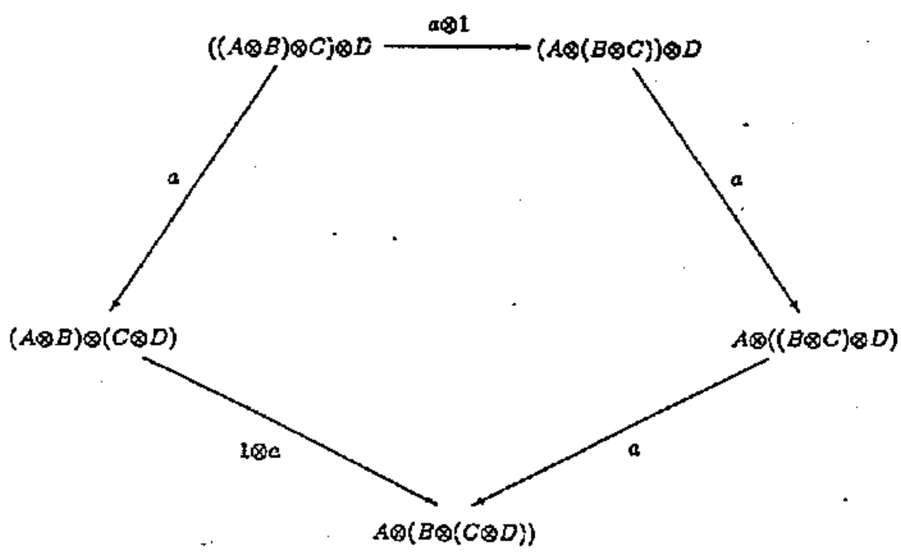
A monoidal category $\mathcal{V}$ consists of an underlying category, also denoted by $\mathcal{V}$, a specified *unit object* I of $\mathcal{V}$, natural *associativity isomorphisms*

$$a_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

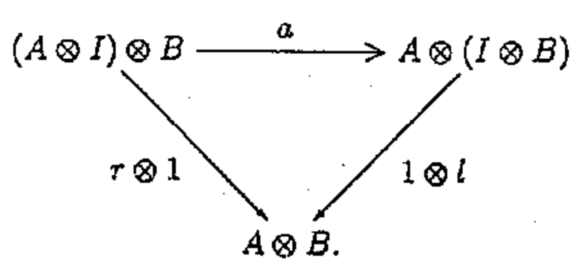
and natural *left* and *right unit isomorphisms*

$$l_A: I \otimes A \to A, \qquad r_A: A \otimes I \to A,$$

subject to the *coherence conditions* that the following diagrams commute:



and



A monoidal category can the thought of as a particular kind of "bicategory", a concept introduced in Chapter 1. Various results which appear in that chapter are therefore of relevance to the theory of monoidal categories. In particular, Theorem 1.3 ensures that monoidal categories are "coherent" in the sense that all diagrams built up with a, l, r, their inverses and $\otimes$ (so-called "expanded instances" of a, l and r) must commute. This allows the common practice of omitting labels from composites of expanded instances. For example, one might write

$$(A \otimes B) \otimes ((C \otimes D) \otimes I) \xrightarrow{\cong} (A \otimes (B \otimes I)) \otimes (C \otimes (D \otimes I))$$

with no fear of ambiguity. This coherence theorem for monoidal categories first appeared in Mac Lane [33] and was soon pursued in Kelly [23]. Ever since, results of this type have been central to the study of categories and, increasingly, their higher-dimensional analogues[2]. Kelly [24] is an attempt to formulate "the most general coherence" in terms of a useful functorial calculus, which is described in Kelly [25]. Kelly's approach is based on the theory of "clubs". Clubs[3] form part of the general machinery of "two-dimensional universal algebra", a subject introduced by Blackwell, Kelly and Power [6] as the study of structure borne by categories, and as such do not seem suited to higher dimensional contexts such as the theory of bicategories. At this stage a "most general coherence theorem" is not within our grasp and approaches to the problem seem almost as numerous as authors in the field. As a result, coherence is a recurring theme in this thesis, appearing in a slightly different guise as each new algebraic structure is introduced.

Examples of monoidal categories include **Set** with the Cartesian product of sets, **Cat** with the Cartesian product of categories, **Ab** with the usual tensor product of abelian groups and $\mathbf{Vect}_k$ with the usual tensor product over k.

A *braiding* for a monoidal category $\mathcal{V}$ consists of natural *braid isomorphisms*[4]

$$c_{A,B}: A \otimes B \to B \otimes A$$

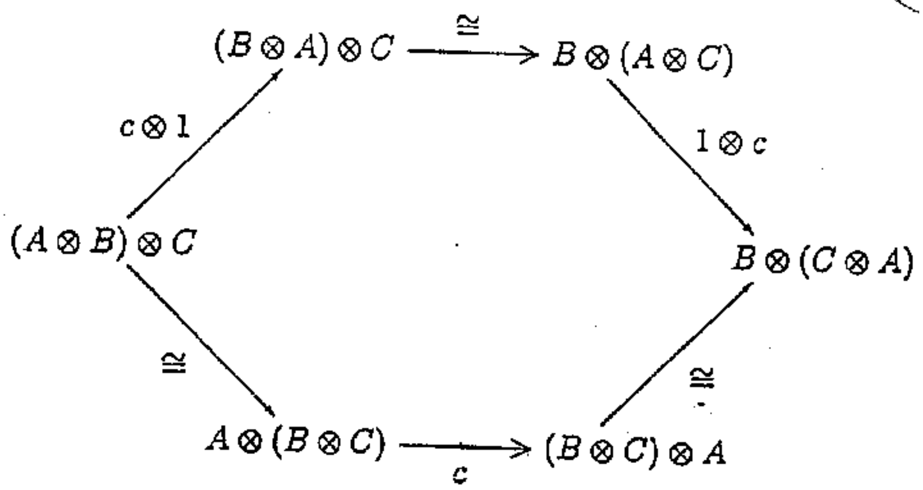
[2] Witness Gordon, Power and Street [15].

[3] Further general observations on clubs appear in Kelly [27].

[4] Kapranov and Voevodsky [22] present a case for relaxing the requirement that braids be isomorphisms. Here we are primarily interested in braids which are in fact *symmetries* and these are certainly isomorphisms, so we ignore weaker notions of braiding.

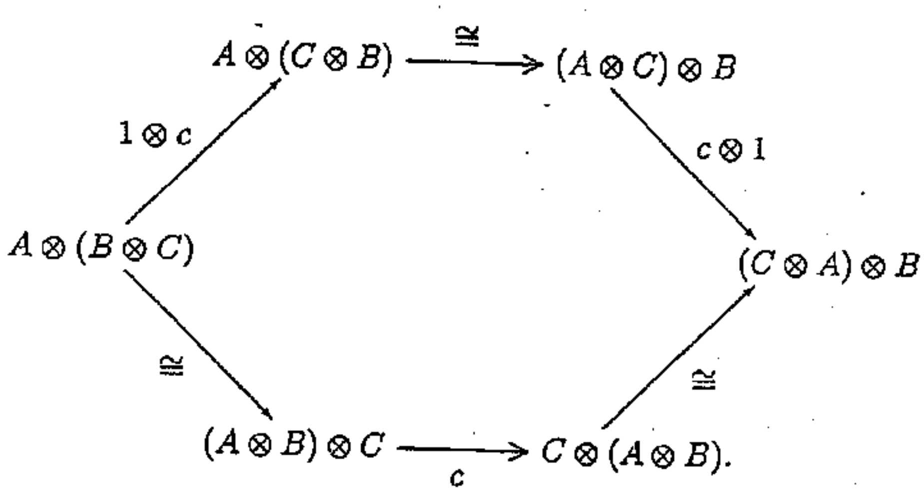
which are subject to the condition that the following two diagrams commute:

(0.1)



and

(0.2)

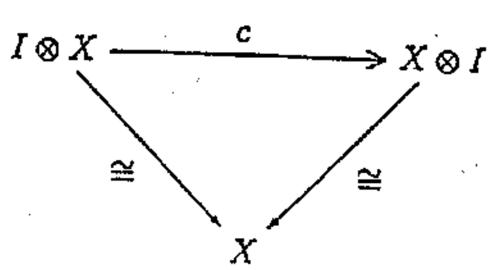


A *braided monoidal category* is a monoidal category equipped with a braiding.

A *symmetry* for a monoidal category is a braiding which satisfies

$$c_{A,B}^{-1} = c_{B,A}.$$

Note that if c is a symmetry then either one of equations (0.1) and (0.2) implies the other. In his definition of a symmetry, Kelly [28] adds the condition that the diagram



commutes. This is unnecessary as the diagram automatically commutes for any braiding, not just symmetries, as is proved in Proposition 2.1 of Joyal and Street [19]. A *symmetric monoidal category* is a monoidal category equipped with a symmetry. Mac Lane [33] proves a coherence theorem for symmetric monoidal categories: all diagrams built up of expanded instances of a, l, r and c must commute. Joyal and Street [19] prove a coherence theorem for braided monoidal theorems which is a little more subtle. To every arrow built of expanded instances of a, l, r and c they associate an *underlying braid*, and any two such arrows are shown to be equal if and only if they have the same underlying braid. While this is a very satisfying result, we have no need for it in this thesis.

Our earlier examples of monoidal categories, namely **Set**, **Cat**, **Ab** and $\mathbf{Vect}_k$ are all symmetric monoidal categories.

0.2 Enriched Categories

Throughout the section $\mathcal{V}$ will denote a fixed monoidal category and we introduce "$\mathcal{V}$-categories" or "categories enriched over the base $\mathcal{V}$". Much of the theory of ordinary categories can be developed for enriched categories, as can be seen in Kelly [28]. In this chapter we concentrate on two aspects of that theory: bimodules over enriched categories and Cauchy complete enriched categories. These notions are of relevance to the study of compact closed categories and bicategories, which we discuss in Chapter 3.

0.2.1 The Definition of an Enriched Category

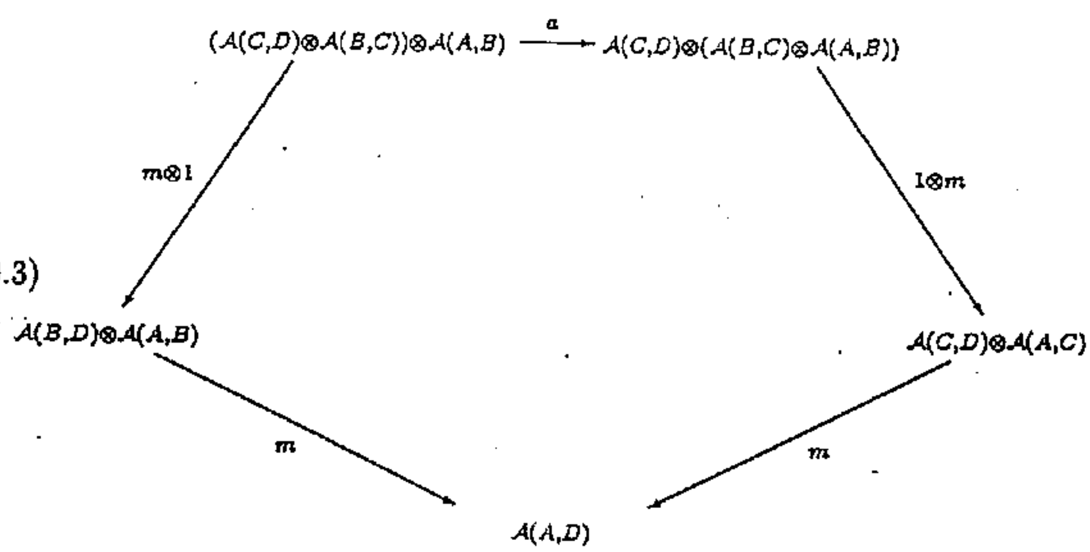
A *$\mathcal{V}$-category* $\mathcal{A}$ consists of a collection of *objects* (A, B, C, ...) of $\mathcal{A}$, which we denote by $\mathcal{A}_0$[5], an assignment of a *hom-object* $\mathcal{A}(A, B)$ of $\mathcal{V}$ to each ordered pair of objects A, B of $\mathcal{A}$, a *composition law*

$$m_{A,B,C}: \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$$

[5] This should be distinguished from the notation used in Kelly [28] where $\mathcal{A}_0$ is used to mean "underlying ordinary category". Kelly uses ob$\mathcal{A}$ to denote the collection of objects of $\mathcal{A}$.

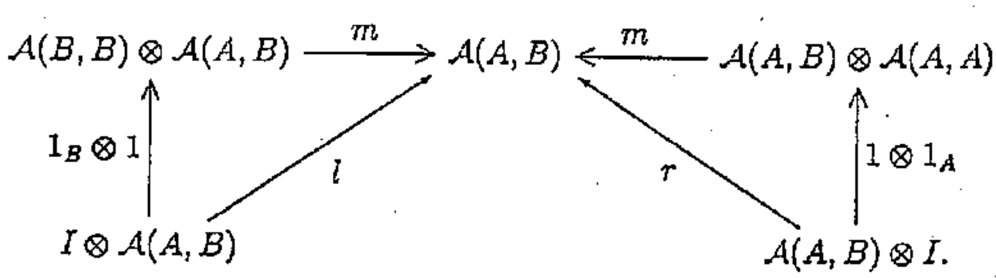
for each ordered triple of objects A, B, C, and an *identity element* $1_A: I \to \mathcal{A}(A, A)$ for each object A. These data are subject to the conditions that the following diagrams commute in $\mathcal{V}$:

(0.3)



and

(0.4)



Categories enriched over Set are ordinary categories, those enriched over Cat are 2-categories. Ab-categories are usually called *additive categories*, while Vect_k-categories are called *k-linear categories* and include $\mathrm{Rep}(G)$, the category whose objects are k-linear representations of a group G with morphisms the intertwining operators. To gain some insight into these examples, it often pays to consider the simple case of enriched categories with only one object. An additive category with one object consists of an abelian group R equipped with a homomorphism of abelian groups $m: R \otimes R \to R$. By definition of the tensor product of abelian groups, this corresponds to a binary operation $\cdot: R \times R \to R$ that distributes over addition and condition (0.3) forces this operation, which we naturally call multiplication, to be associative. The identity element gives a distinguished element 1 of R and condition (0.4) ensures that 1 is a left and right multiplicative identity. In other words, a one-object additive category is a ring. In the same way, a k-linear category with only one object is simply a k-algebra.

Notice in all of our examples, the base category is in fact enriched over itself: Set is a category, Cat is a 2-category, Ab is an additive category and Vect_k is a k-linear category. This is due to the fact that each of these base categories is "closed". A monoidal category $\mathcal{V}$ is *closed* with respect to its tensor product (on the right) if for each object B of $\mathcal{V}$, the functor $- \otimes B \colon \mathcal{V} \to \mathcal{V}$ has a right adjoint $[B, -]$, which is to say that there is a bijection

$$\frac{A \otimes B \;\to\; C}{A \;\to\; [B,C]}$$

which is natural in A and C. The counit for this adjunction, which we denote by ϵ_B, has C-components $[B,C] \otimes B \to C$. We refer to this counit as *evaluation* and it allows us to enrich $\mathcal{V}$ over itself. For objects A and B of $\mathcal{V}$ we define a hom-object by $[A, B]$, the so-called *internal hom*. The composition law

$$[B,C] \otimes [A,B] \longrightarrow [A,C]$$

is defined to be the arrow which corresponds under the adjunction to

$$([B,C] \otimes [A,B]) \otimes A \cong [B,C] \otimes ([A,B] \otimes A) \xrightarrow{1 \otimes \epsilon_A} [B,C] \otimes B \xrightarrow{\epsilon_B} C.$$

The identity $I \to [A,A]$ is defined as the arrow which corresponds to the isomorphism $I \otimes A \to A$. It is straightforward to establish that these definitions do indeed exhibit $\mathcal{V}$ as enriched over itself in the manner suggested by our examples above. Note also that if $\mathcal{V}$ is closed and *symmetric*, then the functor $B \otimes - : \mathcal{V} \to \mathcal{V}$ also has a right adjoint, namely $[-, B]$.

The notion of "enriched sub-category" is rather more subtle than the corresponding notion of ordinary subcategory. For our purposes it suffices to define a particular type of enriched sub-category. We say that $\mathcal{B}$ is a *full enriched sub-category* of $\mathcal{A}$ if every object of $\mathcal{B}$ is an object of $\mathcal{A}$ and $\mathcal{B}(A, A') = \mathcal{A}(A, A')$.

0.2.2 Enriched Functors and Natural Transformations

Ordinary categories, functors and natural transformations make up the 2-category structure Cat. Enriched categories are similarly the objects of a two-dimensional structure as it is possible to define analogues of functors and natural transformations which respect the enriched structure.

For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a *$\mathcal{V}$-functor* $F: \mathcal{A} \to \mathcal{B}$ consists of a map of objects $F: \mathcal{A}_0 \to \mathcal{B}_0$ and an assignment of an arrow $F_{A,B}: \mathcal{A}(A, B) \to \mathcal{B}(FA, FB)$ to each ordered pair A, B of objects of $\mathcal{A}$. These data are required to respect composition and identities in the sense that the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{A}(B,C) \otimes \mathcal{A}(A,B) & \xrightarrow{\quad m \quad} & \mathcal{A}(A,C) \\
{\scriptstyle F \otimes F}\downarrow & & \downarrow{\scriptstyle F} \\
\mathcal{B}(FB,FC) \otimes \mathcal{B}(FA,FB) & \xrightarrow{\quad m \quad} & \mathcal{B}(FA,FC)
\end{array}$$

and

$$\begin{array}{ccc}
 & & \mathcal{A}(A,A) \\
 & \nearrow^{1_A} & \\
I & & \downarrow{\scriptstyle F} \\
 & \searrow_{1_{TA}} & \\
 & & \mathcal{B}(TA,TA).
\end{array}$$

Given $\mathcal{V}$-functors $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ there is an obvious composite $\mathcal{V}$-functor $G \circ F: \mathcal{A} \to \mathcal{C}$. Composition of $\mathcal{V}$-functors is associative and each $\mathcal{V}$-category $\mathcal{A}$ has an identity $\mathcal{V}$-functor $1_{\mathcal{A}}: \mathcal{A} \to \mathcal{A}$ given by

$$1_{\mathcal{A}(A,B)}: \mathcal{A}(A, B) \to \mathcal{A}(A, B).$$

Thus $\mathcal{V}$-categories and $\mathcal{V}$-functors form a category.

When we enrich over Set we clearly recover the notion of ordinary functor, while a Cat-functor is the same as the usual notion of 2-functor. Continuing with the terminology of our other examples, an Ab-functor is called an *additive functor* and a Vect_k-functor is called a *k-linear functor.* Returning to the special case of enriched categories with only one object, it is clear that an additive functor between rings is simply a ring homomorphism and a k-linear functor between k-algebras is a homomorphism of algebras.

For $\mathcal{V}$-functors F, $G: \mathcal{A} \to \mathcal{B}$, a *$\mathcal{V}$-natural transformation* $\alpha: F \to G$ consists of a family of *components* $\alpha_A: I \to \mathcal{B}(FA, GA)$ indexed by objects A of $\mathcal{A}$ which is *$\mathcal{V}$-natural*, which is to say that the following diagram commutes:

$$\begin{array}{ccccc}
I \otimes \mathcal{A}(A,B) & \xrightarrow{\alpha_B \otimes F} & \mathcal{B}(FB, GB) \otimes \mathcal{B}(FA, FB) & & \\
 & & & \searrow^{m} & \\
\cong \Big\downarrow & & & & \mathcal{B}(FA, GB) \\
 & & & \swarrow_{m} & \\
\mathcal{A}(A,B) \otimes I & \xrightarrow[G \otimes \alpha_A]{} & \mathcal{B}(GA, GB) \otimes \mathcal{B}(FA, GA). & &
\end{array}$$

Note that in labelling the left-hand arrow of this diagram simply as an isomorphism, we are making our first appeal to the coherence theorem for monoidal categories.

In a manner exactly analogous to ordinary natural transformations, $\mathcal{V}$-natural transformations can be composed both vertically and horizontally. Given two $\mathcal{V}$-natural transformations $\alpha: F \to G$ and $\beta: G \to H$, where F, G, $H: \mathcal{A} \to \mathcal{B}$, we define the *vertical composite* $\beta\alpha: F \to H$ in terms of its the A-component

$$I \cong I \otimes I \xrightarrow{\beta_A \otimes \alpha_A} \mathcal{B}(GA, HA) \otimes \mathcal{A}(FA, GA) \xrightarrow{m} \mathcal{B}(FA, HA).$$

Given $\mathcal{V}$-natural transformations $\alpha: F \to G$ and $\alpha': F' \to G'$, where F and $G: \mathcal{A} \to \mathcal{B}$ and F' and $G': \mathcal{B} \to \mathcal{C}$, we define the A-component of the *horizontal composite* $\alpha' \circ \alpha: F' \circ F \to G' \circ G$ to be either of the two boundary composites of the diagram

$$
\begin{array}{ccc}
I \cong I \otimes I & \xrightarrow{\quad 1 \otimes \alpha_A \quad} & I \otimes \mathcal{B}(FA, GA) \\
\Big\downarrow{\scriptstyle \alpha_A \otimes 1} & \swarrow{\scriptstyle \cong} & \Big\downarrow{\scriptstyle \alpha'_{GA} \otimes F'} \\
\mathcal{B}(FA, GA) \otimes I & & \mathcal{C}(F'GA, G'GA) \otimes \mathcal{C}(F'FA, F'GA) \\
\Big\downarrow{\scriptstyle G' \otimes \alpha'_{FA}} & & \Big\downarrow{\scriptstyle m} \\
\mathcal{C}(G'FA, G'GA) \otimes \mathcal{C}(F'FA, G'FA) & \xrightarrow[\quad m \quad]{} & \mathcal{C}(F'FA, G'GA).
\end{array}
$$

Note that the $\mathcal{V}$-naturality of α' ensures that this diagram commutes. Relatively straightforward diagram chasing verifies that these vertical and horizontal composites are indeed $\mathcal{V}$-natural transformations. Also, the vertical composition law is associative and each $\mathcal{V}$-functor $F: \mathcal{A} \to \mathcal{B}$ has an identity $\mathcal{V}$-natural transformation $1_F: F \to F$ which has components $1_{FA}: I \to \mathcal{B}(FA, FA)$. We denote by $\mathcal{V}\text{-Cat}(\mathcal{A}, \mathcal{B})$ the resulting category of $\mathcal{V}$-natural functors $\mathcal{A} \to \mathcal{B}$ and $\mathcal{V}$-natural transformations with vertical composition. The horizontal composition law is also associative, it has the same identities as the vertical law and the two composition laws satisfy the interchange law. Thus $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations form a 2-category, which we denote by $\mathcal{V}$-Cat.

0.2.3 Underlying Categories

In all our examples, the objects of the base category have underlying sets, so we can ignore the "extra structure" of the enriched categories and be left with an ordinary category. To see how this can be done in the abstract setting, recall that there is a bijection between the elements of an abelian group A and homomorphisms $\mathbb{Z} \to A$, there is a bijection between elements of a vector space V over k and linear maps $k \to V$ and a bijection between objects in a category and functors from $\mathbf{1} \to \mathbf{C}$. In each case, elements of the "underlying set" of an object A of the base category $\mathcal{V}$ amount to arrows $I \to A$ in $\mathcal{V}$. Equipped with this insight we are able to associate an "underlying" ordinary category to any enriched category.

Given a $\mathcal{V}$-category $\mathcal{A}$, we define the *underlying category* $U\mathcal{A}$ to be the category with the same objects as $\mathcal{A}$ and with arrows $f: A \to B$ given by arrows $f: I \to \mathcal{A}(A, B)$ in $\mathcal{V}$. The composite gf of arrows $f: A \to B$ and $g: B \to C$ is given by

$$I \cong I \otimes I \xrightarrow{g \otimes f} \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \xrightarrow{m} \mathcal{A}(A, C).$$

By way of example, consider a category $\mathcal{A}$ enriched over Cat, which is simply a 2-category. The underlying category $U\mathcal{A}$ is the category obtained from $\mathcal{A}$ by ignoring the 2-cells.

We can go further and give a 2-functor $U: \mathcal{V}\text{-Cat} \to \text{Cat}$ and associate underlying functors and natural transformations to $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. To this end, we introduce a $\mathcal{V}$-category $\mathcal{I}$ which has one object, 0 and has the hom-object $\mathcal{I}(0, 0) = I$. The composition law in $\mathcal{I}$ is the isomorphism $I \otimes I \to I$ and the identity element $I \to I$ is the identity arrow of I in $\mathcal{V}$. Note that a $\mathcal{V}$-functor $\mathcal{I} \to \mathcal{A}$ simply amounts to an object of $\mathcal{V}$. We now define U to be the representable 2-functor $\mathcal{V}\text{-Cat}(\mathcal{I}, -)$. Explicitly, the underlying functor $UT: U\mathcal{A} \to U\mathcal{B}$ associated to the $\mathcal{V}$-functor $F: \mathcal{A} \to \mathcal{B}$ takes objects A to FA and arrows $f: A \to B$ in $U\mathcal{A}$ to the composite

$$I \xrightarrow{f} \mathcal{A}(A, B) \xrightarrow{F_{A,B}} \mathcal{B}(FA, FB).$$

The underlying natural transformation $U\alpha$ associated to a $\mathcal{V}$-natural transformation $\alpha: F \to G$ is even simpler to describe. The A-component of $U\alpha$ is the arrow $FA \to GA$ in $U\mathcal{B}$ given by $\alpha_A: I \to \mathcal{B}(FA, GA)$.

The amount of information lost in the passage from enriched categories to their underlying categories depends very much on the base.

0.3 Enriched Bimodules

In ring theory, there is a well-known notion of "bimodule". Given rings R and S, an (R,S)-*bimodule* consists of an abelian group M which is a left R-module and a right S-module such that the R and S actions commute. In detail this means we are a left action of R on M and a right action of S on M, both of which we denote by $\cdot$. These actions yield group homomorphisms

$$R \otimes M \to M \tag{0.5}$$

$$M \otimes S \to M. \tag{0.6}$$

The two actions are subject to the condition that they commute, that is

$$(r \cdot m) \cdot s = r \cdot (m \cdot s) \tag{0.7}$$

for all $r \in R$, $s \in S$ and $m \in M$. Given two (R,S)-bimodules M and N, the obvious notion of a bimodule map is a group homomorphism $f: M \to N$ which respects the R and S actions, in the sense that

$$f(r \cdot m) = r \cdot f(m) \tag{0.8}$$

$$f(m \cdot s) = f(m) \cdot s. \tag{0.9}$$

An important feature of bimodule maps is that not only do they give (R,S)-bimodules a category structure, but this category is itself an *additive* category.

There is another equivalent formulation of these data which is of particular interest. An (R,S)-bimodule can be considered to be an additive functor

$$S^{op} \otimes R \to \mathrm{Ab} \tag{0.10}$$

where S^{op} is the opposite[6] of the ring S. Described in these terms, a bimodule map is an additive natural transformation.

[6] S^{op} has elements $\overline{s}$ for each $s \in S$ and has addition defined as in S, but has multiplication reversed: $\overline{s}_1\overline{s}_2 = \overline{s_2 s_1}$.

Probably the most important feature of bimodules is that they can be "composed" using the operation of tensor product over a ring: given an (R,S)-bimodule M and an (T,R)-bimodule N, we can give their tensor product $N \otimes_R M$ the structure of a (T,S) bimodule in the obvious way. A very suggestive notation is to write an (R,S)-bimodule as an arrow $S \twoheadrightarrow R$. Then when $M\colon S \twoheadrightarrow R$ and $N\colon R \twoheadrightarrow T$ as above, we have $N \otimes_R M\colon S \twoheadrightarrow T$. Up to isomorphism, this tensor product of bimodules is associative. Furthermore any ring R can itself be considered as a bimodule $R \twoheadrightarrow R$, and as such is an identity for the tensor product, at least up to isomorphism, in the sense that $R \otimes_R M \cong M$ and $N \otimes_R R \cong N$ for any left R-module M and right R-module N. Composition laws such as this, which fall short of yielding categories only in that the associativity and identity laws are isomorphisms rather than equalities, are the subject of Chapter 1.

A bimodule is an object of Ab and rings are additive categories, so it is natural to expect it to be possible to develop a similar notion of bimodule enriched over base categories other than Ab. By analogy with equation (0.10), given $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ it would seem reasonable to define a $\mathcal{V}$-bimodule $\mathcal{A} \twoheadrightarrow \mathcal{B}$ as a $\mathcal{V}$-functor

$$\mathcal{A}^{op} \otimes \mathcal{B} \to \mathcal{V}.$$

Of course, this necessitates defining the opposite of a $\mathcal{V}$-category and the tensor product of $\mathcal{V}$-categories. In making these definitions, we will make use of a symmetry for $\mathcal{V}$. Maps of enriched bimodules can then be defined as $\mathcal{V}$-natural transformations.

Generalizing composition of bimodules is slightly more complicated. Recall that the tensor product of modules over a ring can be expressed as a coequalizer. Explicitly, if M is a left R-module and N a right R-module, then the following is a coequalizer diagram in Ab:

$$N \otimes R \otimes M \rightrightarrows N \otimes M \longrightarrow N \otimes_R M. \tag{0.11}$$

Here $\otimes$ is tensor product of abelian groups and the parallel arrows are given by the

R-action on M and the R-action on N. If composition of enriched bimodules is to be defined in a similar fashion then we will require the existence of coequalizers in the base category. In fact, coproducts are also needed, so we will insist that our base is cocomplete. Following the example of ordinary bimodules, it is reasonable to expect that $\mathcal{V}$-bimodules $\mathcal{A} \nrightarrow \mathcal{B}$ can be given the structure of a $\mathcal{V}$ category. To do so will require the formation of limits in $\mathcal{V}$, which imposes another condition on our base. We will therefore eventually insist that $\mathcal{V}$ is a symmetric, complete and cocomplete closed monoidal category. Note that **Set**, **Cat**, **Ab** and **Vect**$_k$ all satisfy these requirements.

0.3.1 Tensor Products and Opposites

In this section, $\mathcal{V}$ is only assumed to be a symmetric monoidal category. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, we define their *tensor product* $\mathcal{A} \otimes \mathcal{B}$ to be the $\mathcal{V}$-category with objects $\mathcal{A}_0 \times \mathcal{B}_0$ and hom-objects

$$(\mathcal{A} \otimes \mathcal{B})\Big((A, B), (A', B')\Big) = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B').$$

The composition law is given by the composite

$$\begin{array}{c}
\big(\mathcal{A}(A',A'')\otimes\mathcal{B}(B',B'')\big)\otimes\big(\mathcal{A}(A,A')\otimes\mathcal{B}(B,B')\big) \\
\cong \Big\downarrow \\
\big(\mathcal{A}(A',A'')\otimes\mathcal{A}(A,A')\big)\otimes\big(\mathcal{B}(B',B'')\otimes\mathcal{B}(B,B')\big) \\
m\otimes m \Big\downarrow \\
\mathcal{A}(A',A'')\otimes\mathcal{B}(B,B'')
\end{array}$$

for which we make our first appeal to the coherence theorem for *symmetric* monoidal categories: the arrow labelled simply as an isomorphism can be any suitable expanded instance of a and c. The identity element for the object (A, B) is given by the composite

$$I \cong I \otimes I \xrightarrow{1_A \otimes 1_B} \mathcal{A}(A, A) \otimes \mathcal{B}(B, B).$$

It is easy to check that this definition of $\mathcal{A}\otimes\mathcal{B}$ yields a $\mathcal{V}$-category. The $\mathcal{V}$-category $\mathcal{I}$, which was introduced in Section 0.2.3, is a *unit* for the tensor product of $\mathcal{V}$-categories since there are isomorphisms[7] $\mathcal{A}\otimes\mathcal{I}\cong\mathcal{A}\cong\mathcal{I}\otimes\mathcal{A}$. It can also be shown that up to isomorphism the tensor product for $\mathcal{V}$-categories is associative, that is to say $(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}\cong\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C})$. Indeed $\mathcal{V}$-Cat is itself a monoidal category.

By way of example, consider the case of k-linear categories. Recall that a k-linear category with only one object is a k-algebra. The unit $\mathcal{I}$ in this case is simply k considered as an algebra over itself. If A and B are two k-algebras, their tensor product as k-linear categories is the vector space tensor product $A\otimes_k B$ with the algebra structure defined by $(a\otimes b)(a'\otimes b')=aa'\otimes bb'$.

Given a $\mathcal{V}$-category $\mathcal{A}$, we define its *opposite* $\mathcal{A}^{op}$ to be the $\mathcal{V}$-category with the same objects as $\mathcal{A}$ but with hom-objects given by $\mathcal{A}^{op}(A,B)=\mathcal{A}(B,A)$. Composition is given by

$$\mathcal{A}(C,B)\otimes\mathcal{A}(B,A)\cong\mathcal{A}(B,A)\otimes\mathcal{A}(C,B)\xrightarrow{m}\mathcal{A}(C,A)$$

and identity elements are the same as for $\mathcal{A}$. This definition coincides with the usual notion of opposite in the case of ordinary categories and when rings are considered to be additive categories it coincides with the usual opposite of a ring. We are now in in a position to define bimodules enriched over a base $\mathcal{V}$.

0.3.2 The Definition of Enriched Bimodules

From this point on, we assume that $\mathcal{V}$ is a symmetric, complete and cocomplete closed monoidal category. A $\mathcal{V}$*-bimodule* M is a $\mathcal{V}$-functor

$$\mathcal{A}^{op}\otimes\mathcal{B}\to\mathcal{V}$$

where $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories. We also say that M is a *left* $\mathcal{B}$*-, right* $\mathcal{A}$*-module* and write $M\colon\mathcal{A}\twoheadrightarrow\mathcal{B}$. Some authors call bimodules *profunctors*, *distributeurs* or

[7]More precisely, the isomorphisms are 2-natural isomorphisms. See chapter 1 for more on two-dimensional algebra.

distributors. There is an alternative more intuitive formulation of this definition which resembles the ordinary notion of bimodule even more closely. An enriched bimodule $M: \mathcal{A} \longrightarrow\!\!\!\!\!+ \ \mathcal{B}$ consists of a collection of objects $M(A, B) \in \mathcal{V}$, indexed by objects of $\mathcal{A}$ and $\mathcal{B}$, together with arrows

$$M(A,B) \otimes \mathcal{A}(A', A) \;\to\; M(A', B) \tag{0.12}$$

$$\mathcal{B}(B, B') \otimes M(A, B) \;\to\; M(A, B') \tag{0.13}$$

in $\mathcal{V}$ which arrows are required to be *actions*, in the sense that they both satisfy axioms of associativity and identity, and are required to commute. We denote both actions by $\cdot$. The axiom of associativity for the $\mathcal{A}$ action states that the diagram

$$\begin{array}{ccc}
M(A,B) \otimes \big(\mathcal{A}(A',A) \otimes \mathcal{A}(A'',A')\big) & \xrightarrow{1 \otimes m} & M(A,B) \otimes \mathcal{A}(A'',A) \\
\downarrow \cong & & \searrow \\
 & & M(A'',B) \\
 & & \nearrow \\
\big(M(A,B) \otimes \mathcal{A}(A',A)\big) \otimes \mathcal{A}(A'',A') & \xrightarrow[\cdot \otimes 1]{} & M(A',B) \otimes \mathcal{A}(A'',A')
\end{array}$$

commutes, which is a generalization of the condition on right actions for ordinary bimodules that $(m \cdot r) \cdot r' = m \cdot (rr')$. The identity axiom for the $\mathcal{A}$ action states that the diagram

$$\begin{array}{ccc}
M(A,B) \otimes I & & \\
{\scriptstyle 1 \otimes 1_A} \downarrow & \searrow \cong & \\
M(A,B) \otimes \mathcal{A}(A,A) & \longrightarrow & M(A,B)
\end{array}$$

commutes and this generalizes the condition for ordinary bimodules that $m \cdot 1 = m$. The axioms of associativity and identity for the action of $\mathcal{B}$, which generalize the corresponding axioms satisfied by the left actions for ordinary bimodules, are very similar and so we omit them here. The requirement that the $\mathcal{A}$ and $\mathcal{B}$ actions commute with each other is a generalization of equation (0.7) for ordinary bimodules. It is expressed by the commutative diagram

$$\begin{array}{ccccc}
\mathcal{B}(B,B')\otimes\big(M(A,B)\otimes\mathcal{A}(A',A)\big) & \xrightarrow{1\otimes\cdot} & \mathcal{B}(B,B')\otimes M(A',B) & & \\
 & & & \searrow^{\cdot} & \\
\Big\downarrow\cong & & & & M(A',B') \\
 & & & \nearrow_{\cdot} & \\
\big(\mathcal{B}(B,B')\otimes M(A,B)\big)\otimes\mathcal{A}(A',A) & \xrightarrow[\cdot\otimes 1]{} & M(A,B')\otimes\mathcal{A}(A',A). & &
\end{array}$$

The equivalence of these two definitions of bimodules is straightforward to establish. Given actions as above, composing either leg of diagram (0.3.2) with an appropriate expanded instance of a and c yields an arrow

$$\big(\mathcal{A}(A',A)\otimes\mathcal{B}(B,B')\big)\otimes M(A,B)\to M(A',B')$$

which corresponds under the internal hom adjunction to an arrow

$$M_{(A,B),(A',B')}\colon \mathcal{A}(A',A)\otimes\mathcal{B}(B,B')\to\big[M(A,B),M(A',B')\big]$$

and the associativity, identity and commutativity axioms for the actions ensure that $M_{(A,B),(A',B')}$ gives the data for a $\mathcal{V}$-functor $\mathcal{A}^{op}\otimes\mathcal{B}\to\mathcal{V}$. Conversely given such a functor, we can obtain a $\mathcal{B}$ action as the composite

$$\begin{array}{c} \mathcal{B}(B,B') \otimes M(A,B) \\ \Big\downarrow \cong \\ \big(I \otimes \mathcal{B}(B,B')\big) \otimes M(A,B) \\ \Big\downarrow (1_A \otimes 1) \otimes 1 \\ \big(\mathcal{A}(A,A) \otimes \mathcal{B}(B,B')\big) \otimes M(A,B) \\ \Big\downarrow \tilde{M} \\ M(A,B') \end{array}$$

where $\tilde{M}$ corresponds to $M_{(A,B),(A,B')}$ under the internal hom adjunction. A similar composite yields the $\mathcal{A}$ action. The description of bimodules in terms of $\mathcal{A}$ and $\mathcal{B}$ actions will be useful in describing the operation of composition of bimodules. Henceforth we will move freely between the two formulations of enriched bimodules.

Since enriched bimodules are $\mathcal{V}$-functors, one should clearly consider $\mathcal{V}$-natural transformations between them. A *$\mathcal{V}$-bimodule map* $M \to M'$ between enriched bimodules M, $M': \mathcal{A} \dashrightarrow \mathcal{B}$ is a $\mathcal{V}$-natural transformation $M \to M'$. The interpretation of this definition when bimodules are treated in terms of $\mathcal{B}$ and $\mathcal{A}$ actions is straightforward. A bimodule map $\alpha: M \to M'$ consists of a collection of arrows $\alpha_{A,B}: M(A,B) \to M'(A,B)$ in $\mathcal{V}$ which respect the $\mathcal{A}$ and $\mathcal{B}$ actions. Explicitly, saying α respects the $\mathcal{A}$ action means that all diagrams of the form

$$\begin{array}{ccc} M(A,B) \otimes \mathcal{A}(A',A) & \longrightarrow & M(A',B) \\ \Big\downarrow \alpha_{A,B} \otimes 1 & & \Big\downarrow \alpha_{A',B} \\ M'(A,B) \otimes \mathcal{A}(A',A) & \longrightarrow & M'(A',B) \end{array}$$

commute. Similar commuting diagrams express the fact that α respects the $\mathcal{B}$ action. Enriched bimodule maps between bimodules $\mathcal{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{B}$ can be composed in the obvious way and this composition is clearly associative and has identities for each bimodule. We thus have a category, which we denote by $\mathcal{V}$-*Mod* $(\mathcal{A}, \mathcal{B})$, of bimodules $\mathcal{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{B}$ and bimodule maps between them. It is also possible to construct a $\mathcal{V}$-category whose objects are bimodules $\mathcal{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{B}$, however we defer this construction to the next section.

Lawvere [32] introduces the notion of $\mathcal{V}$-module as a "generalized $\mathcal{V}$-functor" and indeed any $\mathcal{V}$-functor $F: \mathcal{A} \to \mathcal{B}$ gives rise to a bimodule $F_*: \mathcal{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{B}$ defined by

$$F_*(A, B) = \mathcal{B}(FA, B)$$

and another bimodule $F^*: \mathcal{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{A}$ which is defined by

$$F^*(B, A) = \mathcal{B}(B, FA).$$

The $\mathcal{B}$ action in both cases is given by composition in $\mathcal{B}$, while the $\mathcal{A}$ action on F_* is defined by

$$\mathcal{B}(FA, B) \otimes \mathcal{A}(A, A') \xrightarrow{1 \otimes F} \mathcal{B}(FA, B) \otimes \mathcal{B}(FA, FA') \xrightarrow{m} \mathcal{B}(FA', B)$$

and the $\mathcal{A}$ action on F^* by a similar composite. Associativity of composition for enriched categories ensures that F_* and F^* are indeed bimodules. The bimodules associated in this way to an enriched functor are central to Cauchy completeness, the subject of the final section of this chapter. Given a $\mathcal{V}$-natural transformation $\alpha: F \to G$ we can define a map of bimodules $\alpha_*: G_* \to F_*$ which has (A, B)-component given by the composite

$$\begin{array}{c} \mathcal{B}(FA, B) \\ \Big\downarrow \cong \\ \mathcal{B}(FA, B) \otimes I \\ \Big\downarrow 1 \otimes \alpha_A \\ \mathcal{B}(FA, B) \otimes \mathcal{B}(GA, FA) \\ \Big\downarrow m \\ \mathcal{B}(GA, FA). \end{array}$$

Similarly, we can define a bimodule map $\alpha^*: F^* \to G^*$. Clearly these definitions will respect composition, so we have in fact defined functors

$$(0.14) \qquad (\ )_*: \mathcal{V}\text{-Cat}(\mathcal{A}, \mathcal{B})^{op} \to \mathcal{V}\text{-}Mod(\mathcal{A}, \mathcal{B})$$

$$(0.15) \qquad (\ )^*: \mathcal{V}\text{-Cat}(\mathcal{A}, \mathcal{B}) \to \mathcal{V}\text{-}Mod(\mathcal{B}, \mathcal{A}).$$

We now turn to composition of enriched bimodules. Composition of ordinary bimodules is defined through the operation of tensor product over a ring, which is expressed in categorical terms as the coequalizer in diagram (0.11). Since we are working over a cocomplete base, a similar construction is possible for enriched bimodules. Consider $\mathcal{V}$ bimodules $M: \mathcal{A} \to\!\!\!\!\!+ \mathcal{B}$ and $N: \mathcal{B} \to\!\!\!\!\!+ \mathcal{C}$. We define their composite $N \circ M$ by specifying $N \circ M(A, C)$ as a coequalizer in $\mathcal{V}$ of the diagram

$$\sum_{B_1, B_2 \in \mathcal{B}} (N(B_1, C) \otimes \mathcal{B}(B_2, B_1)) \otimes M(A, B_2) \rightrightarrows \sum_{B \in \mathcal{B}} N(B, C) \otimes M(A, B)$$

where the parallel arrows are obtained from the $\mathcal{B}$ actions on M and N. Explicitly, one of these arrows is defined by the commuting diagram

$$\begin{array}{ccc}
(N(B_1,C)\otimes \mathcal{B}(B_2,B_1))\otimes M(A,B_2) & \xrightarrow{\ \cdot\otimes 1\ } & N(B_2,C)\otimes M(A,B_2) \\
\Big\downarrow{\scriptstyle i_{(B_1,B_2)}} & & \Big\downarrow{\scriptstyle i_{B_2}} \\
\sum\limits_{B_1,B_2\in\mathcal{B}} (N(B_1,C)\otimes \mathcal{B}(B_2,B_1))\otimes M(A,B_2) & \longrightarrow & \sum\limits_{B\in\mathcal{B}} N(B,C)\otimes M(A,B)
\end{array}$$

where the the arrows labelled by i are coproduct injections. The other arrow is given by a similar commuting diagram which also involves the isomorphism

$$\big(N(B_1,C)\otimes \mathcal{B}(B_2,B_1)\big)\otimes M(A,B_2)\cong N(B_1,C)\otimes\big(\mathcal{B}(B_2,B_1)\otimes M(A,B_2)\big).$$

With this definition, $N\circ M$ clearly inherits an $\mathcal{A}$ action from M and a $\mathcal{C}$ action from N and is thus a bimodule $\mathcal{A}\,\mathrel{\to\!\!\!\!\!\!+}\,\mathcal{C}$. This composite is sometimes also written as $N\otimes_{\mathcal{B}} M$. We will also write $N\circ M(A,C)=N(B,C)\otimes_{\mathcal{B}} M(A,B)$.

Like the tensor product of ordinary bimodules, the composition law for enriched bimodules is associative up to isomorphism. For any $\mathcal{V}$-category $\mathcal{C}$, the hom-objects $\mathcal{C}(C,C')$ clearly define a module $\mathcal{C}\,\mathrel{\to\!\!\!\!\!\!+}\,\mathcal{C}$ which we denote by $1_{\mathcal{C}}$. These bimodules serve as identities for composition of bimodules, since $1_{\mathcal{B}}\circ M\cong M\cong M\circ 1_{\mathcal{A}}$ for any bimodule $M:\mathcal{A}\to\mathcal{B}$. We refer again to Chapter 1 where this composition law for bimodules is seen to yield a "bicategory". In this chapter we have already been using the coherence result of Theorem 1.3 in the context of monoidal categories. Henceforth we shall also be applying it to composition of bimodules: we will not label any bimodule isomorphisms built up from associativity and identity isomorphisms.

We have already described the bimodules associated to an enriched functor. The relationship between composition of such bimodules and composition of the corresponding functors is anticipated by the notation $(\ )_*$ and $(\ )^*$: it can be shown that for $\mathcal{V}$-functors $F:\mathcal{A}\to\mathcal{B}$ and $G:\mathcal{B}\to\mathcal{C}$ there are bimodule isomorphisms

(0.16) $$(GF)_*\ \cong\ G_*\circ F_*$$

$$(GF)^* \cong F^* \circ G^*. \tag{0.17}$$

Recalling diagrams (0.14) and (0.15), there is clearly a sense in which $(\ )_*$ is covariant on $\mathcal{V}$-functors and contravariant on $\mathcal{V}$-natural transformations, while for $(\ )^*$ the reverse in true. This assertion will be made precise in Chapter 1.

Composition of bimodules can also be defined in terms of the $\mathcal{V}$-functor formulation of bimodules. Kelly [28] does this in terms of "indexed colimits" and although indexed colimits will not be discussed in detail, it is worth noting here that they can in turn be defined in terms of composition of bimodules. An index is a $\mathcal{V}$-functor $F: \mathcal{K}^{op} \to \mathcal{V}$, which we interpret as a bimodule $\mathcal{K} \mathrel{-\!\!\!\rightarrow\!\!\!\rightarrow} \mathcal{I}$, and given a $\mathcal{V}$-functor $G: \mathcal{K} \to \mathcal{B}$, the *colimit of G indexed by F* consists of an object $F \otimes G$ of $\mathcal{B}$ which is characterised by a condition which amounts to an isomorphism of bimodules

$$(F \otimes G)^* \cong F \otimes_{\mathcal{B}} G^*$$

where the object $F \otimes G$ is treated here as a $\mathcal{V}$-functor $\mathcal{I} \to \mathcal{B}$.[8] "Indexed limits" can also be described in terms of bimodules in a similar manner. Much of the theory of enriched categories can be developed either in terms of bimodules and their composition law or in terms of functors and indexed limits and colimits.

0.3.3 Adjoints, Density and the Yoneda Embedding

A concept central to the treatment of Cauchy completeness given in the next section is that of "adjoint bimodules". Adjunction can be defined in general for arrows in a bicategory, but here we phrase the definition in terms of bimodules. An *adjunction* $M \dashv N$ between bimodules $M: \mathcal{A} \mathrel{-\!\!\!\rightarrow\!\!\!\rightarrow} \mathcal{B}$ and $N: \mathcal{B} \mathrel{-\!\!\!\rightarrow\!\!\!\rightarrow} \mathcal{A}$ consists of a *unit* $\eta: 1_{\mathcal{A}} \to N \circ M$ and a *counit* $\epsilon: M \circ N \to 1_{\mathcal{B}}$ such that the following

[8] Note that Kelly [28] writes $F \star G$ in place of $F \otimes G$.

adjunction triangles commute:

$$\begin{array}{ccc} & (N \circ M) \circ N \cong N \circ (M \circ N) & \\ {}^{\eta \circ 1_N} \nearrow & & \searrow^{1_N \circ \epsilon} \\ 1_{\mathcal{A}} \circ N \cong N & \xrightarrow[1_N]{\hspace{4em}} & N \cong N \circ 1_{\mathcal{B}} \end{array}$$

$$\begin{array}{ccc} & M \circ (N \circ M) \cong (M \circ N) \circ M & \\ {}^{1_M \circ \eta} \nearrow & & \searrow^{\epsilon \circ 1_M} \\ M \circ 1_{\mathcal{A}} \cong M & \xrightarrow[1_M]{\hspace{4em}} & M \cong 1_{\mathcal{B}} \circ M. \end{array}$$

We say that M is a *left adjoint* of N, and N is a *right adjoint* of B. Here we are, of course, appealing to coherence for bicategories. We have already encountered one important source of bimodule adjunctions, namely the bimodules associated to enriched functors.

Proposition 0.1 *If $F: \mathcal{A} \to \mathcal{B}$ is $\mathcal{V}$-functor, then there is an adjunction $F_* \dashv F^*$ between the bimodules induced by F.*

Another feature of the bimodules induced by enriched functors is that they can be used to characterise or define properties of the functor. A $\mathcal{V}$-functor $F: \mathcal{A} \to \mathcal{B}$ is said to be *fully faithful* if the arrows $F_{A,A'}: \mathcal{A}(A, A') \to \mathcal{B}(FA, FA')$ are isomorphisms in $\mathcal{V}$. This property is characterised in terms of the bimodules associated to F by the following result.

Proposition 0.2 *A $\mathcal{V}$-functor $F: \mathcal{A} \to \mathcal{B}$ is fully faithful if and only if there is a bimodule isomorphism $F^* \circ F_* \cong 1_{\mathcal{A}}$.*

Proof. This follows immediately from the observation that

$$F^* \circ F_*(A, A') \cong \mathcal{B}(FA, FA').$$

□

A $\mathcal{V}$-functor $F: \mathcal{A} \to \mathcal{B}$ is *dense* if $F_* \circ F^* \cong 1_{\mathcal{B}}$. The most important example of a dense $\mathcal{V}$-functor is given by the "Yoneda embedding", but before describing this embedding, we must construct the *enriched* category of bimodules $\mathcal{A} \nrightarrow \mathcal{B}$. We first construct enriched functor categories.

For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ we define a $\mathcal{V}$-category $[\mathcal{A}, \mathcal{B}]$ whose objects are $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{B}$ and the hom-object $[\mathcal{A}, \mathcal{B}](F, G)$ for $\mathcal{V}$-functors F, $G: \mathcal{A} \to \mathcal{B}$ is an equalizer of the diagram

$$\prod_{A \in \mathcal{A}} \mathcal{B}(FA, GA) \rightrightarrows \prod_{A, A' \in \mathcal{A}} [\mathcal{A}(A, A'), \mathcal{B}(FA, GA')]$$

for which one of the parallel arrows is characterised as the top arrow of the commuting diagram

$$\begin{array}{ccc} \prod_{A \in \mathcal{A}} \mathcal{B}(FA, GA) & \longrightarrow & \prod_{A, A' \in \mathcal{A}} [\mathcal{A}(A, A'), \mathcal{B}(FA, GA')] \\ {\scriptstyle \pi_{A'}} \downarrow & & \downarrow {\scriptstyle \pi_{A,A'}} \\ \mathcal{B}(FA', GA') & \longrightarrow & [\mathcal{A}(A, A'), \mathcal{B}(FA, GA')] \end{array}$$

where arrows labelled π are product projections and the bottom arrow corresponds under the internal hom adjunction to the arrow

$$\mathcal{B}(FA', GA') \otimes \mathcal{A}(A, A') \xrightarrow{1 \otimes F} \mathcal{B}(FA', GA') \otimes \mathcal{B}(FA, FA') \xrightarrow{m} \mathcal{B}(FA, GA').$$

The other arrow in diagram (0.3.3) is obtained in a similar manner using G. Given $\mathcal{V}$-functors F, G and $H: \mathcal{A} \to \mathcal{B}$, the arrow

$$[\mathcal{A}, \mathcal{B}](G, H) \otimes [\mathcal{A}, \mathcal{B}](F, G) \to \prod_{A} (FA, HA)$$

with A-component given by the composite

$$\begin{array}{c} [\mathcal{A},\mathcal{B}](G,H)\otimes[\mathcal{A},\mathcal{B}](F,G) \\ \downarrow \\ \prod_{A_1}\mathcal{B}(GA_1,HA_1)\otimes\prod_{A_2}\mathcal{B}(FA_2,GA_2) \\ \Big\downarrow{\scriptstyle \pi_A\otimes\pi_A} \\ \mathcal{B}(GA,HA)\otimes\mathcal{B}(FA,GA) \\ \Big\downarrow{\scriptstyle m} \\ \mathcal{B}(FA,HA) \end{array}$$

equalizes the appropriate arrows in the definition of $[\mathcal{A},\mathcal{B}](F,H)$ and thus yields an arrow $[\mathcal{A},\mathcal{B}](G,H)\otimes[\mathcal{A},\mathcal{B}](F,G)\to[\mathcal{A},\mathcal{B}](F,H)$, which gives the composition law for $[\mathcal{A},\mathcal{B}]$. The collection of all identity elements $1_{FA}: I\to\mathcal{B}(FA,FA)$ gives an arrow $I\to\prod_A\mathcal{B}(FA,FA)$ which in turn yields an arrow $I\to[\mathcal{A},\mathcal{B}](F,F)$ which gives the identity element for F in $[\mathcal{A},\mathcal{B}]$. That these definitions of composition and identity elements satisfy the enriched category associativity and identity laws follows directly from the corresponding axioms for $\mathcal{B}$.

Note that the arrows $F\to G$ in the underlying category $U[\mathcal{A},\mathcal{B}]$ are of the form $I\to[\mathcal{A},\mathcal{B}](F,G)$ and thus correspond to arrows $I\to\prod_A\mathcal{B}(FA,GA)$ subject to a condition which mean that the corresponding families of arrows $I\to\mathcal{B}(FA,GA)$ constitute $\mathcal{V}$-natural transformations. Thus the underlying category of $[\mathcal{A},\mathcal{B}]$ is $\mathcal{V}\text{-Cat}(\mathcal{A},\mathcal{B})$.

The key property of enriched functor categories is that they provide a right adjoint for the tensor product of enriched categories in the sense that there is a bijection

(0.18) $$\frac{\mathcal{A}\otimes\mathcal{B}\ \longrightarrow\ \mathcal{C}}{\mathcal{A}\ \longrightarrow\ [\mathcal{B},\mathcal{C}]}.$$

We now define the enriched category ${}_{\mathcal{B}}Mod_{\mathcal{A}}$ of bimodules $\mathcal{A} \longrightarrow \mathcal{B}$ to be the enriched functor category $[\mathcal{A}^{op} \otimes \mathcal{B}, \mathcal{V}]$. Of course this enriched category can also be defined directly in terms of the action formulation of bimodules. Of particular interest is the enriched category of modules $\mathcal{A} \longrightarrow \mathcal{V}$ which we denote by $Mod_{\mathcal{A}}$. The bijection (0.18) becomes in this instance a bijection

$$\frac{\mathcal{A} \;\;\longrightarrow\;\; \mathcal{B}}{\mathcal{B} \;\;\rightarrow\;\; Mod_{\mathcal{A}}.} \tag{0.19}$$

Therefore, there is a $\mathcal{V}$-functor $i: \mathcal{A} \rightarrow Mod_{\mathcal{A}}$ which corresponds under this bijection to $1_{\mathcal{A}}: \mathcal{A} \longrightarrow \mathcal{A}$. This functor is referred to as the *Yoneda embedding*. The importance of the Yoneda embedding hinges on the follow result, which generalizes the classical Yoneda Lemma.

Proposition 0.3 (Yoneda Lemma) *For any enriched category $\mathcal{A}$ and right $\mathcal{A}$-module $N \in Mod_{\mathcal{A}}$ there is a module isomorphism*

$$N \cong i_*(-, N).$$

Proof. Proving that

$$NA \cong Mod_{\mathcal{A}}(\mathcal{A}(-, A), N)$$

for each object A of $\mathcal{A}$ is simply a matter of diagram chasing to confirm that NA is a coequalizer of the appropriate diagram from the definition of hom-objects for $Mod_{\mathcal{A}}$. The result then follows by checking that these isomorphisms respect the $\mathcal{A}$ action. □

Our main application of the Yoneda Lemma is in the proof of the following result.

Theorem 0.4 *The Yoneda embedding is dense and fully faithful.*

Proof. Setting $N = \mathcal{A}(-, A')$ in Yoneda Lemma, we obtain

$$\mathcal{A}(A, A') \cong \mathrm{Mod}_{\mathcal{A}}(\mathcal{A}(-, A), \mathcal{A}(-, A'))$$

and hence i is fully faithful. To prove that i is dense, we first observe that the Yoneda Lemma also implies that

$$Mod_{\mathcal{A}}(\mathcal{A}(-,A),N) \otimes_{\mathcal{A}} Mod_{\mathcal{A}}(M,\mathcal{A}(-,A)) \cong NA \otimes_{\mathcal{A}} Mod_{\mathcal{A}}(M,\mathcal{A}(-,A))$$

and then chase diagrams to confirm that this is isomorphic to $\mathrm{Mod}_{\mathcal{A}}(M,N)$. □

We now return to the bijection (0.19). Consider $M\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to Mod_{\mathcal{A}}$ which correspond under this bijection, so that $M(A,B) = F(B)(A)$. Using the fact that the Yoneda embedding is dense as well as the Yoneda Lemma, we deduce

$$\begin{aligned} F^*(N,B) &= Mod_{\mathcal{A}}(N, M(-,B)) \\ &\cong Mod_{\mathcal{A}}(\mathcal{A}(-,A), M(-,B)) \otimes_{\mathcal{A}} Mod_{\mathcal{A}}(N,\mathcal{A}(-,A)) \\ &\cong MA \otimes_{\mathcal{A}} Mod_{\mathcal{A}}(N,\mathcal{A}(-,A)) \\ &\cong M \circ i^*(N,B). \end{aligned}$$

We conclude that for every $\mathcal{V}$-bimodule $M\colon \mathcal{A} \to\!\!\!\!+ \mathcal{B}$ there exists a corresponding $\mathcal{V}$-functor $F\colon \mathcal{B} \to Mod_{\mathcal{A}}$ such that

$$F^* \cong M \circ i^*. \tag{0.20}$$

This characterisation of bimodules will be useful in our treatment of Cauchy completeness.

0.4 Cauchy Completeness

Cauchy completeness for enriched categories is very closely related to the usual notion of Cauchy completeness for metric spaces. To see this connection we first describe a base for enriched categories which has not arisen in previous examples.

We denote by $\mathbf{R}$ the category whose objects are non-negative extended real numbers (this includes ∞), an arrow $a \to b$ is a relation $a \geq b$ and tensor product is given by the operation of addition. The unit for this tensor product is 0. Consider *truncated subtraction*, defined by

$$[a,b] = \begin{cases} b-a & b \geq a \\ 0 & a \geq b \end{cases}$$

where we use the definitions

$$\infty - \infty = 0 \tag{0.21}$$

$$\infty - a = \infty \qquad \text{if } a \neq \infty \tag{0.22}$$

$$b - \infty = 0. \tag{0.23}$$

We denote also denote $[a,b]$ by $b-a$. This truncated subtraction gives an internal-hom for $\mathbf{R}$, since there is a double implication

$$\frac{a+b \;\geq\; c}{a \;\geq\; c-b.}$$

$\mathbf{R}$ is complete and cocomplete with limits given by suprema and colimits given by infima.

A category X enriched over $\mathbf{R}$ consists of a collection of objects, which we call *points*, together with an assignment to each ordered pair a, b of points of an element $X(a,b)$ of $\mathbf{R}$. This assignment is subject to the conditions

$$X(b,c) + X(a,b) \;\geq\; X(a,c) \tag{0.24}$$

$$0 = X(a,a). \tag{0.25}$$

Note that the second condition arises from $0 \geq X(a,a)$. If $X(a,b)$ is thought of as the distance from the point a to b then to some extent X resembles a metric space. Certainly any metric space can be considered to be an $\mathbf{R}$-category, however in general the conditions imposed on categories enriched over $\mathbf{R}$ are weaker than the axioms for a metric space. More specifically, distances may be infinite, distance need not be symmetric and distances between distinct points may be zero. Lawvere [32] calls $\mathbf{R}$-categories *generalized metric spaces* and cites as an example "$X(a,b)$ = work required to get from a to b in a mountainous region X".

An enriched functor between R-categories is a *distance-decreasing function*, that is a function $f: X \to Y$ satisfying

$$X(x, x') \geq Y(fx, fx')$$

for all x, $x' \in X$. An R-bimodule $X \dashrightarrow X$ is therefore a function $X \times Y \to \mathrm{R}$ which is distance-increasing in the first variable and distance decreasing in the second. The importance of R-bimodules lies in the following result concerning adjoint bimodules which appears in Lawvere [32].

Proposition 0.5 *A metric space Y is Cauchy complete if and only if every adjunction $M \dashv M^*$ of R-bimodules $M: X \dashrightarrow Y$ and $M^*: Y \dashrightarrow X$ is induced by an R-functor $f: X \to Y$.*

Proof. For R-bimodules, the definition of an adjunction $M \dashv M^*$ becomes

$$X(x, x') \geq \inf_y (M^*(y, x) + M(x', y))$$

$$\inf_x (M(x, y') + M^*(y, x)) \geq Y(y, y').$$

In particular, for a fixed $x \in X$,

$$0 = \inf_y (M^*(y, x) + M(x, y))$$

$$M(x, y') + M^*(y, x) \geq Y(y, y').$$

We can therefore choose a sequence (y_n) of points in Y such that

$$M(x, y_n) + M^*(y_n, x) \leq \frac{1}{n} \tag{0.26}$$

and therefore

$$Y(y_n, y_m) \leq \frac{1}{n} + \frac{1}{m}$$

so that (y_n) is a Cauchy sequence. If (y'_n) is another sequence satisfying (0.26) then

$$Y(y_n, y'_n) \leq \frac{2}{n}$$

so (y_n) and (y'_n) are equivalent Cauchy sequences. Thus, associated to each point $x \in X$ there is an equivalence class of Cauchy sequences. If Y is Cauchy complete and $\lim_{n\to\infty} y_n = f(x)$ then (0.26) implies that $M(x, f(x)) = M^*(f(x), x) = 0$. Thus the adjunction condition yields $M(x,y) \leq Y(f(x), y)$ and the fact that M is distance-decreasing in its second variable implies that $M(x,y) \geq Y(f(x), y)$. Hence $M(x,y) = Y(f(x), y)$ and similarly $M^*(y,x) = Y(y, f(x))$, in other words, $M = f_*$ and $M^* = f^*$.

Conversely, given any Cauchy sequence, (y_n), the definitions

$$\begin{aligned} M(y) &= \lim_{n\to\infty} Y(y_n, y) \\ M^*(y) &= \lim_{n\to\infty} Y(y, y_n) \end{aligned}$$

yield an adjunction $M \dashv M^*$ where $M: \mathcal{I} \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ and $M^*: Y \longrightarrow\!\!\!\!\!\!\!+\;\, \mathcal{I}$. If this adjunction is induced by $f: \mathcal{I} \to Y$, then $f(0)$ is the limit of the Cauchy sequence, so if every adjunction is induced by a R-functor then Y is Cauchy complete. □

Prompted by the above result, we say that a $\mathcal{V}$-category is *Cauchy complete* if every adjunction $M \dashv M^*$ of $\mathcal{V}$-bimodules where $M: \mathcal{C} \longrightarrow\!\!\!\!\!\!\!+\;\, \mathcal{A}$ is induced by an $\mathcal{V}$-functor $f: \mathcal{C} \to \mathcal{A}$. Recall that associated to any metric space Y there is a Cauchy complete metric space $\overline{Y}$ called the "Cauchy completion" of Y whose points are equivalence classes of Cauchy sequences. In the above proof we observed that Cauchy sequences in Y amounted to adjunctions $M \dashv M^*$ where $M: \mathcal{I} \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ and this prompts the following definition. Given a $\mathcal{V}$-category $\mathcal{A}$, its *Cauchy completion* $\overline{\mathcal{A}}$ is the full enriched sub-category of $Mod_{\mathcal{A}}$ determined by those bimodules $M: \mathcal{I} \longrightarrow\!\!\!\!\!\!\!+\;\, \mathcal{A}$ with a left adjoint. If $\mathcal{A}$ is Cauchy complete then clearly $\overline{\mathcal{A}}$ is equivalent to $\mathcal{A}$, so Cauchy completion is a closure operation up to equivalence, and most importantly, we can prove

Proposition 0.6 *The Cauchy completion of an enriched category is Cauchy complete.*

Proof. First observe the Yoneda embedding factors through $j: \mathcal{A} \to \overline{\mathcal{A}}$ which is also dense and fully faithful. Moreover, we can say further to equation (0.20) that if $M: \mathcal{A} \nrightarrow \mathcal{B}$ has a left adjoint, then the associated functor $\mathcal{A} \to Mod_{\mathcal{A}}$ factors through $\overline{\mathcal{A}}$, thus

$$F^* \cong M \circ j^*$$

Now consider an adjunction $N_* \dashv N$ where $N: \overline{\mathcal{A}} \nrightarrow \mathcal{B}$. The composite $N \circ j_*$ has a left adjoint, namely $j^* \circ N_*$, and thus there exists a $\mathcal{V}$-functor $F: \mathcal{B} \to \overline{\mathcal{A}}$ such that

$$F^* \cong (N \circ j_*) \circ j^*$$

and therefore

$$F^* \cong N$$

and so $\overline{\mathcal{A}}$ is Cauchy complete. □

We now interpret Cauchy completeness for base categories **Set** and **Ab**.

Proposition 0.7 *An ordinary category is Cauchy complete if and only if all idempotents split.*

Proof. Since the tensor product for **Set** is Cartesian product, to establish the Cauchy completeness of a category **C**, it suffices to consider adjunctions $M_* \dashv M$ where $M: \mathbf{C} \nrightarrow \mathbf{1}$. Now

$$M \circ M_* = \left(\sum_C M(C) \times M_*(C) \right) / \sim$$

and

$$M_* \circ M(C, C') = M_*(C') \times M(C)$$

where $\sim$ is the equivalence relation generated by $(y \cdot f, \lambda) \sim (y, f \cdot \lambda)$ for $y \in M(C)$, $\lambda \in M_*(C')$ and $f: C \to C'$ in **C**. We denote the equivalence class of (x, λ) by $[x, \lambda]$. The data for an adjunction $M_* \dashv M$ is thus an equivalence class $[x_0, \lambda_0]$, where $x_0 \in M(C_0)$ and $\lambda_0 \in M_*(C_0)$ for some object C_0 of **C**, and functions

$$\epsilon_{C,C'}: M_*(C') \times M(C) \longrightarrow \mathbf{C}(C, C')$$

which respect the C actions and satisfy the conditions

$$(0.27) \qquad x = x_0 \cdot \epsilon(\lambda_0, x)$$

$$(0.28) \qquad \lambda = \epsilon(\lambda, x_0) \cdot \lambda_0$$

for all $x \in M(C)$ and $\lambda \in M_*(C)$. Since ϵ respects the C actions we conclude that

$$(0.29) \qquad \epsilon(\lambda_0, x) = \epsilon(\lambda_0, x_0,)\epsilon(\lambda_0, x)$$

$$(0.30) \qquad \epsilon(\lambda, x_0) = \epsilon(\lambda, x_0)\epsilon(\lambda_0, x_0)$$

and therefore $e = \epsilon(\lambda_0, x_0): C_0 \to C_0$ is an idempotent and furthermore, the assignments $x \mapsto \epsilon(\lambda_0, x)$ and $\lambda \mapsto \epsilon(\lambda, x_0)$ give module isomorphisms

$$MC \cong \{ef \mid f: C \to C_0\}$$

$$M_*C \cong \{fe \mid f: C_0 \to C\}.$$

If the idempotent e splits, these modules are isomorphic to $\mathcal{C}(C, C_1)$ and $\mathcal{C}(C_1, C)$ respectively for some C_1. Conversely, for an arbitrary idempotent e in C, the right-hand sides of the above isomorphisms define adjoint modules, and if these are to be induced by an object of C then e must split. □

Proposition 0.8 *The Cauchy completion of an additive category $\mathcal{A}$ consists of finitely generated projective modules over $\mathcal{A}$.*

Proof. Examining Ab-bimodule adjunctions $M_* \dashv M$ where $M: \mathcal{I} \to \mathcal{A}$ proceeds in a similar fashion to the proof of the previous proposition. Elements of $M_* \circ M$ are written as finite sums of terms of the form $x \otimes \lambda$ which satisfy $(x \cdot f) \otimes \lambda = x \otimes (f \cdot \lambda)$. The data for the adjunction gives an element $\sum_{i=1}^{n} x_i \otimes \lambda_i$, where $x_i \in MA_0$ and $\lambda_i \in M_*A_0$ for some object A_0 of $\mathcal{A}$, and group homomorphisms

$$\epsilon_{A,A'}: M_*(A') \otimes M(A) \longrightarrow \mathcal{A}(A, A')$$

which satisfy the conditions

$$\begin{aligned} x &= \sum_i x_i \cdot \epsilon(\lambda_i \otimes x) \\ \lambda &= \sum_i \epsilon(\lambda \otimes x_i) \cdot \lambda_i \end{aligned}$$

for all $x \in MA$ and $\lambda \in M_*A$. Thus M is finitely generated by $x_1, x_2, \ldots, x_n$. Now consider a module map $f: M \to N_2$ and a surjective module map $g: N_1 \to N_2$. We can define $h: M \to N_1$ by

$$h(x) = \sum_i y_i \cdot \epsilon(\lambda_i \otimes x)$$

where y_i is chosen such that $g(y_i) = f(x_i)$ and then

$$\begin{aligned} gh(x) &= \sum_i g(y_i) \cdot \epsilon(\lambda_i \otimes x) \\ &= \sum_i f(x_i) \cdot \epsilon(\lambda_i \otimes x) \\ &= f\left(\sum_i x_i \cdot \epsilon(\lambda_i \otimes x)\right) \\ &= f(x). \end{aligned}$$

Thus f factors through g and so M is projective. □

Chapter 1

Bicategories

Viewing the operation of "gluing" of cobordisms along common boundaries as a composition law is central to the study of topological quantum field theories. The fact that this composition law is only associative up to diffeomorphism means that *n-Cobord* is not a category but a higher dimensional structure. It is in fact a "bicategory". Bicategories were first defined almost thirty years ago in Bénabou [4] but have only relatively recently gained widespread attention from researchers outside category theory. As well as being used directly in the study of topological quantum field theories, bicategories have surfaced in the study of Zamolodchikov tetrahedra equations, which are of course closely related to field theories. The work of Kapranov and Voevodsky [22] is a good example of this tendency. We begin this chapter with a discussion of a framework for higher-dimensional algebra and then review some of the basic theory of bicategories.

1.1 Multiplicative Graphs

When attempting to generalize the notion of a category to higher dimensions, the basic data requirements (n-cells, composition, identities) are fairly clear, but the axioms these data should satisfy are more elusive and indeed there is no unique choice. At the two-dimensional level, varying the strictness of associativity results in the definition of either 2-categories[1] or bicategories and moving to three dimensions, 3-categories, Gray-enriched categories and tricategories[2] have all been studied. In this chapter we only consider 2-categories, bicategories and one-object tricategories (tensor bicategories). Nevertheless is is useful to introduce a framework within which to speak of the data of higher dimensional categories and to this end we introduce the notion of "multiplicative graphs" of arbitrary dimension.

1.1.1 The Definition of a Multiplicative Graph

A multiplicative graph consists of the same data as a category, but with no associativity and identity laws imposed on the composition. More precisely, a *multiplicative graph* (G_0, G_1, c, d, i, m) consists of a set G_0 of *objects*, a set G_1 of *arrows*, *domain* and *codomain* functions

$$\begin{aligned} d\colon G_1 &\longrightarrow G_0 \\ c\colon G_1 &\longrightarrow G_0 \end{aligned}$$

respectively, an *identity function*

$$i\colon G_0 \longrightarrow G_1$$

and a *composition law*

$$m\colon G_1 \times_{G_0} G_1 \longrightarrow G_1$$

where $G_1 \times_{G_0} G_1 = \{(f,g) \mid d(f) = c(g)\}$. These data are subject to the conditions

$$\begin{aligned} di(a) = a &= ci(a) \\ d(f \circ g) &= d(g) \\ c(f \circ g) &= c(f), \end{aligned}$$

[1]For the basic theory of 2-categories see Kelly and Street [30].
[2]See Gordon, Power and Street [15] on Gray-enriched categories and tricategories.

where we write $f \circ g$ for $m(f, g)$.

The notion of multiplicative graph appears in Ehresmann [11] under the name "graphe multiplicatif". We can associate a (directed, reflexive) graph to any multiplicative graph by forgetting the composition law. A *homomorphism of multiplicative graphs* is a morphism of the associated underlying graphs while a *strict* homomorphism of multiplicative graphs preserves composition and identities. Multiplicative graphs and homomorphisms form a category, which we denote by μGraph[3].

A *multiplicative 2-graph* G consists of sets G_0, G_1 and G_2 and multiplicative graphs $(G_0, G_1, c_0, d_0, i_0, m_0)$ and $(G_1, G_2, c_0, d_1, i_1, m_1)$ together with an additional composition law

$$m: G_2 \times_{G_0} G_2 \to G_2$$

where $G_2 \times_{G_0} G_2 = \{(y, x) \mid d_0 d_1(x) = c_0 c_1(y)\}$. We will write $m_0(f, g) = f \circ g$, $m_1(x, x') = x \cdot x'$ and $m(x, y) = x \circ y$ and adopt the usual practice of referring to $\circ$ as *horizontal composition* and to $\cdot$ as *vertical composition*. These data are subject to the conditions

$$\begin{aligned} d_0 d_1 &= d_0 c_1 \\ c_0 c_1 &= c_0 d_1 \end{aligned}$$

and

$$\begin{aligned} d_1(y \circ x) &= d_1(y) \circ d_1(x) \\ c_1(y \circ x) &= c_1(y) \circ c_1(x). \end{aligned}$$

Elements of G_0 are called *objects* or *0-cells*, elements of G_1 are *arrows* or *1-cells* and elements of G_2 are *2-cells*. A *(strict) homomorphism of multiplicative 2-graphs* $\mathrm{G} \to \mathrm{G}'$ maps objects to objects, arrows to arrows and 2-cells to 2-cells in such a way as to give (strict) homomorphisms of the two constituent multiplicative

[3] We chose not to call this category MGraph to avoid confusion with the very different notion of "multi-graph".

graphs. We denote the category of multiplicative 2-graphs and their homomorphisms by 2-μGraph.

A *multiplicative r-graph* consists of a sequence $G_0, G_1, \ldots, G_r$ of sets together with multiplicative graph structures $(G_k, G_{k+1}, c_k, d_k, i_k)$ for $k = 0, \ldots, r-1$, and for $k = 0, \ldots r-2$, multiplicative 2-graph structures on $(G_k, G_{k+1}, c_k, d_k, i_k)$ and $(G_k, G_{k+2}, c_{k+1}, d_{k+1}, i_{k+1})$. The elements of G_n are referred to as *n-cells* or *cells of dimension n*. Cells of dimension n in a multiplicative r-graph are said to have *codimension* $(r-n)$. A *(strict) homomorphism of multiplicative r-graphs* maps n-cells to n-cells so as to give (strict) multiplicative 2-graph morphisms at each dimension. We denote the category of multiplicative r-graphs and their homomorphisms by r-μGraph.

A *multiplicative ω-graph* consists of a sequence of sets $(G_k)_{k \in \mathbb{N}}$ together with multiplicative graph structures $(G_k, G_{k+1}, c_k, d_k, i_k)$ for all $k > 0$, and multiplicative 2-graph structures on $(G_k, G_{k+1}, c_k, d_k, i_k)$ and $(G_k, G_{k+2}, c_{k+1}, d_{k+1}, i_{k+1})$. A *(strict) homomorphism of multiplicative ω-graphs* maps n-cells to n-cells so as to give (strict) multiplicative 2-graph morphisms at each dimension. We denote the category of multiplicative ω-graphs and their homomorphisms by ω-μGraph. Note that there is a functor

$$\omega\text{-}\mu\text{Graph} \longrightarrow r\text{-}\mu\text{Graph}$$

obtained by "forgetting" cells of dimension higher than r. This functor has a left adjoint D and a right adjoint C. These are the usual "discrete" and "chaotic" constructions: given a multiplicative r-graph G, the only n-cells that DG has for $n > r$ are identity cells, while CG has a single n-cell between each pair of $(n-1)$-cells. Henceforth, when we introduce properties in terms of multiplicative ω-graphs, we will understand a multiplicative r-graph G to have that property if and only if DG does.

Any r-dimensional analogue of a category, however one chooses to define it, will necessarily have associated to it an underlying multiplicative r-graph. For example, we will be able to associate an underlying multiplicative 2-graph to a bicategory. In the next section we define higher order equivalences. These and other concepts are meaningful in a "higher-dimensional category" and are interpreted in the underlying multiplicative graph. This should go some way to indicate the value of the notion of multiplicative graph.

An important example of a multiplicative ω-graph arises when one considers topological spaces, continuous functions between spaces, homotopies between functions, homotopies between these homotopies, and so on. Motivated by this example, we define a *category of homotopies* to be a multiplicative ω-graph for which the 0-cells and 1-cells are the objects and arrows of a category. Note that we could equivalently define a category of homotopies to be a category enriched over ω-μGraph.

1.1.2 Higher Order Equivalences

One use of the notion of multiplicative ω-graphs is to subsume the notions of equality, isomorphism and equivalence into the general notion of "r-equivalence". Our definition of r-equivalence is an inductive one, and is based on a similar definition for ω-categories in Street [45]. Two cells a and b of a multiplicative ω-graph are *0-equivalent* if they are equal. They are *r-equivalent* if there exist cells $f: a \to b$ and $g: b \to a$ such that $g \circ f$ is $(r-1)$-equivalent to 1_a and $f \circ g$ is $(r-1)$-equivalent to $1_b$and we write $a \stackrel{r}{\simeq} b$. We call f an r-equivalence and g the *inverse up to r-equivalence* of f or simply the *inverse* of f when r-equivalence is clear from the context.

A natural question to ask at this point is whether r-equivalence is an equivalence relation. In general it is not. Consider $a \stackrel{r}{\simeq} b$ and $b \stackrel{r}{\simeq} c$, exhibited by r-equivalences $f: a \to b$ and $f': b \to c$ with inverses g and g' respectively. To establish $a \stackrel{r}{\simeq} c$, the only possible candidate for an r-equivalence $a \to c$ is $f' \circ f$, with inverse $g \circ g'$. However, to show that $(g \circ g') \circ (f' \circ f)$ was $(r-1)$-equivalent to 1_a and $(f' \circ f) \circ (g \circ g')$ was $(r-1)$-equivalent to 1_b would require associativity and identity laws at least up to $(r-1)$-equivalence, which we do not have in general. As a example, consider objects in a bicategory (see Section 1.2). It is perfectly possible to define both isomorphism and equivalence of objects, but it turns out that equivalence yields an equivalence relation while isomorphism need not. In general, r-equivalence is the appropriate notion for comparing n-cells in a multiplicative $(r+n)$-graph and any reasonable associativity and identity law axioms will ensure it is an equivalence relation[4]. Isomorphism is 1-equivalence and equivalence is 2-equivalence, while 3-equivalence is usually referred to as biequivalence (being the appropriate notion for comparing bicategories).

Another curious feature of r-equivalence is that $a \overset{r-1}{\simeq} b$ does not imply $a \overset{r}{\simeq} b$. In the case $r = 1$, this means that an object need not be isomorphic to itself. Although perhaps counter-intuitive at first, this is a basic feature of higher dimensional algebra. It is quite possible, for example, to construct a bicategory with an object A which is not isomorphic to itself. Of course, A will be *equivalent* to itself.

1.2 Bicategories

Fifteen years ago, Street [44] observed that "since the paper Bénabou [4] in which bicategories were introduced, little has been published on them explicitly." The same observation could be made today and indeed as yet no one publication brings together the definitions of bicategories, morphisms, transformations and modifications. Perhaps the forthcoming book by Kapranov and Voevodsky [21] will do so. In the meantime, we present all the definitions here. Bénabou [4] and Street [44] and, to a lesser extent, Mac Lane and Paré [35] and Gray [16] all had a part to play in shaping our treatment of the subject.

[4]Thus, for example, 2-equivalence is an equivalence relation on objects of both 2-categories and bicategories.

1.2.1 The Definition of a Bicategory

A *bicategory* $\mathcal{B}$ consists of a collection of *objects* (or *zero-cells*) $A, B, C, \ldots$ and for each pair of objects, A and B, a *hom-category* $\mathcal{B}(A,B)$. The objects $f, g, h, \ldots$ of $\mathcal{B}(A,B)$ are the *arrows* (or *one-cells*) of $\mathcal{B}$ and we write $f\colon A \to B$. The arrows of $\mathcal{B}(A,B)$ are the *two-cells* of $\mathcal{B}$. The collection of objects of $\mathcal{B}$ is denoted by $\mathcal{B}_0$, and the oriented graph with vertices the objects and edges the arrows of $\mathcal{B}$ is denoted by $\mathcal{B}_1$. For any three objects A, B, C, there is given a *composition* functor

$$m_{A,B,C}\colon \mathcal{B}(B,C) \times \mathcal{B}(A,B) \to \mathcal{B}(A,C)$$

the effect of which will be denoted by $\circ$, as in $m_{A,B,C}(g,f) = g \circ f$. It follows immediately from functoriality that for all arrows f and g we have the equality

$$1_g \circ 1_f = 1_{g \circ f}$$

and for all 2-cells $\alpha_1, \alpha_2, \beta_1, \beta_2$ we have

$$(\alpha_2\alpha_1) \circ (\beta_2\beta_1) = (\alpha_2 \circ \beta_2)(\alpha_1 \circ \beta_1)$$

whenever these composites are defined. The second of these equalities is called the *interchange law.* Composition is further required to be unitary and associative up to coherent isomorphisms. Explicitly, the associativity means that there are specified *associativity isomorphisms*

$$a_{h,g,f}\colon (h \circ g) \circ f \to h \circ (g \circ f)$$

which are natural in f, g and h, and to say m is unitary means that for each object A there is a specified arrow $1_A\colon A \to A$ and for each arrow $f\colon A \to B$, there are *identity law isomorphisms*

$$l_f\colon 1_B \circ f \to f, \qquad r_f\colon f \circ 1_A \to f$$

which are natural in f. When it is clear which object is referred to, we often write 1 for 1_A. These isomorphisms are subject to the condition that for all arrows f,

g, h and k, the diagrams,

$$(1.1)\qquad \begin{array}{ccc}
((k\circ h)\circ g)\circ f & \xrightarrow{a_{k\circ h,g,f}} & (k\circ h)\circ(g\circ f) \\
{\scriptstyle a_{k,h,g}\circ 1_f}\downarrow & & \\
(k\circ(h\circ g))\circ f & & \downarrow{\scriptstyle a_{k,h,g\circ f}} \\
{\scriptstyle a_{k,h\circ g,f}}\downarrow & & \\
k\circ((h\circ g)\circ f) & \xrightarrow[1_k\circ a_{h,g,f}]{} & k\circ(h\circ(g\circ f))
\end{array}$$

and

$$(1.2)\qquad \begin{array}{ccc}
(g\circ 1_B)\circ f & \xrightarrow{a_{g,1_B,f}} & g\circ(1_B\circ f) \\
{\scriptstyle r_g\circ 1_f}\searrow & & \swarrow{\scriptstyle 1_g\circ l_f} \\
& g\circ f, &
\end{array}$$

commute whenever the appropriate composites are defined. We will refer to the first of these commuting diagrams as the pentagon condition. The obvious underlying multiplicative 2-graph of $\mathcal{B}$ is denoted by $\mathcal{B}_2$.

The following is a simple extension to bicategories of a result due to Kelly [23] on monoidal categories. The proof given here is closely based on the treatment of Kelly's result which appears in Joyal and Street [19].

Proposition 1.1 *For every object A of a bicategory $\mathcal{B}$,*

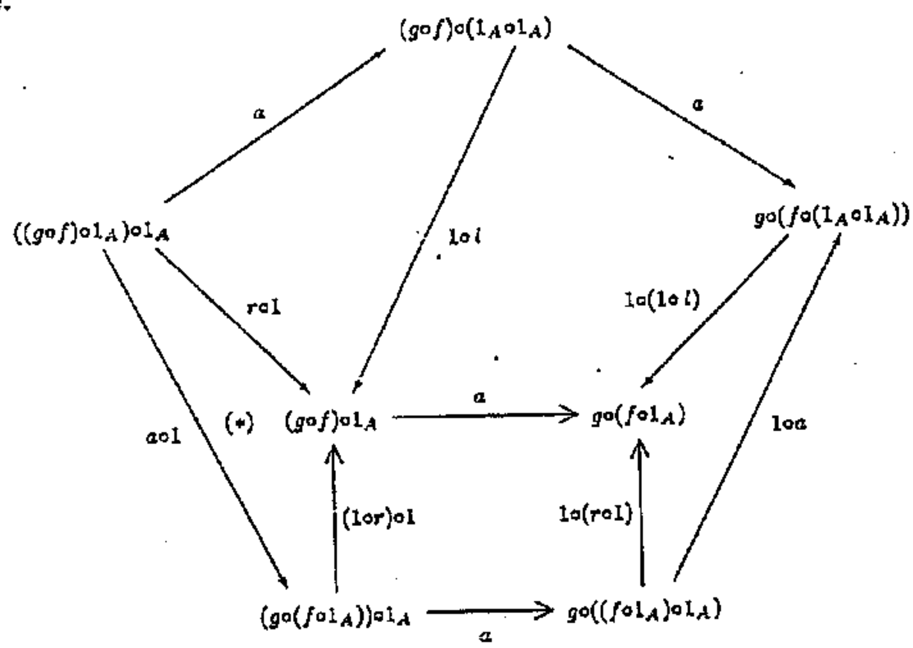
$$l_{1_A} = r_{1_A}$$

and for all arrows $f\colon A\to B$ and $g\colon B\to C$, the following diagrams in $\mathcal{B}(A,C)$ commute:

$$(1.3)\qquad \begin{array}{ccc}
(g\circ f)\circ 1_A & \xrightarrow{a_{g,f,1_A}} & g\circ(f\circ 1_A) \\
{\scriptstyle r_{g\circ f}}\searrow & & \swarrow{\scriptstyle 1_g\circ r_f} \\
& g\circ f &
\end{array}$$

$$
(1.4)\qquad
\begin{array}{ccc}
(1_C \circ g) \circ f & \xrightarrow{a_{1_C,g,f}} & 1_C \circ (g \circ f) \\
 & {\scriptstyle l_g \circ 1_f} \searrow \quad \swarrow {\scriptstyle l_{g\circ f}} & \\
 & g \circ f. &
\end{array}
$$

Proof. First note that naturality and invertibility of r ensures that to establish the equality of 2-cells $\alpha, \beta: h \to k$ where $h, k: A \to B$, it suffices to check that $\alpha \circ 1_{1_A} = \beta \circ 1_{1_A}: h \circ 1_A \to k \circ 1_A$. To prove the commutativity of the first triangle (1.3) of the proposition it is thus enough to show that the region marked $(*)$ in the diagram below commutes. This is indeed the case as all the arrows are invertible, and naturality of a along with (1.2) ensure that all the other regions commute.



A similar argument proves the commutativity of the second triangle (1.4).

Since r is natural, the following diagram commutes:

$$
\begin{array}{ccc}
(1_A \circ 1_A) \circ 1_A & \xrightarrow{r} & 1_A \circ 1_A \\
{\scriptstyle r \circ 1} \downarrow & & \downarrow {\scriptstyle r} \\
1_A \circ 1_A & \xrightarrow[r]{} & 1_A.
\end{array}
$$

But r is invertible, so $r = r \circ 1 : (1_A \circ 1_A) \circ 1_A \to 1_A \circ 1_A$ (or in full subscript glory, $r_{1_A \circ 1_A} = r_{1_A} \circ 1_{1_A}$). Now take $g = f = 1_A$ in the first triangle (1.3) and compare the result to (1.2) to obtain $(1 \circ l)\, a = (1 \circ r)\, a$ and hence $1 \circ l = 1 \circ r$ by invertibility of a. Using naturality and invertibility of l we finally obtain $r_{1_A} = l_{1_A}$ as required. □

1.2.2 Morphisms, Transformations and Modifications

Given bicategories $\mathcal{B}$ and $\mathcal{C}$, a *morphism of bicategories*, $\Phi: \mathcal{B} \to \mathcal{C}$ takes the form $\Phi = (F, \phi)$, where F consists of maps for each of the three levels of structure and ϕ specifies the extent to which F respects composition. More precisely, a morphism of bicategories consists of:

(i) a map on objects $F: \mathcal{B}_0 \to \mathcal{C}_0$;

(ii) a collection of functors $F_{A,B}: \mathcal{B}(A, B) \to \mathcal{C}(FA, FB)$, also denoted by F;

(iii) a collection of two-cells $\phi_A: 1_{FA} \to F1_A$;

(iv) a collection of two-cells $\phi_{g;f}: Fg \circ Ff \to F(g \circ f)$, natural in f and g.

These data are required to satisfy the following coherence conditions. For all arrows f, g and h in $\mathcal{B}$, the diagram

$$\begin{array}{ccc}
(Fh \circ Fg) \circ Ff & \xrightarrow{a_{Fh,Fg,Ff}} & Fh \circ (Fg \circ Ff) \\
\Big\downarrow{\scriptstyle \phi_{h,g} \circ 1_{Ff}} & & \Big\downarrow{\scriptstyle 1_{Fh} \circ \phi_{g,f}} \\
F(h \circ g) \circ Ff & & Fh \circ F(g \circ f) \\
\Big\downarrow{\scriptstyle \phi_{h \circ g, f}} & & \Big\downarrow{\scriptstyle \phi_{h, g \circ f}} \\
F((h \circ g) \circ f) & \xrightarrow[Fa_{h,g,f}]{} & F(h \circ (g \circ f))
\end{array}$$

commutes whenever the composites are defined and for $f\colon A \to B$ in $\mathcal{B}$ the following diagrams commute:

$$\begin{array}{ccc}
Ff \circ F1_A & \xrightarrow{\phi_{f,1_A}} & F(f \circ 1_A) \\
{\scriptstyle 1_{Ff} \circ \phi_A}\uparrow & & \downarrow{\scriptstyle Fr_f} \\
Ff \circ 1_{FA} & \xrightarrow[r_{Ff}]{} & Ff
\end{array}$$

and

$$\begin{array}{ccc}
F1_B \circ Ff & \xrightarrow{\phi_{1_B,f}} & F(1_B \circ f) \\
{\scriptstyle \phi_B \circ 1_{Ff}}\uparrow & & \downarrow{\scriptstyle Fl_f} \\
1_{FB} \circ Ff & \xrightarrow[l_{Ff}]{} & Ff.
\end{array}$$

If all the two-cells specified by ϕ are isomorphisms (respectively identities), Φ is called a *homomorphism* (respectively a *strict homomorphism*) *of bicategories.* When composition of morphisms is defined in the obvious way, it can be shown using the coherence conditions that bicategories and their morphisms form a category. Also, the composite of two homomorphisms is a homomorphism, so we have a subcategory of bicategories and homomorphisms, which is denoted by *Bicat*. We denote by *Bicat$_s$* the category of bicategories and *strict* homomorphisms. Clearly *Bicat$_s$* is a subcategory of *Bicat*.

It should be noted at this point that bicategories are essentially algebraic structures, in the sense that they can be thought of as models for a finite limit theory. The "global definition" of bicategories in Bénabou [4] amounts to a description of the appropriate theory. Taking this finite limit theory perspective, the morphisms which arise between bicategories are strict homomorphisms. Since bicategories are essentially algebraic, we can conclude[5] that the category *Bicat$_s$*

[5] The general result for categories of models of finite limit theories appear on p.147 of Barr and Wells [3].

has arbitrary limits and filtered colimits and furthermore, the inclusion functor of $Bicat_s$ in $Bicat$ commutes with limits and filtered colimits. This result also appears in Bénabou [4] (at least for limits). Of particular importance for our purposes is the special case of products. Given bicategories $\mathcal{B}$ and $\mathcal{C}$, there is a category $\mathcal{B} \times \mathcal{C}$ with objects $\mathcal{B}_0 \times \mathcal{C}_0$ and hom-categories

$$\mathcal{B} \times \mathcal{C}\left((B_1, C_1), (B_2, C_2)\right) = \mathcal{B}(B_1, B_2) \times \mathcal{C}(C_1, C_2)$$

which is the categorical product of $\mathcal{B}$ and $\mathcal{C}$ in $Bicat$. The projection homomorphisms are strict. Binary products of bicategories will be used in the definition of "monoidal bicategories" in the next chapter.

When the bicategories involved are in fact 2-categories, these notions correspond to standard 2-categorical ones: a morphism corresponds to a lax functor, a homomorphism corresponds to a pseudo-functor and a strict homomorphism corresponds to a 2-functor.

We usually refer to a homomorphism $\Phi = (F, \phi)$ simply as F. This practice is justified by the coherence results of section 1.2.6. A (strict) homomorphism $F: \mathcal{B} \to \mathcal{C}$ of bicategories has an obvious underlying (strict) homomorphism $F_2: \mathcal{B}_2 \to \mathcal{C}_2$ of multiplicative 2-graphs.

A *transformation* $\eta: F \to G$ between morphisms $F, G: \mathcal{B} \to \mathcal{C}$ of bicategories, consists of the data in the diagrams

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \eta_A}\downarrow & \Downarrow \eta_f & \downarrow{\scriptstyle \eta_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

as $f: A \to B$ varies over the arrows in $\mathcal{B}$. These data are subject to the condition that

$$\begin{array}{ccc} \eta_B \circ Ff & \xrightarrow{1_{\eta_B} \circ F\alpha} & \eta_B \circ Fg \\ {\scriptstyle \eta_f}\downarrow & & \downarrow{\scriptstyle \eta_g} \\ Gf \circ \eta_A & \xrightarrow[G\alpha \circ 1_{\eta_A}]{} & Gg \circ \eta_A \end{array}$$

commutes for each two-cell $\alpha: f \to g$ in $\mathcal{B}$. If η_f is an isomorphism, then we say η is a *strong transformation* or *pseudo-natural transformation*. A strong transformation which is an equivalence is often called a *pseudo-natural equivalence*. A *modification* $\rho: \eta \to \zeta$ of transformations $\eta, \zeta: F \to G$ consists of a collection of two-cells $\rho_A: \eta_A \to \zeta_A$, subject to the condition that

$$\begin{array}{ccc} \eta_B \circ Ff & \xrightarrow{\rho_B \circ 1_{Ff}} & \zeta_B \circ Ff \\ {\scriptstyle \eta_f}\downarrow & & \downarrow{\scriptstyle \zeta_f} \\ Gf \circ \eta_A & \xrightarrow[1_{Gf} \circ \rho_A]{} & Gf \circ \zeta_A \end{array}$$

commutes for each arrow $f: A \to B$. When composition of transformations and modifications is defined in the obvious way, and for each homomorphism F, a modification 1_F is defined by

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle 1_{FA}}\downarrow & \Downarrow r_{Ff}^{-1} l_{Ff} & \downarrow{\scriptstyle 1_{FB}} \\ FA & \xrightarrow[Gf]{} & FB, \end{array}$$

the coherence conditions ensure that the result is a bicategory $Bicat(\mathcal{B}, \mathcal{C})$ of homomorphisms, strong transformations and modifications[6]. If $\mathcal{C}$ is a 2-category

[6]Clearly there is also a bicategory of morphisms from $\mathcal{B}$ to $\mathcal{C}$, transformations and modifications, but this seems to be of less interest in applications.

then so is $Bicat(\mathcal{B}, \mathcal{C})$.

1.2.3 Examples of Bicategories

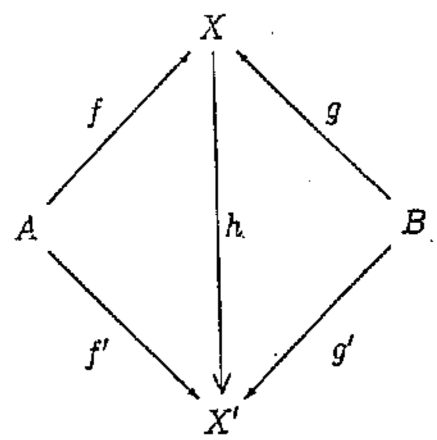
(i) *Monoidal Categories.* Consider a monoidal category C with tensor product $\otimes$. We define a bicategory $\mathcal{B}$ with one object $*$ by $\mathcal{B}(*,*) = \mathrm{C}$. We set $m_{*,*,*} = \otimes$ and a, l and r respectively are given by the associativity and left and right unit law isomorphisms for $\otimes$. Clearly the definition of a monoidal category ensures that $\mathcal{B}$ is indeed a bicategory. Conversely any such one-object bicategory yields a monoidal category, indeed for any object A of a bicategory $\mathcal{B}$, $m_{A,A,A}$ gives $\mathcal{B}(A, A)$ the structure of a monoidal category. This phenomenon should be compared to the connection between monoids and one-object categories and is referred to as *suspension* by Baez and Dolan [2]. In terms of this correspondence between one-object bicategories and monoidal categories, a morphism of bicategories corresponds to the usual notion of monoidal functor, while a homomorphism corresponds to what is often called a "strong monoidal functor"[7].

(ii) *Spans and Cospans.* We describe bicategories of cospans, which are related to *n-Cobord.* Given objects A and B of a category C, a *cospan* (f, X, g) from A to B is a diagram

$$A \xrightarrow{f} X \xleftarrow{g} B$$

[7]Joyal and Street [19] use the term "tensor functor" rather than "strong monoidal functor".

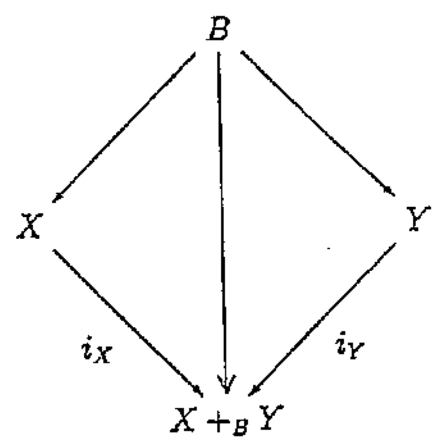
and a morphism of spans $(f, X, g) \to (f', X', g')$ is an arrow $h: X \to X'$ in C such that the diagram



commutes. This clearly defines a category Cospan $\mathrm{C}(A, B)$ of cospans from A to B. If C has pushouts and for each diagram

$$X \longleftarrow B \longrightarrow Y$$

we make a choice of pushout



then we can define a composition law

$$\text{Cospan } \mathrm{C}(B, C) \times \text{Cospan } \mathrm{C}(A, B) \to \text{Cospan } \mathrm{C}(A, C)$$

using pushouts. Explicitly, if (u, Y, v) is a cospan from B to C and (f, X, g) is a cospan from A to B, then their composite, denoted $(u, Y, v) \circ (f, X, g)$, is given by $(i_X f, X +_B Y, i_Y v)$. Further, if $h: (f, X, g) \to (f', X', g')$ and $k: (u, Y, v) \to (u', Y', v')$ are cospan morphisms, then

$$i_{X'} hg = i_{X'} g' = i_{Y'} u' = i_{X'} ku$$

and so there is a unique arrow $k \circ h: X +_B Y \to X' +_B Y'$ such that

(1.5) $$(k \circ h)i_X = i_{X'}h$$

(1.6) $$(k \circ h)i_Y = i_{Y'}k$$

since $X +_B Y$ is a pushout. It then follows that

(1.7) $$(k \circ h)i_X f = i_{X'} f'$$

(1.8) $$(k \circ h)i_Y v = i_{Y'} v'.$$

Thus $k \circ h: (i_X, X +_B Y, i_Y) \to (i_{X'}, X' +_B Y', i_{Y'})$ is a morphism of cospans. If we call $(1_A, A, 1_A)$ the identity span on A, then isomorphisms arising from the properties of pushouts ensure that the composition defined here is associative and unitary up to coherent isomorphisms. Thus Cospan C becomes a bicategory, called the *bicategory of cospans in* C. The *bicategory of spans in* C is easily defined as the dual notion, using pullbacks instead of pushouts.

(iii) Reversing the arrows of $\mathcal{B}$ yields a bicategory denoted by $\mathcal{B}^{op}$, reversing the two-cells yields a bicategory $\mathcal{B}^{co}$, and reversing both one and two-cells yields a bicategory $\mathcal{B}^{coop}$.

(iv) A 2-category is an example of a bicategory in which the isomorphisms a, l, r are all identities. Note that if $\mathcal{B}$ is a 2-category, then the composition in $\mathcal{B}$ gives $\mathcal{B}_1$ the structure of a category.

(v) In Chapter we defined enriched bimodules. Recall that given bimodules $M: \mathcal{A} \to\!\!\!\!\!\!+\; \mathcal{B}$ and $N: \mathcal{B} \to\!\!\!\!\!\!+\; \mathcal{C}$ we defined their composite $N \circ M$ by declaring that $N \circ M(A, C)$ be a coequaliser in $\mathcal{V}$ of a diagram

$$\sum_{B_1, B_2} (N(B_1, C) \otimes \mathcal{B}(B_2, B_1)) \otimes M(A, B_2) \rightrightarrows \sum_B N(B, C) \otimes M(A, B).$$

If $R: \mathcal{C} \relbar\joinrel\mapstochar\rightarrow \mathcal{D}$, then the associativity isomorphism for the tensor product in $\mathcal{V}$ yields an isomorphism

$$\Sigma_{B,C}\big(R(C,D)\otimes N(B,C)\big)\otimes M(A,B) \longrightarrow \Sigma_{B,C}\, R(C,D)\otimes\big(N(B,C)\otimes M(A,B)\big)$$

and routine diagram chasing will verify that this arrow equalises the parallel arrows in the diagram defining $(R \circ N) \circ M(A, D)$ and we therefore have a module isomorphism

$$a_{R,N,M}: (R \circ N) \circ M(A, D) \longrightarrow R \circ (N \circ M)(A, D).$$

The isomorphisms $a_{R,N,M}$ are natural in R, N and M and satisfy equation (1.1), the pentagon condition. In a similar fashion, the left and right unit isomorphisms

$$I \otimes M(A, B) \rightarrow M(A, B), \qquad M(A, B) \otimes I \rightarrow M(A, B)$$

yield module isomorphisms

$$l_M: 1_B \circ M \rightarrow M, \qquad r_M: M \circ 1_A \rightarrow M$$

which are natural in M and satisfy equation (1.2). Thus, $\mathcal{V}$-*Mod* is a bicategory, as promised in Chapter .

1.2.4 Bicategories of Cobordisms

In this section we make the structure of n-*Cobord* precise and confirm that it is indeed a bicategory. If we were only to consider *topological* manifolds, n-*Cobord* could be defined as a sub-bicategory of the bicategory of cospans in the category of topological manifolds. Since we are dealing with *smooth* manifolds, the situation is a little more complicated but n-*Cobord* is nevertheless close in spirit to a bicategory of cospans.

Given a smooth manifold M with boundary ∂M, a *collar on* M is an embedding

$$c: \partial M \times [0, \infty) \rightarrow M$$

such that $c(x,0) = x$. For our purposes, it is convenient to reformulate this definition slightly. Given a diffeomorphism $\phi: S \to \partial M$, a *ϕ-collar on M* is an embedding

$$c: S \times [0,\infty) \to M$$

or

$$c: S \times (-\infty, 0] \to M$$

such that $c(x,0) = \phi(x)$. A standard result says that every smooth manifold with boundary has a collar[8]. A smooth manifold together with a collar is referred to as a *collared manifold.*

If M and N are collared manifolds with diffeomorphic boundaries, one can "glue" them together along their common boundary as follows. Consider diffeomorphisms $\phi_M: S \to \partial M$ and $\phi_N: S \to \partial N$ and collars

$$\begin{aligned} c_M: S \times (-\infty, 0] &\to M \\ c_N: S \times [0,\infty) &\to N \end{aligned}$$

where $c_M(x,0) = \phi_M(x)$ and $c_N(x,0) = \phi_N$. Thinking of M, N and S simply as topological manifolds, there is a topological manifold $W = M +_S N$ such that

$$\begin{array}{ccc} S & \xrightarrow{\phi_N} & N \\ {\scriptstyle \phi_M}\downarrow & \searrow^{\phi} & \downarrow{\scriptstyle i_N} \\ M & \xrightarrow[i_M]{} & W \end{array}$$

is a pushout diagram in the category of topological manifolds. We therefore have maps

$$\begin{aligned} d_M = i_M \cdot c_M: S \times (-\infty, 0] &\to W \\ d_N = i_M \cdot c_N: S \times [0,\infty) &\to W \end{aligned}$$

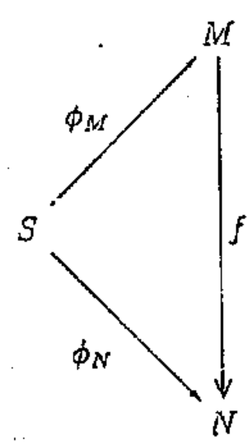
[8]See Hirsch [17] for a proof of this result.

such that $d_M(x,0) = i_M \cdot \phi_M(x) = i_N \cdot \phi_N(x) = d_N(x,0)$, and hence a map

$$d: S \times (-\infty, \infty) \to W$$

such that $d\,|_{S\times(-\infty,0]} = d_M$ and $d\,|_{S\times[0,\infty)} = d_N$. We now define the result of "gluing" M and N along S to be the smooth manifold obtained by giving W the unique differential structure which makes i_M, i_N and d diffeomorphisms. Different choices of collars for M and N may or may not result in the same differential structure on W. It is possible to define the "germ of a collar" and then two different collars for M will yield the same differential structure for W if and only if they have the same germ. This discussion of gluing carries through to oriented manifolds if ϕ_M is an orientation-reversing diffeomorphism and all the other maps are orientation preserving.

If $\phi_M: S \to \partial M$ and $\phi_N: S \to \partial N$ are diffeomorphisms, and c_M and c_N are ϕ_M and ϕ_N-collars respectively, then a smooth map $f: M \to N$ is said to *respect the collars c_M and c_N* if the diagram



commutes and the map $1 +_S f$ is a diffeomorphism. Here 1 denotes the identity map on the manifold $S \times (-\infty, 0]$, equipped with the obvious collar, and $1 +_S f$ is the unique map in the category of topological manifolds such that the following diagram commutes:

$$
\begin{array}{ccccc}
 & & S \times (-\infty, 0] +_S M & \xleftarrow{\;i_M\;} & M \\
 & \nearrow^{i_{S\times(-\infty,0]}} & & & \\
S \times (-\infty, 0] & & \Big\downarrow 1 +_S f & & \Big\downarrow f \\
 & \searrow_{i_{S\times(-\infty,0]}} & & & \\
 & & S \times (-\infty, 0] +_S N & \xleftarrow[\;i_N\;]{} & N.
\end{array}
$$

We can now define the bicategory *n-Cobord*. The objects of *n-Cobord* are compact $(n-1)$-dimensional smooth oriented manifolds. An object of the category $n-Cobord(S_1, S_2)$ is a compact n-dimensional smooth oriented manifold M equipped with diffeomorphism $\phi_M \colon \overline{S}_1 + S_2 \to \partial M$ and a ϕ_M-collaring on M, which we will denote by c_M. Here $+$ denotes disjoint union and $\overline{S}_1$ is the manifold obtained from S_1 by reversing the orientation. This manifold M is called a *cobordism from* S_1 *to* S_2. An arrow $M \to N$ in $n-Cobord(S_1, S_2)$ is a diffeomorphism $M \to N$ which respects the collars c_M and c_N. If M is an object of $n-Cobord(S_1, S_2)$ and N an object of $n-Cobord(S_2, S_3)$ then we can obviously extend our notion of gluing to allow M and N to be glued along S_2, and as a simple consequence of the definition of respecting collars, this gluing operation yields a functor

$$n-Cobord(S_2, S_3) \times n-Cobord(S_1, S_2) \to n-Cobord(S_1, S_3).$$

Given an object S in $n-Cobord$, we define 1_S to be the manifold $S \times [0,1]$, equipped with the obvious collar. It is straightforward to identify the associativity and identity isomorphisms and verify that these data do constitute a bicategory $n-Cobord$.

1.2.5 A Remark on Notation

A shorthand notation commonly used in category theory is to refer to the identity arrow of an object A simply as A rather than 1_A. If a functor F is introduced then $F(1_A) = 1_{FA}$ and so this arrow may safely be called FA. However, when moving to higher dimensions, the situation changes. If A is now an object of a bicategory and F is a bicategory homomorphism then $F(1_A)$ is only isomorphic to 1_{FA} but were the shorthand to be employed, both arrows would be denoted by FA. Exactly this ambiguity appears in Gordon, Power and Street [15] in which $\otimes$ is a homomorphism and $u \otimes v$ is used to denote both $1_u \otimes 1_v$ and $1_{u\otimes v}$. Although the coherence theorems of the next section do away with the need to give such isomorphisms explicit names, it nevertheless seems dangerous to use a notation which fails to distinguish distinct arrows. Using the language of multiplicative graphs, given a cell A of a multiplicative r-graph, it is safe to refer its identity cell as A *only* if A has codimension one (in other words, A must be an $(r-1)$-cell). Other common practices, such as referring to the identity arrow 1_A simply as 1, which has already been done in this dissertation, do not create difficulties in higher dimensions. As a rule of thumb, a labelling convention is "safe" if the label attached to a cell together with the domain and codomain data for that cell are sufficient to identify the cell uniquely. In later sections, we will be using shorthand devices which are safe in this sense.

1.2.6 Coherence for Bicategories

The coherence theorems given in this section assert that certain diagrams involving the constraint isomorphisms of bicategories and homomorphisms of bicategories will always commute. These theorems avoid the necessity of explicitly naming these constraints (which is particularly useful in later sections on monoidal bicategories and enriched bicategories) and also ensure that the pasting diagrams commonly used in working with 2-categories can also be used for bicategories. Modifying the approach of Joyal and Street [19], as suggested in Gordon, Power and Street [15], much of the content of the coherence theorems is in fact concentrated in a bicategorical generalization of the Yoneda Lemma.

Proposition 1.2 (Yoneda Lemma) *Given a homomorphism $F: \mathcal{B} \to \mathrm{Cat}$ and an object A of $\mathcal{B}$, evaluation at the identity provides an equivalence of categories*

$$Bicat(\mathcal{B}, \mathrm{Cat})\big(\mathcal{B}(A, -), F\big) \longrightarrow FA.$$

From this lemma, one is able to prove the following standard coherence results:

Theorem 1.3 (Coherence for Bicategories) *In a bicategory, every 2-cell diagram made up of expanded instances of a, l, r and their inverses must commute.*

Theorem 1.4 (Coherence for Homomorphisms) *If $F: \mathcal{B} \to \mathcal{C}$ is a homomorphism of bicategories, then every 2-cell diagram in $\mathcal{C}$ made up of expanded instances of a, l, r and their inverses, Fa, Fl, Fr[9] and their inverses and the constraints ϕ and their inverses must commute.*

For more on "expanded instances", see MacLane [33]. In outlining the proof of these results, Gordon, Power and Street [15] appeal to the notion of a "category enriched graph", the details of which would take us too far afield from the domain of this dissertation. We will continue to make repeated (usually implicit) appeals to these coherence theorems.

[9]Notice that we are using the same symbols to denote the constraints of both $\mathcal{B}$ and $\mathcal{C}$. This should not cause any confusion.

Chapter 2

Monoidal Bicategories and Enriched Bicategories

In Chapter we introduced monoidal categories, which are categories equipped with a tensor product. Many bicategories are also very naturally equipped with a tensor product. For example, the tensor product of abelian groups provides a tensor product for the bicategory of ordinary bimodules. While the notion of monoidal bicategory has been well-known in principle for a long time in category theory circles, no explicit definition had been published until very recently. A definition of monoidal bicategory does appear in Carboni and Walters [8], but only for the special case of locally posetal bicategories[1]. Gordon, Power and Street [15] define "tricategories" and observe that a monoidal bicategory can be defined to be a one-object tricategory. Kapranov and Voevodsky [22] also give a definition of monoidal bicategories, however their treatment of the subject is flawed. We discuss some of their errors in Appendix A. In this chapter we give our own detailed definition of a monoidal bicategory.

In Chapter we also introduced enriched categories, which have hom-*objects* rather than hom-*sets*. Once again, there is a corresponding notion for bicategories. Instead of hom-*categories*, an enriched bicategory should have hom-*objects*. For example, we have already seen how to construct enriched categories

[1] A locally posetal bicategory is a bicategory in which all the hom-categories are posetal.

${}_B Mod_A$ and these can be considered to be hom-objects for $\mathcal{V}$-*Mod*. We can therefore think of $\mathcal{V}$-*Mod* as being enriched over $\mathcal{V}-\mathsf{Cat}$. If we are to generalise bicategories in this way, we also need to have a composition law

$$\mathcal{B}(B,C) \times \mathcal{B}(A,B) \to \mathcal{B}(A,C)$$

and associativity and identity law isomorphisms, and so the base for enrichment must be two-dimensional. Of course, it must also be monoidal. The natural choice for a base is therefore a monoidal bicategory, and in this chapter we will give the definition of a bicategory enriched over a monoidal bicategory. The notion of enriched bicategory is new, and should be carefully distinguished from that of a "category enriched in a bicategory", which was introduced by Betti, Carboni, Street and Walters [5]. Throughout this section $\mathcal{W}$ will be used to denote a given monoidal bicategory.

2.1 Monoidal Bicategories

A *monoidal bicategory* is a bicategory $\mathcal{W}$ equipped with a *tensor product* which is a homomorphism of bicategories, is coherently associative and has an identity object I. Explicitly, a monoidal bicategory $\mathcal{W}$ consists of the following data:

(1) an underlying bicategory, also denoted by $\mathcal{W}$,

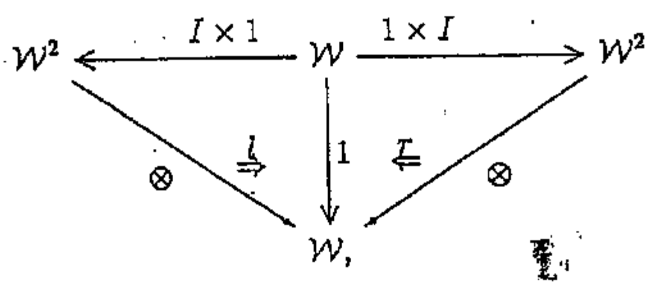
(2) a homomorphism $\otimes : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$,

(3) a homomorphism $I: \mathbf{1} \to \mathcal{W}$, , where $\mathbf{1}$ denotes a bicategory with one object, one arrow and one 2-cell, so that I amounts to an object of $\mathcal{W}$, which is also denoted by I and is known as the "unit", and an arrow $I \to I$ which is isomorphic to 1_I and can in fact be assumed to be 1_I without loss of generality,

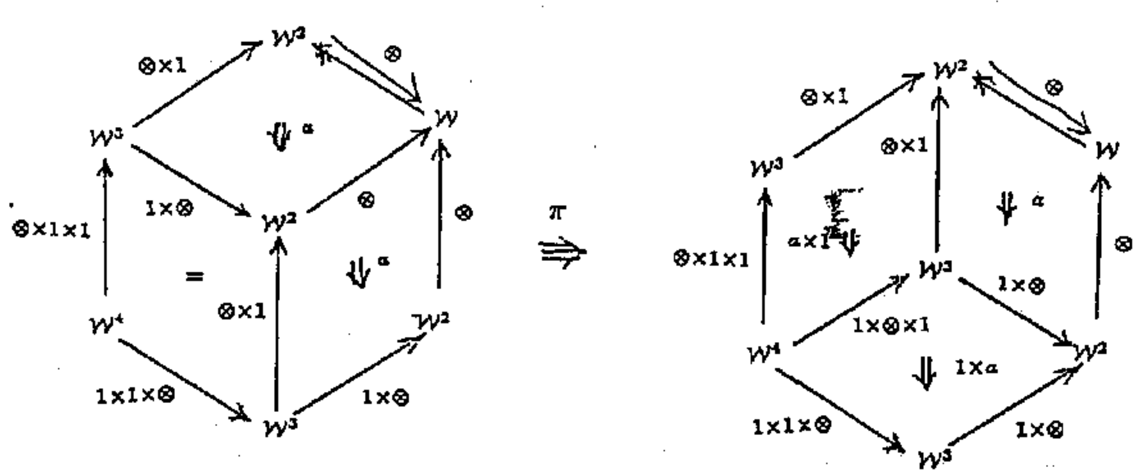
(4) a pseudo-natural equivalence a:

$$\begin{array}{ccc} \mathcal{W}^3 & \xrightarrow{\otimes \times 1} & \mathcal{W}^2 \\ {\scriptstyle 1\times\otimes}\downarrow & \Downarrow a & \downarrow{\scriptstyle \otimes} \\ \mathcal{W}^2 & \xrightarrow[\otimes]{} & \mathcal{W}, \end{array}$$

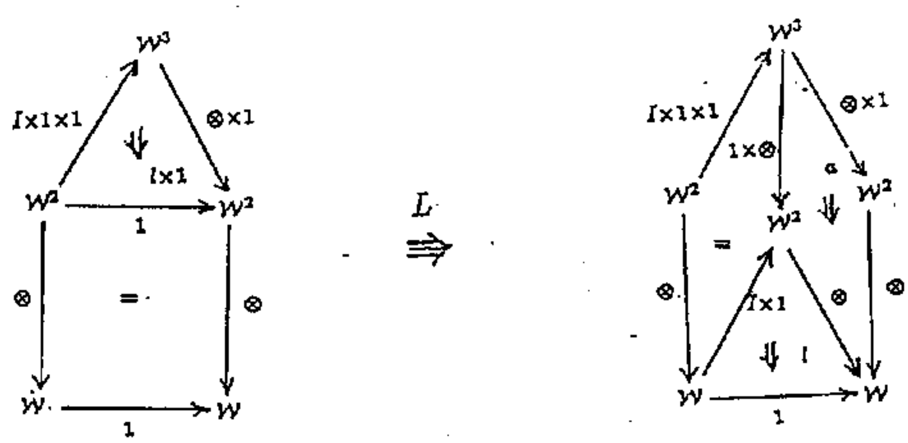
(5) pseudo-natural equivalences l and r:

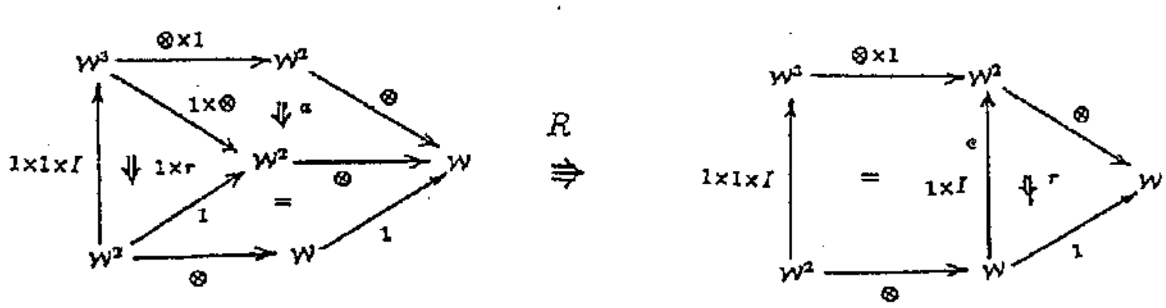


(6) an invertible modification π:



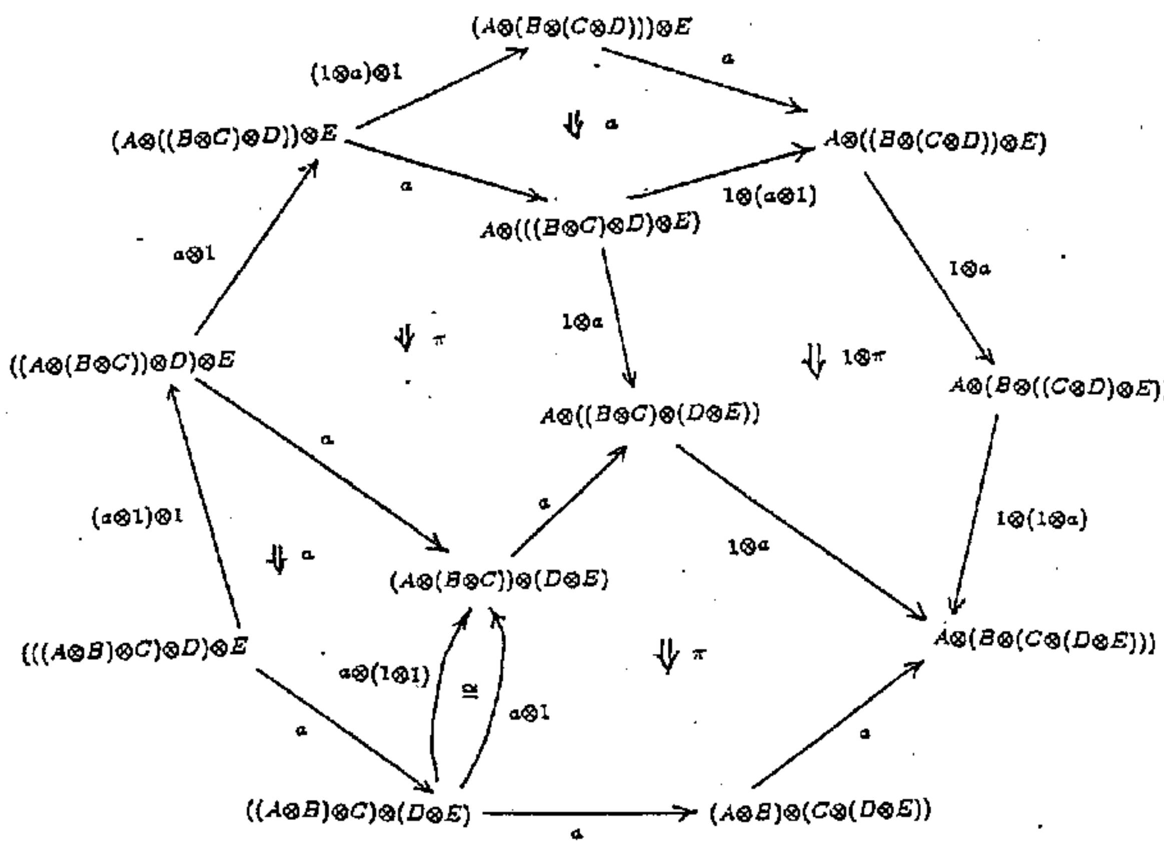
(7) and invertible modifications μ, L and R:





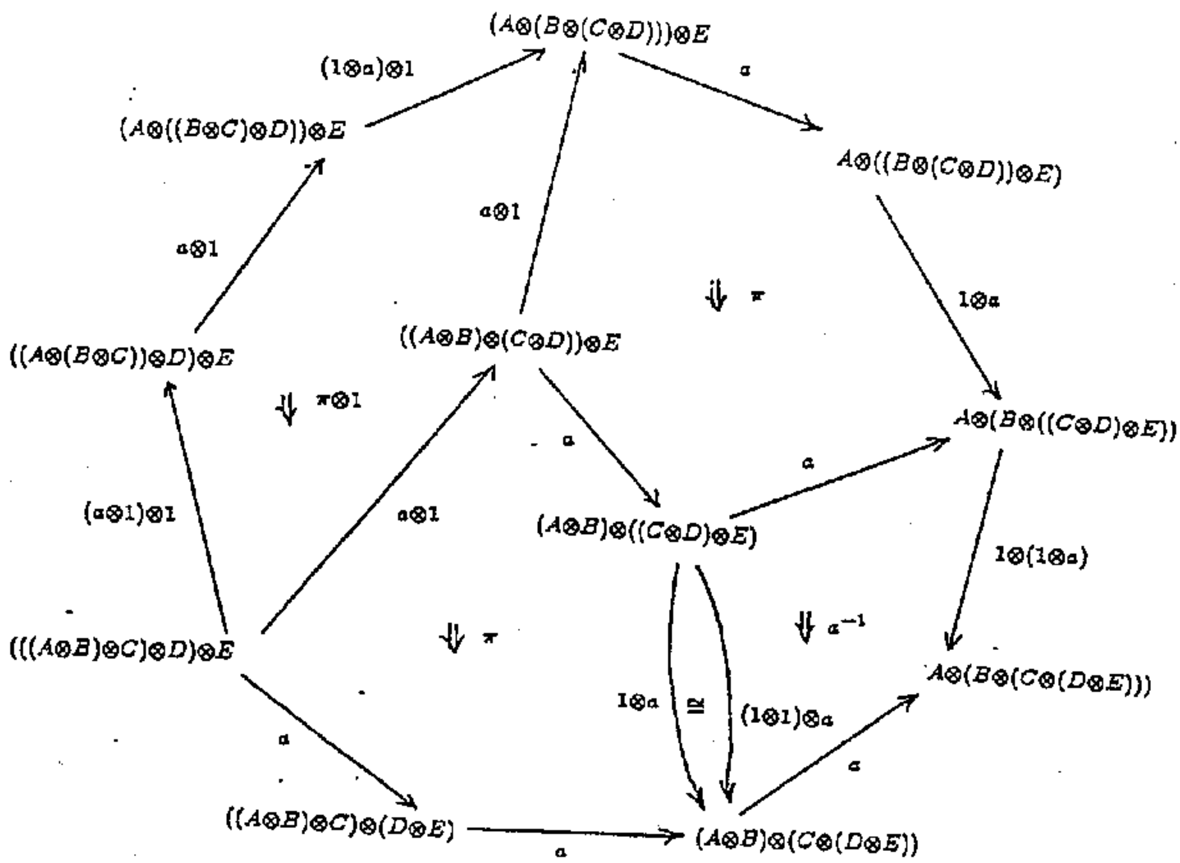
Before presenting the axioms these data satisfy, we should observe that the modification π could be considered to have components $((1 \otimes a) \circ a) \circ (a \otimes 1) \to a \circ a$ or $(1 \otimes a) \circ (a \circ (a \otimes 1)) \to a \circ a$. The coherence theorem for bicategories ensures that it does not matter which choice we make. We are also appealing to the coherence theorem when we present the axioms as pasting diagrams, which are only uniquely defined up to a choice of bracketing of the 1-cell composites. We now list the axioms, using the terminology of Gordon, Power and Street [15] [2].

(1) (non-abelian 4-cocycle condition)

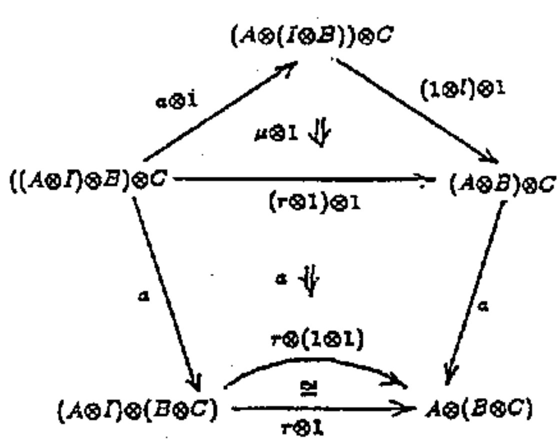


[2]See Joyal and Street [19] p.46 ff for an idea of how this terminology arises from important parallels with cohomology. See also Street [45].

is equal to



(2) (left normalization)



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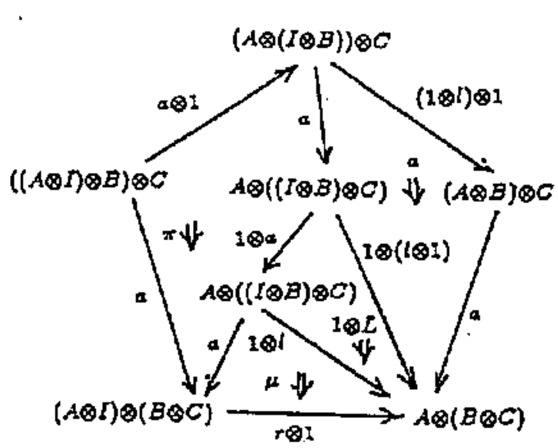
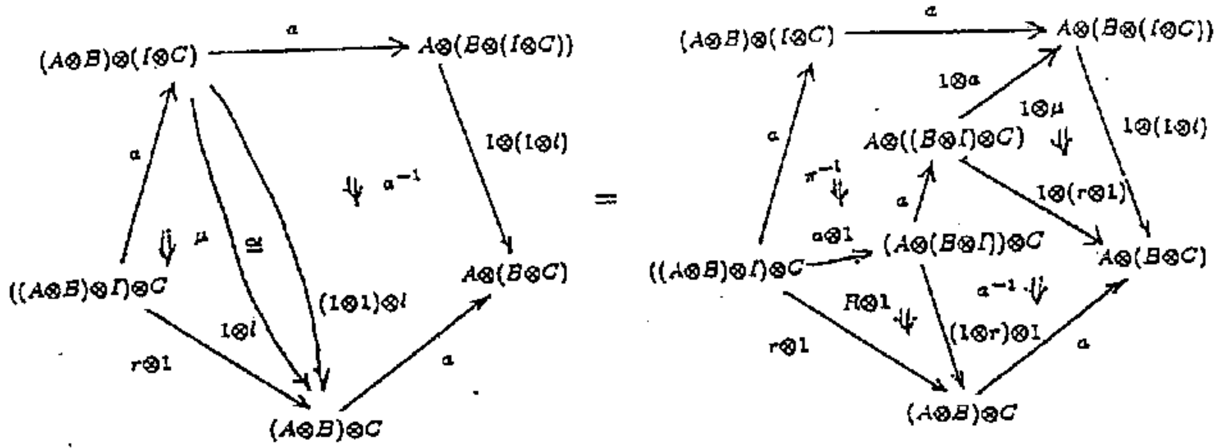


(3) (right normalization)



There is a coherence theorem for monoidal bicategories, but it is beyond the scope of this dissertation. We refer instead to Gordon, Power and Street [15] where a coherence theorem for tricategories is proved, which includes monoidal bicategories as a special case. We also refer to Gordon, Power and Street for a definition of homomorphism of monoidal bicategories as a special case of the notion of tricategorical homomorphism.

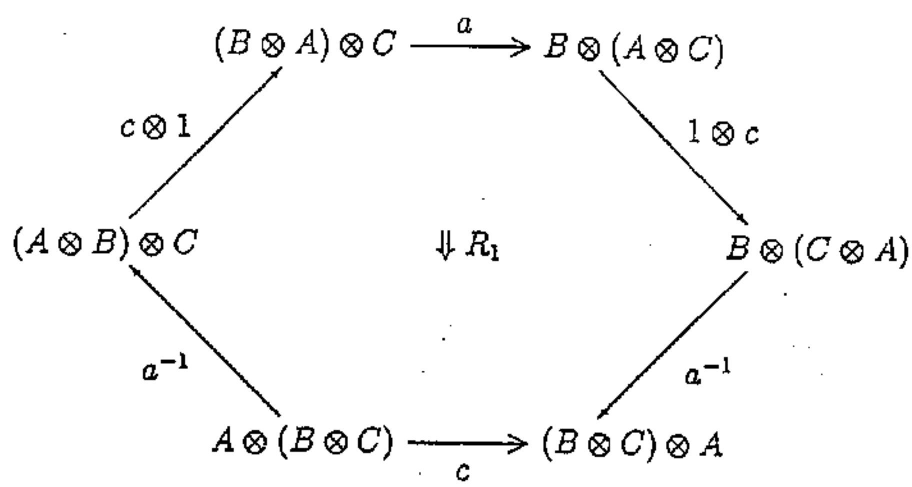
Note that Gordon, Power and Street point out that L and R are uniquely determined by the remaining data and axioms.

A straightforward example of a monoidal bicategory is *n-Cobord* with the tensor product given by disjoint union of manifolds. The bicategory $\mathcal{V}$*-Mod* is also a monoidal bicategory. Just as we defined the tensor product of enriched categories using the tensor product in $\mathcal{V}$, we can also define the tensor product of enriched bimodules. Given bimodules $M: A \twoheadrightarrow B$ and $N: A' \twoheadrightarrow B'$, we define $M \otimes N: A \otimes A' \twoheadrightarrow B \otimes B'$ by setting $M \otimes N\big((A, A'), (B, B')\big) = M(A, B) \otimes M(A', B')$.

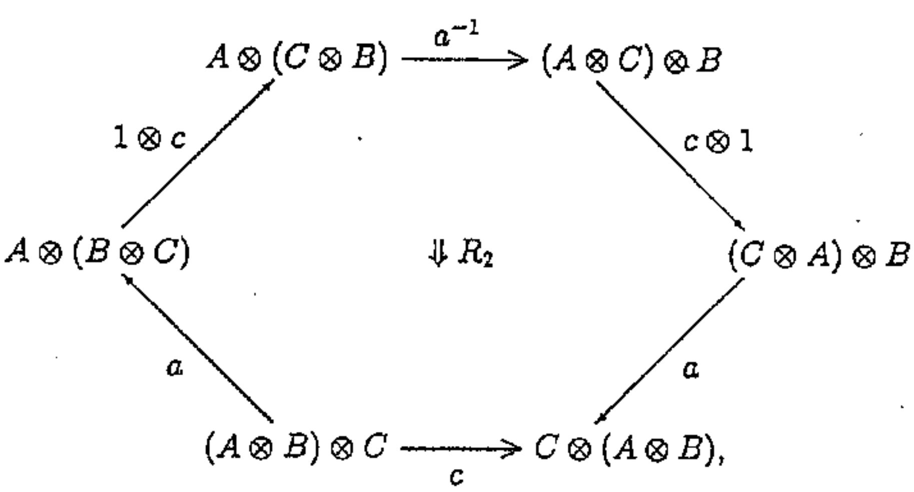
We can continue the analogy with monoidal categories and make the following definition. A *braiding* for a monoidal category $\mathcal{W}$ consists of *pseudo-natural braid equivalences*

$$c_{A,B}: A \otimes B \to B \otimes A$$

and invertible modifications R_1 and R_2 with components



and



where a^{-1} denotes the inverse equivalence of a. These data are subject to a coherence condition which we omit here. A special case of this condition appears in Kapranov and Voevodsky [22] for when $\mathcal{W}$ a 2-category.

A *braided monoidal bicategory* is a monoidal bicategory equipped with a braiding. As yet, no-one has proved a coherence theorem for braided monoidal bicategories. A *symmetry* for a monoidal bicategory is a braiding c, such that $c_{B,A}$ is the equivalence inverse of $c_{A,B}$ and a *symmetric monoidal bicategory* is a monoidal bicategory equipped with a braiding. Note that *n-Cobord* and $\mathcal{V}$*-Mod* are both symmetric monoidal bicategories.

2.2 Enriched Bicategories

Throughout this section, $\mathcal{W}$ is used to denote a fixed monoidal bicategory and we introduce "$\mathcal{W}$-bicategories" or "bicategories enriched over the base $\mathcal{W}$". A $\mathcal{W}$-category $\mathcal{A}$ consists of the following data:

(1) a collection $\mathcal{A}_0$ of *objects*,

(2) a *hom-object* $\mathcal{A}(A, B)$ for each pair of objects of $\mathcal{A}$ (in the diagrams below this is abbreviated to AB),

(3) a *composition law*

$$m = m_{A,B,C} \colon \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$$

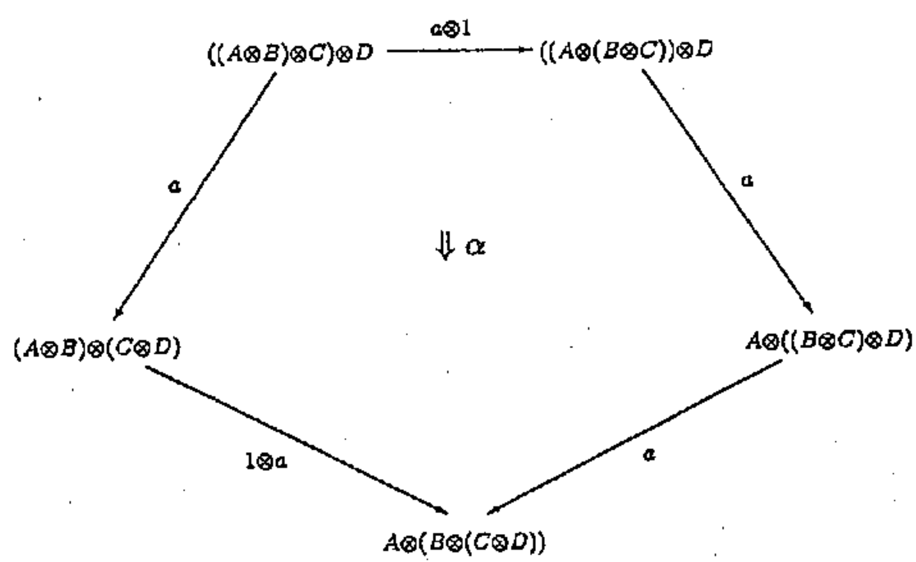
in $\mathcal{W}$ for each triple of objects of $\mathcal{A}$,

(4) an *identity element*

$$1_A \colon I \longrightarrow \mathcal{A}(A, A)$$

for each object of $\mathcal{A}$,

(5) *associativity isomorphisms* $\alpha = \alpha_{A,B,C,D}$



for each quadruple of objects of $\mathcal{A}$

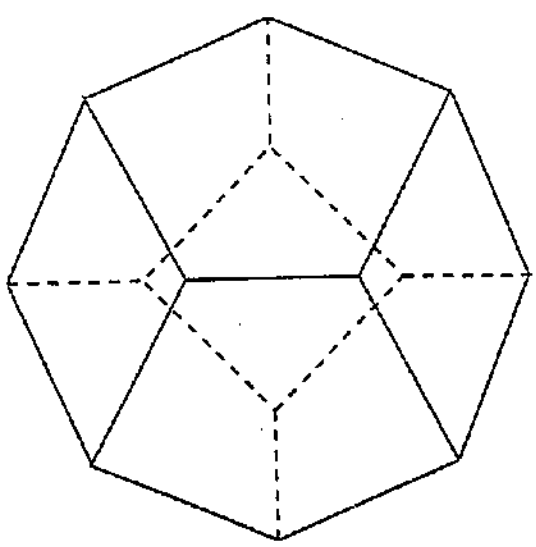
(6) and *identity isomorphisms* $\lambda = \lambda_{A,B}$ and $\rho = \rho_{A,B}$

$$\begin{array}{ccc} & AB \otimes AA & \\ {}^{1 \otimes 1_A}\nearrow & \Downarrow \rho & \searrow^{m} \\ AB \otimes I & \xrightarrow[r]{} & AB \end{array}$$

$$\begin{array}{ccc} & BB \otimes AB & \\ {}^{1_B \otimes 1}\nearrow & \Downarrow \lambda & \searrow^{m} \\ I \otimes AB & \xrightarrow[r]{} & AB \end{array}$$

for each pair of objects of $\mathcal{A}$.

These data are subject to two compatibility conditions, we comment on the geometry of these conditions. In the theory of monoidal categories and enriched categories, "pentagon conditions" abound. Moving up a dimension to monoidal bicategories and enriched bicategories, the key geometric figure is the *Stasheff polytope*:



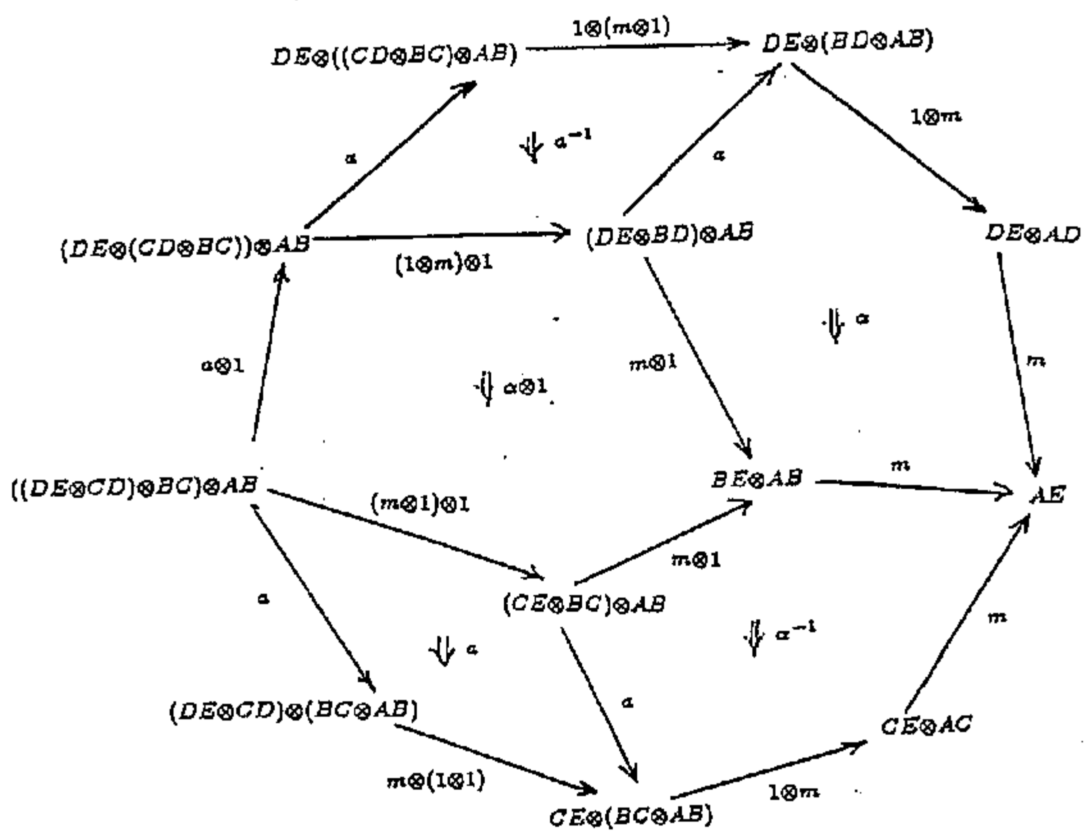
The coherence conditions for enriched bicategories, and indeed the earlier conditions for monoidal bicategories should be thought of as (possibly degenerate)

No, they are more complex than the Stasheff polytope!

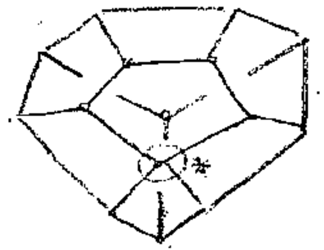
Stasheff polytopes which have been prised apart. We now give the coherence conditions for enriched bicategories.

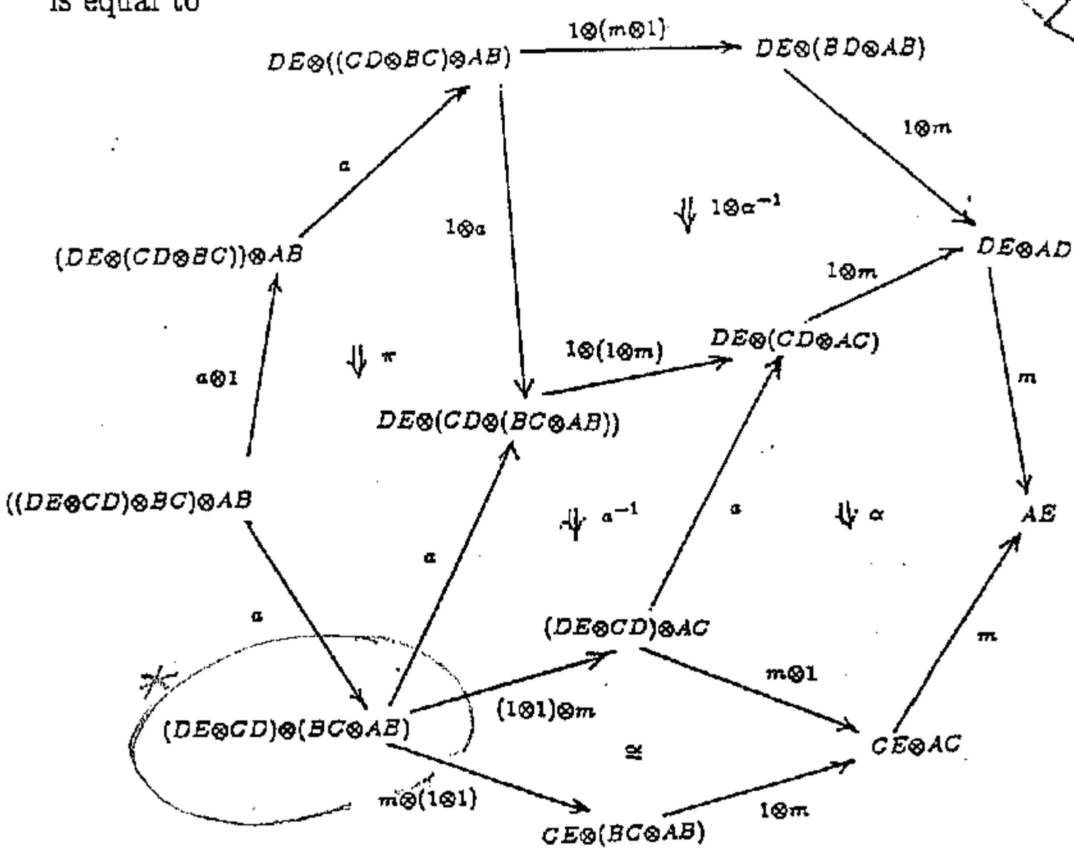
(1) the Stasheff condition

(actually a degenerate multiplihedron)



is equal to

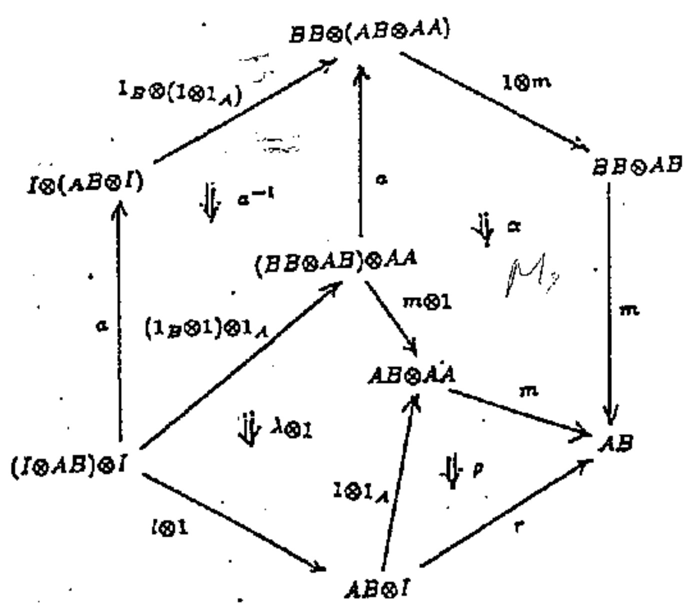


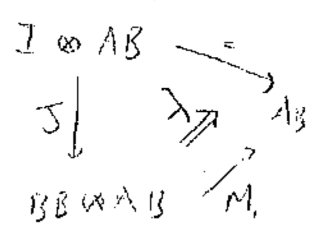


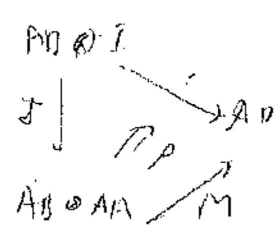
Composihedron!!

(10 facets, not 9)

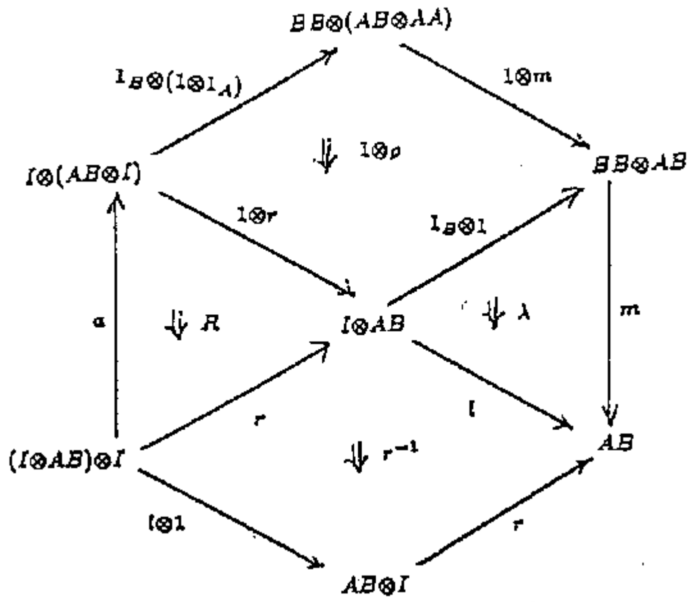
(2) the degenerate Stasheff condition







is equal to



It is possible to go on and define enriched homomorphisms, yielding a category $\mathcal{W}$-*Bicat*, and also enriched transformations and modifications. Enriched homomorphisms $\mathcal{A} \to \mathcal{B}$ are then the objects of a bicategory, which we denote by $\mathcal{W}$-*Bicat*$(\mathcal{A}, \mathcal{B})$. If we define an enriched bicategory $\mathcal{I}$ with a single object 0, and set $\mathcal{I}(0,0) = I$, then we can associate to any enriched category $\mathcal{A}$ a bicategory $\mathcal{W}$-*Bicat*$(\mathcal{I}, \mathcal{A})$, which we call the *underlying bicategory of* $\mathcal{A}$.

When $\mathcal{W}$ is Cat, which is a monoidal bicategory (in fact a 2-category) with tensor product given by Cartesian product, we recover the usual notion of bicategory. A more substantial example of an enriched bicategory is $\mathcal{V}$-*Mod*, which is enriched over the monoidal 2-category $\mathcal{V}$-Cat. Another example is n-*Cobord*, which is enriched over the monoidal 2-category of categories of homotopies. Although we will not be developing the theory of enriched bicategories any further, we believe many of the structures of higher dimensional algebra can be expressed in terms of enriched bicategories. One such structure is a "2-vector space".

The notion 2-vector spaces arose first in the study of the Zamolodchikov tetrahedra equations and was subsequently adopted as an ideal codomain for complicated topological quantum field theories. Freed [13] only gives a "heuristic treatment" of 2-vector spaces, arguing by analogy to suggest some of the features they should have rather than giving an explicit definition. This approach is typical and only Kapranov and Voevodsky [22] attempt greater precision. In their initial definition, 2-vector spaces are "co-ordinatised", in the same sense that a "co-ordinatised" approach to vector spaces and linear maps is to work with spaces of column vectors and matrices. Kapranov and Voevodsky then seek a "co-ordinate-free" definition in terms of the notion of "ring category" developed by Kelly [26] and Laplaza [31] and modules over ring categories. In the same spirit as thinking of a ring as a one-object **Ab**-category, we conjecture that a ring-category can be thought of as a one-object bicategory enriched over the 2-category of symmetric monoidal categories, monoidal functors and natural transformations. Taking a cue from terminology now in vogue, we might call the latter 2-category 2-**Ab**, and then suggest that a 2-vector space be thought of as an enriched homomorphism from a ring category into 2-**Ab**.

Although the theory of enriched bicategories is a potentially powerful tool for handling such higher-dimensional structures as 2-vector spaces, such considerations are beyond the intended scope of this thesis. The study of topological quantum field theories in later chapters in firmly rooted in the simpler setting of vector spaces. It is crucial to develop a rigourous treatment of the simpler, more familiar theories before pursuing more ambitious ends.

Chapter 3

Compact Closed Bicategories

Many important monoidal categories have the property that each object has a "dual". The canonical example is, of course, finite dimensional vector spaces. Joyal and Street [19] call such monoidal categories "autonomous"[1]. Autonomous *symmetric* monoidal categories are usually called "compact closed categories". In this chapter we generalize, introducing "compact closed bicategories".

3.1 Compact Closed Categories

The description of compact closed categories given in this section is based on Kelly and Laplaza [29] and Joyal and Street [19].

3.1.1 The Definition of a Compact Closed Category

An *adjunction* $B \dashv A$ between objects A and B of a monoidal category C consists of a *unit* $\eta\colon I \to A \otimes B$ and a *counit* $\epsilon\colon B \otimes A \to I$ such that the following *adjunction triangles* commute:

[1] More precisely, they introduce *left* autonomous categories, in which every object has a left dual, and similarly *right* autonomous categories, while in an autonomous category every object has both a left and a right dual .

$$
\begin{array}{ccc}
 & (A \otimes B) \otimes A \cong A \otimes (B \otimes A) & \\
{\scriptstyle \eta \otimes 1_A} \nearrow & & \searrow {\scriptstyle 1_A \otimes \epsilon} \\
I \otimes A \cong A & \xrightarrow[1_A]{\hspace{6em}} & A \cong A \otimes I
\end{array}
$$

$$
\begin{array}{ccc}
 & B \otimes (A \otimes B) \cong (B \otimes A) \otimes B & \\
{\scriptstyle 1_B \otimes \eta} \nearrow & & \searrow {\scriptstyle \epsilon \otimes 1_B} \\
B \otimes I \cong B & \xrightarrow[1_B]{\hspace{6em}} & B \cong I \otimes B.
\end{array}
$$

We say that B is a *left adjoint* or *left dual* of A, and A is a *right adjoint* or *right dual* of B. Thinking of a monoidal category as a one-object bicategory, this is in fact a special case of the notion of adjunction for arrows in a bicategory.

An arrow $\epsilon: B \otimes A \to I$ is known as a *pairing*. A pairing induces a function

$$\mathbf{C}(X, B \otimes Y) \to \mathbf{C}(A \otimes X, Y)$$

which is natural in X and Y and takes $f: X \to B \otimes Y$ to the composite

$$A \otimes X \xrightarrow{1 \otimes f} A \otimes (B \otimes Y) \cong (A \otimes B) \otimes Y \xrightarrow{\epsilon \otimes 1} I \otimes Y \cong Y.$$

If this function is an isomorphism, we call the pairing *exact*. It is straightforward to prove

Proposition 3.1 *The following conditions on objects A and B of a monoidal category are equivalent:*

1. *there is an adjunction $B \dashv A$*
2. *there is an exact pairing $B \otimes A \to I$*
3. *there is an adjunction of functors $B \otimes - \dashv A \otimes -$*

4. *there is an adjunction of functors* $- \otimes A \dashv - \otimes B$.

Note that if we have a natural isomorphism $-\otimes B \cong -\otimes B'$, then the I-component of this transformation yields a canonical isomorphism $B \cong B'$. As any two right adjoints to $A \otimes -$ are canonically isomorphic, it follows that any two left duals of A are canonically isomorphic. The same clearly holds for right duals.

It is easy to verify that if F is a strong monoidal functor and $\epsilon: B \otimes A \to I$ is an exact pairing then the composite

$$FB \otimes FA \xrightarrow{\cong} F(B \otimes A) \xrightarrow{F\epsilon} FI \xrightarrow{\cong} I$$

is also an exact pairing. Thus strong monoidal functors preserve duals.

A *compact closed category* is a *symmetric* monoidal category in which every object A has a left dual (and hence, using the symmetry, a right dual). Note that left adjoints are only unique up to isomorphism, while for our purposes it is preferable that compact closed categories are models of a finite limit theory. To achieve this we henceforth insist that each object A has an *assigned* left adjoint A^* with unit η_A and counit ϵ_A. Since strong monoidal functors preserve duals, there is no need for an additional notion of "morphism of compact closed categories".

3.1.2 Examples of Compact Closed Categories

(i) As already mentioned, the archetypal example of a compact closed category is given by finite dimensional vector spaces over a field k. The specified dual of a vector space V is the space of linear functionals. Given a basis $(e_1, \ldots, e_n)$ for V, if we denote the corresponding dual basis for V^* by $(e_1^*, \ldots, e_n^*)$ then the unit (also known as insertion of coordinates) is given by:

$$k \to V \otimes V^* \tag{3.1}$$

$$1 \mapsto \sum_{i=1}^{n} e_i \otimes e_i^*. \tag{3.2}$$

Note that this map is independent of the choice of basis. The counit $V^* \otimes V \to k$ corresponds to the bilinear evaluation map $V^* \times V \to k$ which takes (λ, v) to $\lambda(v)$.

(ii) A direct generalization of the previous example is the category of finitely generated projective modules over a commutative ring R.

(iii) The category **Rel** of sets and relations is a symmetric monoidal category with the tensor product given by Cartesian product. The unit for this tensor product is the set 1 which has a single element, denoted by $\bullet$. The category **Rel** is also compact closed with each object its own dual. For a set X, the unit $\eta_X : 1 \twoheadrightarrow X \times X$ is the relation given by $(x, y)\, \eta_X \bullet$ if and only if $x = y$, and the counit $\epsilon_X : X \times X \twoheadrightarrow 1$ is given by $\bullet\, \epsilon_X\, (x, y)$ if and only if $x = y$. In other words, both η_X and ϵ_X are essentially the diagonal relation on X.

3.1.3 Contravariant and Covariant Dual Functors

In a compact closed category **C**, there is a bijection between arrows $f : A \to B$ and $f^* : B^* \to A^*$ which is determined by any one of the following four commutative diagrams:

(3.3)
$$\begin{array}{ccc} I & \xrightarrow{\eta_A} & A \otimes A^* \\ {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle f \otimes 1} \\ B \otimes B^* & \xrightarrow[1 \otimes f^*]{} & B \otimes A^* \end{array}$$

(3.4)
$$\begin{array}{ccc} A \otimes B^* & \xrightarrow{f \otimes 1} & B \otimes B^* \\ {\scriptstyle 1 \otimes f^*}\downarrow & & \downarrow{\scriptstyle \epsilon_B} \\ A \otimes A^* & \xrightarrow[\epsilon_A]{} & I \end{array}$$

$$
\begin{array}{ccc}
A \cong I \otimes A & \xrightarrow{\eta_B \otimes 1} & (B \otimes B^*) \otimes A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (1 \otimes f^*) \otimes 1} \\
B \cong B \otimes I & \xleftarrow[1 \otimes \epsilon_A]{} & B \otimes (A^* \otimes A) \cong (B \otimes A^*) \otimes A
\end{array}
$$

$$
\begin{array}{ccc}
B^* \cong B^* \otimes I & \xrightarrow{\eta_A \otimes 1} & B^* \otimes (A \otimes A^*) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle 1 \otimes (f \otimes 1)} \\
A^* \cong I \otimes A^* & \xleftarrow[1 \otimes \epsilon_B]{} & (B^* \otimes B) \otimes A^* \cong B^* \otimes (B \otimes A^*).
\end{array}
$$

Clearly any one of these diagrams ensures that the definition of f^* yields a functor $*: \mathbf{C}^{op} \to \mathbf{C}$. Note that (3.3) and (3.4) also ensure that defining $*$ in this way makes η and ϵ "natural transformations" in the generalized sense of Eilenberg and Kelly [12]. The functorial calculus developed in this paper is taken further in Kelly [25].

We have thus established that any compact closed category is equipped with a canonical contravariant endofunctor whose value at an object is its dual. Note that we will refer to any endofunctor whose value at an object is its dual as a *dual functor*. We now turn to *covariant* dual functors. Consider the category of finite dimensional *Hilbert* spaces. The inner product on V gives an exact pairing

$$\langle\ ,\ \rangle : \overline{V} \otimes V \to \mathbb{C},$$

where $\overline{V}$ has the same elements and and abelian group structure as V, but conjugate scalar multiplication. Thus $\overline{V}$ is a dual of V. The importance of this dual is that it is readily extended to a *covariant* endofunctor. Authors such as Baez and Dolan [2] do not speak of covariant dual endofunctors, but introduce a contravariant functor † which is the identity on objects. It is possible to translate this into our terminology by defining $\overline{f} = (f^*)^\dagger$, which yields the desired covariant dual functor.

3.1.4 Traces, Feedback and Inner Products

There are a number of category theoretic definitions of "trace" which generalise the classical trace of endomorphisms of finite dimensional vector spaces. The following definition is taken from Kelly and Laplaza [29]. A *trace* τ for a category C which takes values in a set X consists of a collection of functions $\tau_A \colon End(A) \to X$, indexed by objects of C, such that $\tau_B(fg) = \tau_A(gf)$ for any $f \colon A \to B$ and $g \colon B \to A$ in C. Every compact closed category is equipped with a canonical trace, as we will now demonstrate.

Given $f \colon A \to B$ and $g \colon B \to A$ in a compact closed category C, we define $\delta(f, g)$ as the composite

$$I \xrightarrow{\eta_A} A \otimes A^* \xrightarrow{f \otimes g^*} B \otimes B^* \cong B^* \otimes B \xrightarrow{\epsilon_B} I.$$

The key property of δ is given by the following

Proposition 3.2 *Given f, g and $\delta(f, g)$ as above, we have $\delta(f, g) = \delta(g, f)$.*

Proof. As a result of the generalized naturality of η (3.3), we obtain the equality $(f \otimes g^*)\,\eta_A = (fg \otimes 1_B^*)\,\eta_B$ and so $\delta(f, g) = \delta(fg, 1_B)$. Similarly, the naturality of ϵ (3.4) implies that $\delta(f, g) = \delta(gf, 1_A)$ and hence the required result. □

We now define functions $\tau_A \colon End(A) \to End(I)$ by $\tau_A(h) = \delta(h, 1_A)$ for $h \colon A \to A$. The above result ensures that this defines a trace on C as desired In fact we have that $\tau_A(gf) = \delta(g, f)$. It is straightforward to check that in the case of finite dimensional vector spaces this notion coincides with the usual trace operation.

A more sophisticated definition of trace appears in Joyal, Street and Verity [20]. They define a trace for a balanced[2] monoidal category C as a natural family of functions

$$\mathrm{Tr}^A_{B,C} \colon \mathrm{C}(B \otimes A, C \otimes A) \longrightarrow \mathrm{C}(B, C)$$

[2]The concept of a balanced monoidal category appears in Joyal and Street [18].

subject to a number of axioms which we do not list here. Street and Verity prove that for a compact closed category, a construction very similar to the one above gives a canonical trace. We will describe that canonical trace, which we refer to as "feedback" or "contraction".

Given an arrow $f: (A \otimes X) \otimes B \to (C \otimes X) \otimes D$, we define the *feedback* or *contraction of f along X* to be the arrow $c_X(f): A \otimes B \to C \otimes D$ given by the composite

$$\begin{aligned}
A\otimes B \cong (A\otimes B)\otimes I &\xrightarrow{1\otimes\eta_X} (A\otimes B)\otimes(X\otimes X^*) \cong ((A\otimes X)\otimes B)\otimes X^* \\
&\xrightarrow{f\otimes 1} ((C\otimes X)\otimes D)\otimes X^* \cong (C\otimes D)\otimes(X^*\otimes X) \\
&\xrightarrow{1\otimes\epsilon_X} (C\otimes D)\otimes I \cong C\otimes D.
\end{aligned}$$

Clearly the bracketing of such terms as $(A \otimes X) \otimes B$ is not significant in this definition, and so we will understand an obvious notion of feedback along X for arrows $A \otimes (B \otimes C) \to C \otimes (X \otimes D)$ and indeed arrows $A \otimes X \to C \otimes X$ and so on.

Feedback generalises both trace and composition for a compact closed category. For $f: A \to A$, it is clear that feedback along A yields the trace of f, that is

$$\tau_A(f) = c_A(f).$$

Now consider arrows $f: A \to B$ and $g: B \to C$ in a compact closed category. It follows from the triangular equations that the composite

$$A \cong I \otimes A \xrightarrow{\eta_B \otimes f} (B \otimes B^*) \otimes B \cong B \otimes (B^* \otimes B) \xrightarrow{g \otimes \epsilon_B} C \otimes I \cong C$$

is simply the composite $gf: A \to C$. We can therefore conclude that

$$gf = c_B(g \otimes f).$$

This operation of contraction will be very useful in our later study of two-dimensional topological quantum field theories.

If C is a compact closed category and $\overline{}: \mathrm{C} \to \mathrm{C}$ is a covariant dual functor, we can construct an operation closely related to feedback. Given f, $g: A \to B$, we define $\langle f, g\rangle$ as the composite

$$I \xrightarrow{\eta_A} A \otimes \overline{A} \xrightarrow{f \otimes \overline{g}} B \otimes \overline{B} \cong \overline{B} \otimes B \xrightarrow{\epsilon_B} I.$$

We call $\langle f, g\rangle$ the *inner-product* of f and g. An arrow $h: B \to C$ in C is said to be an *isometry* if

$$\langle hf, hg\rangle = \langle f, g\rangle$$

for all $f, g: A \to B$. In the case of finite-dimensional Hilbert spaces, if elements v and w of V are thought of as linear maps $\mathbb{C} \to V$ then $\langle v, w\rangle$ clearly coincides with the original inner-product on V, which motivates our terminology.

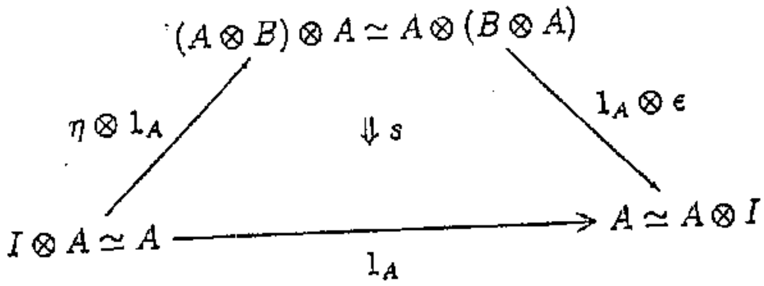
3.2 Compact Closed Bicategories

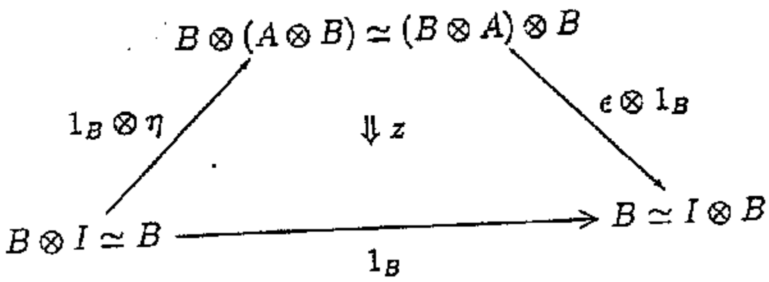
When giving examples of compact closed categories, Kelly and Laplaza [29] observe that "the 'category' of small categories and profunctors[3] fails to be a compact closed bicategory only because it fails to be an honest category with associative composition." Examples such as these serve to motivate a definition for compact closed bicategories.

[3]In our terminology this is simply $\mathcal{V}$-*Mod*.

3.2.1 The Definition of a Compact Closed Bicategory

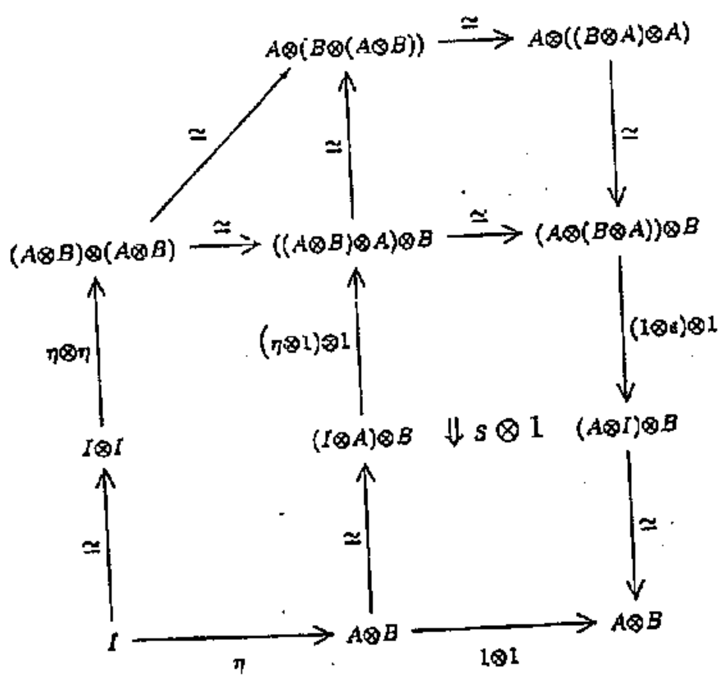
An adjunction $B \dashv A$ between objects A and B of a monoidal bicategory $\mathcal{W}$ consists of a *unit* $\eta: I \to A \otimes B$ and a *counit* $\epsilon: B \otimes A \to I$ and *triangular* 2-cells



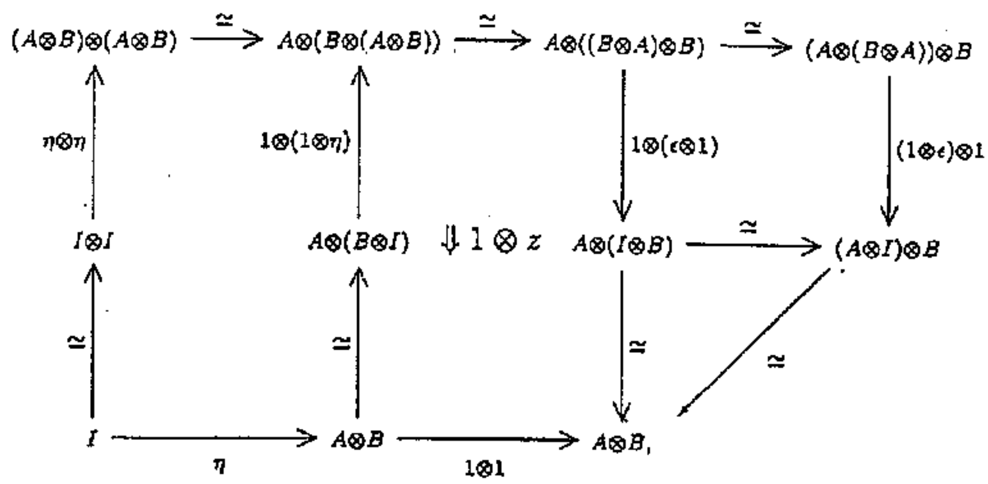


subject to the following two conditions :

(i) the 2-cell

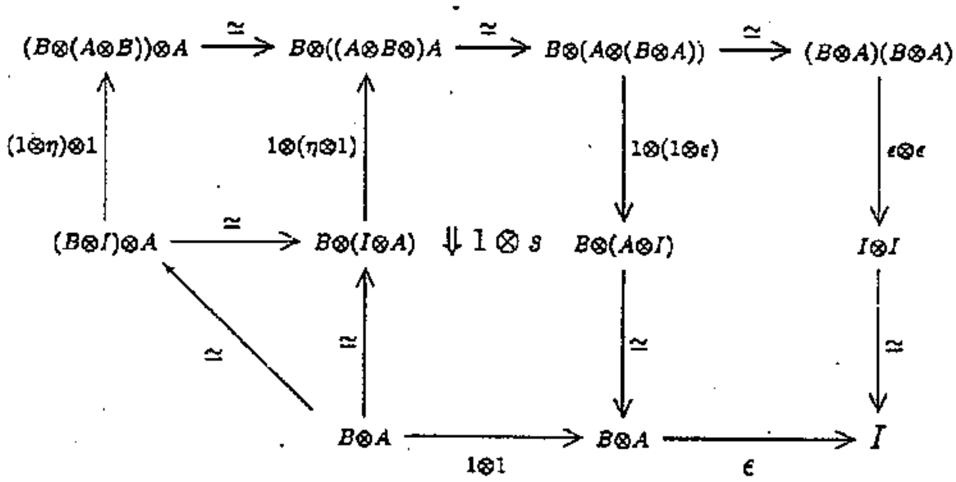


is equal to

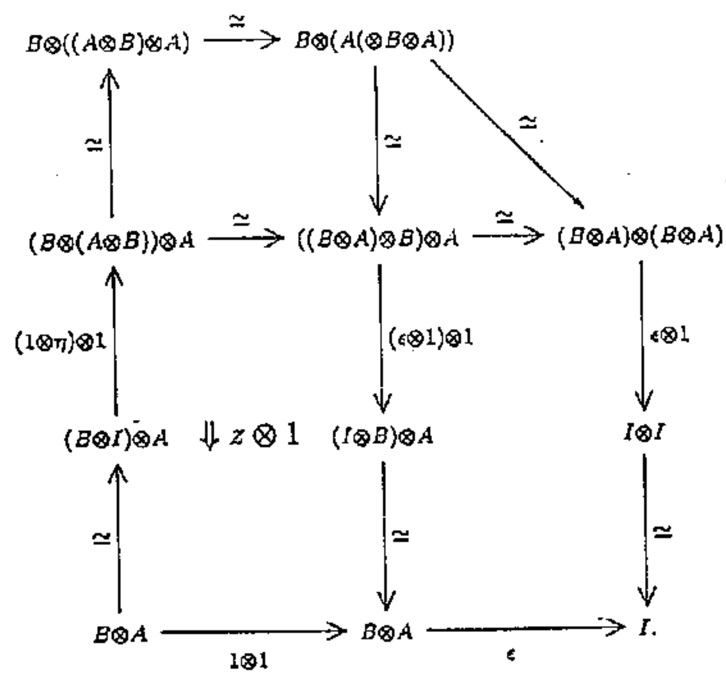


where all empty cells denote canonical isomorphisms,

(ii) the 2-cell



is equal to



As in the category case, we say that B is a *left adjoint* or *left dual* of A, and A is a *right adjoint* or *right dual* of B.

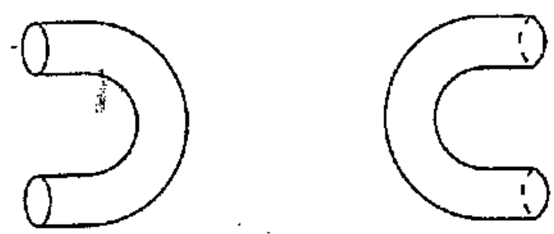
A *compact closed bicategory* is a symmetric monoidal bicategory in which every object has a left dual (and hence a right dual by symmetry). Once again, our two main examples are $\mathcal{V}$-*Mod* and n-*Cobord*.

In $\mathcal{V}$-*Mod*, the dual of a $\mathcal{V}$-category $\mathcal{A}$ is $\mathcal{A}^{op}$. Since the definitions of tensor product and opposite for enriched categories ensure that $(\mathcal{A}^{op})^{op} = \mathcal{A}$ and $(\mathcal{A} \otimes \mathcal{B})^{op} = \mathcal{A}^{op} \otimes \mathcal{B}^{op}$, we have a correspondence

$$\frac{\mathcal{A} \otimes \mathcal{B} \;\longrightarrow\; \mathcal{C}}{\mathcal{A} \;\longrightarrow\; \mathcal{B}^{op} \otimes \mathcal{C}}.$$

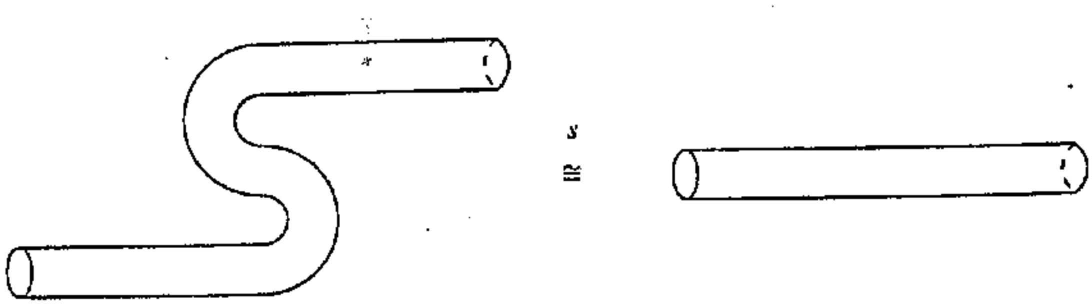
Recalling the definition of composition of bimodules, it is relatively straightforward to see that this bijection is pseudo-natural in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. The unit and counit modules $\eta_{\mathcal{A}}: \mathcal{A} \otimes \mathcal{A}^{op} \longrightarrow \mathcal{I}$ and $\epsilon_{\mathcal{A}}: \mathcal{I} \longrightarrow \mathcal{A}^{op} \otimes \mathcal{A}$, which exhibit $\mathcal{A}^{op}$ as the dual of $\mathcal{A}$, both correspond under this bijection to the identity module $1_{\mathcal{A}}: \mathcal{A} \longrightarrow \mathcal{A}$. Of course we are also using the fact that $\mathcal{I}$ is the unit for the tensor product in $\mathcal{V}$-*Mod*.

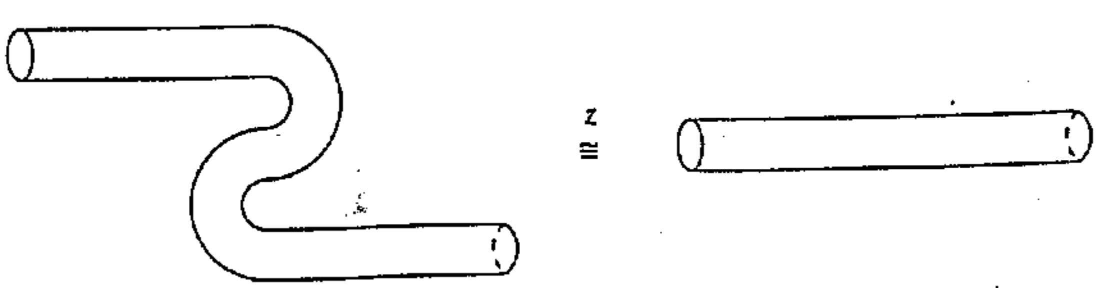
The dual of an object S of *n-Cobord* is $\overline{S}$, the same manifold but with orientation reversed. Since $\partial(S \times I) \cong S + \overline{S}$, the manifold $S \times I$ can be thought of as a cobordism $S \to S$ or $\to \overline{S} + S$ or $S + \overline{S} \to$. These last two interpretations yield the unit and counit which exhibit $\overline{S}$ as the dual of S. Intuitively, we think of a cobordism which "bends back on itself". The following pictures illustrate the counit and unit for the manifold S_1 in 2-*Cobord* (the pictures are progressive from left to right).



ϵ η

We also give pictures to illustrate the diffeomorphisms s and z. In fact, these pictures motivated the names "s" and "z"[4].





[4]This follows a suggestion by Peter Johnstone.

Reversing the orientation of manifolds provides duals for objects of n-*Cobord*, but one can also reverse the orientation of the cobordisms themselves. If $M: S_1 \to S_2$, then we can consider $\overline{M}$ to be a cobordism $\overline{S_1} \to \overline{S_2}$. This process of reversing the orientation of cobordisms corresponds to the covariant dual functors introduced in the context of compact closed categories. Since we have not defined monoidal bicategory homomorphisms, we do not make this analogy precise, but will be content to observe that the importance of considering covariant dual functors is precisely so as to be able to mimic orientation reversal in an algebraic setting.

With the algebraic structures we have introduced so far in this dissertation, we are in a position to make our first excursion into topological quantum field theories in the next chapter.

Chapter 4

Topological Quantum Field Theories

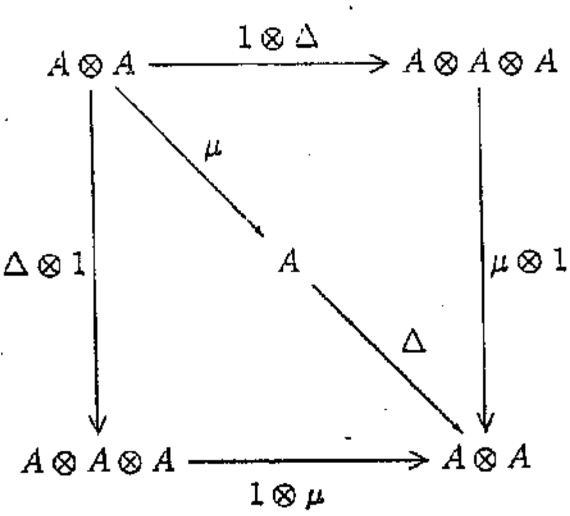
We have seen that n-*Cobord* is a compact closed bicategory. Since topological quantum field theories are "algebraic representations" of n-Cobord, our first approximation of a formal definition of an n-dimensional topological field theory is that it should be a monoidal homomorphism $n\text{-}\mathit{Cobord} \to \mathcal{W}$ where $\mathcal{W}$ is a monoidal bicategory. In this chapter we discuss simple two-dimensional topological quantum field theories. They are simple in the sense that $\mathcal{W}$ is in fact a monoidal *category*. The key algebraic structure arising in this discussion is known as a "Frobenius algebra", so we begin with a definition of Frobenius algebras. Our definition is somewhat non-standard, but is nevertheless equivalent to the more commonly seen definitions. We proceed directly to 2-dimensional theories as 1-dimensional theories are particularly trivial, amounting simply to a choice of object in the codomain bicategory.

4.1 Frobenius Algebras

A *Frobenius algebra* over a commutative ring k is a k-module A such that A is both an algebra and a co-algebra over k. The algebra and co-algebra structures are subject to the condition that if A has multiplication μ and co-multiplication Δ

then the diagram

(4.1)

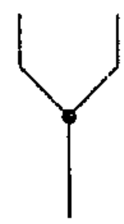


commutes. In this diagram $\otimes$ denotes tensor product over k and we have omitted the obvious associativity isomorphisms. In the words of Joyal and Street [18], "we avoid putting brackets on n-fold tensor products when clarity is gained and rigour preserved." For a more conventional formulation of the definition of a Frobenius algebra, see Drozd and Kirichenko [9].

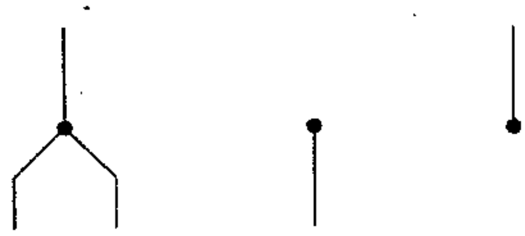
We will also give the axioms for a Frobenius algebra in terms of a diagrammatic notation due originally to Penrose [39] and formalised for tensor categories by Joyal and Street [18] under the name "tensor schemes". For reasons that will become apparent, this notation will be very useful in our discussion of topological quantum field theories.

The following brief description of Penrose diagrams, while far from rigourous, should suffice for the interpretation of the diagrams used in this chapter. Diagrams are to be read progressively down the page and are built up from elementary diagrams, which consist either of single line or a node at which a number of lines meet. A line denotes an identity arrow, while a node denotes a particular arrow in a monoidal category. A diagram with n lines at the top and m lines at the bottom has domain an n-fold and codomain an m-fold tensor product of A. This includes the situation when n and m can be zero, in which case the zero-fold tensor product of A is interpreted as the unit for the tensor product.

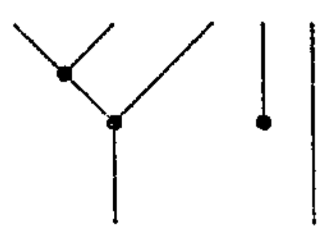
As an example of an elementary node, consider the following diagram which is to represent multiplication $\mu: A \otimes A \to A$ for a Frobenius algebra:



For general tensor schemes, lines are labelled to allow for tensor products built up from more than a single object A. The other basic diagrams we will be using are the co-multiplication $\Delta: A \to A \otimes A$, the unit $\mathbf{1}: k \to A$ and the co-unit $\sigma: A \to k$ respectively:



as well as single lines, which denote the identity on A. More complicated diagrams are then built up by means of juxtaposition, which corresponds to tensor product, and concatenation, which corresponds to composition. Consider the following diagram, constructed from the basic diagrams illustrated above:



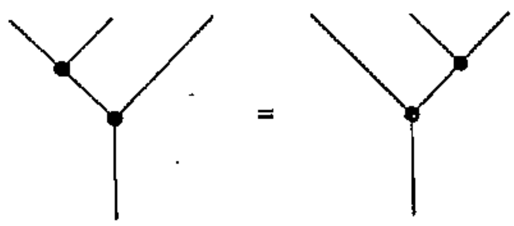
This diagram represents the arrow

$$(\mu \otimes 1 \otimes 1 \otimes 1) \circ (\mu \otimes \sigma \otimes 1): A \otimes A \otimes A \otimes A \otimes A \to A \otimes A.$$

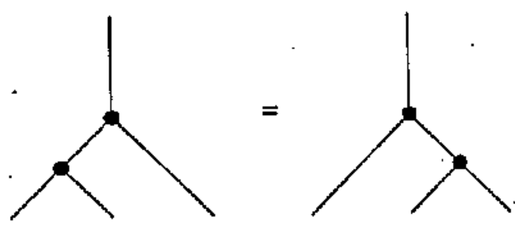
Joyal and Street [18] prove that any such diagram always have a well-defined "valuation" as an arrow and that this is invariant under deformation of the diagram. There is a sense in which these Penrose diagrams are planar duals of the ordinary diagrams of category theory.

The axioms for a Frobenius algebra are now given by a series of diagram equalities. The "interaction" axiom corresponds to diagram (4.1).

Associativity

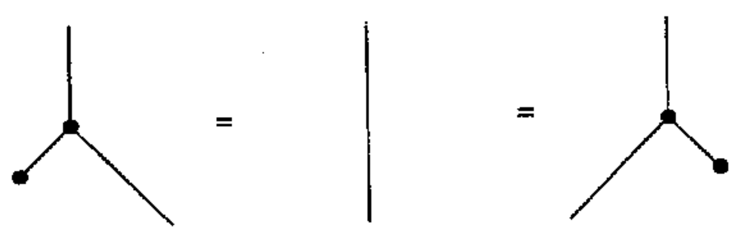


Co-associativity

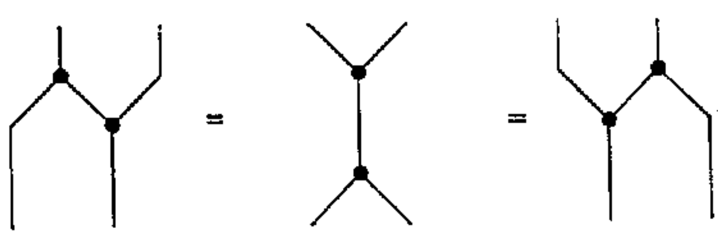


Unit

Co-unit



Interaction



Note that this presentation of the definition of a Frobenius algebra can be interpreted in any monoidal category, so we in fact have a notion of a *Frobenius object* in a monoidal category.

Another point worth noting is that a Frobenius object A is its own dual. If we define $\epsilon = \sigma \circ \mu$ and $\eta = \Delta \circ 1$, then it is easy to check that ϵ and η satisfy the triangular equalities, and are thus the co-unit and unit respectively for an adjunction $A \dashv A$. In particular, this means that if A is a Frobenius algebra, then it must be a finitely generated, projective k-module. This observation provides a connection with a definition of a Frobenius algebra that is perhaps a little better known.

Proposition 4.1 *A finite-dimensional algebra A equipped with a non-degenerate bilinear form T which satisfies the "inner A-bilinearity" condition $T(x \cdot y, z) = T(x, y \cdot z)$, where $\cdot$ is used to denote the algebra multiplication, is a Frobenius algebra.*

Proof. The bilinear form T determines a linear map $\epsilon: A \otimes A \to k$. Since T is non-degenerate, there must be a linear map $\eta: k \to A \otimes A$ such that η and ϵ are the unit and counit of an adjunction $A \dashv A$. This map can be given explicitly in terms of a basis $(e_1, e_2, \ldots, e_k)$ for A as follows. Non-degeneracy ensures the existence of elements $(e'_1, e'_2, \ldots, e'_k)$ such that $\epsilon(e'_i \otimes e_j) = \delta_{ij} 1$ and we then set

$$\eta(\lambda) = \lambda \left(\sum_{i=1}^{k} e_i \otimes e'_i \right).$$

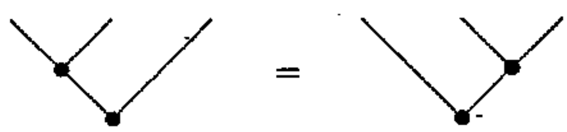
Omitting the obvious isomorphisms, inner A-bilinearity implies that the following diagram commutes:

$$(4.2) \qquad \begin{array}{ccc} A & \xrightarrow{\eta \otimes 1} & A \otimes A \otimes A \\ {\scriptstyle 1 \otimes \eta} \downarrow & & \downarrow {\scriptstyle 1 \otimes \mu} \\ A \otimes A \otimes A & \xrightarrow[\mu \otimes 1]{} & A \otimes A. \end{array}$$

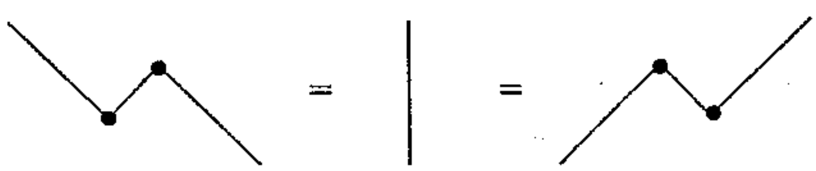
We denote this composite by $\Delta\colon A \to A \otimes A$. As an example of the usefulness of tensor scheme diagrams, we use them to prove that (4.2) commutes. We denote ϵ and η by the diagrams below.

ϵ η

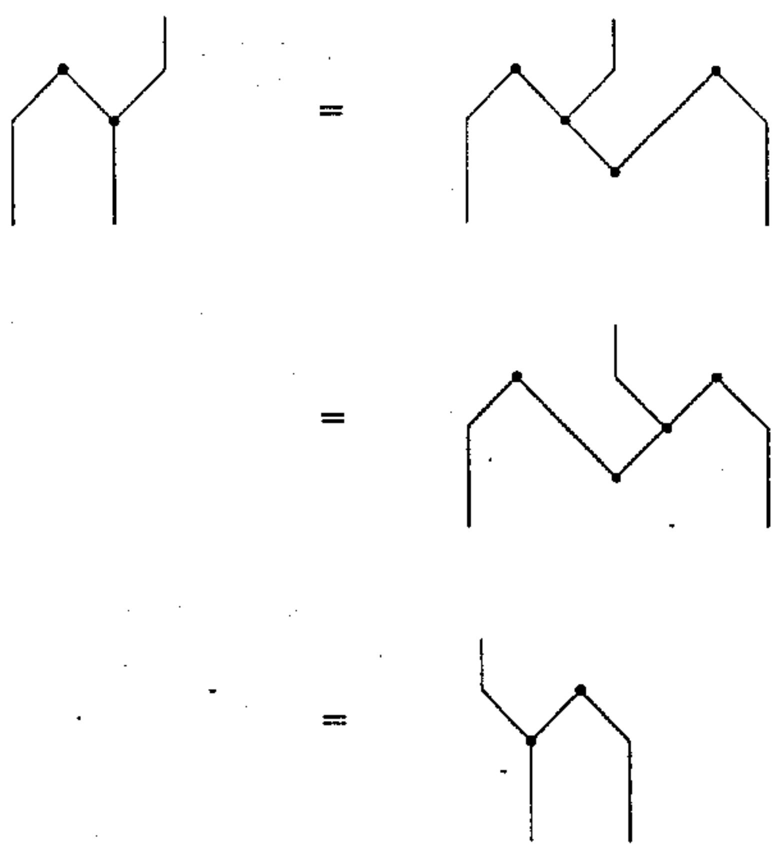
Expressed in terms of these diagrams, inner A-bilinearity is given by the equality



and the adjunction equations are given by the equalities



We now deduce (4.2) using inner-A bilinearity and the adjunction equations as follows.



Similar methods can be used to show that the following diagram also commutes:

$$(4.3) \qquad \begin{array}{ccc} A & \xrightarrow{1_A \otimes 1} & A \otimes A \\ {\scriptstyle 1 \otimes 1_A}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ A \otimes A & \xrightarrow[\epsilon]{} & k. \end{array}$$

Care should be taken here not to confuse identity arrows with the unit 1. We denote this composite by $\sigma: A \to k$. Using (4.2) and (4.3) it is now straightforward to show that Δ and σ give A the structure of a Frobenius algebra. □

We now sketch the notion of a particular type of "free" category. Given a symmetric monoidal category $\mathcal{V}$, there is a category $\mathit{Frob}(\mathcal{V})$ whose objects are

Frobenius objects in $\mathcal{V}$ and whose morphisms are arrows $f\colon A \to A'$ such that

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

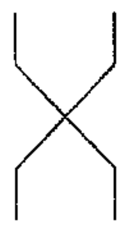
commutes as do similar diagrams involving the co-multiplication, unit and co-unit. If A is a Frobenius object of $\mathcal{V}$, and $F\colon \mathcal{V} \to \mathcal{W}$ is a morphism of monoidal categories, then FA is a Frobenius object in the obvious way. Furthermore, given a transformation $\alpha\colon F \to F'$ and a morphism $f\colon A \to A'$ in $\mathit{Frob}(\mathcal{V})$, then $\alpha_{A'} \cdot Ff$ is a morphism $FA \to F'A'$ in $\mathit{Frob}(\mathcal{V})$. Thus, writing $[\mathcal{V}, \mathcal{W}]$ for the category of morphisms of monoidal categories and transformations between them, we have a "composition" operation

$$\circ\colon [\mathcal{V}, \mathcal{W}] \times \mathit{Frob}(\mathcal{V}) \longrightarrow \mathit{Frob}(\mathcal{W}).$$

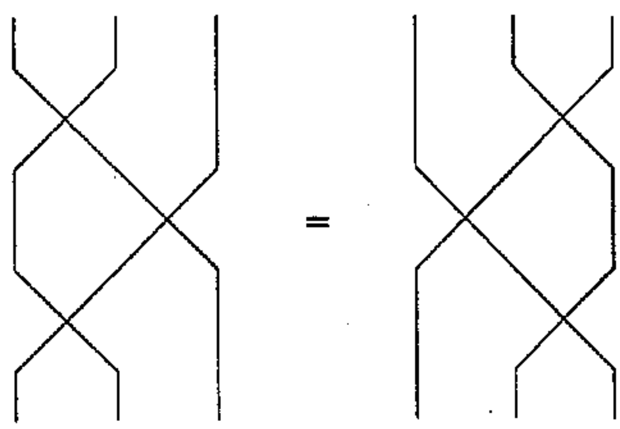
A symmetric monoidal category $\mathcal{F}$ is said to be a *free symmetric monoidal category with a Frobenius algebra* if there is an object A of $\mathit{Frob}(\mathcal{F})$ such that

$$- \circ A\colon [\mathcal{F}, \mathcal{V}] \to \mathit{Frob}(\mathcal{V})$$

is an equivalence of categories. This definition is a special case of the notion of "free categories on a tensor scheme" introduced by Joyal and Street [18]. Although it can also be done algebraically, Joyal and Street prove geometrically that such an $\mathcal{F}$ exists and is determined up to equivalence of monoidal categories. Their construction yields a *strict* monoidal category which we denote by **Frob**. It has the objects the natural numbers and the tensor product on objects is given by addition. There are arrows $\mu\colon 2 \to 1$, $1\colon 0 \to 1$, $\Delta\colon 1 \to 2$ and $\sigma\colon 1 \to 0$ which make 1 a Frobenius object. We will refer to this structure on 1 as the *canonical Frobenius object structure on 1*. There is also an arrow $c\colon 2 \to 2$, which we denote by the diagram



and satisfies conditions expressed by the following diagram equalities:



Setting $c_k = 1_{k-1} \otimes c \otimes 1_{n-k-1}$, we have arrows $c_k\colon n \to n$ for $1 \leq k \leq n-1$ which satisfy the relations

$$
\begin{aligned}
s_k \circ s_k &= 1_n \\
s_k \circ s_{k+1} \circ s_k &= s_{k+1} \circ s_k \circ s_{k+1} && \text{for } 1 \leq k \leq n-2 \\
s_k \circ s_l &= s_l \circ s_k && \text{for } |k-l| > 1.
\end{aligned}
$$

These equations are the same as the relations for a standard presentation of the symmetric group S_n, so we in fact have arrows $c_\sigma\colon n \to n$ in Frob for each permutation $\sigma \in S_n$, such that $\sigma \mapsto c_\sigma$ is a representation of S_n in $End(n)$.

If $\mathcal{V}$ is a symmetric monoidal category with a Frobenius object A, we say A is *commutative* if

$$\mu \circ c_A = \mu$$

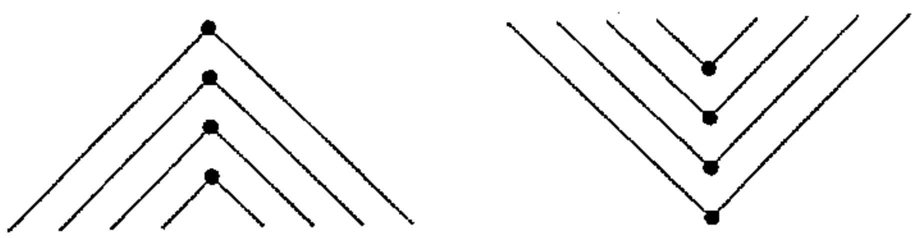
and *co-commutative* if

$$c_A \circ \Delta = \Delta.$$

There is a notion of *free symmetric monoidal category with a commutative and co-commutative Frobenius algebra*, defined in a similarly to the non-commutative case. Again such a free category exits. We will denote by Frob_c the free category with objects the natural numbers and arrows labelled as in Frob. The object 1 of Frob_c is a commutative, co-commutative Frobenius object, and once again we refer to this structure as the *canonical Frobenius object structure on 1*. Note also that Frob_c is compact closed: each object is its own dual and $\eta = \Delta \circ 1$ and $\epsilon = \sigma \circ \mu$ are the unit and counit respectively for the adjunction $1 \dashv 1$. As before these are represented diagrammatically as:

ϵ η

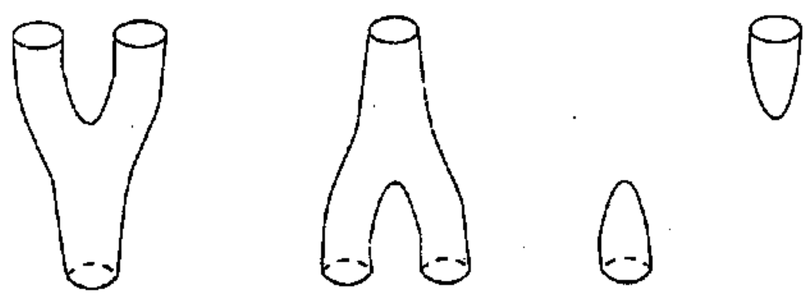
Units and counits for other objects are built from η and ϵ respectively, as illustrated in the diagram below.



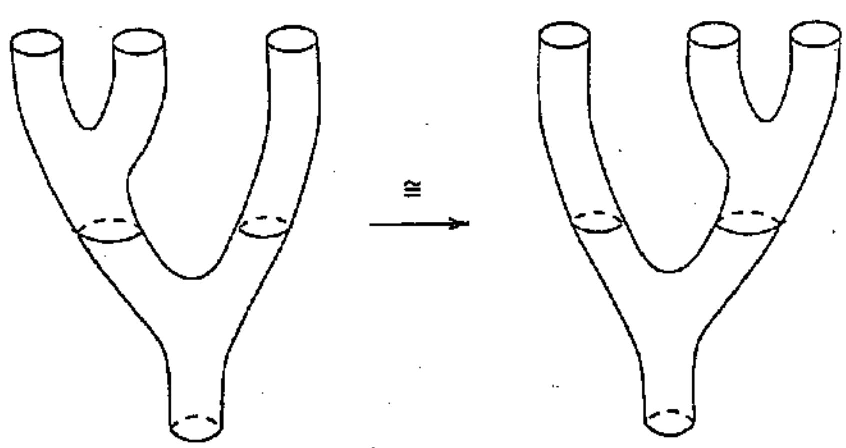
unit counit

4.2 Two-Dimensional Field Theories

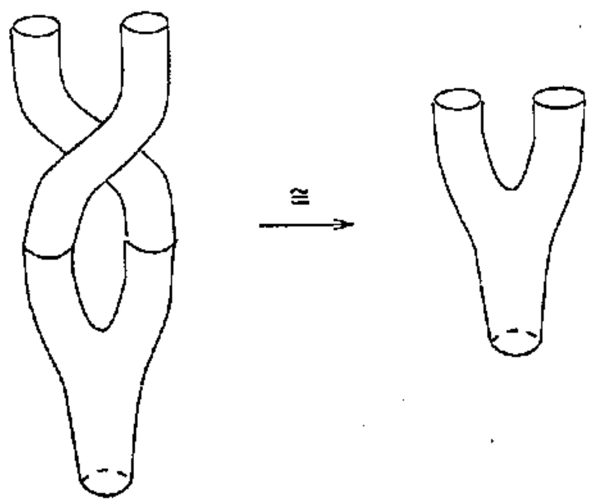
In this section we describe the standard 2-dimensional topological field theories, for which we represent cobordisms in the monoidal category $\mathbf{Mod}_k$. For an example of a similar treatment of 2-dimensional theories, without the language of monoidal bicategories, see Freed [13] or [14]. Consider a homomorphism of monoidal bicategories $F\colon 2\text{-}\mathit{Cobord} \to \mathrm{Mod}_k$. Such a homomorphism associates a k-module to each one-dimensional manifold and k-linear maps to cobordisms between one-dimensional manifolds. Because the 2-cells in $\mathbf{Mod}_k$ are all trivial, diffeomorphic cobordisms induce the same k-linear map. Let $A = F(S^1)$ and consider the following cobordisms, which are progressive down the page.



Since $F(S^1 + S^1) \cong F(S^1) \otimes F(S^1)$ and $F(\emptyset) \cong k$, the images of these manifolds under F yield k-linear maps $\mu: A \otimes A \to A$, $\Delta: A \to A \otimes A$, $1: k \to A$ and $\sigma: A \to k$ respectively. Furthermore, it is easy to check that these maps give A the structure of a Frobenius algebra. For example, there is a (boundary-preserving) diffeomorphism



and since F maps this diffeomorphism into an equality, the multiplication μ must be associative. The other axioms for a Frobenius algebra follow similarly. In addition, A is commutative and commutative. It is commutative because there is a diffeomorphism



A similar diffeomorphism assures co-commutativity. In short, a 2-dimensional topological field theory yields a commutative, co-commutative Frobenius algebra. The same arguments we have outlined here also show that a 2 dimensional theory $F: 2\text{-}\mathit{Cobord} \to \mathcal{V}$ where $\mathcal{V}$ is an arbitrary braided monoidal category determines a commutative, co-commutative Frobenius object.

Conversely, from a commutative, co-commutative Frobenius algebra, we can construct a topological quantum field theory. We first prove

Proposition 4.2 *There is a homomorphism of monoidal bicategories*

$$F: 2\text{-}\mathit{Cobord} \to \mathrm{Frob}_c$$

such that $F(S^1) = 1$ and the Frobenius object structure on 1 determined by F is the canonical one.

Proof. We inductively define iterated multiplication and co-multiplication maps in Frob_c. We define $\mu_{(n)}: n \to 1$ by

$$\begin{aligned} \mu_{(0)} &= 1 \\ \mu_{(1)} &= 1 \\ \mu_{(n+1)} &= \mu \circ (1 \otimes \mu_{(n)}). \end{aligned}$$

Similarly, we define $\Delta_{(n)}: 1 \to n$ by

$$\begin{aligned}\Delta_{(0)} &= \sigma \\ \Delta_{(1)} &= 1 \\ \Delta_{(n+1)} &= (1 \otimes \Delta_{(n)}) \circ \Delta,\end{aligned}$$

where we write 1 for 1_1. Note that associativity and co-associativity respectively imply

$$\begin{aligned}\mu_{(m+n)} &= \mu \circ (\mu_{(m)} \otimes \mu_{(n)}) \\ \Delta_{(m+n)} &= (\Delta_{(m)} \otimes \Delta_{(n)}) \circ \Delta.\end{aligned}$$

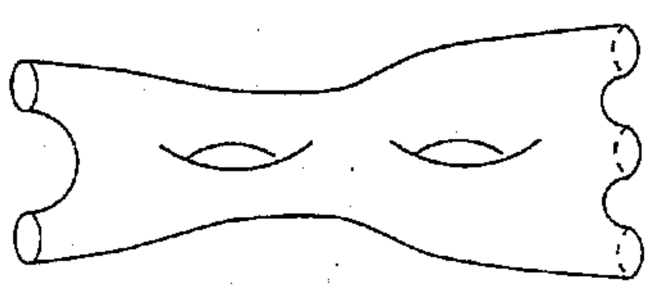
Commutativity and co-commutativity imply

$$\mu_{(n)} \circ c_\sigma = \mu_{(n)} \tag{4.4}$$

$$c_\sigma \circ \Delta_{(n)} = \Delta_{(n)}. \tag{4.5}$$

We also define $T = \mu \circ \Delta: 1 \to 1$.

Now any *connected* cobordism $M: S \to S'$ in 2-*Cobord* is determined up to (boundary preserving) diffeomorphism by the number copies of S^1 in S, which we denote by $n = n(M)$, the number of copies of S_1 in S', which we denote by $m = m(M)$ and the genus of M, which we denote by $g = g(M)$. The genus for a manifold with boundary is simply the genus of the closed manifold obtain by "gluing disks into the holes". For example, in the case of the manifold in the picture below, $n = 2$, $m = 3$ and $g = 2$ (the picture is to read from left to right).

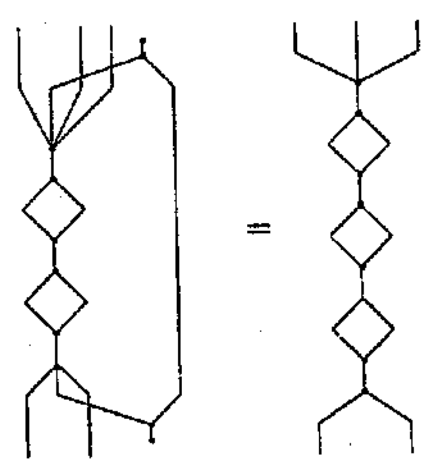


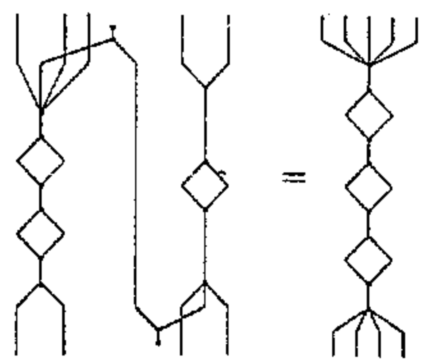
For an object S of $2-Cobord$ which is diffeomorphic to n copies of S^1, we set $F(S) = n$. For connected cobordisms M, with n and m as above, we define $F(M) = \Delta_{(m)} \circ T^g \circ \mu_{(n)}$. The most general cobordism $M: S \to S'$ in $2-Cobord$ can be obtained from a cobordism of the form $M_1 + \ldots + M_k$, where $M_1, \ldots, M_k$ are connected cobordisms, by permuting the boundary components. We define

$$F(M) = c_\tau \circ (F(M_1) \otimes \ldots \otimes F(M_k)) \circ c_\sigma$$

where σ is the permutation of the domain and τ the permutation of the codomain circles. It now remains to check that this definition of F respects composition in 2-*Cobord*.

Since 2-*Cobord* and Frob_c are compact closed, composition can be expressed in terms of feedback loops. We can apply induction over the number of connected components in the manifolds we are composing and over the number of boundary circles we are "feeding back", so we in fact only need to consider a single feedback loop. We must distinguish two cases: a loop from a connected cobordism to itself and a loop which joins two connected cobordisms. In the first case, the genus will increase by one and in the second case, the genus of the new connected cobordism is the sum of the genuses of the two original cobordisms. The following diagrams give examples of the corresponding equalities in Frob_c we must prove.





We now prove the general case algebraically. As before, we set $\epsilon \doteq \sigma \circ \mu$ and $\eta = \Delta \circ \mathbf{1}$. Assuming equations (4.4) and (4.5), the equalities we must prove, corresponding to the two cases we have identified, are

$$(1_{m-1} \otimes \epsilon) \circ (F(M) \otimes 1) \circ (1_{n-1} \otimes \eta) = \Delta_{(m-1)} \circ T^{g+1} \circ \mu_{(n-1)}$$

and

$$(1_m \otimes \epsilon \otimes 1_{m'-1}) \circ (F(M) \otimes 1 \otimes F(M')) \circ (1_{n-1} \otimes \eta \otimes 1_{n'})$$
$$= \Delta_{(m+m'-1)} \circ T^{g+g'} \circ \mu_{(n+n'-1)}.$$

where

$$F(M) = \Delta_{(m)} \circ T^g \circ \mu_{(n)}$$
$$F(M') = \Delta_{(m')} \circ T^{g'} \circ \mu_{(n')}.$$

Simple induction arguments, using the interaction axiom, establish the equalities

$$(4.6) \qquad (\mu_{(n)} \otimes 1_1) \circ (1_{n-1} \otimes \eta) = \Delta \circ \mu_{(n-1)}$$

$$(4.7) \qquad (1_{m-1} \otimes \mu) \circ (\Delta_{(m)} \otimes 1_1) = \Delta_{(m-1)} \circ \mu.$$

The interaction axiom also implies

$$(4.8) \qquad \mu \circ (1 \otimes T) = T \circ \mu = \mu \circ (T \otimes 1)$$

$$(4.9) \qquad (1 \otimes T) \circ \Delta = \Delta \circ T = (T \otimes 1) \circ \Delta,$$

from which we can immediately deduce

$$\mu \circ (T^g \otimes 1) \circ \Delta = T^{g+1}.$$

Combining this with equations (4.6) and (4.7), establishes our first result. Also using equations (4.8) and (4.9) and the interaction axiom, we have

$$\begin{aligned}(1 \otimes \mu) \circ (T^g \otimes 1 \otimes T^{g'}) \otimes (\Delta \otimes 1) &= (T^g \otimes (T^{g'} \circ \mu)) \circ (\Delta \otimes 1) \\ &= (T^g \otimes T^{g'}) \circ (1 \otimes \mu) \circ (\Delta \otimes 1) \\ &= (T^g \otimes T^{g'}) \circ \Delta \circ \mu \\ &= \Delta \circ T^{g+g'} \circ \mu.\end{aligned}$$

Along with equations (4.6) and (4.7), this establishes the second result. □

For an alternative approach to this result in terms of Cerf theory, the study of generic paths between Morse functions, Baez and Dolan [2] refer to an unpublished article by Sawin [41]. However, we have not yet seen Sawin's work.

The defining property of $\mathbf{Frob}_c$ ensures that given a symmetric monoidal $\mathcal{V}$ with a commutative and co-commutative Frobenius object A, there is a monoidal functor $F\colon \mathrm{Frob}_c \to \mathcal{V}$ such that $F(1) = A$. Of course this functor is not uniquely determined, as $F(n)$ may be any of the (canonically isomorphic) n-fold tensor products of A. In any event, this we can now deduce the more conventional result

Proposition 4.3 *If $\mathcal{V}$ is a symmetric monoidal category and $F\colon$ 2-Cobord $\to \mathcal{V}$ is a 2-dimensional topological field theory, then $F(S^1)$ is a commutative, co-commutative Frobenius object. Conversely, given a commutative, co-commutative Frobenius object A of $\mathcal{V}$, there is a field theory F such that the Frobenius object $F(S^1)$ is A.*

Once again, we point out that the field theory is not uniquely determined, but the values it can take on any given object are canonically isomorphic. A detailed study of the extent to which the field theory is determined does not seem to be of great interest.

Although compact closed bicategories have a great deal of structure, we have not yet reached our desired definition of a cobordism category. There is one important aspect of n-*Cobord* which we have not yet mentioned that is traditionally exploited in topological quantum field theories. The objects of n-*Cobord* are $(n-1)$-dimensional manifolds and as well as cobordisms between $(n-1)$-dimensional manifolds we can consider *diffeomorphisms* between them. These two types of "morphism", cobordism and diffeomorphism, are in some sense "orthogonal" to each other. The aim of the next chapter is to combine both into one algebraic structure and to do so we turn to the notion of "double structures".

Chapter 5

Cobordism Categories

§5.1 Double Structures

Usually finite limit theories are modelled in Set but they can also be modelled in any finitely complete category. In particular they can be modelled in a category which is itself the category of models for another theory. We refer to the resulting structures as *double structures*. More precisely, if S and T are finite limit theories, then an (S, T)*-model* is a model of S in $\text{Lex}(T, \text{Set})$. Note that

$$\text{Lex}(S, \text{Lex}(T, \text{Set})) \cong \text{Lex}(T, \text{Lex}(S, \text{Set}))$$

as both are isomorphic to $\text{Lex}(S \times T, \text{Set})$ since limits in functor categories are constructed pointwise. Thus (T, S)-models are essentially the same as (S, T)-models. When S and T are algebraic theories, our notion of an (S, T)-model coincides with that of an (S, T)-bialgebra as discussed in Manes [36].

As a simple example of a double structure, consider (T, T)-models when T is the theory of groups. A well-known argument[1] shows that when a set is equipped with two compatible[2] group multiplication laws, these laws coincide and are commutative. Hence a (T, T)-model is simply an abelian group.

One particular type of double structure will be central to our definition of a quantum field theory. We shall consider (S, T)-models where S is the theory of

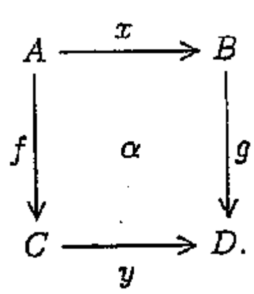
[1] For the original, more general argument, see Eckmann and Hilton [10].

[2] "Compatible" is used in the usual sense in this context, and amounts to saying that the set is equipped with the structure of a (T, T)-model.

groupoids and T the theory of compact closed bicategories. These (S,T)-models admit a more elementary description which is given in this chapter. To motivate this elementary description, we first discuss a double structure that is well-known in category theory: the "double category".

5.1.1 Double Categories

A *double category* is a (T,T)-model where T is the theory of categories. In other words, a double category is a model of the theory of categories in Cat, which some authors refer to as a "category object" in Cat. Double categories were introduced by Ehresmann, although not in these terms. His approach can be found in Ehresmann [11], while the elementary description we give here is taken from Kelly and Street [30]. Double categories are also discussed in Gray [16] and Palmquist [38]. A double category $\mathcal{D}$ consists of *objects* $(A, B, C, \ldots)$, *vertical arrows* $(f, g, h, \ldots)$, *horizontal arrows* $(x, y, z, \ldots)$ and *square cells* $(\alpha, \beta, \gamma, \ldots)$. A typical square cell α is depicted as



A square has *vertical domain* and *codomain arrows*, which will always be drawn on the left and right hand sides respectively, and it also has *horizontal domain* and *codomain arrows*, which will be drawn on the top and bottom respectively. The objects and vertical arrows form a category $\downarrow\!\mathcal{D}$ with identities $\downarrow\! 1_A$, while the objects and horizontal arrows form a category $\overrightarrow{\mathcal{D}}$ with identities $\overrightarrow{1_A}$. The square cells can be composed horizontally,

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ f\Big\downarrow & \alpha & \Big\downarrow g & \beta & \Big\downarrow h \\ D & \xrightarrow[x']{} & E & \xrightarrow[y']{} & F \end{array} \quad \mapsto \quad \begin{array}{ccc} A & \xrightarrow{y \circ x} & C \\ f\Big\downarrow & \beta \circ \alpha & \Big\downarrow h \\ D & \xrightarrow[y' \circ x']{} & F, \end{array}$$

and they can also be composed vertically,

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ f\Big\downarrow & \alpha & \Big\downarrow f' \\ C & \xrightarrow{y} & D \\ g\Big\downarrow & \beta & \Big\downarrow g' \\ E & \xrightarrow[z]{} & F \end{array} \quad \mapsto \quad \begin{array}{ccc} A & \xrightarrow{x} & C \\ g \cdot f\Big\downarrow & \beta \cdot \alpha & \Big\downarrow g' \cdot f' \\ E & \xrightarrow[z]{} & F. \end{array}$$

With these composition laws, the square cells form a category both horizontally and vertically with identities respectively of the form

$$\begin{array}{ccc} A & \xrightarrow{\vec{1}_A} & A \\ f\Big\downarrow & 1_f & \Big\downarrow f \\ B & \xrightarrow[\vec{1}_B]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{x} & B \\ \downarrow 1_A\Big\downarrow & 1_x & \Big\downarrow \downarrow 1_B \\ A & \xrightarrow[x]{} & B. \end{array}$$

Vertical and horizontal composition of square cells satisfy the *interchange law:*

$$(\alpha \circ \beta) \cdot (\gamma \circ \delta) = (\alpha \cdot \gamma) \circ (\beta \cdot \delta)$$

whenever both sides are defined. Also *double identities* coincide:

$$1_{\vec{1}_A} = 1_{\downarrow 1_A}.$$

The connection to the abstract definition of a double category is straightforward. The data for a model of the theory of categories in Cat consists of a diagram

$$\mathrm{C}_2 \xrightarrow{\ m\ } \mathrm{C}_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow\, id\,}{\rightrightarrows}}} \mathrm{C}_0 \tag{5.1}$$

in Cat where C_2 is a pullback of the diagram

$$\begin{array}{ccc} & & \mathrm{C}_1 \\ & & \downarrow{\scriptstyle d_0} \\ \mathrm{C}_1 & \xrightarrow[d_1]{} & \mathrm{C}_0 \end{array}$$

and these data are subject to the usual associativity and identity laws. In terms of our elementary description, C_1 is the vertical category of square cells, while m gives the horizontal composition law. C_0 is the vertical category $\downarrow\mathcal{D}$. Applying the object functor $ob\colon \mathrm{Cat} \to \mathrm{Set}$ to (5.1) yields the data for the category $\vec{\mathcal{D}}$. Functoriality of m gives the interchange law and functoriality of id ensures that double identities coincide. Of course the symmetry of (T, T)-models means that this is only one of two possible interpretations: "horizontal" and "vertical" can be transposed.

A 2-category is a degenerate double category in which the vertical category is discrete. In fact some authors define 2-categories in this way.

Another simple example of a double category is given by taking square cells to be all commuting squares

$$\begin{array}{ccc} A & \xrightarrow{\ x\ } & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\ y\ ]{} & D \end{array}$$

in a category B where f and g are arrows in a given subcategory A of B. Note that the vertical category of square cells is a subcategory of the category of arrows[3] of B. When B is the category of finite dimensional Hilbert spaces and linear maps and A is the subcategory of finite dimensional Hilbert spaces and isometries, this example is relevant to the study of topological quantum field theories.

5.1.2 Double Multiplicative Graphs

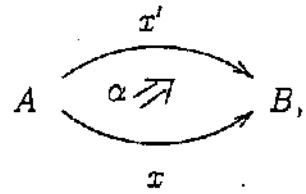
Before proceeding to a definition of cobordism category, we consider a second simple example of a double structure which was also introduced by Ehresmann [11]. A *double multiplicative graph* is a (T,T)-model where T is the theory of multiplicative graphs. A double multiplicative graph has the same underlying structure of objects, vertical and horizontal arrows and square cells as a double category, however the vertical and horizontal composition laws only yield multiplicative graphs, not categories. Vertical and horizontal composition of square cells do satisfy the interchange law and double identities do coincide. Although it is possible to weaken these last two conditions, the resulting structure would of course no longer be a (T,T)-model.

5.2 Cobordism Categories

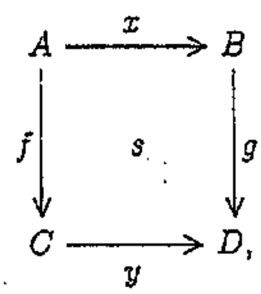
The following definition of a cobordism category is the culmination of the algebraic structures developed so far in this dissertation. Although it comprises a great deal of structure, we do not expect it to be the final word in the characterisation of the algebraic structure of *n-Cobord*. A *cobordism category* is an (S,T)-model where S is the theory of groupoids and T the theory of compact closed bicategories. Here we give an elementary description of cobordism categories, which follows the approach of the previous section. A cobordism category C consists of *objects* $(A,B,C,\ldots)$, *vertical arrows* $(f,g,h,\ldots)$, *horizontal arrows*

[3]See Mac Lane [34] on categories of arrows.

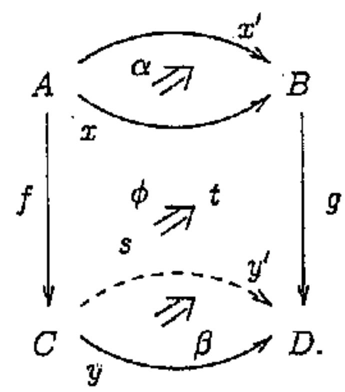
$(x, y, z, \ldots)$, *2-cells* $(\alpha, \beta, \ldots)$ between the horizontal arrows, depicted as



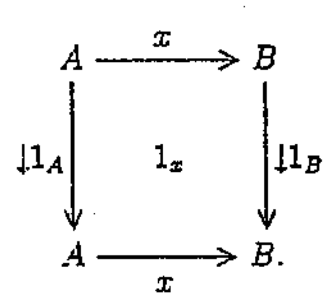
square cells $(s, t, \ldots)$, depicted as



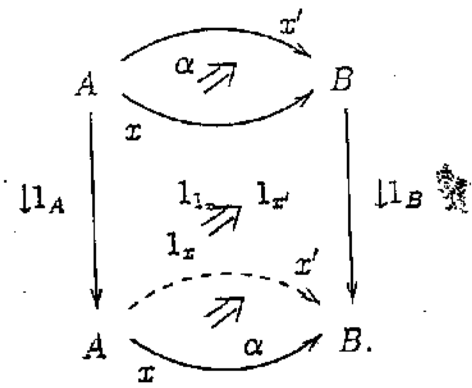
and *3-cells* $(\phi, \psi, \ldots)$ between square cells, depicted as



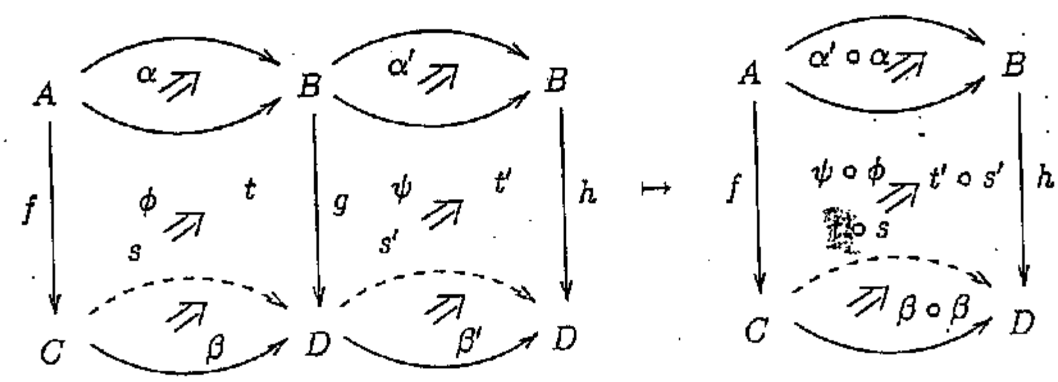
The objects and vertical arrows form a groupoid $\downarrow C$ with identity arrows $\downarrow 1_A$, while the objects, horizontal arrows and 2-cells form a compact closed bicategory $\vec{C}$ with identity arrows $\vec{1_A}$. As in the case of double categories and multiplicative graphs, square-cells can be composed vertically, and form a groupoid with identities of the form



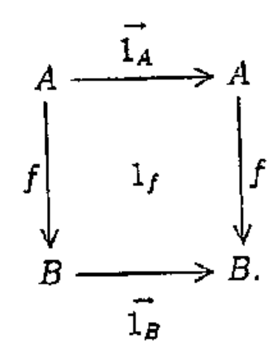
3-cells can be composed vertically and form a groupoid with identities of the form



Also, square-cells and 3-cells can be composed horizontally,



Note that we have omitted the labels for the horizontal arrows to prevent the diagrams from becoming too cluttered. With this composition law, vertical arrows, square cells and 3-cells form a compact closed bicategory with identity square cells of the form



In this compact closed bicategory, the dual of a vertical arrow $f: A \to B$ is an arrow $f^*: A^* \to B^*$ where A^* is the dual of A in $\vec{C}$. Horizontal and vertical composition for both square-cells and 3-cells satisfy interchange laws and double identities coincide.

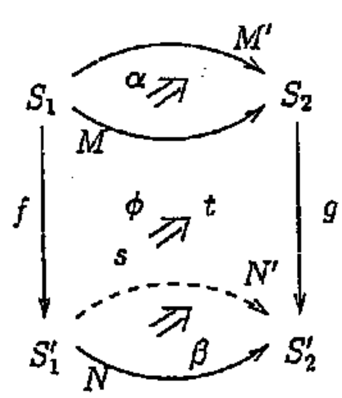
The motivating example of a cobordism category is, of course, *n-Cobord*. The compact closed structure for *n-Cobord* which we have already described is the horizontal compact closed bicategory of a cobordism category. The vertical arrows are diffeomorphisms and a square cell

$$\begin{array}{ccc} S_1 & \xrightarrow{\ M\ } & S_2 \\ {\scriptstyle f}\downarrow & s & \downarrow{\scriptstyle g} \\ S_1' & \xrightarrow[\ N\ ]{} & S_2', \end{array}$$

is a diffeomorphism $s: M \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} \overline{S_1} + S_2 & \xrightarrow{\ \phi_M\ } & M \\ {\scriptstyle \overline{f}+g}\downarrow & & \downarrow{\scriptstyle s} \\ \overline{S_1'} + S_2' & \xrightarrow{\ \phi_N\ } & N. \end{array}$$

A 3-cell



is a commuting diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \alpha\ } & M' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ N & \xrightarrow[\ \beta\ ]{} & N'. \end{array}$$

We have thus reached our destination: an algebraic structure which incorporates both cobordisms and diffeomorphisms.

Earlier we gave an example of a double category which is also a simple cobordism category. The horizontal compact closed category is the compact closed *category* of finite-dimensional Hilbert spaces and the vertical groupoid is the subcategory given by the isometries. Square cells are commuting squares and 3-cells are trivial. In Chapter 3 we defined a general notion of "isometry" for any compact closed category with a covariant dual. Such a category gives a cobordism category in exactly the same way as we have just described for the category of finite dimensional Hilbert spaces.

A topological quantum field theory should be a morphism of cobordism categories with codomain *n-Cobord*, however we will not give a general definition of cobordism category morphisms here. We are content to return to simple two-dimensional field theories and for these theories not all of the data for cobordism category morphisms is necessary. The only data required in this context are mappings of objects to objects, horizontal arrows to horizontal arrows, vertical arrows to vertical arrows, squares to squares, 2-cells to 2-cells and 3-cells to 3-cells which preserve the obvious incidence relations. In a more general setting one would consider additional data which measure the extent to which these mappings preserve the various composition laws. One would also impose various coherence conditions on the data.

We can now consider a simple two-dimensional topological quantum field theory to be a morphism (or at least some of the data for a morphism) F from *2-Cobord* to the cobordism category of finite-dimensional Hilbert spaces outlined above (we will denote this cobordism category by $\mathrm{Hilb}_\square$). Consider two homotopic diffeomorphisms $f, g\colon S \to S'$. Given any homotopy $H\colon S \times I \to S'$ such that

$$H(x, 0) \;=\; f(x)$$

$$h(x,1) \;=\; g(x)$$

and $H(-,t)$ is a diffeomorphism for each t, we can define a diffeomorphism

$$\alpha: S \times I \to S' \times I$$

by setting $\alpha(x,t) = (H(x,t),t)$. Now α is a square cell in the cobordism category 2-*Cobord* and is therefore mapped by F into a commuting square:

$$\begin{array}{ccc} F(S) & \xrightarrow{1} & F(S) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Fg} \\ F(S') & \xrightarrow[1]{} & F(S'). \end{array}$$

Thus $Ff = Fg$. Up to homotopy there is only one orientation-preserving diffeomorphism $\overline{S^1} \to S^1$; in other words up to homotopy there is only one orientation-*reversing* diffeomorphism $r: S^1 \to S^1$ and furthermore r^2 is homotopic to the identity. Therefore F yields a linear *isometry* $\overline{A} \to A$, that is an *anti*-linear isometry

$$c: A \to A$$

such that $c^2 = 1$. Such an isometry allows us to define a real structure on A in the usual way: we set

$$A_{\mathbb{R}} = \{a \in A \mid c(a) = a\}.$$

The same arguments given in the previous chapter indicate that A has the structure of a Frobenius algebra. Now consider an orientation-reversing diffeomorphism of the "pair of pants" manifold

Appendix A

Kapranov and Voevodsky

Long before it was published, Kapranov and Voevodsky's paper *2-categories and the Zamolodchikov tetrahedra equations* [22] became very influential in the study of higher-dimensional algebra. It is a long paper, a fact explained by their remark

> ... lax n-categories (in situations when it is possible to define them) have a lot of structure which is not all apparent at first glance. This is the origin of many difficulties in the subject and the necessary length of every "honest" exposition.

Most authors are not "honest" in this sense, preferring "a heuristic treatment, not a rigorous one" (Freed [13]), and refer to Kapranov and Voevodsky for the details. Unfortunately although "honest", Kapranov and Voevodsky are not always correct or complete and in this Appendix we outline some of their errors and omissions.

Before listing these errors and omissions, some general remarks should be made. Kapranov and Voevodsky refer to bicategories as "2-categories", a common practice amongst authors on topological quantum field theory. They also change the meaning of other long-standing terms from category theory. They use "lax" to mean "pseudo" and for them a "braiding" for a monoidal category need not consist of isomorphisms, but if it does it is referred to as an "isobraiding". Changing the meaning of existing terminology in this way is always likely to be a source of confusion and ambiguity. The use of the term "n-categories to mean "lax" n-categories is particularly dangerous, since it encourages the assumption that such things exist, when it is not yet clear that a most general lax n-category can be defined. Traditional category theory terminology, "category", "bicategory" and "tricategory" only gives names to those higher-dimensional structures for which precise definitions have been given. This seems to be a far safer approach.

Kapranov and Voevodsky do not define bicategorical homomorphisms, transformations and modifications, preferring to defer that material to a forthcoming book [21] so as to "reduce to a minimum the parts related to "pure" theory of 2-categories". The drawback of this approach is that when it comes to defining monoidal 2-categories, the tensor product cannot be introduced as a homomorphism of bicategories, nor can the various constraints be said to be transformations or modifications. Instead, all the structure and constraints must be given explicitly and so definitions are made unnecessarily cumbersome. Indeed the definition of monoidal categories appearing in *2-categories and the Zamolodchikov tetrahedra equations* runs to some twelve pages. Furthermore, without the machinery of homomorphisms, transformations and modifications, it is harder to be systematic and the chance of forgetting some data or axioms increases.

We now turn to specifics. In the list that follows, we refer to the section numbering of *2-categories and the Zamolodchikov tetrahedra equations*. For ease of reference, we also use Kapranov and Voevodsky's hieroglyphic notation. We do not claim that this list is exhaustive.

(§2.2) Kapranov and Voevodsky remark that the axioms labelled $(1 \otimes \bullet \otimes \bullet)$ and $(\bullet \otimes \bullet \otimes 1)$ follow from $(\bullet \otimes 1 \otimes \bullet)$, as shown by Kelly [23]. They do not point out that Kelly's result also gives axiom $(1 \otimes 1)$ as a consequence of $(1 \otimes \bullet \otimes \bullet)$ and $(\bullet \otimes \bullet \otimes 1)$.

(§2.7) Expressed in their terms, the definition of a lax 2-category lacks axioms along the lines of

$$(\beta *_0 u) *_1 (\alpha *_0 u) = (\beta *_1 \alpha) *_0 u$$

and

$$(v *_0 \beta) *_1 (v *_0 \alpha) = v *_0 \ (\beta *_1 \alpha).$$

Without such axioms it is impossible to derive the interchange law for the operations $*_0$ and $*_1$ as defined on 2-morphisms.

(§2.7) The axioms labelled $\left(\begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix} I\right)$ and $\left(I \begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix}\right)$ are clearly incorrect and should be replaced by the requirement that l and r be natural isomorphisms. Consider $\left(\begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix} I\right)$. If $\alpha: u \Rightarrow u'$, then $\mathrm{Id}_B *_0 \alpha$ has source $\mathrm{Id}_B *_0 u$, which is only isomorphic to u, and target $\mathrm{Id}_B *_0 u'$, which is only isomorphic to u'. It does not make sense to equate 2-morphisms with different sources and targets, so instead the axiom should become

$$\alpha *_1 l_u = l_{u'} *_1 (\mathrm{Id}_B *_0 \alpha).$$

A similar condition involving r should replace axiom $\left(I \begin{smallmatrix}\to\\ \downarrow\\ \to\end{smallmatrix}\right)$.

(§2.7) It has been observed that naturality conditions on l and r are omitted. Also omitted is the requirement that $a_{u,v,w}$ be natural in u, v and w.

(§2.7) In their earlier definition of a monoidal structure on a category, they observe that the axioms labelled $(1{\otimes}{\bullet}{\otimes}{\bullet})$ and $({\bullet}{\otimes}{\bullet}{\otimes}1)$ follow from $({\bullet}{\otimes}1{\otimes}{\bullet})$. They fail to observe the corresponding result for bicategories, namely that the axioms labelled $(\to\to I)$ and $(I \to\to)$ follow from $(\to I \to)$. This fact is proved in Proposition (1.1) of this thesis.

(§2.8) While it is not unreasonable to ignore set-theoretic foundational complexities, it is unfortunately not the case that "the introduction of the 2-categorical structure on Cat removes these difficulties for good"!

(§3.9) It should be emphasised that this Proposition only applies to *iso*braided monoidal categories. The same is true of the coherence theorem for braided monoidal categories, which is also mentioned in this section. The potential ambiguity here highlights the danger of non-standard terminology.

(§4.1) To introduce *isomorphisms* $a_{A,B,C}: A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ is to violate Kapranov and Voevodsky's "Main Principle of Category Theory". They should instead be equivalences. More precisely, along with the data given in $(\to \otimes \bullet \otimes \bullet)$, $(\bullet \otimes \to \otimes \bullet)$ and $(\bullet \otimes \bullet \otimes \to)$, they should constitute a pseudo-natural equivalence $a: \otimes \cdot (1 \times \otimes) \to \otimes \cdot (\otimes \times 1)$.

(§4.1) The preceding remark applies equally to the morphisms $l_A: 1 \otimes A \to A$ and $r_A: A \otimes 1 \to A$. They should not be isomorphisms, but part of the data for pseudo-natural equivalences l and r.

(§4.1) The list of data for a monoidal 2-category does not include an isomorphism $1_{A \otimes B} \cong 1_A \otimes 1_B$. This omission is doubtless due to Kapranov and Voevodsky's practice of denoting both of these 1-morphisms by $A \otimes B$. This notation trap is discussed in Section (1.2.5) of this dissertation.

(§4.1) The notation trap just mentioned is the cause of error as well as omission. When giving the conditions that the data for a monoidal 2-category must satisfy, 1-morphisms are not labelled, the implication being that there is only ever one possible label for any 1-morphism drawn in the diagrams. Unfortunately, this is not always the case. Consider the condition expressed by the Stasheff polytope diagram $(\bullet \otimes \bullet \otimes \bullet \otimes \bullet)$. There are two candidates for 1-morphisms $(A(BC))(DE) \to ((AB)C)(DE)$, namely $a_{A,B,C} \otimes (1_D \otimes 1_E)$ and $a_{A,B,C} \otimes 1_{D \otimes E}$. Both of these should appear in the diagram, with the appropriate isomorphism included between them. This situation also arises when considering 1-morphisms $(AB)(C(DE)) \to (AB)((CD)E)$ in the same diagram, and 1-morphisms $(1A)(BC) \to A(BC)$ in the Kelly polytope diagram $(1 \otimes \bullet \otimes \bullet \otimes \bullet)$.

(§4.1) The proof of Proposition 1.1 can be modified, inserting 2-cell isomorphisms in place of commuting regions, to show the existence of a 2-isomorphism $r_1 \to l_1$. The piece of data labelled $(1 \otimes 1)$ is therefore redundant, as is the axiom labelled $(1 \otimes 1 \otimes \bullet)$ which it satisfies.

(§4.1) Since the axioms labelled $(\bullet \otimes 1 \otimes \bullet \otimes \bullet)$ and $(\bullet \otimes \bullet \otimes 1 \otimes \bullet)$ determine λ and ρ, as we remarked in Section 2.1 of this dissertation (using the notion L and R), the axioms $(1 \otimes \bullet \otimes \bullet \otimes \bullet)$ and $(\bullet \otimes \bullet \otimes \bullet \otimes 1)$ are redundant.

(§4.1) No axioms are given which ensure that μ, λ and ρ are modifications.

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