67-17,664

MANES, Ernest Gene, 1943-
A TRIPLE MISCELLANY: SOME ASPECTS OF THE THEORY OF ALGEBRAS OVER A TRIPLE.

Wesleyan University, Ph.D., 1967
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

A TRIPLE MISCELLANY
SOME ASPECTS OF THE THEORY OF ALGEBRAS OVER A TRIPLE

by

Ernest Gene Manes

A dissertation
in Mathematics
submitted in partial fulfillment
of the requirements for
the Degree of Doctor of Philosophy
at Wesleyan University

1967

for Eileen

INTRODUCTION

The standard definition of what "universal algebra" should mean was given in the 1930's by G. D. Birkhoff who, realizing that certain theorems about groups, rings and lattices have a common proof, studied the category of algebras that such examples suggest: algebras are sets with a set of finitary operations satisfying a set of equations and homomorphisms are functions that commute with the operations. Such categories of algebras have been much studied. See the recent book of P. M. Cohn ([5]) and the bibliography there.

In much of the literature cited above, one senses a strong feeling that anything "algebraic" should be "finitary". In [27], Słomiński generalized Birkhoff's schema and considered sets with a set of infinitary operations satisfying a set of equations. (These are the "equationally defineable classes" we define in 1.1.7.) Słomiński's paper has been largely ignored. In this paper we will study universal algebra in a language that makes no mention of "operations" (see the paragraph after next) and for which Słomiński's categories of algebras are models as valid as Birkhoff's. We hope that Słomiński's importance in the history of universal algebra will become more apparent because of our work here.

It has long been known how to construct a free algebra functor $\mathcal{S} \xrightarrow{F} \mathcal{E}$ from the category, $\mathcal{S}$, of sets to an equationally defineable class $\mathcal{E}$ whose operations are finitary (see 1.1.7). It was known, too, that the class of algebras free on finitely many generators contains all the information. Lawvere ([20]) developed "algebraic theories" in order to abstract the "free part" of a category of algebras; we describe them briefly (introducing some inessential modifications for the sake of clarity.) Starting with F as above, the induced functor $\mathcal{S}_0 \xrightarrow{F} \mathbb{T}$, where $\mathcal{S}_0$ is the category of finite sets and $\mathbb{T}$ is the full subcategory of $\mathcal{E}$ generated by algebras free on finitely many generators, is called the <u>algebraic theory of</u> $\mathcal{E}$. $\mathcal{E}$ is recovered as the category of functors $\mathbb{T}^{op} \xrightarrow{X} \mathcal{S}$ such that $\mathcal{S}_0^{op} \xrightarrow{F^{op}} \mathbb{T}^{op} \xrightarrow{X} \mathcal{S}$ is representable, the homomorphisms being natural transformations. $\mathcal{S}_0 \xrightarrow{F} \mathbb{T}$ preserves coproducts and establishes a bijection between the objects of $\mathcal{S}_0$ and the objects of $\mathbb{T}$; conversely, any functor $\mathcal{S}_0 \xrightarrow{F} \mathbb{T}$ with these properties is the algebraic theory of some equationally defineable class whose operations are finitary. Słomiński ([27]) established free functors for equationally defineable classes with infinitary operations. Linton combined and generalized the works of Lawvere and Słomiński in [23]. An algebraic theory there is (essentially) the same as above but replacing $\mathcal{S}_0$ with $\mathcal{S}$. This leads to the following gradation of universal algebra: "finitary" (Birkhoff, Lawvere); "with a rank" (Słomiński); "without a rank" (Linton). In a sense, a theory without a rank corresponds to an equationally defineable class whose class of operations has unbounded arity, but additional conditions are required to prove the converse since the famous theorem of Gaifmann shows that there is no free functor for complete Boolean algebras. A discussion of rank (but not in the language of theories) appears in 2.2.5 - 2.2.6. We will not discuss algebraic theories further in this paper, preferring to use instead the equivalent notion of algebras over a triple.

$\mathbb{T} = (T,\eta,\mu)$ is a <u>triple in a category</u> $\mathcal{K}$ if $\mathcal{K} \xrightarrow{T} \mathcal{K}$, $1_{\mathcal{K}} \xrightarrow{\eta} T$, $TT \xrightarrow{\mu} T$ such that $T\eta.\mu = 1_T = \eta T.\mu$ and $T\mu.\mu = \mu T.\mu$. A <u>$\mathbb{T}$-algebra</u> is a pair (X,ξ) with X a $\mathcal{K}$-object and $XT \xrightarrow{\xi} X$ a $\mathcal{K}$-morphism such that $X\eta.\xi = 1_X$ and $X\mu.\xi = \xi T.\xi$. A <u>$\mathbb{T}$-homomorphism from</u> (X,ξ) to (Y,θ) is a $\mathcal{K}$-morphism $X \xrightarrow{f} Y$ such that $\xi.f = fT.\theta$. It has only recently been realized that the category of algebras over a triple provides an excellent setting for universal algebra. Linton has shown (unpublished, but see [23]) how to define "algebraic theory over" $\mathcal{K}$ so that theories over $\mathcal{K}$ are coextensive with triples in $\mathcal{K}$. In particular, the categories that arise as the algebras over a theory over sets (as in the preceding paragraph) are the categories that arise as the algebras over a triple in sets. A systematic study of "algebras over a triple" (in which we do not include triple cohomology) has not yet been made. This paper is an attempt to begin such a study.

Chapter 1 contains a large fraction of the current folklore if "folklore" can be defined to be what the author has learned in seminar and conversations with Michael Barr, Jon Beck, Bill Lawvere and Fred Linton during the past year; needless to say, the author is deeply indebted to these men. Triples in a category were invented by Godement ([12]) under the name "standard construction"; the motivation for the definition was not 1.1.2. Indeed, algebras over a triple first appear in Eilenberg and Moore ([6]), where special cases of 1.2.1 and 1.2.7 are proved. The relationship between triples and pairs of adjoint functors has been studied by Eilenberg-Moore ([6]), Huber ([16]), Kleisli ([19]) and Maranda ([25]). Maranda obtained for triples what Lawvere called "structure-semantics" theory in [20]. We generalize Maranda's result using more general algebraic functors which compare categories of algebras over triples in different categories (see 1.4.3). These more general algebraic functors were considered by Appelgate in [1]. Appelgate defined morphisms of triples that correspond contravariantly to his algebraic functors; we introduce "intrastructures" which yield a covariant correspondence (see 1.4.4, 1.4.5). A version of Jon Beck's tripleability theorem ([3]) appears in 1.2.9. Linton's conditions for a category of algebras to have $\varinjlim$'s are given in 1.3. The remaining parts of 1.1 - 1.5 are folklore. We introduce the notion of "regular triple" (1.2.5) to abstract certain properties of triples in the category of sets. Many well-known theorems in universal algebra are true for the category of algebras over a regular triple. For instance, a triple-theoretic version of Birkhoff's characterization of varieties in an equationally defineable class ([4], or [5, IV.3.1]) is true in such a situation, see 1.6.6. In 1.7, we consider conditions on triples (T,η,μ), (T',η',μ') in $\mathcal{K}$ such that $(TT',\eta\eta', ?)$ may be completed to a "composite" triple. In [2], Barr defined "distributive laws" to do this. We prove a converse and obtain four equivalent conditions in 1.7.2. We also characterize the composite algebras in terms of the original algebras in 1.7.6.

In Chapter 2 we specialize to triples in the category of sets. This comes close to being ordinary universal algebra, but we emphasize the "infinitary" and "no rank" cases. We discuss "operations" in the language of triples in 2.2. We prove by a direct construction in 2.3.3 that compact T2 spaces is a category of algebras. A corollary is that the usual category of compact algebras induced by a category of algebras (operations are continuous) is itself a category of algebras (2.3.4). In particular, compact topological dynamics is algebraic; this is proved in 2.3.6 with a Birkhoff subcategory argument. For much recent research in topological dynamics it has been assumed that the phase space is compact T2; we can prove theorems of [7], [8] and [9] algebraicly (see 2.4, 2.5), which might help to explain this. The search for examples of non trivial minimal orbit closures should perhaps be conducted in wider spheres that topological dynamics. In 2.6 we generalize Lawvere's characterization of abelian categories of algebras ([20]) to the "no rank" case with the corollary that any additive algebraic category is abelian (2.6.3).

In Chapter 2 we make crucial use of the fact that, in $\mathcal{S}$, a model for the cartesian power X^n is the set of functions from n to X. In Chapter 3 we study a class of categories of "sets with structure" called *lattice fiberings over* $\mathcal{S}$. If $\mathcal{E}$ is a lattice fibering over $\mathcal{S}$, $\mathcal{S}$ sits as a subcategory of $\mathcal{E}$ in such a way that for each $\mathcal{E}$-object X and for each set n, the set of $\mathcal{E}$-morphisms from n to X has a canonical $\mathcal{E}$-structure which is a model for X^n in $\mathcal{E}$. Each triple $\mathbb{T}$ in $\mathcal{S}$ and each lattice fibering $\mathcal{E}$ over $\mathcal{S}$ induces, by a Birkhoff subcategory argument (3.4.5), a regular triple $\tilde{\mathbb{T}}$ in $\mathcal{E}$ whose algebras may be thought of as sets together with $\mathcal{E}$-structure and $\mathbb{T}$-algebra structure "compatible" in the sense that $\mathbb{T}$-operations are $\mathcal{E}$-morphisms. In this way, the study of $\mathbb{T}$ generalizes to $\tilde{\mathbb{T}}$. $\mathcal{E}$ = topological spaces is a lattice fibering over $\mathcal{S}$. If $\mathbb{T}$-algebras = groups that $\widetilde{\prod}$-algebras = topological groups; implicit in this is the construction of the free topological group over a topological space (3.4.8). By a Birkhoff subcategory argument, we prove (3.4.9) that [topological linear spaces] is the category of algebras over a triple in topological spaces.

The author is grateful to the National Science Foundation for financial support during the preparation of this paper, as well as to the Forschungsinstitut für Mathematik of the Eidgenössische Technische Hochschule in Zürich, Switzerland for providing an office.

The author expresses his heartfelt thanks to Fred Linton for help mathematical and non-mathematical without which this paper could not have been written.

TABLE OF CONTENTS

Introduction.

CHAPTER 0. CATEGORY THEORY

0.1 Preliminaries 1

0.2 Yoneda Lemma 4

0.3 Adjoint functors 4

0.4 Epimorphisms 7

0.5 Regular categories 9

0.6 Reflexive pairs 11

0.7 Contractible pairs 12

0.8 Creation of constructions 14

CHAPTER 1. TRIPLES IN A CATEGORY

1.1 Algebras over a triple 15

1.2 Properties of $U^{\mathbb{T}}$ 24

1.3 $\varinjlim$'s in $\mathcal{K}^{\mathbb{T}}$ 32

1.4 Algebraic functors and morphisms of triples 35

1.5 Adjoints of algebraic functors 44

1.6 Birkhoff subcategories for regular triples 50

1.7 Composite triples 60

1.8 Subalgebras for regular triples 70

CHAPTER 2. TRIPLES IN SETS

2.1 Some properties of $\mathcal{S}^{\mathbb{T}}$ 74

2.2 Operations 78

2.3 Compact algebras 90

2.4 The enveloping semigroup of an algebra 99

2.5 Almost periodicity 114

2.6 Tripleable abelian categories 124

CHAPTER 3. TRIPLES IN A LATTICE FIBERING

3.1 Lattice fiberings over a category 130

3.2 Examples of lattice fiberings 141

3.3 Lattice fiberings over sets 145

3.4 Triples in a lattice fibering 151

References.

CHAPTER 0. CATEGORY THEORY

The language of this paper is that of "naive category theory", but all that we do can be interpreted in a category of categories satisfying Lawvere's axioms; for the formalities involved see [21]; this justifies our fearless use of certain "large" subcategories of "the" category of categories and functors. We assume the reader is conversant with elementary category theory at the level of, say, the first five chapters of [26]. The main requirements here are listed in 0.1 - 0.3. 0.4 - 0.8 deal with elementary topics that are not, as far as we know, easily found in the literature. More specialized topics are reviewed at various points throughout the paper.

§0.1 Preliminaries.

If f,g are morphisms in a category, we compose first on the left so that fg = $\xrightarrow{f}\xrightarrow{g}$. Other notations in lieu of fg are f.g and f∘g. If f is a function and if x is an element of the domain of f, x evaluated under f is denoted "xf" or "<x,f>". We go as far as to write "$(X,Y)\mathcal{K}$" for the class of $\mathcal{K}$-morphisms from X to Y and "(H,H')n.t." for the class of natural transformations from H to H', but it would seem too stilted to write $A_i \bigcup_i$, let alone $iA\bigcup_i$, for $\bigcup_i A_i$ and we violate our conventions on such accounts. We sometimes use "$=_{df}$" for "is defined to mean" and "$=_{dn}$" for "is denoted to be", devices we learned from Gottschalk, [14]. The symbol [] is used for "end of proof". A function is bijective $=_{df}$ it is 1-to-1 and onto. We write " $\xrightarrow{f}$ " (with tail, $\rightarrowtail$) to assert that the morphism f is mono; " $\xrightarrow{f}\!\!\!\rightarrow$ "

for epi; ("mono" and "epi" are defined in 0.4.1). Let $\mathcal{K}$ be a category. Either obj $\mathcal{K}$ or $|\mathcal{K}|$ $=_{dn}$ the class of $\mathcal{K}$-objects. For every X in obj $\mathcal{K}$ $1_X =_{dn}$ the identity morphism of X; we also write "$X \xrightarrow{1} X$". $\mathcal{K}^{op} =_{dn}$ the dual category of $\mathcal{K}$. $\mathcal{S} =_{dn}$ the category of sets and functions. $\mathcal{K}$ is legitimate $=_{df}$ for every X, Y in obj $\mathcal{K}$, $(X,Y)\mathcal{K}$ is a set; in this case $\mathcal{K}$ induces a set-valued functor $\mathcal{K}^{op} \times \mathcal{K} \xrightarrow{(-,-)\mathcal{K}} \mathcal{S}$. $\mathcal{K}^{\rightarrow} =_{dn}$ the category whose objects are $\mathcal{K}$-morphisms and such that a $\mathcal{K}^{\rightarrow}$-morphism from $X \xrightarrow{f} Y$ to $X' \xrightarrow{f'} Y'$ is a pair of $\mathcal{K}$-morphisms $(X \xrightarrow{a} X', Y \xrightarrow{b} Y')$ with $af' = fb$. $\mathcal{K}$ is small $=_{df}$ obj $\mathcal{K}^{\rightarrow}$ is a set. A class $\mathcal{F}$ of $\mathcal{K}$-morphisms has a representative set $=_{df}$ there exists a set $\mathcal{R}$ of $\mathcal{K}$-morphisms such that for every f in $\mathcal{F}$ there exists r in $\mathcal{R}$ such that f is isomorphic to r in $\mathcal{K}^{\rightarrow}$.

If D is a $\mathcal{K}$-valued functor, the inverse limit of D (determined only within isomorphism if one exists at all) $=_{dn}$ $\varprojlim$ D, or more precisely $\varprojlim D \longrightarrow D$. We establish notation for some special $\varprojlim$'s The i^{th} projection of a product $=_{dn}$ $\prod X_i \xrightarrow{pr_i} X_i$. The equalizer of a pair (f,g) of $\mathcal{K}$-morphisms $=_{dn}$ eq(f,g). The (dual of the) standard construction of $\varprojlim$'s from products and equalizers is recorded in 0.6.2. The $\varprojlim$ of a family of form $(X_i \xrightarrow{f_i} X : i \in I)$ $=_{df}$ its collective pullback, $=_{dn}$ pullback (f_i); we reserve the term "pullback" for the case crd I = 2. If $X \xrightarrow{f} Y$ is a $\mathcal{K}$-morphism, the kernel pair of f, $=_{dn}$ ker pair f, $=_{df}$ the pullback of f with itself. Note that terminal object = empty product = empty $\varprojlim$. Dually we have $D \longrightarrow \varinjlim D$, $X_i \xrightarrow{in_i} \coprod X_i$, coeq (f,g), collective pushout, cok pair f and initial object = empty $\varinjlim$. In $\mathcal{S}$, products are as usual, eq(f,g) = subset on which f and g agree, pullback $(f_i) = [(x_i)$: for all i, j $x_i f_i = x_j f_j]$, terminal object = 1-point set, coproducts = disjoint unions, coeq(f,g) is obtained by dividing out by the equivalence relation generated by [(xf,xg) : x ε domain f = domain g], pushout (f_i) is defined similarly as a quotient of $\coprod$ range f_i and the empty set is the initial object. In the category of categories, $\varprojlim$'s are essentially the same as in the category of sets, but $\varinjlim$'s are very complicated to describe. If Δ is a category, $\mathcal{K}$ has $\varprojlim$'s of type Δ $=_{df}$ every functor $\Delta \xrightarrow{D} \mathcal{K}$ has a $\varprojlim$; special cases are " $\mathcal{K}$ has equalizers", etc.. $\mathcal{K}$ has $\varprojlim$'s $=_{df}$ $\mathcal{K}$ has $\varprojlim$'s of type Δ whenever Δ is a small category. Make similar definitions for $\varinjlim$'s.

Let $\mathcal{K} \xrightarrow{H} \mathcal{L}$ be a functor. H is faithful $=_{df}$ for every pair (f,g) : X → X' in $\mathcal{K}$, fH = gH implies f = g. If H is faithful and if f is a $\mathcal{K}$-morphism then fH mono implies f mono and fH epi implies f epi. If $\mathcal{L} \xrightarrow{H'} \mathcal{M}$ with HH' faithful then H is faithful. H is full $=_{df}$ for every $\mathcal{L}$-morphism of form $XH \xrightarrow{f} X'H$ there exists a $\mathcal{K}$-morphism $X \xrightarrow{f_0} X'$ such that $f_0 H = f$. H is an isomorphism of categories iff H is full, faithful and bijective on objects. We conclude this section with Godement's "cinq règles" found in [12] very heavy use of which will be implicitly made throughout this paper. Suppose that W, X, Y, Z are functors and that a is a natural transformation from X to Y. Natural transformations $WX \xrightarrow{Wa} WY$ and $XZ \xrightarrow{aZ} YZ$ are induced by defining K(Wa) = (KW)a and K(aZ) = (Ka)Z for every object K. The five rules concerning these operations are (WX)a = W(Xa) : WXY → WXZ; a(YZ) = (aY)Z : WYZ → XYZ; WaZ $=_{df}$ (Wa)Z =W(aZ) : WXZ → WYZ; V(a.b)Z = VaZ.VbZ : VWZ → VYZ; ab $=_{df}$ aY.Xb =

$Wb.aZ : WY \to XZ.$

§0.2 The Yoneda Lemma.

Let $\mathcal{K} \xrightarrow{H} \mathcal{S}$ be a set-valued functor, and let X be a $\mathcal{K}$-object such that $(X,-)\mathcal{K}$ is set-valued. Then the passages

$$((X,-)\mathcal{K},H)\text{n.t.} \longrightarrow XH, \qquad XH \longrightarrow ((X,-)\mathcal{K},H)\text{n.t.}$$

$$a \longmapsto \langle 1_X, Xa\rangle \qquad x \longmapsto (X,-)\mathcal{K} \xrightarrow{a} H$$

$$(X,Y)\mathcal{K} \xrightarrow{Ya} YH$$

$$f \longmapsto \langle x, fH\rangle$$

are mutually inverse. In particular, $((X,-)\mathcal{K},H)$n.t. is a set. For a proof see [10, pp. 112-114], or [26, pp. 97-99]. A set-valued functor is representable $=_{df}$ there exists a $\mathcal{K}$-object X such that $(X,-)\mathcal{K}$ is set-valued and naturally equivalent to H; in this case X is the <u>representing object of</u> H.

§0.3 Adjoint functors.

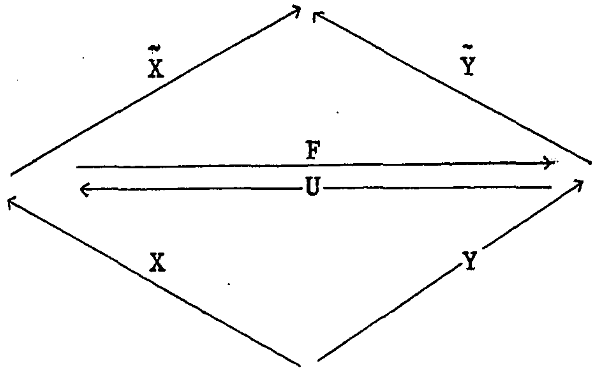
Let $\mathcal{L} \xrightarrow{i} \mathcal{K}$ be a (not necessarily full) subcategory of $\mathcal{K}$, and let X be a $\mathcal{K}$-object. A reflection of X in $\mathcal{L}$ $=_{df}$ a $\mathcal{K}$-morphism $X \xrightarrow{X\eta} X_{\mathcal{L}}$ such that $X_{\mathcal{L}} \in \text{obj}\,\mathcal{L}$ and such that whenever $X \xrightarrow{f} L$ $\in \mathcal{K}$ with $L \in \text{obj}\,\mathcal{L}$ then there exists unique $X_{\mathcal{L}} \xrightarrow{\hat{f}} L$ $\in$ L such that $X\eta\hat{f} = f$. If every $\mathcal{K}$-object has a reflection in $\mathcal{L}$ then $\mathcal{L}$ is a <u>reflective subcategory of</u> $\mathcal{K}$ and there is a <u>reflector functor</u> $\mathcal{K} \xrightarrow{R} \mathcal{L}$ defined so as to make $1 \xrightarrow{\eta} Ri$ natural. R is determined within natural equivalence. $\mathcal{L}$ is full iff R may be chosen with $iR = 1_{\mathcal{L}}$. The definition of reflectors requires a suitable axiom of choice. If $\mathcal{L}$ is a full reflective subcategory and if $\Delta \xrightarrow{D} \mathcal{L}$ with $Di \xrightarrow{a} \varinjlim Di$ in $\mathcal{K}$ then $D \xrightarrow{aR} (\varinjlim Di)R = \varinjlim D$ in $\mathcal{L}$. Dually, define *coreflection*, *coreflective subcategory*, *coreflector*; full coreflective subcategories inherit $\varinjlim$'s.

A *left adjointness* consists of functors $\mathcal{K} \xrightarrow{F} \mathcal{A}$, $\mathcal{A} \xrightarrow{U} \mathcal{K}$ and natural transformations $UF \xrightarrow{\varepsilon} 1_{\mathcal{L}}$, $1_{\mathcal{K}} \xrightarrow{\eta} FU$ (called *adjunctions*) subject to the *adjointness axioms* $F \xrightarrow{\eta F} FUF \xrightarrow{F\varepsilon} F = 1_F$, $U \xrightarrow{U\eta} UFU \xrightarrow{\varepsilon U} U = 1_U$. We denote this by "$F \dashv U$", read "F is left adjoint to U" and let ε, η be understood. U has a left adjoint $=_{df}$ there exists $F \dashv U$. If $\mathcal{K}$ is legitimate, U has a left adjoint iff for every $\mathcal{K}$-object X the functor $(X, (-)U)\mathcal{K}$ is representable. If both $\mathcal{K}$ and $\mathcal{A}$ are legitimate then a left adjointness may be expressed in terms of a natural equivalence $((-)F,-)\mathcal{A} \xrightarrow{\alpha} (-,(-)U)\mathcal{K}$ where $\langle f, (X,A)\alpha\rangle = X\eta.fU$, $\langle g, (X,A)\alpha^{-1}\rangle = gF.A\varepsilon$ and conversely $X\eta = \langle 1_{XF}, (X,XF)\alpha\rangle$, $A\varepsilon = \langle 1_{AU}, (AU,A)\alpha^{-1}\rangle$.

Let $F \dashv U$. U preserves monos and $\varprojlim$'s (for the definition of "preserve" see 0.8). U is faithful iff ε is pointwise $\mathcal{A}$-epi, and U is full and faithful iff ε is a natural equivalence. If $\tilde{F} \dashv U$ then F and $\tilde{F}$ are naturally equivalent. The subcategory generated by the image of U is a reflective subcategory of $\mathcal{K}$, $X \xrightarrow{X\eta} XFU$ being the reflection. Conversely, a subcategory is reflective iff its inclusion functor has a left adjoint. Notice that a subcategory inclusion i is a full reflective subcategory iff there exists $R \dashv i$ with $iR \xrightarrow{\varepsilon} 1$ a natural equivalence.

If $X \dashv Y$ and $X' \dashv Y'$ then $X'X \dashv YY'$; the adjunctions are $1 \xrightarrow{\eta'} X'Y' \xrightarrow{X'\eta Y'} X'XYY'$ and $YY'X'X \xrightarrow{Y\varepsilon' X} YX \xrightarrow{\varepsilon} 1$.

Consider a not necessarily commutative diagram of functors



with $F \dashv U$. Then Γ, Γ^{-1} and $\tilde{\Gamma}$, $\tilde{\Gamma}^{-1}$ are respectively mutually inverse pairs, defined by

$$(XF,Y)\text{n.t.} \xrightarrow{\Gamma} (X,YU)\text{n.t.}, \qquad (X,YU)\text{n.t.} \xrightarrow{\Gamma^{-1}} (XF,Y)\text{n.t.}$$

$$XF \xrightarrow{\sigma} Y \mapsto X \xrightarrow{X\eta} XFU \xrightarrow{\sigma U} YU \qquad X \xrightarrow{\psi} YU \mapsto XF \xrightarrow{\psi F} YUF \xrightarrow{Y\varepsilon} Y$$

$$(U\tilde{X},\tilde{Y})\text{n.t.} \xrightarrow{\tilde{\Gamma}} (\tilde{X},F\tilde{Y})\text{n.t.}, \qquad (\tilde{X},F\tilde{Y})\text{n.t.} \xrightarrow{\tilde{\Gamma}^{-1}} (U\tilde{X},\tilde{Y})\text{n.t.}$$

$$U\tilde{X} \xrightarrow{\tilde{\sigma}} \tilde{Y} \mapsto \tilde{X} \xrightarrow{\eta\tilde{X}} FU\tilde{X} \xrightarrow{F\tilde{\sigma}} F\tilde{Y} \qquad \tilde{X} \xrightarrow{\tilde{\psi}} F\tilde{Y} \mapsto U\tilde{X} \xrightarrow{U\tilde{\psi}} UF\tilde{Y} \xrightarrow{\varepsilon\tilde{Y}} \tilde{Y}$$

This form of the theorem appears in [22, p. 321].

Finally, we state the *adjoint functor theorem* first proved by Freyd. Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor. U *satisfies the solution set condition* if for every $K \in \text{obj}\,\mathcal{K}$ the class $[f \in \mathcal{K} : K \xrightarrow{f} AU$ for some A in obj $\mathcal{A}$] has a representative set. (Such a representative set is called a *solution set for* K). Let $\mathcal{A}$, $\mathcal{K}$ be legitimate and assume $\mathcal{A}$ has $\varprojlim$'s. The adjoint functor theorem says: there exists $F \dashv U$ iff U preserves $\varprojlim$'s and satisfies the solution set condition.

For the rest of Chapter 0 fix a category $\mathcal{K}$.

§0.4 Epimorphisms.

0.4.1 Definition. Let $A \xrightarrow{f} B$ be a $\mathcal{K}$-morphism. f is a split epimorphism if there exists $B \xrightarrow{\tilde{f}} A \in \mathcal{K}$ with $\tilde{f}f = 1_B$. f is a coequalizer if there exist g,h in $\mathcal{K}$ with f = coeq(g,h). Define reg(f) = [$A \xrightarrow{g} Y \in \mathcal{K}$: for every (a,b) : X → A, af = bf implies ag = bg]. f is a regular epimorphism if for every g in reg(f) there exists a unique $\tilde{g}$ in $\mathcal{K}$ with $f\tilde{g} = g$. f is an epimorphism if for every (a,b) : B → X in $\mathcal{K}$, fa = fb implies a = b. Dually, we have split monomorphism, equalizer, regular monomorphism, monomorphism.

0.4.2 Proposition. Let f : A → B ε $\mathcal{K}$. Then f split epi implies f coequalizer implies f regular epi implies f epi.

Proof. If $\tilde{f}f = 1_B$, f = coeq(1_A, $f\tilde{f}$). If there exists (a,b) whose coequalizer is f then for every g in reg(f) we have ag = bg so that the coequalizer property induces unique $\tilde{g}$ with $f\tilde{g} = g$. Finally, suppose f is regular epi and that fa = fb. Defining g $=_{df}$ fa, g ε reg(f) so there exists unique $\tilde{g}$ with $f\tilde{g} = g$, and a = $\tilde{g}$ = b. []

0.4.3 Proposition. Let f : A → B ε $\mathcal{K}$. If ker pair (f) exists then f coequalizer iff f regular epi.

Proof. If (a,b) = ker pair (f) and if f regular epi, it is not hard to show that f = coeq(a,b). []

0.4.4 Proposition. Let $A \xrightarrow{f} B \xrightarrow{g} C$ ε $\mathcal{K}$. f {split} epi and g {split} epi implies fg {split} epi. fg {split} epi implies g split epi. []

<u>0.4.5 Proposition.</u> Let $f : A \to B \in \mathcal{K}$. f iso iff f regular epi and mono.

<u>Proof.</u> [Iso] implies [split epi and mono] implies [regular epi and mono]. Conversely, if f is regular epi and mono, 1_A is in reg(f) and so induces $\tilde{f}$ with $f\tilde{f} = 1_A$. As $f\tilde{f}f = f$ and f is epi, $\tilde{f}f = 1_B$. []

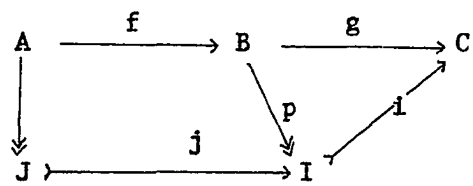
<u>0.4.6 Definition.</u> Let $f : A \to B \in \mathcal{K}$. A <u>regular coimage factorization</u> of f is a factorization $f = A \xrightarrow{p}\!\!\!\!\to Q \rightarrowtail^{i} B$ with p regular epi and i mono. $\mathcal{K}$ <u>has regular coimage factorizations</u> if every $\mathcal{K}$-morphism admits a regular coimage factorization. The dual notion is <u>regular image factorization</u>.

<u>0.4.7 Proposition.</u> Regular coimage factorizations are unique within isomorphism.

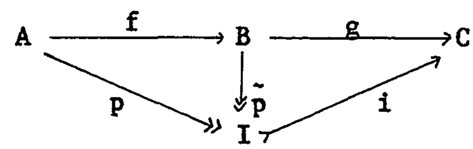
<u>Proof.</u> Suppose p,p' are regular epis and i,i' are monos with pi = p'i'. p' is in reg(p) as i' is mono, so h is uniquely induced with ph = p'. hi' = i because p is epi. h^{-1} is induced similarly. []

<u>0.4.8 Proposition.</u> Assume $\mathcal{K}$ has regular coimage factorizations. Let $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{K}$. Then f,g regular epi implies fg regular epi. fg regular epi implies g regular epi. The hypothesis on $\mathcal{K}$ is necessary in both cases.

<u>Proof.</u> Suppose fg is regular epi. Factoring first g, then fp, we have from 0.4.7 that ji is an isomorphism:



Hence i is mono and split epi and therefore an isomorphism, which proves g is regular epi. Now suppose f,g are regular epi, and factor fg:



As i is mono, p is in reg(f) inducing $\tilde{p}$ such that $f\tilde{p} = p$. As just proved above, i is regular epi; as i is also mono, i is iso and fg is regular epi. The four object category

with $a \neq b$, $af = bf$ is such that $fg = \text{coeq}(a,b)$ but g is not regular epi. Using similar constructions one can show that the composition of a spit epi and a coequalizer (in either order) need <u>not</u> be regular epi. []

<u>§0.5 Regular categories.</u>

<u>0.5.1 Definition.</u> The category $\mathcal{K}$ is <u>regular</u> if it satisfies the following four axioms.

REG 1. $\mathcal{K}$ has regular coimage factorizations.

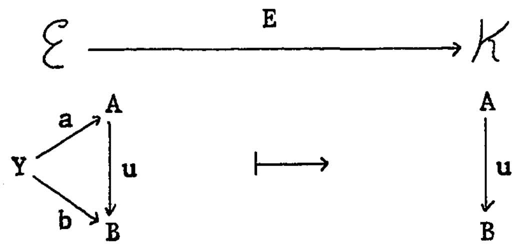
REG 2. $\mathcal{K}$ has $\underleftarrow{\lim}$'s.

REG 3. $\mathcal{K}$ is legitimate.

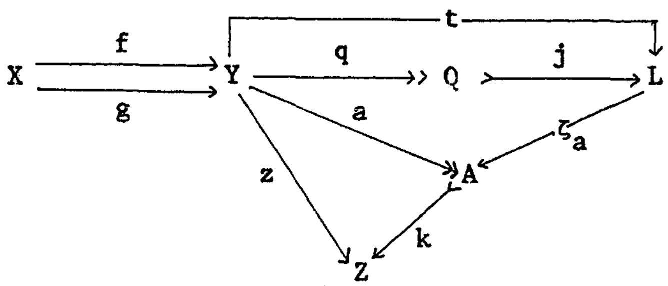
REG 4. For every X in obj $\mathcal{K}$ the class of regular epimorphisms with domain X has a representative set.

<u>0.5.2 Proposition.</u> Let $\mathcal{K}$ be regular. Then $\mathcal{K}$ has coequalizers.

<u>Proof.</u> The proof will not require REG 3. Let $(f,g) : X \to Y \in \mathcal{K}$, and let $\mathcal{R}$ be a representative set of regular epimorphisms with domain Y. Define a category $\mathcal{E}$ with objects $\{Y \xrightarrow{a} A \in \mathcal{R} : fa = ga\}$, and such that a morphism from a to b is a $\mathcal{K}$-morphism u with au = b. With the evident composition, $\mathcal{E}$ is a small category and



is a functor. Construct $\varprojlim E = L \xrightarrow{\zeta_a} A$. As $Y \xrightarrow{a} A$ is natural there exists unique $Y \xrightarrow{t} L$ with $t\zeta_a = a$ for every $a \in$ obj $\mathcal{E}$. We construct the regular coimage factorization $t = Y \xrightarrow{q} Q \xrightarrow{j} L$, and show $q = \text{coeq}(f,g)$. For each a, $fqj\zeta_a = ft\zeta_a = fa = ga = gqj\zeta_a$. Therefore $fqj = gqj$ and then $fq = gq$ as j is mono. Now suppose



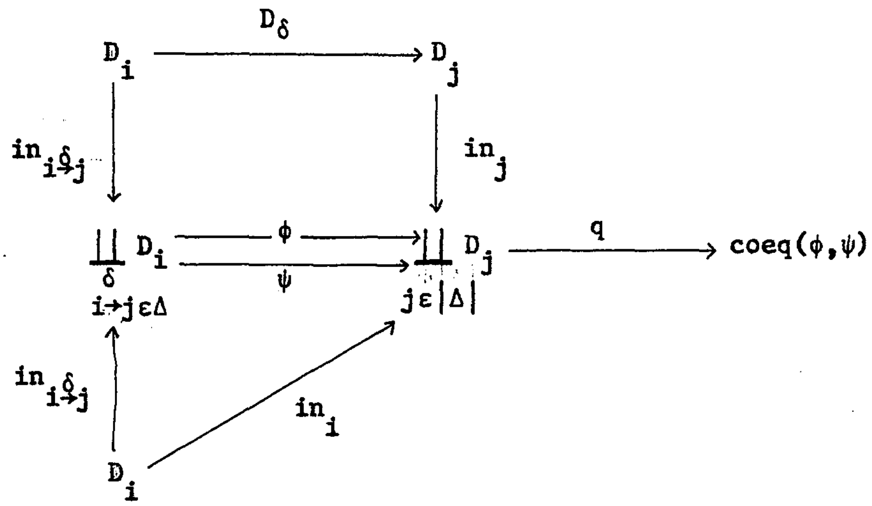
$fz = gz$. There exists a regular coimage factorization $z = ak$ with a in obj $\mathcal{E}$. $q(j\zeta_a k) = t\zeta_a k = ak = z$. Since q is epi, $j\zeta_a k$ is unique with this property. []

§0.6 Reflexive pairs.

0.6.1 Definition. Let $(f,g) : A \to B$ be a pair of $\mathcal{K}$-morphisms. (f,g) is reflexive if there exists $B \xrightarrow{d} A$ $\mathcal{K}$ with $df = 1_B = dg$. (The origin of the terminology lies in the fact that when $\mathcal{K} = \mathcal{S}$, (f,g) is reflexive iff the image of the induced map $A \xrightarrow{(f,g)} B \times B$ contains the diagonal of B.)

0.6.2 Proposition. If $\mathcal{K}$ has coproducts and if every reflexive pair of $\mathcal{K}$-morphisms has a coequalizer, then $\mathcal{K}$ has all $\varinjlim$'s.

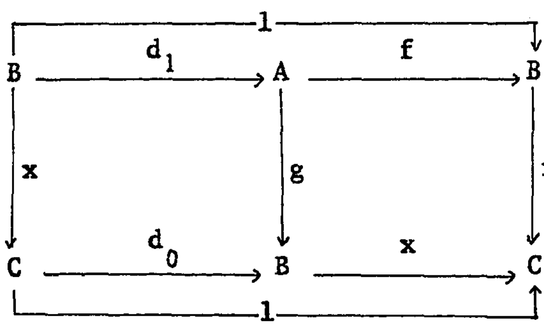
Proof. We recall the classical construction of $\varinjlim$'s from coproducts and coequalizers. Let $\Delta \xrightarrow{D} \mathcal{K}$. If $i \xrightarrow{\delta} j \in \Delta$ write $D_i \xrightarrow{D\delta} D_j$ instead of $iD \xrightarrow{\delta D} jD$ (we often do this for diagrams D) and define maps ϕ,ψ,q by



Then $D_i \xrightarrow{in_i} \coprod_{j \in |\Delta|} D_j \xrightarrow{q} \mathrm{coeq}(\phi,\psi) = \varinjlim D$. We only observe that (Φ,Ψ) is a reflexive pair. Define d by $in_j.d = in_{j \to j} 1_j$. It is trivial to check that $d\phi = 1 = d\psi$.

§0.7 Contractible pairs.

<u>0.7.1 Definition.</u> Let $(f,g) : A \to B$, $x : B \to C$ be $\mathcal{K}$-morphisms. (f,g) is <u>contractible</u> if there exists $d : B \to A$ such that $df = 1_B$ and $fdg = gdg$. (f,g,x) is a <u>contractible</u> <u>coequalizer</u> if $g \xrightarrow{(f,x)} x$ is a split epimorphism in $\mathcal{K}^{\to}$, that is if there exists $C \xrightarrow{d_0} B$, $B \xrightarrow{d_1} A$ such that



commutes. The theory of this section is due to Jon Beck, see [3].

<u>0.7.2 Proposition.</u> Let $(f,g) : A \to B$, $x : B \to C$ be $\mathcal{K}$-morphisms. The following statements are equivalent.

a. (f,g,x) is a contractible coequalizer.

b. (f,g) is contractible and $x = \text{coeq}(f,g)$

<u>Proof.</u> <u>a implies b.</u> By hypothesis $fx = gx$ and there exists (d_0,d_1) with $d_1g = xd_0$, $d_1f = 1$. As $fd_1g = fxd_0 = gxd_0 = fd_1g$, (f,g) is contractible. Now suppose $y : B \to Y$ with $fy = gy$. $\tilde{y} =_{df} d_0y$. Then

$x\tilde{y} = xd_0y = d_1gy = d_1fy = y$. $\tilde{y}$ is unique with this property as x is epi.

b implies a. By hypothesis there exists $B \xrightarrow{d_1} A$ with $d_1f = 1$ and $fd_1g = gd_1g$, and $x = \text{coeq}(f,g)$. We have at once that there exists $C \xrightarrow{d_0} B$ with $xd_0 = d_1g$. As x is epi and $xd_0x = d_1gx = d_1fx = x$, $d_0x = 1$. []

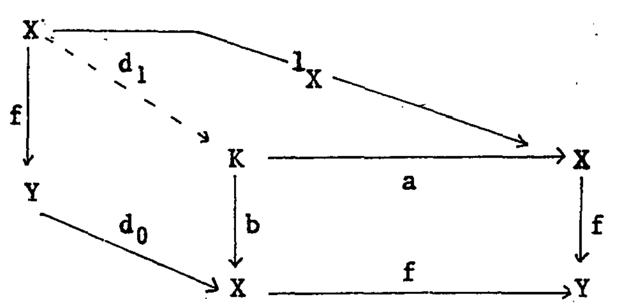
0.7.3 Corollary. Every functor preserves coequalizers of contractible pairs. []

0.7.4 Proposition. If $\mathcal{K}$ has equalizers then every contractible pair of $\mathcal{K}$-morphisms has a coequalizer.

Proof. Let $B \xrightarrow{d_1} A \underset{g}{\overset{f}{\rightrightarrows}} B$ with $d_1f = 1$, $fd_1g = gd_1g$. Set $C \xrightarrow{d_0} B =_{df} \text{eq}(1_B, d_1g)$. As $d_1g.d_1g = d_1f.d_1g = d_1g$ there exists unique $B \xrightarrow{x} C$ with $xd_0 = d_1g$. As d_0 is mono and $fxd_0 = fd_1g = gd_1g = gxd_0$, $fx = gx$. As d_0 is mono and $d_0xd_0 = d_0d_1g = d_0$, $xd_0 = 1$. It follows from 0.7.2 that $x = \text{coeq}(f,g)$. []

0.7.5 Proposition. If $\mathcal{K}$ has kernel pairs then every split epi in $\mathcal{K}$ is a contractible coequalizer.

Proof. Let $Y \xrightarrow{d_0} X \xrightarrow{f} Y = 1$. $(a,b) : K \to X =_{df}$ ker pair (f). As $fd_0f = f$, there exists d_1:



[]

§0.8 Creation of constructions.

Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor and let $\mathfrak{F}$ be a class of $\mathcal{A}$-valued functors. U {weakly} preserves $\varinjlim$'s of elements of $\mathfrak{F}$ $=_{df}$ $D \varepsilon \mathfrak{F}$ and $D \xrightarrow{\kappa} A = \varinjlim D$ {and $\varinjlim DU$ exists} implies $\kappa U = \varinjlim DU$. U detects $\varinjlim$'s of elements of $\mathfrak{F}$ $=_{df}$ $D \varepsilon \mathfrak{F}$ and $\varinjlim DU$ exists implies $\varinjlim D$ exists. U reflects $\varinjlim$'s of elements of $\mathfrak{F}$ $=_{df}$ $D \varepsilon \mathfrak{F}$ and $D \xrightarrow{\kappa} A$ natural with $A \varepsilon$ obj $\mathcal{A}$ (we identify A with the appropriate constant functor) such that $\kappa U = \varinjlim DU$ implies $\kappa = \varinjlim D$. U constructs $\varinjlim$'s of elements of $\mathfrak{F}$ $=_{df}$ $D \varepsilon \mathfrak{F}$ and $DU \xrightarrow{\kappa} X = \varinjlim DU$ implies there exists $D \xrightarrow{\tilde{\kappa}} A$ with $\tilde{\kappa} U = \kappa$ and $\tilde{\kappa} = \varinjlim D$. U creates $\varinjlim$'s of elements of $\mathfrak{F}$ $=_{df}$ $D \varepsilon \mathfrak{F}$ and $DU \xrightarrow{\kappa} X = \varinjlim DU$ implies there exists unique $D \xrightarrow{\tilde{\kappa}} A \varepsilon \mathcal{A}$ with domain D such that $\tilde{\kappa} U = \kappa$; moreover $\tilde{\kappa} = \varinjlim D$. Observe that "creates" implies all of the others. U creates isomorphisms $=_{df}$ $A \varepsilon$ obj $\mathcal{A}$ and $AU \xrightarrow{\kappa} X$ a $\mathcal{K}$-isomorphism implies there exists unique $A \xrightarrow{\tilde{\kappa}} B \varepsilon \mathcal{A}$ with domain A such that $\tilde{\kappa} U = \kappa$; moreover $\tilde{\kappa}$ is an isomorphism. (Observe that U creates isomorphisms iff U creates $\varinjlim$'s of elements of $\mathcal{A}^{\mathbb{1}}$, where $\mathbb{1}$ is the one-morphism category). An observation of Linton is: U creates $\varinjlim$'s of elements of $\mathfrak{F}$ iff U weakly preserves, detects and reflects $\varinjlim$'s of elements of $\mathfrak{F}$ and U creates isomorphisms. An important definition for Chapter 1 is "U creates coequalizers of U-contractible pairs" which arises from $\mathfrak{F}$ $=_{df}$ all U-contractible pairs, that is all functors from $\bullet \rightrightarrows \bullet$ to $\mathcal{A}$ U of which are contractible. We let the reader formulate "U reflects epis", "U creates regular coimage factorizations," etc..

CHAPTER 1. TRIPLES IN A CATEGORY

§1.1 Algebras over a triple (cf. [3], [6], [25]).

1.1.1 Definitions. Let $\mathcal{K}$ be a category. $\mathbb{T} = (T,\eta,\mu)$ is a triple in $\mathcal{K}$ with unit η and multiplication μ if $\mathcal{K} \xrightarrow{T} \mathcal{K}$ is a functor and if $1 \xrightarrow{\eta} T$, $TT \xrightarrow{\mu} T$ are natural transformations subject to the three axioms:

$\mathbb{T}$-unitary axioms.

$$\begin{array}{ccccc} T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta T} & T \\ & {\searrow}^{1} & \downarrow{\scriptstyle\mu} & {}^{1}{\swarrow} & \\ & & T & & \end{array}$$

$\mathbb{T}$-associativity axiom.

$$\begin{array}{ccc} TTT & \xrightarrow{\mu T} & TT \\ \downarrow{\scriptstyle T\mu} & & \downarrow{\scriptstyle\mu} \\ TT & \xrightarrow{\mu} & T \end{array}$$

Let $\mathbb{T} = (T,\eta,\mu)$ be a triple in $\mathcal{K}$. A $\mathbb{T}$-algebra $=_{df}$ a pair (X,ξ) with $X \in \text{obj}\ \mathcal{K}$, $XT \xrightarrow{\xi} X \in \mathcal{K}$ subject to the two axioms:

$$\begin{array}{ccc} X & \xrightarrow{X\eta} & XT \\ & {\searrow}^{1} & \downarrow{\scriptstyle\xi} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} XTT & \xrightarrow{\xi T} & XT \\ \downarrow{\scriptstyle X\mu} & & \downarrow{\scriptstyle\xi} \\ XT & \xrightarrow{\xi} & X \end{array}$$

ξ-unitary axiom. $\qquad$ ξ-associativity axiom.

X is the underlying $\mathcal{K}$-object of (X,ξ) and ξ is the structure map of (X,ξ). If (X,ξ) and (Y,θ) are $\mathbb{T}$-algebras, a $\mathbb{T}$-homomorphism,

$(X,\xi) \xrightarrow{f} (Y,\theta)$, <u>from</u> (X,ξ) <u>to</u> (Y,θ) is a $\mathcal{K}$-morphism $X \xrightarrow{f} Y$ subject to the

$\mathbb{T}$-homomorphism axiom.

$$\begin{array}{ccc} XT & \xrightarrow{fT} & YT \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \theta} \\ X & \xrightarrow{f} & Y \end{array}$$

$\mathcal{K}^{\mathbb{T}} =_{dn}$ the resulting category of $\mathbb{T}$-algebras. $U^{\mathbb{T}} =_{dn}$ the faithful underlying $\mathcal{K}$-object functor .

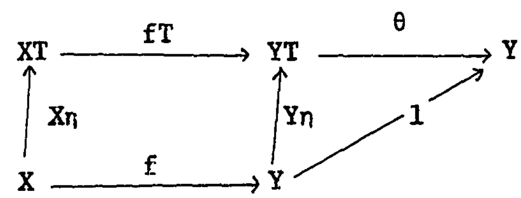
A functor $\mathcal{A} \xrightarrow{U} \mathcal{K}$ is <u>tripleable</u> if there exists a triple $\mathbb{T}$ in $\mathcal{K}$ and an isomorphism of categories $\mathcal{A} \xrightarrow{\Phi} \mathcal{K}^{\mathbb{T}}$ such that $\Phi U^{\mathbb{T}} = U$.

<u>**1.1.2 Heuristics in $\mathcal{K}^{\mathbb{T}}$.**</u> In the course of this paper it will become clear that categories of algebras that exist in nature are tripleable. Right now, we show that, conversely, the category of algebras over a triple has certain properties expected of a "real" category of algebras. Fix a triple $\mathbb{T} = (T,\eta,\mu)$ in a category $\mathcal{K}$. There are free $\mathbb{T}$-algebras. $U^{\mathbb{T}}$ has a canonical left adjoint $\mathcal{K} \xrightarrow{F^{\mathbb{T}}} \mathcal{K}^{\mathbb{T}}$, defined by $(X \xrightarrow{f} Y)F^{\mathbb{T}} = (XT,X\mu) \xrightarrow{fT} (YT,Y\mu)$. That $F^{\mathbb{T}}$ is well-defined follows from the diagrams:

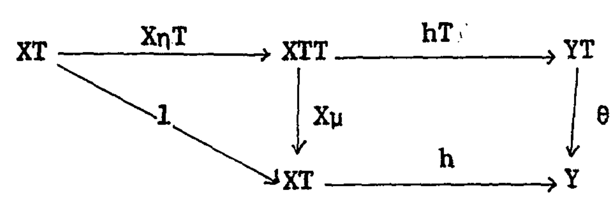
$$\begin{array}{ccc} XT & \xrightarrow{XT\eta} & XTT \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle X\mu} \\ & & XT \end{array} \qquad \begin{array}{ccc} XTTT & \xrightarrow{X\mu T} & XTT \\ \downarrow{\scriptstyle XT\mu} & & \downarrow{\scriptstyle X\mu} \\ XTT & \xrightarrow{X\mu} & XT \end{array} \qquad \begin{array}{ccc} XTT & \xrightarrow{fTT} & YTT \\ \downarrow{\scriptstyle X\mu} & & \downarrow{\scriptstyle Y\mu} \\ XT & \xrightarrow{fT} & YT \end{array}$$

and we prove below that $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ with adjunctions $1_{\mathcal{K}} \xrightarrow{\eta} T$, $U^{\mathbb{T}}F^{\mathbb{T}} \xrightarrow{\varepsilon^{\mathbb{T}}} 1_{\mathcal{K}^{\mathbb{T}}}$ where $(X,\xi)\varepsilon^{\mathbb{T}} =_{df} (XT,X\mu) \xrightarrow{\xi} (X,\xi)$. Thinking of

$X \xrightarrow{X\eta} XT$ as "inclusion of the generators", the axioms defining a $\mathbb{T}$-algebra say that a structure map is a homomorphic extension of the identity map on generators. Hence $\mathcal{K}^{\mathbb{T}}$ parodies classical universal algebra in the sense that algebras are canonically quotients of frees. This model is entirely satisfactory once we point out that free algebras are literally free in the usual sense. Suppose (Y,θ) is a $\mathbb{T}$-algebra and suppose $X \xrightarrow{f} Y \in \mathcal{K}$. The diagram

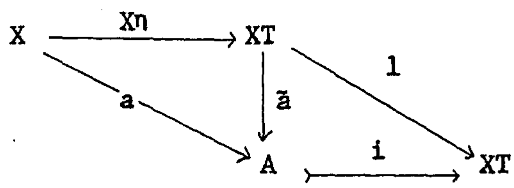


shows that there exists $(XT,X\mu) \xrightarrow{\tilde{f}} (Y,\theta) \in \mathcal{K}^{\mathbb{T}}$ with $X\eta.\tilde{f} = f$, namely $\tilde{f} =_{df} fT.\theta$. Moreover, $\tilde{f}$ is unique with this property; if $(XT,X\mu) \xrightarrow{h} (Y,\theta) \in \mathcal{K}^{\mathbb{T}}$ with $X\eta.h = f$ then the diagram



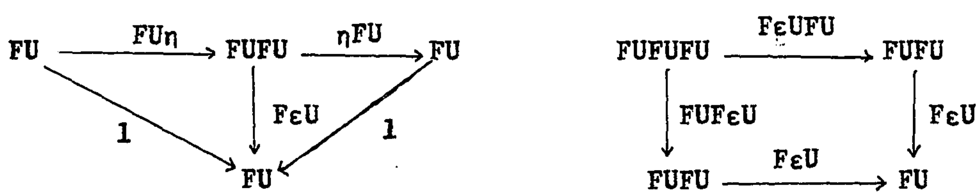
shows that $h = X\eta T.hT.\theta = (X\eta.h)T.\theta = fT.\theta = \tilde{f}$. (This argument is most of the promised proof that $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$; the reader may complete the verification.)

A reasonable definition of "subalgebra" is "monomorphism in $\mathcal{K}^{\mathbb{T}}$". This is equivalent to the definition we will introduce in 1.2.2 below. We observe now that the generators "generate" $(XT,X\mu)$, i.e. whenever $(A,\xi) \rightarrowtail^{i} (XT,X\mu)$ is a subalgebra and A "contains" the generators in the sense that there exists $X \xrightarrow{a} A \in \mathcal{K}$ with $X\eta = a.i$, then i is an isomorphism. To prove it,



$\tilde{a}$ is induced in $\mathcal{K}^{\mathbb{T}}$ with $X\eta.\tilde{a} = a$. 1_{XT}, $\tilde{a}.i$ are $\mathbb{T}$-homomorphisms agreeing on generators, and hence are equal. Applying 0.4.5, i is mono and split epi, and therefore iso.

<u>1.1.3 The triple induced by a pair of adjoint functors.</u> As was first pointed out in [16], if $\mathcal{A} \xrightarrow{U} \mathcal{K} \xrightarrow{F} \mathcal{A}$ with $F \dashv U$ via adjunctions $1_{\mathcal{K}} \xrightarrow{\eta} FU$, $UF \xrightarrow{\varepsilon} 1_{\mathcal{A}}$, then $(FU, \eta, F\varepsilon U)$ is a triple in $\mathcal{K}$. The proof is easy:



It is equally easy to check that if $\mathbb{T}$ is a triple in $\mathcal{K}$, then the triple induced by $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ is just $\mathbb{T}$ itself; all triples arise in this way. A more complete study of this construction will appear in 1.4.

<u>1.1.4 Example; closure operators.</u> Let the category $\mathcal{K}$ be a quasi-ordered class, that is if $X, Y \in \text{obj } \mathcal{K}$ there is at most one morphism from X to Y (in which case we write "$X \leq Y$"). All diagrams in $\mathcal{K}$ are commutative, and hence a triple in $\mathcal{K}$ is an object function

$\mathcal{K} \xrightarrow{T} \mathcal{K}$ such that $X \leq Y$ implies $XT \leq YT$ (T is a functor), $X \leq XT$ ($X\eta$) and $XTT = XT$ ($XT\eta$ and $X\mu$); said differently, a triple in $\mathcal{K}$ is just a closure operator T. For such a triple T, the algebras are precisely the T-closed elements, as is easy to show.

<u>1.1.5 Example; full reflective subcategories.</u> Let $\mathcal{K}$ be a category. There is at least one triple in $\mathcal{K}$, namely the identity triple $\mathbb{T} = (1,1,1)$, with $U^{\mathbb{T}} = 1_{\mathcal{K}}$. Less trivially and more generally let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a full reflective subcategory with reflector F such that $UF = 1_{\mathcal{A}}$. The adjunctions are $1 \xrightarrow{\eta} FU$ and $UF \xrightarrow{1} 1$, $X\eta$ being the reflection $X \to X_{\mathcal{A}}$. The induced triple, $\mathbb{T}$, is $(FU, 1 \xrightarrow{\eta} FU, FU \xrightarrow{1} FU)$. (X,ξ) is a $\mathbb{T}$-algebra iff $X\eta.\xi = 1_X$; in that case, $X\eta.\xi.X\eta = X\eta$ so that $\xi.X\eta = 1_{XFU}$ by the uniqueness of reflection-induced maps. Hence $\mathcal{K}^{\mathbb{T}}$ is the full subcategory generated by all objects isomorphic in $\mathcal{K}$ to some object in $\mathcal{A}$. In particular, if obj $\mathcal{K}$ is a union of $\mathcal{K}$-isomorphism classes, U is tripleable.

<u>1.1.6 Example; triples vs. monoid objects.</u> $(\mathcal{K},\Lambda,*)$ is a <u>category with multiplication</u> if $\mathcal{K}$ is a category, Λ is a $\mathcal{K}$-object and $\mathcal{K} \times \mathcal{K} \xrightarrow{*} \mathcal{K}$ is a functor satisfying the axioms: $* \times 1 \,.\, * = 1 \times * \,.\, *$ (* is associative) and $\Lambda * - = 1_G = - * \Lambda$ (is a *-unit). If $(\mathcal{K},\Lambda,*)$ is a category with multiplication, (G,e,m) is a <u>$(\mathcal{K},\Lambda,*)$-monoid</u> if G is a $\mathcal{K}$-object, $\Lambda \xrightarrow{e} G$, $G * G \xrightarrow{m} G$ are $\mathcal{K}$-morphisms satisfying the axioms: $m * 1 \,.\, m = 1 * m \,.\, m$, $1 * e \,.\, m = 1_G = e * 1 \,.\, m$. (Note: if Cat is the category of categories, if $\mathbb{1}$ is the one-morphism category (so that for any category $\mathcal{K}$, obj $\mathcal{K}$ =

functors $\mathbb{1} \to \mathcal{K}$) and if Cat × Cat $\xrightarrow{\times}$ Cat is cartesian product of categories the (Cat, $\mathbb{1}$,×) is a category with multiplication whose monoids are precisely the categories with multiplication.)

Let $\mathcal{K}$ be a category. $\mathcal{K}^{\mathcal{K}} =_{dn}$ the usual functor category of functors from $\mathcal{K}$ to $\mathcal{K}$ and natural transformations. Let $\mathcal{K}^{\mathcal{K}} \times \mathcal{K}^{\mathcal{K}} \xrightarrow{\circ} \mathcal{K}^{\mathcal{K}}$ be composition. Then ($\mathcal{K}^{\mathcal{K}}$, $1_{\mathcal{K}}$, ∘) is a category with multiplication whose monoids are precisely the triples in $\mathcal{K}$.

Turning in another direction, let (G,e,m) be a ($\mathcal{K}$,Λ, *)-monoid. Define a triple $\mathbb{T}$ = (T,η,μ) in $\mathcal{K}$ by $T =_{df} -*G$, $X\eta =_{df} X \xrightarrow{1*e} X*G$, $X\mu =_{df} X*G*G \xrightarrow{1*m} X*G$. It is easy to check that $\mathbb{T}$ is a triple. If $\mathcal{K} = \mathcal{S}$, crd Λ = 1 and * = × then (G,e,m) is an ordinary monoid and $\mathcal{S}^{\mathbb{T}}$ = G-sets. If $\mathcal{K}$ = topological spaces, crd Λ = 1 and * = × then $\text{Top}^{\mathbb{T}}$ = topological transformation semigroups with topological phase semigroup G. If $\mathcal{K}$ = abelian groups, Λ = $\mathbb{Z}$ and * = $\otimes_{\mathbb{Z}}$ then G is a ring and $\text{Ab}^{\mathbb{T}}$ = G-modules. If $\mathcal{K}$ = Λ-modules (for Λ a commutative ring), Λ = Λ and * = $\otimes_\Lambda$ then G is a Λ-algebra and $\Lambda\text{-mod}^{\mathbb{T}}$ = G-modules.

<u>1.1.7 Example; equationally defineable classes.</u> Let Ω be a set and let $\Omega \xrightarrow{a} \text{obj}\,\mathcal{S}$ be a function (called <u>arity</u>). An Ω-algebra $=_{df}$ a set X together with an ωa-ary operation $X^{\omega a} \xrightarrow{\omega} X$ for every ω∈Ω, and an Ω-homomorphism is a function $X \xrightarrow{f} Y$ such that $\omega . f = f^{\omega a}.\omega$ for all ω. Classically, (dating back to G. D. Birkhoff ca. 1930 but equally so in the recent book of Cohn, [5]), one assumes further that each ωa is finite. In this case the free Ω-algebra XF on a set

X is constructed recursively as a word algebra: (x) is a word for every $x \in X$; if $\omega \in \Omega$ and if $W_1, \ldots, W_{\omega a}$ are words, so is $W_1 W_2 \ldots W_{\omega a-1} W_{\omega a}\, \omega$. The ω's induce operations on XF via concatenation. An equation is then defined to mean a pair of elements in the underlying set of some free Ω-algebra. If E is a set of equations, the category of (Ω,E)-algebras is the full subcategory of those Ω-algebras X such that whenever $(e,f) \in E \cap YF^2$ and whenever $YF \xrightarrow{h} X$ is an Ω-homomorphism, then $eh = fh$. A category arising as (Ω,E)-algebras for some (Ω,E) is an equationally defineable class. Four facts ((i), (ii) by [27], (iii), (iv) by Linton unpublished) are: (i) Foregoing the requirement that ωa be finite, the underlying set functor U from Ω-algebras still has a left adjoint, namely $X \mapsto (U^X, U)$n.t. ("U^X" is defined in 2.2.1 below). (ii) Using (i), (Ω,E)-algebras can still be defined, and then the underlying set functor U_E from (Ω,E)-algebras has a left adjoint. (iii) U_E is tripleable. (iv) The triples rising from (iii) are exactly those that have a rank (as in 2.2.6 below). We will not prove these theorems here.

1.1.8 Example; sets with base point. $XT =_{df} X \coprod \{\infty\}$. $X\eta =_{df} X \xrightarrow{in_X} XT$. $X\mu =_{df} XTT \to XT$ via collapsing two ∞'s to one. The algebras are sets with base point.

1.1.9 Example; abelian groups. Let Ab be the category of abelian groups and let F, U be the usual free and underlying functors. The adjunctions are "inclusion of the generators" $X \xrightarrow{X\eta} XFU$ and "addition" $(X,+)UF \xrightarrow{(X,+)\varepsilon} (X,+)$, $(x_1)\ldots(x_n) \longmapsto x_1 + \ldots + x_n$. Defining $\mathbb{T} =_{df} (FU,\eta,F\mu U)$ it is not hard to see directly that $\mathcal{S}^{\mathbb{T}} \approx$ Ab via an underlying-respecting isomorphism of categories. If (X,ξ) is a $\mathbb{T}$-algebra, "+" may be recovered by $x+y =_{df} (x)(y)\xi$. This approach to abelian groups is "presentation" invariant. For example, an abelian group could be defined as a set X with binary operation $X \times X \xrightarrow{-} X$ satisfying the equation $x - ((y-z) - (y-x)) = z$ (due to Higman and Neumann, see [5, p. 165, ex. 6].

<u>1.1.10 Example; complete semilattices.</u> By a <u>complete semilattice</u> we mean a partially ordered set X in which every subset $A \subset X$ has a supremum sup A in X. In particular, sup ϕ is the least element. Notice that the map $2^X \xrightarrow{\sup} X$ completely determines the structure since $x \leq y$ iff sup $[x,y] = y$. We will construct a triple $\mathbb{T} \doteq (T,\eta,\mu)$ in $\mathcal{S}$ with $U^{\mathbb{T}}$ isomorphic to $\mathcal{A} \xrightarrow{U} \mathcal{S}$ where $\mathcal{A}$ is the category of complete semilattices and sup-preserving maps and U is the underlying set functor; the structure maps will indeed be the sup maps. Let T be the power-set functor, sending X to 2^X, and defined on morphisms via direct images. $X\eta$ sends x to [x] and $X\mu$ assigns to a family its union. If X is a complete semilattice let $X\Phi =_{df} (X, \sup)$. The verification that $\mathbb{T}$ is a triple and that Φ is well-defined on objects may be safely left to the reader. If X,Y are complete semilattices and if $X \xrightarrow{f} Y$ is a function then f is a $\mathbb{T}$-homomorphism from (X, sup) to (Y, sup) iff $\sup.f = fT.\sup$ iff f is sup-preserving. Hence Φ is full and faithful and $\Phi U^{\mathbb{T}} = U$. As we argued earlier, Φ is 1-to-1 on objects. We prove in detail that Φ is onto on objects. Let (X,ξ) be a $\mathbb{T}$-algebra. For x,y in X, define $x \leq y =_{df} [x,y]\xi = y$. As

$x = [x]\xi$, $x \leq x$. If $x \leq y$ and $y \leq x$ then $x = [y,x]\xi = [x,y]\xi = y$. Suppose $x \leq y$ and $y \leq z$. Then $[x,z]\xi = [[x]\xi, [y,z]\xi]\xi =$ $[[x], [y,z]]\xi T.\xi = [[x], [y,z]]X\mu.\xi = [x,y,z]\xi = [[x,y], [z]]X\mu.\xi$ $[[x,y]\xi, [z]\xi]\xi = [y,z]\xi = z$, and $x \leq z$. Now observe that $A \subset B$ implies $A\xi \leq B\xi$; for $[A\xi, B\xi]\xi = [A, B]\xi T.\xi = [A, B]X\mu.\xi = (A \cup B)\xi$ $= B\xi$. Let $A \subset X$. For every $a \in A$, $a = [a]\xi \leq A\xi$ because $[a] \subset A$. To see $A\xi$ is minimal with this property, suppose $x \in X$ and $a \leq x \leq A\xi$ for every $a \in A$. Then $[A\xi, x]\xi = [A, [x]]\xi T.\xi = (A \cup [x])\xi =$ $(\bigcup_{a \in A} [a,x])\xi = [[a,x] : a \in A]X\mu.\xi = [[a,x]\xi : a \in A]\xi = [x]\xi = x$ thus proving $A\xi \leq x$ as desired. The proof that Φ is an isomorphism is complete.

Notice that if T were redefined by $XT =_{df} [A \subset X : A \text{ finite}]$ then, since a finite union of finite sets is finite, the above argument works verbatim to produce partially ordered sets with finite sups. A similar discussion holds for "countable". Hence, while the original $\mathbb{T}$ has no rank (seen easily from the free algebras), $\mathbb{T}$ admits "truncations" with a rank. In fact all triples in $\mathcal{S}$ admit truncations of rank $\aleph$ if $\aleph$ is a regular cardinal, see [23]. (For the definition of rank see 2.2.6).

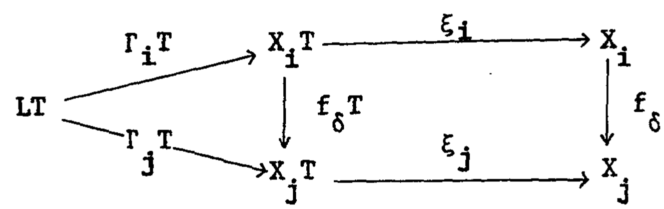
Also, we should point out that the category of complete lattices is not the same as the category of complete semilattices; in the former, homomorphisms must also preserve infs. If X is a topological space, the inclusion of the open sets in 2^X is a sup-preserving inf-destroying map.

§1.2 Properties of $U^{\mathbb{T}}$.

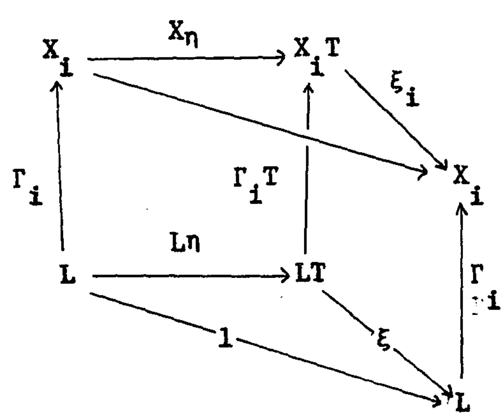
Fix a triple $\mathbb{T} = (T,\eta,\mu)$ in a category $\mathcal{K}$.

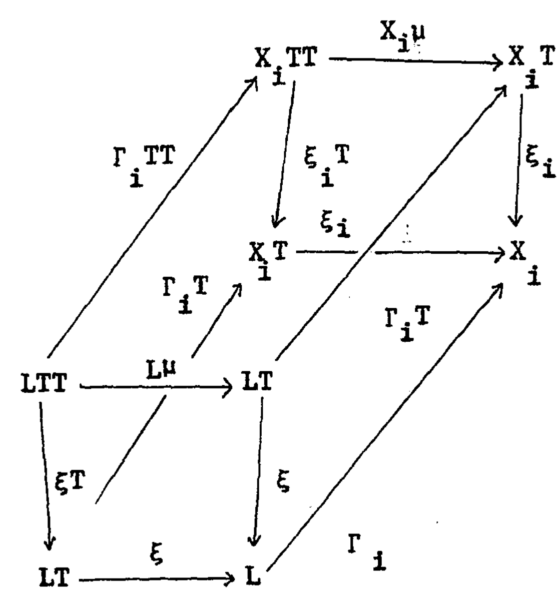
1.2.1 Proposition. $U^{\mathbb{T}}$ creates $\varprojlim$'s.

Proof. Suppose $\Delta \xrightarrow{D} \mathcal{K}^{\mathbb{T}}$ is a functor and $L \xrightarrow{\Gamma_i} X_i$ is a model for $\varprojlim DU^{\mathbb{T}}$. For every $i \overset{\delta}{\to} j \in \Delta$ we have



which induces a unique $\mathcal{K}$-morphism ξ such that $\Gamma_i T.\xi_i = \xi.\Gamma_i$ for all i. We have



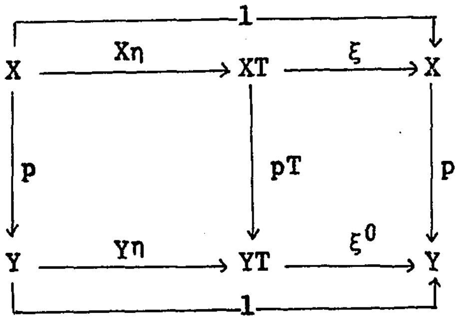


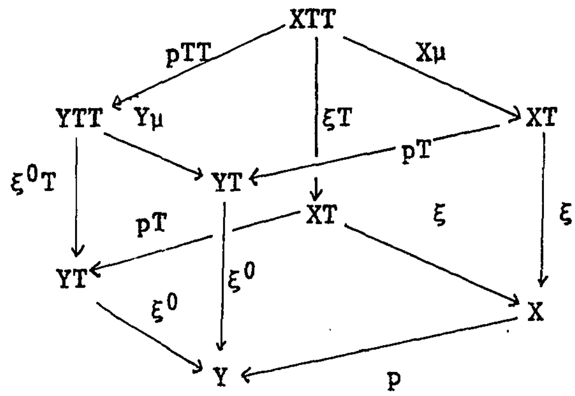
where all commutes except possibly the front faces which then commute since they do so followed by each Γ_i. This proves (L,ξ) is a $\mathbb{T}$-algebra, and each Γ_i is a $\mathbb{T}$-homomorphism. The same sort of argument shows that if (Y,θ) is a $\mathbb{T}$-algebra and if $Y \xrightarrow{f} L$ is a $\mathcal{K}$-morphism then f is a $\mathbb{T}$-homomorphism iff $f.\Gamma_i$ is a $\mathbb{T}$-homomorphism for all i, from which it follows that $(L,\xi) \xrightarrow{\Gamma} D = \varprojlim D$ in $\mathcal{K}^{\mathbb{T}}$. To complete the proof, suppose $(\tilde{L},\tilde{\xi}) \xrightarrow{\tilde{\Gamma}} D$ is a natural transformation with $\tilde{\Gamma}U^{\mathbb{T}} = \Gamma$, and show $\tilde{\Gamma} = (L,\xi) \xrightarrow{\Gamma} D$. As $U^{\mathbb{T}}$ is faithful it is enough to show $\tilde{\xi} = \xi$. But this is clear from the definition of ξ and the fact that $(L,\tilde{\xi}) \xrightarrow{\Gamma_i} D_i$ is a $\mathbb{T}$-homomorphism for all i. []

<u>1.2.2 Subalgebras.</u> Let (X,ξ) be a $\mathbb{T}$-algebra and let $A \rightarrowtail^{i} X$ be a $\mathcal{K}$-monomorphism. Say that i (or by abuse of language, A) is a <u>subalgebra of</u> (X,) if there exists a $\mathcal{K}$-morphism $AT \xrightarrow{\xi_0} A$ such that $\xi_0.i = iT.\xi$, and denote this by "$(A,\xi_0) \leq (X,\xi)$". Clearly such ξ_0 is unique when it exists. To prove that (A,ξ_0) is a $\mathbb{T}$-algebra, and hence that i is a $\mathbb{T}$-homomorphism, use the same diagrams as in 1.2.1 replacing Γ_i by i. As $U^{\mathbb{T}}$ is faithful, $U^{\mathbb{T}}$ reflects monomorphisms; as $U^{\mathbb{T}}$ has a left adjoint, $U^{\mathbb{T}}$ preserves monomorphisms; therefore a subalgebra is precisely a $\mathcal{K}^{\mathbb{T}}$-monomorphism.

<u>1.2.3 Quotient algebras.</u> If $(X,\xi) \xrightarrow{p} (Y,\theta)$ is a $\mathbb{T}$-homomorphism, say that p (or by abuse of language (Y,θ)) is a <u>quotient of</u> (X,ξ) if $X \xrightarrow{p} \!\!\!\!\rightarrow Y$ is a $\mathcal{K}$-epimorphism. This implies that $(X,\xi) \xrightarrow{p} \!\!\!\!\rightarrow (Y,\theta)$ is a $\mathcal{K}^{\mathbb{T}}$-epimorphism, but the converse is false; indeed the inclusion map of the natural numbers in the integers is an epimorphism in the category of monoids, as is easy to show. Various classifications of $\mathcal{K}$-epimorphisms induce corresponding notions of quotient algebras such as "regular quotient", "split quotient", etc..

Given (X,ξ) in obj $\mathcal{K}^{\mathbb{T}}$, and $X \xrightarrow{p} Y$ $\mathcal{K}$-epi, we cannot in general say that p is or is not a quotient accordingly as there exists ξ^0 such that $pT.\xi^0 = \xi.p$. For one thing, it is not clear that ξ^0 would be unique, although it would be clear if pT were epi. If such ξ^0 does exist, then we have from the diagrams

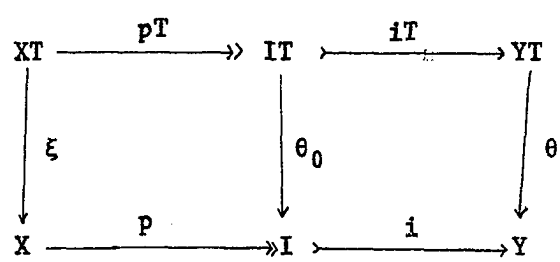




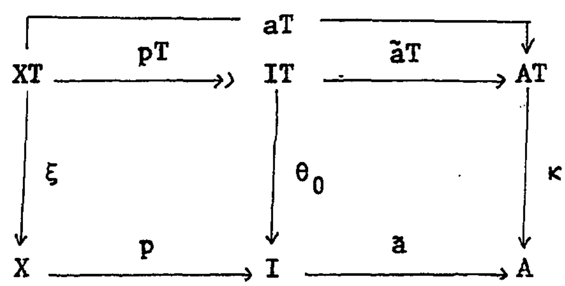
that (X,ξ^0) is a $\mathbb{T}$-algebra providing pTT is epi. Hence the situation for quotients is as well behaved as for subalgebras providing T preserves epimorphisms.

<u>1.2.4 Proposition.</u> Let T preserve regular coimage factorizations. Then $U^{\mathbb{T}}$ creates regular coimage factorizations.

<u>Proof.</u> Let $(X,\xi) \xrightarrow{f} (Y,\theta)$ be a $\mathbb{T}$-homomorphism, and suppose $X \xrightarrow{f} Y$ has regular coimage factorization $f = X \xrightarrow{p} I \rightarrowtail{i} Y$. By hypothesis, $fT = XT \xrightarrow{pT} IT \xrightarrow{iT} YT$ is a regular coimage factorization.



Since $\xi.f = fT.\theta$ and i is mono, $\xi.p$ is in reg(pT) which induces unique θ_0 with $pT.\theta_0 = \xi.p$. $\theta_0.i = iT.\theta$ as pT is epi. We have $(X,\xi) \xrightarrow{p} (I,\theta_0) \xrightarrow{i} (Y,\theta)$ and that θ_0 is unique with this property. To complete the proof we have only to show that $(X,\xi) \xrightarrow{p} (I,\theta_0)$ is regular in $\mathcal{K}^{\mathbb{T}}$. Let $(X,\xi) \xrightarrow{a} (A,\kappa) \in \text{reg}_{\mathbb{T}}(p)$. Suppose $(\zeta,\chi) : B \longrightarrow X$ are $\mathcal{K}$-morphisms with $\zeta.p = \chi.p$. Let $\tilde{\zeta},\tilde{\chi}$ be the induced homomorphic extensions. Since $\tilde{\zeta}.p$, $\tilde{\chi}.p$ are homomorphisms agreeing on generators, $\tilde{\zeta}.p = \tilde{\chi}.p$. By the hypothesis on a, $\tilde{\zeta}.a = \tilde{\chi}.a$, so $\zeta.a = B\eta.\tilde{\zeta}.a = B\eta.\tilde{\chi}.a = \chi.a$. This proves $a \in \text{reg}_{\mathcal{K}}(p)$. As $X \xrightarrow{p} I$ is a regular epimorphism in $\mathcal{K}$ there exists unique $\mathcal{K}$-morphism $\bar{a}$ with $p.\bar{a} = a$. Consulting the diagram,



since a is a $\mathbb{T}$-homomorphism and pT is epi, $\tilde{a}$ is a $\mathbb{T}$-homomorphism. []

<u>1.2.5 Definition.</u> $\mathbb{T}$ is a <u>regular</u> triple if $\mathcal{K}$ is a regular category and if T preserves regular coimage factorizations.

Most of the triples that we consider in this paper are regular.

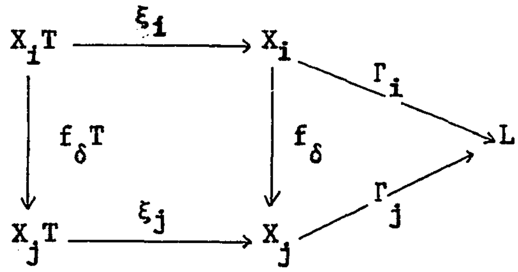
<u>1.2.6 Proposition.</u> If $\mathbb{T}$ is a regular triple then $\mathcal{K}^{\mathbb{T}}$ is a regular category.

<u>Proof.</u> For REG 1, use 1.2.4; for REG 2 use 1.2.1; for REG 3, $U^{\mathbb{T}}$ is faithful; since T preserves regular epis, the reasoning of 1.2.3

induces an injection [regular quotients of (X,ξ)] $\rightarrowtail$ [regular quotients of X], which takes care of REG 4. []

<u>1.2.7 Proposition.</u> Let $\mathcal{F}$ be a class of $\mathcal{K}^{\mathbb{T}}$-valued functors such that T preserves $\varinjlim$'s of elements of $\mathcal{F}U^{\mathbb{T}}$. Then $U^{\mathbb{T}}$ creates $\varinjlim$'s of elements of $\mathcal{F}$.

<u>Proof.</u> Let $\Delta \xrightarrow{D} \mathcal{K}^{\mathbb{T}} \in \mathcal{F}$. Suppose $X_i \xrightarrow{\Gamma_i} L = \varinjlim DU^{\mathbb{T}}$. By hypothesis, $X_iT \xrightarrow{\Gamma_i T} LT = \varinjlim DU^{\mathbb{T}}T$. For every $i \xrightarrow{\delta} j \in \Delta$ we have



which induces a unique $\mathcal{K}$-morphism ξ such that $\Gamma_iT.\xi = \xi_i.\Gamma_i$ for all i. The proof that (L,ξ) is a $\mathbb{T}$-algebra uses the same reasoning as "(X,ξ^0) is a $\mathbb{T}$-algebra" in 1.2.3. That ξ is the only structure map making each Γ_i a $\mathbb{T}$-homomorphism is clear. To complete the proof we must show $(X_i,\xi_i) \xrightarrow{\Gamma_i} (L,\xi) = \varinjlim D$. A natural transformation upstairs induces a map downstairs which is a $\mathbb{T}$-homomorphism using the same reasoning as "a is a $\mathbb{T}$-homomorphism" in 1.2.4. []

<u>1.2.8 Proposition.</u> Let $X \in$ obj $\mathcal{K}$, $XT \xrightarrow{\xi} X \in \mathcal{K}$. The following statements are pairwise equivalent.

a. (X,ξ) is a $\mathbb{T}$-algebra.

b. $(X\mu, \xi T, \xi)$ is a contractible coequalizer in $\mathcal{K}$.

c. $\xi = \text{coeq}(X\mu, \xi T)$ in $\mathcal{K}$.

<u>Proof.</u> <u>a implies b.</u>

$$\begin{array}{ccccc} & & \overset{1}{\overbrace{\qquad\qquad\qquad\qquad}} & & \\ XT & \xrightarrow{XT\eta} & XTT & \xrightarrow{X\mu} & XT \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi T} & & \downarrow{\scriptstyle\xi} \\ X & \xrightarrow{X\eta} & XT & \xrightarrow{\xi} & X \\ & & \underset{1}{\underbrace{\qquad\qquad\qquad\qquad}} & & \end{array}$$

<u>b implies c.</u> This follows from 0.7.2.

<u>c implies a.</u> $X\mu.\xi = \xi T \circ \xi$ by hypothesis. We have all of the diagram of "a implies b" except $X\eta.\xi = 1$ which then follows because ξ is epi. []

<u>1.2.9 Precise tripleability theorem (Jon Beck, [3]).</u> Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be a functor. The following statements are equivalent.

a. U is tripleable.

b. U has a left adjoint and U creates coequalizers of U-contractible pairs.

<u>Proof.</u> <u>a implies b.</u> We may assume without loss of generality that $U = U^{\mathbb{T}}$. $U^{\mathbb{T}}$ has left adjoint $F^{\mathbb{T}}$. It is immediate that $U^{\mathbb{T}}$ creates coequalizers of $U^{\mathbb{T}}$-contractible pairs from 0.7.3 and 1.2.7.

<u>b implies a.</u> There exists $F \dashv U$ with adjunctions $1_{\mathcal{K}} \xrightarrow{\eta} FU$, $FU \xrightarrow{\varepsilon} 1_{\mathcal{A}}$, and induced triple $\mathbb{T} \doteq (T, \eta, \mu) = (FU, \eta, F\varepsilon U)$. Define a functor $\mathcal{A} \xrightarrow{\Phi} \mathcal{K}^{\mathbb{T}}$ by $(A \xrightarrow{f} B)\Phi =_{df} (AU, A\varepsilon U) \xrightarrow{fU} (BU, B\varepsilon U)$. That Φ is a well-defined functor such that $\Phi U^{\mathbb{T}} = U$, follows from the three diagrams:

$$\begin{array}{ccc} AUFUFU & \xrightarrow{AUF\varepsilon U} & AUFU \\ \downarrow{\scriptstyle A\varepsilon UFU} & & \downarrow{\scriptstyle A\varepsilon U} \\ AUFU & \xrightarrow{A\varepsilon U} & AU \end{array} \qquad \begin{array}{ccc} AUFU & \xrightarrow{fUFU} & BUFU \\ \downarrow{\scriptstyle A\varepsilon U} & & \downarrow{\scriptstyle B\varepsilon U} \\ AU & \xrightarrow{fU} & BU \end{array} \qquad \begin{array}{ccc} AU & \xrightarrow{AU\eta} & AUFU \\ & \searrow{\scriptstyle 1} & \downarrow{\scriptstyle A\varepsilon U} \\ & & AU \end{array}$$

Define $\mathcal{K}^{\mathbb{T}} \xrightarrow{\Phi^{-1}} \mathcal{A}$ as follows. Let $(X,\xi) \in$ obj $\mathcal{K}^{\mathbb{T}}$. We have

$$(XFUF \underset{\xi F}{\overset{XF\varepsilon}{\rightrightarrows}} F)U = XTT \underset{\xi T}{\overset{X\mu}{\rightrightarrows}} XT$$

so that by 1.2.8, $(XF\varepsilon, \xi F)$ is a U-contractible pair, U of which has as coequalizer $XFU \xrightarrow{\xi} X$. By the hypothesis on U there exists a unique $\mathcal{A}$-morphism $XF \xrightarrow{\tilde{\xi}} (X,\xi)\Phi^{-1}$, U of which is ξ; moreover, $\tilde{\xi} = \text{coeq}(XF\varepsilon, \xi F)$. Before defining Φ^{-1} on morphisms, we verify that Φ^{-1} is indeed inverse to Φ on objects. If $A \in$ obj $\mathcal{A}$, the fact that $AUF \xrightarrow{A\varepsilon} A$ is an $\mathcal{A}$-morphism U of which is $AUFU \xrightarrow{A\varepsilon U} AU$ proves that $A\Phi\Phi^{-1} = A$. Now let $(X,\xi) \in \mathcal{K}^{\mathbb{T}}$. Because $\Phi U^{\mathbb{T}} = U$ we have $(XF \xrightarrow{\tilde{\xi}} (X,\xi)\Phi^{-1})\Phi =$ $(XT, X\mu) \xrightarrow{\xi} (X, (X,\xi)\Phi^{-1}\varepsilon U)$. But as $(XT,X\mu) \xrightarrow{\xi} (X,\xi)$ is a created coequalizer (by 1.2.8 and "a implies b") we must have $(X,\xi)\Phi^{-1}\varepsilon U = \xi$, and $(X,\xi)\Phi^{-1}\Phi = (X,\xi)$. Now we define Φ^{-1} on morphisms. Let $(X,\xi) \xrightarrow{f} (Y,\theta)$ be a $\mathbb{T}$-homomorphism.

$$\begin{array}{ccccc} XFUF & \underset{\xi F}{\overset{XF\varepsilon}{\rightrightarrows}} & XF & \xrightarrow{\tilde{\xi}} & (X,\xi)\Phi^{-1} \\ & & \downarrow{\scriptstyle fF} & & \downarrow{\scriptstyle f\Phi^{-1}} \\ & & YF & \xrightarrow{\tilde{\theta}} & (Y,\theta)\Phi^{-1} \end{array}$$

$fT.\theta = \xi.f$, and therefore $(XF\varepsilon.fF.\tilde{\theta})U = X\mu.fT.\theta = X\mu.\xi.f = \xi T.\xi.f =$ $\xi T.fT.\theta = (\xi F.fF.\tilde{\theta})U$. Now in the proof that $\Phi^{-1} = 1$ on objects we in fact proved that AUF $\xrightarrow{A\varepsilon}$ A = coeq(AUFε,AεUF) in $\mathcal{A}$ for all $\mathcal{A}$-objects A. In particular the adjunction $UF \xrightarrow{\varepsilon} 1_{\mathcal{A}}$ is pointwise $\mathcal{A}$-epi, or equivalently, U is faithful. Hence it follows that $XF\varepsilon.fF.\tilde{\theta} = \xi F.fF.\tilde{\theta}$. $f\Phi^{-1}$ is then induced by the coequalizer property, and this clearly makes Φ^{-1} into a functor. The fact that ε is natural:

$$\begin{array}{ccc} AUF & \xrightarrow{A\varepsilon} & A \\ \Big\downarrow{\scriptstyle fUF} & & \Big\downarrow{\scriptstyle f} \\ BUF & \xrightarrow{B\varepsilon} & B \end{array}$$

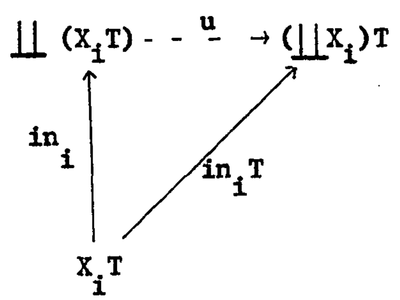
proves $\Phi\Phi^{-1} = 1_{\mathcal{A}}$ on morphisms. Summarizing, we have so far proved that Φ is bijective on objects, full and that $\Phi U^{\mathbb{T}} = U$. Since U is faithful so is Φ, and this completes the proof. []

§1.3 $\varinjlim$'s in $\mathcal{K}^{\mathbb{T}}$.

Fix a triple $\mathbb{T} = (T,\eta,\mu)$ in $\mathcal{K}$.

1.3.1 Proposition (Linton). Let $\mathcal{K}$ have coproducts and let every reflexive pair in $\mathcal{K}^{\mathbb{T}}$ have a coequalizer. Then $\mathcal{K}^{\mathbb{T}}$ has $\varinjlim$'s.

Proof. By 0.6.2 it is sufficient to show $\mathcal{K}^{\mathbb{T}}$ has coproducts. If ϕ is an initial object in $\mathcal{K}$, then $(\phi T,\phi\mu)$ is initial in $\mathcal{K}^{\mathbb{T}}$ with no assumptions needed; this takes care of the empty coproduct. Now let $[(X_i,\xi_i) : i \in I]$ be a non-empty set of $\mathbb{T}$-algebras. Define a $\mathcal{K}$-morphism u and $\mathbb{T}$-homomorphisms ζ,χ by



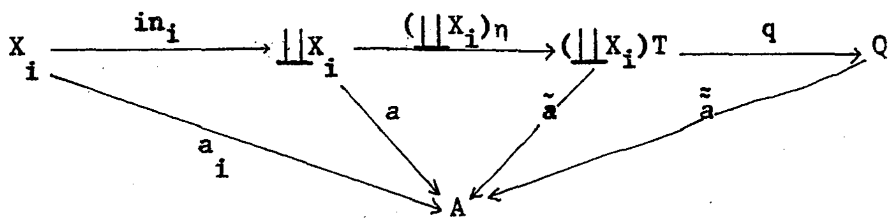
$$\zeta = [\textstyle\coprod(X_iT)]T \xrightarrow{uT} (\coprod X_i)TT \xrightarrow{(\coprod X_i)\mu} (\coprod X_i)T$$

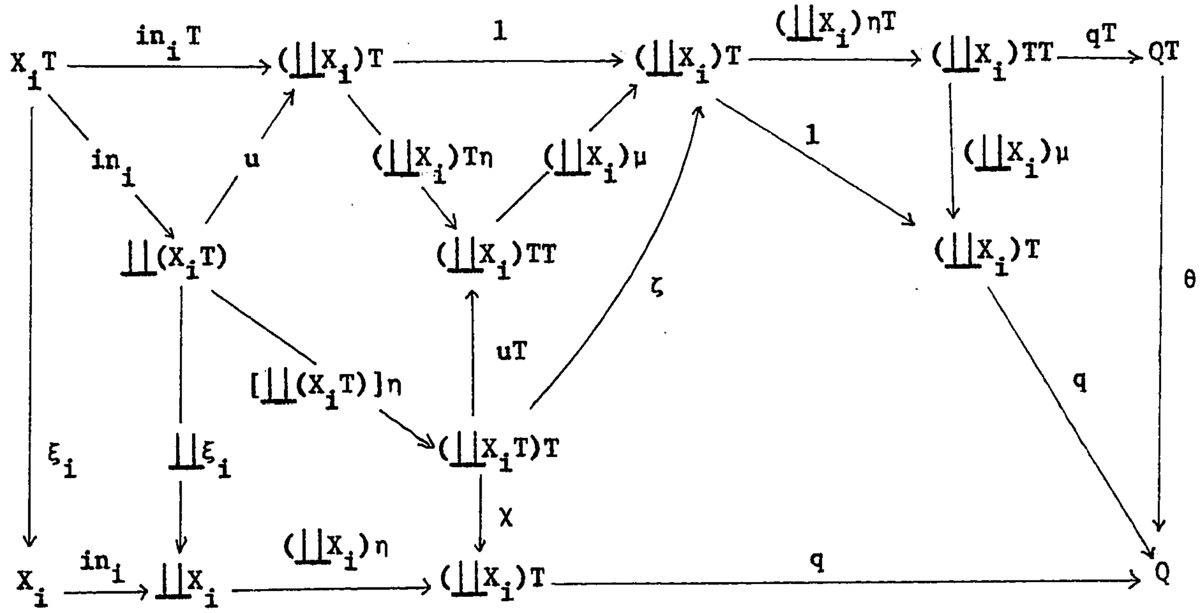
$$\chi = [\textstyle\coprod(X_iT)]T \xrightarrow{(\coprod\xi_i)T} (\coprod X_i)T$$

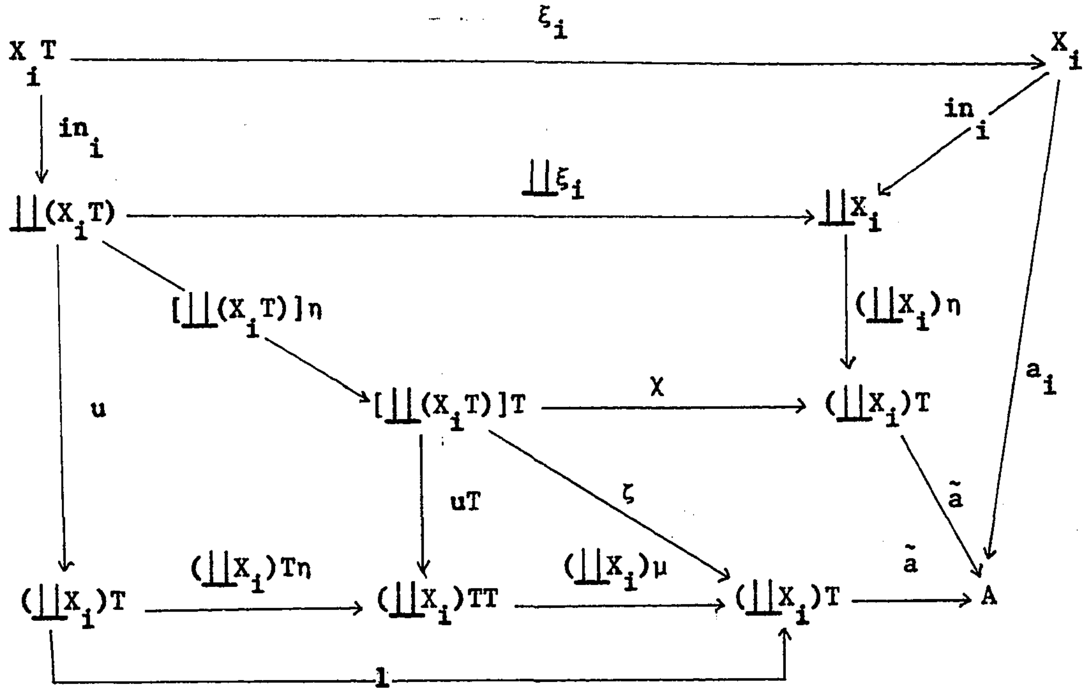
It is easy to see that $\coprod(X_i\eta).u = (\coprod X_i)\eta$ and then that (ζ,χ) is a reflexive pair with $d =_{df} (\coprod X_i)T \xrightarrow{[\coprod(X_i\eta)]T} [\coprod(X_iT)]T$. By hypothesis, let $((\coprod X_i)T, (\coprod X_i)\mu) \xrightarrow{q} (Q,\theta) = \mathrm{coeq}(\zeta,\chi)$ in $\mathcal{K}^{\mathbb{T}}$. We will show $(Q,\theta) = \coprod(X_i,\xi_i)$ with injections

$$X_i \xrightarrow{in_i} \textstyle\coprod X_i \xrightarrow{(\coprod X_i)\eta} (\coprod X_i)T \xrightarrow{q} Q$$

Consider the diagrams:







The second diagram proves that the injections are, in fact, $\mathbb{T}$-homomorphisms. Now suppose given $\mathbb{T}$-homomorphisms $((X_i,\xi_i) \xrightarrow{a_i} (A,\rho) : i \in I)$, and refer to the first diagram. A unique $\mathcal{K}$-morphism is induced with $in_i . a = a_i$ for all i. Let $\tilde{\tilde{a}}$ be the homomorphic extension of a. To complete the proof we have only to show $\zeta.\tilde{a} = \chi.\tilde{a}$. Noting that $in_i.u.\tilde{a} = in_i T.aT.\rho = a_i T.\rho = \xi_i.a_i$ for all i, this follows at once from the third diagram. []

<u>1.3.2 Corollary.</u> If $\mathcal{K}$ has $\varinjlim$'s and if T preserves coequalizers of reflexive pairs, then $\mathcal{K}^{\mathbb{T}}$ has $\varinjlim$'s.

<u>Proof.</u> Use 1.2.7 and 1.3.1. []

<u>1.3.3 Corollary.</u> If $\mathbb{T}$ is a regular triple and if $\mathcal{K}$ has coproducts, then $\mathcal{K}^{\mathbb{T}}$ has $\varinjlim$'s.

<u>Proof.</u> Use 1.2.6, 0.5.2 and 1.3.1. []

<u>1.3.4 Proposition.</u> Let $\Delta \xrightarrow{D} \mathcal{K}$ be a diagram with $X_i \xrightarrow{e_i} L = \varinjlim D$. Then $(X_i T, X_i \mu) \xrightarrow{e_i T} (LT, L\mu) = \varinjlim DF^{\mathbb{T}}$.

<u>Proof.</u> $F^{\mathbb{T}}$ preserves $\varinjlim$'s because it has $U^{\mathbb{T}}$ for a right adjoint. []

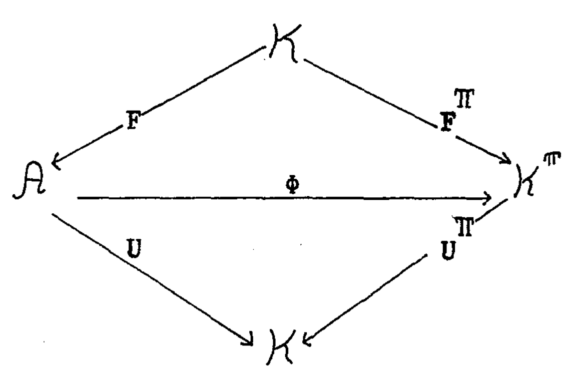
§1.4 Algebraic functors and morphisms of triples.

In this section we generalize the "structure-semantics" theorems of [20], [25] using the triple maps of [1].

1.4.1 Definitions. The category of adjoint pairs, denoted "AD", has as its objects functors $\mathcal{A} \xrightarrow{U} \mathcal{K}$ together with specified left adjointnesses $\mathcal{K} \xrightarrow{F} \mathcal{A}$, $1_{\mathcal{K}} \xrightarrow{\eta} FU$, $UF \xrightarrow{\varepsilon} 1_{\mathcal{A}}$, whereas a map from $\mathcal{A} \xrightarrow{U} \mathcal{K}$ to $\mathcal{A}' \xrightarrow{U'} \mathcal{K}'$ (the remaining data being understood) is a pair of functors $(H,\bar{H})$ yielding a commutative square: $\bar{H}U' = UH$. With the evident composition, AD is a category. The category of algebraic categories, denoted "AL", is the full subcategory of AD generated by objects of form $\mathcal{K}^{\mathbb{T}} \xrightarrow{U^{\mathbb{T}}} \mathcal{K}$, $\mathcal{K} \xrightarrow{F^{\mathbb{T}}} \mathcal{K}^{\mathbb{T}}$, $1_{\mathcal{K}} \xrightarrow{\eta} T$, $UF \xrightarrow{\varepsilon^{\mathbb{T}}} 1_{\mathcal{K}^{\mathbb{T}}}$ for some triple $\mathbb{T}$ in some category $\mathcal{K}$.

If $\mathcal{K}$ is a category, AD($\mathcal{K}$) $=_{df}$ the subcategory of AD whose morphisms are of form $(1_{\mathcal{K}},\bar{H})$, and then AL($\mathcal{K}$) $=_{df}$ the subcategory AL $\cap$ AD($\mathcal{K}$). Loosely speaking, AD($\mathcal{K}$) is the fiber over $\mathcal{K}$ in AD, and AL($\mathcal{K}$) similarly. IF $(H,\bar{H})$ is an AL-morphism, $\bar{H}$ is called an H-algebraic functor. If $(1_{\mathcal{K}},\bar{H})$ is a morphism in AL($\mathcal{K}$), $\bar{H}$ is called an algebraic functor.

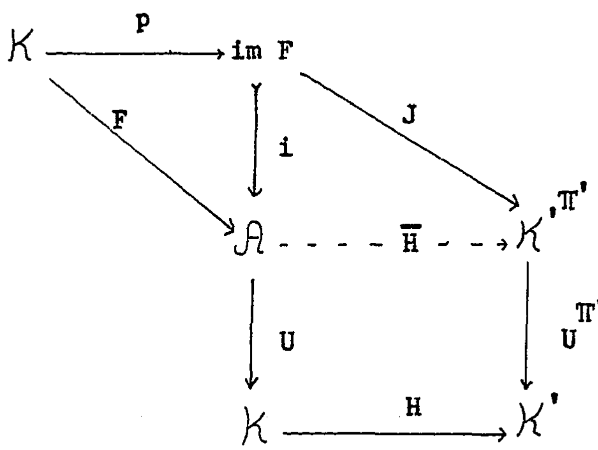
Let $\mathcal{A} \xrightarrow{U} \mathcal{K} \varepsilon$ obj AD, and let $\mathbb{T} = (T,\eta,\mu) = (FU,\eta,F\varepsilon U)$ be the induced triple. We have the functor Φ:



$A\Phi = (AU, A\varepsilon U)$

used in the proof of 1.2.9; it was proved there that Φ is well-defined and that $\Phi U^{\mathbb{T}} = U$; it is obvious in fact that $F\Phi = F^{\mathbb{T}}$. The AD-morphism $(1_{\mathcal{K}}, \Phi)$ from U to $U^{\mathbb{T}}$ is called the <u>canonical reflection of</u> U <u>in</u> AL. We will prove that it is a reflection in 1.4.3.

<u>1.4.2 Proposition.</u> Suppose given a commutative diagram of functors



with $F \dashv U$ in obj AD, $U^{\mathbb{T}'}$ in obj AL and $\mathrm{im}\, f \overset{i}{\rightarrowtail} \mathcal{A}$ the full subcategory of $\mathcal{A}$ generated by objects $[XF \,;\, X \in |\mathcal{K}|]$. Then there exists a unique functor $\bar{H}$ such that $\bar{H}U^{\mathbb{T}'} = UH$ and $i\bar{H} = J$.

<u>Proof.</u> Let $\mathbb{T} = (T, \eta, \mu) = (FU, \eta, F\varepsilon U)$ be the triple in $\mathcal{K}$ induced by $F \dashv U$. Let $A \in \mathrm{obj}\ \mathcal{A}$. Since $(AU, A\varepsilon U)$ is a $\mathbb{T}$-algebra, and in view of 1.2.8,

$$(AUFUF \underset{A\varepsilon UF}{\overset{AUF\varepsilon}{\rightrightarrows}} AUF \xrightarrow{A\varepsilon} A)U = AUTT \underset{A\varepsilon UT}{\overset{AUF\varepsilon U}{\rightrightarrows}} AUT \xrightarrow{A\varepsilon U} AU$$

is a contractible coequalizer in $\mathcal{K}$. By 0.7.2, 0.7.3 and the fact that $iUH = JU^{\mathbb{T}'}$ we have that $AUTH \xrightarrow{A\varepsilon UH} AUH$ is the coequalizer of the $U^{\mathbb{T}'}$-contractible pair $AUFUpJ \underset{A\varepsilon UpJ}{\overset{AUF\varepsilon J}{\rightrightarrows}} AUpJ$. It follows from 1.2.9 that there exists a unique $\mathbb{T}'$-homomorphism $AUpJ \xrightarrow{a} A\bar{H}$ with

$aU^{\mathbb{T}'} = AUTH \xrightarrow{A\varepsilon UH} AUH$; moreover, $a = \mathrm{coeq}(AUF\varepsilon J, A\varepsilon UpJ)$. If $A \xrightarrow{f} B$ is an $\mathcal{A}$-morphism, because ε is natural and a,b are coequalizers (see the diagram below) there exists unique $f\bar{H}$ with $a.f\bar{H} = fUpJ.b$, which makes $\bar{H}$ a well-defined functor.

$$\begin{array}{ccccc}
AUFUpJ & \overset{AUF\varepsilon J}{\underset{A\varepsilon UpJ}{\rightrightarrows}} & AUpJ & \xrightarrow{\;a\;} & A\bar{H} \\
\Big\downarrow{\scriptstyle fUFUpJ} & & \Big\downarrow{\scriptstyle fUpJ} & & \Big\downarrow{\scriptstyle f\bar{H}} \\
BUFUpJ & \overset{BUF\varepsilon J}{\underset{B\varepsilon UpJ}{\rightrightarrows}} & BUpJ & \xrightarrow{\;b\;} & B\bar{H}
\end{array}$$

Since $aU^{\mathbb{T}} = A\varepsilon UH$ is epi and both of the diagrams

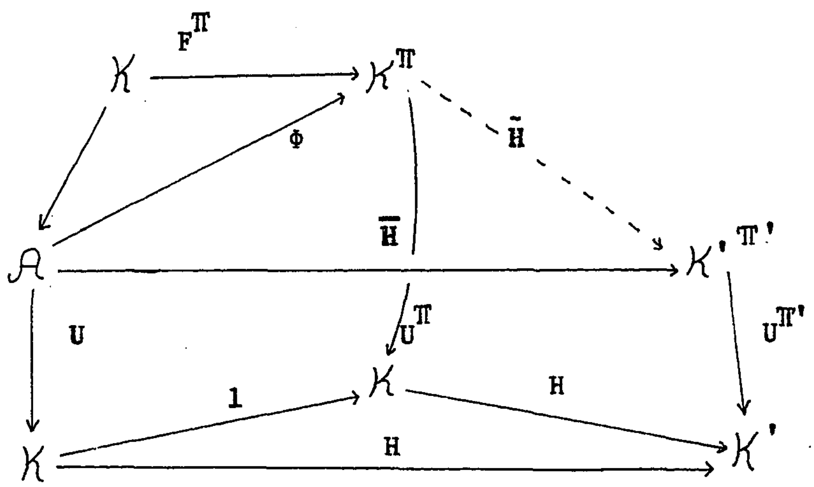
$$\begin{array}{ccc}
AUFUH & \xrightarrow{A\varepsilon UH} & AUH \\
\Big\downarrow{\scriptstyle fUFUH \,=\, fUpJU^{\mathbb{T}'}} & & \Big\downarrow{\scriptstyle f\bar{H}U} \\
BUFUH & \xrightarrow{B\varepsilon UH} & BUH
\end{array}
\qquad\qquad
\begin{array}{ccc}
AUFUH & \xrightarrow{A\varepsilon UH} & AUH \\
\Big\downarrow{\scriptstyle fUFUH} & & \Big\downarrow{\scriptstyle fUH} \\
BUFUH & \xrightarrow{B\varepsilon UH} & BUH
\end{array}$$

commute, $f\bar{H}U^{\mathbb{T}} = fUH$ for arbitrary $A \xrightarrow{f} B \in \mathcal{A}$, that is $\bar{H}U^{\mathbb{T}} = UH$. Let $X \in \mathrm{obj}\,\mathcal{K}$. $XFUpJ \xrightarrow{x} XF\bar{H}$ is the unique $\mathbb{T}'$-homomorphism with domain $XFUpJ$ such that $xU^{\mathbb{T}'} = XFUFUH \xrightarrow{XF\varepsilon UH} XFUH = XFUFUH \xrightarrow{X\mu H} XFUH$. So in particular, $XFJ = XF\bar{H}$. This proves $i\bar{H} = J$ on objects. Since $i\bar{H}U^{\mathbb{T}'} = JU^{\mathbb{T}'}$ and $U^{\mathbb{T}'}$ is faithful, it follows that $i\bar{H} = J$. This completes the proof of existence. To prove uniqueness, suppose $i\tilde{H} = J$, $UH = \tilde{H}U^{\mathbb{T}'}$ and show $\bar{H} = \tilde{H}$. As in the preceding paragraph, we need only show $\bar{H} = \tilde{H}$ on objects. Let $A \in \mathrm{obj}\,\mathcal{A}$. Then $(AUF\tilde{H} \xrightarrow{A\varepsilon\tilde{H}} A\tilde{H})U^{\mathbb{T}'} = AUFUH \xrightarrow{A\varepsilon UH} AUH$. But $AUF\tilde{H} = AUpJ$, and therefore $A\tilde{H} = A\bar{H}$. []

The next proposition is the main idea of "structure-semantics" theory. In our context, the inclusion AL $\rightarrow$ AD is the "semantics" functor and the reflector functor AD $\rightarrow$ AL resulting from passing to the canonical reflection is the "structure" functor.

1.4.3 Proposition. Let $\mathcal{A} \xrightarrow{U} \mathcal{K} \in$ obj AD with induced triple $\mathbb{T}$ and canonical reflection Φ. Then $U \xrightarrow{(1_{\mathcal{K}},\Phi)} U^{\mathbb{T}}$ is indeed a reflection of U in AL.

Proof. Suppose $U^{\mathbb{T}'} \in$ obj AL, $U \xrightarrow{(H,\overline{H})} U^{\mathbb{T}'} \in$ AD. We must prove there exists unique $\tilde{H}$ such that



commutes. The existence proof is much like that of 1.4.2. Let (X,ξ) be a $\mathbb{T}$-algebra. $(X\mu,\xi T,\xi)$ is a contractible coequalizer in $\mathcal{K}$, and hence $(X\mu H,\xi TH,\xi H)$ is a contractible coequalizer in $\mathcal{K}'$, and so $XTH \xrightarrow{\xi H} XH$ is the coequalizer of the $U^{\mathbb{T}'}$-contractible pair $XFUF\overline{H} \underset{\xi F\overline{H}}{\overset{XF\varepsilon\overline{H}}{\rightrightarrows}} XF\overline{H}$, and there exists unique $\mathbb{T}'$-homomorphism $XF\overline{H} \xrightarrow{x} (X,\xi)\tilde{H}$ with domain $XF\overline{H}$ and such that $xU^{\mathbb{T}'} = \xi H$; further, $x = \text{coeq}(XF\varepsilon\overline{H},\xi F\overline{H})$ so that each $\mathbb{T}$-homomorphism $(X,\xi) \xrightarrow{f} (Y,\theta)$ induces unique $f\tilde{H}$ such that $x.f\tilde{H} = fF\overline{H}.y$, as in the prove of 1.4.2;

also use the reasoning of 1.4.2 to prove that $\tilde{H}U^{\mathbb{T}'} = U^{\mathbb{T}}H$. Now consider:

$$\begin{array}{ccc} \operatorname{im} F^{\mathbb{T}} & \xrightarrow{\quad i^{\mathbb{T}} \quad} & \mathcal{K}^{\mathbb{T}} \\ \uparrow {\scriptstyle \Phi_0} & & \uparrow {\scriptstyle \Phi} \\ \operatorname{im} F & \xrightarrow{\quad i \quad} & \mathcal{A} \end{array}$$

Since $XF.\Phi = (XT, X\mu)$ for every $X \in \text{obj } \mathcal{K}$, Φ maps im F into im $F^{\mathbb{T}}$. Therefore, $\Phi\tilde{H}$ and $\bar{H}$ agree on im F and, by 1.4.2, indeed $\Phi\tilde{H} = \bar{H}$. This proves existence. To prove uniqueness, suppose $\Phi\hat{H} = \bar{H}$, $\hat{H}U^{\mathbb{T}} = U^{\mathbb{T}}H$. Then $\Phi_0 i^{\mathbb{T}}\tilde{H} = \Phi_0 i^{\mathbb{T}}\hat{H}$. As Φ_0 is onto on objects, $i^{\mathbb{T}}\tilde{H} = i^{\mathbb{T}}\hat{H}$ on objects. But $i^{\mathbb{T}}\tilde{H}U^{\mathbb{T}'} = i^{\mathbb{T}}\hat{H}U^{\mathbb{T}'}$ and $U^{\mathbb{T}'}$ is faithful, so $i^{\mathbb{T}}\tilde{H} = i^{\mathbb{T}}\hat{H}$, and by 1.4.2 we have that $\tilde{H} = \hat{H}$. []

<u>1.4.4 Definitions.</u> Let $\mathbb{T}$, $\mathbb{T}'$ be triples in $\mathcal{K}$, $\mathcal{K}'$. $(H,\lambda) : \mathbb{T}' \longrightarrow \mathbb{T}$ is a <u>triple map</u> (or λ is an <u>H-triple map</u>) <u>from</u> $\mathbb{T}'$ <u>to</u> $\mathbb{T}$ if $H : \mathcal{K} \to \mathcal{K}'$ is a functor and $HT' \xrightarrow{\lambda} TH$ is a natural transformation satisfying TM1, TM2:

$$\begin{array}{ccc} HT' & \xrightarrow{\quad\lambda\quad} & TH \\ & \nwarrow {\scriptstyle H\eta'} \quad \text{TM1} \quad {\scriptstyle \eta H} \nearrow & \\ & H & \end{array} \qquad \begin{array}{ccccc} HT'T' & \xrightarrow{\lambda T'} & THT' & \xrightarrow{T\lambda} & TTH \\ \downarrow {\scriptstyle H\mu'} & & \text{TM2} & & \downarrow {\scriptstyle \mu H} \\ HT' & & \xrightarrow{\qquad\lambda\qquad} & & TH \end{array}$$

$(H,\Gamma) : \mathbb{T} \longrightarrow \mathbb{T}'$ is an <u>intrastructure</u> (or Γ is an <u>H-intrastructure</u>) <u>from</u> $\mathbb{T}$ <u>to</u> $\mathbb{T}'$ if $H : \mathcal{K} \longrightarrow \mathcal{K}'$ is a functor and $THT' \xrightarrow{\Gamma} TH$ is a natural transformation satisfying IN1, IN2, IN3:

$$
\begin{array}{ccc}
TH & \xrightarrow{TH\eta'} & THT' \\
& \searrow^{1} \;\; \text{IN1} & \downarrow \Gamma \\
& & TH
\end{array}
\qquad
\begin{array}{ccc}
THT'T' & \xrightarrow{TH\mu'} & THT' \\
\downarrow \Gamma T' & \text{IN2} & \downarrow \Gamma \\
THT' & \xrightarrow{\Gamma} & TH
\end{array}
\qquad
\begin{array}{ccc}
TTHT' & \xrightarrow{\mu HT'} & THT' \\
\downarrow T\Gamma & \text{IN3} & \downarrow \Gamma \\
TTH & \xrightarrow{\mu H} & TH
\end{array}
$$

(that is, $(XTH, X\Gamma)$ is a $\mathbb{T}'$-algebra and $X\mu H$ is a $\mathbb{T}'$-homomorphism). The <u>category of triples and triple maps</u>, denoted "OPTR" ("OP" because maps go backwards, cf. 1.4.5 below), has triples for objects, triple maps for morphisms, and composition $\mathbb{T}'' \xrightarrow{(H',\lambda')} \mathbb{T}' \xrightarrow{(H,\lambda)} \mathbb{T}$ $=_{df}$ $(HH',\ HH'T'' \xrightarrow{H\lambda'} HT'H' \xrightarrow{\lambda H'} THH')$. The <u>category of triples and intrastructures</u>, denoted "TR", has triples for objects, intrastructures for morphisms and composition $\mathbb{T} \xrightarrow{(H,\Gamma)} \mathbb{T}' \xrightarrow{(H',\Gamma')} \mathbb{T}''$ $=_{df}$ $(HH',\ THH'T'' \xrightarrow{TH\eta H'T''} THT'H'T'' \xrightarrow{TH\Gamma'} THT'H' \xrightarrow{\Gamma H'} THH')$. If $\mathcal{K}$ is a category, the subcategories OPTR($\mathcal{K}$), TR($\mathcal{K}$) are defined by considering only morphisms of form $(1_{\mathcal{K}}, \lambda)$, $(1_{\mathcal{K}}, \Gamma)$.

"Q_0" of the following proposition can be found in [1].

<u>1.4.5 Proposition.</u> TR, OPTR are, in fact, categories. The passages

$$
\begin{array}{ccc}
(\text{OPTR})^{op} & \xrightarrow{Q_0} & \text{AL} \\
(H,\lambda) & \longmapsto & (H, \overline{H}_\lambda)
\end{array}
\quad \text{where} \quad
\begin{array}{ccc}
\mathcal{K}^{\mathbb{T}} & \xrightarrow{\overline{H}_\lambda} & \mathcal{K}^{\mathbb{T}'} \\
(X,\xi) & & (XH, X\lambda.\xi H) \\
\downarrow f & \longmapsto & \downarrow fH
\end{array}
$$

$$
\begin{array}{ccc}
\text{AL} & \xrightarrow{Q_1} & \text{TR} \\
(H,\overline{H}) & \longmapsto & (H, \Gamma_{\overline{H}})
\end{array}
\quad \text{where} \quad (XTH, X\Gamma_{\overline{H}}) =_{df} (XT, X\mu)\overline{H}
$$

$$\mathrm{TR} \xrightarrow{\;Q_2\;} (\mathrm{OPTR})^{op}$$

$$(H,\Gamma) \longmapsto (H,\lambda_\Gamma) \quad \text{where} \quad \lambda_\Gamma =_{df} \eta HT'.\Gamma,$$

are cyclically-inverse (meaning all cycles = id) isomorphisms of categories. For each category $\mathcal{K}$ the Q_i's establish, by restriction, isomorphisms $\mathrm{OPTR}(\mathcal{K})^{op} \cong \mathrm{AL}(\mathcal{K}) \cong \mathrm{TR}(\mathcal{K})$.

<u>Proof.</u> The program for the proof is:

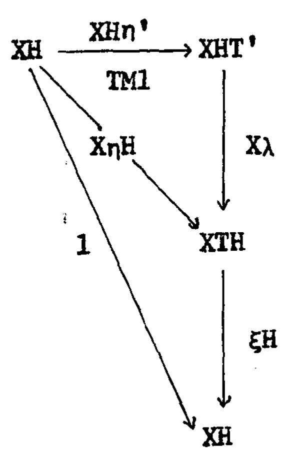
(a) Prove that the Q_i's are well-defined in the sense that $(H,\overline{H}_\lambda) \in |\mathrm{AL}|$, $(H,\Gamma_{\overline{H}}) \in |\mathrm{TR}|$, $(H,\lambda_\Gamma) \in |\mathrm{OPTR}|$;

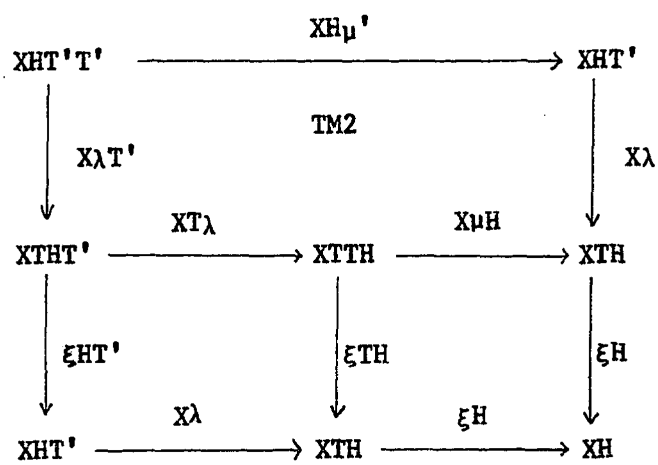
(b) prove $Q_iQ_{i+1}Q_{i+2} = \mathrm{id}$;

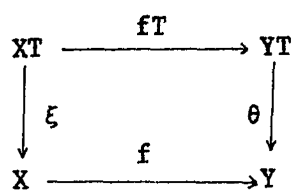
(c) prove $(HH',\overline{HH'})\,Q_1 = (H,\Gamma_{\overline{H}})\,(H',\Gamma_{\overline{H'}})$ and $(H,\Gamma_{\overline{H}})\,(H',\Gamma_{\overline{H'}})Q_2 = (H',\lambda_{\Gamma_{\overline{H'}}})\,(H,\lambda_{\Gamma_{\overline{H}}})$;

for the remaining details are clear.

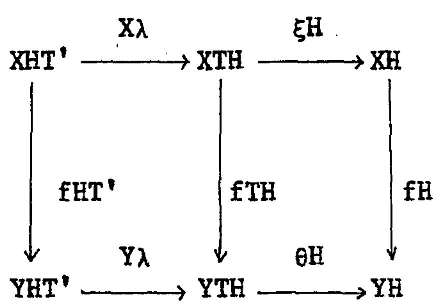
<u>Q_0 well-defined.</u>





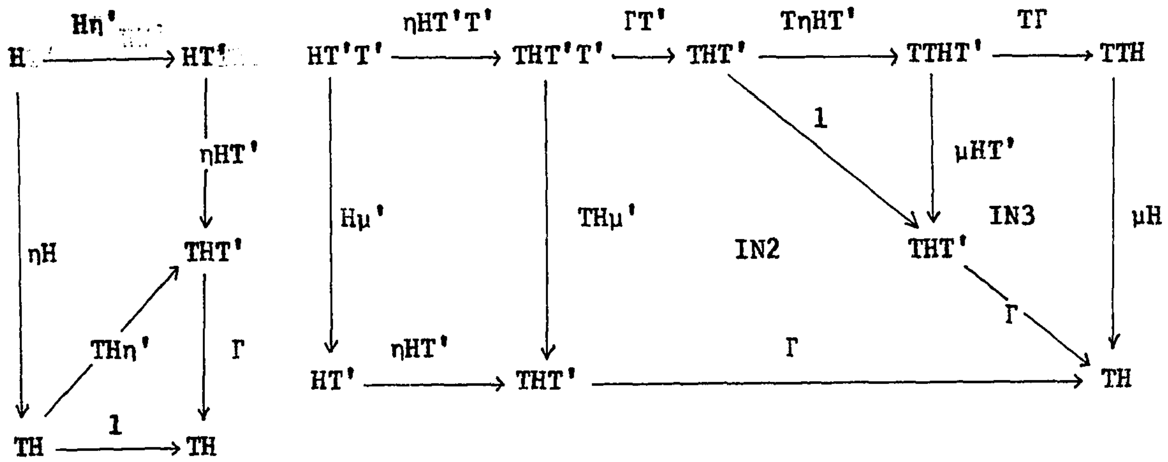


implies

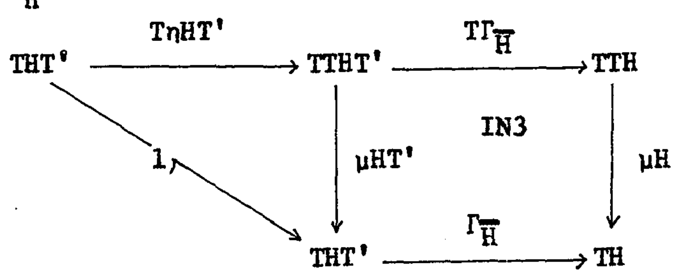


<u>Q_1 well-defined.</u> If $X \xrightarrow{f} Y \ \varepsilon\ \mathcal{K}$, $(XT,X\mu)\overline{H} \xrightarrow{f T\overline{H}} (YT,Y\mu)\overline{H}$ is in $\mathcal{K}^{'\mathbb{T}'}$, that is $fTHT'.Y\Gamma_{\overline{H}} = X\Gamma_{\overline{H}}.fTH$, and $\Gamma_{\overline{H}}$ is natural. IN1, IN2 and IN3 are clear (for IN3, notice that $X\mu\overline{H}U^{\mathbb{T}'} = X\mu H$).

<u>Q_2 well-defined.</u>



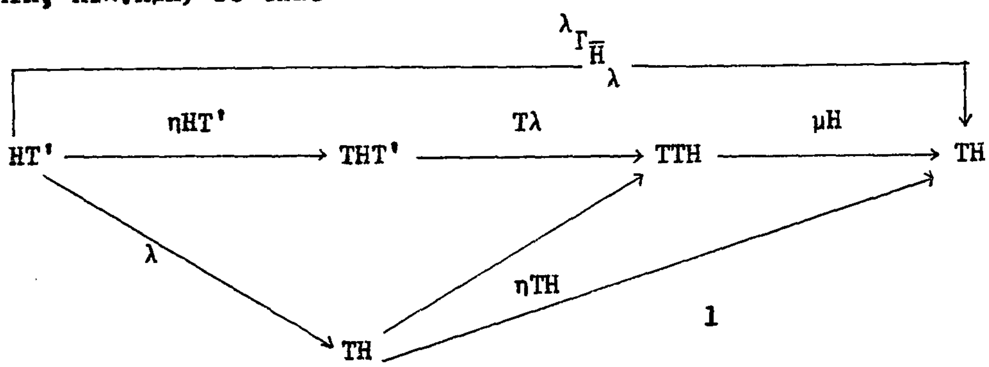
<u>$Q_1Q_2Q_0$ = id.</u> Let $(H,\overline{H})\ \varepsilon$ obj AL and let $X\ \varepsilon$ obj $\mathcal{K}$. $<(XT,X\mu),\ \overline{H}_{\lambda_{\Gamma_{\overline{H}}}}> = (XTH,\ XT\lambda_{\Gamma_{\overline{H}}}.X\mu H) = (XTH,\ XT\eta HT'.XT\Gamma_{\overline{H}}.X\mu H)$. But we have



therefore $<(XT,X\mu),\ \overline{H}_{\lambda_{\Gamma_{\overline{H}}}}> = (XTH,\ X\Gamma_{\overline{H}}) = (XT,X\mu)\overline{H}$. It follows from 1.4.2 that $\overline{H}\ Q_1Q_2Q_0 = \overline{H}$.

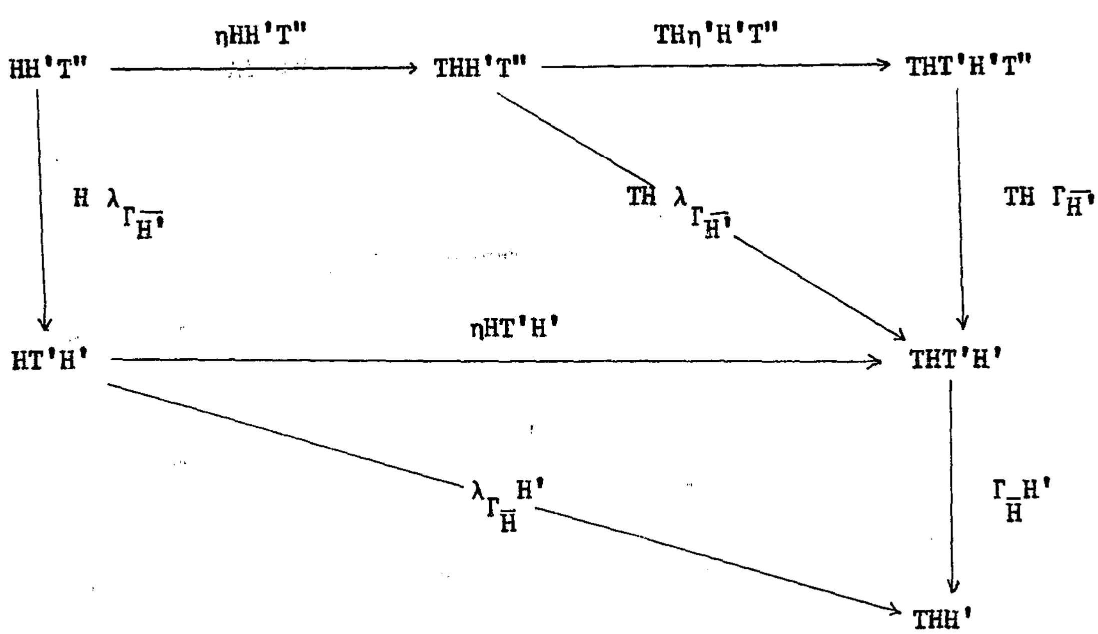
<u>$Q_2Q_0Q_1$ = id.</u> Let $(H,\Gamma)\ \varepsilon$ obj TR, and let $X\ \varepsilon$ obj $\mathcal{K}$. $(XT,X\mu)\overline{H}_{\lambda_\Gamma} = (XTH,\ X\lambda_\Gamma.X\mu H) = (XTH,\ XT\eta HT'.XT\Gamma.X\mu H)$ so that $\Gamma_{\overline{H}_{\lambda_\Gamma}} = T\eta HT'.T\Gamma.\mu H =$ (as just shown above) Γ.

<u>$Q_0Q_1Q_2 = id.$</u> Let $(H,\lambda) \in$ obj OPTR. $(XTH, X\Gamma_{\overline{H}_\lambda}) = (XT, X\mu)\overline{\overline{H}}_\lambda =$
$(XTH, XT\lambda . X\mu H)$ so that



<u>$(HH', \overline{HH'})Q_1 = (H, \Gamma_{\overline{H}})\ (H', \Gamma_{\overline{H'}})$.</u> $(XTHH', X\Gamma_{\overline{HH'}}) = (XT, X\mu)\overline{\overline{HH'}} =$
$(XTH, X\Gamma_{\overline{H}})\overline{\overline{H'}} = (XTHH', XTH\eta'H'T''.XTH\Gamma_{\overline{H'}}.X\Gamma_{\overline{H}}H')$.

<u>$(H, \Gamma_{\overline{H}})\ (H', \Gamma_{\overline{H'}})\ Q_2 = (H', \lambda_{\Gamma_{\overline{H'}}})\ (H, \lambda_{\Gamma_{\overline{H}}})$.</u>



[]

§1.5 Adjoints of algebraic functors.

1.5.1 Proposition. Let $\mathbb{T}$ be a triple in $\mathcal{K}$ such that $\mathcal{K}^{\mathbb{T}}$ has coequalizers of reflexive pairs. Let $\mathcal{K} \xrightarrow{H} \mathcal{K}'$ be a functor having a left adjoint $\hat{H}$ with adjunctions $1_{\mathcal{K}'} \xrightarrow{e} \hat{H}H$, $H\hat{H} \longrightarrow 1_{\mathcal{K}}$. Let $\mathbb{T}'$ be a triple in $\mathcal{K}'$ and let $\bar{H}$ be an H-algebraic functor. The following statements are valid.

a. $\bar{H}$ has a left adjoint.

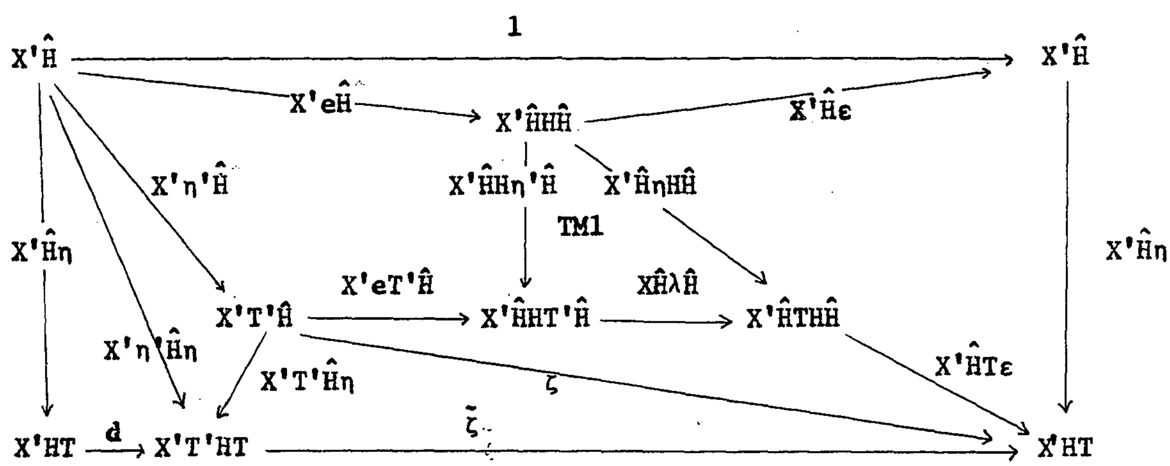
b. If $U^{\mathbb{T}}H$ is tripleable, $\bar{H}$ is tripleable.

Proof. a. Fix a $\mathbb{T}'$-algebra (X',ξ'). We must show that $((X',\xi'), (-)\bar{H})\mathcal{K}^{'\mathbb{T}'} : \mathcal{K}^{\mathbb{T}} \longrightarrow \mathcal{S}$ is representable. Let λ be the H-triple map corresponding to $\bar{H}$ via the isomorphisms of 1.4.5. Define $\mathcal{K}$-morphisms ζ, χ, by

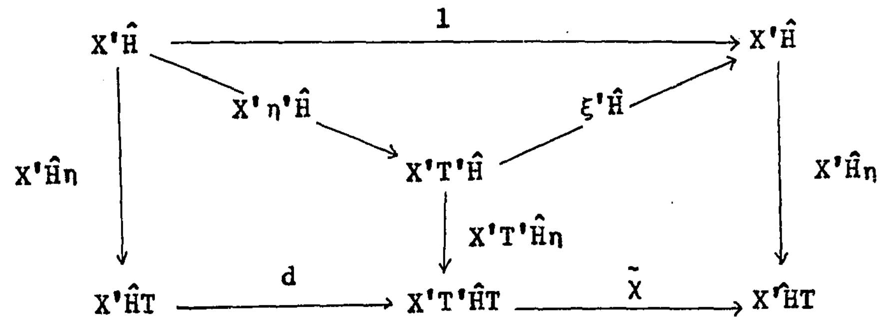
$$\zeta = X'T'\hat{H} \xrightarrow{X'eT'\hat{H}} X'\hat{H}HT'\hat{H} \xrightarrow{X'\hat{H}\lambda\hat{H}} X'\hat{H}TH\hat{H} \xrightarrow{X'\hat{H}T\varepsilon} X'\hat{H}T$$

$$\chi = X'T'\hat{H} \xrightarrow{\xi'\hat{H}} X'\hat{H} \xrightarrow{X'\hat{H}\eta} X'\hat{H}T$$

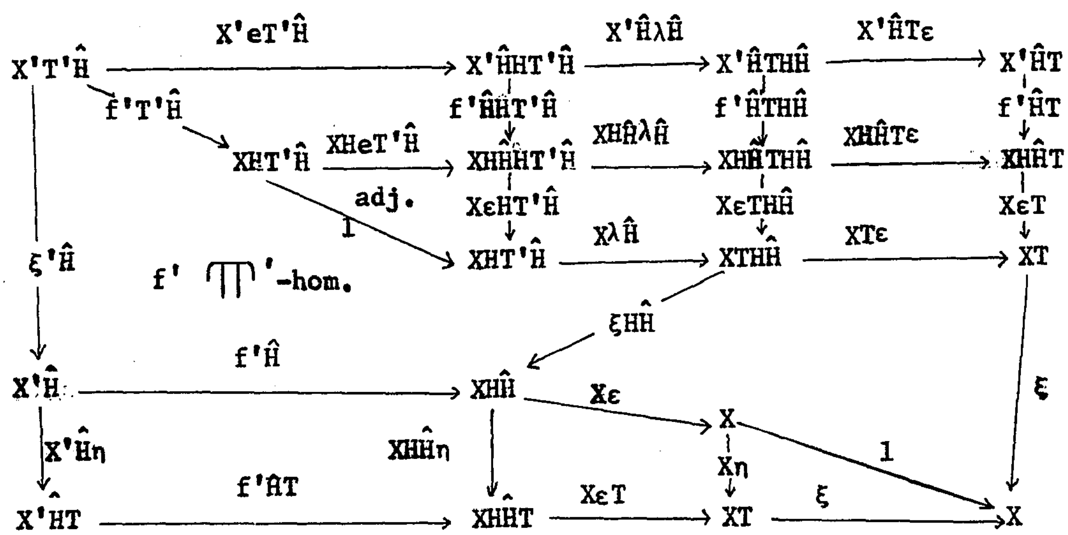
Let $\tilde{\zeta},\tilde{\chi}: X'T'\hat{H}T \to X'\hat{H}T$ be the corresponding homomorphic extensions. $(\tilde{\zeta},\tilde{\chi})$ is a reflexive pair in $\mathcal{K}^{\mathbb{T}}$. To prove it, let $X'\hat{H}T \xrightarrow{d} X'T'\hat{H}T$ be the homomorphic extension of $X'\hat{H} \xrightarrow{X'\eta'\hat{H}\eta} X'T'\hat{H}T$. The commutativity of the diagram



proves that $d\tilde{\zeta} = 1_{X'\hat{H}T'}$. Similarly,



proves that $d\tilde{\chi} = 1_{X'\hat{H}T}$. Let $X'\hat{H}T \xrightarrow{q} Q =_{df} \mathrm{coeq}(\tilde{\zeta},\tilde{\chi})$ in $\mathcal{K}^{\mathbb{T}}$. We will show Q is the representing object. Let $(X,\xi) \in \mathcal{K}^{\mathbb{T}}$, and let $(X',\xi') \xrightarrow{f'} (X,\xi)\overline{H}$ be a $\mathbb{T}'$-homomorphism. We have that $\zeta.f'\hat{H}T.X\varepsilon T.\xi = \chi.f'\hat{H}T.X\varepsilon T.\xi$. To prove this, first observe that $(X,\xi)\overline{H}$ $= (XH,\ XHT' \xrightarrow{X\lambda} XTH \xrightarrow{\xi H} XH)$ by 1.4.5, and then use the diagram:



Therefore, each (X,ξ) in obj $\mathcal{K}^{\mathbb{T}}$ induces a function $((X',\xi'),\ (X,\xi)\overline{H})\mathcal{K}^{\mathbb{T}'} \xrightarrow{(X,\xi)\alpha} (Q,\ (X,\xi))\mathcal{K}^{\mathbb{T}}$ sending $(X',\xi') \xrightarrow{f'} (X,\xi)\overline{H}$ to the unique $\mathbb{T}$-homomorphism from Q to (X,ξ) which when preceded by q equals $f'\hat{H}T.X\varepsilon T.\xi$. (To do this, notice that $f'\hat{H}T.X\varepsilon T.\xi$ is a $\mathbb{T}$-homomorphism). We will show that α is a natural equivalence. To see α is natural, let $(X,\xi) \xrightarrow{g} (Y,\theta) \in \mathcal{K}^{\mathbb{T}}$, $(X',\xi') \xrightarrow{f'} (X,\xi)\bar{H} \in \mathcal{K}^{\mathbb{T}'}$. The diagram on the right follows from the diagram on the left because $q.\langle f', -\circ g\bar{H}.(Y,\theta)\alpha\rangle = q.\langle f'.g\bar{H}, (Y,\theta)\alpha\rangle$

$$
\begin{array}{ccccc}
XH\hat{H}T & \xrightarrow{X\varepsilon T} & XT & \xrightarrow{\xi} & X \\
\downarrow{\scriptstyle gH\hat{H}T} & & \downarrow{\scriptstyle gT} & & \downarrow{\scriptstyle g} \\
YH\hat{H}T & \xrightarrow[Y\varepsilon T]{} & YT & \xrightarrow[\theta]{} & Y
\end{array}
\qquad
\begin{array}{ccc}
((X',\xi'),(X,\xi)\bar{H})\mathcal{K}^{\mathbb{T}'} & \xrightarrow{(X,\xi)\alpha} & (Q,(X,\xi))\mathcal{K}^{\mathbb{T}} \\
\downarrow{\scriptstyle -\circ g\bar{H}} & & \downarrow{\scriptstyle -\circ g} \\
((X',\xi'),(Y,\theta)\bar{H})\mathcal{K}^{\mathbb{T}'} & \xrightarrow[(Y,\theta)\alpha]{} & (Q,(Y,\theta))\mathcal{K}^{\mathbb{T}}
\end{array}
$$

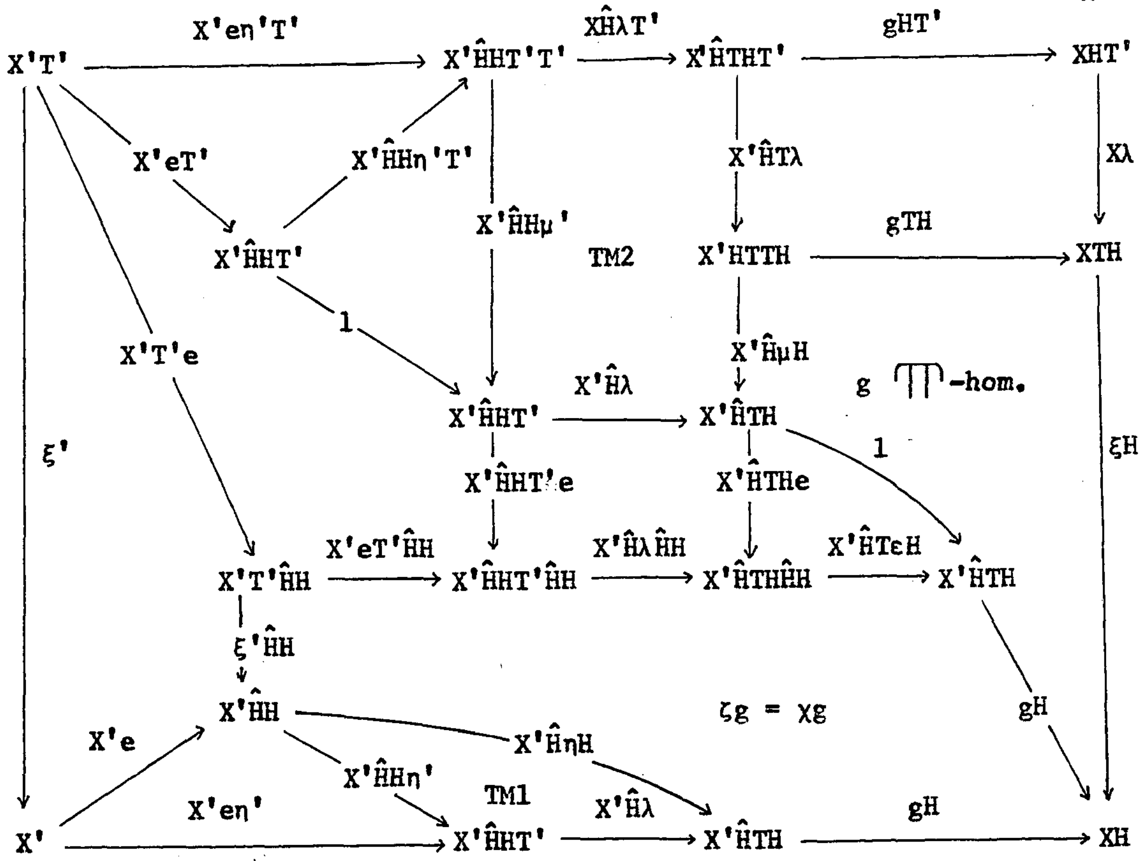
$= f'\hat{H}T.g\bar{H}\hat{H}T.Y\varepsilon T.\theta = f'\hat{H}T.X\varepsilon T.\xi.g = q.\langle f',(X,\xi)\alpha\rangle.g = q.\langle f',(X,\xi)\alpha.-\circ g\rangle.$

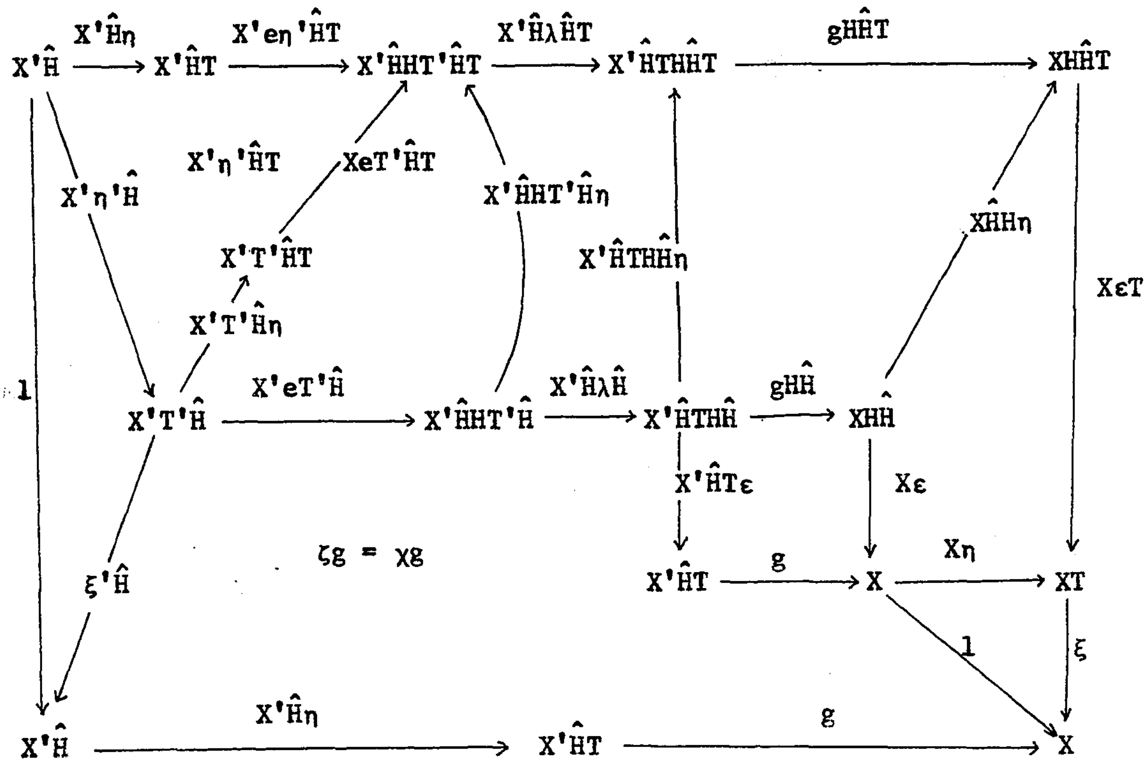
To see that $(X,\xi)\alpha$ is 1-to-1, the diagram:

$$
\begin{array}{ccccccc}
X' & \xrightarrow{f'} & XH & & & & \\
\downarrow{\scriptstyle X'e} & & \downarrow{\scriptstyle XHe} & \overset{1}{\searrow} & & & \\
X'\hat{H}H & \xrightarrow{f'\hat{H}H} & XH\hat{H}H & \xrightarrow{X\varepsilon H} & XH & & \\
\downarrow{\scriptstyle X'\hat{H}\eta H} & & \downarrow{\scriptstyle XH\hat{H}\eta H} & & \downarrow{\scriptstyle X\eta H} & \overset{1}{\searrow} & \\
X'\hat{H}TH & \xrightarrow{f'\hat{H}TH} & XH\hat{H}TH & \xrightarrow{X\varepsilon TH} & XTH & \xrightarrow{\xi H} & XH
\end{array}
$$

recovers f' from $\langle f',(X,\xi)\alpha\rangle$.

Finally, let $X'\hat{H}T \xrightarrow{g} (X,\xi)$ be a $\mathbb{T}'$-homomorphism with $\zeta g = Xg$. Define $f' =_{df} X' \xrightarrow{X'e\eta'} X'\hat{H}HT' \xrightarrow{X'\hat{H}\lambda} X'\hat{H}TH \xrightarrow{gH} XH$. To complete the proof of (a) we show that $(X',\xi') \xrightarrow{f'} (X,\xi)\bar{H}$ is a $\mathbb{T}'$-homomorphism, and that $g = f'\hat{H}T.X\varepsilon T.\xi$. The first statement follows from the diagram at the top of the next page, and the second statement follows from the diagram at the bottom of the next page (which says that g, $f'\hat{H}T.X\varepsilon T.$ agree on the generators).


X'T'
X'eη'T'
X'ĤHT'T'
XĤλT'
X'ĤTHT'
gHT'
XHT'
X'eT'
X'ĤHη'T'
X'ĤTλ
Xλ
X'ĤHT'
X'ĤHμ'
TM2
X'HTTH
gTH
XTH
1
X'T'e
X'ĤμH
X'Ĥλ
g ∏-hom.
ξ'
X'ĤHT'
X'ĤTH
1
ξH
X'ĤHT'e
X'ĤTHe
X'eT'ĤH
X'ĤλĤH
X'ĤTεH
X'T'ĤH
X'ĤHT'ĤH
X'ĤTHĤH
X'ĤTH
ξ'ĤH
X'ĤH
ζg = χg
gH
X'e
X'ĤηH
X'ĤHη'
TM1
X'Ĥλ
gH
X'eη'
X'
X'ĤHT'
X'ĤTH
XH


X'Ĥη
X'eη'ĤT
X'ĤλĤT
gHĤT
X'Ĥ
X'ĤT
X'ĤHT'ĤT
X'ĤTHĤT
XHĤT
X'η'ĤT
XeT'ĤT
X'η'Ĥ
X'ĤHT'Ĥη
XĤHη
X'T'ĤT
X'ĤTHĤη
XεT
X'T'Ĥη
1
X'eT'Ĥ
X'ĤλĤ
gHĤ
X'T'Ĥ
X'ĤHT'Ĥ
X'ĤTHĤ
XHĤ
X'ĤTε
Xε
ζg = χg
ξ'Ĥ
Xη
X'ĤT
g
X
XT
1
ξ
X'Ĥη
g
X'Ĥ
X'ĤT
X

b. We use 1.2.9. Let $(f,g) : (X,\xi) \longrightarrow (Y,\theta)$ be $\mathbb{T}$-homomorphisms with

$$(X,\xi)\overline{H} \underset{gH}{\overset{fH}{\rightrightarrows}} (Y,\theta)\overline{H} \xrightarrow{\quad q \quad} (Q,\xi')$$

a contractible coequalizer in $\mathcal{K}^{\mathbb{T}'}$. Therefore (f,g) is a $U^{\mathbb{T}}H$-contractible pair with coequalizer $YH \xrightarrow{Q} Q$. By hypothesis, there exists unique $(Y,\theta) \xrightarrow{\hat{q}} \hat{Q}$ in $\mathcal{K}^{\mathbb{T}}$, with domain (Y,θ) such that $\hat{q}U^{\mathbb{T}}H =$ $YH \xrightarrow{q} Q$; further, $\hat{q} = \text{coeq}(f,g)$. Since $\hat{q}\overline{H}U^{\mathbb{T}'} = YH \xrightarrow{q} Q$ and $U^{\mathbb{T}'}$ is tripleable, necessarily $\hat{q}\overline{H} = (Y,\theta)\overline{H} \xrightarrow{q} (Q,\xi')$. Suppose also $(Y,\theta) \xrightarrow{\tilde{q}} \tilde{Q}$ is a $\mathbb{T}$-algebra with $\tilde{q}\overline{H} = (Y,\theta)\overline{H} \xrightarrow{q} (Q,\xi')$. Then $\tilde{q}U^{\mathbb{T}}H = YH \xrightarrow{q} Q$, and so $\tilde{q} = \hat{q}$. []

1.5.2 Corollary. If $H \in AL(\mathcal{K})$

$$\begin{array}{ccc} \mathcal{K}^{\mathbb{T}} & \xrightarrow{\quad H \quad} & \mathcal{K}^{\mathbb{T}'} \\ & {\scriptstyle U^{\mathbb{T}}} \searrow \quad \swarrow {\scriptstyle U^{\mathbb{T}'}} & \\ & \mathcal{K} & \end{array}$$

and if $\mathcal{K}^{\mathbb{T}}$ has coequalizers of reflexive pairs, then H is tripleable. []

1.5.3 Corollary. Let $\mathcal{A}$ be a category with coequalizers of reflexive pairs, let $\mathbb{T}$ be a triple in $\mathcal{K}$ and let $\mathcal{A} \xrightarrow{H} \mathcal{K}^{\mathbb{T}}$ be a functor. Then H has a left adjoint iff $HU^{\mathbb{T}}$ has a left adjoint.

Proof. Since $U^{\mathbb{T}}$ has a left adjoint, H has a left adjoint implies $HU^{\mathbb{T}}$ has a left adjoint on general principles. Conversely, observe that $\mathcal{A} \xrightarrow{1} \mathcal{A}$ is tripleable, and apply 1.5.1 to the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad H \quad} & \mathcal{K} \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle U^{\pi}} \\ \mathcal{A} & \xrightarrow{\quad HU^{\pi} \quad} & \mathcal{K} \end{array}$$

[]

<u>1.5.4 Corollary.</u> Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ ε obj AD, and let $\mathcal{A}$ have coequalizers of reflexive pairs. Then the canonical reflection of U in AL (as defined in 1.4.1) has a left adjoint. []

<u>1.5.5 The algebraic dimension of an adjoint pair.</u> Let $\mathcal{A} \xrightarrow{\Phi_{-1}} \mathcal{K}$ ε obj AD, and let $\mathcal{A}$ have coequalizers of reflexive pairs. 1.5.4 yields a sequence Φ_0, Φ_1 , ... of canonical reflections, $\Phi_{-1} = \Phi_0 . U^{\pi_0}$ $= \Phi_1 . U^{\pi_1} . U^{\pi_0} = \ldots$ which suggests the definition: dim $\Phi_{-1} \leq n =_{df} \Phi_n$ is an isomorphism, or equivalently Φ_{n-1} is tripleable.

Φ_{-1} is tripleable iff dim $\Phi_{-1} = 0$. The demension of a reflective subcategory of a tripleable functor is ≤ 1. The dimension of the lattice fiberings to be studied in Chapter 3 is infinite.

Often objects in a category induce pairs of adjoint functors; e.g. if X is a topological space, the set-valued functor "continuous maps from X" has a left adjoint. We could define dim X $=_{df}$ the algebraic dimension of this adjoint pair.

Apart from these suggestive remarks, we will not study algebraic dimension in this paper.

§1.6 Birkhoff subcategories for regular triples.

1.6.1 Definitions. Let $\mathcal{K}$ be a category and let $\mathcal{B}$ be a full subcategory of $\mathcal{K}$ with inclusion functor $\mathcal{B} \xrightarrow{i} \mathcal{K}$. $\mathcal{B}$ is closed under products if every model for a product in $\mathcal{K}$ of a set of $\mathcal{B}$-objects lies in $\mathcal{B}$. $\mathcal{B}$ is closed under subobjects if every monomorphism in $\mathcal{K}$ with range in $\mathcal{B}$ lies in $\mathcal{B}$. Let $\mathcal{C}$ be any subcategory of $\mathcal{K}$. Define $\widehat{\mathcal{C}} =_{df}$ the intersection of all full subcategories of $\mathcal{K}$ containing $\mathcal{C}$ and closed under products and subobjects. Clearly $\widehat{\mathcal{C}}$ is the smallest full subcategory containing $\mathcal{C}$ that is closed under products and subobjects.

We could easily formulate the above definitions without using full subcategories but the gain in generality would be negligible because of the observation that if $\mathcal{K}$ has finite products every $\mathcal{K}$-morphism factors as a mono followed by a projection: $f = (1,f).pr_2$. Note, too, that if a cartesian product of $\mathcal{K}$-monos is mono then $\widehat{\mathcal{C}}$ = the full subcategory generated by the class of monomorphisms into products of elements of obj $\mathcal{C}$.

Evidently "$\frown$" is a closure operator on the (large) lattice of subcategories of $\mathcal{K}$, and $\mathcal{C} = \widehat{\mathcal{C}}$ iff $\mathcal{C}$ is closed under products and subobjects.

1.6.2 Proposition. Let $\mathcal{K}$ be a regular category, $\mathcal{B} \xrightarrow{i} \mathcal{K}$ a full subcategory. The following statements are equivalent.

a. $\mathcal{B} = \widehat{\mathcal{B}}$

b. $\mathcal{B}$ is a reflective subcategory of $\mathcal{K}$ in such a way that for every $\mathcal{K}$-object X the reflection $X \xrightarrow{X\eta} X_{\mathcal{B}}$ of X in $\mathcal{B}$ is a regular epimorphism; also obj $\mathcal{B}$ is a union of $\mathcal{K}$-isomorphism classes.

<u>Proof.</u> <u>a implies b.</u> Since an isomorphism may be viewed either as a monomorphism or as a unary product, obj $\mathcal{B}$ is a union of $\mathcal{K}$-isomorphism classes. Let X ε obj $\mathcal{K}$ and let $\mathcal{R}$ be a representative set of regular quotients of X. If $X \xrightarrow{f} B$ ε $\mathcal{K}$ with B ε $|\mathcal{B}|$, there exists a regular coimage factorization $f = X \xrightarrow{p} R \xrightarrow{j} B$ with R ε $\mathcal{R}$. As j is mono and as B ε obj $\mathcal{B}$, R ε obj $\mathcal{B}$. Hence i satisfies the solution set condition. But clearly $\mathcal{B}$ has $\varprojlim$'s and i preserves them. It follows from the adjoint functor theorem that i has a left adjoint, that is $\mathcal{B}$ is a reflective subcategory. Now let X ε obj $\mathcal{K}$, and let $X \xrightarrow{X\eta} X_{\mathcal{B}}$ be a reflection of X in $\mathcal{B}$. Form a regular coimage factorization of Xη,

$$\begin{array}{ccc} X & \xrightarrow{X\eta} & X_{\mathcal{B}} \\ {\scriptstyle p}\downarrow & \overset{x}{\swarrow} & \downarrow{\scriptstyle 1} \\ I & \xrightarrow[k]{} & X_{\mathcal{B}} \end{array}$$

Xη = p.k. Since I ε obj $\mathcal{B}$, x is induced with Xη.x = p. As p is epi so is x. As Xη.x.k = p.k = Xη it follows by the uniqueness of reflection-induced maps that x.k = 1. So x is epi and split mono, hence iso, and Xη is a regular epimorphism because p is.

<u>b implies a.</u> Let X be a product in $\mathcal{K}$ of a set of $\mathcal{B}$-objects. Each projection factors through Xη inducing a map $X_{\mathcal{B}} \xrightarrow{a} X$ such that Xη.a = 1_X. Hence Xη is split mono; since we assume Xη is epi, Xη is an isomorphism. Now suppose X is a $\mathcal{K}$-object admitting a monomorphism i to some object in $\mathcal{B}$. Then i factors through Xη, and

hence $X\eta$ is mono. But then $X\eta$ is mono and regular epi and hence iso.

[]

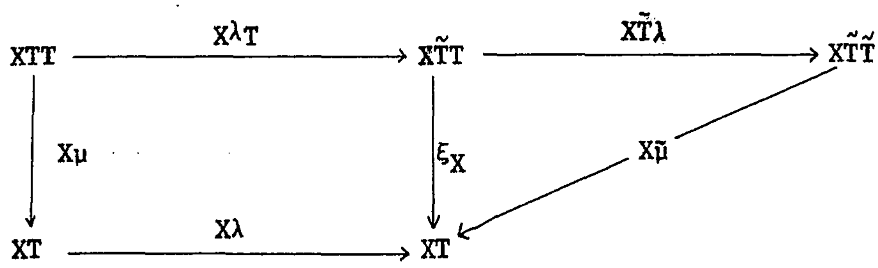
For 1.6.3 - 1.6.6 fix a regular triple $\mathbb{T} = (T,\eta,\mu)$ in a (regular) category $\mathcal{K}$.

__1.6.3 Proposition.__ Let $T \xrightarrow{\lambda} \tilde{T}$ be a pointwise regular epimorphic natural transformation, and suppose further that for every object X in obj $\mathcal{K}$ there exists a $\mathcal{K}$-morphism $X\tilde{\mu}$ such that $X\lambda\lambda.X\tilde{\mu} = X\mu.X\lambda$. Then $\tilde{\mathbb{T}} =_{df} (\tilde{T},\tilde{\eta},\tilde{\mu})$ (where $\tilde{\eta} =_{df} \eta\lambda$) is a triple in $\mathcal{K}$ and $\mathbb{T} \xrightarrow{\lambda} \tilde{\mathbb{T}} \in \mathrm{OPTR}(\mathcal{K})$.

__Proof.__ The fact that $X\lambda$ is epi yields the unitary axioms. It is also so that $X\lambda\lambda$ and $X\lambda\lambda\lambda$ are epi, e.g. $X\lambda\lambda\lambda = X\lambda TT.X\tilde{T}\lambda T.X\tilde{T}\tilde{T}\lambda$ so use 0.4.8 and the fact that T preserves regular epi's. $X\lambda\lambda$ epi implies $\tilde{\mu}$ is natural, and $X\lambda\lambda\lambda$ epi implies the associativity axiom. The reader can provide the requisite diagrams. []

__1.6.4 The regular quotient triple induced by a $\frown$-closed subcategory.__ Let $\mathcal{B} \subset \mathcal{K}^{\mathbb{T}}$ be a subcategory such that $\mathcal{B} = \widehat{\mathcal{B}}$. By 1.2.6 $\mathcal{K}^{\mathbb{T}}$ is a regular category, so that by 1.6.2 $\mathcal{B}$ is a full reflective subcategory with regular epimorphic reflections. In particular, for each $X \in$ obj $\mathcal{K}$ let $(XT,X\mu) \xrightarrow{X\lambda} (X\tilde{T},\xi_X)$ be a regular epimorphic reflection of $(XT,X\mu)$ in $\mathcal{B}$. By the reflection property, each $\mathcal{K}$-morphism $X \xrightarrow{f} Y$ induces unique $f\tilde{T}$ such that $X\lambda.f\tilde{T} = fT.Y\lambda$ which establishes a functor $\mathcal{K} \xrightarrow{\tilde{T}} \mathcal{K}$ and a pointwise regular epimorphic natural transformation $T \xrightarrow{\lambda} \tilde{T}$.

For every $X \in \text{obj}\ \mathcal{K}$, the fact that ξ_X is a $\mathbb{T}$-homomorphism and the reflection property induce $X\tilde{\mu}$:



By 1.6.3, $\tilde{\mathbb{T}} = (\tilde{T}, \eta\lambda, \tilde{\mu})$ is a triple in $\mathcal{K}$ and $\mathbb{T} \xrightarrow{\lambda} \tilde{\mathbb{T}}$ is an OPTR($\mathcal{K}$)-morphism. $\tilde{\mathbb{T}}$ is called the <u>regular quotient triple induced by</u> $\mathcal{B}$.

<u>1.6.5 Definitions.</u> A full subcategory $\mathcal{B}$ of $\mathcal{K}^{\mathbb{T}}$ is closed under $U^{\mathbb{T}}$-contractible coequalizers $=_{df}$ every $\mathcal{K}^{\mathbb{T}}$-morphism expressible as the coequalizer of a pair of $\mathcal{B}$-morphisms, $U^{\mathbb{T}}$of which is contractible in $\mathcal{K}$, lies in $\mathcal{B}$. For each subcategory $\mathcal{C}$ of $\mathcal{K}^{\mathbb{T}}$, define $\hat{\hat{\mathcal{C}}} =_{df}$ the intersection of all subcategories of $\mathcal{K}^{\mathbb{T}}$ containing $\mathcal{C}$ and closed under products, subalgebras ($=_{df}$ subobjects in $\mathcal{K}^{\mathbb{T}}$) and $U^{\mathbb{T}}$-contractible coequalizers. A $\hat{\ }$-closed subcategory of $\mathcal{K}^{\mathbb{T}}$ is called a <u>Birkhoff subcategory of</u> $\mathcal{K}^{\mathbb{T}}$.

In an equationally defineable class, Birkhoff subcategories arise by imposing new equations and conversely; this was proved by G. D. Birkhoff [4], hence the terminology. The next proposition is the triple-theoretic version of this theorem.

<u>1.6.6 Proposition.</u> Let $\mathcal{A}$ be a subcategory of $\mathcal{K}^{\mathbb{T}}$. Set $\mathbb{T} \xrightarrow{\lambda} \tilde{\mathbb{T}}$ to be the regular quotient triple induced by $\hat{\mathcal{A}}$, and

and define $\mathcal{B}$ to be the image (literally) of the induced algebraic functor $\mathcal{K}^{\tilde{\mathbb{T}}} \xrightarrow{\lambda\circ -} \mathcal{K}^{\mathbb{T}}$ in AL($\mathcal{K}$). Define $\mathcal{C}$ to be the full subcategory generated by all coequalizers of $U^{\mathbb{T}}$-contractible pairs of $\hat{\hat{\mathcal{A}}}$-morphisms. Then the following conclusions are valid.

a. $\mathcal{B}$ is a subcategory and $\hat{\hat{\mathcal{A}}} = \mathcal{B} = \mathcal{C}$.

b. $\lambda\circ -$ is an isomorphism onto $\mathcal{B}$; hence the restriction of $U^{\mathbb{T}}$ to any Birkhoff subcategory is tripleable.

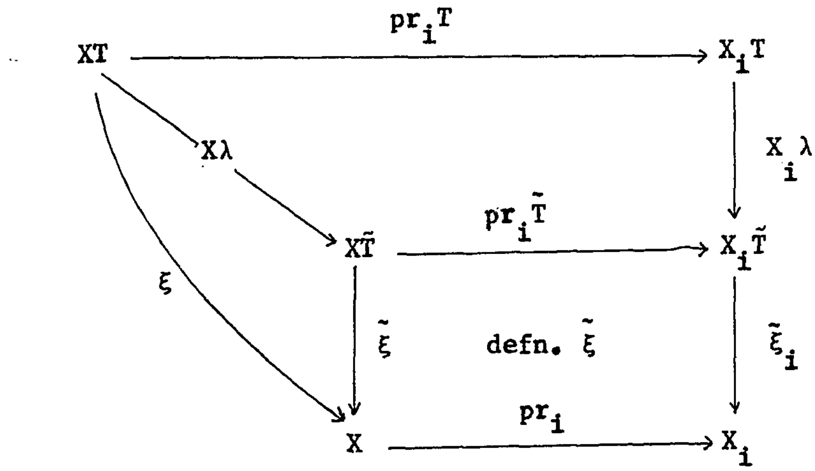
c. If $(B,\xi) \xrightarrow{q} (Q,\alpha) \in \mathcal{K}^{\mathbb{T}}$ with $(B,\xi) \in |\hat{\hat{\mathcal{A}}}|$ and $B \xrightarrow{q} Q$ split epi in $\mathcal{K}$, then $(Q,\alpha) \in |\hat{\hat{\mathcal{A}}}|$.

<u>Proof.</u> Let $(X,\tilde{\xi})$, $(Y,\tilde{\theta}) \in$ obj $\mathcal{K}^{\tilde{\mathbb{T}}}$, and let $X \xrightarrow{f} Y \in \mathcal{K}$.

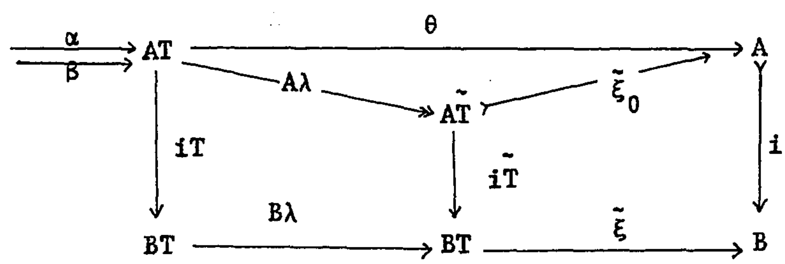
Consider:

$$\begin{array}{ccccc} XT & \xrightarrow{X\lambda} & X\tilde{T} & \xrightarrow{\tilde{\xi}} & X \\ \downarrow{\scriptstyle fT} & & \downarrow{\scriptstyle f\tilde{T}} & & \downarrow{\scriptstyle f} \\ YT & \xrightarrow{Y\lambda} & Y\tilde{T} & \xrightarrow{\tilde{\theta}} & Y \end{array}$$

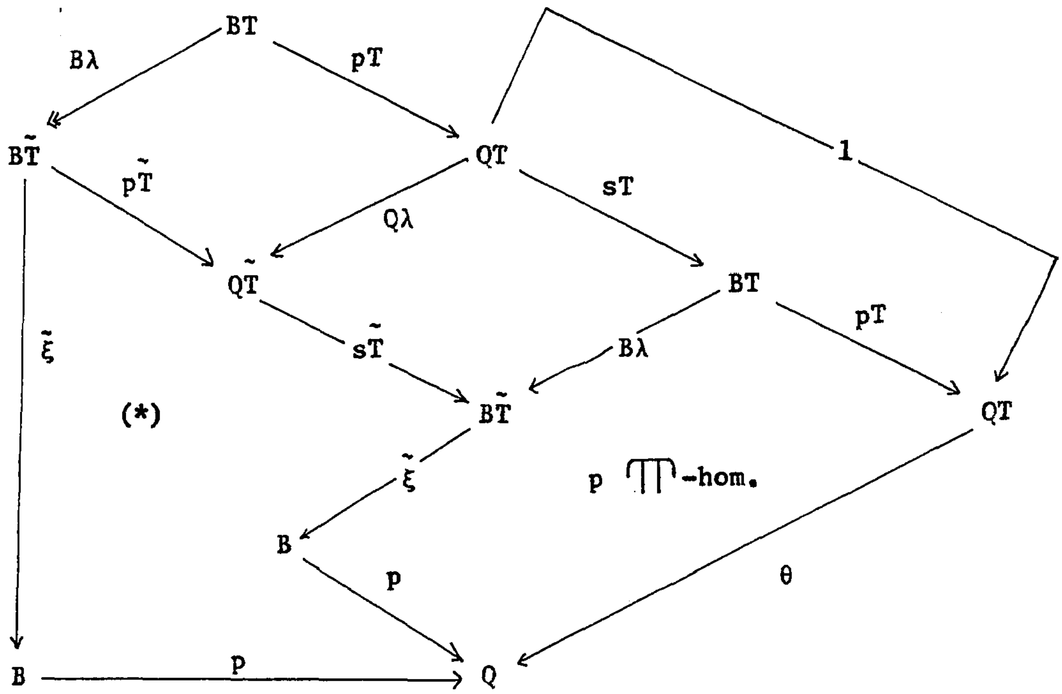
If f is a $\mathbb{T}$-homomorphism then the outer rectangle commutes so that f is a $\tilde{\mathbb{T}}$-homomorphism as $X\lambda$ is epi. Therefore, $\lambda\circ -$ is a full functor. That $X\lambda$ is epi also clearly implies that $\lambda\circ -$ is 1-to-1 on objects. $\lambda\circ -$ is faithful as are all algebraic functors. This proves that $\mathcal{B}$ is a full subcategory of $\mathcal{K}^{\mathbb{T}}$ and that $\lambda\circ -$ is an isomorphism onto $\mathcal{B}$. Next we establish that $\mathcal{B} = \hat{\hat{\mathcal{B}}}$. Let $[(X_i,\tilde{\xi}_i) : i \in I]$ be a set of $\tilde{\mathbb{T}}$-algebras, and set $(X,\tilde{\xi}) =_{df} \prod(X_i,\tilde{\xi}_i)$, $(X,\xi) =_{df} \prod (X_i, X_i\lambda.\tilde{\xi}_i)$. Consider the diagram:



The outer square commutes for all i by the definition of (X,ξ). Hence $\xi.pr_i = X\lambda.\tilde{\xi}.pr_i$ for all i and $\xi = X\lambda.\tilde{\xi}$, that is $(X,\xi) \in \text{obj } \mathcal{B}$. This shows that $\mathcal{B}$ is closed under products. Let $(B,\tilde{\xi})$ be a $\widetilde{\mathbb{T}}$-algebra, and let $(A,\theta) \overset{i}{\rightarrowtail} (B,B\lambda.\tilde{\xi})$ be a subalgebra.



If $\alpha.A\lambda = \beta.A\lambda$ then $\alpha.\theta = \beta.\theta$ since i is mono. Therefore $\theta \in \text{reg}(A\lambda)$ which induces unique $\tilde{\xi}_0$ such that $A\lambda.\tilde{\xi}_0 = \theta$. As $A\lambda$ is epi, $(A,\tilde{\xi}_0) \leq (B,\tilde{\xi})$. This proves $\mathcal{B}$ is closed under subalgebras. More general than showing that $\mathcal{B}$ is closed under $U^{\mathbb{T}}$-contractible coequalizers, we show that $\mathcal{B}$ is closed under $U^{\mathbb{T}}$-split epimorphisms, which will also take care of (c). Let $(B,\tilde{\xi})$ be a $\widetilde{\mathbb{T}}$-algebra and let $(B, B\lambda.\tilde{\xi}) \xrightarrow{p} (Q,\theta)$ be a $\mathbb{T}$-homomorphism such that there exists $Q \xrightarrow{s} B \in \mathcal{K}$ with $s.p = 1_Q$. In the diagram at the top of the next page, all commutes (including the outer figure) except possibly (*). But as $B\lambda$ is epi,



(*) then commutes. Since pT and Qλ are epi, so is $p\tilde T$; similarly, $p\tilde T\tilde T$ is epi. By 1.2.3, $(Q, s\tilde T.\tilde\xi.p)$ is a $\tilde{\mathbb{T}}$-algebra. That $(Q,\theta) = (Q, Q\lambda.s\tilde T.\tilde\xi.p)$ is also clear from the above diagram. Hence $(Q,\theta) \in$ obj $\mathcal{B}$, and $\mathcal{B} = \hat{\hat{\mathcal{B}}}$.

Now suppose $(X,\xi) \in$ obj $\hat{\mathcal{A}}$. The reflection property induces $\tilde\xi$ with $X\lambda.\tilde\xi = \xi$. It is not hard to show that $(X,\tilde\xi)$ is a $\tilde{\mathbb{T}}$-algebra; use the facts that $X\lambda\lambda = XT\lambda.X\lambda\tilde T$, and is epi. We have proved so far that $\hat{\mathcal{A}} \subset \mathcal{B} = \tilde{\mathcal{B}}$, so in fact $\hat{\hat{\mathcal{A}}} \subset \mathcal{B}$. To see that $\mathcal{B} \subset \mathcal{C}$ observe that if $(X,\tilde\xi)$ is a $\tilde{\mathbb{T}}$-algebra then $\tilde\xi$ is the coequalizer of the $U^{\tilde{\mathbb{T}}}$-contractible pair of $\mathcal{K}^{\tilde{\mathbb{T}}}$-morphisms $(X\tilde\mu, \tilde\xi T)$ and hence that $(X\tilde T, XT\lambda.X\tilde\mu) \xrightarrow{\tilde\xi} (X, X\lambda.\tilde\xi)$ is the coequalizer of the $U^{\mathbb{T}}$-contractible pair of $\mathcal{B}$-morphisms $(X\tilde T\tilde\mu, \tilde\xi\tilde T)$. That $\mathcal{C} \subset \mathcal{A}$ is obvious. []

<u>1.6.7 LF-Birkhoff subcategories.</u> There are certain categories $\mathcal{K}$ in which the regular monomorphisms are the natural "subobjects". For

instance if $\mathcal{K}$ = topological spaces, the regular monomorphisms are the relative subspaces (we prove a generalization in 3.1.9) whereas the identity function of a set X from discrete X to indiscrete X is a just plain monomorphism that is surely no subspace. In 3.4.3 we will show that the category of topologized groups, whose objects are sets with both group and topological structures but no relations between these structures and whose morphisms are continuous group homomorphisms, is tripleable over spaces via a regular triple. The full subcategory of topological groups, where now the group operations are continuous, is closed under products, $U^{\mathbb{T}}$-contractible coequalizers and the usual topological subgroups whose inclusion is a relative subspace and not just 1-to-1 continuous; it is not, however, a Birkhoff subcategory in the sense of 1.6.5. The question arises whether we can obtain a theory of Birkhoff subcategories under the transition "subalgebra with mono underlying" $\longmapsto$ "subalgebra with regular mono underlying" by modifying the notion of "regular triple". The answer is in the affirmative and the modification required is slight, as we shall now see. We will use this technique to prove that [topological groups] is tripleable over [topological spaces] in 3.4.4; the level of generality there will allow an arbitrary lattice fibering in place of topological spaces, hence the "LF" in the definitions we now establish.

<u>1.6.7A Definitions.</u> Paralleling 0.5.1, a category $\mathcal{K}$ is <u>LF-regular</u> if it satisfies the following four axioms.

LFR1. $\mathcal{K}$ has regular image factorizations.

LFR2. $\mathcal{K}$ has $\varprojlim$'s

LFR3. $\mathcal{K}$ is legitimate.

LFR4. For each $\mathcal{K}$-object A, the class of epimorphisms with domain A has a representative set.

A triple $\mathbb{T}$ in $\mathcal{K}$ is LF-regular if $\mathcal{K}$ is LF-regular and T preserves epimorphisms. For the rest of this section, fix an LF-regular triple $\mathbb{T}$ in $\mathcal{K}$. An LF-Birkhoff subcategory of $\mathcal{K}^{\mathbb{T}}$ is a full subcategory of $\mathcal{K}^{\mathbb{T}}$ which is closed under products, relative subalgebras ($=_{df}$ subalgebras whose underlying $\mathcal{K}$-morphism is a regular mono) and $U^{\mathbb{T}}$-contractible coequalizers.

Define the LF-modification of a statement by substituting "LF-regular triple" for "regular triple", "epi" for "regular epi" and "regular mono" for "mono". For example, the definition of LF-regular category is the LF-modification of 0.5.1.

The following proposition generalizes [18, Theorem C] where it is proved for topological spaces.

1.6.7B Proposition. Let $\mathcal{B}$ be a full subcategory of $\mathcal{K}$. Then is closed under products and regular monomorphisms iff $\mathcal{B}$ is a reflective subcategory with epimorphic reflections and obj $\mathcal{B}$ is a union of $\mathcal{K}$-isomorphism classes.

Proof. Use the LF-modification of the proof of 1.6.2. []

Note: Clearly $\mathcal{B}$ is closed under $\varprojlim$'s if it is closed under products and regular monomorphisms, but the converse is false. For instance the subcategory $\{P\} \subset \mathcal{S}$ consisting of the 1-point set, P, is closed under $\varprojlim$'s but $\phi \to P$ is a regular mono.

__1.6.7C Proposition.__ The LF-modification of 1.6.3 is true.

__Proof.__ Use 0.4.4 instead of 0.4.8. []

__1.6.7D Proposition.__ The LF-modification of 1.6.4 is valid. []

__1.6.7E Proposition.__ The LF-modification of 1.6.6 is valid. Hence the theory of LF-Birkhoff subcategories for LF-regular triples is as good as the theory of Birkhoff subcategories for regular triples.

__Proof.__ The only part of the proof that is not immediate via LF-modification is, in the language of the proof of 1.6.6, the argument that $\mathcal{B}$ is closed under relative subalgebras. Consulting the corresponding diagram, we have that $A\lambda$ is epi and i is regular mono. Hence $i\tilde{T}.\tilde{\xi} \in reg(i)$ inducing unique $\tilde{\xi}_0$ with $\tilde{\xi}_0.i = i\tilde{T}.\tilde{\xi}$. $A\lambda.\tilde{\xi}_0 = \theta$ as i is mono. []

§1.7 Composite triples.

For this section let $\mathbb{T} = (T,\eta,\mu)$, $\mathbb{H} = (H,e,m)$ be triples in a category $\mathcal{K}$.

1.7.1 Definitions. An $\mathbb{H}$-distributive law on $\mathbb{T}$ is an H-triple map $\mathbb{T} \xrightarrow{(H,\lambda)} \mathbb{T}$ satisfying axioms

$$\begin{array}{ccc} HT & \xrightarrow{\lambda} & TH \\ {\scriptstyle eT}\nwarrow & D1 & \nearrow{\scriptstyle Te} \\ & T & \end{array} \qquad \begin{array}{ccccc} HHT & \xrightarrow{H\lambda} & HTH & \xrightarrow{\lambda H} & THH \\ \downarrow{\scriptstyle mT} & & D2 & & \downarrow{\scriptstyle Tm} \\ HT & & \xrightarrow{\lambda} & & TH \end{array}$$

A lifting of $\mathbb{H}$ over $U^{\mathbb{T}}$ is a triple $\bar{\mathbb{H}} = (\bar{H},\bar{e},\bar{m})$ in $\mathcal{K}^{\mathbb{T}}$ such that $\bar{H}U^{\mathbb{T}} = U^{\mathbb{T}}H$ and such that for every $(X,\xi) \in \mathcal{K}^{\mathbb{T}}$, $(X,\xi)\bar{e}U^{\mathbb{T}} = Xe$ and $(X,\xi)\bar{m}U^{\mathbb{T}} = Xm$. An $\mathbb{H}$-intrastructure on $\mathbb{T}$ is an H-intrastructure $\mathbb{T} \xrightarrow{(H,\Gamma)} \mathbb{T}$ also satisfying

$$\begin{array}{ccc} TT & \xrightarrow{TeT} & THT \\ \downarrow{\scriptstyle \mu} & IN4 & \downarrow{\scriptstyle \Gamma} \\ T & \xrightarrow{Te} & TH \end{array} \qquad \begin{array}{ccccc} THTHT & \xrightarrow{\Gamma HT} & THHT & \xrightarrow{TmT} & THT \\ \downarrow{\scriptstyle TH\Gamma} & & IN5 & & \downarrow{\scriptstyle \Gamma} \\ THTH & \xrightarrow{\Gamma H} & THH & \xrightarrow{Tm} & TH \end{array}$$

A $\mathbb{T}$-$\mathbb{H}$ composite triple is a natural transformation $THTH \xrightarrow{M} TH$ such that $\mathbb{T}\mathbb{H} =_{df} (TH,\eta e,M)$ is a triple in $\mathcal{K}$, such that

$$\mathbb{T} \xrightarrow{Te} \mathbb{T}\mathbb{H} \xleftarrow{\eta H} \mathbb{H}$$

are OPTR($\mathcal{K}$)-morphisms and such that C1 and C2 commute:

$$\begin{array}{ccc} THTHH & \xrightarrow{MH} & THH \\ \downarrow{\scriptstyle THTm} & C1 & \downarrow{\scriptstyle Tm} \\ THTH & \xrightarrow{M} & TH \end{array} \qquad \begin{array}{ccc} TTHTH & \xrightarrow{TM} & TTH \\ \downarrow{\scriptstyle \mu HTH} & C2 & \downarrow{\scriptstyle \mu H} \\ THTH & \xrightarrow{M} & TH \end{array}$$

The respective classes of all such $=_{dn}$ $[\lambda]$, $[\overline{||-|}]$, $[\Gamma]$, $[M]$. R_3^{-1} of the following proposition is found in [2].

<u>1.7.2 Proposition.</u> Define correspondences

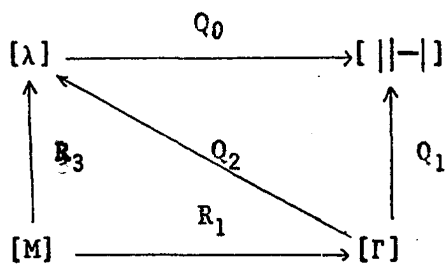
$$[\lambda] \xrightarrow{Q_0} [\overline{||-|}] \xrightarrow{Q_1} [\Gamma] \xrightarrow{Q_2} [\lambda]$$

as in 1.4.5. Also define

$$[\Gamma] \xrightarrow{R_2} [M], \qquad [M] \xrightarrow{R_3} [\lambda].$$

$$THT \xrightarrow{\Gamma} TH \mapsto THTH \xrightarrow{\Gamma H} THH \xrightarrow{Tm} TH \qquad THTH \xrightarrow{M} TH \mapsto HT \xrightarrow{\eta HTe} THTH \xrightarrow{M} TH$$

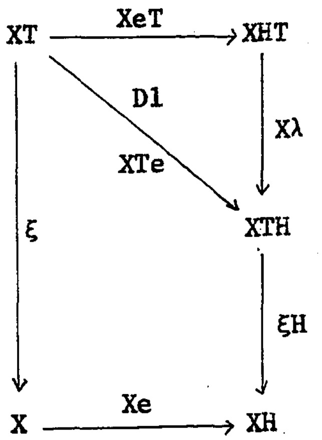
Then Q_0, Q_1, Q_2, R_2, R_3 are all well-defined; further, $R_2 R_3 = Q_2$ and the system

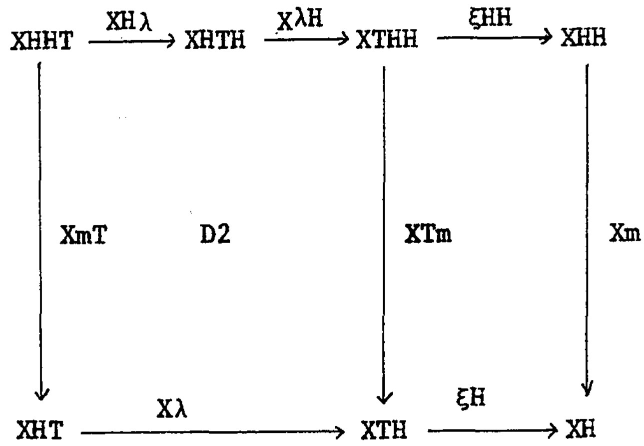


consists of cyclically-inverse bijections.

<u>Proof.</u> We build on the proof of 1.4.5.

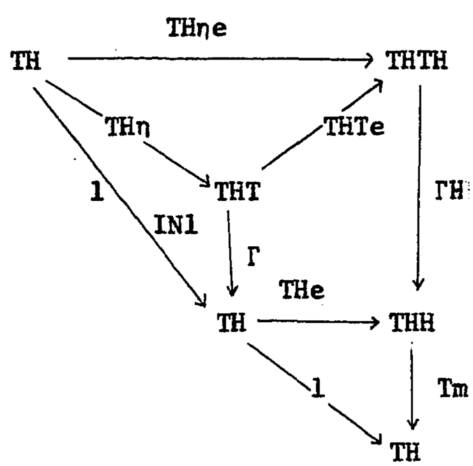
<u>Q_0 well-defined.</u>

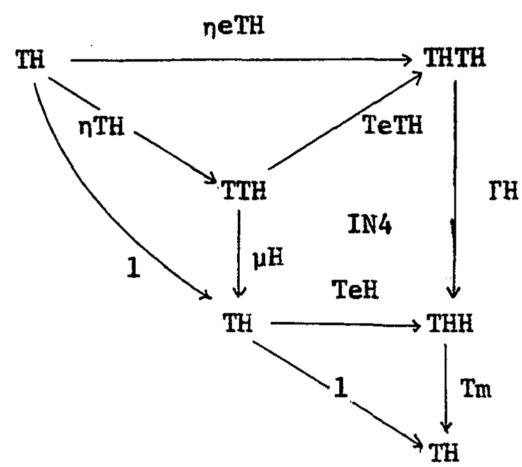


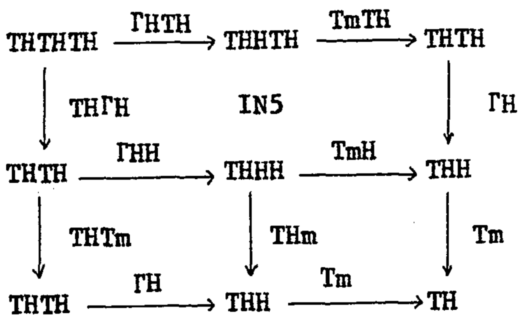


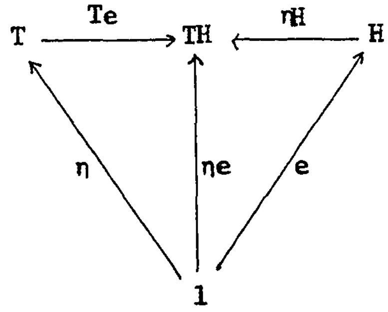
<u>**Q_1 well-defined.**</u> $(XT, X\mu) \xrightarrow{XTe} (XT, X\mu)\bar{H}$ is a $\prod$-homomorphism, that is $XTeT.X\Gamma = X\mu.XTe$ which is IN4. Fix X in obj $\mathcal{K}$ and define ξ by $(XTHH, \xi) =_{df} (XTH, X\Gamma)\bar{H} = (XT, X\mu)\bar{H}\bar{H}$. As $(XTHT, XTH\mu) \xrightarrow{X\Gamma} (XTH, X\Gamma)$ is a $\prod$-homomorphism, so is $(XTHTH, XTH\Gamma) \xrightarrow{X\Gamma H} (XTHH, \xi)$, yielding $X\Gamma HT.\xi = XTH\Gamma.X\Gamma H$. On the other hand, $(XTHH, \xi) \xrightarrow{XTm} (XTH, X\Gamma)$ is a $\prod$-homomorphism by hypothesis so that $XTmT.X\Gamma = \xi.XTm$. Pasting together along ξ proves IN5.

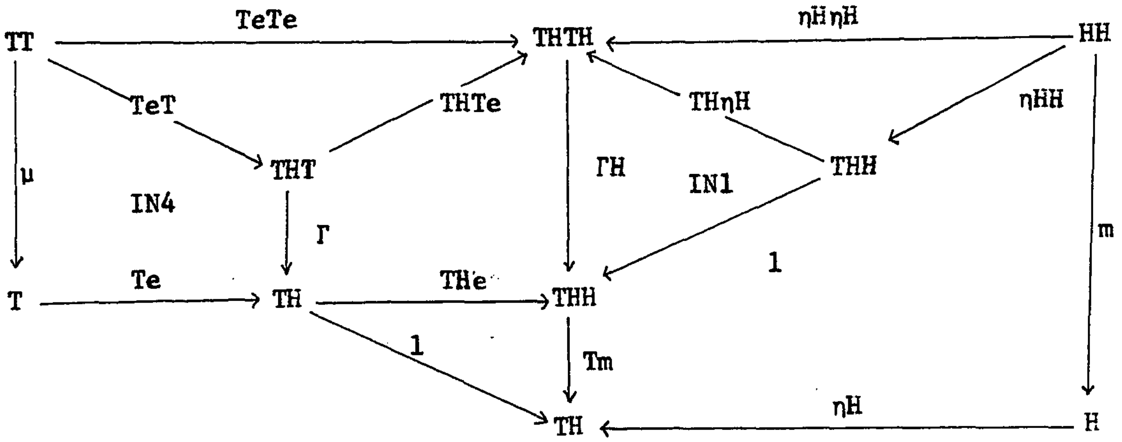
<u>**R_2 well-defined.**</u>

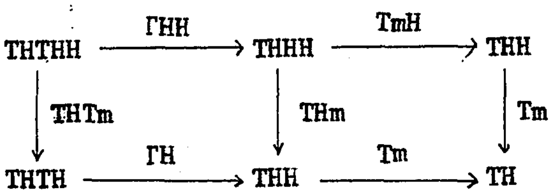


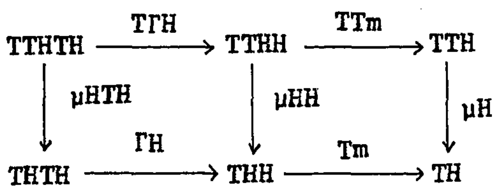




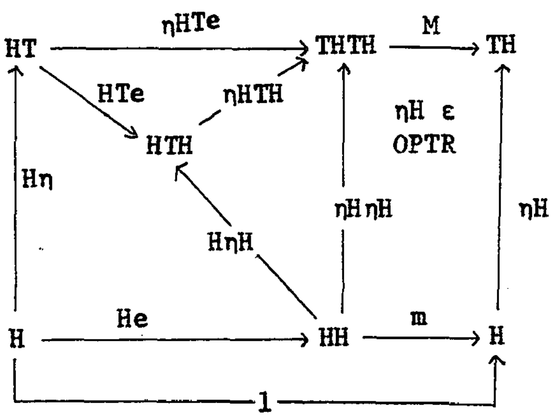


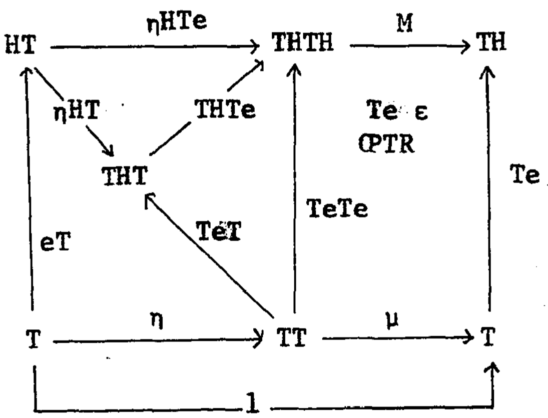


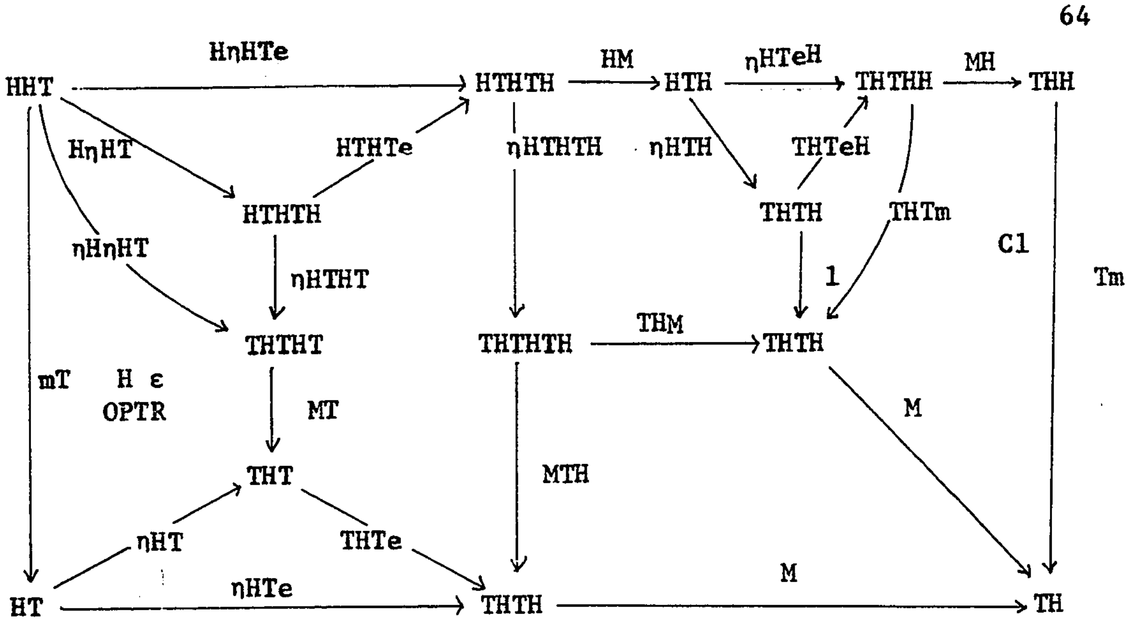


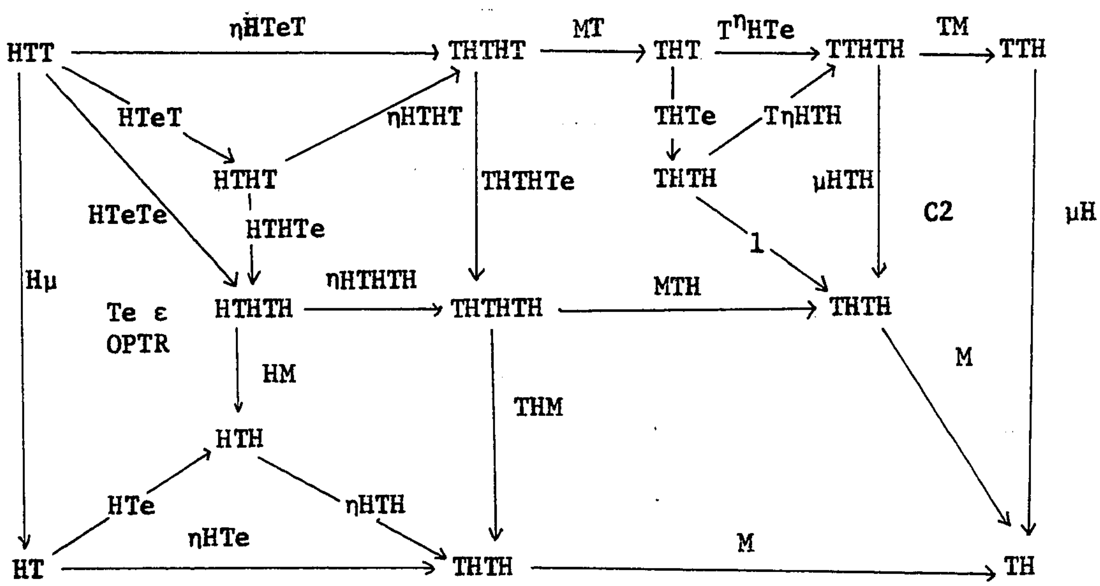


R_3 well-defined.

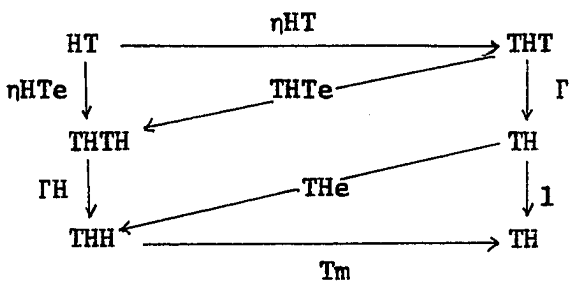




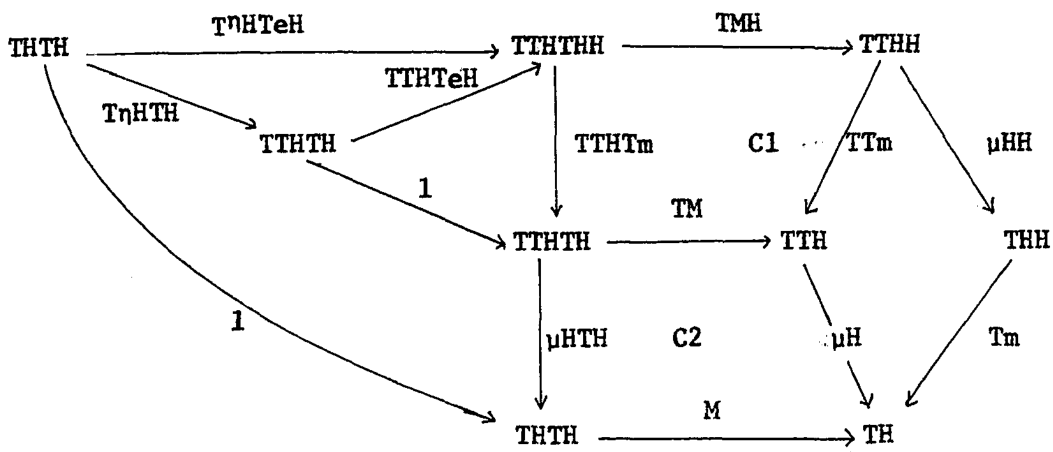




$R_2R_3 = Q_2$. Let Γ ε [Γ]. That ηHTe.ΓH.Tm = ηHT.Γ follows from:

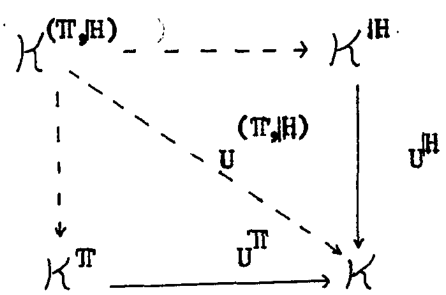


$R_3Q_0Q_1R_2 = \mathrm{id}$. Let $M \in [M]$. If $\Gamma =_{df} MR_3Q_0Q_1$ then $(XTH, X\Gamma) = (XT, X\mu)\, \bar{H}_{\lambda_M} = (XTH, XT\lambda_M . X\mu H) = (XTH,\ XT\eta HTe.XTM.X\mu H)$. Consulting the diagram:



we have $MR_3Q_0Q_1R_2 = T\eta HTe.TMH.\mu HH.Tm = M$. []

1.7.3 Definition. We define a new functor $\underline{U^{(\mathbb{T},\mathbb{H})}}$ by forming the (usual model of the) pullback in the category of categories



.

Hence obj $\mathcal{K}^{(\mathbb{T},\ \mathbb{H})} = [(X,\xi,\alpha) : (X,\xi) \in |\mathcal{K}^{\mathbb{T}}|$ and $(X,\alpha) \in |\mathcal{K}^{\mathbb{H}}|]$, a $\mathcal{K}^{(\mathbb{T},\ \mathbb{H})}$-morphism $(X,\xi,\alpha) \xrightarrow{f} (X',\xi',\alpha')$ is a $\mathcal{K}$-morphism $X \xrightarrow{f} X'$ which is both a $\mathbb{T}$- and an $\mathbb{H}$-homomorphism and $U^{(\mathbb{T},\ \mathbb{H})}$ is the obvious underlying $\mathcal{K}$-object functor.

The proof of the following proposition may be safely left to the reader.

1.7.4 Proposition. The following statements are valid.

a. $U^{(\mathbb{T},\mathbb{H})}$ creates $\varprojlim$'s.

b. $U^{(\mathbb{T},\mathbb{H})}$ creates coequalizers of $U^{(\mathbb{T},\mathbb{H})}$-contractible pairs.

c. $U^{(\mathbb{T},\mathbb{H})}$ is tripleable iff $U^{(\mathbb{T},\mathbb{H})}$ has a left adjoint.

d. If $\mathbb{T}$, $\mathbb{H}$ are regular then $U^{(\mathbb{T},\mathbb{H})}$ creates regular coimage factorizations and $\mathcal{K}^{(\mathbb{T},\mathbb{H})}$ is regular. []

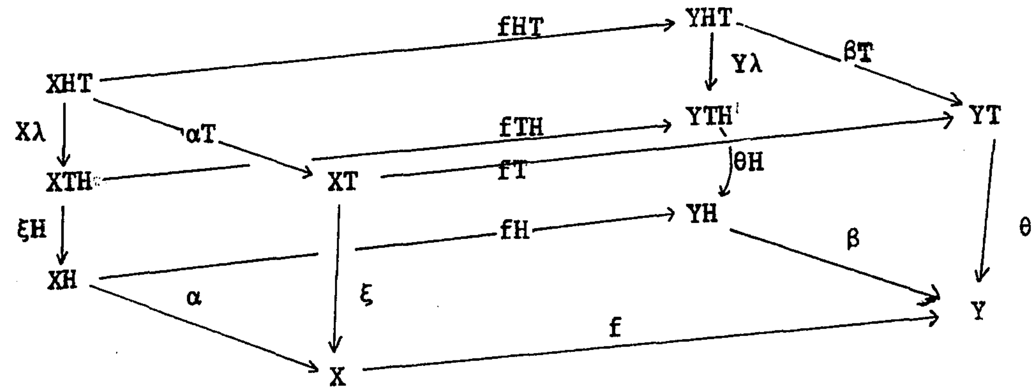
1.7.5 Definition. It is perfectly clear what "subalgebra" and "coequalizer of $U^{(\mathbb{T},\mathbb{H})}$-contractible pair" mean in $\mathcal{K}^{(\mathbb{T},\mathbb{H})}$ (indeed we have already used the latter); define the notion of <u>$\widehat{\ }$-closed subcategory</u> accordingly. The term "Birkhoff subcategory" will be reserved for the regular triple case. Note that if $\mathcal{B} \subset \mathcal{K}^{(\mathbb{T},\mathbb{H})}$ is $\widehat{\ }$-closed and that if $U =_{df}$ the restriction of $U^{(\mathbb{T},\mathbb{H})}$ to $\mathcal{B}$, then it is obvious that U creates coequalizers of U-contractible pairs. Hence U is tripleable iff U has a left adjoint.

1.7.6 Proposition. Let $M \varepsilon [M]$ be a $\mathbb{T}$-$\mathbb{H}$ composite triple, and let λ, $\overline{\mathbb{H}}$, Γ correspond to M under the bijections of 1.7.2. Let be the full subcategory of $\mathcal{K}^{(\mathbb{T},\mathbb{H})}$ generated by those objects (X,ξ,α) satisfying the <u>composite law</u>:

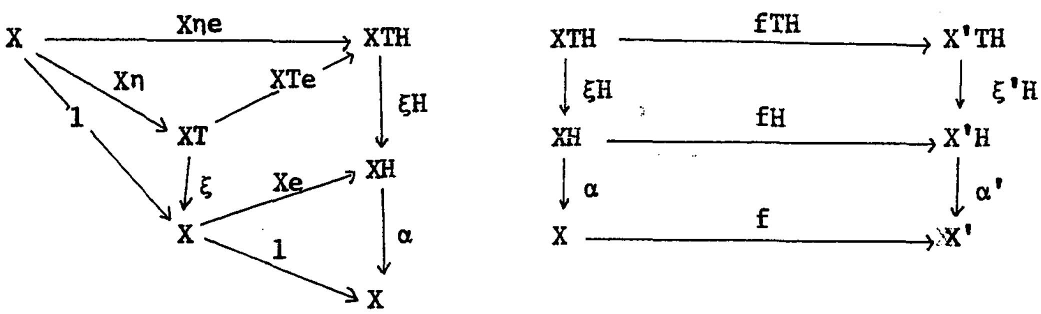
$$\begin{array}{ccc} XHT & \xrightarrow{\alpha T} & XT \\ \downarrow{\scriptstyle X\lambda} & & \\ XTH & CL & \downarrow{\scriptstyle \xi} \\ \downarrow{\scriptstyle \xi H} & & \\ XH & \xrightarrow{\alpha} & X \end{array}$$

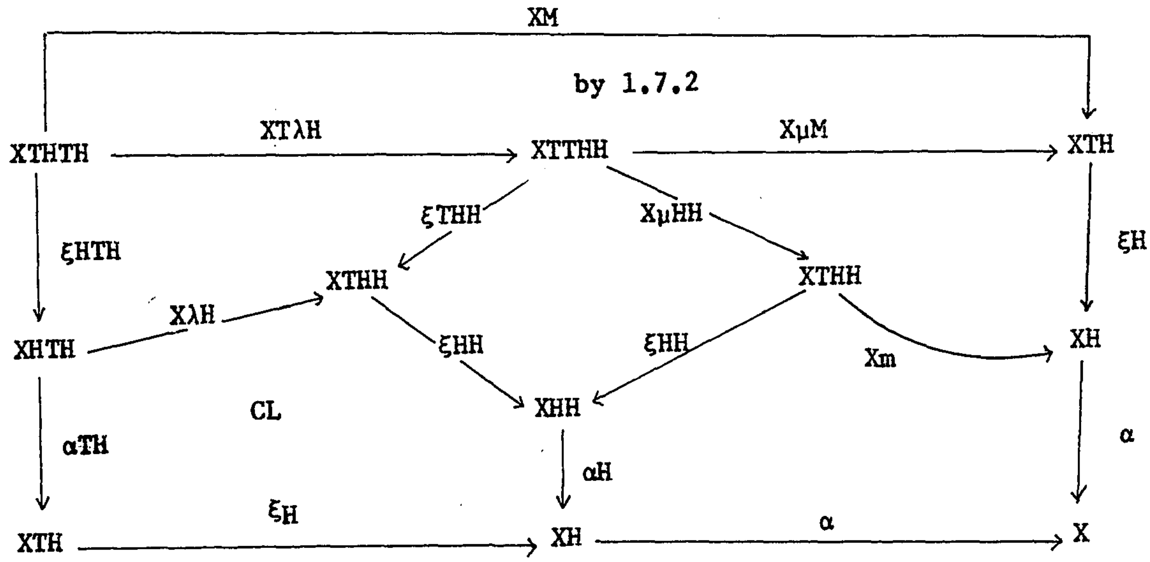
and let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be the restriction of $U^{(\mathbb{T}, \mathbb{H})}$ to $\mathcal{A}$. Define $\mathcal{B} \xrightarrow{V} \mathcal{K} =_{df} ((\mathcal{K}^{\mathbb{T}})^{\bar{\mathbb{H}}} \xrightarrow{U^{\bar{\mathbb{H}}}} \mathcal{K}^{\mathbb{T}} \xrightarrow{U^{\mathbb{T}}} \mathcal{K}$. Define $\mathcal{C} \xrightarrow{W} \mathcal{K} =_{df}$ the underlying $\mathcal{K}$-object functor from $(TH, \eta e, M)$-algebras. Then U, V, W are isomorphic objects in AD($\mathcal{K}$), with W in AL($\mathcal{K}$). Moreover, $\mathcal{A}$ is a $\widehat{\ }$-closed subcategory of $\mathcal{K}^{(\mathbb{T}, \mathbb{H})}$.

Proof. Noting that $(x,\xi)\bar{H} = (XH, X\lambda.\xi H)$, it is trivial to check that $((X,\xi),\alpha) \mapsto (X,\xi,\alpha)$ establishes an isomorphism of U with V. To see that $\mathcal{A}$ is a $\widehat{\ }$-closed subcategory, let $(X,\xi,\alpha) \xrightarrow{f} (Y,\theta,\beta)$ be a morphism in $\mathcal{K}^{(\mathbb{T}, \mathbb{H})}$ and consider the diagram:



All commutes except possibly the left and right faces. If f is mono, right implies left. If f is split epi, so is fHT and left implies right. If f is a typical projection from a product, right's imply left. We turn now to the proof that U is isomorphic with W. Define $\mathcal{A} \xrightarrow{\Psi} \mathcal{C}$ by $(X,\xi,\alpha)\Psi =_{df} (X, \xi H.\alpha)$. The diagrams

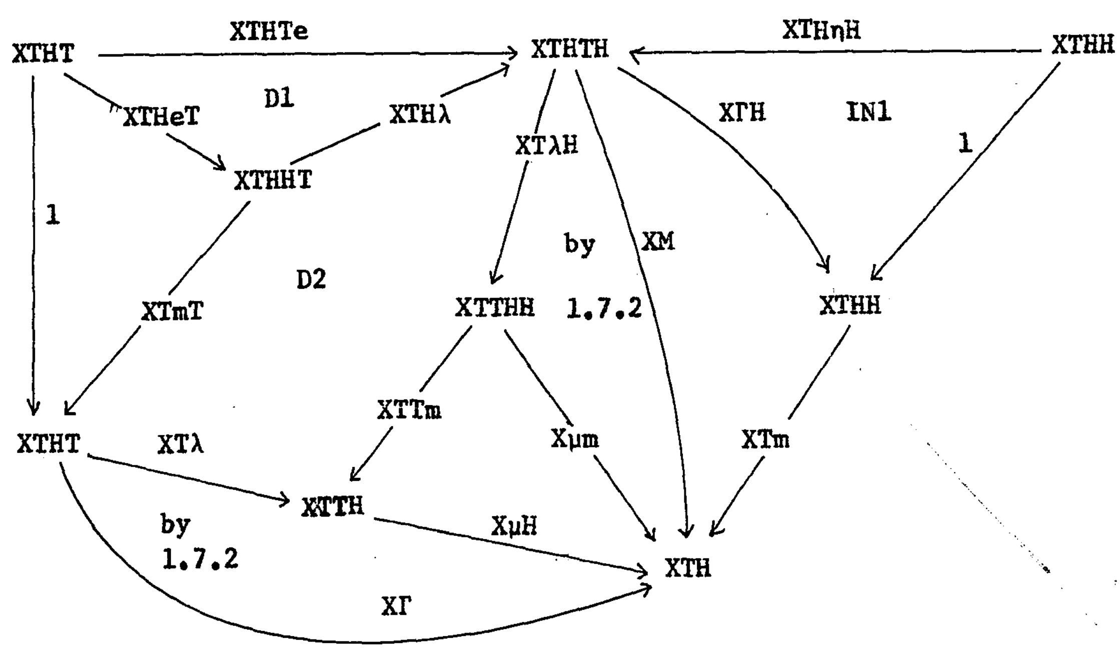




prove that $U \xrightarrow{\Psi} W$ is an AD($\mathcal{K}$)-morphism. The fact that Te, ηH are OPTR-morphisms induces algebraic functors

$$\mathcal{K}^{\mathbb{T}} \xrightarrow{\ Te\circ-\ } \mathcal{C} \xleftarrow{\ \eta H\circ-\ } \mathcal{K}^{\mathbb{H}}$$

and hence a functor $\mathcal{C} \xrightarrow{\tilde{\Psi}} \mathcal{K}^{(\mathbb{T},\, \mathbb{H})}$ defined by $(X,\omega)\tilde{\Psi} =_{df}$ $(X,\ Te.\omega,\ \eta H.\omega)$. $\tilde{\Psi}$ is defined on morphisms by imposing $U^{(\mathbb{T},\, \mathbb{H})} = W$. The first thing to observe is that $(XTH,XM)\tilde{\Psi} = (XTH,X\Gamma,XTm)$ as follows from the diagram:



The same diagram proves that $XTH\lambda . X\Gamma H . XTm = XTmT . X\Gamma$ which shows that $(XTH,XM)\tilde{\Psi} \in \text{obj}\ \mathcal{A}$. Given arbitrary $(X,\omega) \in \text{obj}\ \mathcal{C}$, $(XTH,XM)\tilde{\Psi} \xrightarrow{\omega} (X,\omega)\tilde{\Psi}$ is a $\mathcal{K}^{(\mathbb{T},\mathbb{H})}$-morphism with $XTH \xrightarrow{\omega} X$ split epi in $\mathcal{K}$, so since $\mathcal{A}$ is a $\widehat{\ }$-closed subcategory, $(X,\omega)\tilde{\Psi} \in \text{obj}\ \mathcal{A}$. Hence redefine $\mathcal{C} \xrightarrow{\tilde{\Psi}} \mathcal{A}$, and $\tilde{\Psi}$ is well-defined. Noting that $(XTH,XM)\tilde{\Psi}\Psi = (XTH,X\Gamma,XTm)\Psi = (XTH,X\Gamma H.XTm) = (XTH,XM)$, it follows from 1.4.2 that $\tilde{\Psi}\Psi = 1_{\mathcal{C}}$. Hence Ψ is full and onto on objects. Clearly Ψ is faithful. To complete the proof we show that Ψ is 1-to-1 on objects. Let $(X,\xi,\alpha)\Psi = (Y,\xi,\beta)\Psi$. Clearly $X = Y$ and $(X,\xi H.\alpha) \xrightarrow{1} (X, \theta H.\beta)$ is a homomorphism. As Ψ is full, $(X,\xi) \xrightarrow{1} (X,\theta)$ and $(X,\alpha) \xrightarrow{1} (X,\beta)$ are homomorphisms so that $\xi = \theta$, $\alpha = \beta$. []

<u>**1.7.7 Proposition.**</u> Let $\mathbb{T}$, $\mathbb{H}$ be regular triples, let $M \in [M]$ and let $\mathcal{B}$ be a $\widehat{\ }$-closed subcategory of $\mathcal{K}^{(\mathbb{T},\mathbb{H})}$. Then $\mathbb{T}\mathbb{H}$ is regular and $\mathcal{B} \cap \mathcal{K}^{\mathbb{T}\mathbb{H}}$ is tripleable. []

§1.8 Subalgebras for regular triples.

1.8.1 Definitions. Let $\mathcal{K}$ be a category. If $X \xrightarrow{f} Y$ is a $\mathcal{K}$-morphism and if $f = X \xrightarrow{p} I \rightarrowtail^{i} Y$ is a regular coimage factorization of f, we introduce the notations coim f $=_{dn}$ $X \xrightarrow{p} I$ and im f $=_{dn}$ $I \xrightarrow{i} Y$. If $A \xrightarrow{i} X \xrightarrow{f} Y$ we also denote im i.f by "$Af \rightarrowtail Y$". If $X \xrightarrow{f} Y \xleftarrow{j} B$, we denote the pullback of j along f by "$Bf^{-1} \rightarrowtail X$". That $Bf^{-1} \rightarrow X$ is a monomorphism is easily verified. Depending on $\mathcal{K}$, such constructions may not exist; if $\mathcal{K} = \mathcal{S}$ the ordinary images and inverse images work. If $A \xrightarrow{i} X \xleftarrow{j} B$ in $\mathcal{K}$, $A \subset B =_{df}$ i factors (necessarily uniquely and by a mono) through B. Given any family of monomorphisms into an object, the collective pullback is called the intersection of the family, and we use the symbol "$\bigcap$". Again, it is easily verifies that the intersection is a subobject. When $\mathcal{K} = \mathcal{S}$ this construction is the ordinary intersection of subsets.

Let $\mathbb{T}$ be a triple in $\mathcal{K}$, let (X,ξ) be a $\mathbb{T}$-algebra and let $A \xrightarrow{i} X \in \mathcal{K}$. The subalgebra of (X,ξ) generated by A, $=_{dn}$ $\langle A\rangle \rightarrowtail (X,\xi)$, is defined to be the intersection :

$$\bigcap [(D,\alpha) \leq (X,\xi) : A \subset D] \rightarrowtail (X,\xi)$$

When $\langle A\rangle$ exists it is in fact the smallest subalgebra of (X,ξ) containing A.

For the rest of this section, let $\mathbb{T}$, $\mathbb{H}$ be regular triples in a category $\mathcal{K}$.

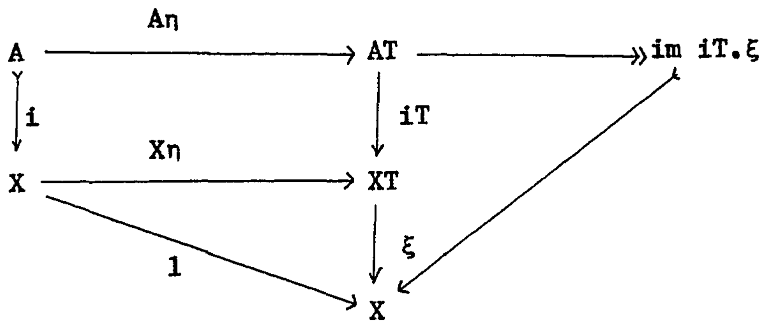
1.8.2 Proposition. Let (X,ξ) be a $\mathbb{T}$-algebra, and let $A \xrightarrow{i} X$.

The following statements are valid.

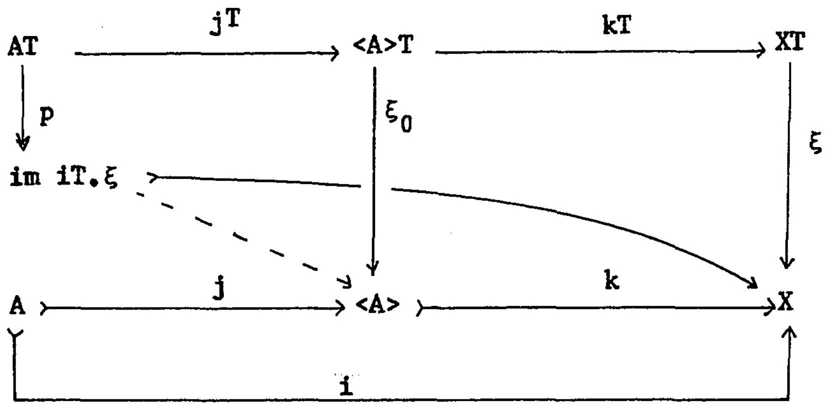
a. $\langle A\rangle = \text{im } iT.\xi$.

b. If $(X,\xi) \xrightarrow{f} (Y,\theta) \in \mathcal{K}^{\mathbb{T}}$, $\langle A\rangle f = \langle Af\rangle$.

__Proof.__ __a.__ The diagram

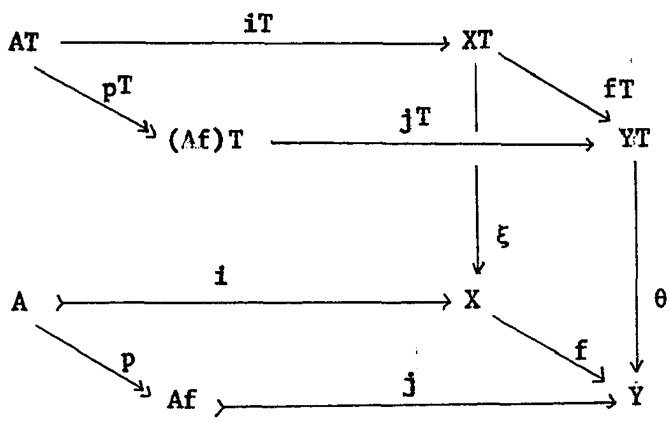


proves that $A \subset \text{im } iT.\xi$, and hence $\langle A\rangle \subset \text{im } iT.\xi$. Conversely, consider



$jT.\xi_0 \in \text{reg}(p)$ because k is mono. Therefore, $\text{im } iT.\xi \subset \langle A\rangle$.

__b.__



$\langle Af\rangle = \text{im } jT.\theta = \text{im } pT.jT.\theta = \text{im } iT.\xi.f = \langle A\rangle f.$ []

Note: Observe, in the above, that $iT.\xi$ is the homomorphic extension of i.

<u>1.8.3 Definition.</u> $(X,\xi,\alpha) \in \text{obj } \mathcal{K}^{(\mathbb{T},\mathbb{H})}$ is a <u>$\mathcal{T}$-$\mathcal{H}$ quasicomposite algebra</u> if for every $\mathcal{K}$-monomorphism $A \overset{i}{\rightarrowtail} X$, the $\mathcal{K}^{(\mathbb{T},\mathbb{H})}$-subalgebra generated by A is $\langle\langle A\rangle_{\mathbb{T}}\rangle_{\mathbb{H}}$. (What we mean by "subalgebra generated by" here is clear.) Equivalently, if A is a $\mathcal{T}$-subalgebra, so is $\langle A\rangle_{\mathbb{H}}$.

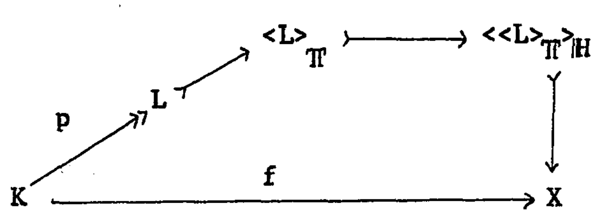
<u>1.8.4 Proposition.</u> Let $\mathcal{T}$ $\mathcal{H}$ be a $\mathcal{T}$-$\mathcal{H}$ composite triple and let $(X,\xi,\alpha) \in \text{obj } \mathcal{K}^{\mathbb{T}\mathbb{H}}$. Then (X,ξ,α) is a $\mathcal{T}$-$\mathcal{H}$ quasicomposite algebra.

<u>Proof.</u> By 1.7.6, (X,ξ,α) qua algebra over the triple $(\mathbb{T}\mathbb{H},\eta e,M)$ is $(X,\xi H.\alpha)$. Let $A \overset{i}{\rightarrowtail} X$. By 1.8.2, $\langle A\rangle_{\mathbb{T}\mathbb{H}} = \text{im } i\mathbb{T}\mathbb{H}.\xi H.\alpha$. Similarly, we construct $\langle A\rangle_{\mathbb{T}}$ from the coimage factorization $iT.\xi = AT \overset{p}{\twoheadrightarrow} \langle A\rangle_{\mathbb{T}} \overset{j}{\rightarrowtail} X$. Hence we have $ATH \overset{pH}{\twoheadrightarrow} \langle A\rangle_{\mathbb{T}}H \overset{jH}{\longrightarrow} XH \overset{\alpha}{\longrightarrow} X$, so that $\langle\langle A\rangle_{\mathbb{T}}\rangle_{\mathbb{H}} = \text{im } jH.\alpha = \text{im } pH.jH.\alpha = \text{im } i\mathbb{T}H.\xi H.\alpha = \langle A\rangle_{\mathbb{T}\mathbb{H}}$. []

<u>1.8.5 Proposition.</u> Let $\mathcal{B}$ be a $\widehat{\ }$-closed subcategory of $\mathcal{K}^{(\mathbb{T},\mathbb{H})}$ consisting of $\mathcal{T}$-$\mathcal{H}$ quasicomposite algebras. Then $U =_{df} U^{(\mathbb{T},\mathbb{H})}$ restricted to $\mathcal{B}$ is tripleable.

<u>Proof.</u> By 1.7.4, 1.7.5 we need only show U has a left adjoint. Since $\mathcal{B}$ is closed under $\varprojlim$'s, $\mathcal{B}$ has and U preserves $\varprojlim$'s. Since $\mathcal{K}^{(\mathbb{T},\mathbb{H})}$ is legitimate, the adjoint functor theorem applies. We need only show U satisfies the solution set condition. Let $K \in \text{obj } \mathcal{K}$.

Let $\mathcal{S}_1$ be a representative set of regular epimorphisms with domain K. Let $\mathcal{S}_2$ be a representative set of split epimorphisms with domain of form LT for some L in $\mathcal{S}_1$. Let $\mathcal{S}_3$ be a representative set of split epimorphisms with domain of form LH for some L in $\mathcal{S}_2$. Now suppose $(X,\xi,\alpha) \in \text{obj}\, \mathcal{K}^{(\mathbb{T},\mathbb{H})}$ and $K \xrightarrow{f} X \in \mathcal{K}$. There exists $L \in \mathcal{S}_1$ with $f = K \xrightarrow{p} L \rightarrowtail{i} X$. There exists a model for $\langle L\rangle_{\mathbb{T}}$ such that the canonical split epimorphism $LT \xrightarrow{\theta} \langle L\rangle_{\mathbb{T}}$ is in $\mathcal{S}_2$; (we can always transport a structure map through a $\mathcal{K}$-isomorphism). Similarly there exists a split epimorphism $\langle L\rangle_{\mathbb{T}}H \xrightarrow{\beta} \langle\langle L\rangle_{\mathbb{T}}\rangle_{\mathbb{H}} \in \mathcal{S}_3$. Hence we have



proved that f factors through a set of objects $[\langle\langle L\rangle_{\mathbb{T}}\rangle_{\mathbb{H}}]$. The crucial point is our hypothesis which says that $\langle\langle L\rangle_{\mathbb{T}}\rangle_{\mathbb{H}}$ is in $|\mathcal{B}|$. []

1.8.5 will be used to construct compact algebras in 2.3.4.

CHAPTER 2. TRIPLES IN SETS

§2.1 Some properties of $\mathcal{S}^{\mathbb{T}}$.

2.1.1 Proposition. In the category of sets, $\mathcal{S}$, the following notions are equivalent: contractible coequalizer, split epimorphism, coequalizer, regular epimorphism, onto function.

Proof. Fix $X \xrightarrow{f} Y \in \mathcal{S}$. First suppose f is an epimorphism. If there exists $y \in Y - \text{im } f$ it is clear how to define $(a,b) : Y \to \{0,1\}$ with $f.a = f.b$ but $a \neq b$; hence f is onto. If f is onto, then for every $y \in Y$ there exists $yd \in X$ with $\langle yd,f\rangle = y$, so that f is split epi. Equivalently, f is a contractible coequalizer by 0.7.5. []

2.1.2 Proposition. Every triple in sets is regular.

Proof. That $\mathcal{S}$ is a regular category is well-known; ordinary image factorizations provide the regular coimage factorizations by 2.1.1; they are also, in fact, regular image factorizations. Now let $\mathbb{T} = (T,\eta,\mu)$ be a triple in $\mathcal{S}$. Clearly T preserves all epimorphisms and all monomorphisms with non-empty domain since these are split. To complete the proof we must show $(\phi \overset{i}{\rightarrowtail} X)T$ is mono. If $\phi T = \phi$ this is clear. Otherwise, there exists a function $X \xrightarrow{f} \phi T$. Since $(\phi T,-)\mathcal{S}^{\mathbb{T}} \simeq (\phi,(-)U^{\mathbb{T}})\mathcal{S} \simeq P$ (from now on we will use "P" to denote the 1-point set), ϕT is an initial object in $\mathcal{S}^{\mathbb{T}}$. Therefore, $\phi T \xrightarrow{iT} XT \xrightarrow{fT} \phi TT \xrightarrow{\phi\mu} \phi T = 1_{\phi T}$ and iT is mono. []

<u>2.1.3 Proposition.</u> Define functors $\mathcal{S} \xrightarrow{T_i} \mathcal{S}$ $(i = 1,2)$ by $(X \xrightarrow{f} Y)T_1 =_{df} P \xrightarrow{1} P$ and $(X \xrightarrow{f} Y)T_2 =_{df} \phi \to \phi$ (if $X = \phi = Y$), $\phi \to P$ (if $X = \phi \neq Y$) or $P \to P$ (if $X \neq \phi \neq Y$). Let $\mathbb{T} = (T,\eta,\mu)$ be a triple in $\mathcal{S}$. The following statements are equivalent.

a. T is faithful.

b. T is not naturally equivalent to either T_1 or T_2.

c. There exists $(X,\xi) \in \text{obj } \mathcal{S}^{\mathbb{T}}$ with crd $X > 1$.

d. η is pointwise mono.

<u>Proof.</u> <u>a implies b.</u> Clear, as T_1, T_2 are not faithful.

<u>b implies c.</u> As $1_{\mathcal{S}} \cong (P,-)\mathcal{S}$ and $\eta \in (1_{\mathcal{S}},T)$n.t., it follows from the Yoneda lemma that $PT \neq \phi$. For each set X, observe $X \neq \phi$ implies $XT \neq \phi$. Since we assume T is not naturally equivalent to T_1 or T_2 there exists a set X with crd $XT > 1$. But then $(XT, X\mu)$ is a $\mathbb{T}$-algebra.

<u>c implies d.</u> Let Y be a set. By taking sufficiently large powers, the hypothesis guarantees that there is a $\mathbb{T}$-algebra (X,ξ) and a monomorphism $Y \overset{i}{\rightarrowtail} X$. As $X\eta.\xi = 1_X$, $X\eta$ is mono, so $i.X\eta = Y\eta.iT$ is mono and then $Y\eta$ is mono.

<u>d implies a.</u> If $(f,g) : X \to Y$, and if $fT = gT$ then $f.Y\eta = g.Y\eta$ by naturality and then $f = g$ as $Y\eta$ is mono. []

<u>2.1.4 Definition.</u> Let $\mathbb{T}$ be a triple in $\mathcal{S}$. Say that $\mathbb{T}$ is <u>consistent</u> if $\mathbb{T}$ satisfies any of the equivalent conditions of 2.1.3. (This terminology goes back to Lawvere, [20].) In view of (c) in 2.1.3, the inconsistent case is not interesting.

Whenever X,Γ are sets we will always choose as a model for the cartesian power X^{Γ} the set of all functions from Γ to X together with the various γ-evaluation maps. Of course we must invoke special properties of the category of sets to do this.

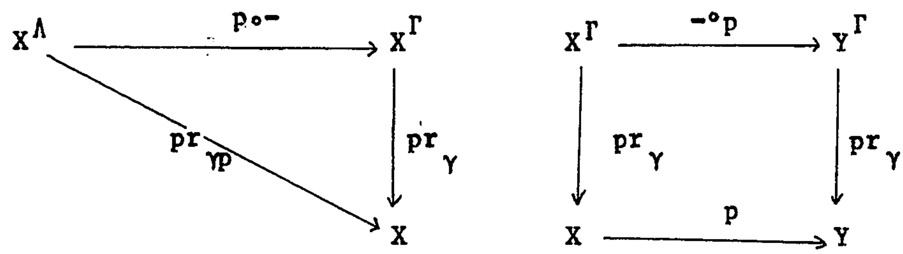
For the balance of this section fix a triple $\mathbb{T} = (T, \eta, \mu)$ in $\mathcal{S}$.

<u>2.1.5 Proposition.</u> The following statements are valid.

a. Let (X, ξ) be a $\mathbb{T}$-algebra and let $\Gamma \xrightarrow{p} \Lambda$ be a function. Then $X^{\Lambda} \xrightarrow{p\circ -} X^{\Gamma}$ is a $\mathbb{T}$-homomorphism.

b. Let $(X, \xi) \xrightarrow{p} (Y, \theta)$ be a $\mathbb{T}$-homomorphism and let Γ be a set. Then $X^{\Gamma} \xrightarrow{-\circ p} Y^{\Gamma}$ is a $\mathbb{T}$-homomorphism.

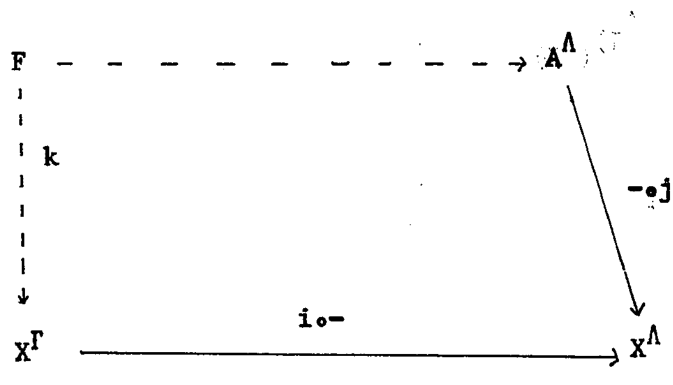
<u>Proof.</u>



[]

<u>2.1.6 Proposition.</u> Let $(A, \xi_0) \xrightarrow{j} (X, \xi)$ be a subalgebra in $\mathcal{S}^{\mathbb{T}}$, and let $\Lambda \xrightarrow{i} \Gamma$ be a 1-to-1 function (= mono in $\mathcal{S}$). $F =_{df} [\Gamma \xrightarrow{f} X : \Lambda f \subset A]$. Then $F \leq X^{\Gamma}$.

<u>Proof.</u> The inclusion map $F \xrightarrow{k} X^{\Gamma}$ arises as the pullback:



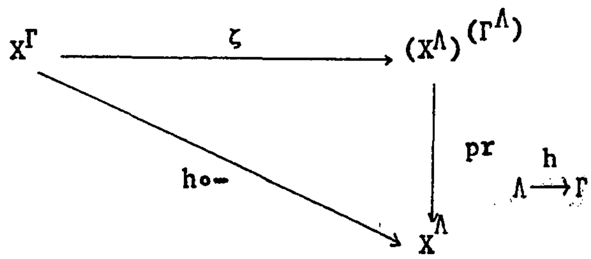
[]

<u>2.1.7 Proposition.</u> Let (X,ξ) be a $\prod$-algebra and let Γ,Λ be sets. The following statements are valid.

a. $X^\Gamma \xrightarrow{\zeta} (X^\Lambda)^{(\Gamma^\Lambda)}$, $p \mapsto -\circ p$, is a $\prod$-monomorphism.

b. Let $F \subset \Gamma^\Lambda$, let $G \leq X^\Lambda$ and define $H =_{df} [p : p \in X^\Gamma$ and $Fp \subset G]$. Then $H \leq X^\Gamma$.

<u>Proof.</u> <u>a.</u> The commutativity of each diagram:



shows that ζ is a $\prod$-homomorphism. Let $\Lambda \xrightarrow{\tilde{\gamma}} \Gamma$ be the constant function induced by γ for each $\gamma \in \Gamma$. If p,q are in X^Γ with $-\circ p = -\circ q$, then for all $\gamma \in \Gamma$ we have $\widetilde{\gamma p} = \tilde{\gamma}.p = \tilde{\gamma}.q = \widetilde{\gamma q}$ so that $p = q$. (The case $\Gamma = \phi$ requires separate proof, but is trivial.)

<u>b.</u> $M =_{df} [\Gamma^\Lambda \xrightarrow{f} X^\Lambda : Ff \subset G]$, $N =_{df} [\Gamma^\Lambda \xrightarrow{-\circ p} X^\Lambda : p \in X^\Gamma]$. Then $M \leq (X^\Lambda)^{(\Gamma^\Lambda)}$ by 2.1.6. $N \leq (X^\Lambda)^{(\Gamma^\Lambda)}$ by (a). The function $M \cap N \longrightarrow X^\Gamma$, $-\circ p \longmapsto p$ is the restriction of ζ^{-1} to $M \cap N$. Its image is H, so $H \leq X$. []

§2.2 Operations.

2.2.1 Definition. Let n be a set. "Raising to the n^{th} power" is a functor

$$\mathcal{S} \xrightarrow{\quad 1^n \quad} \mathcal{S}$$

$$X \xrightarrow{f} Y \quad \longmapsto \quad X^n \xrightarrow{f^n} Y^n$$

More generally, if $\mathcal{A} \xrightarrow{U} \mathcal{S}$ is any set-valued functor, define $\mathcal{A} \xrightarrow{U^n} \mathcal{S} =_{df} U1^n$.

For this section, fix a triple $\mathbb{T} = (T,\eta,\mu)$ in $\mathcal{S}$. For simplicity write $U =_{dn} U^{\mathbb{T}}$, $F =_{dn} F^{\mathbb{T}}$, $\varepsilon =_{dn} \varepsilon^{\mathbb{T}}$.

2.2.2 Proposition. Let n be a set. The following four classes are sets and are isomorphic by the indicated correspondences (in terms of the first set.)

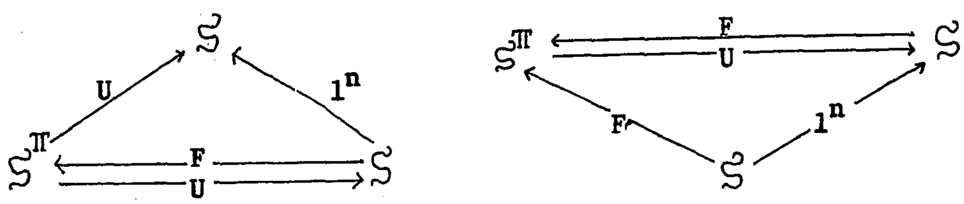
a. $(1^n,T)$n.t. $\qquad 1^n \xrightarrow{g} T$

b. (U^n,U)n.t. $\qquad U^n \xrightarrow{Ug} UT \xrightarrow{\varepsilon U} U$

c. $(1^nF,F)$n.t. $\qquad 1^nF \xrightarrow{gF} TF \xrightarrow{F\varepsilon} F$

d. nT $\qquad \langle 1_n,\ n^n \xrightarrow{ng} nT \rangle$

Proof. Use the diagrams



and the lemma of 0.3 to check (b) ≅ (a) ≅ (c). (a) ≅ (d) is just the Yoneda Lemma. []

2.2.3 Definition. Any of the four sorts of thing in 2.2.2 deserve to be called an <u>n-ary operation</u> of $\mathbb{T}$. For definiteness, define $\mathbb{T}(n)$ for each set n by $\mathbb{T}(n) =_{df} (1^n, T)n.t.$. If (X,ξ) is a $\mathbb{T}$-algebra and if $g \in \mathbb{T}(n)$ define $\xi^g =_{df}$ the function $X^n \xrightarrow{Xg} XT \xrightarrow{\xi} X$, that is ξ^g is the $(X,\xi)^{th}$ component of the natural transformation from U^n to U corresponding to g. ξ^g is called an <u>n-ary operation of</u> (X,ξ) and the set of all such is denoted by "$\mathcal{O}_n(X,\xi)$".

2.2.4 Proposition. Let (X,ξ), (Y,θ) be $\mathbb{T}$-algebras, and let $X \xrightarrow{f} Y$ be a function. The following statements are equivalent.

a. f is a $\mathbb{T}$-homomorphism.

b. For every set n and for every $g \in \mathbb{T}(n)$ the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{f^n} & Y^n \\ {\scriptstyle \xi^g}\downarrow & (*)_g & \downarrow{\scriptstyle \theta^g} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

c. $(*)_g$ commutes for every $g \in \mathbb{T}(X)$.

<u>Proof.</u> <u>a implies b.</u>

$$\begin{array}{ccccc} X^n & \xrightarrow{Xg} & XT & \xrightarrow{\xi} & X \\ \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle fT} & & \downarrow{\scriptstyle f} \\ Y^n & \xrightarrow{Yg} & YT & \xrightarrow{\theta} & Y \end{array}$$

<u>b implies c.</u> obvious

__c implies a.__ Consider the diagram of "a implies b" with $n = X$. Let $x \varepsilon XT$. By the Yoneda Lemma there exists $g \varepsilon \mathfrak{T}(X)$ with $\langle 1_X, Xg\rangle = x$. We have $\langle x, \xi.f\rangle = \langle 1_X, \xi^g.f\rangle = \langle 1_X, f^X.\theta^g\rangle = \langle 1_X, Xg.fT.\theta\rangle = \langle x, fT.\theta\rangle$. []

__2.2.5 Proposition.__ For each set n define $nT_0 =_{df} nT - [\text{im } n\eta]$, and define $\mathfrak{T}_0(n) =_{df}$ the subset of $\mathfrak{T}(n)$ corresponding to nT_0; (if $i \varepsilon n$, the $(X,\xi)^{th}$ component of the corresponding g is the i^{th} projection). Let $\aleph$ be a cardinal number. The following statements are equivalent.

a. For every $\mathfrak{T}$-algebra (X,ξ), set n and $g \varepsilon \mathfrak{T}_0(n)$ there exists a subset $m \xrightarrow{i} n$ and there exists $h \varepsilon \mathfrak{T}(m)$ with crd $m < \aleph$ and $X^n \xrightarrow{\xi^g} X = X^n \xrightarrow{i\circ-} X^m \xrightarrow{\xi^h} X$.

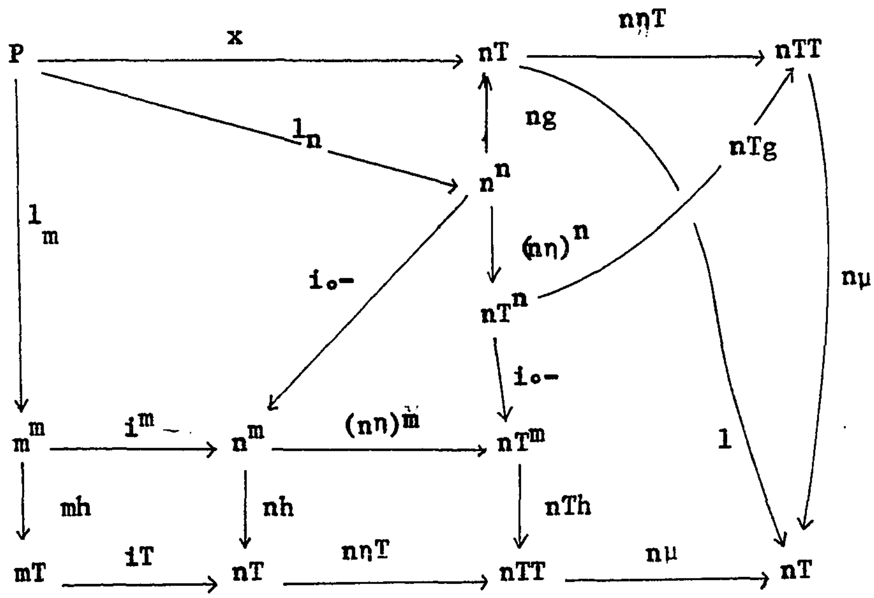
b. If n is a set and if $x \varepsilon nT$ then there exists a subset $m \xrightarrow{i} n$ with crd $m < \aleph$ and $x \varepsilon \text{im}(mT \xrightarrow{iT} nT)$.

c. For every $\mathfrak{T}$-algebra (X,ξ), subset $A \rightarrowtail X$ and $x \varepsilon \langle A\rangle - A$, there exists a subset $F \subset A$ with crd $F < \aleph$ and $x \varepsilon \langle F\rangle$.

__Proof.__ __a implies b.__ Let n be a set, $x \varepsilon nT_0$. There exists unique $g \varepsilon \mathfrak{T}_0(n)$ with $\langle 1_n, ng\rangle = x$. By hypothesis, there exists $m \xrightarrow{i} n$ and $h \varepsilon \mathfrak{T}(m)$ with crd $m < \aleph$ and $nTg.n\mu = i\circ-.nTh.n\mu$. We have the diagram at the top of the next page which shows $x = \langle 1_m, mh.iT\rangle$ as desired.

__b implies c.__ We use 1.8.2. Let (X,ξ) be a $\mathfrak{T}$-algebra, let $A \xrightarrow{i} X$ and let $x \varepsilon \langle A\rangle - A$. Consider

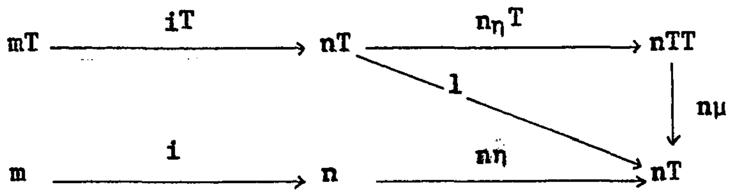
$$\begin{array}{ccccccc}
FT & \overset{kT}{\dashrightarrow} & AT & \xrightarrow{jT} & \langle A\rangle T & \xrightarrow{iT} & XT \\
 & & \uparrow{\scriptstyle A\eta} & & \langle A\rangle\eta \uparrow\downarrow \xi_0 & & \downarrow{\scriptstyle \xi} \\
F & \overset{k}{\dashrightarrow} & A & \xrightarrow{j} & \langle A\rangle & \xrightarrow{i} & X
\end{array}$$



(diagram for "a implies b")

As <A> = im jT.iT.ξ there exists y ε AT with <y,jT.iT.ξ> = x. By hypothesis there exists $F \overset{k}{\rightarrowtail} A$ with crd F < $\aleph$ and y ε im kT. Therefore, x ε im kT.jT.iT.ξ = <F>.

<u>c implies a.</u> Let (X,ξ) be a $\prod$-algebra, let n be a set and let g ε $\prod_0$(n). Set x $=_{df}$ <1_n,ng> ε nT_0. As nT = <im nη>, we have by hypothesis that there exists $m \overset{i}{\rightarrowtail} n$ with crd m < $\aleph$ and x ε <m(nη)>. From the diagram:



we have <m(nη)> = im iT.nηT.nμ = im iT. Hence there exists y ε mT with <y,iT> = x. There exists unique h ε $\prod$(m) with <1_m,mh> = x. Using the Yoneda Lemma, we check that Xg = i∘-.Xh:

$$
\begin{array}{ccc}
n \xrightarrow{f} x & \longmapsto & \langle x,\ nT \xrightarrow{fT} XT \rangle \\
\downarrow & & \| \\
m \xrightarrow{i} n \xrightarrow{f} x & \longmapsto & \langle y,\ mT \xrightarrow{iT} nT \xrightarrow{fT} XT \rangle
\end{array}
$$

In particular, $\xi^g = Xg.\xi = i\circ -.\xi^h$. []

<u>2.2.6 Definition.</u> $\mathfrak{T}$ has a rank $=_{df}$ there exists a cardinal number $\aleph$ with either of the three equivalent properties of 2.2.5. In this case we also say "rnk($\mathfrak{T}$) $\leq \aleph$". If $\mathfrak{T}$ has a rank then there exists a least {regular} $\aleph$ with rnk($\mathfrak{T}$) $\leq \aleph$; it is called the <u>{regular} rank of</u> $\mathfrak{T}$, and is written "{r}rnk($\mathfrak{T}$)". rrnk($\mathfrak{T}$) $\leq \aleph_0$ is classical universal algebra. 2.2.5 (a) says that operations are finitary. (a) iff (c) is a classical theorem for $\aleph = \aleph_0$ which may be found in [5]. For perspective on "rank" see [23].

<u>2.2.7 Example; G-sets.</u> Let G be a monoid, $\mathcal{G} =_{df} (-\times G, e, m)$ the resulting triple in sets as in 1.1.6. Let n be a set and let $(i,g) \in n\times G$. The resulting natural transformation $\zeta \in \mathcal{G}(n)$ has X^{th} component

$$
\begin{array}{ccc}
X^n & \xrightarrow{\qquad X\zeta \qquad} & X \times G \\
n \xrightarrow{f} x & \longmapsto & \langle (i,g),\ n\times G \xrightarrow{f\times 1} X\times G \rangle
\end{array}
$$

that is $\zeta = X^n \xrightarrow{pr_i} X \xrightarrow{(1,g)} X\times G$. It follows at once that rnk($\mathcal{G}$) $= 2$. The only important operations of a G-set $X \times G \xrightarrow{\alpha} X$ are the G-indexed unary operations $\alpha^g = X \xrightarrow{(1,g)} X \times G \xrightarrow{\alpha} X$. Notice, too, that the symbol e may be not-too-ambiguously used as the monoid unit, for they correspond under the Yoneda correspondence; nor are the symbols g ambiguous.

2.2.8 Proposition. Let (X,ξ) be a $\mathbb{T}$-algebra, and let n be a set. The following statements are valid.

a. $\mathcal{O}_n(X,\xi)$ is the subalgebra of $X^{(X^n)}$ generated by the evaluation maps $\{X^n \xrightarrow{ev_i} X : i \in n\}$.

b. $nT \xrightarrow{\zeta} \mathcal{O}_n(X,\xi)$ is a $\mathbb{T}$-homomorphism onto.

$\langle 1_n, ng\rangle \longmapsto \xi g$

Proof. Consider the function $n \xrightarrow{ev} X^{(X^n)}$ sending i to ev_i. For all $f \in X^n$, $g \in \mathbb{T}(n)$ we have the diagram:

$$\begin{array}{ccccc}
n^n & \xrightarrow{\quad ev^n \quad} & [X^{(X^n)}]^n & \xrightarrow{\quad pr_f^{\,n} \quad} & X^n \\
\Big\downarrow{\scriptstyle ng} & & \Big\downarrow{\scriptstyle X^{(X^n)}g} & & \Big\downarrow{\scriptstyle Xg} \\
nT & \xrightarrow{\quad evT \quad} & [X^{(X^n)}]T & \xrightarrow{\quad pr_fT \quad} & XT \\
 & & \Big\downarrow{\scriptstyle \xi^{(X^n)}} & \text{defn. } \xi^{(X^n)} & \Big\downarrow{\scriptstyle \xi} \\
 & & X^{(X^n)} & \xrightarrow{\quad pr_f \quad} & X
\end{array}$$

Since $\langle 1_n, ev^n.pr_f^{\,n}\rangle = f$ this shows that $\zeta = evT.\xi^{(X^n)}$, and hence ζ is a $\mathbb{T}$-homomorphism. Using 1.8.2 we have $\mathcal{O}_n(X,\xi) = \operatorname{im}\zeta = \operatorname{im}(evT.\xi^{(X^n)}) = \langle \operatorname{im} ev\rangle = \langle [ev_i]\rangle$. []

2.2.9 Proposition. Let (X,ξ) be a $\mathbb{T}$-algebra, and let $A \xrightarrow{i} X$. The following statements are valid.

a. $\langle A\rangle = i\,\mathcal{O}_A(X,\xi)$.

b. $A = \langle A\rangle$ iff for every g in $\mathbb{T}(A)$, $A^A \xrightarrow{i^A} X^A \xrightarrow{\xi g} X$ factors through i.

<u>Proof.</u> <u>a.</u> If $a \in A$, $\langle ev_a, X^{(X^A)} \xrightarrow{pr_i} X\rangle = a$. Hence using 2.2.8 we have $\langle A\rangle = \langle[ev_a : a \in A]\ pr_i\rangle = \langle[ev_a]\rangle pr_i = \mathcal{O}_A(X,\xi)\ pr_i$ $= i\ \mathcal{O}_A(X,\xi)$.

<u>b.</u> If $(A,\xi_0) \leq (X,\xi)$ the desired factorization is ξ^g. Conversely,

$$\begin{array}{ccc} X^A & \xrightarrow{\ \xi^g\ } & X \\ \uparrow{\scriptstyle i^A} & & \uparrow{\scriptstyle i} \\ A^A & \dashrightarrow & A \end{array}$$

suppose $i^A.\xi^g$ factors through A for all $g \in \Pi(A)$. Evaluating at $1_A^{\,i}$ we have $i\ \mathcal{O}_A(X,\xi) = \langle A\rangle \subset A$. []

<u>2.2.10 Proposition.</u> Let (X,ξ) be a Π-algebra, let $(A,\xi_0) \overset{i}{\rightarrowtail} (X,\xi)$ be a subalgebra and let n be a set. Then

$$\begin{array}{ccc} \mathcal{O}_n(X,\xi) & \xrightarrow{\ \zeta\ } & \mathcal{O}_n(A,\xi_0) \\ \xi^g & \longmapsto & \xi_0^g \end{array}$$

is a Π-homomorphism onto.

<u>Proof.</u> Let $g \in \Pi(n)$. 2.1.5 and the diagram on the left produce

$$\begin{array}{ccccc} A^n & \xrightarrow{\ Ag\ } & AT & \xrightarrow{\ \xi_0\ } & A \\ \downarrow{\scriptstyle i^n} & & \downarrow{\scriptstyle iT} & & \downarrow{\scriptstyle i} \\ X^n & \xrightarrow{\ Xg\ } & XT & \xrightarrow{\ \xi\ } & X \end{array} \qquad \begin{array}{ccccc} \mathcal{O}_n(A,\xi_0) & & \rightarrowtail & & A^{(A^n)} \\ \uparrow{\scriptstyle \zeta} & & & & \downarrow{\scriptstyle -\circ i} \\ \mathcal{O}_n(X,\xi) & \rightarrowtail & X^{(X^n)} & \xrightarrow{\ i^n\circ -\ } & X^{(A^n)} \end{array}$$

the image factorization of Π-homomorphisms on the right. []

<u>2.2.11 Proposition.</u> Let (Y,θ) be a quotient algebra of (X,ξ). Then

$$\begin{array}{ccc} \mathcal{O}_n(X,\xi) & \xrightarrow{\;\zeta\;} & \mathcal{O}_n(Y,\theta) \\ \xi^g & \longmapsto & \theta^g \end{array}$$

is a $\ulcorner\prod\urcorner$-homomorphism onto.

__Proof.__ There exists $Y \xrightarrow{i} X \xrightarrow{f} Y$ with $i.f = 1_Y$ and with f a $\ulcorner\prod\urcorner$-homomorphism. Let $g \in \ulcorner\prod\urcorner(n)$ and consider the diagram:

$$\begin{array}{ccccc} X^n & \xrightarrow{\;Xg\;} & XT & \xrightarrow{\;\xi\;} & X \\ {\scriptstyle i^n}\uparrow\;\downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle fT} & & \downarrow{\scriptstyle f} \\ Y^n & \xrightarrow{\;Yg\;} & YT & \xrightarrow{\;\theta\;} & Y \end{array}$$

$i^n.\xi^g.f = i^n.f^n.\theta^g = (i.f)^n\theta^g = \theta^g$. Therefore ζ is well-defined and in fact $\zeta = i^n\circ\!-\!\circ f$ which is a $\ulcorner\prod\urcorner$-homomorphism by 2.1.5. []

__2.2.12 Proposition.__ Let $[(X_i,\xi_i) : i \in I]$ be a set of $\ulcorner\prod\urcorner$-algebras and let n be a set. Then $\mathcal{O}_n(\prod X_i,\xi) \xrightarrow{\;\zeta\;} \prod\mathcal{O}_n(X_i,\xi_i)$ defined by $\xi^g \longmapsto \prod\xi_i^g$ is a $\ulcorner\prod\urcorner$-monomorphism.

__Proof.__ Let $g \in \ulcorner\prod\urcorner(n)$. The commutativity of the diagram:

$$\begin{array}{ccccccc} \prod X_i^n & \xrightarrow{\;\chi\;} & (\prod X_i)^n & \xrightarrow{\;(\prod X_i)g\;} & (\prod X_i)T & \xrightarrow{\;\xi\;} & \prod X_i \\ & {\scriptstyle pr_i}\searrow & \downarrow{\scriptstyle pr_i^n} & & \downarrow{\scriptstyle pr_iT} & & \downarrow{\scriptstyle pr_i} \\ & & X_i^n & \xrightarrow{\;X_ig\;} & X_iT & \xrightarrow{\;\xi_i\;} & X_i \end{array}$$

for all i (where χ is the canonical bijection) proves that $\xi^g\zeta = \chi\xi^g$, and hence that ζ is 1-to-1. Since each (X_i,ξ_i) is a quotient algebra of $(\prod X_i,\xi)$, it follows from 2.2.11 that each ζpr_i is a homomorphism, so that in fact ζ is a homomorphism. []

Fix another triple $\tilde{\mathbb{T}} = (\tilde{T},\tilde{\eta},\tilde{\mu})$ in $\mathcal{S}$.

<u>**2.2.13 Proposition.**</u> Let $(X,\xi,\tilde{\xi}) \in \text{obj}\ \mathcal{S}^{(\mathbb{T},\tilde{\mathbb{T}})}$. The following statements are equivalent.

a. ξ^g is a $\tilde{\mathbb{T}}$-homomorphism for all $g \in \mathbb{T}(n)$, for all n.

b. $\tilde{\xi}^h$ is a $\mathbb{T}$-homomorphism for all $h \in \tilde{\mathbb{T}}(m)$, for all m.

<u>Proof.</u> This follows from 2.2.4, 2.2.12 and the symmetry of:

$$\begin{array}{ccccc} & & (X^n)^m & \xrightarrow{(\xi^g)^m} & X^m \\ & \cong\nearrow & & & \downarrow \tilde{\xi}^h \\ (X^m)^n & & & & \\ & \searrow (\tilde{\xi}^h)^n & & & \\ & & X^n & \xrightarrow{\xi^g} & X \end{array} \quad . \quad []$$

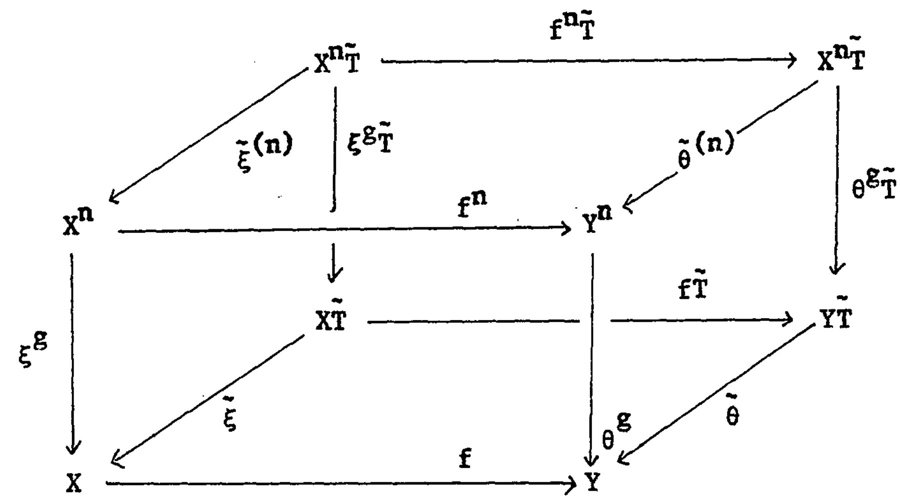
<u>**2.2.14 Definitions.**</u> $(X,\xi,\tilde{\xi}) \in \text{obj}\ \mathcal{S}^{(\mathbb{T},\tilde{\mathbb{T}})}$ is a $\mathbb{T}$-$\tilde{\mathbb{T}}$ <u>bialgebra</u> if it satisfies either of the equivalent conditions of 2.2.13. The full subcategory of $\mathbb{T}$-$\tilde{\mathbb{T}}$ bialgebras will be denoted "$\mathcal{S}^{[\mathbb{T},\tilde{\mathbb{T}}]}$" and the restriction of $U^{(\mathbb{T},\tilde{\mathbb{T}})}$ to bialgebras will be denoted "$U^{[\mathbb{T},\tilde{\mathbb{T}}]}$". If $U^{[\mathbb{T},\tilde{\mathbb{T}}]}$ is tripleable, the resulting triple is called the <u>tensor product of</u> $\mathbb{T}$ and $\tilde{\mathbb{T}}$ and is denoted "$\mathbb{T} \otimes \tilde{\mathbb{T}}$". It is an open question whether or not $\mathbb{T} \otimes \tilde{\mathbb{T}}$ always exists. A constructive proof can be given if both $\mathbb{T}$ and $\tilde{\mathbb{T}}$ have a rank by generalizing Freyd's proof in [11].

<u>**2.2.15 Proposition.**</u> The following statements are valid.

a. $\mathcal{S}^{[\mathbb{T},\tilde{\mathbb{T}}]}$ is a $\frown$-closed subcategory of $\mathcal{S}^{(\mathbb{T},\tilde{\mathbb{T}})}$.

b. [$U^{(\mathbb{T},\tilde{\mathbb{T}})}$ satisfies the solution set condition] iff [$U^{(\mathbb{T},\tilde{\mathbb{T}})}$ is tripleable] implies [$U^{[\mathbb{T},\tilde{\mathbb{T}}]}$ is tripleable] iff [$U^{[\mathbb{T},\tilde{\mathbb{T}}]}$ satisfies the solution set condition].

<u>Proof.</u> The diagram used in 2.2.12 shows "closed under products". Consider the diagram:



If $(X,\xi,\tilde{\xi}) \xrightarrow{f} (Y,\theta,\tilde{\theta})$, all commutes except possibly the left and right sides. Hence if f is mono then right implies left; if f is epi then so is $f^n\tilde{T}$ so left implies right. This proves (a). To prove (b), the adjoint functorem and 1.6.6 apply, so this is summing up the theory of 1.7.4, 1.7.5. []

<u>2.2.16 Proposition.</u> Monoids act algebraically on algebras. More precisely, let G be a monoid with associated triple $\mathcal{G} = (- \times G, e, m)$. Then there exists a $\mathcal{G}$-$\mathbb{T}$ composite triple $\mathcal{G}\mathbb{T}$ with $\mathcal{S}^{\mathcal{G}\mathbb{T}} = \mathcal{S}^{[G,\mathbb{T}]}$. In particular, $\mathcal{G} \otimes \mathbb{T}$ always exists.

<u>Proof.</u> We construct a lifting of $\mathbb{T}$ over $U^{\mathcal{G}}$. If $(X,\alpha) \in \mathcal{S}^{\mathcal{G}}$, define $(X,\alpha)\bar{T} =_{df} (XT, \bar{\alpha})$ where $XT \times G \xrightarrow{\bar{\alpha}} XT$ sends (x,g) to $\langle x, \alpha^g T\rangle$. Since $\bar{\alpha}^e = \alpha^e T = 1_X T = 1_{XT}$ and $\bar{\alpha}^{gh} = \alpha^{gh} T = \alpha^g \alpha^h T = \alpha^g T \alpha^h T = \bar{\alpha}^g \bar{\alpha}^h$, $(XT, \bar{\alpha})$ is a G-set. If $(X,\bar{\alpha}) \xrightarrow{f} (Y,\beta)$ is a $\mathcal{G}$-homomorphism, that is if $f\beta^g = \alpha^g f$ for all g, then we have $fT \times 1.\bar{\beta} = \bar{\alpha}.fT$ and so $\bar{T}$

is a functor such that $\bar{T}U^{\mathbb{G}} = U^{\mathbb{G}}T$. As η is natural, $X\eta \times 1.\bar{\alpha} = \alpha.X\eta$; also, as μ is natural, $X\mu \times 1.\bar{\alpha} = \bar{\bar{\alpha}}.X\mu$. Hence we have a lifting $\bar{\mathbb{T}} = (\bar{T},\bar{\eta},\bar{\mu})$ over $U^{\mathbb{G}}$. Now remembering that, in the language of 1.4.5, $(X,\xi)\bar{H} = (XH,\ X\lambda.\xi H)$, we have from 1.7.6 that $\mathcal{S}^{\mathbb{G}\,\mathbb{T}}$ is the full subcategory of (X,α,ξ)'s in $\mathcal{S}^{(\mathbb{G},\mathbb{T})}$ satisfying CL in the diagram below:

$$
\begin{array}{ccccc}
 & & \bar{\alpha}^g = \alpha^g T & & \\
XT & \xrightarrow{(1,g)} & XT \times G & \xrightarrow{\bar{\alpha}} & XT \\
\downarrow \xi & (*) & \downarrow \xi \times 1 & \text{CL} & \downarrow \xi \\
X & \xrightarrow{(1,g)} & X \times G & \xrightarrow{\alpha} & X
\end{array}
$$

Suppose $(X,\alpha,\xi) \in \mathcal{S}^{\mathbb{G}\,\mathbb{T}}$. Then $(*) \cup \text{CL}$ commutes for every $g \in G$, that is each α^g is a $\mathbb{T}$-homomorphism. It follows from 2.2.7 that every $\mathbb{G}$-operation is a $\mathbb{T}$-homomorphism, and $(X,\alpha,\xi) \in \mathcal{S}^{[\mathbb{G},\mathbb{T}]}$. Conversely, let $(X,\alpha,\xi) \in \text{obj}\ \mathcal{S}^{[\mathbb{G},\mathbb{T}]}$. Then $(*) \cup \text{CL}$ and $(*)$ commute for every g in G. But clearly every element of $XT \times G$ is in the image of some $(1,g)$. Therefore CL commutes. []

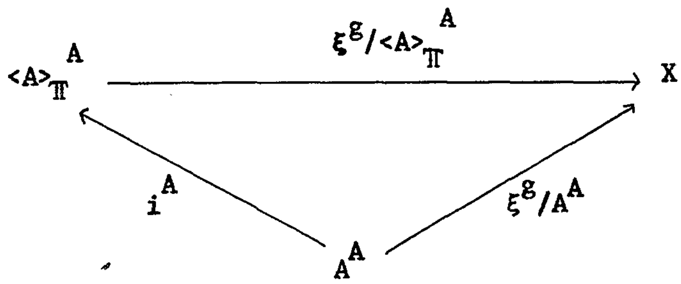
<u>2.2.17 Definition.</u> Let $(X,\xi) \in \mathcal{S}^{\mathbb{T}}$, and let A be a subset of X. Look at the factorization $A \xrightarrow{i} \langle A\rangle \xrightarrow{j} X$. Because $\langle A\rangle^A \xrightarrow{j^A} X^A \leq X^A$ we have a factorization

$$
\begin{array}{ccccccc}
A^A & \xrightarrow{k} & \langle A^A\rangle & \xrightarrow{m} & \langle A\rangle^A & \xrightarrow{j^A} & X^A
\end{array}
\qquad i^A : A^A \to \langle A\rangle^A
$$

Say that <u>subalgebras commute with powers in</u> $\mathcal{S}^{\mathbb{T}}$ if m is always an isomorphism, for all (X,ξ), A.

<u>**2.2.18 Proposition.**</u> Suppose that subalgebras commute with powers in $\mathcal{S}^{\mathbb{T}}$. Then every $\mathbb{T}$-$\tilde{\mathbb{T}}$ bialgebra is a $\mathbb{T}$-$\tilde{\mathbb{T}}$ quasicomposite algebra, and hence $\mathbb{T} \otimes \tilde{\mathbb{T}}$ exists.

<u>Proof.</u> Let $(X,\xi,\tilde{\xi}) \in \mathcal{S}^{[\mathbb{T},\tilde{\mathbb{T}}]}$, and let $(A,\xi_0) \overset{i}{\rightarrowtail} (X,\xi)$ be a subalgebra. For each g in $\mathbb{T}(A)$ consider the diagram:



By 2.2.9, ξ^g/A^A factors through A. Applying our hypothesis, $<A>_{\mathbb{T}}^{A}$ is generated as a $\mathbb{T}$-algebra by i^A and ξ^g is a $\mathbb{T}$-homomorphism. Hence $\mathrm{im}(\xi^g/<A>_{\mathbb{T}}^{A}) \subset <A>_{\tilde{\mathbb{T}}}$ by 1.8.2. It follows from 2.2.9 (b) that $<A>_{\mathbb{T}}$ is a subalgebra. That $\mathbb{T} \otimes \tilde{\mathbb{T}}$ exists now follows from 1.8.5. []

§2.3 Compact algebras.

2.3.1 Filter theory. Let X be a set, $\mathcal{F} \subset 2^X$. $\mathcal{F}^c =_{df} [A \subset X :$ there exists $F \in \mathcal{F}$ with $F \subset A]$. $\mathcal{F}$ is a filter on X if $\mathcal{F} \neq \phi$, $\phi \notin \mathcal{F}$, $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and $\mathcal{F} = \mathcal{F}^c$. An ultrafilter on X is an inclusion maximal filter on X. $X\beta =_{df} [\mathcal{U} : \mathcal{U}$ is an ultrafilter on X]. If $A \subset X$, $\mathcal{F} \wedge A =_{df} [F \cap A : F \in \mathcal{F}]$. If $\mathcal{F}$ is a filter on X, it is trivial to verify that $A \notin \mathcal{F}$ iff $\mathcal{F} \wedge A'$ is a filter on A' iff $(\mathcal{F} \wedge A')^c$ is a filter on X (where $A' =_{dn}$ the complement in X of A.)

2.1.3A Lemma. Let $\mathcal{F}$ be a filter on X. Then $\mathcal{F} \in X\beta$ iff for every subset A of X, $A \in \mathcal{F}$ or $A' \in \mathcal{F}$.

Proof. If $A \notin \mathcal{F}$, $(\mathcal{F} \wedge A')^c$ is a filter finer, hence equal to, $\mathcal{F}$. Therefore $A' \in \mathcal{F}$. Conversely, let $\mathcal{G}$ be a filter containing $\mathcal{F}$. If $G \in \mathcal{G}$, $G' \notin \mathcal{F}$ so that $G \in \mathcal{F}$. []

2.1.3B Lemma. Let $\mathcal{F}$ be a filter on X. Then $\mathcal{F} = \bigcap [\mathcal{U} \in X\beta : \mathcal{F} \subset \mathcal{U}]$.

Proof. Let $A \subset X$, $A \notin \mathcal{F}$. $(\mathcal{F} \wedge A')^c$ is a filter on X. By Zorn's Lemma (a nested union of filters is a filter) every filter is contained in an ultrafilter. Hence there exists $\mathcal{U} \in X\beta$ with $(\mathcal{F} \wedge A')^c \subset \mathcal{U}$. We have $\mathcal{F} \subset \mathcal{U}$, $A \notin \mathcal{U}$ proving $A \notin \bigcap [\mathcal{V} \in X\beta : \mathcal{F} \subset \mathcal{V}]$. []

2.3.2 Topological lemmas. Let $(X, \mathcal{S})$ be a topological space, let $\mathcal{F} \subset 2^X$ and let $x \in X$. Recall that $\mathcal{F}$ converges to $x =_{df}$ $\mathcal{F}^c \supset \mathcal{N}_x$ (where $\mathcal{N}_x =_{dn}$ the neighborhood filter of x), $=_{dn}$ $\mathcal{F} \longrightarrow x$. More generally, if $A \subset X$, $\mathcal{F} \longrightarrow A =_{dn}$ there exists $x \in A$ with $\mathcal{F} \longrightarrow x$. If $X \xrightarrow{f} Y$ is a function, $\mathcal{F}f =_{df}$ the filter $[Ff : F \in \mathcal{F}]^c \subset 2^Y$.

<u>2.3.2A Lemma.</u> This is due to Ellis and Gottschalk; see [7], Lemma 7. Let $(X, \mathcal{S})$, $(X', \mathcal{S}')$ be topological spaces, let $X \xrightarrow{f} Y$ be a function and let $x \in X$. Then f is continuous at x iff for every $\mathcal{U} \in X\beta$, $\mathcal{U} \longrightarrow x$ implies $\mathcal{U}f \longrightarrow xf$.

<u>Proof.</u> Let $\mathcal{U} \in X\beta$, $\mathcal{U} \longrightarrow x$. Let $V \in \mathcal{N}_{xf}$. There exists $W \in \mathcal{N}_x$ with $Wf \subset V$. As $W \in \mathcal{U}$, $V \in \mathcal{U}f$. Now the converse. For every $\mathcal{U} \longrightarrow x$ we have $\mathcal{U}f \supset \mathcal{N}_{xf}$. $\mathcal{U} \supset \mathcal{U}ff^{-1} \supset \mathcal{N}_{xf}f^{-1}$. By 2.3.1B, $\mathcal{N}_x = \bigcap [\mathcal{U} : \mathcal{U} \longrightarrow x] \supset \mathcal{N}_{xf}f^{-1}$. []

<u>2.3.2B Lemma.</u> Let $(X, \mathcal{S})$ be a topological space, and let $A \subset X$. Then A is open iff for every $\mathcal{U} \in X\beta$, $\mathcal{U} \longrightarrow A$ implies $A \in \mathcal{U}$.

<u>Proof.</u> A is open iff $A \in \bigcap_{x \in A} \mathcal{N}_x = \bigcap_{x \in A} \bigcap_{\mathcal{U} \to x} \mathcal{U} = \bigcap_{\mathcal{U} \to A} \mathcal{U}$. []

<u>2.3.2C Lemma.</u> Let $(X, \mathcal{S})$ be a topological space, $X \xrightarrow{f} X'$ an onto function. Let $\mathcal{S}'$ be the quotient topology induced by f. Then if $(X, \mathcal{S})$ is compact T2 and if $(X, \mathcal{S}) \xrightarrow{f} (X', \mathcal{S}')$ is closed then $(X', \mathcal{S}')$ is compact T2.

<u>Proof.</u> This is standard. See [17], chapter 5, theorem 20, p. 148. []

<u>2.3.3 Proposition.</u> Let $\mathcal{C}$ be the category of compact T2 spaces with

underlying set functor $\mathcal{C} \xrightarrow{U} \mathcal{S}$. Then U is tripleable.

Proof. A fairly short proof can be given using 1.2.9. We offer instead a direct construction which makes the triple very explicit and thereby offers an independent definition of a compact T2 space.

If X is a set and if $x \in X$, $A \subset X$, define $\dot{x} =_{df} [B \subset X : x \in B]$, $\mathring{A} =_{df} [\mathcal{U} \in X\beta : A \in \mathcal{U}]$. It is trivial to verify the following: $\dot{x} \in X\beta$, $\dot{\{x\}} = \{\dot{x}\}$, $\mathring{A} \cap \mathring{B} = \overset{\circ}{A \cap B}$, $\mathring{A}' = \mathring{A'}$, $\mathring{\phi} = \phi$. Define $\mathbb{\beta} = (\beta, \eta, \mu)$ by $\mathcal{S} \xrightarrow{\beta} \mathcal{S}$

$$X \xrightarrow{f} Y \mapsto X\beta \xrightarrow{f\beta} Y\beta, \qquad X \xrightarrow{X\eta} X\beta$$

$$\mathcal{U} \mapsto \mathcal{U}f \qquad\qquad x \longmapsto \dot{x}$$

$$X\beta\beta \xrightarrow{X\mu} X\beta$$

$$\mathcal{H} \longmapsto [A \subset X : \mathring{A} \in \mathcal{H}]$$

We will show that $\mathbb{\beta}$ is a triple in $\mathcal{S}$ with $U^{\beta} \approx U$.

Functoriality of β. Let $\mathcal{U} \in X\beta$, $X \xrightarrow{f} Y$. If $A,B \in \mathcal{U}$, $Af \cap Bf \supset (A \cap B)f$, so it is clear that $\mathcal{U}f$ is a filter on Y. If $A \subset Y$, either Af^{-1} or $(Af^{-1})' \in \mathcal{U}$ by 2.3.1A. If $Af^{-1} \in \mathcal{U}$ then $A \supset Af^{-1}f \in \mathcal{U}f$ implies $A \in \mathcal{U}f$; otherwise $Af^{-1\prime} \in \mathcal{U}$ and $A' \supset A'f^{-1}f = (Af^{-1})'f \supset \mathcal{U}f$ implies $A' \in \mathcal{U}f$. By 2.3.1A, $\mathcal{U}f \in Y\beta$. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{S}$, and if $\mathcal{F} \subset 2^X$ then $\mathcal{F}(fg) = (\mathcal{F}f)g$ is immediate, a fact we use implicitly from now on.

Naturality of η. This reduces to the assertion that for each $x \in X$ and each function $X \xrightarrow{f} Y$ we have $[Af : x \in A]^c = [B : xf \in B]$, which is clear.

<u>Naturality of μ.</u> Let $\mathcal{H} \in X\beta\beta$. If $A,B \subset X$ with $\dot{A},\dot{B} \in \mathcal{H}$ then $\dot{(A \cap B)} = \dot{A} \cap \dot{B} \in \mathcal{H}$. If $A \subset X$ with $\dot{A} \in \mathcal{H}$ then $\dot{A'} = \dot{A}' \in \mathcal{H}$. This proves $X\mu$ is well-defined. Now let $X \xrightarrow{f} Y$ be a function, $\mathcal{H} \in X\beta\beta$.

$$
\begin{aligned}
<\mathcal{H}, f\beta\beta.Y\mu> &= [\,\mathcal{L}f\beta : \mathcal{L} \in \mathcal{H}\,]^c\, Y\mu \\
&= [B \subset Y : \dot{B} \in [\,\mathcal{L}f\beta : \mathcal{L} \in \mathcal{H}\,]^c] \\
&= [B \subset Y : \exists \mathcal{L} \in \mathcal{H} .\ \mathcal{L}f\beta \subset \dot{B}] \\
&= [B \subset Y : \exists \mathcal{L} \in \mathcal{H}\ \forall \mathcal{U} \in \mathcal{L} . B \in \mathcal{U}(f\beta)] \\
&= [B \subset Y : \exists \mathcal{L} \in \mathcal{H}\ \forall \mathcal{U} \in \mathcal{L}\ \exists A \in \mathcal{U} . B \supset Af]
\end{aligned}
$$

$$
\begin{aligned}
<\mathcal{H}, X\mu.f\beta> &= [A \subset X : \dot{A} \in \mathcal{H}]\, f\beta \\
&= [Af : A \subset X \,\&\, \dot{A} \in \mathcal{H}]^c \\
&= [B \subset Y : \exists A \subset X . \dot{A} \in \mathcal{H} \,\&\, B \supset Af]
\end{aligned}
$$

Let $B \in <\mathcal{H}, X\mu.f\beta>$. There exists $A \subset X$ with $\dot{A} \in \mathcal{H}$ and $B \supset Af$. $\mathcal{L} =_{df} \dot{A}$. For every $\mathcal{U} \in \dot{A}$, $A \in \mathcal{U}$ and $B \supset Af$. Therefore $B \in <\mathcal{H}, f\beta\beta.Y\mu>$. Two maximal filters are equal if one is contained in the other. Therefore $f\beta\beta.Y\mu = X\mu.f\beta$, that is μ is natural.

<u>Unitary axioms.</u> Let $\mathcal{U} \in X\beta$.

$$
\begin{aligned}
<\mathcal{U}, X\eta\beta.X\mu> &= [AX\eta : A \in \mathcal{U}]^c\, X\mu \\
&= [B \subset X : \exists A \in \mathcal{U} .\ \dot{B} \supset [\dot{x} : x \in A]] \\
&= [B \subset X : \exists A \in \mathcal{U} .\ x \in A \text{ implies } x \in B] \\
&= \mathcal{U}
\end{aligned}
$$

$$
<\mathcal{U}, X\beta\eta.X\mu> = [\,\mathcal{L} \subset X\beta : \mathcal{U} \in \mathcal{L}\,]\, X\mu
$$

$= [A \subset X : \mathcal{U} \in \dot{A}]$

$= \mathcal{U}$

<u>Associativity axiom.</u> Let $\Omega \in X\beta\beta\beta$.

$$<\Omega, X\mu\beta.X\mu> = [\mathcal{A} X\mu : \mathcal{A} \in \Omega]^c X\mu$$
$$= [A \subset X : \exists \mathcal{A} \in \Omega . \mathcal{A} X\mu \subset \dot{A}]$$
$$= [A \subset X : \exists \mathcal{A} \in \Omega \; \forall \mathcal{H} \in \mathcal{A} . \dot{A} \in \mathcal{H}]$$

$$<\Omega, X\beta\mu.X\mu> = [\dot{\mathcal{L}} \subset X\beta : \dot{\mathcal{L}} \in \Omega] X\mu$$
$$= [A \subset X : \dot{\dot{A}} \in \Omega]$$

Let $A \in <\Omega, X\beta\mu.X\mu>$. $\mathcal{A} =_{df} \dot{\dot{A}}$. Then $\mathcal{A} \in \Omega$ and $\mathcal{H} \in \mathcal{A}$ implies $\dot{A} \in \mathcal{H}$. Hence $A \in <\Omega, X\mu\beta.X\mu>$. Therefore $\mu\beta.\mu = \beta\mu.\mu$. This completes the argument that $\mathbb{\beta}$ is a triple.

Define a functor $\mathcal{C} \xrightarrow{\Phi} \mathcal{S}^{\mathbb{\beta}}$ by $[(X, \mathcal{S}) \xrightarrow{f} (X', \mathcal{S}')] \Phi$ $=_{df}$ $(X, \xi_{\mathcal{S}}) \xrightarrow{f} (X', \xi_{\mathcal{S}'})$, where $X\beta \xrightarrow{\xi_{\mathcal{S}}} X$ is the convergence map sending each ultrafilter to the unique point to which it converges. $X\eta.\xi_{\mathcal{S}} = 1$ because $\dot{x} \longrightarrow x$ in all topologies. Now let $(X, \mathcal{S}) \in$ obj $\mathcal{C}$, and let $\mathcal{H} \in X\beta\beta$. $x =_{df} <\mathcal{H}, \xi_{\mathcal{S}}\beta.\xi_{\mathcal{S}}> = [\mathcal{L}\xi_{\mathcal{S}} : \mathcal{L} \in \mathcal{H}]^c \xi_{\mathcal{S}}$. We must show that $<\mathcal{H}, X\mu> = [A \subset X : \dot{A} \in \mathcal{H}]$ $\longrightarrow x$. Let $B^{open} \in \mathcal{N}_x$. There exists $\mathcal{L} \in \mathcal{H}$ such that $[\mathcal{U}\xi_{\mathcal{S}} : \mathcal{U} \in \mathcal{L}] \subset B$. Therefore $\mathcal{U} \in \mathcal{L}$ implies $\mathcal{U}\xi_{\mathcal{S}} \in B$ implies there exists $b \in B$ such that $\mathcal{U} \longrightarrow b$. As $B \in \mathcal{N}_b$, $B \in \mathcal{U}$, so $\mathcal{U} \in \dot{B}$. Therefore $\dot{B} \supset \mathcal{L} \in \mathcal{H}$ and $\dot{B} \in \mathcal{H}$, as we wished to show. Thus far Φ is well-defined on objects. Now suppose $(X, \mathcal{S})$, $(X', \mathcal{S}')$ $\in$ obj $\mathcal{C}$, and let $X \xrightarrow{f} X'$ be a function. f is a $\mathbb{\beta}$-homomorphism iff $f\beta.\xi_{\mathcal{S}'} = \xi_{\mathcal{S}}.f$ iff for every $\mathcal{U} \in X\beta$ and for every $x \in X$,

$\mathcal{U} \longrightarrow x$ implies $\mathcal{U}f \longrightarrow xf$ iff (by 2.3.2A) f is continuous. Therefore Φ is well-defined on morphisms and full. It is also clearly faithful and satisfies $\Phi U^{|\beta} = U$. Moreover it is immediate from 2.3.2B that Φ is 1-to-1 on objects. To complete the proof we show that Φ is onto on objects. Let X be a set, and define a topology $\mathcal{S}_X$ on $X\beta$ by taking $[\dot{A} : A \subset X]$ as a base; we may do this since the $\dot{A}$'s are closed under finite intersections; explicitly, every open set is a union of $\dot{A}$'s and conversely. Let $\mathcal{H} \in X\beta\beta$. $\mathcal{H} \longrightarrow \mathcal{H}X\mu$, because if $\mathcal{H}X\mu = [A \subset X : \dot{A} \in \mathcal{H}] \in \dot{B}$ then $B \in [A \subset X : \dot{A} \in \mathcal{H}]$, that is $\dot{B} \in \mathcal{H}$. Moreover if $\mathcal{U} \in X\beta$ and $\mathcal{H} \longrightarrow \mathcal{U}$ it follows that $\mathcal{U} = \mathcal{H}X\mu$. For if $A \in \mathcal{U}$, then $\mathcal{U} \in \dot{A} \in \mathcal{H}$ and hence $A \in [B \subset X : \dot{B} \in \mathcal{H}] = \mathcal{H}X\mu$. This proves that $(X\beta, \mathcal{S}_X) \in \text{obj}\ \mathcal{C}$ and $(X\beta, \mathcal{S}_X)\Phi = (X\beta, X\mu)$.

Let $\mathcal{L} \overset{i}{\rightarrowtail} X\beta$, and consider the diagram

$$\begin{array}{ccc} \mathcal{L}\beta & \xrightarrow{\quad i\beta \quad} & X\beta\beta \\ \downarrow & & \downarrow X\mu \\ \mathcal{L} & \overset{i}{\rightarrowtail} & X\beta \end{array}$$

One sees immediately that $\mathcal{L}$ is a subalgebra of $(X\beta, X\mu)$ iff every ultrafilter on $\mathcal{L}$ converges in $\mathcal{L}$ iff $\mathcal{L}$ is closed.

Now let (X,ξ) be any $\mathbb{\beta}$-algebra. $(X\beta, X\mu) \overset{\xi}{\twoheadrightarrow} (X,\xi)$ is a homomorphism onto. Let $\mathcal{S}$ be the quotient topology induced by ξ on X. Let $\mathcal{L} \subset X\beta$. $\mathcal{L}$ closed iff $\mathcal{L} \le (X\beta, X\mu)$ implies $\mathcal{L}\xi \le (X,\xi)$ implies $(\mathcal{L}\xi)\xi^{-1} \le (X\beta, X\mu)$ iff $(\mathcal{L}\xi)\xi^{-1}$ is closed in $(X\beta, \mathcal{S}_X)$ iff $\mathcal{L}\xi$ is closed in (X, ξ_X). Therefore ξ is a closed mapping. By 2.3.2C, $(X,\xi) \in \text{obj}\ \mathcal{C}$. Finally, let $\mathcal{U} \in X\beta$ and show $\mathcal{U} \longrightarrow \mathcal{U}\xi$. Let

$\mathcal{U}\xi \in A \in \mathcal{S}$. There exists $B \subset X$ with $\mathcal{U} \in \dot{B} \subset A\xi^{-1}$. For all $b \in B$, $b = \dot{b}\xi \in A\xi^{-1}\xi = A$. Therefore $A \supset B \in \mathcal{U}$ and $A \in \mathcal{U}$. []

2.3.4 Proposition. For every triple $\mathbb{T}$ in sets, every $\mathbb{T}$-β bialgebra is a $\mathbb{T}$-β quasicomposite algebra. In particular $\mathbb{T} \otimes \beta$ always exists.

Proof. Subalgebras = closed sets in $\mathcal{S}^{\beta}$ [the argument we used for free algebras in 2.3.3 is general]. A well known topological theorem is "product of the closures = closure of the product". Now use 2.2.18. []

$\mathbb{T} \otimes \beta$-algebras are called compact $\mathbb{T}$-algebras.

2.3.5 Example; discrete actions with compact phase space. Let G be a discrete monoid, with associated triple $\mathcal{G}$. The category of compact T2 transformation semigroups with phase semigroup G is $\mathcal{S}^{G \otimes \beta}$. We have only to observe that since G is discrete, $X \times G \xrightarrow{\alpha} X$ is continuous iff each $X \xrightarrow{\alpha^g} X$ is continuous.

2.3.6 Proposition. Compact topological dynamics is tripleable. More precisely, let G be a monoid with associated triple $\mathcal{G}$. Let $\mathcal{S}$ be any topology on the underlying set of G. $\mathcal{B} =_{df}$ the full subcategory of $\mathcal{S}^{G \otimes \beta}$ generated by objects (X,α,ξ) such that $(X,\xi) \times (G, \mathcal{S}) \xrightarrow{\alpha} (X,\xi)$ is continuous. Then $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{S}^{G \otimes \beta}$, and in particular is tripleable. (Compact topological dynamics is recovered by insisting that $\mathcal{S}$ be compatible with G; in this case $\mathcal{B} =_{dn} ts_G$ or tg_G accordingly as G is a monoid or a group.)

<u>Proof.</u> Consider a product of $\mathcal{B}$-objects, $(X,\alpha,\xi) = \prod(X_i,\alpha_i,\xi_i)$. Using 1.7.4 it is clear that $(X,\alpha) = \prod(X_i,\alpha_i)$ and $(X,\xi) = \prod(X_i,\xi_i)$. Hence at the level of sets we have

$$\begin{array}{ccc} X \times G & \xrightarrow{\quad\alpha\quad} & X \\ \downarrow{\scriptstyle pr_i \times 1} & & \downarrow{\scriptstyle pr_i} \\ X_i \times G & \xrightarrow{\quad\alpha_i\quad} & X_i \end{array}$$

By the tychonoff theorem, $(X,\xi) = \prod(X_i,\xi_i)$ in the category of all topological spaces. Hence α is continuous as each $\alpha.pr_i$ is, and (X,α,ξ) ε obj $\mathcal{B}$. Next, let $(A,\alpha_0,\xi_0) \xrightarrow{i} (X,\alpha,\xi)$ in $\mathcal{S}^{G\otimes\mathbb{B}}$ with (X,α,ξ) in obj $\mathcal{B}$. We have

$$\begin{array}{ccc} A \times G & \xrightarrow{\quad\alpha_0\quad} & A \\ \downarrow{\scriptstyle i \times 1} & & \downarrow{\scriptstyle i} \\ X \times G & \xrightarrow{\quad\alpha\quad} & X \end{array} .$$

Now all monomorphisms in $\mathcal{S}^{\mathbb{B}}$ become relative subspaces when viewed in the category of all topological spaces because every algebraic monomorphism is an isomorphism into. Therefore α_0 is continuous because $i \times 1.\alpha$ is.

To show that $\mathcal{B}$ is closed under quotients it suffices to prove the following topological lemma: consider the situation

$$\begin{array}{ccc} X \times H & \xrightarrow{\quad a\quad} & X \\ \downarrow{\scriptstyle f \times 1} & & \downarrow{\scriptstyle f} \\ Y \times H & \xrightarrow{\quad b\quad} & Y \end{array}$$

where X, H, Y are topological spaces with X compact and Y T2 and where a is continuous and f is continuous onto. Then b is continuous. To prove it, we use 2.3.2A. Let $\mathcal{U}$ be an ultrafilter on Y × H such that $\mathcal{U} \longrightarrow (y,h) \in Y \times H$. $\mathcal{V} =_{df} \mathcal{U}(f \times 1)^{-1}$. $\mathcal{V}$ is a filter on X × H. If $A \subset X \times H$ such that $A(f \times 1) \notin \mathcal{U}$ then as f × 1 is onto, $A'(f \times 1) \supset [A(f \times 1)]' \in \mathcal{U}$. Therefore $\mathcal{V}$ is an ultrafilter on X × H. As X is compact there exists $x \in X$ such that $\mathcal{V} pr_X \longrightarrow x$. Also, $\mathcal{V} pr_H = \mathcal{V}(f \times 1).pr_H = \mathcal{U} pr_H \longrightarrow (y,h)pr_H = h$. If V, W $\in \mathcal{V}$, $Vpr_X \times Wpr_H \supset (V \cap W)pr_X \times (V \cap W)pr_H \supset V \cap W \in \mathcal{V}$ so that $\mathcal{V} \supset \mathcal{V} pr_X \times \mathcal{V} pr_H \longrightarrow (x,h)$. Therefore $\mathcal{V}(f \times 1) pr_X = \mathcal{U} f.pr_X$ converges both to y and to xf; since Y is T2, xf = y. Hence $\mathcal{U} b = \mathcal{V}(f \times 1).b = \mathcal{V} a.f \longrightarrow (x,h)a.f = (y,h)b$ as desired. []

§2.4 The enveloping semigroup of an algebra.

Let G be a topological group, and let $(X,\alpha,\xi) \in \text{obj } tg_G$. The enveloping semigroup, E, of (X,α,ξ) is defined in [7] to be the pointwise closure in $(X,\xi)^X$ of the transition group $[\alpha^g : g \in G]$. Recalling that subalgebras in tg_G are computed in $\mathcal{S}^{G \otimes \beta}$, (X,α,ξ) is a $\mathcal{G}-\beta$ quasicomposite algebra and $E = \langle\langle 1_X \rangle_G \rangle_\beta = \langle 1_X \rangle_{tg_G}$. This observation suggests that we can always define the enveloping semigroup of an algebra. In the next two sections we enlarge to $\mathcal{S}^{\mathbb{T}}$ the analysis of tg_G of [7], [8] and [9].

For this section fix a consistent triple $\mathbb{T} = (T,\eta,\mu)$ in $\mathcal{S}$. $U =_{dn} U^{\mathbb{T}}$, $F =_{dn} F^{\mathbb{T}}$, $\varepsilon =_{dn} \varepsilon^{\mathbb{T}}$.

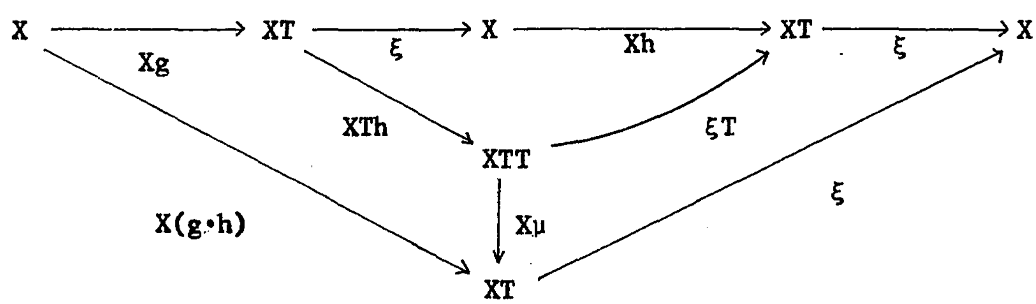
2.4.1 Definition and proposition. $G_{\mathbb{T}}$ (or simply G) $=_{df} \mathbb{T}(P) = (1,T)\text{n.t.}$ Also define

$$G \times G \xrightarrow{\ \bullet\ } G$$

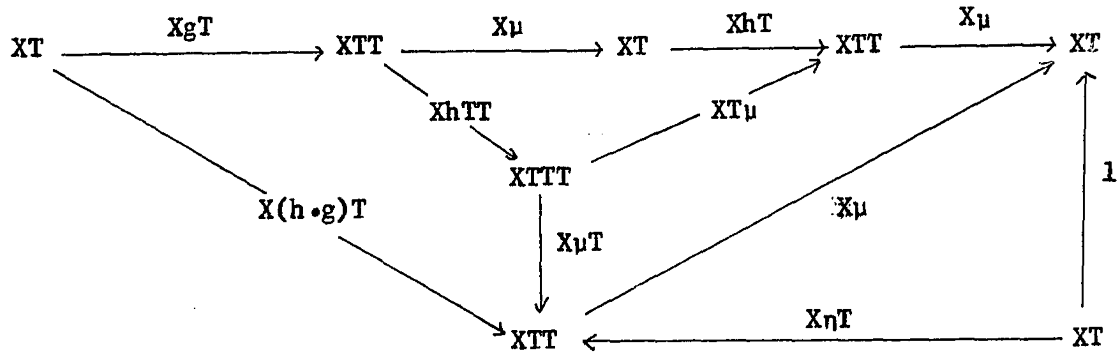
$$(g,h) \longmapsto g\bullet h =_{df} 1 \xrightarrow{gh} TT \xrightarrow{\mu} T.$$

Then $(G,\bullet)$ is a monoid with unit η, and, letting $G \xrightarrow{\psi} (U,U)\text{n.t.}$, $G \xrightarrow{\psi'} (F,F)\text{n.t.}$ be the bijections of 2.2.2, ψ is a monoid isomorphism and ψ' is a monoid antiisomorphism.

Proof. For $(X,\xi) \in |\mathcal{S}^{\mathbb{T}}|$ and $g,h \in G$ we have



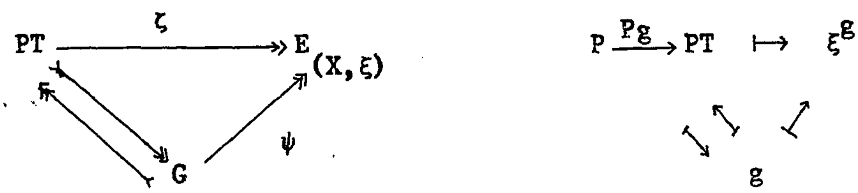
which proves $(g \cdot h)\psi = g\psi . h\psi$, and we have $X\eta . \xi = 1_X$ which proves $\eta\psi = 1_U$. Therefore $(G, \cdot)$ is a monoid and ψ is a monoid isomorphism. We have



Therefore U of the equations $XgF.XF\varepsilon.XhF.XF\varepsilon = (X(h \cdot g)F.XF\varepsilon$, $X\eta F.XF\varepsilon = 1_{XF}$ are valid, and then the equations themselves are valid because U is faithful. Therefore ψ' is a monoid antiisomorphism. []

<u>2.4.2 Proposition and definition.</u> Let (X,ξ) be a $\prod$-algebra. Then $\mathcal{O}_P(X,\xi) = [\xi^g : g \varepsilon G] = \langle 1_X \rangle \subset (X,\xi)^X$. This subset of X^X, both a $\prod$-subalgebra and a monoid under composition, is called the <u>enveloping semigroup of</u> (X,ξ) and is denoted "$E_{(X,\xi)}$", "E_X", or "E".

$E_{(PT,P\mu)}$ "blends" PT and G in the following sense: in the commutative diagram:



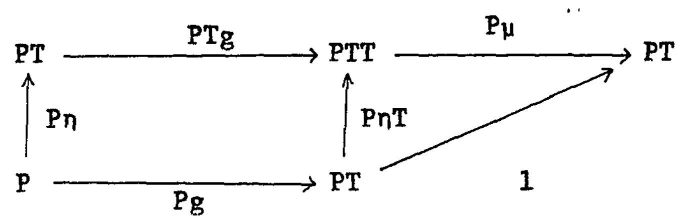
ζ is a $\prod$-homomorphism and ψ is a monoid homomorphism and if $(X,\xi) = (PT,P\mu)$ then ζ,ψ are isomorphisms.

<u>Proof.</u> That $\mathcal{O}_P(X,\xi) = \langle 1_X \rangle$ and that $Pg \mapsto \xi^g$ is a $\prod$-homomorphism were proved in 2.2.8. Two arguments that E is a submonoid of X^X:

(i) $[p \in E : Ep \subset E]$ is a subalgebra of E (by 2.1.7 (b)) containing 1_X so that $EE = E$; more precisely,

(ii) ξ^g is the $(X,\xi)^{th}$ component of the natural transformation $U \to U$ corresponding to g, so by 2.4.1, $\xi^g\xi^h = \xi^{g\cdot h}$.

It follows immediately from (ii) that ψ is a monoid homomorphism. To complete the proof we must show that $G \longrightarrow E_{(PT,P\mu)}$ is 1-to-1. But



expresses Pg in terms of $(P\mu)^g = PTg.P\mu$. []

2.4.3 Proposition. Every $\mathbb{T}$-algebra may be interpreted as a $G_{\mathbb{T}}$-set with algebraic structure. More precisely, there is a forgetful functor $\mathcal{S}^{\mathbb{T}} \xrightarrow{\Phi} \mathcal{S}^{\mathbb{G}}$ which is tripleable, where $\mathbb{G} =_{df}$ the triple associated with $G = G_{\mathbb{T}}$.

Proof. Define $(X,\xi)\Phi = (X,\alpha_\xi)$ where $X \times G \xrightarrow{\alpha_\xi} X$ is defined by $(x,g)\alpha_\xi =_{df} \langle x,\xi^g\rangle$. It follows from 2.4.2 that (X,α_ξ) is a G-set. If $(X,\xi) \xrightarrow{f} (Y,\theta)$ is a $\mathbb{T}$-homomorphism, then in particular f commutes with unary operations (by 2.2.4) and hence $(X,\alpha_\xi) \xrightarrow{f} (Y,\alpha_\theta)$ is equivariant (we sometimes call G-set homomorphisms "equivariant"). Hence Φ is a well-defined functor and $\Phi U^{\mathbb{G}} = U^{\mathbb{T}}$. It follows from 1.5.2 that Φ is tripleable. []

2.4.4 Proposition. Let (X,ξ) be a $\mathbb{T}$-algebra, let Λ be a set and let $(A,\tilde{\xi}_0) \leq (X^\Lambda,\tilde{\xi})$. Then for every g in G, $\tilde{\xi}_0^g : A \to A = -\circ\xi^g$.

<u>Proof.</u> Denoting $pr_\lambda =_{dn} pr_\lambda / A$, we have

$$\begin{array}{ccccc} A & \xrightarrow{Ag} & AT & \xrightarrow{\tilde{\xi}_0} & A \\ \downarrow{\scriptstyle pr_\lambda} & & \downarrow{\scriptstyle pr_\lambda T} & & \downarrow{\scriptstyle pr_\lambda} \\ X & \xrightarrow{Xg} & XT & \xrightarrow{\xi} & X \end{array} .$$

For every $\lambda \in \Lambda$, $a \in A$ and $g \in G$ we have $\langle \lambda, a\tilde{\xi}_0^g \rangle = \langle a, \tilde{\xi}_0^g . pr_\lambda \rangle =$ $\langle a, pr_\lambda . \xi^g \rangle = \langle \lambda a, \xi^g \rangle = \langle \lambda, a.\xi^g \rangle$. []

<u>2.4.5 Proposition.</u> Let (Y,θ) be a quotient algebra of (X,ξ). Then the following statements are valid.

a. $E_Y \xrightarrow{\zeta} (E_X, E_Y)\,\mathcal{S}^{\mathbb{T}}$

$$p \longmapsto \quad \begin{array}{ccc} E_X & \xrightarrow{\zeta p} & E_Y \\ \xi^g & \longmapsto & p.\theta^g \end{array}$$

is a well-defined bijection.

b. Every G-equivariant map $E_X \xrightarrow{f} E_Y$ is a $\mathbb{T}$-homomorphism; indeed $f = \zeta_{\langle 1_X, f \rangle}$.

c. If X is singly generated, X is a quotient of E_X.

d. $\chi : E_X \to E_{E_X}$, $\xi^g \mapsto -\circ\xi^g$ is a $\mathbb{T}$-monoid isomorphism.

<u>Proof.</u> <u>a,b.</u> ζ_{1_Y} is a $\mathbb{T}$-homomorphism by 2.2.11. Hence $\zeta_p =$ $E_X \xrightarrow{\zeta_{1_Y}} E_Y \xrightarrow{p\circ -} E_Y$ is a $\mathbb{T}$-homormophism by 2.1.5. Hence ζ is well-defined. $p = \langle 1_X, \zeta_p \rangle$ proves that ζ is 1-to-1. Now let $E_X \xrightarrow{f} E_Y$ be equivariant (in particular f might be a $\mathbb{T}$-homomorphism.) Using 2.4.4, we have, for each $g \in G$, the diagram:

$$\begin{array}{ccc} E_X & \xrightarrow{\quad f \quad} & E_Y \\ \Big\downarrow{\scriptstyle -\circ\xi^g} & & \Big\downarrow{\scriptstyle -\circ\theta^g} \\ E_X & \xrightarrow{\quad f \quad} & E_Y \end{array}$$

and therefore $<\xi^g,f> = <1_X.\xi^g,f> = <1_X, (-\circ\xi^g).f> = <1_X, f.(-\circ\theta^g)> = <1_X,f>.\theta^g = <\xi^g, \zeta_{1_X f}>$.

c. We assume there exists $x_0 \in X$ with $X = <x_0>$. Then $E_X \xrightarrow{pr_{x_0}} X$ is onto because $x_0 = <1_X, pr_{x_0}>$.

d. χ is a well-defined bijection by 2.4.4. In view of (c) we can apply 2.2.11 to insure that χ^{-1} is a $\mathbb{T}$-isomorphism. But χ^{-1} is a monoid isomorphism because $(-\circ\xi^g).(-\circ\xi^h) = -\circ\xi^g\xi^h$. []

<u>2.4.6 Definitions.</u> Let (X,ξ) be a $\mathbb{T}$-algebra. The least subalgebra of X ($= \bigcap [A : A \leq X] = <\phi>$) will be denoted "$0_{(X,\xi)}$" or "$0_X$" or "0". If $A \leq X$, A is a <u>minimal subalgebra of</u> X if A is an atom in the complete lattice of $\mathbb{T}$-subalgebras of X. If A is either a minimal subalgebra of X or 0_X, say that A is a <u>preminimal subalgebra of</u> X. X is itself minimal or preminimal $=_{df}$ it has such a property qua subalgebra of itself. Clearly a subalgebra is minimal or preminimal iff it has such a property qua algebra.

<u>2.4.7 Proposition.</u> The following statements are valid.

a. $(\phi T,\phi\mu)$ is an initial object in $\mathcal{S}^{\mathbb{T}}$ and for every $\mathbb{T}$-algebra (X,ξ), $0_{(X,\xi)} = \text{im } (\phi T \to (X,\xi))$.

b. If $(X,\xi) \xrightarrow{f} (Y,\theta)$ is a $\mathbb{T}$-homomorphism then $0_X f = 0_Y$. In particular, if $A \leq X$ then $0_A = 0_X$.

c. If (X,ξ) is a $\mathbb{T}$-algebra and if Λ is a non-empty set then

$0_{X\Lambda} = [\Lambda \xrightarrow{\hat{x}} X : x \in 0_X$ and $\hat{x}$ is constantly $x]$, and hence each projection $X^\Lambda \xrightarrow{pr_\lambda} X$ establishes an isomorphism of $0_{X\Lambda}$ with 0_X. Elements of $0_{X\Lambda}$ are said to be constantly zero.

d. Let X be a $\top\!\!\top$-algebra and let $A \leq X$. Then A is preminimal iff for every $x \in A - 0$, $A = \langle x \rangle$.

e. Let $X \xrightarrow{f} Y$ be a $\top\!\!\top$-homomorphism and let $A \leq X$. Then A preminimal implies Af preminimal.

f. Let X be a $\top\!\!\top$-algebra, let $x \in X$, and let $I \leq E_X$. Then I preminimal implies xI preminimal.

Proof. a. For each $\top\!\!\top$-algebra (X,ξ) the unique $\phi \xrightarrow{i} X$ has unique homomorphic extension $\phi T \xrightarrow{iT} XT \xrightarrow{\xi} X$ whose image is $\langle\phi\rangle$ by 1.8.2.

b. If $B \leq Xf$ then $B = Bf^{-1}f$. Hence f induces an order-preserving surjection $[A : A \leq X] \xrightarrow{f} [B : B \leq Bf]$. In particular, $0_X f = 0_{Xf}$. As $Xf \leq Y$, $0_Y \leq Xf$ so that $0_{Xf} \leq 0_Y$. As $0_{Xf} \leq Xf \leq Y$, $0_Y \leq Xf$ so that $0_{Xf} = 0_Y$.

c. For each element x of X denote $\Lambda \xrightarrow{\hat{x}} X$ to be the induced constant function. There exists $\lambda \in \Lambda$ by hypothesis. $[\hat{x} : x \in 0_X] = [\hat{x} : x \in X] \cap 0_X\, pr_\lambda^{-1}$. But $[\hat{x} : x \in X] \leq X^\Lambda$ since it is the collective equalizer of all the projections. Therefore $0_{X\Lambda} \leq [\hat{x} : x \in 0_X] \leq X^\Lambda$. Conversely, $0_{X\Lambda}.pr_\lambda = 0_X$ by (b), proving every $\hat{x}$ with x in 0_X is in $0_{X\Lambda}$.

d. If A is preminimal and if $x \in A - 0$ then $0 \subsetneq \langle x \rangle \subset A$ which implies that $\langle x \rangle = A$. Conversely, if $x \in A - 0$ implies $\langle x \rangle = A$ and if $0 \neq B \leq A$, then there exists $x \in B - 0$, whence $A = \langle x \rangle \subset B \subset A$.

e. Use the order-preserving surjection of (b).

f. pr_X is a homomorphism, so use (e). []

2.4.8 Proposition. The following statements are valid.

a. (i) $\phi T = \phi$ iff (ii) there exists an algebra structure on ϕ iff (iii) for every $\mathbb{T}$-algebra (X,ξ), $0_X = \phi$ iff (iv) there exists a $\mathbb{T}$-algebra (X,ξ) with $0_X = \phi$.

b. $PT = P$ implies $\phi T = \phi$.

c. $\phi T = \phi$ and $(X,\xi) \in \text{obj } \mathcal{S}^{\mathbb{T}}$ implies $E_X \supsetneq 0_X$.

d. If $\phi T \neq \phi$ and if $(X,\xi) \in \text{obj } \mathcal{S}^{\mathbb{T}}$ then $E_X = 0_X$ iff $X = P$.

Proof. a. (i) implies (ii). If $\phi T = \phi$ then $(\phi, 1_\phi)$ is a $\mathbb{T}$-algebra. (ii) implies (iii). For every $\mathbb{T}$-algebra X, 0_X is a quotient of ϕT so that [there exists $\phi T \to \phi$] implies $0_X = \phi$. (iii) implies (iv). This is obvious as there exists at least one $\mathbb{T}$-algebra, $(P, PT \to P)$ for instance. (iv) implies (i). If (X,ξ) is a $\mathbb{T}$-algebra and if $0_X = \phi$ then $\phi \to 0_X$ extends to a homomorphism $\phi T \to 0_X = \phi$, so $\phi T = \phi$.

b. $\phi \to P$ induces a monomorphism $\phi T \rightarrowtail PT$ by 2.1.2, so crd $\phi T \leq 1$. Suppose crd $\phi T = 1$. Then $(\phi T, \phi\mu) = (P, PT \to P)$ and for every $\mathbb{T}$-algebra (X,ξ), crd X = crd $(P,X)\mathcal{S}$ = crd $(\phi T, XT)\mathcal{S}^{\mathbb{T}} \leq 1$, which contradicts our standing hypothesis that $\mathbb{T}$ be consistent.

c. This is clear from (a) as E_X is never empty.

d. Suppose $\phi T \neq \phi$ and X is a $\mathbb{T}$-algebra. By (a), $X \neq \phi$, so we have from 2.4.7 (c) that $E_X = 0$ implies 1_X is constant implies $X = P$. Conversely, $E_P = P$ is obvious, so we must show $\phi T \neq \phi$ implies $P = 0$. This is clear as $\phi T \to P$ is onto. []

__2.4.9 Definitions.__ Let M be an abstract monoid. A right ideal in M $=_{df}$ a non-empty subset $I \subset M$ such that $IM \subset I$. An abstract constant of M $=_{df}$ an element $p \in M$ such that for every $q \in M$, $qp = p$. If I is a right ideal in M then

I is ϕ-minimal $=_{df}$ the set of right ideals contained in $I = \{I\}$;

I is AC-minimal $=_{df}$ the set of right ideals contained in $I = \{[p \in M : p \text{ abstract constant}], I\}$;

$$I \text{ is } \prod\text{-minimal} =_{df} \begin{cases} I \text{ is } \phi\text{-minimal} & (\phi T = \phi) \\ I \text{ is AC-minimal} & (\phi T \neq \phi) \end{cases}$$

__2.4.10 Proposition.__ Let M be an abstract monoid and set $I =_{df} [p \in M : p \text{ abstract constant}]$. Then $I = \phi$ or I is a ϕ-minimal right ideal.

__Proof.__ Suppose $p \in I$, $q \in M$. For every $r \in M$ we have $r(pq) = (rp)q = pq$ so $IM \subset M$. Now suppose $\phi \neq J \subset I$, $JM \subset J$. Let $j \in J$. Then for every $i \in I$, $i = ji \in J$ proves $I \subset J$. []

__2.4.11 Proposition.__ Let (X,ξ) be a $\prod$-algebra, $E =_{df} E_{(X,\xi)}$, $x \in X$, $p \in E$, $I \subset E$. The following statements are valid.

a. $\langle x \rangle = xE$.

b. $\langle p \rangle = pE$.

c. I a right ideal implies pI a right ideal.

d. $\phi \neq I \leq E$ implies I is a right ideal.

__Proof.__ __a.__ $\langle x \rangle = \langle 1_X \, pr_x \rangle = \langle 1_X \rangle \, pr_x = E \, pr_x = xE$.

__b.__ By (a), $\langle p \rangle = pE_E = [p(-\circ\xi^g) : g \in G] = [p\xi^g : g \in G] = pE$.

__c.__ $pIE \subset pI$; $I \neq \phi$ implies $pI \neq \phi$.

__d.__ $[q \in E : Iq \subset I] \leq E$ and contains 1_X. []

2.4.12 Proposition. Suppose $\phi T \neq \phi$. Let X be a $\sqcap\!\sqcap$-algebra, $E =_{df} E_X$. Define $I =_{df} [p \in E : p \text{ abstract constant}]$, $J =_{df} [p \in E : p \text{ constant function}]$. Then $I = 0_E = J$.

Proof. $X \neq \phi$ by 2.4.8 (a) so by 2.4.7 (c) $0_E \subset J$. $J \subset I$ is clear. Now suppose $p \in I$, $\phi \neq A \subset E$. There exists $a \in A$ and, as A is a right ideal (by 2.4.11(d)), we have $p = ap \in A$. Therefore $I \subset \bigcap [A \leq X : A \neq \phi]$ which by 2.4.8 (a) is equal to $\bigcap [A : A \leq X] = 0_E$. []

2.4.13 Proposition. Let X be a $\sqcap\!\sqcap$-algebra, $E = E_X$, $I \subset E$. The following statements are equivalent.

a. I is a minimal subalgebra of E.

b. I is a $\sqcap\!\sqcap$-minimal right ideal in the abstract monoid E.

c. I is a right ideal properly containing 0_E and is minimal with this property.

Proof. By 2.4.8 (a) and 2.4.12 we have $0_E = \phi$ $(\phi T = \phi)$, $0_E =$ [abstract constants] $(\phi T \neq \phi)$. In view of 2.4.10, (b) iff (c) is clear.

a implies c. $0 \subsetneqq I \subset E$ so I is a right ideal by 2.4.11 (d), and I properly contains 0. Suppose $0 \subsetneqq J \subset I$ with $JE \subset J$. Let $p \in J - 0$. Then $0 \subsetneqq pE \subset J$ and $pE \leq J$ (by 2.4.11 (b)) so that $I = pE \subset J \subset I$.

c implies a. As $1_X \in E$, $I = IE = \bigcup_{p \in I} pE$. As $0 \subsetneqq I$ there exists $p \in I$ with $0 \subsetneqq pE \subset I$. Since pE is a right ideal, $pE = I$. By 2.4.11 (b), therefore $I \leq E$. Now suppose $0 \subsetneqq J \leq E$. J is a right ideal by 2.4.11 (d), and hence $J = I$. []

2.4.14 Proposition. If $\phi T = \phi$ then the following are equivalent.

a. Every non-empty $\mathcal{T}$-algebra contains a minimal subalgebra.

b. Every right ideal in $G_{\mathbb{T}} = G$ contains a ϕ-minimal right ideal.

c. G contains a ϕ-minimal right ideal.

<u>Proof.</u> <u>a implies b.</u> By 2.4.2, $G \simeq E_{PT}$ qua monoid. Let I be a right ideal in E_{PT}. Let $p \in I$. $pE_{PT} \subset I$. As $pE_{PT} \leq E_{PT}$ there exists a minimal subalgebra $A \leq pE_{PT}$. By 2.4.13, A is a ϕ-minimal right ideal, and clearly $A \subset I$.

<u>b implies c.</u> Obvious, as $G \neq \phi$.

<u>c implies a.</u> As $G \simeq E_{PT}$ qua monoid, E_{PT} has a ϕ-minimal right ideal, and hence (by 2.4.13) a minimal subalgebra. Let $\phi \neq (X,\xi) \in$ obj $\mathcal{S}^{\mathbb{T}}$. Then there exists $PT \xrightarrow{f} (X,\xi) \in \mathcal{S}^{\mathbb{T}}$. As $PT \simeq E_{PT}$ qua algebra (by 2.4.2) there exists a minimal subalgebra $A \leq PT$. Af is non-empty and preminimal (by 2.4.7 (e)). Since $\phi T = \phi$, "0" means "empty", so indeed Af is minimal. []

Note: 2.4.14 (c implies b) is true for any abstract monoid M. To prove it, observe $M = G_{\mathbb{T}}$ for $\mathcal{T} = (M \times -, \ , \ )$ (the discussion of 2.2.7 essentially proves this) and $\phi \times M = \phi$ so that 2.4.14 applies to $\mathcal{T}$.

The following proposition generalizes the main existence theorem for minimal orbit closures in topological dynamics, namely [13, 2.22].

<u>2.4.15 Proposition.</u> If $\phi T = \phi$ and if $\phi \neq (X,\alpha,\xi) \in$ obj $\mathcal{S}^{\mathbb{T} \otimes \mathbb{B}}$ then there exists a minimal $\mathcal{T} \otimes \mathcal{B}$ -subalgebra of X.

<u>Proof.</u> Since (X,α,ξ) is a $\mathcal{T}$-$\mathcal{B}$ quasicomposite algebra (by 2.3.4), $0_X = \langle\langle\phi\rangle_{\mathbb{T}}\rangle_{\mathbb{B}} = \langle\phi\rangle_{\mathbb{B}} = \phi$. There exists a non-empty subalgebra,

namely X. Every inclusion nest of non-empty subalgebras has non-empty intersection by compactness. By Zorn's Lemma, there exists a minimal subalgebra. []

<u>2.4.16 Definition.</u> Let X be a $\top\!\!\top$-algebra. X is <u>distal</u> if $E_X - 0$ is a subgroup of bijections of X. The full subcategory of distal $\top\!\!\top$-algebras will be denoted "$\mathcal{D}_{\mathbb{T}}$" or "$\mathcal{D}$".

<u>2.4.17 Proposition.</u> If $\phi T = \phi$, $\mathcal{D}$ is a Birkhoff subcategory of $\mathcal{S}^{\mathbb{T}}$.

<u>Proof.</u> Let $X = \prod X_i$ with each $X_i \in$ obj $\mathcal{D}$. For every $g \in G$, $\xi^g = \prod \xi_i^g$ by the proof of 2.2.12 with n = 1. Since "non-zero" means "non-empty", each $(\xi_i^g)^{-1}$ exists in E_{X_i} so that $(\xi^g)^{-1} = \prod (\xi_i^g)^{-1}$ exists in E_X. Hence X is distal. The argument for subalgebras is clear from 2.2.10. The argument for quotients is clear from 2.2.11 and the fact that a monoid quotient of a group is a group. []

<u>2.4.18 Proposition.</u> Let X be a $\top\!\!\top$-algebra with $1_X \notin 0$ (see 2.4.8 (d)), and let $E = E_X$. The following statements are equivalent.

a. X is distal.

b. E is a minimal subalgebra of X^X.

c. For every $p \in E$, $p \notin 0$ implies $pE = E$.

<u>Proof.</u> <u>a implies b.</u> Suppose $0 \subsetneqq K \leq E$. There exists $p \in K$, $p \notin 0$. Therefore p^{-1} exists in E and $1_X \in pE \leq E$ so that $E = pE \subset K \subset E$.

<u>b implies c.</u> This is clear.

<u>c implies a.</u> Since $1_X \notin 0$, $[p \in E : p \notin 0] \neq \phi$. Let $p \in E$, $p \notin 0$. By hypothesis, $pE = E$ and there exists $q \in E$ with $p.q = 1_X$. If $\phi T = \phi$,

$q \notin 0$. If $\phi T \neq \phi$, from 2.4.8 (d) we have crd $X > 1$ so, since q is onto, q is not constant and still $q \notin 0$. But then $qE = E$ and there exists r ε E with $q.r = 1_X$. Therefore q is bijective and $p^{-1} = q \;\varepsilon\; E$. []

<u>2.4.19 Definition.</u> Let $\mathcal{P}$ (for "property") be a full subcategory of $\mathcal{S}^{\Pi}$ whose objects are a union of $\mathcal{S}^{\Pi}$-isomorphism classes. Let U ε obj $\mathcal{S}^{\Pi}$. U is a universal $\mathcal{P}$ minimal algebra $=_{df}$ U satisfies (i)-(iii):

(i) U ε obj $\mathcal{P}$ and U is a minimal Π-algebra.

(ii) Every minimal Π-algebra in $\mathcal{P}$ is a Π-quotient of U.

(iii) If V satisfies (i) and (ii) then $U \cong V$.

When $\mathcal{P} = \mathcal{S}^{\Pi}$, we say simply "universal minimal algebra".

<u>2.4.20 Proposition.</u> Assume $\phi T = \phi$. Let $\mathcal{B}$ be a Birkhoff subcategory of $\mathcal{S}^{\Pi}$, and set U $=_{df}$ the free $\mathcal{B}$-algebra on one generator. Then the following statements are equivalent.

a. U is distal.

b. U is minimal.

c. U is a universal $\mathcal{B}$ minimal algebra.

<u>Proof.</u> We remark that notions such as "subalgebra", "0_X", "singly-generated", "minimal" and "enveloping semigroup" in a Birkhoff subcategory are equally computed in $\mathcal{S}^{\Pi}$ so that we need not specify where U is minimal, etc..

<u>a implies c.</u> Let $\tilde{\Pi}$ be the triple corresponding to $\mathcal{B}$. Since $\phi\tilde{T}$ is a quotient of ϕT, $\phi\tilde{T} = \phi$. Hence $1_U \notin 0$ and 2.4.18 applies to show E_U is minimal. But $U = PT \cong E_U$ by 2.4.2, which proves that U is minimal. Clearly every singly-generated $\mathcal{B}$-algebra, every minimal $\mathcal{B}$-algebra in

particular, is a quotient of U. Now observe that if $U \xrightarrow{f} U$ is a $\mathcal{B}$-homomorphism with non-zero image then f is an isomorphism. Clearly f is onto. To see that f is 1-to-1 let u be the free $\mathcal{B}$-generator of U. There exists $x \in U$ with $xf = u$. Let g be the unique $\mathcal{B}$-homomorphism $U \xrightarrow{g} U$ such that $ug = x$. Since $fg = 1_U$ on generators, $fg = 1_U$ and f is 1-to-1. If V satisfies 2.4.19 (i), (ii) then there exist epimorphisms $U \xrightarrow{\zeta} V \xrightarrow{\chi} U$; as $\zeta\chi$ is an isomorphism, ζ is 1-to-1 and $U \cong V$.

<u>c implies b.</u> This is clear.

<u>b implies a.</u> If U is minimal, so is $E_U \cong PT = U$. By 2.4.18, U is distal. []

<u>2.4.21 Computations in $\mathcal{S}^{\mathbb{H} \otimes \beta}$.</u> Let H be a monoid with associated triple $\|-|$. $G =_{df} G_{\mathbb{H}\otimes\beta}$. $P(H \otimes \beta) = (P \times H)\beta = H\beta$, so elements of G are in bijective correspondence with ultrafilters on H. If $\mathcal{U} \in H\beta$, the X^{th} component of the corresponding natural transformation $g_{\mathcal{U}} \in (1, H \otimes \beta)$n.t. is given by the Yoneda correspondence as

$$X \xrightarrow{\quad Xg_{\mathcal{U}} \quad} (X \times H)\beta$$
$$P \xrightarrow{x} X \quad \longmapsto \quad \langle \mathcal{U}, H\beta \xrightarrow{(x,1)\beta} (X \times H)\beta \rangle$$

that is, $Xg_{\mathcal{U}}$ sends x to the ultrafilter $[\{x\} \times A : A \in \mathcal{U}]^c$. The interested reader may compute the monoid operation $H\beta \times H\beta \xrightarrow{\bullet} H\beta$ as $\mathcal{U} \bullet \mathcal{V} = [A \subset H : \exists V \in \mathcal{V} \; \forall v \in V \; \exists U \in \mathcal{U} \, . \, Uv \subset A]$. Hence $\mathcal{U} \bullet \mathcal{V}$ is a canonical ultrafilter containing the filter $\mathcal{U}\mathcal{V}$; that $\mathcal{U}\mathcal{V}$ is not an ultrafilter was kindly pointed out to us by Robert Ellis.

To compute the general enveloping semigroup, let $(X,\alpha,\xi) \in$ obj $\mathcal{S}^{\mathbb{H}\otimes\beta}$ and recall that the structure map with respect to the composite triple $\|-|\,\beta = \|-| \otimes \beta$ is $(X,\alpha\beta.\xi)$. Therefore $(\alpha\beta.\xi)^{\mathcal{U}}$ is described by $x \mapsto [\{x\} \times U : U \in \mathcal{U}]^c \longmapsto [xU : U \in \mathcal{U}]^c \longmapsto (x\mathcal{U})\xi$. In words, the unary operation induced by $\mathcal{U}$ sends $x \in X$ to the unique point of X to which the ultrafilter $x\mathcal{U}$ on X converges. Notice that if $\mathcal{U} = \dot{h}$, $x \longmapsto (x\mathcal{U})\xi$ is just α^h, that is $E_{(X,\alpha)}$ is a submonoid (though not a subalgebra) of $E_{(X,\alpha,\xi)}$.

It is proved in [7, lemma 4] that the usual notion of "distal" used in topological dynamics coincides with the property that p is 1-to-1 for every $p \in E_X$. That this is the same as our definition will follow from 2.5.18 below.

2.4.22 Open question. If H is a monoid with associated triple $\|-|$, then every Birkhoff subcategory, $\mathcal{B}$, of $\mathcal{S}^{\mathbb{H} \otimes \mathbb{B}}$ has a universal minimal set U. We will prove this in 2.5.16 using methods similar to the proof of Ellis in [9], in the case of $\mathcal{B} = \mathcal{S}^{\mathbb{H} \otimes \mathbb{B}}$. The question arises whether a proof more like that of 2.4.20 can be given, that is whether one could show U were a free algebra on one generator with respect to some triple reasonably associated with $\mathcal{B}$. 2.4.20 (b) shows that Birkhoff subcategory arguments are doomed to failure, for it is known that in $\mathcal{S}^{\mathbb{H} \otimes \mathbb{B}}$ U need not be distal. The case of groups in semigroups shows that good tripleable subcategories need not, in fact, be Birkhoff subcategories; that is, we can add new operations (in this case "inverse") in addition to new equations.

2.4.23 Example; $\phi T \neq \phi$ is necessary in 2.4.12. Consider $\mathcal{S}^{\mathbb{H} \otimes \mathbb{B}}$ where $H = \mathbb{Z}$. Let $S^1 \xrightarrow{\alpha} S^1$ be the homeomorphism induced by

$$[0,1] \longrightarrow [0,1]$$
$$x \longmapsto x^2$$

and identifying $0 \equiv 1$. This induces the discrete flow

$$S^1 \times \mathbb{Z} \longrightarrow S^1$$
$$(x, n) \longmapsto \langle x, \alpha^n \rangle$$

The enveloping semigroup consists of the powers $[\alpha^n : n \in \mathbb{Z}]$ and the constant function $0 \equiv 1$.

§2.5 Almost periodicity.

For this section let $\mathbb{T}$ = (T,η,μ) be a triple in $\mathcal{S}$.

2.5.1 Definitions. Let X, Γ be sets. The discrete topology on X is denoted "$\mathcal{S}_d$", X being understood. If $x \in X$, $\mathcal{S}_{(x)} =_{df}$ the compact T2 topology on X obtained by discretifying $X - \{x\}$ and replacing x with the topology of the 1-point compactification. In the language of 2.3, $(X, \mathcal{S}_{(x)}) = (X,\xi)$ where $X\beta \xrightarrow{\xi} X$, $\mathcal{U}\xi = y$ (if there exists $y \in X$ with $\mathcal{U} = \dot{y}$) and $= x$ (otherwise). If $\mathcal{S}$ is any topology on X, $\mathcal{S}^\Gamma$ $=_{dn}$ the induced cartesian power topology on X^Γ. The fine power topology on $X^\Gamma =_{df}$ the topology $\mathcal{S}_d^\Gamma$. It is clear that if $(\mathcal{S}_\gamma : \gamma \in \Gamma)$ is any Γ-indexed family of topologies on X then $\prod \mathcal{S}_\gamma$ is coarser than $\mathcal{S}_d^\Gamma$. Observe that if $A \subset (X^\Gamma, \mathcal{S}_d^\Gamma)$ and if $x \in X$ then $x \in \bar{A}$ iff for every finite subset F of Γ there exists $a \in A$ with x and a agreeing on F.

$\mathbb{T}$ is a fine-powered triple $=_{dn}$ $\mathbb{T}$ fp triple, $=_{df}$ for every $\mathbb{T}$-algebra (X,ξ) and for every subset $\Gamma \subset X$ and for every subalgebra $A \leq (X,\xi)$, A is closed in the fine power topology on X^Γ. $\mathbb{T}$ is a weakly fine-powered triple, $=_{dn}$ $\mathbb{T}$ wfp triple, $=_{df}$ for every (X,ξ) and Γ as above and for every $x \in X$, $\langle x \rangle$ is closed in the fine power topology on X^Γ. Clearly $\mathbb{T}$ fp triple implies $\mathbb{T}$ wfp triple, but the converse is false for the identity triple.

2.5.2 Remarks. $\mathcal{S}_d^\Gamma$ is canonically a $\varprojlim$ of compact T2 topologies on X^Γ. In fact $\mathcal{S}_d^\Gamma = \sup[\mathcal{S}_{(x)}^\Gamma : x \in X]$ (that supremums are $\varprojlim$'s is a typical lattice fibering property, as is seen from the proof of 3.1.6).

If X is finite, $\mathcal{S}_d^{\Gamma} = \mathcal{S}_{(x)}^{\Gamma}$ for all $x \in X$. Otherwise, assume X is infinite. Surely $\mathcal{S}_{(x)}^{\Gamma} \subset \mathcal{S}_d^{\Gamma}$ for all x. Conversely, let F be a finite subset of Γ, $F \xrightarrow{a} X$ a function. Then $U =_{df} [f \in X^{\Gamma} : f/F = a]$ is a basic open set in $\mathcal{S}_d^{\Gamma}$. There exists $x \in X - \text{im } a$. Since any subset of X not containing x is open in $(X, \mathcal{S}_{(x)})$ we have $U \in \mathcal{S}_{(x)}^{\Gamma}$. This motivates our attempt to use $\mathcal{S}_d^{\Gamma}$ as a "prototype" for topologies of the form $\mathcal{S}^{\Gamma}$ where $\mathcal{S}$ is compact T2. We introduce the notion of jointly almost periodic subset, and prove some theorems following a pattern set by W. H. Gottschalk in [14]. The important 2.5.12 is proved for wfp triples which, in fact, is where the fine power topology comes in. By a method similar to that of Ellis in [9] we show that if $\mathbb{T}$ wfp triple and if $\phi T = \phi$, then any minimal subalgebra of PT is a universal minimal set. The main difference in our method is that we substitute 2.5.15 (a), (b) for compactness arguments. We begin now with some observations that ensure the existence of enough wfp triples to make all this worth while.

<u>2.5.3 Remark.</u> If PT is finite then $\mathbb{T}$ wfp triple. Examples include Boolean algebras, sets, G-sets for finite G, complete semilattices and others. Such triples are unlikely to provide interesting minimal algebras, however.

<u>2.5.4 Proposition.</u> If there exists an algebraic functor $\Phi: \mathcal{S}^{\mathbb{T}} \to \mathcal{S}^{\tilde{\mathbb{T}}}$, $\Phi U^{\tilde{\mathbb{T}}} = U^{\mathbb{T}}$ with $\tilde{\mathbb{T}}$ fp triple, then $\mathbb{T}$ fp triple.

<u>Proof.</u> Let (X,ξ) be a $\mathbb{T}$-algebra, let $\Gamma \subset X$ and let $A \leq (X,\xi)^{\Gamma}$. Then $A \leq (X,\xi)^{\Gamma}\Phi = (X,\xi)\Phi^{\Gamma}$ (noting that Φ preserves $\underleftarrow{\lim}$'s by 1.5.2) and

so A is closed in $\mathcal{S}_d^{\Gamma}$. []

<u>2.5.5 Proposition.</u> Let $\tilde{\mathbb{T}}$ be another triple in $\mathcal{S}$ with $\mathbb{B} \subset \mathcal{S}^{(\mathbb{T},\tilde{\mathbb{T}})}$ a tripleable $\widehat{\ }$-closed subcategory with triple $\mathbb{S}$. Then if either $\mathbb{T}$ fp triple or $\tilde{\mathbb{T}}$ fp triple, then $\mathbb{S}$ fp triple.

<u>Proof.</u> Let $(X,\xi,\tilde{\xi})$ be a $\mathbb{B}$-algebra, and let $\Gamma \subset X$, $A \leq (X,\xi,\tilde{\xi})^{\Gamma}$. Then $A \leq (X,\xi)^{\Gamma}$ and $A \leq (X,\tilde{\xi})^{\Gamma}$. []

<u>2.5.6 Corollary.</u> Any Birkhoff subcategory of $\mathcal{S}^{\mathbb{T} \otimes \mathbb{B}}$ comes from an fp triple.

<u>Proof.</u> By 2.5.5 we need only observe $\mathbb{B}$ fp triple. Indeed, if $(X,\xi) \in \mathcal{S}^{\mathbb{B}}$, $\Gamma \subset X$, $A \leq (X,\xi)^{\Gamma}$ then A is closed in $\mathcal{S}^{\Gamma}$, where $(X,\xi) = (X, \mathcal{S})$, and hence A is closed in $\mathcal{S}_d^{\Gamma}$. []

<u>2.5.7 Definition.</u> Let (X,ξ) be a $\mathbb{T}$-algebra, and let $\Gamma \xrightarrow{i} X \in \mathcal{S}$. i is a jointly almost periodic injection, $=_{dn}$ i jtapi, $=_{df}$ <i> is a minimal subalgebra of X^{Γ}. If $\Gamma \subset X$, Γ is a jointly almost periodic subset, $=_{dn}$ Γ jtaps, $=_{df}$ the inclusion map of Γ is jtapi. We consider the set of isomorphism classes of monomorphisms into X partially ordered by the inclusion relation discussed in 1.8.1. Subsets of X are partially ordered by ordinary inclusion. i mxjtapi $=_{dn}$ i maximally jtapi; Γ mxjtaps $=_{dn}$ Γ maximally jtaps. If $x \in X$, x is an almost periodic point of X, $=_{dn}$ x ap pt, $=_{df}$ $\{x\}$ jtaps.

<u>2.5.8 Proposition.</u> Let (X,ξ) be a $\mathbb{T}$-algebra, let $\Gamma \xrightarrow{i} X$, and let $\Delta \xrightarrow{j} X$. The following statements are valid.

a. i jtapi implies im i jtaps; conversely,

b. Δ jtaps implies j jtapi.

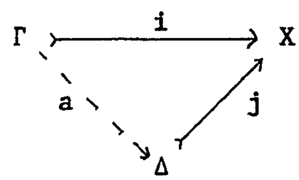
c. i mxjtapi implies im i mxjtaps; conversely,

d. Δ mxjtaps implies j mxjtapi.

Proof. a. Factor $\Gamma \rightarrowtail\!\xrightarrow{p}\!\twoheadrightarrow \text{im } i \rightarrowtail\!\xrightarrow{j} X = i$. Then $X^{\Gamma} \xrightarrow{p^{-1}\circ -} X^{\text{im } i}$ is an isomorphism sending <i> to <j>.

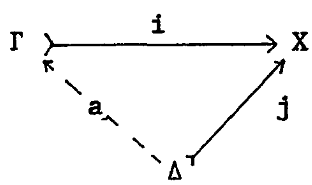
b. This is by definition.

c. Suppose $\Delta \overset{j}{\hookrightarrow} X$ jtaps, im i $\subset$ Δ.



Clearly a exists. As i mxjtapi and j jtapi by (b), a is a bijection and im i = Δ.

d. Suppose $\Gamma \rightarrowtail\!\xrightarrow{i} X$ mxjtapi such that a exists:



Then im i jtaps by (a) and im i $\supset$ Δ implies im i = Δ implies a is a bijection. []

2.5.9 Proposition. Let (X,ξ) be a $\ulcorner\prod\urcorner$-algebra and let $\Lambda \rightarrowtail\!\xrightarrow{i} \Gamma \rightarrowtail\!\xrightarrow{j} X$ with i.j not constantly 0 and j jtapi. Then i.j jtapi.

Proof. Consider $X^{\Gamma} \xrightarrow{i\circ -} X^{\Lambda}$. <i.j> = <j.(i∘−)> = <j>i∘−. Hence <i.j> is preminimal. But i.j $\not\equiv$ 0 implies <i.j> ≠ 0. []

2.5.10 Proposition. Let $(X,\xi) \overset{f}{\rightarrowtail} (Y,\theta) \in \mathcal{S}^{\mathbb{T}}$, $\Gamma \overset{i}{\rightarrowtail} X$ jtapi. Then i.f jtapi.

Proof. $X^{\Gamma} \xrightarrow{-\circ f} Y^{\Gamma}$ is a homomorphism sending $\langle i \rangle$ to $\langle i.f \rangle$. If i is not constant, neither is i.f since f is mono. Otherwise, i is a non-zero element of X. But $0_Y f^{-1} = 0_X f f^{-1} = 0_X$ (both equalities because f is mono) so that $i.f \not\in 0$. Either way, $\langle i.f \rangle \neq 0$. []

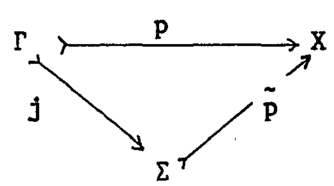
2.5.11 Proposition. Let $\Gamma \overset{i}{\rightarrowtail} (X,\xi)$, $0 \neq p \in \langle i \rangle \subset X^{\Gamma}$. Then the following statements are valid.

a. i jtapi implies p jtapi.

b. i mxjtapi implies p mxjtapi.

Proof. a. As $p \not\in 0$, $\langle p \rangle = \langle i \rangle$. Hence we need only show p is 1-to-1. Suppose $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1.p = \gamma_2.p$. Let $\{\gamma_1,\gamma_2\} \overset{j}{\rightarrowtail} \Gamma$ be inclusion, let D be the diagonal of $X^{\{\gamma_1,\gamma_2\}}$. $D \leq X^{\{\gamma_1,\gamma_2\}}$ on general principles (its inclusion map is categorically induced in $\mathcal{S}^{\mathbb{T}}$). Since $\langle p \rangle j\circ- = \langle j.p \rangle$ and $j.p \in D$, $\langle p \rangle j\circ- \subset D$. On the other hand, $\langle p \rangle j\circ- = \langle i \rangle j\circ- = \langle j.i \rangle$. Therefore $j.i \in D$, and j.i is constant. As i is mono, j is constant, and $\gamma_1 = \gamma_2$.

b. p jtapi by (a). Now suppose



with $\tilde{p}$ jtapi. Consider $X^{\Sigma} \xrightarrow{j\circ-} X^{\Gamma}$. Since $\langle \tilde{p} \rangle j\circ- = \langle j.\tilde{p} \rangle = \langle p \rangle = \langle i \rangle$ there exists $\tilde{i} \in \langle p \rangle$ with $j.\tilde{i} = i$. Clearly i is not constantly 0 and so $\tilde{i}$ is not constantly 0. From (a), $\tilde{i}$ jtapi. Since i mxjtapi and $j.\tilde{i} = i$, j is an isomorphism, as we wished to prove. []

<u>**2.5.12 Proposition.**</u> Assume $\mathbb{T}$ wfp triple. Let (X,ξ) be a $\mathbb{T}$-algebra, let $\Delta \overset{j}{\rightarrowtail} X$ jtaps and let $\Gamma \overset{i}{\rightarrowtail} X$ jtapi. The following statements are valid.

a. Δ extends to a mxjtaps.

b. i extends to a mxjtapi.

<u>Proof.</u> <u>a.</u> $\mathcal{H} =_{df} [\Sigma \subset X : \Sigma \text{ jtaps } \& \ \Sigma \supset \Delta]$. $\mathcal{H} \neq \phi$. Let (Σ_α) be a chain in $\mathcal{H}$, $\Sigma =_{df} \bigcup \Sigma_\alpha$, with inclusions $\Sigma_\alpha \overset{i_\alpha}{\rightarrowtail} X$, $\Sigma \overset{i}{\rightarrowtail} X$. Consider the restriction maps $X^\Sigma \xrightarrow{pr_\alpha} X^{\Sigma_\alpha}$. For all α we have $\langle i\rangle pr_\alpha = \langle i_\alpha\rangle$. It follows at once that $\langle i\rangle \neq 0$ (a chain is never empty). To show $\langle i\rangle$ is minimal, let $p \in \langle i\rangle - 0$ and show $\langle p\rangle = \langle i\rangle$. Since p is not constantly 0 there exist $\sigma_1, \sigma_2 \in \Sigma$ with $p/\{\sigma_1,\sigma_2\}$ not constantly 0. There exists α_0 with $\{\sigma_1,\sigma_2\} \subset \Sigma_{\alpha_0}$ by the nestedness. Therefore $\alpha \geq \alpha_0$ implies $\langle p/\Sigma_\alpha\rangle = \langle i\rangle$. Let $F \subset \Sigma$ be finite. By the nestedness there exists $\alpha \geq \alpha_0$ with $F \subset \Sigma_\alpha$. As $\langle p\rangle pr_\alpha = \langle p/\Sigma_\alpha\rangle = \langle i_\alpha\rangle$ there exists $q \in \langle p\rangle$ with $q/\Sigma_\alpha = i_\alpha$, so in particular, $q/F = i/F$. As F is arbitrary, this proves that i is in the fine power closure $\overline{\langle p\rangle}$ of $\langle p\rangle$. As $\mathbb{T}$ wfp triple, $\overline{\langle p\rangle} = \langle p\rangle$. Therefore, $\langle i\rangle \subset \langle p\rangle \subset \langle i\rangle$. By Zorn's Lemma, $\mathcal{H}$ has a maximal element.

<u>b.</u> We have im i jtaps by 2.5.8, so from (a) there exists Δ mxjtaps with im $i \subset \Delta$. The inclusion map of Δ extends i and is mxjtapi by 2.5.8. []

<u>**2.5.13 Definition.**</u> Let (X,ξ) be a $\mathbb{T}$-algebra and let $I \subset X^X$. $I^\wedge =_{df} [p \in I : pp = p \ \& \ p \not\equiv 0]$.

<u>**2.5.14 Proposition.**</u> Let (X,ξ) be a $\mathbb{T}$-algebra and let I be a

$\sqcap\!\sqcap$-minimal right ideal in E_X. Then for every $p \in I$ and $u \in I^\wedge$, $u.p = p$; said differently, elements in I are determined on im u for any $u \in I^\wedge$.

Proof. Suppose $p \in I$, $u \in I^\wedge$. As $u \not\in 0$ and $uu = u \in uI$, $uI \not\subset 0$. As $uI \subset IE \subset I$ and uI is a right ideal, we conclude $uI = I$. Therefore there exists $q \in I$ with $uq = p$. We have $up = uuq = uq = p$. []

2.5.15 Proposition. Assume $\sqcap\!\sqcap$ wfp triple. Let (X,ξ) be a $\sqcap\!\sqcap$-algebra, let $E = E_X$ and let I be a $\sqcap\!\sqcap$-minimal right ideal in E. Then the following statements are valid.

a. If $XI \not\subset 0_X$ then there exists $u \in I^\wedge$ with $Xu \not\subset 0_X$.

b. The passage $u \mapsto \text{im } u$ establishes a bijection from $[u \in I^\wedge : Xu \not\subset 0_X]$ to $[\Delta \subset X : \Delta \text{ mxjtaps } \& \ \Delta \cap XI \not\subset 0_X]$

c. If Δ mxjtaps with inclusion map i such that $\Delta \cap XI \not\subset 0_X$ then $I \longrightarrow \langle i \rangle \subset X^\Delta$, $p \mapsto p/\Delta$ is a $\sqcap\!\sqcap$-isomorphism.

d. If J is a $\sqcap\!\sqcap$-minimal right ideal in E and if there exists $x \in XI \cap XJ$ with x ap pt, then $I \cong J$ qua $\sqcap\!\sqcap$-algebras.

e. If $XI \not\subset 0_X$ then every $\sqcap\!\sqcap$-endomorphism of I with non-zero image is an isomorphism.

Proof. If $XI \not\subset 0$ there exists $x \in X$ with $xI \not\subset 0$. By 2.4.7 (f), x ap pt. Hence $\{x\}$ extends to a mxjtaps by 2.5.12. Hence whenever $XI \not\subset 0$, $[\Delta \subset X : \Delta \text{ mxjtaps } \& \ \Delta \cap XI \not\subset 0]$ is non-empty.

Now suppose Δ mxjtaps, $\Delta \cap XI \not\subset 0$. Let $\Delta \overset{i}{\rightarrowtail} X$ be inclusion, and let $I \xrightarrow{\zeta} X^\Delta$ be the restriction homomorphism $p \mapsto p/\Delta$. As $X^X \xrightarrow{pr_\Delta} X$ maps $E = \langle 1_X \rangle$ into $\langle i \rangle$, we have $I\zeta \leq \langle i \rangle$. By hypothesis, there exists $x \in \Delta$ and $p \in I$ with $xp \not\in 0$. It follows $p\zeta = p/\Delta$ is not constantly 0. As <i> is minimal and $I\zeta \neq 0$, ζ is onto. This implies that there exists $u \in I$ with $u/\Delta = i$. Clearly $\Delta \subset \text{im } u$; we show in fact that $\Delta = \text{im } u$ as follows. Let $x \in X$. If $x \in \Delta$, surely $xu \in \Delta$. Otherwise, suppose $x \in X - \Delta$. Consider the restriction homomorphism $X^X \xrightarrow{\chi} X^{(\Delta \cup \{x\})}$. As Δ jtaps there exists $y \in \Delta$, $y \notin 0$. As $yu = y$, we have $u \notin 0$, $\langle u \rangle = I$ and $u/(\Delta \cup \{x\}) \notin 0$ from which we derive $0 \neq \langle u/(\Delta \cup \{x\}) \rangle = \langle u \rangle\chi = I\chi$ which proves $\langle u/(\Delta \cup \{x\}) \rangle$ is minimal. Since i has no proper jtap extensions, necessarily $u/(\Delta \cup \{x\})$ fails to be 1-to-1. But $u/\Delta = i$ is 1-to-1. So there exists $\delta \in \Delta$ with $xu = \delta u = \delta \in \Delta$. This shows im $u = \Delta$. Since $u/\Delta = i$ we have in fact that $uu = u$. Therefore $u \in I^\wedge$ and in particular (a) is established.

Now let $u \in I^\wedge$ with $Xu \not\subset 0_X$. $\Delta \overset{i}{\rightarrowtail} X =_{df} \text{im } u$. Consider $I \xrightarrow{\zeta} \langle i \rangle \subset X^\Delta$, $p \mapsto p/\Delta$. Clearly $\langle i \rangle \neq 0$. $uu = u$ implies $u/\Delta = i$ and hence $I\zeta = \langle u \rangle\zeta = \langle i \rangle$ and $\langle i \rangle$ is minimal, which proves Δ jtaps. By 2.5.12 there exists $\tilde{\Delta}$ mxjtaps, $\Delta \subset \tilde{\Delta}$. As proved above, there exists $v \in I^\wedge$ with im $v = \tilde{\Delta}$. For all $x \in X$, $xu \in \Delta \subset \tilde{\Delta}$ so that $uv = u$. By 2.5.14, $uv = v$. Therefore $\Delta = \text{im } u = \text{im } v = \tilde{\Delta}$ and Δ mxjtaps. The proof of (b) is complete.

To prove (c), let $\Delta \overset{i}{\rightarrowtail} X$ mxjtaps with $\Delta \cap XI \not\subset 0$. We have already observed that $I \xrightarrow{\zeta} \langle i \rangle \subset X^\Delta$, $p \mapsto p/\Delta$ is onto, and that there exists $u \in I^\wedge$ with im $u = \Delta$. Hence if $p,q \in I$ then $p/\Delta = q/\Delta$ iff $up = uq$ iff $p = q$ (by 2.5.14) and ζ is an isomorphism. To prove (d) extend $\{x\}$ to a mxjtaps $\Delta \overset{i}{\subset} X$ and observe that $\Delta \cap XI \not\subset 0$ and $\Delta \cap XJ \not\subset 0$ so that by (c) $I \cong \langle i \rangle \cong J$.

Finally, we prove (e). We assume $XI \not\subset 0$ so there exists $\Delta \overset{i}{\subset} X$ mxjtaps with $I \cong \langle i \rangle$. Let $\langle i \rangle \xrightarrow{f} \langle i \rangle$ be a $\prod$-endomorphism which has a non-zero image. Clearly f is onto. We must show f is 1-to-1. In view of 2.4.4, and the facts that f commutes with unary operations and that $E = [\xi^g : g \in G]$ we have commutative diagrams

$$\begin{array}{ccc} \langle i\rangle & \xrightarrow{\;f\;} & \langle i\rangle \\ \downarrow{\scriptstyle -\circ p} & & \downarrow{\scriptstyle -\circ p} \\ \langle i\rangle & \xrightarrow{\;f\;} & \langle i\rangle \end{array}$$

for all $p \in I$. Also there exists unique $u \in I^{\wedge}$ with $\text{im } u = \Delta$. We have $if = (i.u)f = (if).u$ so that $\text{im } if \subset \text{im } u = \Delta$. Since $if \not\leq 0$ it follows from 2.5.11 (b) and 2.5.8 (c) that $\text{im } if = \Delta$. Let $p \in I$. By (c) there exists unique $\tilde{p} \in I$ with $i.\tilde{p} = p$. Therefore $pf = (i.\tilde{p})f = (if).\tilde{p}$. If $q \in I$ with $pf = qf$ then $if.p = if.q$. As $\text{im } if = \Delta$, $p = \tilde{p}/\Delta = \tilde{q}/\Delta = q$. []

<u>2.5.16 Proposition.</u> Assume $\mathbf{T}$ wfp triple such that $\phi T = \phi$. Then if PT has a minimal subalgebra U, U is a universal minimal set.

<u>Proof.</u> Suppose such U exists. If M is a minimal $\mathbf{T}$-algebra there exists $U \rightarrowtail PT \longrightarrow M$ which is necessarily onto since "non-zero" means "non-empty". $PT \simeq E_{PT}$ by 2.4.2. Let $\tilde{U} \leq E_{PT}$ correspond to U. Then $\tilde{U}$ is a minimal subalgebra and hence a $\mathbf{T}$-minimal right ideal by 2.4.13. Clearly $(PT)\tilde{U} \subset PT \neq \phi$ and hence it follows from 2.5.15 (e) that every $\mathbf{T}$-endomorphism of U is an isomorphism. The rest of the details are clear. []

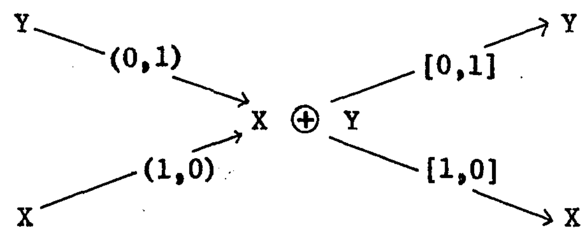
<u>2.5.17 Definition.</u> Say that a $\mathbf{T}$-algebra (X,ξ) is <u>weakly distal</u> if for every $p \in E_X - 0$, p is 1-to-1. Clearly distal implies weakly distal.

<u>2.5.18 Proposition.</u> Assume $\mathbb{T}$ wfp triple. Let (X,ξ) be a $\mathbb{T}$-algebra which is weakly distal. If there exists a $\mathbb{T}$-minimal right ideal $I \subset E_X$ with $XI \not\subset 0_X$, then (X,ξ) is distal.

<u>Proof.</u> Suppose such I exists. By 2.5.15 (a) there exists $u \in I^\wedge$. Since $u \notin 0$, u is 1-to-1. For every $x \in X$, $xuu = xu$ implies $xu = x$ and $u = 1_X$. But by 2.5.15, X = im u is mxjtaps. Therefore E is minimal. As our hypothesis on I makes $1_X = 0$ impossible, by 2.4.18 we are done. []

§2.6 Tripleable abelian categories.

2.6.1 Review of additive and abelian categories. The reader is assumed, in this section, to be familiar with the elementary theory of additive and abelian categories. We sketch here only a few basic definitions; see [10] and [26] for detailed accounts. Let $\mathbb{A}$ be the triple of abelian groups over sets and let $\mathcal{K}$ be a category. $\mathcal{K}$ is additive if $\mathcal{K}$ is legitimate, has finite products and coproducts, and there exists a functor $\mathcal{K}^{op} \times \mathcal{K} \longrightarrow \mathcal{S}^{\mathbb{A}}$ whose composition with $U^{\mathbb{A}}$ is $\simeq (-,-)\mathcal{K}$. The third condition says that each $(X,Y)\mathcal{K}$ is provided with an abelian group structure so that composition distributes over addition on the left and right. If $\mathcal{K}$ is additive it has the following properties. $\mathcal{K}$ has a zero object 0, that is an object which is at the same time initial and terminal. If X, Y ε obj $\mathcal{K}$, the unique zero map $X \xrightarrow{o} Y =_{df} X \rightarrow 0 \rightarrow Y$ coincides with the identity of the abelian group $(X,Y)\mathcal{K}$. That product = coproduct is true in the finite non-empty case too; if X, Y ε obj $\mathcal{K}$ there is a direct sum system



with $X \amalg Y = X \oplus Y = X \times Y$. Injections are defined in terms of projections and vice versa as the notation indicates. If $(x, y) : X \rightarrow Y \;\varepsilon\; \mathcal{K}$ then

$$x + y = X \xrightarrow{(1,1)} X \oplus X \xrightarrow{[x,y]} Y = X \xrightarrow{(x,y)} Y \oplus Y \xrightarrow{[1,1]} Y.$$

$$-x = X \xrightarrow{-1} X \xrightarrow{x} Y = X \xrightarrow{x} Y \xrightarrow{-1} Y, \text{ where } (-1) + 1 = o.$$

Every object in $\mathcal{K}$ is an <u>abelian group object</u>; that is when $X \in \text{obj } \mathcal{K}$ is equipped with addition $X \oplus X \xrightarrow{[1,1]} X$, inversion $X \xrightarrow{-1} X$ and zero $0 \to X$, then the usual diagrams commute (cf. 1.1.6).

$\mathcal{K}$ is <u>abelian</u> if $\mathcal{K}$ has a 0 object, $\mathcal{K}$ has finite products and coproducts, every $\mathcal{K}$-morphism has a kernel ($=_{dn}$ ker f, $=_{df}$ eq(f,o)) and a cokernel ($=_{dn}$ cok f, $=_{df}$ coeq(f,o)), and every {mono}{epi} is normal ($=_{df}$ {=ker f}{= cok f} for some f). Every abelian category is additive.

For the rest of this section fix a triple $\mathbb{T}$ in $\mathcal{S}$, and let $\mathbb{A}$ be the triple of abelian groups.

<u>2.6.2 Proposition.</u> The following statements are valid.

a. $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$ is an abelian category.

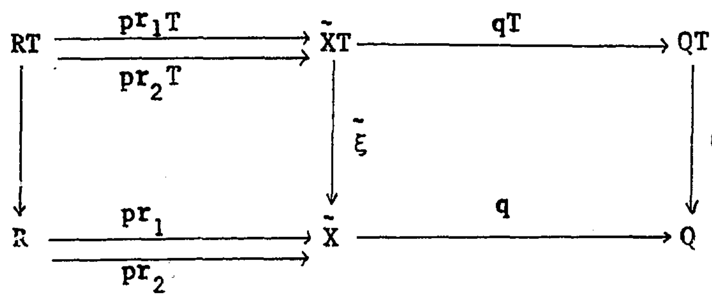
b. The underlying group functor $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]} \xrightarrow{V} \mathcal{S}^{\mathbb{A}}$ creates 0 objects, o maps, cokernels, direct sum systems, exact sequences and in fact all $\varprojlim$'s and finite $\varinjlim$'s.

c. Epimorphisms are onto in $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$.

<u>Proof.</u> $0 =_{df} (P,\ PT \to P,\ PA \to P)$ is a terminal object in $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$. If $(X,\xi,\alpha) \in \text{obj } \mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$, then $0 \xrightarrow{o} (X,\alpha)$ is an $\mathbb{A}$-homomorphism; it is also an $\mathbb{A}$-operation, hence a $\mathbb{T}$-homomorphism. If $0 \xrightarrow{x} (X,\xi,\alpha)$ is a $\mathbb{T}$-$\mathbb{A}$-homomorphism it is an $\mathbb{A}$-homomorphism in particular so that $x = o$. That V creates 0 objects and o maps is now clear.

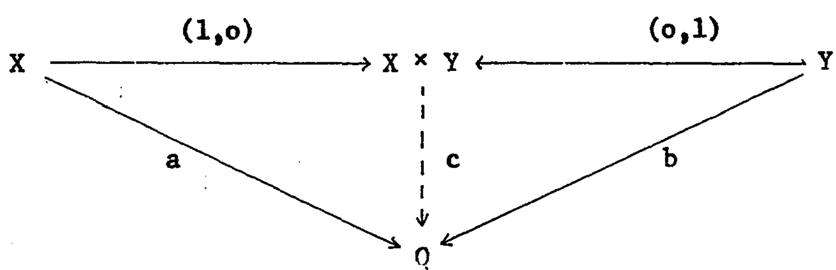
Let $(X,\xi,\alpha) \xrightarrow{f} (\tilde{X},\tilde{\xi},\tilde{\alpha}) \in \mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$, and let $(X,\tilde{\alpha}) \xrightarrow{q} (Q,\omega)$ $=_{df}$ cok f in $\mathcal{S}^{\mathbb{A}}$. From special knowledge of abelian groups, we may write $\tilde{X} \xrightarrow{q} Q = \tilde{X} \xrightarrow{q} \tilde{X}/R$ as the coequalizer of its kernel pair (in $\mathcal{S}$) $R \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} \tilde{X}$ where $R = [(x,y) \in \tilde{X} \times \tilde{X} : x - y \in \text{im } f]$ and pr_i is the i^{th} projection $\tilde{X} \times \tilde{X} \longrightarrow \tilde{X}$ $(i = 1, 2)$. Factoring f =

$(X,\xi) \twoheadrightarrow (\text{im } f, \xi_0) \rightarrowtail (\tilde{X},\tilde{\xi}) \in \mathcal{S}^{\mathbb{T}}$ and observing that the $\mathbb{A}$-operation $X \times X \xrightarrow{-} X$ is a $\mathbb{T}$-homomorphism, we have that R inherits a unique $\mathbb{T}$-structure making pr_1, pr_2 $\mathbb{T}$-homomorphisms because R is just the inverse image of im f under $-$. The top row of the diagram



is a coequalizer in $\mathcal{S}^{\mathbb{T}}$ (by 1.3.4) which uniquely induces θ. Hence there exists unique θ admitting $(\tilde{X},\tilde{\xi},\tilde{\alpha}) \xrightarrow{q} (Q,\theta,\omega) \in \mathcal{S}^{(\mathbb{T},\,\mathbb{A})}$, and $(Q,\theta,\omega) \in \text{obj } \mathcal{S}^{[\mathbb{T},\,\mathbb{A}]}$ by 2.2.15 (a) since q is onto. We also have $q = \text{cok } f$ in $\mathcal{S}^{[\mathbb{T},\,\mathbb{A}]}$ as follows. $fq = o$ in $\mathcal{S}^{[\mathbb{T},\,\mathbb{A}]}$ because $fq = o$ in $\mathcal{S}^{\mathbb{A}}$. Suppose $(\tilde{X},\tilde{\xi},\tilde{\alpha}) \xrightarrow{\tilde{q}} (\tilde{Q},\tilde{\theta},\tilde{\omega}) \in \mathcal{S}^{[\mathbb{T},\,\mathbb{A}]}$ with $f\tilde{q} = o$. Then there exists unique $(Q,\omega) \xrightarrow{t} (\tilde{Q},\tilde{\omega}) \in \mathcal{S}^{\mathbb{A}}$ with $qt = \tilde{q}$. t is also a $\mathbb{T}$-homomorphism because qt is and qT is epi (cf. the third diagram in the proof of 1.2.4). This demonstrates that V creates cokernels.

To see that V creates direct sum systems, let X, Y ε obj $\mathcal{S}^{[\mathbb{T},\,\mathbb{A}]}$ and suppose given a, b



Then there exists unique c $\in \mathcal{S}^{\mathbb{A}}$ such that $(1,o).c = a$ and $(o,1).c = b$, and moreover $c = X \times Y \xrightarrow{a \times b} Q \times Q \xrightarrow{+} Q$. Since the $\mathbb{A}$-operation $+$ is a $\mathbb{T}$-homomorphism, so is c. This shows that V creates direct sum systems.

All remaining details in (b) will follow from standard theorems about abelian categories (and the fact that V creates $\varprojlim$'s and images, reasoning as in 1.7.4), so we will go on to prove (a). We have only to show that monos and epis are normal. Let $X \xrightarrow{f} Y$ be an epimorphism in $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$, and let $Y \xrightarrow{q} Q =_{df}$ cok f in $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$. As fq = o and fo = o, q = o. Since V creates cokernels, cok f = o in $\mathcal{S}^{\mathbb{A}}$, and hence applying a well-known property of the category of abelian groups we see that f is onto (which, in passing, proves (c)). Since V creates kernels and cokernels, the fact that f = cok ker f in $\mathcal{S}^{\mathbb{A}}$ implies that f = cok ker f in $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$. If $X \rightarrowtail^{i} Y$ is mono in $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$, we do not know à priori that i is 1-to-1 since our usual argument that V preserves monos requires V to have a left adjoint. However, i is indeed 1-to-1 and i = ker cok i by dualizing the argument used for epis. []

<u>2.6.3 Proposition.</u> The following statements are equivalent.

a. $\mathcal{S}^{\mathbb{T}}$ is abelian.

b. $\mathcal{S}^{\mathbb{T}}$ is additive.

c. $\mathbb{T} = \mathbb{T} \otimes \mathbb{A}$.

d. There exists a triple $\tilde{\mathbb{T}}$ in $\mathcal{S}$ with $\mathbb{T} = \tilde{\mathbb{T}} \otimes \mathbb{A}$.

<u>Proof.</u> <u>a implies b.</u> Every abelian category is additive.

<u>b implies c.</u> Each $\mathbb{T}$-algebra X is an abelian group object with addition $X \oplus X \xrightarrow{[1,1]} X$, inverse $X \xrightarrow{-1} X$ and zero $0 \xrightarrow{o} X$. The group operations are $\mathbb{T}$-homomorphisms by construction, and

$(f \oplus f).[1,1] = [1,1].f$ for each $\mathbb{T}$-homomorphism f. Hence we have defined a functor $\mathcal{S}^{\mathbb{T}} \xrightarrow{\tilde{\Phi}} \mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$ which commutes with the underlying set functors. Let $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]} \xrightarrow{\Phi} \mathcal{S}^{\mathbb{T}}$ be the underlying $\mathcal{S}^{\mathbb{T}}$-object functor. Clearly $\tilde{\Phi}\Phi = 1_{\mathcal{S}^{\mathbb{T}}}$. Now let $X \in$ obj $\mathcal{S}^{[\mathbb{T}, \mathbb{A}]}$. Let $X \times X \xrightarrow{m} X$, $X \xrightarrow{i} X$, $0 \xrightarrow{e} X$ be the operations corresponding to X qua abelian group. Then m, i, e are $\mathbb{T}$-homomorphisms. As 0 is initial in $\mathcal{S}^{\mathbb{T}}$, necessarily e = o and then $X \xrightarrow{[1,o]} X \times X \xrightarrow{m} X$ $= 1_X = X \xrightarrow{[o,1]} X \times X \xrightarrow{m} X$ is known. Since

$$X \xrightarrow{[1,o]} X \times X \xleftarrow{[o,1]} X$$

$= X \coprod X$ in $\mathcal{S}^{\mathbb{T}}$, we conclude $m = [1,1]$. $i = -1$ by the fact that inverses are unique in a group. This proves $\Phi\tilde{\Phi} = 1$.

<u>c implies d.</u> This is obvious.

<u>d implies a.</u> This follows from 2.6.2. []

<u>2.6.4 Remark.</u> If rnk($\mathbb{T}$) $\leq \aleph_0$, $\mathcal{S}^{\mathbb{T}}$ is abelian iff $U^{\mathbb{T}}$ is the underlying set functor from the category of right modules over the endomorphism ring of PT. This has been observed by Lawvere [20], using [26, 4.1]. Hence if operations are finitary the only tripleable abelian categories are the obvious ones. $\mathbb{\beta} \otimes \mathbb{A}$ = compact abelian groups. A more exotic example is given by:

<u>2.6.5 Example; lattice groups.</u> Let $\mathbb{T}$ be the triple for complete semilattices described in 1.1.10. It is easy to show that if $A \subset (X,\xi) \in$ obj $\mathcal{S}^{\mathbb{T}}$, $\langle A \rangle = [\sup B : B \subset A]$. If $A \subset (X,\alpha,\xi) \in$ obj $\mathcal{S}^{[\mathbb{A},\mathbb{T}]}$ with $A \leq (X,\alpha)$ and if B, C $\subset$ A then sup B - sup C = sup(b - c :

b ε B, c ε C) ε $\langle A \rangle_{\mathbb{T}}$ because "–" is a $\mathbb{T}$-homomorphism. Hence all $\mathbb{A}$-$\mathbb{T}$ bialgebras are $\mathbb{A}$-$\mathbb{T}$ quasicomposite algebras and hence $\mathbb{A} \otimes \mathbb{T} = \mathbb{T} \otimes \mathbb{A}$ exists. rnk $(\mathbb{T} \otimes \mathbb{A}) > \aleph_0$, because $\mathbb{R}$ is a $\mathbb{T} \otimes \mathbb{A}$-algebra.

2.6.6 Remark. 2.6.3 shows that a category which is additive but not abelian is not tripleable over any underlying set functor. For instance, no set-valued functor from the category of torsion-free groups is tripleable.

CHAPTER 3. TRIPLES IN A LATTICE FIBERING

§3.1 Lattice fiberings over a category.

3.1.1 Definitions. Let $\mathcal{E} \xrightarrow{b} \mathcal{K}$ be a functor. An $\mathcal{E}$-morphism $X \xrightarrow{f} Y$ is *cartesian* if the conditions:

$$\begin{array}{ccc} Z & & \\ {\scriptstyle h}\downarrow & \searrow^{g} & \\ X & \xrightarrow{f} & Y \end{array} \quad \text{in } \mathcal{E} \qquad \begin{array}{ccc} Zb & & \\ {\scriptstyle \alpha}\downarrow & \searrow^{gb} & \\ Xb & \xrightarrow{fb} & Yb \end{array} \quad \text{in } \mathcal{K}$$

induce unique $Z \xrightarrow{h} X$ in $\mathcal{E}$ with $hf = g$ and $hb = \alpha$. b is a *fibration* if for every $Y \in \text{obj } \mathcal{E}$, $K \xrightarrow{\alpha} Yb \in \mathcal{K}$, there exists $Y\alpha^{*} \xrightarrow{f} Y \in \mathcal{E}$ with $fb = \alpha$ and f a cartesian morphism. Dually, $X \xrightarrow{f} Y$ in $\mathcal{E}$ is *opcartesian* if $Z \xleftarrow{g} X \xrightarrow{f} Y$ in $\mathcal{E}$ and $fb.\alpha = gb$ in $\mathcal{K}$ induce unique $Y \xrightarrow{h} Z$ in $\mathcal{E}$ with $fh = g$ and $hb = \alpha$, and b is an *opfibration* if $Xb \xrightarrow{\alpha} K$ induces $X \xrightarrow{f} X\alpha_{*}$ opcartesian with $fb = \alpha$. Say that b is a *fibering* if b is both a fibration and an opfibration.

For a comprehensive account of the theory of fibrations see the paper of Gray [15] as well as the references cited there. The sort of fibrations we consider in this chapter are so much simpler than the general case that we give an independent treatment.

3.1.2 Definitions. As noted in 1.1.4, a quasi-ordered set (meaning "$\leq$" is reflexive and transitive) may be thought of as a small category in which all diagrams are commutative. Hence we may ascribe functorial properties to order-preserving maps. POS $=_{df}$ the category of partially ordered sets and order-preserving maps which have a right adjoint. If X, Y ε obj POS and if $X \xrightarrow{f} Y$ is order-preserving, observe that an order preserving map $Y \xrightarrow{g} X$ is right adjoint to f iff for every x ε X, y ε Y it is the case that $x \leq xfg$ and $ygf \leq y$. CSL $=_{df}$ the category of complete semilattices as described in 1.1.10.

<u>3.1.3 Proposition.</u> CSL is a full subcategory of POS.

<u>Proof.</u> Let X, Y ε obj CSL, $X \xrightarrow{f} Y$ order-preserving. If f has a right adjoint then f preserves sups since sups are coproducts. Conversely, suppose f is sup-preserving. Define $Y \xrightarrow{g} X$ by yg $=_{df}$ sup $[x : xf \leq y]$. g is clearly order-preserving. $x \leq$ sup $[x : xf \leq x_0 f] = x_0 fg$ and $ygf = (\sup [x : xf \leq y])f = \sup [xf : xf \leq y] \leq y$. []

<u>3.1.4 Definitions.</u> Let $\mathcal{E} \xrightarrow{b} \mathcal{K}$ be a functor, and let K ε obj $\mathcal{K}$. The fiber over K, $=_{dn}$ K_* or Kb^{-1}, $=_{df}$ the subcategory of all $\mathcal{E}$-morphisms f such that $fb = 1_K$. K_* is always, in fact, a subcategory but may be empty. If b is faithful, K_* is a quasi-ordered class.

$\mathcal{E} \xrightarrow{b} \mathcal{K}$ is an <u>order fibering over</u> $\mathcal{K}$ if b satisfies the the following three axioms.

OF1. b is a fibering.

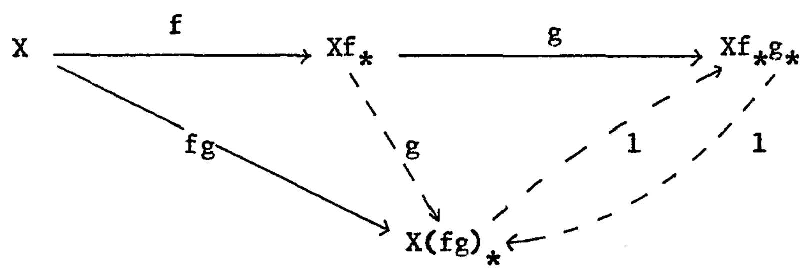
OF2. b is faithful

OF3. For every K ε obj $\mathcal{K}$ the quasi-ordered class K_* is, in fact, a partially ordered set.

In dealing with order fiberings, we think of an $\mathcal{E}$-morphism as a $\mathcal{K}$-morphism which is "admissible", and use the same symbol upstairs and downstairs; e.g. $X \leq Y \in K_*$ iff $X \xrightarrow{1_K} Y \in \mathcal{E}$.

3.1.5 Proposition. Let $\mathcal{K}$ be a category. Then there is a canonical identification [order fiberings over $\mathcal{K}$] $\xleftrightarrow{\tau}$ [functors from $\mathcal{K}$ to POS].

Proof. Let $\mathcal{E} \xrightarrow{b} \mathcal{K}$ be an order fibering over $\mathcal{K}$. Let $K \xrightarrow{f} L \xrightarrow{g} M \in \mathcal{K}$, and let $X \in K_*$. Consider the diagram:



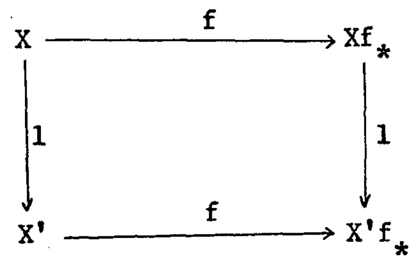
$Xf_* \xrightarrow{g} X(fg)_*$ is induced because $X \xrightarrow{f} Xf_*$ is opcartesian, so that $Xf_*g_* \xrightarrow{1} X(fg)_*$ is induced because $Xf_* \xrightarrow{g} Xf_*g_*$ is opcartesian. Therefore $Xf_*g_* \leq X(fg)_*$. But $X(fg)_* \xrightarrow{1} Xf_*g_*$ is induced as $X \xrightarrow{fg} X(fg)_*$ is opcartesian, so $X(fg)_* \leq Xf_*g_*$. By OF3, $Xf_*g_* = X(fg)_*$. In view of this observation we may define a functor

$$\mathcal{K} \xrightarrow{\quad * \quad} \text{POS}$$

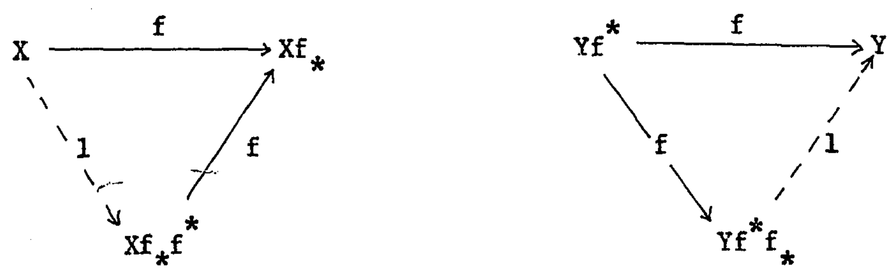
$$K \xrightarrow{f} L \quad \longmapsto \quad K_* \xrightarrow{f_*} L_*$$

$$X \longmapsto Xf_*$$

Xf_* is determined uniquely, not just within isomorphism (for let $g = 1_L$). f_* is order-preserving as $X \leq X' \in K_*$ induces:

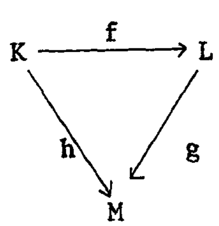


The diagrams

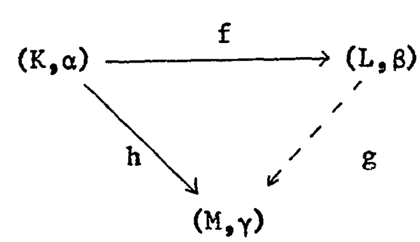


prove $f_* \dashv f^*$. $1_* = 1$ is clear and $(fg)_* = f_*g_*$ has already been proved. This defines τ.

To define τ^{-1}, let $\mathcal{K} \xrightarrow{H} \text{POS}$ be a functor and define $H\tau^{-1}$ as follows. Define a category $\mathcal{E}$ by obj $\mathcal{E} =_{df} [(K,\alpha) : K \in$ obj $\mathcal{K}$ & $\alpha \in KH]$. $(K,\alpha) \xrightarrow{f} (L,\beta)$ is an $\mathcal{E}$-morphism $=_{df}$ $K \xrightarrow{f} L \in \mathcal{K}$ and $\langle\alpha, fH\rangle \leq \beta$. Composition is defined at the level $\mathcal{K}$. $\mathcal{E}$ is a category and there is an obvious faithful functor $\mathcal{E} \xrightarrow{b} \mathcal{K}$. For every $K \in$ obj $\mathcal{K}$, $(K,\alpha) \leq (K,\beta)$ in K_* iff $\langle\alpha, 1_K H\rangle \leq \beta$ iff $\alpha \leq \beta$ so that for all practical purposes $K_* = KH$. So far we have OF2, OF3; we turn now to OF1. Let $K \xrightarrow{f} L \in \mathcal{K}$, and let $(K,\alpha) \in K_*$. $\beta =_{df} \langle\alpha, fH\rangle$. Clearly, $(K,\alpha) \xrightarrow{f} (L,\beta) \in \mathcal{E}$. If

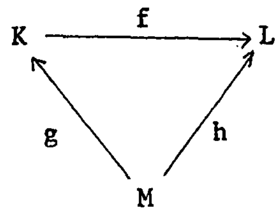


in $\mathcal{K}$, and

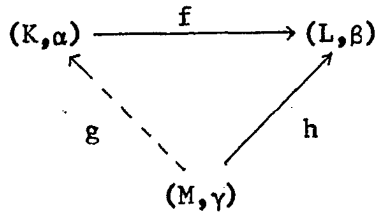


in $\mathcal{E}$

then $\langle\beta, gH\rangle = \langle\langle\alpha, fH\rangle, gH\rangle = \langle\alpha, hH\rangle \leq \gamma$ proves $(L,\beta) \xrightarrow{g} (M,\gamma) \in \mathcal{E}$. Hence b is an opfibration. Now let $(L,\beta) \in L_*$ Let $L_* \xrightarrow{\tilde{f}} K_*$ be a right adjoint to fH. $\alpha =_{df} \langle\beta, \tilde{f}\rangle$. As $\langle\alpha, fH\rangle = \langle\beta, \tilde{f}.fH\rangle \leq \beta$, $(K,\alpha) \xrightarrow{f} (L,\beta) \in \mathcal{E}$. If



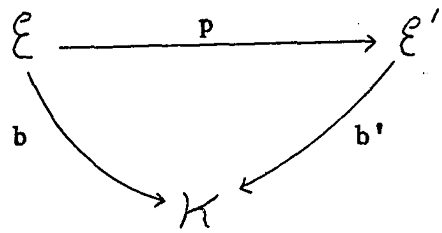
in $\mathcal{K}$, and



in $\mathcal{E}$

then $\langle\gamma, gH\rangle \leq \langle\gamma, gH.fH.\tilde{f}\rangle = \langle\langle\gamma, hH\rangle, \tilde{f}\rangle \leq \langle\beta, \tilde{f}\rangle = \alpha$ which proves $(M,\gamma) \xrightarrow{g} (K,\alpha) \in \mathcal{E}$. This completes the proof that τ^{-1} is well-defined. While τ, τ^{-1} are not quite mutually inverse it is clear that $\tau\tau^{-1}$ and $\tau^{-1}\tau$ differ negligibly from the respective identity functions, which completes the proof. []

Note: τ as above is actually the object function of an equivalence of categories. The range is the usual functor category $POS^{\mathcal{K}}$. The corresponding morphisms of order fiberings are functors p



such that pb' = b and such that p preserves all cartesian and opcartesian morphisms. See [15, 1.9].

<u>3.1.6 Proposition.</u> Let $\mathcal{E} \xrightarrow{b} \mathcal{K}$ be a faithful functor. The following three sets of axioms are equivalent.

<u>Set I.</u> LF1. b is an order fibering.

LF2. K_* is a complete lattice for all $K \in \text{obj } \mathcal{K}$.

<u>Set II.</u> OF3 and

LF3. b constructs $\varprojlim$'s and $\varinjlim$'s (for the definition of "construct" see 0.8).

LF4. b has left and right adjointnesses $\overleftarrow{b} \dashv b \dashv \overrightarrow{b}$ with $\overleftarrow{b}b = 1_{\mathcal{K}} = \overrightarrow{b}b$ and adjunctions $1_{\mathcal{K}} \xrightarrow{1} \overleftarrow{b}b$, $\overrightarrow{b}b \xrightarrow{1} 1_{\mathcal{K}}$, $b\overleftarrow{b} \xrightarrow{1} 1_{\mathcal{E}}$, $1_{\mathcal{E}} \xrightarrow{1} b\overrightarrow{b}$ (that is for every $X \in \text{obj } \mathcal{E}$, $Xb\overleftarrow{b} \leq X \leq Xb\overrightarrow{b}$.)

<u>Set III.</u> LF2 and

LF5. b constructs pullbacks and pushouts.

LF6. Every $\mathcal{K}$-morphism $K \xrightarrow{f} L$ has a lifting to an $\mathcal{E}$-morphism $X \xrightarrow{f} Y$.

<u>Proof.</u> <u>I implies II.</u> OF3 is subsumed in LF1. Let $\Delta \xrightarrow{D} \mathcal{E}$ be a diagram with Δ small, and suppose that $K \xrightarrow{\zeta_i} D_i b = \varprojlim Db$. By OF1 and LF2, define $X =_{df} \sup_{\zeta_i} [D_i\zeta_i^*] \in K_*$. Since each $D_i\zeta_i^* \xrightarrow{\zeta_i} D_i \in \mathcal{E}$, so is each $X \xrightarrow{\zeta_i} D_i$. If $Y \xrightarrow{x_i} D_i$ is natural, there exists unique $Yb \xrightarrow{f} K \in \mathcal{K}$ with $f\zeta_i = x_i$ for all i. Since each $D_i\zeta_i^*$ is cartesian, each $Y \xrightarrow{f} D_i\zeta_i^* \in \mathcal{E}$ and hence $Y \xrightarrow{f} X \in \mathcal{E}$. This proves that b constructs $\varprojlim$'s. That b constructs $\varinjlim$'s is proved dually. Lastly, we show LF4. For each $\mathcal{K}$-object K let $K\overleftarrow{b} =_{df}$ the least (resp., $K\overrightarrow{b} =_{df}$ the greatest) element of K_*. If $K \xrightarrow{f} L \in \mathcal{K}$, $K\overleftarrow{b} \xrightarrow{1} (L\overleftarrow{b})f^* \xrightarrow{f} L\overleftarrow{b} = K\overleftarrow{b} \xrightarrow{f} L\overleftarrow{b} \in \mathcal{E}$ so that $(K \xrightarrow{f} L)\overleftarrow{b} = K\overleftarrow{b} \xrightarrow{f} L\overleftarrow{b}$ is well-defined. Dually $(K \xrightarrow{f} L)\overrightarrow{b} =_{df}$

$K\overrightarrow{b} \xrightarrow{f} L\overrightarrow{b}$ is well-defined. That $\overleftarrow{b}$, $\overrightarrow{b}$ behave as stated in LF4 is clear.

<u>II implies III.</u> LF5 is subsumed in LF3. LF6 is clear from LF4. We show LF2. If $X \in K_*$, $K\overleftarrow{b} \xrightarrow{1} X \xrightarrow{1} K\overrightarrow{b} \in \mathcal{E}$ by LF4, so that K_* has a least element $K\overleftarrow{b}$ and a greatest element $K\overrightarrow{b}$. Let $(X_i : i \in I) \subset K_*$ be a non-empty subset. As $(K \xrightarrow{1} K : i \in I)$ is a model for the collective pullback of b of the $\mathcal{E}$-diagram $(X_i \xrightarrow{1} K\overrightarrow{b} : i \in I)$, there is a constructed pullback $(\sup [X_i] \xrightarrow{1} X_i : i \in I)$. That $\sup [X_i]$ is the supremum in K_* of $[X_i]$ is clear. Inf $[X_i]$ is constructed dually as the collective pushout of $(K\overleftarrow{b} \xrightarrow{1} X_i)$.

<u>III implies I.</u> LF2 and OF2 are standing and OF3 is subsumed in LF2. We must show OF1. Let $K \xrightarrow{f} L \in \mathcal{K}$, $Y \in L_*$. Define $K\overrightarrow{b}$, $L\overrightarrow{b}$ to be the greatest elements of K_*, L_* which we may do by LF2. By LF6, there exsits $X_1 \xrightarrow{f} X_2 \in \mathcal{E}$. The pushout diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ K & \xrightarrow{f} & L \end{array}$$

in $\mathcal{K}$ of b of $K\overrightarrow{b} \xleftarrow{1} X_1 \xrightarrow{f} X_2$ is constructed in $\mathcal{E}$ as

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \downarrow{\scriptstyle 1} & & \vdots{\scriptstyle 1} \\ K\overrightarrow{b} & \dashrightarrow{f} & Q \end{array}$$

Therefore $K\overrightarrow{b} \xrightarrow{f} L\overrightarrow{b} = K\overrightarrow{b} \xrightarrow{f} Q \xrightarrow{1} L\overrightarrow{b} \in \mathcal{E}$. Now b of

$$K\overrightarrow{b} \xrightarrow{f} L\overrightarrow{b} \xleftarrow{1} Y$$

has pullback

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \big\downarrow 1 & & \big\downarrow 1 \\ K & \xrightarrow{f} & L \end{array}$$

which allows us to define $Yf^* \xrightarrow{f} Y$ by the constructed pullback

$$\begin{array}{ccc} Yf^* & \dashrightarrow^{f} & Y \\ \big\downarrow 1 & & \big\downarrow 1 \\ K\vec{b} & \xrightarrow{f} & L\vec{b} \end{array}$$

That $Yf^* \xrightarrow{f} Y$ is cartesian is clear, and b is a fibration. The proof that b is an opfibration is dual. []

<u>3.1.7 Definition and remarks.</u> If a faithful functor $\mathcal{E} \xrightarrow{b} \mathcal{K}$ satisfies any of the three equivalent sets of axioms of 3.1.6, then b is a <u>lattice fibering over</u> $\mathcal{K}$. b is then both a fibration and an opfibration in the sense of [15] and a "pullback stripping functor" in the sense of Kennison [18]; (the latter is true with inessential changes.) Our proof of "I implies LF3" in 3.1.6 can, essentially, be found in [18].

If $(K \xrightarrow{f_i} X_i b : i \in I)$ is given, define $con^*(K \xrightarrow{f_i} X_i b)$ $=_{df} \sup[X_i f_i^* \xrightarrow{f_i} X_i) \in K_*$. It is the smallest element of K_* admitting each f_i, and a map into $con^*(K \xrightarrow{f_i} X_i b)$ is admissible iff it is admissible followed by each $con^*(K \xrightarrow{f_i} X_i b) \xrightarrow{f_i} X_i$. Dually, define $con_*(X_i b \xrightarrow{g_i} L) =_{df} \inf[X_i \xrightarrow{g_i} X_i g_{i*}) \in L_*$.

The identification τ of 3.1.5 sets up an identification between lattice fiberings over $\mathcal{K}$ and functors from $\mathcal{K}$ to CSL, as is immediate from 3.1.3 and 3.1.6 (set I).

For the rest of this section fix a lattice fibering $\mathcal{E} \xrightarrow{b} \mathcal{K}$.

3.1.8 Proposition. The following statements are valid.

a. $\mathcal{E}$ is legitimate iff $\mathcal{K}$ is.

b. For every small category Δ, $\mathcal{E}$ has $\varprojlim$'s (resp., $\varinjlim$'s) of type Δ iff $\mathcal{K}$ does.

c. b preserves and reflects monos and epis.

Proof. $\mathcal{K}$ is a full reflective subcategory of $\mathcal{E}$ with inclusion $\vec{b}$ and reflector b, and a full coreflective subcategory of $\mathcal{E}$ with inclusion $\overleftarrow{b}$ and reflector b.

a. If $\mathcal{E}$ is legitimate, so is $\mathcal{K}$ being a subcategory of $\mathcal{E}$. Conversely, $\mathcal{K}$ legitimate and b faithful implies $\mathcal{E}$ legitimate.

b. $\mathcal{E}$ has implies $\mathcal{K}$ has because full reflective subcategories inherit $\varprojlim$'s and full coreflective subcategories inherit $\varinjlim$'s. The converse is clear from LF3.

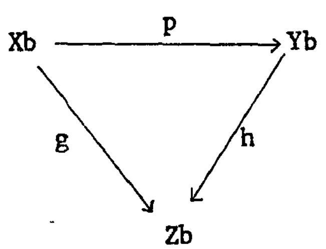
c. b is faithful and has left and right adjoints. []

3.1.9 Proposition. Let $X \xrightarrow{p} Y \in \mathcal{E}$. The following statements are equivalent.

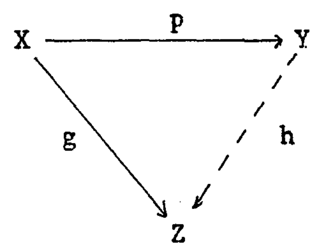
a. $X \xrightarrow{p} Y$ is a regular epi.

b. $X \xrightarrow{p} Y$ is opcartesian and $Xb \xrightarrow{p} Yb$ is a regular epi.

Proof. **a implies b.** Suppose



in $\mathcal{K}$, and



in $\mathcal{E}$

Since $g \in \mathrm{reg}_{\mathcal{E}}(p)$, there exists $Y \xrightarrow{\tilde{h}} Z$ with $p\tilde{h} = g$, and $\tilde{h} = h$ as

$Xb \xrightarrow{p} Yb$ is epi (3.1.8 (c)). Hence $X \xrightarrow{p} Y$ is opcartesian. To see $Xb \xrightarrow{p} Yb$ is regular epi, let $g \in reg_{\mathcal{K}}(p)$. Clearly $X \xrightarrow{g} Xg_*$ $\in reg_{\mathcal{E}}(p)$ so that there exists $Y \xrightarrow{h} Xg_*$ with $ph = g$. $Yb \xrightarrow{h} Zb$ is the only $\mathcal{K}$-morphism with this property since $Xb \xrightarrow{p} Yb$ is epi (by 3.1.8 (c)).

b implies a. Let $X \xrightarrow{g} Z \in reg_{\mathcal{E}}(p)$. Clearly $Xb \xrightarrow{g} Zb$ $\in reg_{\mathcal{K}}(p)$, so there exists unique $Yb \xrightarrow{h} Zb \in \mathcal{K}$ with $ph = g$. $Y \xrightarrow{h} Z \in \mathcal{E}$ as p is opcartesian. []

3.1.10 Proposition. The following statements are valid.

a. b constructs regular image factorizations and regular coimage factorizations.

b. $\mathcal{E}$ is regular iff $\mathcal{K}$ is and $\mathcal{E}$ is LF-regular iff $\mathcal{K}$ is.

Proof. a. Let $X \xrightarrow{f} Y \in \mathcal{E}$ and let $Xb \xrightarrow{p} I \xrightarrow{i} Yb$ $= Xb \xrightarrow{f} Yb$ be a regular coimage factorization in $\mathcal{K}$. Then $X \xrightarrow{f} Y = X \xrightarrow{p} Xp_* \xrightarrow{i} Y$ where $Xp_* \xrightarrow{i} Y \in \mathcal{E}$ because $X \xrightarrow{p} Xp_*$ is opcartesian. i is mono by 3.1.8 (c) and p is regular epi by 3.1.9. The proof that b constructs regular image factorizations is dual.

b. $\mathcal{E}$ satisfies LFR2 iff $\mathcal{K}$ does and $\mathcal{E}$ satisfies LFR3 iff $\mathcal{K}$ does by 3.1.8. By 3.1.8 (c), 3.1.9 and (a), $\mathcal{E}$ satisfies LFR1 iff $\mathcal{K}$ does. Let $X \in obj\ \mathcal{E}$, and let $\mathcal{F}$ be the class of epimorphisms with domain X. As b preserves epimorphisms, $\mathcal{F}b$ is a class of epimorphisms with domain Xb. If $\mathcal{R}_0$ is a representative set for $\mathcal{F}b$, $\mathcal{R} =_{df} [Y \xrightarrow{f} Z \in \mathcal{E} : Yb \xrightarrow{fb} Zb \in \mathcal{R}_0]$ is a representative set for $\mathcal{F}$. It is a set because $\mathcal{R}_0$ is and because each K_* is a set.

If $X \xrightarrow{f} Y \in \mathcal{F}$ there exist $\mathcal{K}$-isomorphisms α, β and $K \xrightarrow{g} L$ $\in \mathcal{R}_0$ with $f.\beta = \alpha.g$. Because $X \xrightarrow{\alpha} X\alpha_*$ and $Y \xrightarrow{\beta} Y\beta_*$ are opcartesian, $X\alpha_* \xrightarrow{\alpha^{-1}} X$, $X\alpha_* \xrightarrow{g} Y\beta_*$ and $Y\beta_* \xrightarrow{\beta^{-1}} Y$ are $\mathcal{E}$-morphisms. It follows that $X \xrightarrow{f} Y$ is isomorphic in $\overleftarrow{\mathcal{E}}$ to $X\alpha_* \xrightarrow{g} Y\beta_* \in \mathcal{R}$. We have shown LFR4 for $\mathcal{K}$ implies LFR4 for $\mathcal{E}$. That REG4 for $\mathcal{K}$ implies REG4 for $\mathcal{E}$ is proved similarly, noting that b preserves regular epimorphisms by 3.1.9. Consider $\mathcal{K}$ as a full subcategory of $\mathcal{E}$ with inclusion $\overleftarrow{b}$. $\mathcal{K}\overleftarrow{b}$ is a union of $\mathcal{E}$-isomorphism classes. As $\overleftarrow{b}b = 1_{\mathcal{K}}$, as every $\mathcal{K}\overleftarrow{b}$-morphism is opcartesian and by 3.1.9, $\overleftarrow{b}$ preserves regular epimorphisms. $\overleftarrow{b}$ preserves epimorphisms as $\overleftarrow{b}b = 1_{\mathcal{K}}$ and b reflects epimorphisms. It is now easy to see that LFR4 for $\mathcal{E}$ implies LFR4 for $\mathcal{K}$ and that REG4 for $\mathcal{E}$ implies REG4 for $\mathcal{K}$. The remaining details are clear. []

§3.2 Examples of lattice fiberings.

3.2.1 Example; trivial lattice fiberings. Let $\mathcal{K}$ be a category and let F be a complete lattice. The constant functor $\mathcal{K} \longrightarrow \mathrm{CSL}$, $K \xrightarrow{f} L \mapsto F \xrightarrow{1} F$ induces the trivial lattice fibering with fiber F, $\mathcal{K} \times F \xrightarrow{pr_1} \mathcal{K}$.

3.2.2 Example; sets and relations. Let k be a non-negative integer and let n be a set. $\mathcal{S}_{(n,k)} =_{df}$ the category whose objects are $[(X, \mathcal{F}) : X \in \text{obj } \mathcal{S},\ \mathcal{F} \in X^n \mathcal{P}^k]$, where $\mathcal{P}^k$ is the k^{th} iterate of the power set operator $\mathcal{P}$, $X \mapsto 2^X$. An $\mathcal{S}_{(n,k)}$-morphism $(X, \mathcal{F}) \xrightarrow{f} (Y, \mathcal{Y})$ is a function $X \xrightarrow{f} Y$ such that $\mathcal{F}(f^n) \subset \mathcal{Y}$. Composition is the obvious one. $\mathcal{S}^{(n,k)} =_{df}$ the category such that obj $\mathcal{S}^{(n,k)}$ = obj $\mathcal{S}_{(n,k)}$, but $(X, \mathcal{F}) \xrightarrow{f} (Y, \mathcal{Y})$ is admissible if $\mathcal{Y}(f^n)^{-1} \subset \mathcal{F}$. There are obvious underlying set functors $\mathcal{S}_{(n,k)} \xrightarrow{b_{(n,k)}} \mathcal{S}$, $\mathcal{S}^{(n,k)} \xrightarrow{b^{(n,k)}} \mathcal{S}$. The proof that these are lattice fiberings is easy; we tabulate the main constructions:

	$b_{(n,k)}$	$b^{(n,k)}$
$(X, \mathcal{F}) \leq (X, \mathcal{Y})$	$\mathcal{F} \subset \mathcal{Y}$	$\mathcal{Y} \subset \mathcal{F}$
sup $[(X, \mathcal{F}_i)]$	$(X, \bigcup \mathcal{F}_i)$	$(X, \bigcap \mathcal{F}_i)$
inf $[(X, \mathcal{F}_i)]$	$(X, \bigcap \mathcal{F}_i)$	$(X, \bigcup \mathcal{F}_i)$
$(X, \mathcal{F})f_*$	$(Y, \mathcal{Y}f^n)$	$(Y, [A : A(f^n)^{-1} \in \mathcal{F}])$
$(Y, \mathcal{Y})f^*$	$(X, \mathcal{Y}(f^n)^{-1})$	$(X, \mathcal{Y}(f^n)^{-1})$

For the rest of this section fix a lattice fibering $\mathcal{E} \xrightarrow{b} \mathcal{K}$.

<u>3.2.3 Proposition.</u> Let $\mathcal{E}_0 \subset \mathcal{E}$ be a full subcategory such that for every $K \in \text{obj}\ \mathcal{K}$, $K_0 =_{df} K_* \cap \mathcal{E}_0$ is, under the subset partial order, a complete lattice with least element $0(K)$ and greatest element $1(K)$. Assume further that for every $\mathcal{K}$-morphism $K \xrightarrow{f} L$ we have that $0(K) \xrightarrow{f} 0(L) \in \mathcal{E}$, and that $1(K) \xrightarrow{f} 1(L) \in \mathcal{E}$. Finally, assume either of the two hypotheses:

a. For every $K \xrightarrow{f} L \in \mathcal{K}$ and for every $Y \in L_0$, $Yf^* \xrightarrow{f} Y \in \mathcal{E}_0$.

b. For every $K \xrightarrow{f} L \in \mathcal{K}$ and for every $X \in L_0$, $X \xrightarrow{f} Xf_* \in \mathcal{E}_0$.

Then $\mathcal{E}_0 \xrightarrow{b_0} \mathcal{K} =_{df} b/\mathcal{E}_0$ is a lattice fibering.

<u>Proof.</u> We prove LF1, LF2. Everything is given except OF1 which we prove now. Let $K \xrightarrow{f} L \in \mathcal{K}$, $Y \in L_0$. If (a) is assumed, then $Yf^* \xrightarrow{f} Y$ is cartesian with respect to b_0 because $\mathcal{E}_0$ is full. Otherwise, assume (b). There exists a lifting $X \xrightarrow{f} Y \in \mathcal{E}_0$, namely $0(K) \xrightarrow{f} 0(L) \xrightarrow{1} Y$. Define $Yf^0 =_{df} \sup_{K_0}[X \in K_0 : X \xrightarrow{f} Y \in \mathcal{E}_0]$. Clearly $Yf^0 \leq Yf^* = \sup_{K_*}[X \in K_* : X \xrightarrow{f} Y \in \mathcal{E}]$, so $Yf^0 \xrightarrow{f} Y = Yf^0 \xrightarrow{1} Yf^* \xrightarrow{f} Y \in \mathcal{E}$. Now suppose

$$\begin{array}{ccc} Zb & & \\ {\scriptstyle g}\downarrow & \searrow{\scriptstyle h} & \\ K & \xrightarrow{f} & L \end{array} \quad \text{in } \mathcal{K}\text{, and} \quad \begin{array}{ccc} Z & & \\ {\scriptstyle g}\,\vdots & \searrow{\scriptstyle h} & \\ Yf^0 & \xrightarrow{f} & Y \end{array} \quad \text{in } \mathcal{E}_0.$$

Consider,

$$\begin{array}{ccccc} Z & \xrightarrow{g} & Zg_* & \xrightarrow{f} & Y \\ & & {\scriptstyle 1}\,\vdots & \nearrow{\scriptstyle f} & \\ & & Yf & & \end{array}$$

$Z \xrightarrow{g} Zg_* \in \mathcal{E}_0$ by hypothesis, and is opcartesian in $\mathcal{E}_0$, so that

$Zg_* \xrightarrow{f} Y \in \mathcal{E}_0$. By the definition of Yf^0, $Zg_* \leq Yf^0$. It follows that $Z \xrightarrow{g} Zg_* \xrightarrow{1} Yf^0 \in \mathcal{E}_0$. Hence b_0 is a fibration. The proof that b_0 is an opfibration is dual. []

<u>3.2.4 Applications of 3.2.3 to 3.2.2.</u>

a. Topological spaces $\subset \mathfrak{S}^{(1,2)}$.

b. Uniform spaces $\subset \mathfrak{S}^{(2,2)}$.

c. Quasi-ordered sets $\subset \mathfrak{S}_{(2,1)}$.

d. Measureable spaces ($=_{df}$ sets with σ-ring thereon) and measureable transformations $\subset \mathfrak{S}^{(1,2)}$.

<u>3.2.5 Proposition.</u> $\mathcal{E}^{op} \xrightarrow{b^{op}} \mathcal{K}^{op}$ is a lattice fibering over $\mathcal{K}^{op}$. []

<u>3.2.6 Proposition.</u> If $\mathcal{E} \xrightarrow{b} \mathcal{F}$, $\mathcal{F} \xrightarrow{c} \mathcal{G}$ are lattice fiberings then so is $\mathcal{E} \xrightarrow{bc} \mathcal{G}$.

<u>Proof.</u> LF3 and that bc is faithful are clear and LF4 is easy with $\overleftarrow{bc} = \overleftarrow{c}\overleftarrow{b}$, $\overrightarrow{bc} = \overrightarrow{c}\overrightarrow{b}$. To show OF3, suppose $Xbc = X'bc$ and $X \leq X' \leq X$. Then $Xb \leq X'b \leq Xb$ in $(Xbc)c^{-1}$ so that $Xb = X'b$; then $X \leq X' \leq X$ in Xbb^{-1} and so $X = X'$. []

<u>3.2.7 Proposition.</u> Let Δ be a small category. Then $\mathcal{E}^{\Delta} \xrightarrow{-\circ b} \mathcal{K}^{\Delta}$ is a lattice fibering over $\mathcal{K}^{\Delta}$.

<u>Proof.</u> LF3 is clear as limits are constructed pointwise in functor categories. LF4 is easy using $-\circ\overleftarrow{b}$, $-\circ\overrightarrow{b}$. If $\alpha,\beta \in (F,G)$n.t. and if $\alpha b = \beta b$ then $\alpha = \beta$ since b is faithful; therefore $-\circ b$ is faithful. To

prove OF3, let $G \in \text{obj } \mathcal{K}^{\Delta}$. G_* is a set because there is an injection $G_* \rightarrowtail \prod_{i \xrightarrow{\delta} j \in \Delta} ([X \in |\mathcal{E}| : Xb = iG] \times [X \in |\mathcal{E}| : Xb = jG])$. Suppose $H, H' \in G_*$ with $H \leq H' \leq H$. For all $i \in \text{obj } \Delta$, $iH \leq iH' \leq iH$, so $H = H'$ on objects. As $H.b = H'.b$ and as b is faithful, $H = H'$. []

<u>**3.2.8 Proposition.**</u> Let $\mathcal{A} \xrightarrow{U} \mathcal{K}$ be any $\mathcal{K}$-valued functor and construct (the usual model in the category of categories of) the pullback

$$\begin{array}{ccc} \mathcal{P} & \overset{\tilde{U}}{\dashrightarrow} & \mathcal{E} \\ {\scriptstyle \tilde{b}} \downarrow & & \downarrow {\scriptstyle b} \\ \mathcal{A} & \underset{U}{\longrightarrow} & \mathcal{K} \end{array}$$

Then $\tilde{b}$ is a lattice fibering over $\mathcal{A}$.

<u>**Proof.**</u> Let $\mathcal{K} \xrightarrow{H}$ CLS be the functor corresponding to b. It is easily checked that $\tilde{b}$ is the lattice fibering corresponding to $\mathcal{A} \xrightarrow{U} \mathcal{K} \xrightarrow{H}$ CLS. []

§3.3 Lattice fiberings over sets.

For this section let $\mathcal{E} \xrightarrow{b} \mathcal{S}$ be a lattice fibering over $\mathcal{S}$.

3.3.1 Proposition. $\mathcal{E}$ is legitimate, has $\varprojlim$'s and $\varinjlim$'s and is regular and LF-regular. Because all epis and monos in $\mathcal{S}$ are regular, for each $\mathcal{E}$-morphism f, f is cartesian mono iff f is an equalizer, and f is opcartesian epi iff f is a coequalizer. []

3.3.2 Definition. If $A \overset{i}{\rightarrowtail} X$ is a "good" subobject $=_{df}$ equalizer = cartesian mono in $\mathcal{E}$ write "A << X." Observe that if X ε obj $\mathcal{E}$ and if $A \overset{i}{\rightarrowtail} Xb$ then A << X canonically via $Ai^* \xrightarrow{i} A$. Think of relativization of subsets of a topological space.

3.3.3 Review of autonomous categories. A set-valued functor $\mathcal{A} \xrightarrow{U} \mathcal{S}$ together with a lifted hom-functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{HOM} \mathcal{A}$ and natural transformation $(-,-)\mathcal{A} \xrightarrow{\gamma} HOM.U$ is autonomous if the following five axioms hold:

A1. $\mathcal{A}$ is legitimate.

A2. U is faithful

A3. γ is a natural equivalence

A4. HOM is coherent in the sense that for every A,B,C ε obj $\mathcal{A}$ the usual bijection $(AU, (BU,CU)\mathcal{S})\mathcal{S} \cong (BU, (AU,CU)\mathcal{S})\mathcal{S}$ in $\mathcal{S}$ sets up by restriction (and through γ) an $\mathcal{A}$-isomorphism $(A, (B,C)HOM)HOM \cong (B, (A,C)HOM)HOM$ natural in A,B,C.

A5. For every A ε obj $\mathcal{A}$, the functor $\mathcal{A} \xrightarrow{(A,-)HOM} \mathcal{A}$

has a <u>strong left adjoint</u> $\mathcal{A} \xrightarrow{-\otimes A} \mathcal{A}$, that is there is an equivalence natural in B and C: ((A,B)HOM,C)HOM ≃ (B,C ⊗ A)HOM.

For an account of the theory and examples see [22]. The only result we mention is the <u>supernaturality lemma</u> of [22, 3.15], which is as follows. Let $\mathcal{A} \xrightarrow{U} \mathcal{S}$ be autonomous and assume further that U preserves $\varprojlim$'s. Let $\mathcal{A} \xrightarrow{H} \mathcal{A}$ be a <u>strong functor</u>, that is for all A,B ε obj $\mathcal{A}$, the function $(A,B)\mathcal{A} \xrightarrow{H_{A,B}} (AH,BH)\mathcal{A}$ induced by H lifts to $(A,B)HOM \xrightarrow{H_{A,B}} (AH,BH)HOM$. Then for all A ε obj $\mathcal{A}$, the inclusion of sets (through γ):

$$((A,-)HOM,H)n.t. \rightarrowtail ((A,-)\mathcal{A},HU)n.t.$$

is onto. Hence ((A,-)HOM, H)n.t. ≃ AHU.

<u>3.3.4 Discussion.</u> Let $\mathcal{A} \xrightarrow{U} \mathcal{S}$ be a set-valued functor. Two questions arise natually at this point, namely: if U is tripleable, when is U autonomous? If U is a lattice fibering, when is U autonomous?

The first question has been answered by Freyd in [11] and Linton in [24]: $U^{\mathbb{T}}$ is autonomous iff $\mathbb{T} = \mathbb{T} \otimes \mathbb{T}$. (These proofs are in the language of equationally defineable classes, but, as indicated in [24], it is still true when $\mathbb{T}$ has no rank.)

The second question has a pleasant answer: always. We prove this shortly. Hence a lattice fibering $\mathcal{E} \xrightarrow{b} \mathcal{S}$ over sets has the following properties:

a. $\mathcal{E}$ is regular.

b. $\mathcal{E}$-morphisms are, in part, functions of sets.

c. (E,F)HOM is at least a subobject of the cartesian power F^{Eb}.

d. There is a Yoneda Lemma for strong functors via supernaturality. This indicates that a large part of the work in Chapter 2 might generalize to certain triples in a lattice fibering over sets. We will make some indications in this direction in the next section.

3.3.5 Proposition. The following statements are valid.

a. If $X \in \text{obj}\ \mathcal{E}$, and if $\Gamma \xrightarrow{p} \Lambda$ is a function of sets then $X^{\Lambda} \xrightarrow{p\circ-} X^{\Gamma} \in \mathcal{E}$.

b. If $X \xrightarrow{p} Y \in \mathcal{E}$ and if Γ is a set then $X^{\Gamma} \xrightarrow{-\circ p} Y^{\Gamma} \in \mathcal{E}$.

Proof. Recall how powers are constructed and use the proof of 2.1.5. []

3.3.6 Proposition. $\mathcal{E} \xrightarrow{b} \mathcal{S}$ is autonomous.

Proof. If $X, Y \in \text{obj}\ \mathcal{E}$ define $(X,Y)\text{HOM} =_{df} (X,Y)\mathcal{E} \ll Y^{Xb}$. If $X' \xrightarrow{f} X$, $Y \xrightarrow{g} Y' \in \mathcal{E}$, then $Y^{Xb} \xrightarrow{f\circ-\circ g} Y'^{X'b} \in \mathcal{E}$ by 3.3.5, and maps $(X,Y)\text{HOM}$ into $(X',Y')\text{HOM}$ so that $(X,Y)\text{HOM} \xrightarrow{f\circ-\circ g} (X',Y')\text{HOM} \in \mathcal{E}$. Therefore $\mathcal{E}^{op} \times \mathcal{E} \xrightarrow{\text{HOM}} \mathcal{E}$ is a well-defined functor and in fact $\text{HOM}.b = (-,-)\mathcal{E}$ so that we take γ to be the identity natural transformation. To prove coherence, let $X,Y,Z \in \text{obj}\ \mathcal{E}$ and define $\psi_{X,Y,Z}$ by

$$\begin{array}{ccc}
(X, (Y,Z)\text{HOM})\text{HOM} & \xrightarrow{\psi_{X,Y,Z}} & (X,Z)\text{HOM}^{Yb} \\
& \searrow^{-\circ pr_y} & \downarrow pr_y \\
& & (X,Z)\text{HOM}
\end{array}$$

We may do this because for every $y \in Yb$, $-\circ pr_y$ is an $\mathcal{E}$-morphism by

3.3.5. If $X \xrightarrow{h} (Y,Z)\mathrm{HOM} \in \mathcal{E}$ then for every $x \in Xb$,

$$Y \xrightarrow{\langle h,\, \psi_{X,Y,Z}\rangle} (X,Z)\mathrm{HOM} \xrightarrow{pr_x} Z$$

$= \langle x,h\rangle \in \mathcal{E}$ and hence, by the definition of HOM, we have $\langle h,\psi_{X,Y,Z}\rangle \in (Y,(X,Z)\mathrm{HOM})\mathrm{HOM}$. Therefore we redefine $\psi_{X,Y,Z}$ by $(X,\ (Y,Z)\mathrm{HOM})\mathrm{HOM} \xrightarrow{\psi_{X,Y,Z}} (Y,\ (X,Z)\mathrm{HOM})\mathrm{HOM} \in \mathcal{E}$. By definition, $\psi_{X,Y,Z}$ is just the usual interchange bijection $(Xb,\ (Yb,Zb)\mathcal{S})\ \mathcal{S}$ $(Yb,\ (Xb,Zb)\mathcal{S})\ \mathcal{S}$ at the level of sets; since the latter is natural, the former is forced to be. $\psi_{X,Y,Z}{}^{-1} = \psi_{Y,X,Z}$ is clear. This demonstrates coherence of HOM.

Fix $A \in \mathrm{obj}\ \mathcal{E}$. For all $X \in \mathrm{obj}\ \mathcal{E}$ define $X \otimes A =_{df}$ $\mathrm{con}_*[(X \xrightarrow{(1,a)} X \times A : a \in A) \bigcup (A \xrightarrow{(x,1)} X \times A : x \in X)]$. Hence a function $X \otimes A \xrightarrow{f} Y$ is admissible in $\mathcal{E}$ iff f is <u>separately admissible</u>, that is each slice ${}_xf = A \xrightarrow{(x,1)} X \times A \xrightarrow{f} Y$, $f_a = X \xrightarrow{(1,a)} X \times A \xrightarrow{f} Y$ is admissible in $\mathcal{E}$; in effect, $X \otimes A$ is a "universal bilinear junction". $\otimes$ is in fact a functor in both variables which we may see as follows. If $X \xrightarrow{f} X'$, $A \xrightarrow{g} A'$ $\in \mathcal{E}$ then $f \otimes g =_{df} f \times g \in \mathcal{E}$ as is seen from

$$\begin{array}{ccccc} X & \xrightarrow{(1,a)} & X \times A & \xleftarrow{(x,1)} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \times g} & & \downarrow{\scriptstyle g} \\ X' & \longrightarrow & X' \times A' & \longleftarrow & A' \end{array}$$

and functoriality is clear.

For each $X,Y \in \mathrm{obj}\ \mathcal{E}$ define $\alpha_{X,Y} \in \mathcal{E}$ by

$$
\begin{array}{ccc}
(X \otimes A,\ Y)\mathrm{HOM} & \xrightarrow{\ \alpha_{X,Y}\ } & (X,\ (A,Y)\mathrm{HOM})\mathrm{HOM} \\
 & \searrow^{(x,1)\circ -} & \downarrow \mathrm{pr}_x \\
 & & (A,Y)\mathrm{HOM}
\end{array}
$$

Clearly $\alpha_{X,Y}$ is well-defined as an $\mathcal{E}$-morphism into $(A,Y)\mathrm{HOM}^{Eb}$. If $h \in (X \otimes A,\ Y)\mathrm{HOM}$ and if $a \in Ab$ then indeed

$$
X \xrightarrow{\langle h,\ \alpha_{X,Y}\rangle} (A,Y)\mathrm{HOM} \xrightarrow{\mathrm{pr}_a} Y = X \xrightarrow{(1,a)} X \otimes A \xrightarrow{h} Y
$$

$\in \mathcal{E}$, and so $\alpha_{X,Y}$ takes values in $(X,\ (A,Y)\mathrm{HOM})\mathrm{HOM}$. For all X,Y $\in$ obj $\mathcal{E}$ define $\beta_{X,Y}$ by

$$
\begin{array}{ccc}
(X,\ (A,Y)\mathrm{HOM})\mathrm{HOM} & \xrightarrow{\ \beta_{X,Y}\ } & (X \otimes A, Y)\mathrm{HOM} \\
\downarrow \mathrm{pr}_x & & \downarrow \mathrm{pr}_{(x,a)} \\
(A,Y)\mathrm{HOM} & \xrightarrow{\ \mathrm{pr}_a\ } & Y
\end{array}
$$

Clearly $\beta_{X,Y}$ is well-defined as an $\mathcal{E}$-morphism into $Y^{Xb \times Ab}$. If $h \in (X,\ (A,Y)\mathrm{HOM})\mathrm{HOM}$ then for all a, x we have

$$
\begin{array}{ccccc}
X & \xrightarrow{\ (1,a)\ } & X \otimes A & \xleftarrow{\ (x,1)\ } & A \\
 & \searrow_{\langle a,\tilde{h}\rangle} & \downarrow \langle h,\beta_{X,Y}\rangle & \swarrow_{\langle x,h\rangle} & \\
 & & Y & &
\end{array}
$$

where $\tilde{h}$ corresponds to h under $(A, (X,Y)HOM)HOM \simeq (X, (A,Y)HOM)HOM$ and therefore $\beta_{X,Y}$ takes values in $(X \otimes A,Y)HOM$. That α and β are natural and mutually inverse follows from the fact that at the level of sets, α, β are the usual equivalences $(Xb, (Ab,Yb)\mathcal{S})\mathcal{S} \simeq (Xb \times Ab, Yb)\mathcal{S}$. []

§3.4 Triples in a lattice fibering.

For this section fix a lattice fibering $\mathcal{E} \xrightarrow{b} \mathcal{K}$ over $\mathcal{K}$, and fix a triple $\mathbb{T} = (T,\eta,\mu)$ in $\mathcal{K}$.

3.4.1 Remark. If b is tripleable then b creates isomorphisms so that for every $K \in$ obj $\mathcal{K}$, $K\overleftarrow{b} \xrightarrow{1} K\overrightarrow{b}$ is an isomorphism. Therefore K_* has only one element, and in all essentials b is the identity functor of $\mathcal{K}$.

3.4.2 Definition. $\mathcal{K}^{(b,\mathbb{T})} \xrightarrow{U^{(b,\mathbb{T})}} \mathcal{E} =_{df}$ the (usual model in the category of categories for the) pullback

$$\begin{array}{ccc} \mathcal{K}^{(b,\mathbb{T})} & \dashrightarrow & \mathcal{K}^{\mathbb{T}} \\ {\scriptstyle U^{(b,\mathbb{T})}} \downarrow & & \downarrow {\scriptstyle U^{\mathbb{T}}} \\ \mathcal{E} & \xrightarrow{\quad b \quad} & \mathcal{K} \end{array} .$$

An object in $\mathcal{K}^{(b,\mathbb{T})}$, then, is a $\mathcal{K}$-object K together with an $\mathcal{E}$-structure and a $\mathbb{T}$-structure, but no relations between them; a $\mathcal{K}^{(b,\mathbb{T})}$-morphism is a $\mathcal{K}$-morphism which is admissible both as an $\mathcal{E}$-morphism and as a $\mathcal{K}^{\mathbb{T}}$-morphism.

3.4.3 Proposition. $\mathcal{K}^{(b,\mathbb{T})} \xrightarrow{U^{(b,\mathbb{T})}} \mathcal{K}$ is tripleable, and $\mathcal{K}^{(b,\mathbb{T})} \longrightarrow \mathcal{K}^{\mathbb{T}}$ is a lattice fibering.

Proof. The second statement follows from 3.2.8. The prove the first, we define the canonical lifting of $\mathbb{T}$ over b to be the triple

$\overline{\mathbb{T}} = (\bar{T},\bar{\eta},\bar{\mu})$ defined as follows. For every $X \in \text{obj } \mathcal{E}$, $X\bar{T} =_{df} \text{con}^*(XbT \xrightarrow{fT} YbT \xrightarrow{\theta} Y : X \xrightarrow{f} Y \in \mathcal{E}\ \&\ (Yb,\theta) \in \text{obj } \mathcal{K}^{\mathbb{T}})$. If $X \xrightarrow{h} X' \in \mathcal{E}$, $(X \xrightarrow{h} X')\bar{T} =_{df} X\bar{T} \xrightarrow{hT} X'\bar{T}$.

$$X\bar{T} \xrightarrow{hT} X'T \xrightarrow{fT} YT \xrightarrow{\theta} Y, \qquad X\bar{T} \xrightarrow{(h.f)T} YT$$

proves that $h\bar{T}$ is well-defined. Therefore $\mathcal{E} \xrightarrow{\bar{T}} \mathcal{E}$ is a functor with $\bar{T}b = bT$. For $X \in \text{obj } \mathcal{E}$ define $X\bar{\eta} =_{df} X \xrightarrow{Xb\eta} X\bar{T}$. The diagram

$$X \xrightarrow{Xb\eta} XT \xrightarrow{fT} YT \xrightarrow{\theta} Y, \qquad X \xrightarrow{f} Y, \qquad Y \xrightarrow{Yb\eta} YT, \qquad Y \xrightarrow{1} Y$$

proves $X\bar{\eta} \in \mathcal{E}$. Noting that $\bar{T}\bar{T}b = bTT$, define $X\bar{\mu} =_{df} X\bar{T}\bar{T} \xrightarrow{Xb\mu} X\bar{T}$. $X\bar{\mu} \in \mathcal{E}$ by the definition of $(X\bar{T})\bar{T}$. That $\overline{\mathbb{T}} = (\bar{T},\bar{\eta},\bar{\mu})$ is a triple in $\mathcal{E}$ is now clear. To complete the proof, observe that the passage $(X, X\bar{T} \xrightarrow{\xi} X) \longmapsto (X, XbT \xrightarrow{\xi} X)$ is an isomorphism of $U^{\overline{\mathbb{T}}}$ with $U^{(b,\mathbb{T})}$. [ ]

For the balance of this section, $\mathcal{K} =_{df} \mathcal{S}$.

<u>3.4.4 Definition.</u> $\mathcal{S}^{[b,\mathbb{T}]} =_{df}$ the full subcategory of $\mathcal{S}^{(b,\mathbb{T})}$ generated by $[(X, XbT \xrightarrow{\xi} X)$: for every set n and for every $g \in \mathbb{T}(n)$, $X^n \xrightarrow{\xi^g} X \in \mathcal{E}$]. $U^{[b,\mathbb{T}]} =_{df} U^{(b,\mathbb{T})}$ restricted to $\mathcal{S}^{[b,\mathbb{T}]}$.

<u>3.4.5 Proposition.</u> $\mathcal{S}^{[b,\mathbb{T}]}$ is an LF-Birkhoff subcategory of $\mathcal{S}^{(b,\mathbb{T})}$, and hence $U^{[b,\mathbb{T}]}$ is tripleable.

<u>Proof.</u> Fix $n \in \text{obj}\ \mathcal{S}$, $g \in \mathbb{T}(n)$. If $(X,\xi) = \prod(X_i,\xi_i)$ in $\mathcal{S}^{(b,\mathbb{T})}$, with each $(X_i,\xi_i) \in \text{obj}\ \mathcal{S}^{[b,\mathbb{T}]}$, then

$$\begin{array}{ccc} X^n & \xrightarrow{\ pr_i{}^n\ } & X_i{}^n \\ \downarrow{\scriptstyle \xi^g} & & \downarrow{\scriptstyle \xi_i{}^g} \\ X & \xrightarrow{\ pr_i\ } & X_i \end{array}$$

shows that $\xi^g . pr_i \in \mathcal{E}$ for all i so that $\xi^g \in \mathcal{E}$. Therefore $\mathcal{S}^{[b,\mathbb{T}]}$ is closed under products. A similar argument using the diagram

$$\begin{array}{ccc} A^n & \xrightarrow{\ i^n\ } & X^n \\ \downarrow{\scriptstyle \xi_0{}^g} & & \downarrow{\scriptstyle \xi^g} \\ A & \xrightarrow{\ i\ } & X \end{array}$$

shows that $\mathcal{S}^{[b,\mathbb{T}]}$ is closed under relative subobjects. Let $(X,\xi) \xrightarrow{p} (Y,\theta) \in \mathcal{S}^{(b,\mathbb{T})}$ with p split spi in $\mathcal{E}$, and with $(X,\xi) \in \mathcal{S}^{[b,\mathbb{T}]}$. There exists $Y \xrightarrow{s} X \in \mathcal{E}$ with $sp = 1_Y$. As $s^n . p^n = 1_{A^n}$, p^n is split epi in $\mathcal{E}$. It follows from 0.4.2 and 3.1.9 that $X^n \xrightarrow{p^n} Y^n$ is opcartesian in $\mathcal{E}$. The diagram

$$\begin{array}{ccc} X^n & \xrightarrow{\ p^n\ } & Y^n \\ \downarrow{\scriptstyle \xi^g} & & \downarrow{\scriptstyle \theta^g} \\ X & \xrightarrow{\ p\ } & Y \end{array}$$

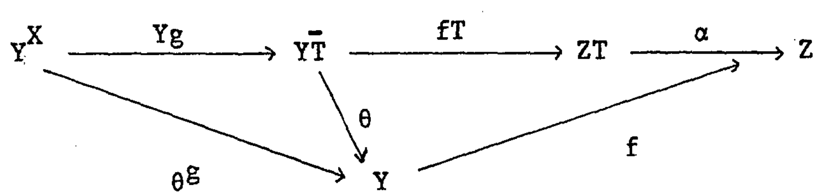
then shows that $(Y,\theta) \in \mathcal{S}^{[b,\mathbb{T}]}$. []

<u>3.4.6 Proposition.</u> Let $\tilde{\mathbb{T}} = (\tilde{T},\tilde{\eta},\tilde{\mu})$ be the triple in $\mathcal{E}$ corresponding to $\mathcal{S}^{[b,\mathbb{T}]}$ via the pointwise onto OPTR-morphism $\bar{\mathbb{T}} \xrightarrow{\lambda} \tilde{\mathbb{T}}$ of 1.6.4. The following statements are valid.

a. $\tilde{T}$ is a strong functor.

b. For every set n the passage $(1_{\mathcal{S}}{}^{n}, T)\text{n.t.} \longrightarrow (1_{\mathcal{E}}{}^{n}, \tilde{T})\text{n.t.}$ defined by $1_{\mathcal{S}}{}^{n} \xrightarrow{g} T \mapsto 1_{\mathcal{E}}{}^{n} \xrightarrow{g} \bar{T} \xrightarrow{\lambda} \tilde{T}$ is onto.

<u>Proof.</u> <u>a.</u> Let $X,Y \in \text{obj } \mathcal{E}$. We must show that $(X,Y)\text{HOM} \xrightarrow{\tilde{T}_{X,Y}} (X\tilde{T},Y\tilde{T})\text{HOM}$ is an $\mathcal{E}$-morphism. Before proving the general case, assume that for some structure map θ, $(Y,\theta) \in \text{obj}$ $\mathcal{S}^{[b,\mathbb{T}]}$. Let $\tilde{x} \in X\tilde{T}$. As $X\bar{T} \xrightarrow{X\lambda} X\tilde{T}$ is onto there exists $x \in X\bar{T}$ with $\langle x, X\lambda\rangle = \tilde{x}$. Let $1_{\mathcal{S}}{}^{X} \xrightarrow{g} T$ be the unique natural transformation such that $\langle 1_X, Xg\rangle = x$. The diagram:



and the hypothesis on Y prove that $Y^X \xrightarrow{Yg} Y\bar{T}$ is an $\mathcal{E}$-morphism. Let $k =_{df}$ the restriction of Yg to (X,Y)HOM. By the Yoneda correspondence and the fact that λ is natural, we have $k.Y\lambda = \langle x, (-)\bar{T}.Y\lambda\rangle = \langle x, X\lambda.(-)\tilde{T}\rangle = \langle \tilde{x}, (-)\tilde{T}\rangle$. It follows at once from the diagram at the top, left of the next page that $\tilde{T}_{X,Y}$ is an $\mathcal{E}$-morphism. The general case then follows from the diagram at the top, right because $(Y\tilde{T},Y\tilde{\mu}) \in \text{obj}$ $\mathcal{S}^{[b,\mathbb{T}]}$.

$$\begin{array}{ccc} (X,Y)\mathrm{HOM} & \xrightarrow{\tilde{T}_{X,Y}} & (X\tilde{T},Y\tilde{T})\mathrm{HOM} \\ \downarrow k & & \downarrow pr_{\tilde{x}} \\ Y\bar{T} & \xrightarrow{Y\lambda} & Y\tilde{T} \end{array} \qquad \begin{array}{ccc} (X,Y)\mathrm{HOM} & \xrightarrow{\tilde{T}_{X,Y}} & (X\tilde{T},Y\tilde{T})\mathrm{HOM} \\ \downarrow -\circ Y\tilde{\eta} & & \uparrow -\circ Y\tilde{\mu} \\ (X,Y\tilde{T})\mathrm{HOM} & \xrightarrow{\tilde{T}_{X,Y\tilde{T}}} & (X\tilde{T},Y\tilde{T}\tilde{T})\mathrm{HOM} \end{array}$$

b. Let n be a set. Clearly $(1_{\mathcal{S}})^n = (n\overleftarrow{b}, -)\mathrm{HOM}$. By (a), we have the supernaturality bijection $(1_{\mathcal{E}}{}^n, \tilde{T})\mathrm{n.t.} \approx n\overleftarrow{b\tilde{T}b}$. The surjection

$$(1_{\mathcal{S}}{}^n, T) \approx nT = n\overleftarrow{b\bar{T}b} \xrightarrow{n\lambda b} n\overleftarrow{b\tilde{T}b} \approx ((1_{\mathcal{E}})^n, \tilde{T})\mathrm{n.t.}$$

is easily checked to be the desired one. []

3.4.7 Discussion. Let $\bar{\mathbb{T}} \xrightarrow{\lambda} \tilde{\mathbb{T}}$ be as in 3.4.6. Let $(X,\xi) \in$ obj $\mathcal{E}^{\bar{\mathbb{T}}}$ and let $\tilde{x} \in X\tilde{T}$. There exists $x \in X\bar{T}$ with $\langle x, X\lambda\rangle = \tilde{x}$ and there exists $1_{\mathcal{E}} \xrightarrow{X} \xrightarrow{g} \bar{T}$ with $\langle 1_X, Xg\rangle = x$. By 3.4.6 (b), $1_{\mathcal{E}} \xrightarrow{X} \xrightarrow{g} \bar{T} \xrightarrow{\lambda} \tilde{T}$ indeed has $\mathcal{E}$-morphisms for components. It follows that there exists $1_{\mathcal{E}} \xrightarrow{X} \xrightarrow{\zeta} \tilde{T}$ with $\langle 1_X, X\zeta\rangle = \tilde{x}$, namely $\zeta = g.\lambda$. This crucial fact sets the stage for generalizing the theory of Chapter 2 to triples of form $\tilde{\mathbb{T}}$. A deeper analysis must wait for a later paper.

3.4.8 Applications of 3.4.5. The forgetful functor from topological groups to topological spaces is tripleable. Notice that we have proved the existence of a free topological group over a topological space. Similarly, the forgetful functor from quasi-ordered groups to quasi-ordered sets is tripleable, and there exists a free quasi-ordered group over each quasi-ordered set.

<u>3.4.9 Example; topological linear spaces.</u> The category, $\mathcal{V}$, of topological linear spaces is in fact the full subcategory of $\mathcal{S}^{[b,\mathbb{T}]}$, for $\mathcal{E}$ = topological spaces and $\mathcal{S}^{\mathbb{T}}$ = real vector spaces, generated by those $V \in \text{obj } \mathcal{S}^{[b,\mathbb{T}]}$ for which the action $\mathbb{R} \times V \xrightarrow{\alpha} V$ is continuous with respect to the usual topology of $\mathbb{R}$. To show that $\mathcal{V}$ is closed under products and relative subobjects use the same diagrams as in 2.3.6. Let $X \xrightarrow{p} Q \in \mathcal{S}^{[b,\mathbb{T}]}$ with p split epi in $\mathcal{E}$, and with X an object in $\mathcal{V}$. We have the diagram:

$$\begin{array}{ccc} \mathbb{R} \times X & \xrightarrow{1 \times p} & \mathbb{R} \times Q \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ X & \xrightarrow{p} & Q \end{array} .$$

As $1 \times p$ is split epi, $1 \times p$ is opcartesian. As α is an $\mathcal{E}$-morphism, so is γ. []

REFERENCES

[1] H. Appelgate, Acyclic models and resolvent functors, dissertation Columbia University, 1965.

[2] Michael Barr, Composite cotriples, mimeographed notes, University of Illinois.

[3] Jon Beck, Triples, algebras and cohomology, to appear.

[4] G. D. Birkhoff, On the structure of abstract algebras, Proc. Cambridge Phil. Soc. vol. 31 (1935), 433-454.

[5] P. M. Cohn, Universal algebra, New York, Harper and Row, 1965.

[6] Samuel Eilenberg and John C. Moore, Adjoint functors and triples, Ill. J. Math., vol. 9 (1965), 381-398.

[7] Robert Ellis and W. H. Gottschalk, Homomorphisms of transformation groups, Trans. Amer. Math. Soc. vol. 94 (1960), 258-271.

[8] Robert Ellis, A semigroup associated with a transformation group, Trans. Amer. Math. Soc. vol. 94 (1960), 272-281.

[9] Robert Ellis, Universal minimal sets, Proc. Amer. Math. Soc. vol. 11 (1960), 540-543.

[10] Peter Freyd, Abelian categories, New York, Harper and Row, 1965.

[11] Peter Freyd, Algebra valued functors in general and tensor products in particular, Colloq. Math. vol. 14 (1966), 89-106.

[12] R. Godement, Theorie des faisceaux, Paris, Hermann, 1958, appendix.

[13] W. H. Gottschalk and G. A. Hedlund, Topological dynamics, Amer. Math. Soc. Colloquium Publications, vol. 36, 1955.

[14] W. H. Gottschalk, course of lectures on topological dynamics given at Wesleyan University, 1965-66.

[15] John Gray, Fibred and cofibred categories, Proceedings of the La Jolla conference on categorical algebra, Berlin, Springer, 1963, 21-83.

[16] P. Huber, Homotopy theory in general categories, Math. Ann. vol. 144 (1961), 361-385.

[17] J. L. Kelley, General topology, New York, Van Nostrand, 1955.

[18] J. F. Kennison, Reflective functors in general topology and elsewhere, Trans. Amer. Math. Soc. vol. 118 (1965), 303-315.

[19] H. Kleisli, Every standard construction is induced by a pair of adjoint functors, Proc. Amer. Math. Soc. vol. 16 (1965), 544-546.

[20] F. W. Lawvere, Functorial semantics of algebraic theories, dissertation, Columbia University, 1963.

[21] F. W. Lawvere, The category of categories as a foundation for mathematics, Proceedings of the La Jolla conference on categorical algebra, Berlin, Springer, 1963, 1-20.

[22] F. E. J. Linton, Autonomous categories and duality of functors, J. of Algebra, vol. 2 (1965), 315-349.

[23] F. E. J. Linton, Some aspects of equational categories, Proceedings of the La Jolla conference on categorical algebra, Berlin, Springer, 1963, 84-94.

[24] F. E. J. Linton, Autonomous equational categories, J. of Math. and Mech., vol. 15 (1966), 637-642.

[25] J.-M. Maranda, On fundamental constructions and adjoint functors, Canad. Math. Bull. vol. 9 (1966), 581-591.

[26] Barry Mitchell, Theory of categories, New York, Acad. press, 1965.

[27] J. Słomiński, The theory of abstract algebras with infinitary operations, Rozprawy Mat. vol. 18 (1959).