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FUNCTOR THEORY

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INTRODUCTION

In 1955 Buchsbaum showed that an additive category in which exact sequences behave reasonably, enjoys many of the same properties as the category of abelian groups [1] and [2]. In particular, he showed that a theory of derived functors could be established for such "Exact" categories. Since that time, a great amount of empirical evidence has been accumulated that virtually any statement about exact diagrams true for abelian groups is true in arbitrary exact categories. It became natural to ask whether this fact could be formalized into a metatheorem. The advantages of such a metatheorem are clear: first, it would eliminate the need for the many laborous categorical proofs that seemingly bear no relation to the classical elemental proofs; and secondly, it would establish in advance an unending supply of lemmas as they become needed.

In Chapter One of this work, we prepare the ground for proving such a metatheorem. We prove that it suffices to construct a group-valued functor from a given category which will carry exact sequences into exact sequences <u>and</u> carry non-exact sequences into non-exact sequences. Several metatheorems for categories which admit such functors are proved, the last of which applies to certain existential theorems. There is considerable overlap between the material in Chapter One and that of Grothendieck [3].

In Chapter Two we prove that every small exact category (one in which the objects form a set) admits the desired type of functor. The proof is reminiscent of the Hurewicz-Wallman proof that finite dimensional spaces can be embedded in Euclidean space [8]. Recall that they considered the space of maps from an n-dimensional X to E^{2n+1}, metrized in such a way as to produce a complete space. They then used the Baire theorem to

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160
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prove that the subset of embeddings is dense, in particular, non-empty. Here, we will consider the category of functors from a given category to the category of groups. The objects are the covariant functors, the maps between the objects are the natural transformations between the functors. We prove that this category is exact and is "complete", that is, every system has a direct and inverse limit. Next we prove that the functor category has a "projective generator", a projective object with a non-zero map to every object. We are then in a position to apply Grothendiek's proof of the existence of injective resolutions, and we show that the injective envelope of the projective generator is the desired functor.

The techniques of Chapter Two will be seen to suggest very strongly that a functor category can be viewed as a category of modules over a ring. We find ourselves, for example, working with certain functors in the same way we work with torsion-free modules. One purpose of the remaining two chapters is to determine just how similar a functor category is to a category of modules.

In Chapter Three we study complete categories and the behavior of functors which preserve limits - i.e. those which have been called continuous functors. We find here a striking similarity with Hilbert space theory. Recall that a Hilbert space is a complete space with an additive structure and a canonical bilinear continuous function to the real numbers. In complete additive categories we also have a canonical bilinear continuous function - the "hom functor". Given a closed subspace H' of a Hilbert space H, and an element $x \in H$ we define the projection $\hat{x}$ of x into H' as the element in H' closest to x. We can then verify that for all $x' \in H'$ it is true that $(x,x')=(\hat{x},x')$, where $(\ ,\ )$ is the inner product. The analogue in categories to closed subspaces is what we have termed "reflective" subcategories. If $\mathcal{A}'$ is a reflective subcategory of $\mathcal{A}$ and A is an object in $\mathcal{A}$ then it has a "reflection" $\hat{A}$ in $\mathcal{A}'$ which, in a sense that will be made clear in Chapter Three, is the closest object to A in $\mathcal{A}'$. We shall find that for any $A' \in \mathcal{A}'$ that $(A,A')_{\mathcal{A}}$ is canonically isomorphic to $(\hat{A},A')_{\mathcal{A}'}$ where $(\hat{A},A')_{\mathcal{A}'}$ is the group of maps from A to A' which are in $\mathcal{A}'$. We show that compactifications, completetions, and tensor products can all be viewed as reflections. After characterizing reflective subcategories and using it to give new proofs o f the existance of the above mentioned examples, we define, after Kan [4], adjoint functors, and characterize functors which have left-adjoints. We use this result to derive another striking analogue to Hilbert space theory: if T is a left-continuous (commutes with left-limits), left-exact functor from a complete category $\mathcal{A}$ onto a reflective subcategory of abelian groups, then there exists an object $A \in \mathcal{A}$ such that T is naturally equivalent to the hom functor $(A,-)$. We use adjoints to define a generalized tensor product: given an abelian group $G \in \mathcal{G}$ an object $A \in \mathcal{A}$, $G \otimes A$ will be an object in $\mathcal{A}$. And again $\mathcal{G}$ plays a role among complete additive categories analogous to the role the space of real numbers play among Hilbert spaces: If T is right-continuous, right exact functor from $\mathcal{G}$ to $\mathcal{A}$ then there exists an object $A \in \mathcal{A}$ such that T is naturally equivalent to the functor $- \otimes A$. We return to functor categories at the end of Chapter Three to prove that if a complete category $\mathcal{B}$ has a projective generator then so does the category of functors from an arbitrary small category $\mathcal{A}$ to $\mathcal{B}$. The full significance of this fact becomes apparent in the fourth chapter where we prove that a complete exact category with a projective generator and in which direct sums are naturally embedded in direct products is representable as a category of modules over a "super-ring", that is, a ring in which certain infinite sums are defined. A module over a super-ring inherits a super-structure from the ring, and the maps we consider between such modules are those which preserve the designated infinite sums. Our final theorem is that the category of n-variable functors from a given set of n small categories to a category of modules over a super-ring is itself representable as a category of modules over a super-ring.

Conventions

A familiarity with the fundamentals of categories and functors will be assumed. By an EXACT CATEGORY is meant a category as defined by Buchsbaum [1] , axioms I through V (direct sums). A SMALL category is a category in which the class of objects is a set. For all categories we shall assume that the class of subobjects and the class of image objects of any object are both sets.

If A and B are two objects in an additive category, we shall denote the group of homomorphisms from A to B by "(A,B)". If we hold the first variable fixed we obtain what we shall call a COVARIANT HOM FUNCTOR and shall denote as " H^A ". To facilitate the description of diagrams, we shall consistantly write operators on the right. Hence (A,B) is isomorphic to BH^A. If the second variable of (A,B) is held fixed, we obtain a CONTRAVARIANT HOM FUNCTOR -- H_B. (A,B) is isomorphic to AH_B. A Functor unless qualified as contravariant will be assumed to be covariant.

CHAPTER ONE

Section 1. Preliminaries

Given an exact sequence $0 \to A' \to A \to A''$ in an exact category $\mathcal{A}$, and an object $B \in \mathcal{A}$ it is easily proved that the induced sequence

$$0 \to (B,A') \to (B,A'')$$

is exact. This property is traditionally described by stating that the covariant hom functor H^B is LEFT EXACT. We can equally well describe this property by saying that the functor H^B preserves kernals — if $A' \to A$ is a kernal of $A \to A''$ then $A'H^B \to AH^B$ is a kernal of $AH^B \to A''H^B$.

As all additive functors, H^B preserves finite direct products. But unlike all additive functors H^B also preserves infinite direct products. In fact, as is proved in [2], H^B preserves inverse limits, and hence is what has been called a LEFT-CONTINUOUS functor.

The left-exactness and left-continuity of the covariant hom functor are, in a sense which we will make clear, very similar properties. To be precise, kernals and direct products are both special cases of a larger concept:

Definition. Given a subcategory $\mathcal{A}'$ of a category $\mathcal{A}$ we say that a family $\{X \to A'\}_{A' \in \mathcal{A}'}$ is a LEFT-COMPATABLE FAMILY from X over $\mathcal{A}'$ if for every $(A_1' \to A_2') \in \mathcal{A}'$ the maps from the family yield a commutative triangle

$$X \begin{smallmatrix} \nearrow A_1' \\ \downarrow \\ \searrow A_2' \end{smallmatrix} .$$

A LEFT-ROOT of $\mathcal{A}'$ is a left-compatable family through which every other left-compatable family uniquely factors: If $\{R \to A'\}_{A' \in \mathcal{A}'}$ is a left-root and $\{X \to A'\}_{A' \in \mathcal{A}'}$ is an arbitrary left-compatable family, then there exists a unique $X \to R$ such that corresponding maps from the two families yield commutative triangles $\begin{matrix} X & \\ \downarrow & \searrow A'. \\ R & \nearrow \end{matrix}$

We can study the left compatable families from an object X by studying the maps from X to R, and conversely. Hence if $\mathcal{A}'$ consists of a set of objects with no maps between them, then a left-root of $\mathcal{A}'$ is precisely the direct product of all the objects in $\mathcal{A}'$. The maps from $\prod_{A' \varepsilon \mathcal{A}'} A'$ into $\mathcal{A}'$ are the projections: $\prod_{A' \varepsilon \mathcal{A}'} A' \xrightarrow{P_{A'}} A'$. A map $X \to \prod_{A' \varepsilon \mathcal{A}'} A'$ is completely identified by the set of maps $\{X \to \prod A' \xrightarrow{P_{A'}} A'\}_{A' \varepsilon \mathcal{A}}$.

We shall speak of these as the coordinate maps of the map $X \to \prod A'$.

If $\mathcal{A}'$ consists of two objects A and A'' and two maps $A \xrightarrow{\alpha} A''$ and the zero-map $A \xrightarrow{0} A''$ then its left-root is the kernal of α. In non-additive categories it is convenient to work with DIFFERENCE KERNALS. Given two maps $\alpha, \beta: A \to A''$ the difference kernal is the left-root of the sub category consisting of $A \xrightarrow{\alpha} A''$ and $A \xrightarrow{\beta} A''$. Without danger of confusion we can label this $\mathrm{Ker}(\alpha - \beta)$.

Intersections may be viewed as left-roots. Given a family of subobjects of an object A, that is, monomorphisms $\{A_i' \to A\}_{i \varepsilon I}$ where I is an indexing set, the intersection can be constructed by taking the left-root of the subcategory $\{A_i' \to A\}_{i \varepsilon I}$. It is easily verified that this construction will agree with the lattice theoretic definition of intersection.

A category is said to be LEFT-COMPLETE if every small subcategory has a left root.

<u>Proposition 1</u>. If a category has infinite direct products and intersections then it is left-complete.

Proof.

Lemma. If a category has finite direct products and intersections then it has difference kernals.

Proof of Lemma. Let $A \xrightarrow{\alpha} B$ and $A \xrightarrow{\beta} B$ be two maps, $A \xrightarrow{(e,\alpha)} A \times B$ the map whose first coordinate is the identity map $A \xrightarrow{e} A$ and whose second coordinate is α, and simularly let $A \xrightarrow{(e,\beta)} A \times B$ be def'ned. The intersection of the two monomorphisms $A \xrightarrow{(e,\alpha)} A \times B$ and $A \xrightarrow{(e,\beta)} A \times B$ when viewed as a subobject of A is the difference kernal of α and β.

Lemma. If a category has difference kernals, infinite direct products and intersections, then it is left-complete.

Proof of lemma. Let $\mathcal{A}'$ be a small subcategory, P the direct product of all its objects. For every $(A_1' \xrightarrow{\alpha} A_2') \in \mathcal{A}'$ we let $K_\alpha \to P$ be the difference kernal of the two maps $P \xrightarrow{p} A_1' \xrightarrow{\alpha} A_2'$ and $P \xrightarrow{p} A_2'$. Let $R \to P$ be the intersection of $\{K_\alpha \to P\}_{\alpha \in \mathcal{A}'}$. Then it is easy to verify that $\{R \to P \xrightarrow{p} A'\}_{A' \in \mathcal{A}'}$ is a left root of $\mathcal{A}'$.

Proposition 2. If an exact category has infinite direct products then it is left-complete.

Proof. By the above proposition we must verify that an exact category with infinite direct products has infinite intersections. We consider a family of monomorphisms $\{A_i' \to A\}_I$. For each $i \in I$ we define $A \to A_i''$ to be the cokernal of $A_i' \to A$. Let $\{A \to \prod_I A_i''\}$ be the map defined by the family $\{A \to A_i''\}_I$. Then the kernal of $A \to \prod_I A''$ is easily seen to be the intersection of $\{A_i' \to A\}$.

Proposition 3. If $\mathcal{A}$ is a left-complete category $\mathcal{B}$ an arbitrary category, then a functor $T: \mathcal{A} \to \mathcal{B}$ is a left-root preserving iff T preserves direct products and intersections. The proof follows quickly from the proof of proposition 1.

Proposition 4. If $\mathcal{A}$ is an exact left-complete category, $\mathcal{B}$ an arbitrary category, then a functor $T: \mathcal{A} \to \mathcal{B}$ is left-root preserving iff it is left-exact, and preserves direct products.

Proposition 5. If $\mathcal{A}$ is an exact category, $\mathcal{B}$ an additive category, then a functor $T: \mathcal{A} \to \mathcal{B}$ preserves left-roots of finite sub-categories (is finitely left-root preserving) iff it is left-exact and additive.

Proof. Additivity can quickly be seen to be equivalent with the property of preserving finite direct products. The remainder of the proof follow from the proof of proposition 2.

The dual concept of RIGHT-ROOT will not be separately defined. We point out only that cokernals, direct sums, and direct limits are examples of right-roots. Note that in an exact category finite direct products are isomorphic to finite direct sums.

Proposition 6. If $\mathcal{A}$ is an exact category, $\mathcal{B}$ an additive category, then a functor $T: \mathcal{A} \to \mathcal{B}$ is finitely root-preserving (both left and right) iff it is exact and additive.

We shall frequently use the following:

Definition. A rectangle $\begin{matrix} P_1 & \to & A \\ \downarrow & & \downarrow \\ B & \to & P_2 \end{matrix}$ is said to be a PULLBACK diagram if the maps from P_1 constitute a left-root for the maps between A, B and

P_2 ; it is a PUSHOUT diagram if the maps into P_2 constitute a right-root for the maps between P_1 , A , and B. Given A, B, P_2 and the maps between them, we can construct the pullback by first taking the direct sum $A \oplus B$ and then the difference kernal of the two maps from $A \oplus B$ to P_2 .

<u>Proposition 7</u>. If $\begin{array}{ccc} P_1 & \to & A \\ \downarrow & & \downarrow \\ B & \to & P_2 \end{array}$ is a pullback diagram in an exact category and $B \to P_2$ is an epimorphism, then $P_1 \to A$ is an epimorphism. Dually if the diagram is a pushout and $P_1 \to A$ is a monomorphism then $B \to P_2$ is a monomorphism.

<u>Proof</u>. We prove the second part directly. Letting $\alpha = P_1 \to A$ and $\beta = P_1 \to B$ we consider the diagram

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & B & & & \\ & & & \downarrow{\scriptstyle i_2} & & & \\ 0 \to & P_1 & \xrightarrow{(\alpha i_1 - \beta i_2)} & A \oplus B & \longrightarrow & P_2 & \to 0 \\ & & \searrow^{\alpha} & \downarrow{\scriptstyle P_1} & & & \\ & & & A & & & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

where i_1 i_2, P_1 are the cannonical map associated with direct sums

The Vertical sequence is exact, and the horizontal sequence, by construction of P_2 , is right-exact. Since the diagonal map is a monomorphism, $(\alpha i_1 - \beta i_2)$ is a monomorphism and the horizontal sequence is exact. Considering P_1 and B as subobjects of $A \oplus B$ we note that their intersection must be trivial, again because the diagonal map is monomorphic. Hence $B \xrightarrow{i_2} A \oplus B \to P_2$ is monomorphic which is precisely what we wished to prove.

A number of similar theorems may be easily proven for pullback and pushout diagrams. The above seems to be the most used and the most difficult.

Proposition 8. If $\begin{array}{ccc} P_1 & \rightarrow & A \\ \downarrow & & \downarrow \\ B & \rightarrow & P_2 \end{array}$ is a pullback diagram and $B \rightarrow P_2$ is an epimorphism then the diagram is also a pushout diagram, and dually

Proof. We consider the diagram

$$\begin{array}{cccc} K & & \searrow & \\ & P_1 & \rightarrow & A \\ & \downarrow & & \downarrow \\ & B & \rightarrow & P_2 \end{array}$$

Where $K \rightarrow B$ is a kernal of $B \rightarrow P_2$, $K \rightarrow A$ is the zero map and $K \rightarrow P_1$ is the induced pullback map. If $\begin{array}{cccc} P_1 & \rightarrow & A & \\ \downarrow & & \downarrow & \searrow \\ B & \rightarrow & P_2 & \searrow \\ & & & X \end{array}$ commutes, then $K \rightarrow B \rightarrow X$ is the zero map and there exists a unique factorization through the cokernal of $K \rightarrow B$, i.e. through $B \rightarrow P_2$. The commutativity of the triangle from A is forced by the epimorphism $P_1 \rightarrow A$

Section 2. Embeddings

Definition. An EMBEDDING functor is one which takes non-zero maps into non-zero maps. Note that it need not be one-to-one on the class of objects.

Theorem 9. If $\mathcal{A}$ and $\mathcal{B}$ are exact categories and T is an additive functor from $\mathcal{A}$ to $\mathcal{B}$ then the following conditions are equivalent

(a) T is an embedding functor

(b) The natural map from (A,B) to (AT,BT) is monomorphic

(c) T carries non-commutative diagrams into non-commutative diagrams

(d) T carries non-exact sequences into non-exact sequences

(e) If $\mathcal{A}'$ is a finite subcategory of $\mathcal{A}$ and $\{R \rightarrow A'\}_{A' \in \mathcal{A}'}$

is <u>not</u> a left-root of $\mathcal{A}$, then $\{RT \to A'T\}_{A'T \in (\mathcal{A}')T}$ is not a left-root of $(\mathcal{A}'')T$, and dually.

<u>Proof</u>: The equivalence of (a), (b) and (c) is clear. To prove that (d) implies (a) we consider a non-zero map $A \xrightarrow{\alpha} B$. The sequence $A \xrightarrow{e} A \xrightarrow{\alpha} B$ is not exact, hence by (d) the sequence $AT \xrightarrow{e} AT \xrightarrow{(\alpha)T} BT$ is not exact and $(\alpha)T$ is non-zero.

In proving that (a) implies (d) we consider the two ways in which $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ can fail to be exact:

First $\alpha\beta \neq 0$ Then $(\alpha)T(\beta)T \neq 0$ and $A'T \xrightarrow{(\alpha)T} AT \xrightarrow{(\beta)T} A''T$ is not exact.

Second $\alpha\beta = 0$ but $Im(\alpha)$ is properly contained in $Ker(\beta)$. Letting $K \to A = Ker(\beta)$ and $A \to F = Cok(\alpha)$ we can restate the failure of $A' \xrightarrow{\alpha} A \xrightarrow{\beta} A''$ to be exact as $K \to A \to F \neq 0$. Applying T we have

$$\begin{array}{ccccc} & & KT & & \\ & & \downarrow & \searrow^{0} & \\ A'T & \xrightarrow{(\alpha)T} & AT & \xrightarrow{(\beta)T} & A''T \\ & \searrow_{0} & \downarrow & & \\ & & FT & & \end{array}$$

The diagonal maps are zero because T, as do all additive functors, takes zero maps into zero maps. Hence $Im(KT \to AT) \subset Ker(\beta)T$ and $Im(\alpha)T \subset Ker(AT \to FT)$. Now if the horizontal sequence were exact then

$$Ker(\beta)T = Im(\alpha)T \qquad \text{and}$$

$$Im(KT \to AT) \subset Ker(\beta)T = Im(\alpha)T \subset Ker(AT \to FT)$$

which implies that the vertical map is zero which contradicts the assumption of (a).

It is clear that (e) implies (d). We shall use (a), (b), (c) to imply (e). Suppose that $\{R \to A'\}_{A' \in \mathcal{A}'}$ is not a left-root of finite $\mathcal{A}'$. There are three ways in which it can fail to be the left-root:

First, $\{R \to A'\}_{A' \in \mathcal{A}'}$ is not a compatable family over $\mathcal{A}'$. Condition (c) insures that $\{RT \to A'T\}$ is not a compatable family over $(\mathcal{A}')T$.

Second, there exists a compatable family $\{X \to A'\}_{A' \in \mathcal{A}'}$ which factors through $\{R \to A'\}_{A' \in \mathcal{A}'}$, but not uniquely. Condition (b) implies that $\{XT \to A'T\}_{A'T \in (\mathcal{A}')T}$ also has two distinct factorizations through $\{RT \to A'T\}_{A'T \in (\mathcal{A}')T}$.

Third, $\{R \to A'\}_{A' \in \mathcal{A}'}$ is a compatable family, and for any other compatable family there exists at most one factorization through R, but for at least one $\{X \to A'\}_{A' \in \mathcal{A}'}$ there is no factorization. We consider the finite direct sum $\sum_{A' \in \mathcal{A}'} A'$ and the maps $R \to \Sigma A'$ and $X \to \Sigma A'$ defined by the families $\{R \to A'\}$ and $\{X \to A'\}$. The uniqueness of the factorizations is equivalent to $R \to \Sigma A'$ being monomorphic. The non-existence of a factorization for X is thus equivalent to the map $X \to \Sigma A' \to F \neq 0$, where F is the cokernal of $R \to \Sigma A'$. The functor T preserves finite direct sums; if XT can be factored through RT then the image of $XT \to \Sigma A'T$ would be in the image of $RT \to \Sigma A'T$ which is in the kernal of $\Sigma A'T \to FT$ proving that $XT \to \Sigma A'T \to FT = 0$ a contradiction.

<u>Theorem 10:</u> If $\mathcal{A}'$ is a subcategory of an additive category $\mathcal{A}$, then $\{R \to A'\}_{A' \in \mathcal{A}'}$ is a left-root of $\mathcal{A}'$ iff $\{RH^B \to (A')H^B\}_{A'H^B \in (\mathcal{A}')H^B}$ is a left-root of $(\mathcal{A}')H^B$ for every $B \in \mathcal{A}$.

<u>Proof.</u> The proof is very similar to the preceeding. It will be noticed in the preceeding proof that we could restrict the use of the fact

$(\alpha)T \neq 0$ to single map α. Given any non-zero map $A \xrightarrow{\alpha} A''$ we can choose a B such that $(\alpha)H^B \neq 0$, namely by letting $B = A$.

An object $G \in \mathcal{A}$ is a GENERATOR iff for every non-zero map $A \to A''$ there exists a map $G \to A$ such that $G \to A \to A''$ is non-zero. Equivalently, G is a generator iff the functor H^G is an embedding. Co-generators are defined dually.

An object $P \in \mathcal{A}$ is PROJECTIVE iff for every epimorphism $A \to A''$ and every $P \to A''$ there is a map $P \to A$ such that $\begin{array}{ccc} & & A \\ P & \nearrow & \downarrow \\ & \searrow & A'' \end{array}$ commutes. Equivalently P is projective iff the functor H^P is exact.

An object $Q \in \mathcal{A}$ is INJECTIVE iff for every monomorphism $A' \to A$ and every $A' \to Q$ there is a map $A \to Q$ such that $\begin{array}{ccc} A' & & \\ \downarrow & \searrow & Q \\ A & \nearrow & \end{array}$ commutes. Equivalently Q is injective iff the functor H_Q is exact.

Embeddings and Exact functors are in a sense converses of each other: given a sequence $0 \to A' \to A \to A'' \to 0$ and a functor T, then

if T is an embedding

$$0 \to (A')T \to (A)T \to (A'')T \to 0 \text{ exact implies that}$$

$$0 \to A' \to A \to A'' \to 0 \text{ is exact}$$

if T is exact

$$0 \to A' \to A \to A'' \to 0 \text{ exact implies that}$$

$$0 \to (A')T \to (A)T \to (A'')T \to 0 \text{ is exact}$$

This conversal relation is also apparent between projectives and generators, most clearly seen in:

<u>Proposition 11.</u> In the category of left R modules, an object A is a :

Projective iff A appears as a direct summand of a direct sum of copies of R

Generator iff R appears as a direct summand of a direct sum of copies of A

Proposition 12. An exact functor T is an embedding iff T carries non-zero objects into non-zero objects.

Proof. For non-zero $A \xrightarrow{\alpha} B$ we consider the factorization $A \xrightarrow{\alpha} B = A \to I \to B$ where $A \to I$ is epimorphic and $I \to B$ is monomorphic. Hence $AT \to IT$ is epimorphic and $IT \to BT$ is monomorphic and IT is not zero.

Thus $(\alpha)T = AT \to IT \to BT \neq 0$.

Corollary 13. A projective object is a generator iff it has a non-trivial image in every object.

Section 3. METATHEOREMS

The great majority of categories that have been studied to date possess a projective generator or its dual, an injective cogenerator. In A category of modules over a ring, the ring itself is a projective generator. Grothendieck has shown that his categories of sheaves possess an injective co-generator. Such categories thus have exact group-valued embeddings. In this section we shall prove three metatheorems for exact categories which admit exact group-valued embeddings. In the next chapter, we prove that all small exact categories admit exact group-valued embeddings, whether or not they possess projective generators or their duals. For the purposes of this section we shall define a VERY EXACT category as an exact category which admits an exact group-valued embedding.

Metatheorem 14. If a theorem is of the form "P implies Q" where P states that a certain diagram is commutative and exact in certain places and Q states that the same diagram is commutative and exact in certain other places, and if the theorem is provable in the category of groups, then the theorem is true in every very exact category.

Metaproof: We shall prove the contrapositive. Suppose that in a very exact category $\mathcal{A}$ there is a diagram that satisfies P but not Q i.e. P does not imply Q in $\mathcal{A}$. Applying the exact group-valued embedding we would obtain a diagram in $\mathcal{G}$ that satisfies P but not Q, since the exact embedding will preserve the exactnesses and commutativities specified in P, and the failures of exactness, commutativity, and rootedness as specified by Q. Hence P does not imply Q in $\mathcal{G}$. ≠

We can immediately elaborate the metatheorem as follows:

Definition: A statement about a diagram is CATEGORICAL if it states that the diagram is commutative, and exact in certain places and that certain parts are left or right roots of certain other finite parts of the diagram, or negatively, that the diagram is not commutative or exact in certain places, and that certain parts are not left or right roots of certain other finite parts of the diagram, or any combination of such positive and negative assertions.

Metatheorem 15. If a theorem is of the form "P implies Q" where P and Q are categorical statements about the same diagram, and if the theorem is true in the category of abelian groups then it is true in every very exact category. ≠

Frequently in homological algebra a diagram is given and a map is constructed from the diagram. The most notable example is the connecting homomorphism:

If the following diagram of abelian groups is exact and commutative

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K' & \longrightarrow & K & \longrightarrow & K'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A'_1 & \longrightarrow & A_1 & \longrightarrow & A''_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & F' & \longrightarrow & F & \longrightarrow & F'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

then there exists a map $K'' \to F'$ such that the sequence $K' \to K \to K'' \to F' \to F \to F''$ is exact. The map $K'' \to F'$ is defined as $K'' \to A''_1$ followed by the inverse of $A_1 \to A''_1$ followed by $A_1 \to A$ followed by the inverse of $A' \to A$ followed by $A' \to F'$. It is proved that this composition of relations is a function (that it is everywhere defined, and that it is well defined), that the function is a homomorphism, and finally that $K \to K'' \to F' \to F$ is exact.

The existance of a map with the desired properties was insured by the exactness and commutativity of the diagram. Our question is whether the same diagram in a category admitting an exact group-valued embedding insures the existance of such a map. We can rephrase the question. We define an EXACT GROUP CATEGORY as a subcategory of the category of groups closed under the operations of taking roots of finite subcategories, that is, one for which the inclusion map is exact. The question then becomes whether the diagram being in an exact group subcategory implies that the map constructed from the diagram is still in that same subcategory. We need a few definitions.

Definition. Let $\mathcal{A}$ be a subcategory of $\mathcal{G}$. A and $B \in \mathcal{A}$

An $\mathcal{A}$-RELATION from A to B is a relation from the set of elements of A to the set of elements of B that can be represented as a composition of $\mathcal{A}$-maps and inverses of $\mathcal{A}$-maps. An $\mathcal{A}$-FUNCTION is an $\mathcal{A}$-relation that is a function. The questions in the above paragraph are affirmatively answered by

<u>Theorem 16.</u> For $\mathcal{A}$ an exact group category, all $\mathcal{A}$-functions are $\mathcal{A}$-maps.

To prove the theorem we need one more definition: A LEFT-SIMPLE $\mathcal{A}$-relation is an $\mathcal{A}$-relation that can be represented as the inverse of an $\mathcal{A}$-map followed by an $\mathcal{A}$-map.

<u>Lemma.</u> All $\mathcal{A}$-relations are left-simple $\mathcal{A}$-relations.

<u>Proof of lemma.</u> It is clear that all relations are compositions of left-simple relations. Hence by induction we need only prove that the composition of two left-simple relations is left-simple. We have the following situation

$$\begin{array}{ccccc} & & C & \longrightarrow & A \\ & & \downarrow & & \\ E & \longrightarrow & D & & \\ \downarrow & & & & \\ B & & & & \end{array}$$

a left-simple relation from A to D followed by a left-simple relation from D to B .

We let $\begin{array}{ccc} P & \rightarrow & C \\ \downarrow & & \downarrow \\ E & \rightarrow & D \end{array}$ be a pullback diagram.

Then the left-simple relation given by

$$\begin{array}{ccccc} P & \rightarrow & C & \rightarrow & A \\ \downarrow & & & & \\ E & & & & \\ \downarrow & & & & \\ B & & & & \end{array}$$

is the same as the above relation. Indeed, the left-simple relation given by

$\begin{array}{ll} P \to C \\ \downarrow \\ E \end{array}$ is the same as that given by $\begin{array}{cc} & C \\ & \downarrow \\ E \to & D \end{array}$.

<u>Lemma 2.</u> If a left-simple $\mathcal{A}$ relation is an $\mathcal{A}$ function then it is an $\mathcal{A}$ map.

<u>Proof of lemma 2.</u> Let a left-simple relation be given by $\begin{array}{ll} C \to A \\ \downarrow \\ B \end{array}$.

Its being defined everywhere is equivalent to $C \to A$ being epimorphic. Let $K \to C = \mathrm{Ker}(C \to A)$. That the left-simple relation is well defined is equivalent to $K \to C \to B = 0$. In particular for $k \in K \subset C$, we note that 0 in A is related to the image of k in B, hence image of k in B is 0 . By the exactness of the inclusion map of $\mathcal{A}$ into $\mathcal{G}$, and the fact that $C \to A$ is a cokernal of $K \to C$ there exists a map from A to B such that $\begin{array}{ll} C \to A \\ \downarrow \swarrow \\ B \end{array}$ commutes. Since $C \to A$ is onto in the set theoretic sense, the map $A \to B$ is equal to the relation.

The two lemmas immediately prove the theorem. $\neq$

If we define CONSTRUCTION BY DIAGRAM CHASING as the process of defining a map by composing the maps and inverses of the maps of a diagram, we can state the following, the proof of which follows quickly from the preceeding

<u>Metatheorem 17.</u> If a theorem is of the form "P implies Q" where P states that a certain diagram is commutative and exact in certain places and Q states that certain additional maps exist between designated objects such that the resulting diagram is exact and commutative in certain places, and if in the category of groups the theorem can be proved by constructing the maps through diagram chasing, then the theorem is true for very exact categories.

CHAPTER TWO

Functor Categories

Section 1. Preliminaries

Given two additive categories $\mathcal{A}$ and $\mathcal{B}$, $(\mathcal{A}, \mathcal{B})$ shall denote the CATEGORY OF COVARIANT FUNCTORS from $\mathcal{A}$ to $\mathcal{B}$. The objects of $(\mathcal{A}, \mathcal{B})$ are the covariant additive functors from $\mathcal{A}$ to $\mathcal{B}$, and the maps are the natural transformations between the functors.

We recall that a natural transformation between two functors T', T: $\mathcal{A} \to \mathcal{B}$ is a function η from the objects of $\mathcal{A}$ to the maps of $\mathcal{B}$ such that $\eta_A \in (AT', AT)$ and all rectangles of the form

$$\begin{array}{ccccc} & A_1T' & \xrightarrow{(\alpha)T'} & A_2T' & \\ \eta_{A_1} & \downarrow & & \downarrow & \eta_{A_2} \\ & A_1T & \xrightarrow{(\alpha)T} & A_2T & \end{array} \qquad \text{commute.}$$

A natural equivalence is a natural transformation with an inverse, hence one for which η_A is an isomorphism for every $A \in \mathcal{A}$. $\mathcal{A}$ will be assumed to be small.

If $\mathcal{B}$ is an exact category then so is $(\mathcal{A}, \mathcal{B})$. Given a natural transformation $\eta : T_1 \to T_2$ we can define its kernal $K \to T_1$ by

$$(A)K \to (A)T = \text{Ker}\,(\eta_A)$$

Note that a natural transformation η is a monomorphism as a map in $(\mathcal{A}, \mathcal{B})$ iff η_A is a monomorphism for every $A \in \mathcal{A}$. In general, a diagram in $(\mathcal{A}, \mathcal{B})$ is exact or commutative iff it is <u>pointwise</u> exact and commutative, that is, if an exact or commutative diagram results when the functors and natural transformations are evaluated throughout for any A. We can formalize this by introducing the EVALUATION FUNCTORS $E_A: (\mathcal{A},\mathcal{B}) \to \mathcal{B}$. $(T)E_A = (A)T$, $(\eta)E_A = \eta_A$. A diagram in $(\mathcal{A},\mathcal{B})$ is exact or commutative iff its image under all evaluation functors is exact or commutative.

Allowing ourselves a little set theoretical license, we can say that the evaluation functors provide a cononical embedding

$E: \mathcal{A} \to ((\mathcal{A},\mathcal{B}),\mathcal{B})$. $(A)E = E_A$ for $\alpha: A \to B$ $(\alpha)E_A$ for a functor T is equal to $(\alpha)T$.

<u>Section 2.</u> Let R be a ring with identity. It may be considered a category with one object (the ring of endomorphisms of that object is R). $(R,\mathcal{G})$ for $\mathcal{G}$ the category of abelian groups is then equal to the category of right modules over R: Every functor from R to $\mathcal{G}$, sends the unique object of R to an abelian group, and the endomorphisms of that object into the endomorphisms of the chosen group. Hence the objects of $(R,\mathcal{G})$ can be viewed as ordered pairs $(G, R \to \mathrm{End}(G))$ where G is an abelian group and $R \to \mathrm{End}(G)$ isa unitary ring homomorphism from R to the ring of endomorphisms of G. A map from $(G, R \to \mathrm{End}(G))$ to $(G', R \to \mathrm{End}(G'))$ is a group homomorphism $G \to G'$ which commutes with designated endomorphisms, or more conventionally, with ring multiplications. This is precisely the category of right R modules. In this and certain other cases, we shall write $(R,\mathcal{G})$ as $\mathcal{G}^R$.

Note that $(R_1,(R_2,\mathcal{G}))$ is equal to $(R_1 \otimes R_2,\mathcal{G})$ and that $(R_1 \oplus R_2,\mathcal{G})$ is equal to $(R_1,\mathcal{G}) \oplus (R_2,\mathcal{G})$.

$$(\mathcal{G}^{R_1})^{R_2} = \mathcal{G}^{R_1 \otimes R_2} \qquad \mathcal{G}^{R_1 \oplus R_2} = \mathcal{G}^{R_1} \oplus \mathcal{G}^{R_2}$$

Section 3. A contravariant functor T from $\mathcal{A}$ to $\mathcal{B}$, can be factored uniquely through $\mathcal{A}^*$. $\mathcal{A} \xrightarrow{T} \mathcal{B} = \mathcal{A} \xrightarrow{D} \mathcal{A}^* \xrightarrow{T^D} \mathcal{B}$ where D if the contravariant functor from $\mathcal{A}$ to its dual, and T^D is a covariant functor. A natural transformation from T to T' yields a natural transformation from T^D to T'^D. Hence the category of contravariant functors from $\mathcal{A}$ to $\mathcal{B}$ is canonically isomorphic to $(\mathcal{A}^*, \mathcal{B})$.

A contravariant functor T may also be uniquely factored through $\mathcal{B}^*$. $\mathcal{A} \xrightarrow{T} \mathcal{B} = \mathcal{A} \xrightarrow{T_D} \mathcal{B}^* \xrightarrow{D^{-1}} \mathcal{B}$ T_D is again a covariant functor. A natural transformation from T to T' yields a natural transformation from T'_D to T_D. The twist arises from the twisting of the maps of $\mathcal{B}$. Since the natural transformations are functions from objects of $\mathcal{A}$ to maps of $\mathcal{B}$ the process of dualizing affects the pointwise direction of the natural transformations only when the range is twisted. Hence, the category of contravariant functors from $\mathcal{A}$ to $\mathcal{B}$ is isomorphic to $(\mathcal{A}^*, \mathcal{B})$ and dual to $(\mathcal{A}, \mathcal{B}^*)$.

$$(\mathcal{A}^*, \mathcal{B}) \text{ is dual to } (\mathcal{A}, \mathcal{B}^*)$$

Rephrasing: $(\mathcal{A}, \mathcal{B})^* = (\mathcal{A}^*, \mathcal{B}^*)$ we obtain immediately

Proposition 1. The dual of $\mathcal{G}^R = (R, \mathcal{G})$ is equal to $(R^*, \mathcal{G}^*)$. Since $\mathcal{G}^*$ is representable as the category of compact abelian groups the latter can be interpreted as the left compact R modules : Compact abelian groups on which the ring operates continuously from the left. $\neq$

Section 4. Group valued functors

Yoneda has proved that the group of natural transformations from H^A to T is isomorphic to AT. [5] The covariant hom functors yield a canonical contravariant embedding of $\mathcal{A}$ into $(\mathcal{A}, \mathcal{G})$ - an object is sent into the functor H^A. If we repeat this process we find a covariant embedding $\mathcal{A} \to (\mathcal{A}, \mathcal{G}) \to ((\mathcal{A}, \mathcal{G})\ \mathcal{G})$ which turns out to be naturally equivalent to E, the evaluation embedding mention in § 1.

For an epimorphic natural transformation $T \to T''$ we have $(H^A, T) \to (H^A, T'') = AT \to AT''$ epimorphic, hence H^A is projective in $(\mathcal{A}, \mathcal{G})$ and $\Sigma_{A\in\mathcal{A}} H^A$ is a projective generator. This is destined to be confused with the separate fact that $\Sigma_{A\in\mathcal{A}} H^A$ is an embedding of $\mathcal{A}$ into $\mathcal{G}$.

<u>Theorem 2.</u> If T is an injective object of $(\mathcal{A}, \mathcal{G})$, then T is right exact.

<u>Proof:</u> Given a right exact sequence $A' \to A \to A'' \to 0$ we consider the left-exact sequence of functors $0 \to H^{A''} \to H^A \to H^{A'}$. If T is injective, then the sequence $(H^{A'}, T) \to (H^A, T) \to (H^{A''}, T) \to 0$ is right exact. But this last sequence is equivalent to $A'T \to AT \to A''T \to 0$, hence T is right exact.

Since $(\mathcal{A}, \mathcal{G})$ satisfies Grothendieck's axiom AB5 and has a generator, every object can be embedded in an injective object. [3]. Using the methods of Eckmann [6] we can construct an injective envelope for any object, that is, an embedding $T \to Q$, Q injective, such that if $Q' \subset Q$ and Q' meets T trivially, then Q' is trivial. Mitchell has refined a method which constructs the injective envelope directly without first constructing injectives [7].

<u>Theorem 3.</u> If $\mathcal{A}$ is a small exact category, the injective envelope of $\Sigma_{A\in\mathcal{A}'} H^A$ is an exact embedding, hence the metatheorems of chapter one hold for $\mathcal{A}$.

Proof. If Q is an injective envelope of $\sum_{A \in \mathcal{A}} \Pi^A$ then it is right exact, and an embedding since it contains an embedding. We must prove that it carries monomorphisms into monomorphisms, that is, is a MONO FUNCTOR.

Note that in the case $(R, \mathcal{G})$ a monofunctor is a torsion-free module for R a domain. We can construct the torsion-free image of a module by factoring out the torsion module. In essence, we repeat this process for $(\mathcal{A}, \mathcal{G})$.

Given $T \in (\mathcal{A}, \mathcal{G})$ we define $AK \subset AT$ as the elements of AT which are killed by $(\alpha)T$ for monomorphic α. $AK = \{x \in AT :$ there exists monomorphic $A \xrightarrow{\alpha} B$ such that $x(\alpha)T = 0.\}$

Lemma 1. AK is carried into BK by $(\beta)T$ for any $A \xrightarrow{\beta} B$.

Proof of lemma 1. Let $x \in AK$ and $A \xrightarrow{\gamma} C$ a monomorphism such that $x(\gamma)T = 0$. We construct the pushout diagram
$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\bar{\gamma}} \\ C & \xrightarrow[\bar{\beta}]{} & P \end{array}$$
which according to Proposition 7 is such that $\bar{\gamma}$ is monomorphic. Applying T we obtain
$$\begin{array}{ccc} AT & \rightarrow & BT \\ \downarrow & & \downarrow \\ CT & \rightarrow & PT \end{array}$$
. Starting with x in the northeast corner, we arrive at $0 \in PT$ travelling counter-clockwise. Hence $x(\beta)T$ is killed by $(\bar{\gamma})T$.

Lemma 2. AK is a subgroup of AT.

Proof of Lemma 2. Given x, y in AK, monomorphisms $A \xrightarrow{\beta} B$, $A \xrightarrow{\gamma} C$ such that $x(\beta)T = 0$, $y(\gamma)T = 0$, we repeat the above proof:
$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\bar{\gamma}} \\ C & \xrightarrow[\bar{\beta}]{} & P \end{array}$$
is a pushout diagram in which all the maps are monomorphisms.

Hence the monomorphism $A \xrightarrow{\beta\bar{\gamma}} P$ is such that $(x + y)(\beta\bar{\gamma})T = 0$.

K is therefore a subfunctor of T , trivial iff T is mono. Every non-trivial subfunctor of K fails to be a monofunctor. Hence if $T \to Q$ is an injective envelope of a monofunctor , then Q is a monofunctor, since the K defined from Q must meet T trivially (every subfunctor of a monofunctor is mono), hence K is trivial, hence Q is a monofunctor.

CHAPTER THREE

Section 1. Reflections

We consider a category $\mathcal{A}$, a subcategory $\mathcal{A}'$ and an object $A \epsilon \mathcal{A}$. A REFLECTION of A in $\mathcal{A}'$ is an object $\hat{A} \epsilon \mathcal{A}'$ together with a map $A \to \hat{A}$ such that every map from A to an object in $\mathcal{A}'$ uniquely factors through $A \to \hat{A}$, i.e. with the property that for every $A' \epsilon \mathcal{A}'$ and every map $A \to A'$ there exists a unique map $(\hat{A} \to A') \epsilon \mathcal{A}'$ such that

$$\begin{array}{ccc} & & \hat{A} \\ & \nearrow & \downarrow \\ A & & \\ & \searrow & A' \end{array}$$

commutes. We can study the maps from A into $\mathcal{A}'$ simply by studying the maps in $\mathcal{A}'$ from $\hat{A}$. That is, $(\hat{A}, A')_{\mathcal{A}'}$ is sent isomorphically onto $(\hat{A}, A')_{\mathcal{A}}$ by $A \to \hat{A}$, where $(\quad,\quad)_{\mathcal{A}'}$ indicates the maps from $\hat{A}$ to A' which are in $\mathcal{A}'$.

Some examples of reflections are the following:

I We consider $\mathcal{T}$ the category of topological spaces, $\mathcal{C}$ the subcategory of compact spaces. If A is a Tychonoff space, then its reflection in $\mathcal{C}$ is its Stone-Ceck compactification.

II Let $\mathcal{M}$ be the category of metric spaces and distance decreasing maps (weakly decreasing), $\mathcal{C}$ the subcategory of complete spaces. Then the reflection of a metric space A in $\mathcal{C}$ is its completetion.

III Let $\mathcal{B}$ be the category of ordered pairs and singletons of abelian groups. An object of $\mathcal{B}$ is either a group G or an ordered pair of groups (G,H). The maps will be linear and bilinear: a map between two singletons is linear, a map from a pair to a singleton is bilinear, there are no maps into pairs. Let $\mathcal{G}$ be the subcategory or singletons and linear maps. The reflection of a pair in $\mathcal{G}$ is then its tensor product.

IV Let $\mathcal{Q}$ be the category of R modules and R' homomorphisms, where R' is a subring of R. Let $\mathcal{G}^R$ be the subcategory of R homomorphisms. The reflection of A in $\mathcal{G}^R$ is what has been called its covariant extension.

If $\mathcal{A}'$ is such that every object in $\mathcal{A}$ has a reflection in $\mathcal{A}'$, we say that $\mathcal{A}'$ is a REFLECTIVE SUBCATEGORY. Such subcategories play a central role in this paper. A functor $R: \mathcal{A} \to \mathcal{A}' \subset \mathcal{A}$ which sends each object into its reflection, and for which there exists a natural transformation

$$r: I \to R$$

where I is the identity functor on $\mathcal{A}$, such that $A \xrightarrow{r_A} AR$ is the canonical map, is called a REFLECTOR of $\mathcal{A}$ into $\mathcal{A}'$.

Section 2.

Characterization of reflective subcategories. In order to facilitate the characterization of reflective subcategories we introduce the notion of an object in the larger category GENERATING an object in the smaller category. For $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$, A is said to generate A' if there exists a map $A \to A'$ such that there exist no non-trivial monomorphisms $(A_{\cdot} \to A') \in \mathcal{A}'$ through which $A \to A'$ can be factored. We say that the map $A \to A'$ is a GENERATING MAP. Given any map $A \to A'$, where $A' \in \mathcal{A}'$ we define its $\mathcal{A}'$-IMAGE as an object $I \in \mathcal{A}'$ together with maps $A \to I$ and $(I \to A') \in \mathcal{A}'$ such that $A \to I$ is a generating map.

We note that if $\mathcal{A}'$ is a reflective subcategory of $\mathcal{A}$ then every object of $\mathcal{A}$ generates at most a set of objects in $\mathcal{A}'$. Indeed, if A in $\mathcal{A}$ generates A' in $\mathcal{A}'$ then A' is an image of $\hat{A}$.

We say that a subcategory $\mathcal{A}'$ is CLOSED ON THE LEFT or LEFT-CLOSED if all left roots definable from subcategories of $\mathcal{A}'$ are in $\mathcal{A}'$. We shall prove later that a reflective subcatefory of a complete category is left-closed.

<u>Theorem 1.</u> A subcategory $\mathcal{A}'$ of $\mathcal{A}$ is reflective if

(1) It is left-complete and left-closed.

(2) Every object in $\mathcal{A}$ generates at most a set of distinct objects in $\mathcal{A}'$.

<u>Proof:</u> The proof will be in two parts. First we shall show under the hypotheses, that for every $A \in \mathcal{A}$ there exists a NEAR-REFLECTION in $\mathcal{A}'$, that is, an object $N \in \mathcal{A}'$ together with a map $A \to N$ such that every map from A into $\mathcal{A}'$ factors through $A \to N$, but not necessarily uniquely. Then secondly, that if an object has a near-reflection in a left-closed subcategory that it has a reflection.

<u>Preliminary Lemma:</u> If $A \to A'$ is any map, $A' \in \mathcal{A}'$ and $\mathcal{A}'$ is left-closed, then it has an $\mathcal{A}'$-image.

<u>Proof:</u> We intersect all the $\mathcal{A}'$-subobjects of A' through which $A \to A'$ can be factored. Since $\mathcal{A}'$ is left-closed, the intersection I is in $\mathcal{A}'$. Hence we have the maps $A \to I$ and $(I \to A') \in \mathcal{A}'$, the composition of which is $A \to A'$. The first map clearly is a generating map.

<u>Near-Reflection Lemma:</u> If $A \in \mathcal{A}$, then there exists an object $N \in \mathcal{A}'$ together with a map $A \to N$ such that for every $A' \in \mathcal{A}'$ and $A \to A'$ there exists a map $(N \to A') \in \mathcal{A}'$ such that the triangle $\begin{array}{ccc} & & N \\ & \nearrow & \downarrow \\ A & \searrow & A' \end{array}$ commutes.

<u>Proof of Lemma:</u> Let $\mathcal{S}$ be a complete set of distinct $\mathcal{A}'$-objects generated by A . (That such a <u>set</u> exists is a consequence of condition 2 of the hypothesis .) We define $N = \prod_{S \in \mathcal{S}} \prod_{(A,S)} S$ and observe that condition 1 of the hypothesis insures that $N \in \mathcal{A}'$. Let $A \to \prod_{S \in \mathcal{S}} \prod_{(A,S)} S = N$ be the map whose (S,α) coordinate is $A \xrightarrow{\alpha} S$.

To prove that N is a near-reflection we consider an arbitrary map $A \to A'$, $A' \in \mathcal{A}'$. By the preliminary lemma it has an $\mathcal{A}'$-image. That is $A \to A' = A \to I \to A'$ where $A \to I$ is a generating map and $I \to A'$ is in $\mathcal{A}'$. Hence there exists an $S \in \mathcal{S}$ which is isomorphic to I, i.e. there exists an isomorphism $(S \to I) \in \mathcal{A}'$. Letting $\alpha = A \to I \to S$ we have that $A \to A' = A \to N = \Pi_{S \in \mathcal{S}(A,S)} \Pi \; S \to S \to I \to A'$.

Final Lemma: If A has a near-reflection in $\mathcal{A}'$, and $\mathcal{A}'$ is left-closed, then A has a reflection in $\mathcal{A}'$.

Proof: Suppose that $A \to N$ is a near reflection. We let R be its $\mathcal{A}'$-image, $A \to N = A \to R \to N$. Then R is still a near reflection. We prove the uniqueness of the induced maps by considering the case $A \to R \xrightarrow{\alpha} A' = A \to R \xrightarrow{\beta} A'$ where both α and β are in $\mathcal{A}'$. Since $\mathcal{A}'$ is left-closed we can take the difference kernal K of α and β: $K \to R \xrightarrow{\alpha} A' = K \to R \xrightarrow{\beta} A'$ where $A \to R = A \to K \to R$ for some suitable $A \to K$. The important point is that $(K \to R) \in \mathcal{A}'$, which when combined with the fact that $A \to R$ is a generating map, yields the fact that $K \to R$ is epimorphic, hence that $\alpha = \beta$. $\neq$

In application we generally show the second condition of the above theorem to be satisfied by a two step program: first we find a function from the objects of $\mathcal{A}$ to the class of cardnal numbers that has the property that there are at most a set of non-isomorphic objects in $\mathcal{A}$ with the same cardinal value. Such would be called a 'Cardinality functor. Secondly, we verify that there is an upper bound on the cardinalities of the objects in $\mathcal{A}'$ generated by a given object in $\mathcal{A}$.

Thus we prove the existence of the reflections exemplified in section 2 :

I. We take the obvious cardinality function in the category of topological spaces $\mathcal{T}$ — the number of elements of a space. We can count the different topologies that can be put on a set S by observing that a family of open sets is a subset of the family of subsets of S. Hence there are no more than 2^{2^S} different topologies on S. If a space T is the same cardinality as S, there is a one-to-one correspondence $S \to T$ which we use to put a topology on S. Hence T is homomorphic to one of 2^{2^S} spaces.

If monomorphic $S \to T$ generates compact T then the image of $S \to T$ is dense in T and for every pair of points in the compliment of the image there is a filter in S that converges to one but not the other. If this last condition were not true, we could adjoin to S one point from each class of points in T which are limits of the same nets in S, and obtain a compact subset. Hence the cardinality of T is not more than the number of filters in S which is not more than 2^{2^S}. That $\mathcal{G}$ is left-closed is a direct consequence of the Tychonoff Product theorem.

II. We change the problem by introducing base points. $\mathcal{M}$ shall be the category of metric spaces with base points and distance decreasing maps that take base points into base points. The cardinality of a metric space shall be its number of elements. The number of metrics that can be put on a set M is not more than the number of functions from M x M to the real numbers. Hence, as above there is at most a set of non-isomorphic metric spaces of a given cardinality. Since metric spaces are also topological spaces, we can use the same arguement as above to prove that a space generates at most a set of non-isomorphic complete spaces.

The left-closure of the subcategory of complete spaces requires verification. First we must examine the construction of products in the category $\mathcal{M}$. Given a set of metric spaces $\{M_i\}_I$, with O_i the base point of M_i , and $|x-y|_i$ the distance between two points in M_i , we define $\prod_I M_i$ as

$$\{f : I \to \bigcup_I M_i \mid (i)f \in M_i \text{ for all } i \in I, \text{ and such that there is an upper bound for } \{|(i)f-o_i|\ i\}_{i\in I}\}$$

The distance between two elements, f and g in $\prod M_i$ is defined as $\sup_I \{|(i)f-(i)g|_i\}$. The ith projection from $\prod M_i$ to M_i is defined by $(f)p_i = (i)f$. That this construction yields the proper mapping properties, and that the product of complete spaces is complete, is straight-forward.

III. For $\mathcal{B}$ the category of linear and bilinear maps, cardinality is again taken as the number of elements of a singleton, and the number of elements of the cartesian product of the groups in a pair. That this cardinality function has the desired properties, and that a finite set generates only countable groups, and infinite sets have the same cardinality as the groups they generate is straight-forward.

<u>Section 3</u>. ADJOINT FUNCTORS

We shall speak of natural transformations between bifunctors of two variables. The definition is the obvious generalization of that for functors

of one variable. Let T_1, T_2 be functors from $\mathcal{A}$, $\mathcal{B}$ to $\mathcal{C}$.
A natural transformation η from T_1 to T_2 is a function from pairs of objects (A,B) $A\epsilon\ \mathcal{A}$, $B\epsilon\ \mathcal{B}$, to maps in $\mathcal{C}$ such that all diagrams of the form

$$\begin{array}{ccccc} & (A',B')T_1 & \xrightarrow{(\alpha,\beta)T_1} & (A,B)T_1 & \\ \eta A',\theta' & \downarrow & & \downarrow & \eta A,B \\ & (A',B')T_2 & \xrightarrow[(A\alpha,\beta)T_2]{} & (A,B)T_2 & \end{array}$$

commute. To verify naturality it suffices to consider the cases where either α or β is the identity map.

Throughout this section the categories will be assumed to be additive and complete.

A pair of functors $T^R: \mathcal{A} \to \mathcal{B}$ and $T^L: \mathcal{B} \to \mathcal{A}$ are said to be ADJOINT if the bifunctors $(T^L, -): \mathcal{A} \times \mathcal{B} \to \mathcal{G}$ and $(-,T^R): \mathcal{A} \times \mathcal{B} \to \mathcal{G}$ are naturally equivalent. That is, for every pair $(A,B)\epsilon\ \mathcal{A} \times \mathcal{B}$ there is an isomorphism $(AT^L,B) \xrightarrow{\eta_{A,B}} (A,BT^R)$ which satisfies the conditions of naturallity. T^L is said to be the left-adjoint of T^R. A pair of contravariant functors $T: \mathcal{A} \to \mathcal{B}$ and $T^*: \mathcal{B} \to \mathcal{A}$ are said to be ADJOINT ON THE RIGHT if the contravariant bifunctors $(-,T): \mathcal{B} \times \mathcal{A} \to \mathcal{G}$ and $(-,T^*): \mathcal{A} \times \mathcal{B} \to \mathcal{G}$ are naturally equivalent. They are said to be ADJOINT ON THE LEFT if the co-variant bifunctors $(T,-): \mathcal{A} \times \mathcal{B} \to \mathcal{G}$ and $(T^*,-): \mathcal{B} \times \mathcal{A} \to \mathcal{G}$ are naturally equivalent.

The best known pair of adjoint functors are $H^G: \mathcal{G} \to \mathcal{G}$ and $-\otimes G: \mathcal{G} \to \mathcal{G}$. Note that H_G is adjoint to itself on the right.

The reflector of a subcategory is adjoint to the inclusion functor of the subcategory. Given a reflective subcategory $\mathcal{A}'$ of $\mathcal{A}$, and a reflector $R: \mathcal{A} \to \mathcal{A}'$ together with the natural transformation r from the Identity functor to R, we construct the natural transformation η from $(AR, A')_{\mathcal{A}'}$ to $(A, A')_{\mathcal{A}}$ by letting $\eta_{A,A'}$ be the map induced by $A \xrightarrow{r_A} AR$. We know this to be an isomorphism. Its naturality is easily verified.

<u>Theorem 2.</u> A functor $T: \mathcal{A} \to \mathcal{B}$ has a left adjoint iff T is left root preserving and has a reflective subcategory as an image.

<u>Proof:</u> Sufficiency. Suppose $T^L: \mathcal{B} \to \mathcal{A}$ is a left adjoint of T and η is the natural equivalence from $(T^L, -)$ to $(-, T)$. For fixed $B \epsilon \mathcal{B}$ the functor H^{BT^L} if left-root preserving. H^{BT^L} is naturally equivalent through η to the functor $(B, (-)T)$ which is therefore left-root preserving, and hence by Theorem 1.10 T is left-root preserving.

We shall prove that $(\mathcal{A})T$ is a reflective subcategory by proving that for any $B \epsilon \mathcal{B}$, that BT^LT is its reflection in $(\mathcal{A})T$. We consider the map $(BT^L, BT^L) \xrightarrow{\eta_{B,BT^L}} (B, T^LT)$ and define $r_B = (e_{BT^L})\eta_{B,BT^L}$

<u>Lemma 1.</u> Given $B \xrightarrow{x} AT$ there exists a unique $(y)T \epsilon (\mathcal{A})T$, such that the triangle

$$\begin{array}{ccc} B & \xrightarrow{r_B} & BT^LT \\ & \searrow & \downarrow (y)T \\ & & AT \end{array}$$

commutes. $(y \epsilon (BT^L, A))$.

<u>Proof of Lemma 1.</u> Consider the commutative diagram

$$\begin{array}{ccc} (BT^L, BT^L) & \xrightarrow{(e_B T^L, y)} & (BT^L, A) \\ \downarrow \eta_{B,BT^L} & & \downarrow \eta_{B,A} \\ (B, BT^LT) & \xrightarrow{(e_B, yT)} & (B, AT) \end{array}$$

If we start with the element e_{BT^L} in the northwest corner and travel clockwise we obtain the element $y\eta_{B,A}$ in the southeast corner, if we travel counterclockwise we obtain $(r_B)(yT)$. Hence we have Equation one: $y\eta_{B,A} = (r_B)(yT)$. Since $\eta_{B,A}$ is an isomorphism there exists one and only one $y \in (BT^L, A)$ such that $x = y\eta_{B,A} = (r_B)(yT)$.

Lemma 2. $T^LT: \mathcal{B} \to (\mathcal{A})T$ is a reflector of $(\mathcal{A})T$ and the transformation $r: I \to T^LT$ defined as above is natural.

Proof. We wish to prove that the rectangle

$$\begin{array}{ccc} B' & \xrightarrow{\quad x \quad} & B \\ \downarrow r_{B'} & & \downarrow r_B \ X \\ B'T^LT & \xrightarrow[\quad xT^LT \quad]{} & BT^LT \end{array}$$

is commutative. That is that $xr_B = (r_{B'})(xT^LT)$. By equation one above, the right hand side of this equation is equal to $(xT^L)\eta_{B,BT^L}$. Hence we wish to show that $xr_B = (xT^L)\eta_{B',BT^L}$.

To that end we consider the commutative rectangle

$$\begin{array}{ccc} (BT^L, BT^L) & \xrightarrow{\quad (xT^L, e_{BT^L}) \quad} & (B'T^L, BT^L) \\ \eta_{B,BT^L} \downarrow & & \downarrow \eta_{B',BT^L} \\ (B, BT^LT) & \xrightarrow[\quad (x(e_{BT^L})T) \quad]{} & (B', BT^LT) \end{array}$$

Again starting with e_{BT^L} in the northwest corner and traveling clockwise we obtain $(xT^L)\eta_{B',BT^L}$, traveling counterclockwise we obtain $(x e_{BT^L} \eta_{B,BT^L}) = xr_b$. Hence Equation two:

$$xr_B = (xT^L)\,\eta_{B',BT^L}$$

which when combined with equation one yields

$$xr_B = (r_{B'})(xT^LT).$$

Necessity:

We suppose that T is a left-root preserving functor and has a reflective image. We let $R: \mathcal{B} \to (\mathcal{A})T$ be a reflector. We shall construct T^L as follows: for $B \in \mathcal{B}$ we let $A \in \mathcal{A}$ be such that $AT = BR$. Next we intersect all subobjects $\{A'\}$ of A such that $B \xrightarrow{r} BR$ factors through $A'T \to AT$. Since T is left-root preserving this intersection still is such that r_B factors through it. We shall define BT^L to be this intersection, and observe that BT^LT is still a reflection of B. BT^L has the property that if $BT^L \xrightarrow{\alpha} A$ is any non-zero-map then $(\alpha)T$ is non-zero, since for $K \to BT^L$ the kernal of α, our construction of BT^L insures that KT is a proper subobject of BT^LT.

Given $B' \xrightarrow{\beta} B$ we consider the map between their reflections $(B'T^LT \to BT^LT) \in (\mathcal{A})T$ and define $(\beta)T^L$ to be the pre-image under T. Since $B'T^L$ has the property described above, there is only one pre-image.

That this definition behaves well with composition follows from the uniqueness of maps induced through reflections. The natural transformation needed to complete the proof is defined as the composition $(BT^L,A) \to (BT^LT,AT) \xrightarrow{\approx} (B,AT)$. The first map, which is induced by T, is one-to-one because of the special choice of BT^L. The second map is the natural equivalence that arises from reflectors.

There are <u>three</u> more theorems that can be obtained from this one by applying duality.

<u>Theorem 3.</u> A contravariant functor $T: \mathcal{A} \to \mathcal{B}$ has an adjoint on the right iff it carries right roots into left-roots and its image is reflective.

<u>Proof:</u> We transform the problem into a problem about covariant

functors by replacing T: with DT: $\mathcal{A}^* \xrightarrow{D} \mathcal{A} \to \mathcal{B}$.

By using the dual of $\mathcal{B}$ we obtain

<u>Theorem 4.</u> A contravariant functor T: $\mathcal{A} \to \mathcal{B}$ has an adjoint on the left iff it carries left-roots into right-roots and its image is co-reflective. And by using the duals of both $\mathcal{A}$ and $\mathcal{B}$ we obtain

<u>Theorem 5.</u> A covariant functor T: $\mathcal{A} \to \mathcal{B}$ has a right adjoint iff it is right-root preserving and its image is co-reflective.

<u>Section 4.</u> Transformation adjoints

<u>Theorem 6.</u> If T_1 and T_2 are functors from $\mathcal{A}$ to $\mathcal{B}$ with left-adjoints T^L_1 and T^L_2 respectively, and α is a natural transformation from T_1 to T_2, there exist a unique transformation α^L from T^L_2 to T^L_1 such that all rectangles of the form

$$\begin{array}{ccc} (BT^L_1, A) & \xrightarrow{(\alpha^L_B, e)} & (BT^L_2, A) \\ \downarrow \eta_1 & & \downarrow \eta_2 \\ (B, AT_1) & \xrightarrow[(e, \alpha_B)]{} & (B, AT_2) \end{array}$$

are commutative.

<u>Proof.</u> For $B \in \mathcal{B}$ we define α^L_B by considering the rectangle

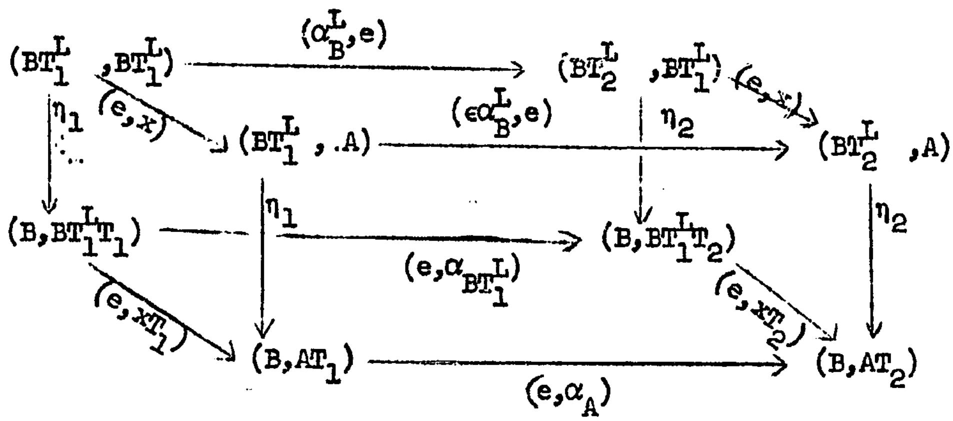
$$\begin{array}{ccc} (BT^L_1, BT^L_1) & & (BT^L_2, BT^L_1) \\ \downarrow \eta_1 & & \downarrow \eta_2 \\ (B, BT^L_1 T_1) & \xrightarrow[(e, \alpha_{BT^L_1 T})]{} & (B, BT^L_1 T_2) \end{array}$$

Starting with e in the northwest corner we obtain

$$\begin{array}{ccc} e & & \alpha^L_B \\ \downarrow \eta_1 & & \uparrow \eta_2 \\ r_{1B} & \longrightarrow & r_{1B}\alpha_{BT^L_1} \end{array}$$

where the northeast element is defined as $(r_{1B} \circ \alpha_{BT^L_1})\eta_2^{-1}$.

We prove the commutativity of the rectangle of the theorem by superimposing it on the above rectangle. Let $x \in (BT^L_1 \quad ,A)$. Consider



The top face is clearly commutative, the bottom face is commutative because α is a natural transformation, the two side faces are commutative since η_1 and η_2 are natural transformations. The back face is commutative as far as the element $e \in (BT^L_1 , BT^L_1)$ is concerned. Hence the front face is commutative as far as the element $x \in (BT^L_1 , A)$ is concerned.

That α is a natural transformation follows from the defining identity identity $(\alpha^L_B)\eta_2 = r_{1B} \circ \alpha_{BT^L_1}$ and the two equations in the proof of theorem 2

$$1) \quad (y)\eta = r_B \circ (yT_1)$$

$$2)\quad xr_B = (xT^L)\eta$$

plus one more derived from equation 2

$$3)\quad (y_1 \circ y_2)\eta = (y_1)\eta \circ (y_2 T)$$

We have established a duality between functors with left-adjoints from $\mathcal{A}$ to $\mathcal{B}$ and functors with right-adjoints from $\mathcal{B}$ to $\mathcal{A}$. Set theoretical difficulties prevent us from formalizing it as a functor.

<u>Theorem 7</u>. Let T_1, T_2, and T_3 be three functors from $\mathcal{A}$ to $\mathcal{B}$ with left adjoints T_1^L, T_2^L, and T_3^L respectively. Then $0 \to T_1 \xrightarrow{\alpha} T_2 \xrightarrow{\beta} T_3$ is exact iff $T_3^L \xrightarrow{\beta^L} T_2^L \xrightarrow{\alpha^L} T_1^L \to 0$ is exact.

<u>Proof</u>. The theorem follows quickly from the commutative diagram and Theorem 1.10

$$\begin{array}{ccccc}
0 \to (BT_1^L, A) & \xrightarrow{(\alpha^L, e)} & (BT_2^L, A) & \xrightarrow{(\beta^o, e)} & (BT_3^L, A) \\
\downarrow \eta_1 & & \downarrow \eta_2 & & \downarrow \eta_3 \\
MO \to (B, AT_1) & \xrightarrow[(e,\alpha)]{} & (B, AT_2) & \xrightarrow[(e,\beta)]{} & (B, AT_3)
\end{array}$$

<u>Section 5</u>. Characterization of functors naturally equivalent to a covariant hom functor.

<u>Theorem 8.</u> A group-valued functor $T: \mathcal{A} \to \mathcal{G}$ is naturally equivalent to H^C for some C iff it is left-root presrving and has a reflective image.

<u>Proof</u>. If $T: \mathcal{A} \to \mathcal{G}$ is left-root presrving and has a reflective image, we let T^L be its adjoint. Then for Z the group of integers in $\mathcal{G}$, we $C = ZT^L$ then [illegible].

$$AH^C = (ZT^L, A) = (Z, AT) = AT$$

Conversely, to show that H^A has a reflective image, we let $G \in \mathcal{G}$ and (A,B) an object in $(\mathcal{A})H^A$ generated by $G \xrightarrow{F} (A,B)$. Using the set of elements of G as an indexing set, we consider the map $\sum_G A \to B$ whose g'th coordinate is gF. Letting $B' \subset B$ be its image, we note that $G \xrightarrow{F} (A,B)$ factors through $(A,B') \to (A,B)$, hence $B' = B$, and the number of distinct objects in $(\mathcal{A})H^A$ that G can generate is not more than the number of images of $\sum_G A$, hence by the previous characterization of reflective subcategories, $(\mathcal{A})H^A$ is reflective.

<u>Corollary 9</u>. A group-valued contravariant functor is naturally equivalent to H_A for some A iff it carries right-roots into left-roots and has a reflective image.

The above theorems suggest the definition of $-\otimes A: \mathcal{G} \to \mathcal{A}$ and $\overline{\overline{H}}_A: \mathcal{G} \to \mathcal{A}$ as the adjoints of H^A and H_A. $G \otimes A$ is called the tensor product of G and A, and $G\overline{\overline{H}}_A$ the symbolic hom functor $\overline{(G,A)}$. The two are related through duality

$$G \otimes A = \overline{(G,A^*)}^*$$

$$\overline{(G,A)} = (G \otimes A^*)^*.$$

Since $-\otimes A$ is right-root preserving and $\overline{\overline{H}}_A$ is a contravariant functor that carries right-roots into left-roots, we can prove easily

<u>Theorem 10</u>. If $T: \mathcal{G} \to \mathcal{A}$ is right-root preserving then it is naturally equivalent to $-\otimes \mathbb{Z}T$

Theorem 11. If a contravariant functor $T: \mathcal{C} \to \mathcal{A}$ carries right-roots into left-roots then it is naturally equivalent to $\bar{H}_{ZT}$.

Hence this new definition of tensor product agree with the old, where defined. The many canonical natural transformations between compositions of hom, symbolic hom, and tensor products are all provable in the general case. Using section 4 we can easily show the bifunctor properties of tensor products and symbolic hom functors.

Section 6. Applications to functor Categories

Theorem 12. An evaluation functor $E_A: (\mathcal{A}, \mathcal{B}) \to \mathcal{B}$ where $A \in \mathcal{A}$ and $\mathcal{B}$ is complete, has a left-adjoint E_A^L

Proof. We pointed out in chapter two that E_A is root-preserving, hence by the characterization of functors with adjoints we need only prove that the image of E_A is reflective. We consider a map $B \xrightarrow{F} TE_A$. We shall construct its $(\mathcal{A})E_A$-image as follows. For $A_2 \in \mathcal{A}$, let $(A_2)I$ be the subobject of $(A_2)T$ generated by the images of all maps of the form $B \xrightarrow{F} AT \xrightarrow{(x)T} (A_2)T$ for $x \in (A,A_2)$, i.e. $(A_2)I$ is the image of the map $\Sigma_{(A,A_2)} B \to (A_2)T$, whose x'th coordinates is $F(x)T$. Then for any $A_1 \xrightarrow{y} A_2$ it follows that $(y)T$ carries $(A_1)I$ into $(A_2)I$, and I is a subfunctor of T. It is clear that $B \xrightarrow{F} AT = B \to AI \to AT$ and that $B \to AI$ is a generating map. Our construction of I shows that it is naturally equivalent to a functor whose image is contained in the subcategory $\mathcal{B}'$ consisting of a set of objects representing the images of $\sum_{(A,A_2)} B$ for all $A_2 \in \mathcal{A}$. $\mathcal{B}'$ is a small category and thus $(\mathcal{A}, \mathcal{B}')$ is small, contains a complete set of $(\mathcal{A}, \mathcal{B})E_A$ objects generated by B.

Note that E_A^L has functors as values. If $B \in \mathcal{B}$ then BE_A^L is a functor.

Theorem 13. If P is projective in $\mathcal{B}$ then PE_A^L is projective in $(\mathcal{A}, \mathcal{B})$ for any A.

Proof. We consider an epimorphism $T \to T''$ in $(\mathcal{A}, \mathcal{B})$. $(PE_A^L, T) \to (PE_A^L, T'')$ is naturally equivalent to $(P, AT) \to (P, AT'')$ which is epimorphic

Theorem 14. If $\mathcal{B}$ has a projective generator G then $(\mathcal{A}, \mathcal{B})$ has a projective generator namely $\sum_{A \in \mathcal{A}} (GE_A^L)$

Proof. By proposition 1.13 it suffices to prove that the projective functor $\sum_{A \in \mathcal{A}} (GE_A^L)$ has a non-zero image in every non-zero functor.

$$\left(\sum_{A \in \mathcal{A}} (GE_A^L), T\right) = \prod_{A \in \mathcal{A}} (GE_A^L, T) = \prod_{A \in \mathcal{A}} AT$$

the last is zero only if T is the zero functor.

The next theorem eliminates the necessity of the chapter two citation of Grothendieck's proof of the existence of injective resolutions.

Theorem 15. If $\mathcal{B}$ has as injective co-generator then so does $(\mathcal{A}, \mathcal{B})$

Proof. It is not necessary to prove directly that the evaluation functors have right adjoints. This theorem is seen to be the dual of the proceeding theorem by recalling that $(\mathcal{A}, \mathcal{B})$ is dual to $(\mathcal{A}^*, \mathcal{B}^*)$. It is possible to define E_A^R as the functor into $(\mathcal{A}, \mathcal{B})$ that corresponds to $E_{A^*}^L$ into $(\mathcal{A}^*, \mathcal{B}^*)$.

CHAPTER FOUR

Section 1. Representations

A category $\mathcal{A}$ is <u>represented</u> by a category $\mathcal{B}$ if there exists functors $T_1: \mathcal{A} \to \mathcal{B}$, $T_2: \mathcal{B} \to \mathcal{A}$ such that T_1T_2 and T_2T_1 are both naturally equivalent to the identity functors. If T_1T_2 and T_2T_1 are actually the identity functors we say that $\mathcal{A}$ is isomorphic to $\mathcal{B}$.

In the first case we say that T_1 is a representation and in the second, an isomorphism.

$\mathcal{A}'$ is a FULL SUBCATEGORY of $\mathcal{A}$ if all $\mathcal{A}$ maps between any two $\mathcal{A}'$-objects are in $\mathcal{A}'$. It is REPRESENTATIVE SUBCATEGORY if it is full and includes an object from every class of isomorphic objects in $\mathcal{A}$ It is easily verified that a full subcategory that contains a representative subcategory is a representative subcategory.

If $T_1: \mathcal{A} \to \mathcal{B}$ and $T_2: \mathcal{B} \to \mathcal{A}$ are representations T_1T_2 and T_2T_1 naturally equivalent to the identity functors — then both are embeddings and have representative subcategories as images. T_1 is an embedding since T_1T_2 is an embedding, T_1 has a full image since T_2T_1 does, and it contains a representative subcategory for the same reason. These two properties characterize representations:

<u>Theorem 1</u>. If $T: \mathcal{A} \to \mathcal{B}$ is an embedding and has a representative subcategory as an image then T is a representation.

<u>Proof</u>: It will suffice to prove that the inclusion map of a representative subcategory is a representation and that an onto embedding is a representation.

<u>Lemma 1</u>. The inclusion map of a representative subcategory is a

representation.

Proof of Lemma 1. Let $\mathcal{A}'$ be a representative subcategory of $\mathcal{A}$. For every $A \in \mathcal{A}$ we choose an isomorphic copy $AT \in \mathcal{A}'$ and an isomorphism $\eta_A \colon A \to AT$. If $A \in \mathcal{A}'$ we let $AT = A$ and η_A the identity map. For a map $A_1 \overset{\alpha}{\to} A_2$ we let $(\alpha)T$ be the unique map such that

$$\begin{array}{ccc} A_1 & \overset{\alpha}{\to} & A_2 \\ \eta_{A_1}\downarrow & & \downarrow\eta_{A_2} \\ (A_1)T & \underset{(\alpha)T}{\to} & (A_2)T \end{array}$$

commutes. Thus $T\colon \mathcal{A} \to \mathcal{A}' \subset \mathcal{A}$ is naturally equivalent by η to the identity functor, and $T|\mathcal{A}'$ (T restricted to $\mathcal{A}'$) is the identity functor.

Lemma 2. An onto embedding is a representation

Proof of Lemma 2. We let $T\colon \mathcal{B} \to \mathcal{A}$ be an onto embedding. For each $A \in \mathcal{A}$ we choose an inverse image $AT^* \in \mathcal{B}$, that is $AT^*T = A$. Given a $A_1 \overset{\alpha}{\to} A_2$ we let (α) be the map that corresponds to α under the natural isomorphism $(A_1T^*, A_2T^*) \to (A_1T^*T, A_2T^*T) = (A_1, A_2)$ known to be monomorphic since T is an embedding, and epimorphic since T is onto. That T^* commutes with composition, that is, is a functor, is a result of T being an embedding. T^*T is the identity functor on $\mathcal{A}$. To construct the naturally equivalence η from the identity functor on $\mathcal{B}$ to TT^* we let η_A be the map in (A, ATT^*) that corresponds under the natural isomorphism to the identity map in (AT, AT).

Note that if $\mathcal{A}$ is represented by $\mathcal{B}$ then they have isomorphic representative subcategories. Virtually all categorical properties of interest are preserved under representations. Since representations are root-preserving embeddings the properties discussed in the metatheorems of the chapter one are preserved, even with the finiteness conditions removed. Furthermore, since representations have full images, an almost unrestricted existential metatheorem could be proved for the theorems true in a category, and a category it represents.

Section 2. We shall prove that a complete exact category with a projective generator in which direct sums are naturally embedded in direct products is representable as a full category of modules over a ring with a superstructure, that is, a super-ring. The triviality of the super-structure, in cases of interest, will be found to be verifiable categorically.

Definition. A SUPER-STRUCTURE on an infinite ring R is a sub-bimodule Q of $\prod_I R$, where I is an indexing set with the same cardnality as R, together with a bimodule homomorphism $\Sigma: Q \to R$ satisfying the following axioms.

SR1 If $(r_i)_I \in \prod_I R$, and $r_i = 0$ for all but a finite number of $i \in I$, then $(r_i) \in Q$ and $\Sigma(r_i)$ is equal to the sums of the non-zero terms.

SR2 Commutativity. If $(r_i)_I \in Q$ then for any permutation π on I, $(r_{i\pi}) \in Q$ and $\Sigma(r_i) = \Sigma(r_{i\pi})$.

SR3 If $(r_i) \in Q$ and $(s_i) \in \prod R$, then $(r_i s_i) \in Q$.

The commutativity axioms allows us to define $\sum_J r_j$ as equal to

$\Sigma\,(r_{if})_I$ for f a onto-one correspondence from I to J. SR3 implies among other things that for $I' \subset I$ we can define $\sum_{I'}(r_i)$ as equal to $\Sigma(s_i)_I$ where $s_i = r_i$ for $i \in I'$ and $s_i = 0$ otherwise. Hence we can easily define $\sum_J$ for any set J whose cardnality is not greater than that of I.

SR4 ASSOCIATIVITY. If $\sum_{J \times K} r_{j,k}$ is defined then it is equal to $\sum_J [\sum_K r_{j,k}]$

SR5 If for every $j \in J, (r_{i,j}) \in Q$ and if $(s_i) \in Q$ then $\sum_{I \times J} s_i r_{i,j}$ is defined.

A SUPER-MODULE is a left R-module M, together with a left R-homomorphism $\Sigma_M\colon Q \otimes_R \prod_I M \to M$ (recall that Q is a bimodule), with the properties:

SM1 If $((r_i) \otimes (x_i))$ and $((r_i') \otimes (x_i))$ in $Q \otimes \Pi M$ are such that $r_i x_i = r_j' x_j'$ for $j = i\pi$ where π is a permutation of I, then Σ_M sends both into the same element in M.

SM2 If all x_i's are equal then $\Sigma_M((r_i) \otimes (x_i)) = [\Sigma(r_i)]\, x$.

In view of the above axioms we shall write $\Sigma\, r_i x_i$ for $\Sigma_M((r_i) \otimes (x_i))$ and if $y_i = r_i x_i$ all $i \in I$, then $\Sigma\, y_i$ for $\Sigma\, r_i x_i$. As before we define $\sum_J y_j$ for set J whose cardnality is not greater than I.

SM3 If $\sum_{J \times K} y_{j,k}$ is defined then it is equal to $\sum_J [\Sigma_K\, y_{i,j}]$

Finally, a super-homomorphism $(M' \xrightarrow{\alpha} M)$ is a homomorphism of left R-modules which yields a commutative diagram:

SH1

$$\begin{array}{ccc} Q \otimes \Pi M' & \xrightarrow{(e \otimes \Pi \alpha)} & Q \otimes \Pi M \\ \Sigma_{M'} \downarrow & & \downarrow \Sigma_M \\ M' & \xrightarrow[\alpha]{} & M \end{array}$$

The super-ring R is itself a super-module; and right multiplications by elements of R are all super-homomorphisms. In general, given a super-module M and an element $x \in M$, the map $R \to M$ which sends $r \in R$ into $rx \in M$, is easily seen to be a super-homomorphism. Thus the category of modules over a super-ring has a projective generator. The exactness of the category can be verified directly.

The super-structure on a product of supermodules is the obvious: an I-sequence of elements in ΠM_j is summable iff it is summable in each co-ordinate. Before examining direct sums we define $\sum_J x_j$ for sets larger than I. $\sum_J x_j$ is defined if there exists a one-to-one map $f: I \to J$ such that if an element j is not in its image then $x_j = 0$. $\sum_J x_j = \sum_I x_{if}$.

We now claim that the direct sum of a family of supermodules $\{M_j\}$ is the subobject of $\prod_J M_j$ generated by the images of the injection maps, that is, the elements which are zero everywhere except for one coordinate. Axioms SR5 tells us that ΣM_j would thus consist precisely of all elements which can be written in the form $\Sigma r_i x_i$ where $(r_i) \in Q$ and each x_i is an element zero everywhere except for one coordinate. Given a collection of maps $\{f_j: M_j \to X\}$ we define $f: \sum_J M_j \to X$ as the map which sends $\Sigma r_i x_i$ into $\Sigma r_i (x_i) f_j \in X$ where j is the index of the non-zero coordinate of x_i. Axiom SM3 is used in proving that f is a super-homomorphism.

SR4 and SR5 together prove that $\sum_I R$ is precisely Q, and that for larger J, $\sum_J R$ consists of all elements in $\sum_J R$ which are zero except on a set of coordinates of cardnality I, and such that if the zero co-ordinates were "thrown away" an element of Q would be left.

We can show, then, that the category of super-modules over a super-ring is an exact complete category with a projective generator and such that direct sums are naturally embedded in direct products.

There is a natural super-structure on a product of super-rings: a sequence is summable if it is summable in each coordinate. The product of an infinite number of super-rings, even with trivial super-structures, always has a non-trivial super-structure. (A super-structure will be called trivial if it yields an ordinary ring.) Given a family of super-rings R_i, it can be verified that the product of their categories (of Modules) is isomorphic to the category of modules over their product with the natural super-structure.

<u>Section 3</u>. Characterization of Categories representable as the category of modules over a super-ring.

<u>Theorem 2</u>. A category is representable as the category of super-modules over a super-ring iff

1) It is exact and complete.

2) Direct sums are naturally embedded in direct products.

3) It has a projective generator.

<u>Proof</u>. The necessity is obvious by the previous discussion. Let $\mathcal{A}$ satisfy the three conditions, $P \in \mathcal{A}$ a projective generator. We

define R to be ring of endomorphisms of P and observe that for $A \in \mathcal{A}$ (P,A) can be made into a left R-module in a natural way. Given $A \xrightarrow{\alpha} B$ the induced map $(P,A) \xrightarrow{(P,\alpha)} (P,B)$ id a R-homomorphism.

Assuming R is infinite we define the super-structure on R by letting $Q = (P, \sum_I P)$, and Σ: $Q \to R$ the map induced by $\sum_I P \xrightarrow{\sigma} P$ all of whose coordinates are the identity maps. Axioms SR1 follows quickly, SR2 from the fact that $\sum_I R \xrightarrow{\pi} \sum_I R \xrightarrow{\sigma} R = \sum_I R \xrightarrow{\sigma} R$ for π any permutation of the coordinates. SR3 follows from the fact that $P \xrightarrow{(r_i)} \sum_I P \xrightarrow{(s_i)} \sum_I P = P \xrightarrow{(r_i s_i)} \sum_I P$ for any I-sequence (s_i). SR4 and SR5 are straightforward, applications of the associativity of direct sums.

We make (P,A) into a super-module by defining $\Sigma_M((r_i) \otimes (x_i))$ as $P \xrightarrow{(r_i)} \sum_I P \xrightarrow{(x_i)} A$. i.e. the composition map from $(P, \sum_I P) \otimes (P,A)$ to (P,A). We can prove SM1 and SM2 by noticing that $\Sigma_M((r_i) \otimes (x_i))$ is equivalently defined as $P \xrightarrow{(r_i)} \sum_I P \xrightarrow{(x_i)} \sum_I A \xrightarrow{\sigma} A$. SM3 follows from the associativity of direct sums.

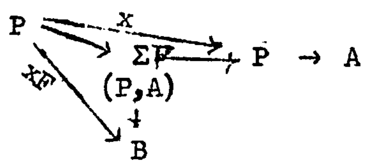
That the induced map $(P,A) \xrightarrow{(P,\alpha)} (P,B)$ is a super-homomorphism follows from the fact that $\sum R \xrightarrow{(x_i)} A \xrightarrow{\alpha} B = \sum_I R \xrightarrow{(x_i\alpha)} B$.

Hence we define $\bar{H}^P$: $\mathcal{A} \to \mathcal{S}^R$, the category of super-modules over the super-ring R, as the functor that sends $A \in \mathcal{A}$ into the super-module (P,A). To prove that $\bar{H}^P$ is a representation we show first

<u>Lemma 1</u> $\bar{H}^P$ has a full image

<u>Proof of Lemma 1</u>. Let F: $(P,A) \to (P,B)$ be a super-homomorphism. We let $\sum_{(P,A)} P \to A$ be the map whose x'th coordinate is the map $P \xrightarrow{x} A$.

$\sum_{(P,A)} P \to B$ the map whose x'th coordinate is $(x)F$. Hence for any $x \in (P,A)$ we have the commutative diagram



where $P \to \sum_{(P,A)} P$ is the x'th injection map

The horizontal map is onto since P is a generator. We let $K \to \sum_{(P,A)} P$ be its kernal. If we can prove that $K \to \sum_{(P,A)} P \to B$ is zero, then we know that there is a map from A to B which makes the above diagram commutative, and hence, which induces the map F. For that purpose, we enlarge the indexing set (P,A) so that it is at least as large as I; the extra coordinate maps will all be zero.

If $K \to \sum_{(P,A)} P \to B$ were not zero, then since P is a generator, there exists a map $P \to \sum_{(P,A)} P$ which when followed by the map to B would be non-zero and when followed by the map to A would be zero. The coordinates of the map $P \to \sum_{(P,A)} P$ must be zero except on a set of cardnality I. Otherwise there would be an infinite set of coordinates all of which would be the same set of non-zero maps. This latter situation quickly leads to a contradiction.

Hence there is a one-to-one function $g: I \to (P,A)$ such that $P \to \sum_{(P,A)} P$ can be factored through $\sum_I P \to \sum_{(P,A)} P$ and we now have a map $(P \to \sum_I^I P) \in Q$ which when followed by $\sum_I P \xrightarrow{(x_i)} A$ is zero and when followed by $\sum_I P \xrightarrow{(x)F} B$ is non-zero $x_i = (i)g$. But this contradicts the assumption that F is a super-homomorphism.

Barry Mitchell has informed the writer that the above proof correctly modified will work with-out assuming F to be a super-homomorphism if $\mathcal{A}$ satisfies Grothendieck's axioms AB5.

<u>Lemma 2.</u> The image of $\overline{\overline{H}}^P$ includes an isomorphic copy any super-module over R.

<u>Proof of Lemma 2.</u> Given a super-module M we let $\sum_J R \to \sum_K R \to M \to 0$ be an exact sequence, J and K at least as large as I. The previous discussion of free modules over super-rings, gives the fact that $\sum_J R \cong (P, \sum_J P)$. The fullness and right-exactness of $\overline{\overline{H}}^P$ suffice to construct a copy of M.

Recalling that $\overline{\overline{H}}^P$ is an embedding, the two lemmas prove that it is a representation.

Section 4. Applications

<u>Definition:</u> An object A in a complete category will be said to be ABSTRACTLY FINITE if every map $A \to \sum_I A$ factors through a finite subsum: i.e. there exists a finite $I' \subset I$ and a map $A \to \sum_{I'} A$ such that

$$A \to \sum_I A = A \to \sum_{I'} A \to \sum_I A \,.$$

<u>Theorem 3.</u> A category is representable as a category of modules over a ring iff it is

(1) exact and complete

(2) has an abstractly finite projective generator

Proof: The necessity of the two conditions is clear. In the preceeding theorem the only use of the fact that direct sums were naturally embedded in direct products was for the case $\Sigma_I P \to \Pi_I P$ where P is the chosen projective generator. If P is abstractly finite, the fact follows quickly. Hence a complete exact category $\mathcal{A}$ with an abstractly finite projective generator P, is representable as the category of super-modules over End(P) . But the condition of abstract finiteness insures a trivial super-structure on End(P) , and in fact is clearly equivalent with the triviality of the super-structure.

Theorem 4. Any representation from a category $\mathcal{A}$ to a category of super-modules is naturally equivalent to a functor of the form $\bar{H}^P$ as described above. The proof follows quickly from the fact that a representation is right root preserving. P is defined to be such that its image is isomorphic to the super-ring.

Corollary 5. The category of super-modules over R_1 is representable as the category of super-modules over R_2 iff R_2 appears as the super-ring of endomorphisms of a projective generator over R_1 .

Corollary 6. If Two rings produce equivalent categories, they have isomorphic centers. In particular different commutative rings produce different categories.

Theorem: 7 A projective generator over a ring R has trivial super-structure in its ring of endomorphisms iff it is finitely generated.

Proof: If P is an image of R^n the condition of theorem is easily verified. Conversely if the ring of endomorphisms of P has trivial super-structure we consider the set J of endomorphisms $P \xrightarrow{\alpha} P$ which can be factored through R $P \xrightarrow{\alpha} P = P \to R \to P$ for suitable maps $P \to R$

and $R \to P$. Then $\sum_J P \to P$ is onto since P and R are generators. Since P is projective there exists a map $P \to \sum_J P$ which when followed by $\sum_J P \to P$ is the identity. Since the ring of endomorphisms of P has trivial super-structure, $P \to \sum_J P$ can be factored through a finite subsum.

Corollary 8. The category of modules over a matrix ring can be represented as the category of modules over the ground ring.

Corrolary 9. The global dimension of a matrix ring is equal to the global dimension of the ground ring. Mitchell has extended these techniques and has obtained results for triangular matrices and other modifications of the matrix rings [7].

Theorem 10. If $\mathcal{B}$ is a category of modules over a super-ring, then the category of n-variable functors from $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ to $\mathcal{B}$ is represented by a category of modules over a super-ring.

Proof: We proved that $(\mathcal{A}_i, \mathcal{B})$ has a projective generator. That $(\mathcal{A}_i, \mathcal{B})$ inherits the desired properties of limits is verified through a pointwise examination. The corollary can be proved inductively by noticing that the bifunctors from $\mathcal{A}_1, \mathcal{A}_2$ to $\mathcal{B}$ are representable as the functors from $\mathcal{A}_1$ to $(\mathcal{A}_2, \mathcal{B})$.

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