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BIALGEBRAS IN LOCALLY PRESENTABLE CATEGORIES

by

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Preface

This paper constitutes a second part to "Lokal präsentierbare Kategorien" by P.Gabriel and the author (Lecture Notes vol. 221). The reader need only be familiar with the basic facts about locally presentable categories. The relevant material is collected in § 2.

The starting point was marked by unsolved problems in the first part. They were successfully tackled step by step in [31], [32], [33] and [34]. In the process the notion of a bialgebra - generalizing bialgebras over a commutative ring - emerged as a unifying concept. We give here a systematic treatment of bialgebras in locally presentable categories and then apply the results to the above mentioned problems and to problems in other areas as well, in particular to Hopfalgebras, bialgebras, coalgebras over a commutative ring and to descent data,etc.

This material was first presented in seminars of H.Schubert in the summer semesters 1975 and 1976. I profited a great deal from the lively discussions with the participants. I also would like to thank A.Kock and W.Wischnewsky for discussions in Amiens about Hopfalgebras, bialgebras, coalgebras, comodules etc. over a commutative ring. Without their disbelieve I would not have tried to prove that these categories are locally presentable. I am indebted to M.Barr and T.Fox for discussions later on in Zurich about problems associated with Props. This put me on the track to look for something better (namely bialgebras). Finally M.Tierney raised at the AMS meeting in Toronto the question of the relationship between bialgebras and sections (resp. cartesian closed sections) of a fibration. This turned out to be very fruitful.

I am particularly indebted to Christa Becker and Heidi Paulus for typing a difficult manuscript and to Lothar Schumacher for proof reading. I am grateful to the vice chancellor of the University of

Wuppertal Walter Liesenhoff for providing secretarial help and I am less grateful to the chancellor Klaus Peters for taking it away when it was mot needed. In order to get the whole manuscript typed I had to extend the sectretary's contract on my own. The chancellor prevented the mathematics department from doing so with one of its vacant secretarial positions. His ingenuity in making life difficult at a university with plenty of jobs and funds is only exceeded by his endurance in doing so.

CONTENTS

Introduction

The methods developped in "Lokal präsentierbare Kategorien" (L.N. vol. 221) were not sufficient to decide whether any of the following categories were locally presentable: the category of functors on a small category $\underline{U}$ with values in a locally presentable category which preserve a given class of colimits in $\underline{U}$, the category of cosheaves on a site with values in a locally presentable category, the categories of coalgebras, bialgebras, Hopfalgebras ... over a commutative ring Λ and likewise the category of comodules (resp. bimodules) over a Λ-coalgebra (resp. Λ-bialgebra), the category $\underline{A}_{\mathbb{G}}$ of $\mathbb{G}$-coalgebras, where $\mathbb{G}$ is a cotriple with rank in a locally presentable category $\underline{A}$, the category $\mathrm{Adj}(\underline{A},\underline{B})$ of adjoint functors between two locally presentable categories $\underline{A}$ and $\underline{B}$, etc. These questions were solved affirmatively in [31], [32], [33] and [34] by new techniques. In the process the notion of a bialgebra in a category - generalizing the notion of a bialgebra over a commutative ring - emerged as the unifying concept from the point of view of the constructions on which the proofs were based. The basic problem in all cases involved the construction of generators in the category under consideration which in turn lead to the following general question: <u>Given an object A in a category $\underline{A}$ equipped with a structure $\mathcal{H}$ and given a subobject U of A in $\underline{A}$. How does one construct a subobject U' with structure $\mathcal{H}'$ containing U such that the inclusion $U' \hookrightarrow A$ is compatible with the structures $\mathcal{H}'$ and $\mathcal{H}$ and such that U' is not much bigger than U?</u> The complexity of this problem is perhaps best illustrated by two seemingly unrelated examples: Given a Hopfalgebra H over a commutative ring Λ and a Λ-submodule U of H. Find a sub-Hopfalgebra U' of H containing U such that the underlying Λ-module of U' is not much bigger than U; or more specifically, that the size of U' depends only on U but not on H. Clearly U' is - if it exists - not unique because there is no such thing as "the" sub-Hopfalgebra "generated" by U. (For coalgebras over a commutative ring the corresponding problem was investigated by M.Barr [1] using purity.) On the other hand consider an object A equipped with a descent datum φ_A and a subobject U of A. Find a subobject U' containing U and a descent datum $\varphi_{U'}$ on U' such that $\varphi_{U'}$ is compatible with φ_A and U' is not much bigger than U. A construction of $(U',\varphi_{U'})$ was given by Grothendieck and Verdier in SGA 4 (p. 138-179) in a more general context. But the proof has a gap and their size estimate of U' is false.

Our main results consist in 1) making precise what an object with structure is - this is done by the notion of a bialgebra in a category 2) solving the above mentioned problem for bialgebras in locally presentable categories under appropriate conditions and 3) establishing size estimates for the constructed sub-bialgebras which in most cases are the best possible (cf. 3.1, 3.8, 3.22). With the exception of §5 all our results in §3 - §6 are applications of this.

Roughly speaking a bialgebra in a category $\underline{A}$ with respect to a given set M of operations and a set R of relations consists of an object $A \in \underline{A}$ together with a structure morphism μ_A for every $\mu \in M$ and a functorial diagramm for every $r \in R$ which commutes. In the literature so far a structure morphism μ_A on an object A is a morphism like $A \times A \longrightarrow A$, $A \longrightarrow A \amalg A$, $A \otimes A \longrightarrow A$, $A \longrightarrow A$, $A \otimes A \longrightarrow A \otimes A$, etc. In contrast we allow it to be a morphism $FA \longrightarrow F'A$, where F and F' is any pair of functors with domain $\underline{A}$ and a common codomain (the latter can depend on μ). F is called the domain of μ and F' the codomain. Likewise a relation is normally given by diagrams such as $A \times \ldots \times A \rightrightarrows A$, $A \rightrightarrows A \amalg \ldots \amalg A$, $A \otimes \ldots \otimes A \rightrightarrows A$, $A \rightrightarrows A \otimes \ldots \otimes A$, $A \rightrightarrows A$, $A \otimes \ldots \otimes A \rightrightarrows A \otimes \ldots \otimes A$, etc. which are built up of structure morphisms $\mu_A, \mu \in M$, and canonical morphisms (like twisting, etc.)

The relevant aspect here is that these diagrams are natural with respect to those morphisms $A \longrightarrow A'$ in $\underline{A}$ which are compatible with the given operations. Therefore we define a relation r to be a map which assigns to every object A equipped with structure morphisms μ_A, $\mu \in M$, a diagram $GA \rightrightarrows G'A$ which is natural in the sense just mentioned and where G and G' is any pair of functors with domain $\underline{A}$ and common codomain (the latter can depend on r). An object equipped with structure morphisms is said to satisfy the relation r if the corresponding diagram commutes. In this way one obtains the category $\mathrm{Bialg}(\underline{A})$ of bialgebras in $\underline{A}$ with respect to specified operations M and relations R. The morphisms in $\mathrm{Bialg}(\underline{A})$ are those morphisms in $\underline{A}$ which are compatible with the operations in M.

The notion of a bialgebra covers a wide range of examples, e.g. universal algebras resp. coalgebras in a category with finite products resp. coproducts in the sense of Lawvere [21] or Birkhoff [2], coalgebras over an arbitrary commutative ring Λ and likewise Λ-Hopfalgebras resp. Λ-bialgebras in the usual sense (more generally tensor product preserving functors on a Prop in the sense of MacLane [24]), comodules over a Λ-coalgebra, bimodules over a Λ-bialgebra, algebras over a triple, coalgebras over a cotriple, données de recollements, descent data and more generally sections (resp. cartesian closed sections) with respect to a fibration, functors on a small category which preserve a given class of limits resp. colimits, sheaves and cosheaves on a site, pairs of adjoint functors between locally presentable categories and more generally Σ-continuous resp. Σ-cocontinuous functors on a small category $\underline{U}$ with respect to an arbitrary class Σ of morphisms in the set valued functor category $[\underline{U}^o, \underline{\mathrm{Sets}}]$, and finally Σ-closed objects in a category $\underline{A}$ with respect to a bifunctor $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ and a class Σ of morphisms in $\underline{B}$ (Recall that $A \in \underline{A}$ is called Σ-closed with respect to T if

$T(\sigma,A)$ is an isomorphism for every $\sigma \in \Sigma$).

A bialgebra in a category $\underline{A}$ is denoted with (A,M,R), where $A \in \underline{A}$ is the underlying object and M and R refer to the specified operations and relations. Given a subobject U of the underlying object A we are concerned with the construction of sub-bialgebras (U',M,R) of (A,M,R) containing U such that U' is "as small as possible", in particular the construction should provide effective size estimates for U' in terms of U, M and R (but not A). More generally given a bialgebra (A,M,R) and an object U we investigate factorizations of a morphism $f : U \longrightarrow A$ into a morphism $U \to U'$ and a bialgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that U' is not much bigger than U and its size can be estimated in terms of U, M and R. It is obvious that without conditions on M, R and on the underlying category $\underline{A}$ no reasonable answers can be expected. In order to elaborate on these conditions we recall a few basic facts about locally α-presentable categories.

Let $\alpha \geq \aleph_0$ be a regular cardinal. A directed set is called α-filtered if every subset with less than α elements has an upper bound. A functor F is said to preserve α-filtered colimits if the domain of F has colimits over α-directed sets and F preserves them. An object A in a category $\underline{A}$ is called α-presentable (resp. α-generated) if the hom functor $[A,-]: \underline{A} \longrightarrow \underline{\text{Sets}}$ preserves α-filtered colimits (resp. preserves those α-filtered colimits whose transition morphisms are monomorphic). For instance, if $\underline{A}$ is the category of groups, rings, modules over a ring, etc., then $A \in \underline{A}$ is α-presentable (resp. α-generated) iff A admits a presentation in the usual sense by <u>less</u> than α generators and <u>less</u> than α relations (resp. <u>less</u> than α generators). In particular $\aleph_0$-presentable (resp. $\aleph_0$-generated) is equivalent with finitely presentable (resp. finitely generated) and likewise $\aleph_1$-presentable (resp. $\aleph_1$-generated) with countably presentable (resp.

countably generated). A category $\underline{A}$ is called locally α-presentable if it has colimits (i.e. sums and cokernels) and a set M of α-presentable generators. (It is called locally presentable if it is locally α-presentable for some α.) In a locally α-presentable category $\underline{A}$ every object is α'-presentable for some regular cardinal α' and, roughly speaking, for $\beta \geq \alpha$ an object $A \in \underline{A}$ is β-presentable iff it is the cokernel of two morphisms $\coprod_{j \in J} U_j \rightrightarrows \coprod_{i \in I} U_i$, where $U_i, U_j \in M$ and J and I have less than β elements. Moreover $\underline{A}$ has limits (= inverse limits), is cowellpowered and α-filtered direct limits commute with kernels and products with less than α factors. Also a functor F between locally presentable categories preserves γ-filtered colimits for some γ provided it has either a left or right adjoint. The class of locally presentable categories is larger than one might expect and includes the categories of sets, groups, rings, modules and more generally universal algebras, the category of set (group, ring ...) valued sheaves on a small category with respect to a Grothendieck topology, the category of set (group, ring ...) valued functors on a small category $\underline{U}$ which preserve a given set of limits in $\underline{U}$ (e.g. the category $\underline{\text{Cat}}$ of small categories and other "universal algebras" with partial operations), the dual category $\underline{\text{Comp}}^o$ of compact spaces, etc. In contrast the categories $\underline{\text{Comp}}$ and $\underline{\text{Top}}$ of (compact) topological spaces and other related categories are not locally presentable.

For the above mentioned construction of sub-bialgebras of a bialgebra containing a given subobject (resp. the decomposition of a morphism into a morphism and a bialgebra morphism) we need the following.

1) the underlying category $\underline{A}$ and the categories occuring in the definition of the operations and relations are locally presentable (or more generally "catégories localisables" in the sense of Y.Diers [5]).

2. the operations M and relations R form a set and the functors which are domain or codomain of either an operation or a relation

preserve β-filtered colimits for some cardinal β.

Then there are cardinals γ such that a bialgebra (X,M,R) is γ-presentable in $\mathrm{Bialg}(\underline{A})$ iff its underlying object X is γ-presentable in $\underline{A}$ (cf. 3.8). Moreover for a bialgebra (A,M,R) and a γ-presentable object $U \in \underline{A}$ every morphism $f : U \longrightarrow A$ admits a decomposition into a morphism $U \rightarrow U'$ and a bialgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that U' is again γ-presentable (cf. 3.8). The class of all such γ's is cofinal in the class of all cardinals. Of special interest is the smallest possible γ. Estimates are given in terms of $\underline{A}$, M and R. (The analogue assertions concerning the existence and size estimates of sub-bialgebras containing a given subobject are discussed later on.) We illustrate the above with some examples.

a) For Hopfalgebras over a commutative ring Λ one can choose for γ any cardinal $\geq \aleph_1$ (cf. 4.4). In particular every Λ-homomorphism $U \longrightarrow H$ from a countably presentable Λ-module U to an arbitrary Λ-Hopfalgebra H admits a decomposition into a Λ-homomorphism $U \rightarrow U'$ and a Hopfalgebra morphism $U' \longrightarrow H$ such that the underlying Λ-module of U' is again countably presentable (the corresponding assertion for finitely presentable Λ-modules is obviously false). Moreover the Λ-Hopfalgebras whose underlying Λ-module is countably presentable form a "set" of dense generators in the category of Λ-Hopfalgebras (i.e. the equivalence classes of such Hopfalgebras form a set).

The same holds for Λ-bialgebras, Λ-coalgebras etc. (cf. 4.3-4.7). Moreover the following categories are locally $\aleph_1$-presentable: commutative Λ-Hopfalgebras, cocommutative Λ-Hopfalgebras, bicommutative Λ-Hopfalgebras, Λ-bialgebras, commutative Λ-bialgebras, cocommutative Λ-bialgebras, bicommutative Λ-bialgebras, Λ-coalgebras, cocommutative Λ-coalgebras, comodules over a Λ-coalgebra, bimodules over a Λ-bialgebra, etc (cf. 4.3-4.9).

b) Let $\mathfrak{F}$ be a fibration with base $\underline{C}$ and let $\alpha : S_0 \longrightarrow S$ be a morphism in $\underline{C}$ such that 1) the fibres $\underline{\mathfrak{F}}_{S_0}$ and $\underline{\mathfrak{F}}_S$ over S_0 and S are locally countably presentable categories and 2) the inverse image functors α^*, p_1^*, p_2^* and p_{31}^* preserve filtered colimits and take countably presentable objects into countably presentable objects (cf. Grothendieck [16], also for the notation). Then for an object $A \in \underline{\mathfrak{F}}_{S_0}$ with descent datum φ_A and a countably presentable object $U \in \underline{\mathfrak{F}}_{S_0}$ every morphism $f : U \longrightarrow A$ admits a factorization into a morphism $U \longrightarrow U'$ and a morphism $(U', \varphi_{U'}) \longrightarrow (A, \varphi_A)$ between descent data such that U' is again countably presentable (cf. 4.14, 4.15). As a consequence the category $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_0})$ of descent data is locally $\aleph_1$-presentable and the forgetful functor $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_0}) \longrightarrow \underline{\mathfrak{F}}_S$ cotripleable provided the inverse image functors preserve colimits (cf. 4.15). Likewise $\mathrm{Desc}(\underline{\mathfrak{F}}_{S_0})$ is a Grothendieck category (resp. a topos) provided the fibres are, and the inverse image functors preserve colimits and finite limits (4.16).

If $\alpha : S_0 \longrightarrow S$ is of $\mathfrak{F}$-descent type (cf. Grothendieck [17] 1.7), then the above implies that every descent datum on objects of $\underline{\mathfrak{F}}_{S_0}$ is effective providdd every descent datum on countably presentable objects is effective (cf. 4.18).

Similar assertions hold for sections and cartesian closed sections with respect to a fibration (cf. 4.19-4.26).

c) Let $\mathbb{G} = (G, \varepsilon, \delta)$ be a cotriple in a locally α-presentable category $\underline{A}$ and assume that $G : \underline{A} \longrightarrow \underline{A}$ preserves β-filtered colimits for some β. Let $\gamma \geq \sup(\aleph_1, \alpha, \beta)$. Then for a $\mathbb{G}$-coalgebra (A, ξ) and a γ-presentable object $U \in \underline{A}$ every morphism $f : U \longrightarrow A$ admits a factorization into a morphism $U \longrightarrow U'$ and a $\mathbb{G}$-coalgebra morphism $(U', \xi') \longrightarrow (A, \xi)$ such that $U' \in \underline{A}$ is again γ-presentable (cf. 4.10). This implies that the category $\underline{A}_{\mathbb{G}}$ of $\mathbb{G}$-coalgebras is locally $\sup(\aleph_1, \alpha, \beta)$-presentable and that a $\mathbb{G}$-coalgebra is γ-presentable iff

its underlying object is. Moreover $\underline{A}_G$ is a topos (resp. a Grothendieck category) provided $\underline{A}$ is and $G : \underline{A} \longrightarrow \underline{A}$ preserves finite limits (cf. 4.11).
Applications of this are given for comodules over a Λ-coalgebra (cf. 4.8) and for bimodules over a Λ-bialgebra (cf. 4.9).

d) Let $\underline{U}$ be a small category and Σ a set of morphisms in $[\underline{U}^o, \underline{Sets}]$. Let $\underline{X}$ be a locally α-presentable category and let $Cc_\Sigma[\underline{U},\underline{X}]$ be the category of all Σ-cocontinuous functors. For instance if Σ is given by a set K of colimits in $\underline{U}$ (resp. by a Grothendieck topology τ) then the Σ-cocontinuous functors $\underline{U} \longrightarrow \underline{X}$ are exactly the K-colimit preserving functors on $\underline{U}$ (resp. the τ-cosheaves on $\underline{U}$). Let γ be any regular cardinal such that $\alpha \leq \gamma \geq \aleph_1$ and $\gamma > \operatorname{card}(\Sigma)$, $\gamma > \operatorname{card}(d\sigma(U))$, $\gamma > \operatorname{card}(r\sigma(u))$ for every $\sigma \in \Sigma$ and $U \in \underline{U}$, where $d\sigma$ and $r\sigma$ denote the domain and codomain of σ. Then for a Σ-cocontinuous functor $t : \underline{U} \longrightarrow \underline{X}$ and a γ-presentable functor $s \in [\underline{U},\underline{X}]$ every natural transformation $s \longrightarrow t$ admits a decomposition $s \longrightarrow s' \longrightarrow t$ such that $s' : \underline{U} \longrightarrow \underline{X}$ is Σ-cocontinuous and again γ-presentable in $[\underline{U},\underline{X}]$. This implies that the category $Cc_\Sigma[\underline{U},\underline{X}]$ of Σ-cocontinuous functors is locally γ-presentable and that the inclusion $Cc_\Sigma[\underline{U},\underline{X}] \xrightarrow{\subset} [\underline{U},\underline{X}]$ has a right adjoint. The latter has been a long outstanding problem in category theory.
The above can be generalized to a class Σ of morphisms whose codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set (modulo equivalence). Therefore we can also consider functors, which preserve a given class of colimits in $\underline{U}$ (in particular one can choose all existing colimits in $\underline{U}$). The above size estimates for γ however have to be replaced by more elaborate ones. The apparatus needed for the generalization to a class Σ is substantial (the entire chapter § 5 concerning purity and a good deal of § 6). Further generalizations concern the replacement of $\underline{X}$ by a topological category over $\underline{X}$ (cf. 6.21).

e) The category $\mathrm{Adj}(\underline{A},\underline{B})$ of adjoint fuctors between locally presentable categories $\underline{A}$ and $\underline{B}$ can be shown to be equivalent with the category of Σ-cocontinuous functors $\underline{U} \to \underline{B}$ for an appropriate small category $\underline{U}$ and a set Σ of morphisms in $[\underline{U}^o, \underline{Sets}]$ (cf.6.19). Thus by d) above $\mathrm{Adj}(\underline{A},\underline{B})$ is again locally presentable. In contrast if $\underline{A}$ and $\underline{B}$ are Grothendieck categories (or topoi), then $\mathrm{Adj}(\underline{A},\underline{B})$ need not be so. A surprising counter example is the following. Let $\underline{A}$ be the category of abelian p-groups for some prime p and $\underline{B} = \underline{Ab}.\underline{Gr}.$ the category of all abelian groups. Then $\mathrm{Adj}(\underline{A},\underline{B})$ can be shown to be equivalent with the category of p-adic complete abelian groups (cf. 6.25 c)).

f) Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor between locally presentable categories and let Σ be a set of morphisms in $\underline{B}$. Let $\underline{A}_{\Sigma,T}$ be the full subcategory of $\underline{A}$ consisting of all $X \in \underline{A}$ such that $T(\sigma,X)$ is an isomorphism for every $\sigma \in \Sigma$. For example T can be $\otimes_\Lambda$, $\mathrm{Tor}_n^\Lambda(-,-)$, $[-,-]$, $\mathrm{Ext}_\Lambda^n(-,-)$ etc. and Σ the inclusion of a set $\mathfrak{F}$ of right ideals in the ring Λ. Assume that for every $B \in \underline{B}$ there is a cardinal β_B such $T(B,-)$ preserve β_B-filtered colimits (which is obviously the case for the above examples). Then there are cardinals γ such that every morphism $f : U \to A$ with $A \in \underline{A}_{\Sigma,T}$ and U γ-presentable in $\underline{A}$ admits a decomposition $U \to U' \to A$ with $U' \in \underline{A}_{\Sigma,T}$ and U' being again γ-presentable in $\underline{A}$ (cf. 6.2). For instance if T is as above, $\mathfrak{F}$ is countable and the ideals $I \in \mathfrak{F}$ countably presentable, then one can choose for γ any cardinal $\geq \aleph_1$.

If $T = \otimes_\Lambda$, then $\underline{A}_{\Sigma,\otimes_\Lambda}$ consists of modules which are uniquely divisible by the ideals of $\mathfrak{F}$. For instance, let $\underline{A}$ and $\underline{B}$ be Grothendieck categories and $U \in \underline{A}$ a generator with endomorphism ring Λ. Then the category $\mathrm{Adj}(\underline{A},\underline{B})$ of adjoint functors between $\underline{A}$ and $\underline{B}$ is equivalent with the full subcategory of ${}_\Lambda\underline{B}$ consisting of those left Λ-objects which are uniquely divisible by the Gabriel filter $\mathfrak{F}$ in Λ associated with $\underline{A}$ (cf. 6.25 b)).

We now return to the problem of constructing sub-bialgebras (U',M,R) of a bialgebra (A,M,R) which contain a given subobject $U \subset A$ such that U' is not much bigger than U . This can be done under the conditions as above (cf. 1) and 2)) but the size estimates for U' are different, in general less effective. They are best stated in terms of noetherian conditions. The details are too involved to be given here (cf. 3.22, 3.23). and we illustrate them with an example. Let Λ be a commutative noetherian ring (or more generally a $\aleph_1$-noetherian ring which means that every countably generated ideal is countably presentable). Then every countably generated Λ-submodule of a Λ-Hopfalgebra is contained in a sub-Hopfalgebra whose underlying Λ-module is again countably generated. The same holds for Λ-bialgebras, Λ-coalgebras etc. If Λ is not $\aleph_1$-noetherian this need not be so. However there is always a cardinal γ such that Λ is γ-noetherian (i.e. every γ-generated ideal is γ-presentable). Then the above holds for γ-generated Λ-submodules of Λ-Hopfalgebras, etc. The same phenomenon happens for locally presentable categories. By Gabriel-Ulmer [13], 13.3 a locally α-presentable category is locally γ-noetherian for some $\gamma \geq \alpha$ The increase of γ over α accounts for the less effective size estimates for the constructed sub-bialgebras.

The basic idea for the construction of sub-bialgebras I got in a seminar of the University of Zurich 1974/75 in which Kaplansky's decomposition of projective modules into a direct sum of countably generated projective modules was studied (among other things). The parallel may be still apparent in § 1 in which an "elementwise" exposition of the basic techniques is given. The incentive to study sub-bialgebras "generated" by a subobject resulted from a problem which was given to us (= a group of students) in Heidelberg in 1964 by A.Dold. He suggested to investigate the category of cocontinuous abelian group valued functors on a Grothendieck category $\underline{A}$ in terms of a generator $U \in \underline{A}$ and its endomorphism ring Λ. We didn't get anywhere with it at the time but I kept it in the back of my mind and worked on it from time to time without much success. The turning point was the discovery that in the special case $\underline{A}$ = abelian p-groups the category of cocontinuous functor is equivalent with the category of p-adic complete abelian groups.

§ 2 Review of locally presentable categories

2.1 Definition ([13] 5.1) Let $\alpha \geq \aleph_0$ be a regular cardinal. A small category $\underline{D}$ is called α-filtered if

a) for every family $(D_\iota)_{\iota \in I}$ of objects in $\underline{D}$ with card(I) $< \alpha$ there exists an object $D \in \underline{D}$ together with a morphism $D_\iota \longrightarrow D$ for every $\iota \in I$.

b) for every family $(D_0 \xrightarrow{\gamma_\iota} D_1)_{\iota \in I}$ of morphisms in $\underline{D}$ with card(I) $< \alpha$ there exists a morphism $\gamma : D_1 \longrightarrow D_2$ such that $\gamma \circ \gamma_\iota = \gamma \circ \gamma_\varkappa$ for every pair $\iota, \varkappa \in I$.

For $\alpha = \aleph_0$ this specializes to the usual definition of filtered colimits (resp. direct limits).

A functor $F : \underline{A} \to \underline{B}$ is said to preserve α-filtered colimits if it preserves colimits over α-filtered categories. The least regular cardinal α with this property is called the presentation rank of F and denoted by $\pi(F)$. Examples are functors $F : \underline{A} \longrightarrow \underline{B}$ which have a right adjoint or - somewhat surprisingly - functors $F : \underline{A} \longrightarrow \underline{B}$ between locally presentable categories (2.3) which have a left adjoint, in particular underlying or forgetful functors (cf. 2.9, 3.4 c)). Likewise a functor $F : \underline{A} \longrightarrow \underline{B}$ is said to preserve monomorphic α-filtered colimits if it preserves colimits over α-filtered categories whose transition morphisms in $\underline{A}$ are monomorphic. (This does not mean that F preserves monomorphisms.) The least regular carinal α with this property is called the generation rank and denoted with $\varepsilon(F)$.

2.2 Definition (cf [13] 6.1) Let $\alpha \geq \aleph_0$ be a regular cardinal and let $\underline{A}$ be a category with α-filtered colimits. An object $A \in \underline{A}$ is called α-presentable (resp. α-generated) if the hom-functor $[A,-] : \underline{A} \longrightarrow \underline{\text{Sets}}$ preserves α-filtered colimits (resp. monomorphic α-filtered colimits). The least regular cardinal $\alpha \geq \aleph_0$ with this property is called the presentation rank (resp. generation rank) and denoted with $\pi(A)$ (resp. $\varepsilon(A)$). Clearly $\pi(A) \geq \varepsilon(A)$.

It may appear that this definition is stronger than the one given in the introduction. This is however not the case, at least in practise. First by Swan every $\aleph_0$-filtered category admits a cofinal directed set. Hence for $\alpha = \aleph_0$ the two notions coincide. Second for $\alpha > \aleph_0$ the two definitions are equivalent in a locally α-presentable category. Moreover they lead to the same notion of a locally α-presentable category in 2.3 below. This can be shown by going over the proofs of § 7 in Gabriel-Ulmer [13].

2.3 Definition (cf [13] 7.1, 9.1) Let $\alpha \geq \aleph_0$ be a regular cardinal. A category $\underline{A}$ is called locally α-presentable if $\underline{A}$ has colimits and a set M of α-presentable generators (M is a set of generators means: A morphism $f : A \longrightarrow A'$ is an isomorphism iff $[U,f]$ is a bijection for every $U \in M$).
Likewise a category $\underline{A}$ is called locally α-generated if $\underline{A}$ has colimits and a set M of α-generated generators such that every coproduct $\coprod_{\iota \in I} U_\iota$ with $U_\iota \in M$ and $\mathrm{card}(I) < \alpha$ has only a set of proper quotients. (Recall that an epimorphism $p : X \longrightarrow Y$ is called proper if it does not factor through a proper subobject of Y.) The least regular cardinal $\alpha \geq \aleph_0$ with this property is called the presentation rank of $\underline{A}$ (resp. the generation rank) and denoted with $\pi(\underline{A})$ (resp. $\varepsilon(\underline{A})$)
2.4 A category is called locally presentable (resp. locally generated) if it is locally α-presentable (resp. locally α-generated) for some α.

2.5 A locally α-presentable category is locally α-generated ([13] 6.6c)).

Surprisingly there is a converse: A locally α-generated category is locally β-presentable for some $\beta \geq \alpha$ (cf [13] 9.8, 9.10).

2.6 A locally α-presentable category has limits ([13] 1.12) and is cowellpowered ([13] 7.14; i.e. every object has only a set of quotients).

Moreover α-filtered colimits commute with α-limits (cf [13] 7.12; recall that $\varprojlim(\underline{D} \xrightarrow{H} \underline{A})$ is called an α-limit if $\underline{D}$ has less than α morphisms, [13] § 0).

2.7 In a locally presentable category $\underline{A}$ with a set M of α-presentable generators an object $A \in \underline{A}$ is β-generated for some $\beta \geq \alpha$ iff there is a proper epimorphism $\coprod_{\iota \in I} U_\iota \longrightarrow A$ with $U_\iota \in M$ and card(I) < β (cf [13] 9.3). If moreover M is a set of regular generators, then $A \in \underline{A}$ is β-presentable iff there is a cokernel diagram

$$\coprod_{j \in J} U_j \rightrightarrows \coprod_{i \in I} U_i \longrightarrow Y$$

with U_i, $U_j \in M$ and card(J) < β > card(I) such that A is a retract of Y (cf [13] 7.6). (Recall that M is called regular if for every $A \in \underline{A}$ there is a cokernel diagram $K \rightrightarrows \coprod_\nu U_\nu \longrightarrow A$ with $U_\nu \in M$.) Moreover there is a regular cardinal δ such that every δ-generated object in $\underline{A}$ is δ-presentable and δ can be chosen so as to exceed any given cardinal (cf [13] 13.3).

2.8 In a locally α-presentable category $\underline{A}$ the full subcategory $\underline{A}(\alpha)$ of all α-presentable objects is small and closed in $\underline{A}$ under α-colimits. The same holds for the full subcategory $\tilde{\underline{A}}(\alpha)$ of all α-generated objects ([13] 6.2). In particular for every $A \in \underline{A}$ the category $\underline{A}(\alpha)/A$ of α-presentable objects over A is small and α-filtered, and the colimit of the forgetful functor $\underline{A}(\alpha)/A \longrightarrow \underline{A}$, $(U \longrightarrow A) \rightsquigarrow U$, is A (cf [13] 2.6, 7.4, 3.1). The same holds for the category of α-generated subobjects of A (cf [13] 9.5). The functor

$$\underline{A} \longrightarrow [\underline{A}(\alpha)^o, \underline{\text{Sets}}], \; A \rightsquigarrow [-, A]$$

induces an equivalence between $\underline{A}$ and the full subcategory of $[\underline{A}(\alpha)^o, \underline{\text{Sets}}]$ consisting of all functors $\underline{A}(\alpha)^o \longrightarrow \underline{\text{Sets}}$ which take α-colimits in α-limits ([13] 7.9, for the corresponding assertion

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with $U_i, U_j \in M$ and $card(J) < \beta > card(I)$ such that A is a retract of Y (cf [13] 7.6). (Recall that M is called regular if for every $A \in \underline{A}$ there is a cokernel diagram $K \rightrightarrows \coprod_\nu U_\nu \longrightarrow A$ with $U_\nu \in M$.) Moreover there is a regular cardinal δ such that every δ-generated object in $\underline{A}$ is δ-presentable and δ can be chosen so as to exceed any given cardinal (cf [13] 13.3).

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induces an equivalence between $\underline{A}$ and the full subcategory of $[\underline{A}(\alpha)^o, \underline{Sets}]$ consisting of all functors $\underline{A}(\alpha)^o \longrightarrow \underline{Sets}$ which take α-colimits in α-limits ([13] 7.9, for the corresponding assertion for $\tilde{\underline{A}}(\alpha)$ see [13] 9.1o).

2.9 By the special adjoint functor theorem every colimit preserving functor between locally presentable categories has a right adjoint. By [13] 14.6 a limit preserving functor $S : \underline{A} \longrightarrow \underline{B}$ between locally presentable categories admits a left adjoint iff S has rank (cf. 2.1), i.e. iff S preserves α-filtered colimits for some cardinal $\alpha \geq \aleph_0$.

2.1o Let $\underline{U}$ be a small category and let Σ be a class of morphisms in $[\underline{U}^o, \underline{Sets}]$. Recall that a functor $t : \underline{U} \longrightarrow \underline{X}$ (resp. $s : \underline{U}^o \longrightarrow \underline{X}$) is called Σ-cocontinuous (resp. Σ-continuous) if for every $X \in \underline{X}$ and every $\sigma \in \Sigma$ the map $[\sigma, [t-, X]]$ (resp. $[\sigma, [X, s-]]$) is bijective. If $\underline{X}$ is cocomplete (resp. complete), then there is a tensor product bifunctor (resp. symbolic hom)

$$\otimes \quad : \quad [\underline{U}^o, \underline{Sets}] \times [\underline{U}, \underline{X}] \longrightarrow \underline{X}$$

$$[-,-] \quad : \quad [\underline{U}^o, \underline{Sets}] \times [\underline{U}^o, \underline{X}] \longrightarrow \underline{X}$$

defined by

$$[R \otimes t, X] \cong [R, [t-, X]]$$

$$[X, [R, s]] \cong [R, [X, s-]]$$

for all $X \in \underline{X}$, $R \in [\underline{U}^o, \underline{Sets}]$, $t \in [\underline{U}, \underline{X}]$ and $s \in [\underline{U}^o, \underline{X}]$, cf Gabriel-Ulmer [13] 8.1. Hence $t : \underline{U} \longrightarrow \underline{X}$ (resp. $s : \underline{U}^o \longrightarrow \underline{X}$) is Σ-cocontinuous (resp. Σ-continuous) iff $\sigma \otimes t$ (resp. $[\sigma, s]$) is an isomorphism for every $\sigma \in \Sigma$. The full subcategory of $[\underline{U}, \underline{X}]$ consisting of all Σ-cocontinuous functors is denoted with $Cc_\Sigma[\underline{U}, \underline{X}]$. Likewise $C_\Sigma[\underline{U}^o, \underline{X}]$ denotes the full subcategory of all Σ-continuous functors.
Examples for Σ-continuous (resp. Σ-cocontinuous) functors are sheaves (resp. cosheaves) with respect to a Grothendieck topology and functors which take a given class of colimits into limits (resp. colimits) etc, see § 6.

A class Σ of morphisms in $[\underline{U}^o, \underline{Sets}]$ is called <u>closed</u> if

1) Σ contains all isomorphisms 2) Σ is closed unter colimits
3) if $\rho = \sigma\tau$ and two of the morphisms ρ, σ, τ belong to Σ, then so does the third.

For instance, if T is a class of functors $\underline{U} \longrightarrow \underline{X}$, then the class Ω of all morphisms ω such that $\omega \otimes t$ is an isomorphism for every $t \in T$ is closed.

The closure $\bar{\Sigma}$ of a class Σ is the smallest closed class containing Σ. Hence a Σ-cocontinuous functor $\underline{U} \longrightarrow \underline{X}$ is also $\bar{\Sigma}$-cocontinuous.

Let Σ be a class of morphisms in $[\underline{U}^o, \underline{Sets}]$, where $\underline{U}$ is a small category, such that the codomains $r\sigma$, $\sigma \in \Sigma$, form a set (modulo equivalence). Then by [13] 8.11*) the inclusion $C_\Sigma[\underline{U}^o, \underline{Sets}] \hookrightarrow [\underline{U}^o, \underline{Sets}]$ has a left adjoint and a morphism τ in $[\underline{U}^o, \underline{Sets}]$ belongs to $\bar{\Sigma}$ iff $[\tau, t]$ is a bijection for every $t \in C_\Sigma[\underline{U}^o, \underline{Sets}]$.

2.11 A category $\underline{A}$ is locally presentable iff there is a small category $\underline{U}$ together with a set Σ of morphisms in $[\underline{U}^o, \underline{Sets}]$ such that $\underline{A} \cong C_\Sigma[\underline{U}^o, \underline{Sets}]$, cf. [13] 8.5, 8.6 c). Moreover if $\underline{B}$ is any locally presentable category and $\underline{U}$ and Σ are as above, then $C_\Sigma[\underline{U}^o, \underline{B}]$ is again locally presentable and

$$\pi(C_\Sigma[\underline{U}^o, \underline{B}]) \leq \sup_{\sigma \in \Sigma}{}^{*} \left(\pi(\underline{B}), \pi(d\sigma), \pi(r\sigma)\right)$$

where $d\sigma$ (resp. $r\sigma$) denotes the domain (resp. codomain) of $\sigma \in \Sigma$ and $\sup^*(\ )$ denotes the least regular cardinal $\geq \sup(\ )$, cf. [13] 8.7.

2.12 Let $\mathbb{T} = (T, u, \mu)$ be a triple in a locally presentable category $\underline{A}$. Then by [13] §10 the category of $\mathbb{T}$-algebras $\underline{A}$ is locally presentable iff T has rank (2.1). Moreover if T has rank, then

$$\pi(\underline{A}^{\mathbb{T}}) \leq \sup \{\pi(\underline{A}), \pi(T)\}$$

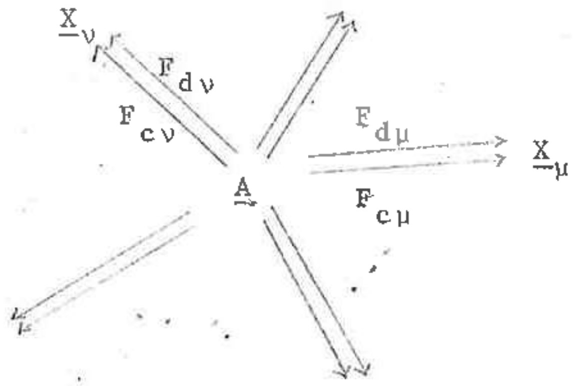
*) The proofs of [13] 8.10 and 8.11 have a gap: On p.99 it is used that in $\underline{A}_{\Sigma 2}$ every object has only a set of proper quotients. This may not be the case unless $\underline{A}$ has additional properties. The easiest way out is to assume that $\underline{A}$ is locally presentable ... .

§ 3 Bialgebras in locally presentable categories

In this chapter some questions of "Universal bialgebra" in a locally presentable category $\underline{A}$ are investigated. Our definition of bialgebras in a category $\underline{A}$ is fairly broad and includes universal algebras and coalgebras in the sense of Birkhoff [2] or Lawvere [21], Σ-continuous and Σ-cocontinuous functors in the sense of Gabriel-Ulmer [13] 8.1, algebras, coalgebras, Hopfalgebras and bialgebras in the usual sense over a commutative ring Λ , or more generally bialgebras with respect to some tensor product and an arbitrary Prop [24], coalgebras over a cotriple with rank in $\underline{A}$ (e.g. comodules over a Λ-coalgebra), algebras over a triple with rank in $\underline{A}$, the category of descent data with respect to fibrations and more generally sections and cartesian closed sections with respect to a fibration or cofibration, etc. Roughly speaking a bialgebra (A,M,R) consists of an object A in $\underline{A}$ together with a set M of operations which satisfy certain relations R . An operation is represented by a morphism $\mu(A) : FA \longrightarrow F'A$ for some pair of functors $F,F' : \underline{A} \rightrightarrows \underline{X}$, and a relation by a morphism pair $r(A,M) : HA \rightrightarrows H'A$ for some pair of functors $H,H' : \underline{A} \rightrightarrows \underline{Y}$ (for details see 3.1). Given a bialgebra (A,M,R) and a subobject $U \subset A$ in the underlying category $\underline{A}$, which we assume to be locally noetherian for the moment, we are concerned with sub-bialgebras (U',M,R) of (A,M,R) containing U such that U' is not much bigger than U . We give in 3.22 a construction and size estimates for U' which in many cases are the best possible. For instance, if Λ is a commutative noetherian ring, then it follows that any countably generated submodule of a Hopfalgebra (resp. coalgebra, bialgebra, comodule over a fixed Λ-coalgebra) is contained in a sub-Hopfalgebra (resp. sub-coalgebra,...) whose underlying Λ-module is again countably generated regardless of the size of A . If the category $\underline{A}$ is not locally noetherian, the situation is different and the question should be put instead as follows: Given a bialgebra (A,M,R) and a morphism

$f : U \longrightarrow A$ in the underlying category $\underline{A}$ with U being a γ - presentable object, does then f factor through a bialgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that U' is γ' - presentable and γ' is not much bigger than γ ? In 3.8 we give a construction and size estimates for U' similar to the noetherian case which in particular implies the existence of dense generators in the above mentioned examples. If Λ is any commutative ring and (A,M,R) is a Λ-Hopfalgebra (resp. coalgebra, bialgebra, comodule over a fixed Λ-coalgebra), then by 3.8 any homomorphism $f : U \longrightarrow A$ with U being countably presentable factors through a Hopfalgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ (resp. coalgebra morphism ...) such that U' is again countably presentable. Also 3.8 implies that every descent data is effective provided every descent data on "small" objects is effective. For modules "small" means countably presentable. More generally for a fibration with countably presentable fibres "small" means countably presentable provided either the inverse image functors have right adjoints which preserve countably filtered direct limits or the inverse image functors take countably presentable objects into countably presentable objects and preserve filtered direct limits. The main results of this chapter are 3.8, 3.9, 3.22, 3.24 and 3.28. The last two concern conditions which guarantee that the category Bialg($\underline{A}$) of bialgebras in a locally γ-presentable category $\underline{A}$ is locally γ'-presentable and that γ' is not much bigger than γ . For instance, if Λ is any commutative ring, then they imply that the categories of commutative Λ-Hopfalgebras, cocommutative Λ-Hopfalgebras, Λ-coalgebras, Λ-bialgebras, comodules over a fixed Λ-coalgebra etc. are locally countably presentable, regardless of the size of Λ . Also if $\mathbb{G}$ is a cotriple with rank α in a locally γ-presentable category $\underline{A}$, then the category $\underline{A}_{\mathbb{G}}$ of $\mathbb{G}$-coalgebras in $\underline{A}$ is locally γ'-presentable, where $\gamma' = \sup(\aleph_1, \gamma, \alpha)$.

3.1 In this paragraph we give the basic definitions. Let $\underline{A}$ be a category. Let M be a set (or class) and assume that with each $\mu \in M$ there is associated an ordered pair of functors $F_{d\mu} : \underline{A} \longrightarrow \underline{X}_{\mu}$ and $F_{c\mu} : \underline{A} \longrightarrow \underline{X}_{\mu}$. Note that the domain is always $\underline{A}$ and that each pair has the same codomain (which can vary from pair to pair)



Also note that the assignment $\mu \rightsquigarrow (F_{d\mu}, F_{c\mu})$ need not be injective. A pre-bialgebra $(A,\mu(A))_{\mu \in M}$ in $\underline{A}$ with respect to M is an object $A \in \underline{A}$ together with a morphism $\mu(A) : F_{d\mu}A \longrightarrow F_{c\mu}A$ for every $\mu \in M$. We say that an element $\mu \in M$ is an operation and $\mu(A)$ is the structure morphism on A associated with μ . A morphism $(A,\mu(A))_{\mu \in M} \longrightarrow (A',\mu(A'))_{\mu \in M}$ between pre-bialgebras is a morphism $f : A \longrightarrow A'$ in $\underline{A}$ which is compatible with the structure morphisms, i.e. for every $\mu \in M$ the diagram

$$\begin{array}{ccc} F_{d\mu}A & \xrightarrow{\mu(A)} & F_{c\mu}A \\ {\scriptstyle F_{d\mu}f}\downarrow & & \downarrow{\scriptstyle F_{c\mu}f} \\ F_{d\mu}A' & \xrightarrow[\mu(A')]{} & F_{c\mu}A' \end{array}$$

commutes. The category of pre-bialgebras is denoted with $\text{P-Bialg}_M(\underline{A})$. Let $V : \text{P-Bialg}_M(\underline{A}) \longrightarrow \underline{A}$ denote the (faithful) forgetful functor $(A,\mu(A))_{\mu \in M} \rightsquigarrow A$. If it is clear which M we are referring to we write $\text{P-Bialg}(\underline{A})$ instead of $\text{P-Bialg}_M(\underline{A})$. Further we abbreviate $(A,\mu(A))_{\mu \in M}$ to (A,M) in order to avoid expressions of extreme complexity in the following. This notation does no longer distinguish between pre-bialgebras with the same underlying object. The reader should keep this in mind.

Clearly in practice one is not interested in all pre-bialgebras but only in those which satisfy certain given relations. The relations are normally expressed in terms of diagrams which have to commute. The diagrams are constructed from the structure morphisms and other canonical morphisms. However there is a great deal of variety and a scheme of sufficient generality to cover the above mentioned examples becomes hopelessly involved. Surprisingly it turned out – after many attempts – that the explicit description of the relations in terms of structure and canonical morphisms is not needed to establish the main results of this chapter. Instead the following common features suffice : 1) for every pre-bialgebra the diagrams expressing the relations are given in some way 2) the diagrams are natural with respect to pre-bialgebra morphisms.

More precisely by a <u>relation</u> r <u>on</u> $P\text{-Bialg}_M(\underline{A})$ we mean a pair of functors $F_{dr} : \underline{A} \longrightarrow \underline{X}_r$ and $F_{cr} : \underline{A} \longrightarrow \underline{X}_r$ together with a pair of natural transformations $F_{dr} \circ V \rightrightarrows F_{cr} \circ V$. Explicitely with every pre-bialgebra (A,M) there is associated a pair of morphisms $r(A,M) : F_{dr}A \rightrightarrows F_{cr}A$ in such a way that for every pre-bialgebra morphism $f: (A,M) \longrightarrow (A',M)$ the diagram

$$\begin{array}{ccc} F_{dr}A & \overset{r(A,M)}{\rightrightarrows} & F_{cr}A \\ {\scriptstyle F_{dr}f}\downarrow & & \downarrow{\scriptstyle F_{cr}f} \\ F_{dr}A' & \underset{r(A',M)}{\rightrightarrows} & F_{cr}A' \end{array}$$

commutes in the obvious sense (i.e. with respect to both components of r).

Let R be a set (or class) of relations on $P\text{-Bialg}_M(\underline{A})$.

A <u>bialgebra</u> (A,M,R) in $\underline{A}$ with respect to M and R is a pre-bialgebra (A,M) such that for every $r \in R$ the morphisms $r(A,M) : F_{dr}A \rightrightarrows F_{cr}A$ coincide. In other words a bialgebra is a pre-bialgebra satisfying the relations of R. A morphism between bialgebras is a morphism between the underlying pre-bialgebras. The category of bialgebras is denoted with $\mathrm{Bialg}_{M,R}(\underline{A})$. If there is no ambiguity we drop the indices M and R. Clearly Bialg($\underline{A}$) is a full subcategory of P-Bialg($\underline{A}$). The forgetful functor Bialg($\underline{A}$) $\to$ $\underline{A}$, (A,M,R) $\rightsquigarrow$ A, is also denoted with V.

By the support of the operations M and the relations R we mean the set (or class) $\mathbb{F}$ of all functors $F_{d\mu}$, $F_{c\mu}$, F_{dr} and F_{cr}, where μ and r are running through M and R respectively. The subclass of all functors of $\mathbb{F}$ which are the domain of either an operation or a relation is denoted with $\mathbb{F}_d$. Likewise $\mathbb{F}_c$ denotes the subclass of all functors appearing as the codomain of either an operation or a relation. In the following the hypothesis are often stated in terms of $\mathbb{F}$, $\mathbb{F}_d$, and $\mathbb{F}_c$ instead of M and R. It is therefore essential to keep their meanings in mind (d = domain, c = codomain).

3.2 Remarks I) It is easy to express that for some specified operation $\mu \in M$ the structure morphism $\mu(A) : F_{d\mu}A \to F_{c\mu}A$ should be an isomorphism for every pre-bialgebra (A,M) : One has to add an operation $\bar{\mu}$ to M and two relations to R expressing $\bar{\mu}(A)\mu(A) = id_{F_{d\mu}A}$ and $\mu(A)\bar{\mu}(A) = id_{F_{c\mu}A}$ (cf. 3.2 IIId).

II) One can call an operation μ algebraic (resp. coalgebraic) if $F_{d\mu}$ and $F_{c\mu}$ are endofunctors of $\underline{A}$ and $F_{c\mu}$ is the identity of $\underline{A}$ (resp. $F_{d\mu} = id_{\underline{A}}$); and likewise for relations. Typical examples are functors $\underline{A} \to \underline{A}$ which assign to an object A its n-fold product, coproduct or tensor product etc.

III) For examples of bialgebras see § 4 and § 6. It should however be clear at this point how to express some of the examples given in the introduction to § 3 as bialgebras, i.e. how to choose the underlying category $\underline{A}$, the operations M and the relations R such that $\mathrm{Bialg}(\underline{A})$ is

a) the category of groups, rings, ..., cogroups, ... in $\underline{A}$.

b) the category of algebras, coalgebras, bialgebras, Hopfalgebras ... over a commutative ring Λ .

c) the category of $\mathbb{T}$ -algebras (resp. $\mathbb{G}$-coalgebras) for a triple $\mathbb{T}$ (resp. cotriple $\mathbb{G}$) in $\underline{A}$.

d) the category of descent data (or données de recollement) in the standard situation

$$\mathcal{F}_S \xrightarrow{\alpha^*} \mathcal{F}_{S'} \;\underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}}\; \mathcal{F}_{S'\times_S S'} \;\underset{p_{21}^*}{\overset{p_{31}^*,\ p_{32}^*}{\rightrightarrows}}\; \mathcal{F}_{S'\times_S S'\times_S S'} \qquad (\Delta^* : \mathcal{F}_{S'\times_S S'} \to \mathcal{F}_{S'})$$

given by $\alpha : S' \to S$ (cf. Grothendieck [16] Def. 1.4 - Def 1.7).

The reader should be familiar with these examples, in particular know what the functors $F_{d\mu}$, $F_{c\mu}$, F_{dr}, F_{cr} and the natural transformations $F_{dr} \circ V \Longrightarrow F_{cr} \circ V$ look like for every operation $\mu \in M$ and relation $r \in R$. If not, he is advised to first have a look at § 4 because the following is often motivated by these examples.

3.3 We start with some elementary properties of the underlying functor $V: \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ concerning the preservation of limits and colimits.

Lemma Let $H : \underline{D} \longrightarrow \mathrm{Bialg}(\underline{A})$ be a functor such that the limit (resp. colimit) of the composite $V \circ H : \underline{D} \longrightarrow \underline{A}$ exists. Then the following hold:

a) If every $F \in \mathbb{F}_c$ preserves $\varprojlim V \circ H$, then $\varprojlim H$ exists in $\mathrm{Bialg}(\underline{A})$ and $\varprojlim H = (\varprojlim V \circ H, M, R)$.

b) <u>If every</u> $F \in F_d$ <u>preserves</u> $\varinjlim V \circ H$, <u>then</u> $\varinjlim H$ <u>exists in</u> $\mathrm{Bialg}(\underline{A})$ <u>and</u> $\varinjlim H = (\varinjlim V \circ H, M, R)$.

<u>Proof</u> It suffices to consider a) because b) is dual.

By assumption for every operation $\mu \in M$ there is an unique morphism $\mu(\varprojlim V \circ H) : F_{d\mu}(\varprojlim V \circ H) \longrightarrow F_{c\mu}(\varprojlim V \circ H)$ such that for every $D \in \underline{D}$ the diagram

$$\begin{array}{ccccc}
F_{d\mu}(\varprojlim V \circ H) & \xrightarrow{\mu(\varprojlim V \circ H)} & F_{c\mu}(\varprojlim V \circ H) & \xrightarrow{\cong} & \varprojlim (F_{c\mu} \circ V \circ H) \\
\downarrow {\scriptstyle F_{d\mu}(p_D)} & & \downarrow {\scriptstyle F_{c\mu}(p_D)} & & \\
F_{d\mu}((V \circ H)D) & \xrightarrow{\mu((V \circ H)D)} & F_{c\mu}((V \circ H)D) & &
\end{array}$$

commutes, where $p_D : \varprojlim (V \circ H) \longrightarrow (V \circ H)D$ denotes the canonical morphism. Thus $\varprojlim (V \circ H)$ together with $\mu(\varprojlim V \circ H)$, $\mu \in M$, is a pre-bialgebra and p_D is a pre-bialgebra morphism for every $D \in \underline{D}$. Hence for every relation $r \in R$ and every $D \in \underline{D}$ the morphism pair $r(\varprojlim V \circ H, M)$ gives rise to a commutative diagram (with respect to both components of r)

$$\begin{array}{ccccc}
F_{dr}(\varprojlim V \circ H) & \overset{r(\varprojlim V \circ H, M)}{\rightrightarrows} & F_{cr}(\varprojlim V \circ H) & \xrightarrow{\cong} & \varprojlim F_{cr} \circ V \circ H \\
\downarrow {\scriptstyle F_{dr}(p_D)} & & \downarrow {\scriptstyle F_{cr}(p_D)} & & \\
F_{dr}((V \circ H)D) & \overset{r(HD)}{\rightrightarrows} & F_{cr}((V \circ H)D) & &
\end{array}$$

This shows that $r(\varprojlim V \circ H, M)$ is the inverse limit over all "pairs" $r(HD)$, $D \in \underline{D}$. Since the two components of $r(HD)$, $D \in \underline{D}$, coincide, the same holds for $r(\varprojlim V \circ H, M)$ and thus $(\varprojlim V \circ H, M)$ is a bialgebra. One readily checks that the latter together with the bialgebra morphisms $p_D : (\varprojlim V \circ H, M, R) \longrightarrow HD$ is the limit of $H : \underline{D} \longrightarrow \mathrm{Bialg}(\underline{A})$.

3.4 Corollary a) If $\underline{A}$ is complete and every $F \in \mathbb{F}_c$ preserves limits, then Bialg($\underline{A}$) is complete and the forgetful functor V : Bialg($\underline{A}$) $\longrightarrow$ $\underline{A}$ preserves (and creates) limits. Moreover V is tripleable provided it has a left adjoint.

b) Likewise if $\underline{A}$ is cocomplete and every $F \in \mathbb{F}_d$ preserves colimits, then Bialg($\underline{A}$) is cocomplete and the forgetful functor V preserves (and creates) colimits. Moreover V is cotripleable provided it has a right adjoint.

c) If $\underline{A}$ has α-filtered colimits and every $F \in \mathbb{F}_d$ preserves them, then Bialg($\underline{A}$) has α-filtered colimits and V preserves (and creates) them.

As for the tripleability and cotripleability note that by 3.3 a), b) the underlying V always preserves (and creates) both V-contractible kernels and cokernels. The condition c) holds in most examples for an appropriate α. This is not so for a) and b). However a) holds when all operations and relations are algebraic (3.2 II), while condition b) holds when all operations and relations are coalgebraic (3.2 II).

3.5 In order to study the category Bialg($\underline{A}$) from the point of view of locally presentable categories the first question to answer is whether there exist α-presentable objects for sufficiently large α and how they look like. The following and 3.6, 3.7 give a partial answer.

Lemma Let $\underline{A}$ be a category with α-filtered colimits and let M and R be a data for bialgebras (3.1). Assume that card(M) $< \alpha$ and that every $F \in \mathbb{F}$ preserves α-filtered colimits. Then a bialgebra (U,M,R) is α-presentable in Bialg($\underline{A}$) provided $U \in \underline{A}$ and FU are α-presentable *) for every $F \in \mathbb{F}_d$.

Remark If the underlying functor $V : \text{Bialg}(\underline{A}) \longrightarrow \underline{A}$ and the functors $F \in \mathbb{F}_c$ preserve monomorphisms - e.g. in the situation 3.4 a) - then there is an analoguous assertion for α-generated objects: A bialgebra (U,M,R) is α-generated provided 1) card(M) $< \alpha$ and every $F \in \mathbb{F}$ preserves α-filtered colimits 2) U and FU are α-generated for every $F \in \mathbb{F}_d$. The proof is the same as for 3.5.

Proof First note that by 3.4 c) Bialg($\underline{A}$) has α-filtered colimits and $V : \text{Bialg}(\underline{A}) \longrightarrow \underline{A}$ preserves (and creates) them. Let $(X,M,R) = \varinjlim_\nu (X_\nu,M,R)$ be an α-filtered colimit in Bialg($\underline{A}$) and let $f : (U,M,R) \longrightarrow \varinjlim_\nu (X_\nu,M,R)$ be a bialgebra morphism with $\pi(U) \leq \alpha$ and $\pi(FU) \leq \alpha$ for every $F \in \mathbb{F}_d$. Since $\varinjlim_\nu (X_\nu,M,R) = (\varinjlim_\nu X_\nu, M,R)$ and U is α-presentable, the underlying morphism $U \longrightarrow \varinjlim_\nu X_\nu$ of f admits a factorization $U \xrightarrow{f_\nu} X_\nu \xrightarrow{u_\nu} \varinjlim_\nu X_\nu$ for some ν, where u_ν denotes the underlying canonical morphism. In general, f_ν is not a bialgebra morphism because for an operation $\mu \in M$ the morphisms $F_{c\mu}f_\nu \circ \mu(U)$ and $\mu(X_\nu) \circ F_{d\mu}f_\nu$ need not coincide. However they become equal when composed with $F_{c\mu}u_\nu : F_{c\mu}X_\nu \longrightarrow F_{c\mu}\varinjlim X_\nu$ because $f = u_\nu f_\nu$ is a bialgebra morphism. Since $F_{d\mu}U$ is α-presentable and $F_{c\mu} \varinjlim_\nu X_\nu \cong \varinjlim_\nu F_{c\mu}X_\nu$ is an α-filtered colimit, this implies that there is a transition morphism $u : X_\nu \longrightarrow X_{\nu'}$ - depending on μ - such that the diagram

---

*) It is not assumed that the codomain of F, $F \in \mathbb{F}_d$, is locally presentable, but merely that $[FU,-]$ preserves all existing α-filtered colimits.

$$\begin{array}{ccc} F_{c\mu}U & \xrightarrow{F_{c\mu}(u\circ f_\nu)} & F_{c\mu}X_{\nu'} \\ \Big\uparrow{\scriptstyle \mu(U)} & & \Big\uparrow{\scriptstyle \mu(X_{\nu'})} \\ F_{d\mu}U & \xrightarrow[F_{d\mu}(u\circ f_\nu)]{} & F_{d\mu}X_{\nu'} \end{array}$$

commutes. Since $\operatorname{card}(M) < \alpha$ one can find a transition morphism $X_\nu \longrightarrow X_{\nu''}$ which has this property for every $\mu \in M$. This shows that $f : (U,M,R) \longrightarrow \varinjlim_\nu (X_\nu, M, R)$ admits a factorization into bialgebra morphisms $(U,M,R) \longrightarrow (X_{\nu''}, M, R) \longrightarrow \varinjlim_\nu (X_\nu, M, R)$, i.e. the canonical map

$$\varinjlim_\nu \left[(U,M,R),(X_\nu,M,R)\right] \longrightarrow \left[(U,M,R), \varinjlim_\nu (X_\nu,M,R)\right]$$

is surjective. In the same way one can show it is also injective. Hence (U,M,R) is α-presentable in $\mathrm{Bialg}(\underline{A})$. (The former follows also directly from the fact that $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ is faithful and preserves α-filtered colimits and that U is α-presentable in $\underline{A}$).

3.6 Remark One would like to conclude from 3.5 that a bialgebra (U,M,R) is α-presentable in $\mathrm{Bialg}(\underline{A})$ provided its underlying object $U \in \underline{A}$ is. We will show below in 3.7 that this is true provided α is sufficiently large and $\underline{A}$ is locally presentable. If $\underline{A}$ is locally finitely presentable, then 3.7 provides the smallest α for which this is true. In general this is not so and resorting to 3.7 can give poor estimates. However in examples one often knows enough about the functors $F \in \mathbb{F}_d$ (e.g. in 3.2 III) to find out directly what the smallest α is such that $\pi(U) \leq \alpha$ implies $\pi(FU) \leq \alpha$ for every $F \in \mathbb{F}_d$. A particular situation is the following. Assume that the codomain of every $F \in \mathbb{F}_d$ is locally presentable and that every F has a right adjoint G_F. Then by 2.9 there is a (smallest) regular cardinal β such that every G_F preserves β-filtered colimits. Hence $\pi(U) \leq \beta$ implies $\pi(FU) \leq \beta$ for every $F \in \mathbb{F}_d$ because $[FU,-] \cong [U, G_F -]$. Likewise $\varepsilon(U) \leq \beta$ implies $\varepsilon(FU) \leq \beta$ for every $F \in \mathbb{F}_d$ and $U \in \underline{A}$.

3.7 Lemma Let $\underline{A}$ be a locally α-presentable category and let $\mathbb{F}_d$ be a set of functors with domain $\underline{A}$ which preserve α-filtered colimits. Let $\bar{\alpha} \geq \alpha$ be a regular cardinal such that

1) if $X \in \underline{A}$ and $\pi(X) \leq \alpha$, then $\pi(FX) \leq \bar{\alpha}$ for every $F \in \mathbb{F}_d$,

2) if $\rho < \alpha$ and $\beta < \bar{\alpha}$, then $\beta^{\rho} < \bar{\alpha}$.

Then $\pi(U) \leq \bar{\alpha}$ implies $\pi(FU) \leq \bar{\alpha}$ for every $F \in \mathbb{F}_d$ and $U \in \underline{A}$.

Corollary Let $\underline{A}$ be a locally α-presentable category with a data M and R for bialgebras (3.1). Assume that $\mathrm{card}(M) < \alpha$ and that every $F \in \mathbb{F}$ preserves α-filtered colimits. Let $\bar{\alpha} \geq \alpha$ be a cardinal with the above properties 1) and 2). Then a bialgebra (U,M,R) is $\bar{\alpha}$-presentable in Bialg($\underline{A}$) provided U is $\bar{\alpha}$-presentable in $\underline{A}$.

Remarks a) Note that condition 2) is trivially satisfied if either $\alpha = \aleph_0$ or $\bar{\alpha}$ is of the form $(2^{\gamma})^+$ for some $\gamma^+ \geq \alpha$

b) Since the α-presentable objects in $\underline{A}$ form a small subcategory there exists always a cardinal $\bar{\alpha}$ with the properties 1) and 2).

c) Using 5.1 one can prove an assertion analoguous to 3.7 for locally α-generated categories (cf. remark 3.5).

Proof of 3.7 The case $\bar{\alpha} = \alpha$ is trivial and we assume $\bar{\alpha} > \alpha$. Given $U \in \underline{A}$ with $\pi(U) \leq \bar{\alpha}$ we are looking for an α-filtered colimit presentation $U = \varinjlim_K X_K$ such that $\pi(X_K) \leq \bar{\alpha}$ for every K and the cardinality of the index system is strictly smaller than $\bar{\alpha}$. Since $F \in \mathbb{F}_d$ preserves α-filtered colimits, it then follows easily that $\pi(FU) = \pi(\varinjlim_K FX_K) \leq \bar{\alpha}$.

We need some preparation. Let D be a partially ordered set which is α-filtered and let D' be a subset of cardinality $< \bar{\alpha}$. Then D' is contained in an α-filtered subset D'' whose cardinality is also $< \bar{\alpha}$. One constructs D'' by transfinite induction as follows. Let $D'_0 = D'$. If $\lambda < \alpha$ is a successor ordinal then let D'_{λ} be the subset consisting of $D'_{\lambda-1}$ and an upper bound in D for every subset $I \subset D'_{\lambda-1}$ with $\mathrm{card}(I) < \alpha$. If $\lambda < \alpha$ is a limit ordinal let $D'_\lambda = \bigcup_{\rho<\lambda} D_\rho$. In either case it follows from assumption 2) that $\mathrm{card}(D'_\lambda) < \bar{\alpha}$. Clearly $D'' = \bigcup_{\lambda<\alpha} D'_\lambda$ is α-filtered and $\mathrm{card}(D'') < \bar{\alpha}$ because $\alpha < \bar{\alpha}$.

Let $U \in \underline{A}$ be $\bar{\alpha}$-presentable. Then by 1.7 there is a cokernal diagram

$$\coprod_{i\in I} X_i \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{j\in J} X_j \longrightarrow Y$$

such that 1) $\mathrm{card}(I) < \bar{\alpha} > \mathrm{card}(J)$ 2) X_i and X_j are α-presentable for every $i \in I$ and $j \in J$ and 3) U is a retract of Y . We will show that $\pi(FY) \leq \bar{\alpha}$ for every $F \in \mathbb{F}_d$. Since this implies $\pi(FU) \leq \bar{\alpha}$, we can assume without loss of generality that $Y = U$.

Let D be the partially ordered set consisting of quadruples (I_K, J_K, f_K, g_K) , where $I_K \subset I$, $J_K \subset J$ and $f_K, g_K : \coprod_{i\in I} X_i \rightrightarrows \coprod_{j\in J_K} X_j$ are morphisms such that $\mathrm{card}(I_K) < \alpha > \mathrm{card}(J_K)$ and the canonical diagram

$$\begin{array}{ccc} \coprod_{i\in I_K} X_i & \overset{f_K}{\underset{g_K}{\rightrightarrows}} & \coprod_{j\in J_K} X_j \\ \downarrow & & \downarrow \\ \coprod_{i\in I} X_i & \overset{f}{\underset{g}{\rightrightarrows}} & \coprod_{j\in J} X_j \end{array}$$

commutes. The ordering is given by inclusion, i.e. $K < K'$ provided $I_K \subset I_{K'}$, $J_K \subset J_{K'}$, and the induced diagram

$$\begin{array}{ccc} \coprod_{i\in I_K} X_i & \overset{f_K}{\underset{g_K}{\rightrightarrows}} & \coprod_{j\in J_K} X_j \\ \downarrow & & \downarrow \\ \coprod_{i\in I_{K'}} X_i & \overset{f_{K'}}{\underset{g_{K'}}{\rightrightarrows}} & \coprod_{j\in J_{K'}} X_j \end{array}$$

commutes. Note that in both diagrams the vertical morphisms need not be monomorphic! (This complicates the proof considerably). Using that $\coprod_{i\in I_K} X_i$ is α-presentable and $\coprod_{j\in J} X_j$ is the α-filtered colimit of its subcoproducts with less than α summands, it is routine to verify that D is α-filtered. Let D' be the subset of D obtained in the following way: For every pair $I'\subset I$ and $J'\subset J$ with $\mathrm{card}(I') < \alpha > \mathrm{card}(J')$ pick one element (I_K, J_K, f_K, g_K) of D with the property $I' = I_K$ and $J' = J_K$ (provided there is such an element, there may be many or none with this property). Clearly condition 2) and $\mathrm{card}(I) < \bar\alpha > \mathrm{card}(J)$ imply $\mathrm{card}(D') < \bar\alpha$. Given $\coprod_{i\in I'} X_i$ with $\mathrm{card}(I') < \alpha$ there is an element (I_K, J_K, f_K, g_K) with $I' = I_K$ because $\coprod_{i\in I'} X_i$ is α-presentable and $\coprod_{j\in J} X_j$ is the α-filtered colimit of its subcoproducts with less than α summands. Likewise given $\coprod_{j\in J'} X_j$ with $\mathrm{card}(J') < \alpha$ one can find an element (I_K, J_K, f_K, g_K) such that $J'\subset J_K$. From this it follows that D' is not empty and that the colimits of $D'' \longrightarrow \underline{A}$, $K \leadsto \coprod_{i\in I_K} X_i$ and $D'' \longrightarrow \underline{A}$, $K \leadsto \coprod_{j\in J_K} X_j$, are $\coprod_{i\in I} X_i$ and $\coprod_{j\in J} X_j$ respectively (for D'' see above). Whence the colimit of $D'' \longrightarrow \underline{A}$, $K \leadsto X_K = \mathrm{coker}(f_K, g_K)$ is U. Note that D'' is α-filtered and $\mathrm{card}(D'') < \bar\alpha$. Since $X_K = \mathrm{coker}(f_K, g_K)$ is α-presentable, by condition 1) FX_K is $\bar\alpha$-presentable for every $F\in\mathbb{F}_d$. Summarizing we obtain

$$\pi(FU) = \pi(F \varinjlim_{K\in D''} X_K) = \pi(\varinjlim_{K\in D''} FX_K) \leq \bar\alpha$$

because an $\bar\alpha$-colimit of $\bar\alpha$-presentable objects is again $\bar\alpha$-presentable. This completes the proof.

3.8 Theorem Let $\underline{A}$ be a locally presentable category and let M, R and $\mathbb{F}$ be a data for bialgebras (cf. 3.1). Assume there is a regular cardinal β such that every $F\in\mathbb{F}$ preserves β-filtered colimits. Let $\gamma > \beta$ be any regular cardinal such that

a) $\mathrm{card}(M) < \gamma > \mathrm{card}(R)$ and $\underline{A}$ is locally γ-presentable.

b) *if* $U \in \underline{A}$ *is* γ*-presentable, then* FU *is* γ*-presentable for every* $F \in \mathbb{F}_d$ (cf. 3.6, 3.7 for $\gamma = \hat{\alpha}$).

Let (A,M,R) *be a bialgebra and let* $U \in \underline{A}$ *be a* γ*-presentable object. Then every morphism* $f : U \longrightarrow A$ *admits a factorization into a morphism* $U \longrightarrow U'$ *and a bialgebra morphism* $(U',M,R) \longrightarrow (A,M,R)$ *such that* $U' \in \underline{A}$ *is again* γ*-presentable. Moreover a bialgebra* (X,M,R) *is* γ*-presentable in* Bialg$(\underline{A})$ *iff* X *is* γ*-presentable in* $\underline{A}$.

Remark Note that γ has to be strictly bigger than β ; hence $\gamma \geq \aleph_1$. If the codomain of every $F \in \mathbb{F}_d$ is locally presentable, then by 3.7 there is always a cardinal $\gamma > \beta$ such that the above conditions a) and b) hold. The point is of course to choose γ as small as possible. The moste useful situation seems $\gamma = \aleph_1$ and $\beta = \aleph_0$. This happens in any of the following cases

I card(M) $\leq \aleph_0 \geq$ card(R) , $\pi(\underline{A}) = \aleph_0$, every $F \in \mathbb{F}$ preserves filtered colimits, and every $F \in \mathbb{F}_d$ takes finitely presentable objects into countably presentable objects (cf. 3.7).

II card(M) $\leq \aleph_0 \geq$ card(R), $\pi(\underline{A}) \leq \aleph_1$, every $F \in \mathbb{F}$ preserves filtered colimits, and every $F \in \mathbb{F}_d$ takes countably presentable objects into countably presentable objects.

III card(M) $\leq \aleph_0 \geq$ card(R), $\pi(\underline{A}) \leq \aleph_1$, every $F \in \mathbb{F}$ preserves filtered colimits and every $F \in \mathbb{F}_d$ has a right adjoint G_F which preserves countably filtered colimits (cf. 3.6).

3.9 *Corollary* *Let* $\underline{Y}(\gamma)$ *be the full subcategory of* $\underline{Y}$ = Bialg$(\underline{A})$ *consisting of all* γ*-presentable objects. Then for every* $Y \in \underline{Y}$ *the category* $\underline{Y}(\gamma)/Y$ *is* γ*-filtered and the colimit of the forgetful functor* $\underline{Y}(\gamma)/Y \longrightarrow \underline{Y}$ *is* Y ; i.e. *the inclusion* $\underline{Y}(\gamma) \hookrightarrow \underline{Y}$ *is dense* (cf.[13]3.1).

Definition *A set valued functor on a small category is called* γ*-flat if it is a* γ*-filtered colimit of representable functors.*

3.10 *Corollary* *Let* $\text{Flat}_\gamma[\underline{Y}(\gamma)^{o},\underline{\text{Sets}}]$ *denote the full subcategory* of $[\underline{Y}(\gamma)^{o},\underline{Sets}]$ consisting of all γ-flat functors. Then the functor

$$\underline{Y} \longrightarrow \mathrm{Flat}_{\gamma}[\underline{Y}(\gamma)^{o},\underline{Sets}]\ ,\ Y \rightsquigarrow [-,Y]$$

is an equivalence.

3.11 Remarks I One can view 3.10 as a "generalization" of [13] 7.9 The latter asserts that a locally γ-presentable category $\underline{X}$ is of the form $\underline{X} \xrightarrow{\simeq} \mathrm{St}_{\gamma}[\underline{X}(\gamma)^{o},\underline{Me}]$. Thus, if $\underline{Y}(\gamma)$ is γ-cocomplete, then by [13] 5.4 a functor $\underline{Y}(\gamma)^{o} \longrightarrow \underline{Sets}$ is γ-flat iff it is γ-continuous, i.e. $\mathrm{Flat}_{\gamma}[\underline{Y}(\gamma)^{o},\underline{Sets}] = \mathrm{St}_{\gamma}[\underline{Y}(\gamma)^{o},\underline{Me}]$ (cf. [13] 7.9).

II It will be apparent from the proofs of 3.8 - 3.10 that the hypotheses have not been fully used; in particular the existence of arbitrary colimits in $\underline{A}$. Besides b) and $\mathrm{card}(M) < \gamma > \mathrm{card}(R)$ only the following properties are used

a) $\underline{A}$ has β-filtered colimits for some $\beta < \gamma$ and every $F \in \mathbb{F}$ preserves them,

b) for every $A \in \underline{A}$ the category $\underline{A}(\gamma)/A$ of γ-presentable objects over A is γ-filtered and A is the colimit of $\underline{A}(\gamma)/A \longrightarrow \underline{A}$, $(U \longrightarrow A) \rightsquigarrow U$ (cf. 2.8).

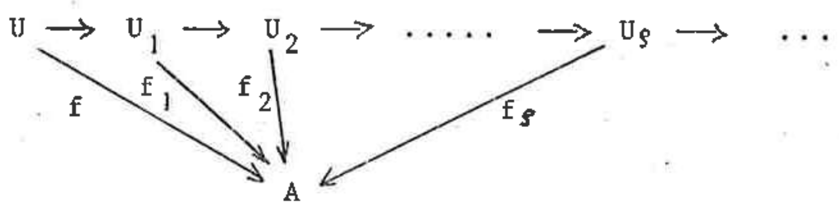
In general $\mathrm{Bialg}(\underline{A})$ is not locally presentable but it has again β-filtered colimits and by 3.9 it inherits property b). For instance the category of flat left Λ-modules over a ring Λ need not be locally presentable, but has filtered colimits and satisfies property b) for every γ . An important class of categories which are not locally presentable but for which 3.8 - 3.10 applies are the "catégories localisables" recently introduced by Y. Diers [5].

Proof of 3.10. By 3.4 $\underline{Y}$ has γ-filtered colimits. The functor $\underline{Y} \longrightarrow [\underline{Y}(\gamma)^{o},\underline{Sets}]$, $Y \rightsquigarrow [-,Y]$ is full and faithful because the inclusion $\underline{Y}(\gamma) \xrightarrow{\subset} \underline{Y}$ is dense, cf [ ] 3.4. Moreover it preserves and reflects γ-filtered colimits because the objects of $\underline{Y}(\gamma)$ are γ-presentable in $\underline{Y}$. Also its values are in Flat $[\underline{Y}(\gamma)^{o},\underline{Sets}]$

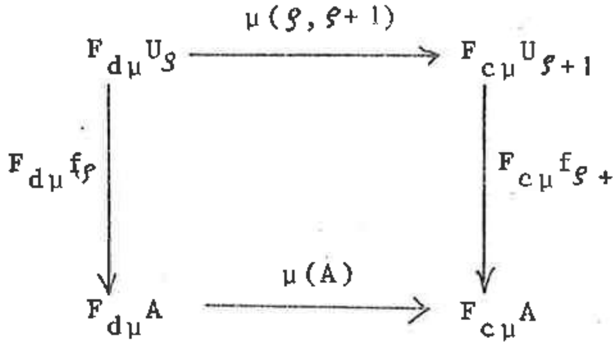
because by 3.9 $\underline{Y}(\gamma)/Y$ is γ-filtered for every $Y \in \underline{Y}$. Hence the factorization $\underline{Y} \longrightarrow \mathrm{Flat}_{\gamma}\ [\underline{Y}(\gamma)^{o}, \underline{Sets}]$, $Y \rightsquigarrow [-,Y]$ is a full embedding which preserves γ-filtered colimits. If $F : \underline{Y}(\gamma)^{o} \longrightarrow \underline{Sets}$ is γ-flat, let $\underline{D}$ be a γ-filtered category together with a functor $\underline{D} \longrightarrow \underline{Y}$, $D \rightsquigarrow Y_D$ such that $F = \varinjlim_{D \in \underline{D}} [-,Y_D]$. By the above $\varinjlim_{D \in \underline{D}} [-,Y_D] \cong [-, \varinjlim_{D \in \underline{D}} Y_D]$, whence $F \cong [-,Y]$ for $Y = \varinjlim_{D \in \underline{D}} Y_D$. This completes the proof.

<u>Proof of 3.8 and 3.9</u> Since the proof is fairly involved and technical we first give a sketch.

In a first step (3.12 - 3.17) we construct factorizations of $f: U \longrightarrow V(A,M,R)$

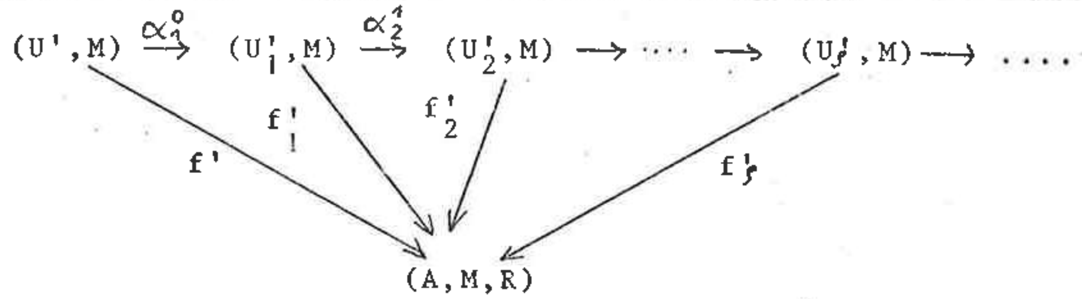


for every ordinal $\varrho < \beta$ such that $\pi(U_\varrho) \leq \gamma$ and for every operation $\mu \in M$ there is a morphism $\mu(\varrho,\varrho+1) : F_{d\mu}U_\varrho \longrightarrow F_{c\mu}U_{\varrho+1}$ making the diagram

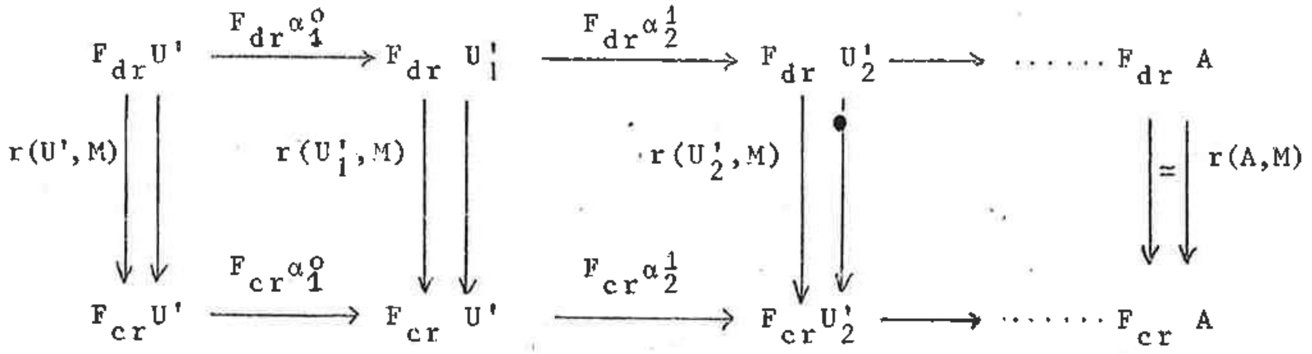


commutative. Using that $F_{d\mu}$ and $F_{c\mu}$ preserve β-filtered colimits, we obtain in the limit a morphism $F_{d\mu}(\varinjlim_{\varrho<\beta} U_\varrho) \longrightarrow F_{c\mu}(\varinjlim_{\varrho<\beta} U_{\varrho+1})$ for every $\mu \in M$. The latter make $\varinjlim_{\varrho<\beta} U_\varrho$ into a pre-bialgebra and $\varinjlim_{\varrho<\beta} f_\varrho : \varinjlim_{\varrho<\beta} U \longrightarrow A$ into a pre-bialgebra morphism. Since $\beta < \gamma$ the colimit $U' = \varinjlim_{\varrho<\beta} U_\varrho$ is again γ-presentable. In this way one obtains a factorization of $f : U \longrightarrow A$ into a morphism $U \longrightarrow U'$ and a pre-bialgebra morphism $f' = \varinjlim_{\varrho<\beta} f_\varrho : (U',M) \longrightarrow (A,M)$ with (U',M) being γ-presentable in P-Bialg($\underline{A}$) (cf. 3.5).

In a second step (3.18-3.2o) we construct factorizations



in the category P-Bialg($\underline{A}$) for every $\rho < \beta$ such that $\pi(U'_\rho) \leq \gamma$ and for every relation $r \in R$ the vertical morphism pairs in the diagram



become equal when composed with the adjacent horizontal morphism (Note that the two components of r(A,M) coincide). Since relations on P-Bialg($\underline{A}$) commute with β-filtered colimits passing to the limit yields a pre-bialgebra $(U'',M) = \varinjlim_{\rho<\beta} (U'_\rho, M)$ which satisfies the relations. Since $\beta < \gamma$, one has also $\pi(U'') \leq \gamma$. Thus the induced factorization $(U',M) \longrightarrow (U'',M) \longrightarrow (A,M,R)$ of $f' : (U',M) \longrightarrow (A,M,R)$ together with the one from the first step yields the desired decomposition of $f : U \longrightarrow V(A,M,R)$.

Finally to show that a γ-presentable bialgebra (X,M,R) has a γ-presentable underlying object X we study the category of those bialgebras (U,M,R) over (X,M,R) whose underlying object U is γ-presentable. We show that (X,M,R) is the colimit in Bialg($\underline{A}$) of these bialgebras and that this (comma) category is γ-filtered. Thus the identity of (X,M,R) admits a factorization $(X,M,R) \longrightarrow (U,M,R) \longrightarrow (X,M,R)$ with $\pi(U) \leq \gamma$. Hence X is a retract of U and thus also γ-presentable. Conversely, if X is γ-presentable, then by 3.5 (X,M,R) is likewise in $\mathrm{Bialg}(\underline{A})$.

3.12 Let $R = \emptyset$ and let (A,M) be a bialgebra and $f : U \longrightarrow A$ a morphism with $\pi(U) \leq \gamma$. Let $(U_t,M)_{t\in T}$ be a family of bialgebras with $\mathrm{card}(T) < \gamma$ and let $(h_t : U_t \longrightarrow U)_{t\in T}$ be [redacted]family of morphisms such that U_t is γ-presentable and $f\circ h_t : (U_t,M) \longrightarrow (A,M)$ is a bialgebra morphism for every $t\in T$. We will show that f admits a factorization into a morphism $g' : U \longrightarrow U'$ and a bialgebra morphism $f' : (U',M) \longrightarrow (A,M)$ such that U' is γ-presentable and $g'\circ h_t : (U_t,M) \longrightarrow (U',M)$ is a bialgebra morphism for every $t\in T$.

3.13 Let $\underline{D}_{A,f}$ be the category whose objects are factorizations $i = (U \xrightarrow{g_i} U_i \xrightarrow{f_i} A)$ of f with $\pi(U_i) \leq \gamma$ and whose morphisms $i \longrightarrow j$ are morphisms $\alpha^i_j : U_i \longrightarrow U_j$ in $\underline{A}$ with $\alpha^i_j g_i = g_j$ and $f_j\alpha^i_j = f_i$. Since $\underline{D}_A = \underline{A}(\gamma)/A$ is γ-filtered (cf. 2.8), it easily follows that the functor

$$\underline{D}_{A,f} \longrightarrow \underline{D}_A , \quad (U \xrightarrow{g_i} U_i \xrightarrow{f_i} A) \rightsquigarrow (U_i \xrightarrow{f_i} A)$$

is cofinal and that $\underline{D}_{A,f}$ is also γ-filtered. Since $\beta < \gamma$ the category $\underline{D}_A$ has β-wellordered colimits which are computed pointwise. Hence the same holds for $\underline{D}_{A,f}$. For an ordinal $\lambda \leq \beta$ let $\underline{I}^-_\lambda$ (resp. $\underline{I}_\lambda$) denote the wellordered set of all ordinals $\rho < \lambda$ (resp. $\rho \leq \lambda$). By transfinite induction we will construct a functor

$$\Phi : \underline{I}^-_\beta \rightarrow \underline{D}_{A,f} , \quad \lambda \rightsquigarrow (U \xrightarrow{g_\lambda} U_\lambda \xrightarrow{f_\lambda} A)$$

with $$\Phi(0) = (U \xrightarrow{id} U \xrightarrow{f} A) \quad \text{such that}$$

$$\varinjlim \Phi = (U \rightarrow \varinjlim_{\lambda<\beta} U_\lambda \xrightarrow{\varinjlim f_\lambda} A)$$

is a factorization of $f : U \rightarrow A$ with the properties stated in 3.12. For $\rho < \tau$ in $\underline{I}^-_\beta$ the transition morphism $\Phi(\rho) \rightarrow \Phi(\tau)$ is denoted with α^ρ_τ .

3.14 The induction hypothesis for an ordinal λ is as follows.

There is a functor

(*) $\qquad \underline{I}_\lambda^- \longrightarrow \underline{D}_{A,f}, \quad \rho \rightsquigarrow (U \xrightarrow{g_\rho} U_\rho \xrightarrow{f_\rho} A)$

whose value at 0 is $(U \xrightarrow{id} U \xrightarrow{f} A)$ together with a morphism

(**) $\qquad \mu(\rho,\rho+1) : F_{d\mu}U_\rho \longrightarrow F_{c\mu}U_{\rho+1}$

for every $\mu \in M$ and every ρ such that $\rho+1 < \lambda$ subject to the following condition: For every $t \in T$ and for every pair ρ,τ with $\rho+1 < \tau+1 < \lambda$ the diagrams

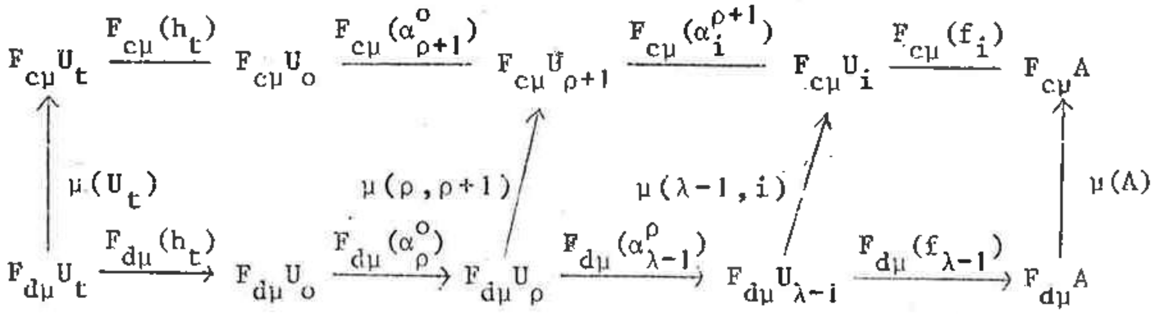
(***)
$$\begin{array}{ccccccccccc}
F_{c\mu}U_t & \xrightarrow{F_{c\mu}(h_t)} & F_{c\mu}U_o & \xrightarrow{F_{c\mu}(\alpha^o_\rho)} & F_{c\mu}U_\rho & \xrightarrow{F_{c\mu}(\alpha^\rho_{\rho+1})} & F_{c\mu}U_{\rho+1} & \xrightarrow{F_{c\mu}(\alpha^{\rho+1}_{\tau+1})} & F_{c\mu}U_{\tau+1} & \xrightarrow{F_{c\mu}(f_{\tau+1})} & F_{c\mu}A \\
\uparrow{\scriptstyle \mu(U_t)} & & & & & \nearrow{\scriptstyle \mu(\rho,\rho+1)} & & \nearrow{\scriptstyle \mu(\tau,\tau+1)} & & & \uparrow{\scriptstyle \mu(A)} \\
F_{d\mu}U_t & \xrightarrow{F_{d\mu}(h_t)} & F_{d\mu}U_o & \xrightarrow{F_{d\mu}(\alpha^o_\rho)} & F_{d\mu}U_\rho & \xrightarrow{F_{d\mu}(\alpha^\rho_\tau)} & F_{d\mu}U_\tau & & \xrightarrow{F_{d\mu}(f_\tau)} & & F_{d\mu}A
\end{array}$$

commute.

3.15 For $\lambda=1$ we put $\bar{\Phi}(0) = (id_U, f)$ and (*) trivially holds whereas (**) and (***) are vacuous. If λ is a limit ordinal $< \beta$, the functor $\bar{\Phi} : \underline{I}_\lambda^- \longrightarrow \underline{D}_{A,f}$ given by induction hypothesis has to be extended to $\underline{I}_\lambda$. Since $\underline{D}_{A,f}$ is γ-filtered, the image of $\bar{\Phi} : \underline{I}_\lambda^- \longrightarrow \underline{D}_{A,f}$ has an upper bound in $\underline{D}_{A,f}$; i.e. there is an object $(f_{\lambda'}, g_{\lambda'}) \in \underline{D}_{A,f}$ together with a morphism $\alpha^\tau_{\lambda'} : (f_\tau, g_\tau) \longrightarrow (f_{\lambda'}, g_{\lambda'})$ for every $\tau < \lambda$. Since $\underline{D}_{A,f}$ is γ-filtered, there is moreover an object $(f_\lambda, g_\lambda) \in \underline{D}_{A,f}$ together with a morphism $\alpha^{\lambda'}_\lambda : (f_{\lambda'}, g_{\lambda'}) \longrightarrow (f_\lambda, g_\lambda)$ such that for every pair $\rho < \tau$ in $\underline{I}_\lambda^-$ the equation $\alpha^\rho_\lambda = \alpha^\tau_\lambda \circ \alpha^\rho_\tau$ holds, where $\alpha^\tau_\lambda = \alpha^{\lambda'}_\lambda \circ \alpha^\tau_{\lambda'}$ and $\alpha^\rho_\lambda = \alpha^{\lambda'}_\lambda \circ \alpha^\rho_{\lambda'}$. Therefore we can define $\bar{\Phi}(\lambda) = (f_\lambda, g_\lambda)$ and $\bar{\Phi}(\rho<\alpha) = \alpha^\rho_\lambda$ and obtain an extension $\bar{\Phi} : \underline{I}_\lambda \longrightarrow \underline{D}_{A,f}$. Note that (**) and (***) hold trivially for every ρ with $\rho+1 \leq \lambda$ and every pair ρ,τ with

$\rho+1 < \tau+1 \leq \lambda$ because λ is a limit ordinal.

If λ is not a limit ordinal, then by assumption there is a functor $\Phi : \underline{I}_{\lambda-1} \to \underline{D}_{A,f}$ together with a morphism $\mu(\rho,\rho+1) : F_{d\mu}U_\rho \to F_{c\mu}U_{\rho+1}$ satisfying (***) for every $\mu \in M$ and every ρ with $\rho+1 < \lambda$. It suffices to construct $\alpha_\lambda^{\lambda-1} : (f_{\lambda-1}, g_{\lambda-1}) \to (f_\lambda, g_\lambda)$ and $\mu(\lambda-1,\lambda) : F_{d\mu}U_{\lambda-1} \to F_{c\mu}U_\lambda$ such that the conditions (*), (**) and (***) hold. For an operation $\mu \in M$ it follows from $F_{c\mu}A = \varinjlim_{i \in \underline{D}_{A,f}} F_{c\mu}U_i$ and $\pi(F_{d\mu}U_{\lambda-1}) \leq \gamma \geq \pi(F_{d\mu}U_\rho)$ and $\lambda < \gamma$ that there is an object $i = i(\mu)$ in $\underline{D}_{A,f}$ together with morphisms $\alpha_i^{\lambda-1} : (f_{\lambda-1}, g_{\lambda-1}) \to (f_i, g_i)$ and $\mu(\lambda-1,i) : F_{d\mu}U_{\lambda-1} \to F_{c\mu}U_i$ such that for every $t \in T$ and every $\rho < \lambda-1$ the two squares on the right in the diagram

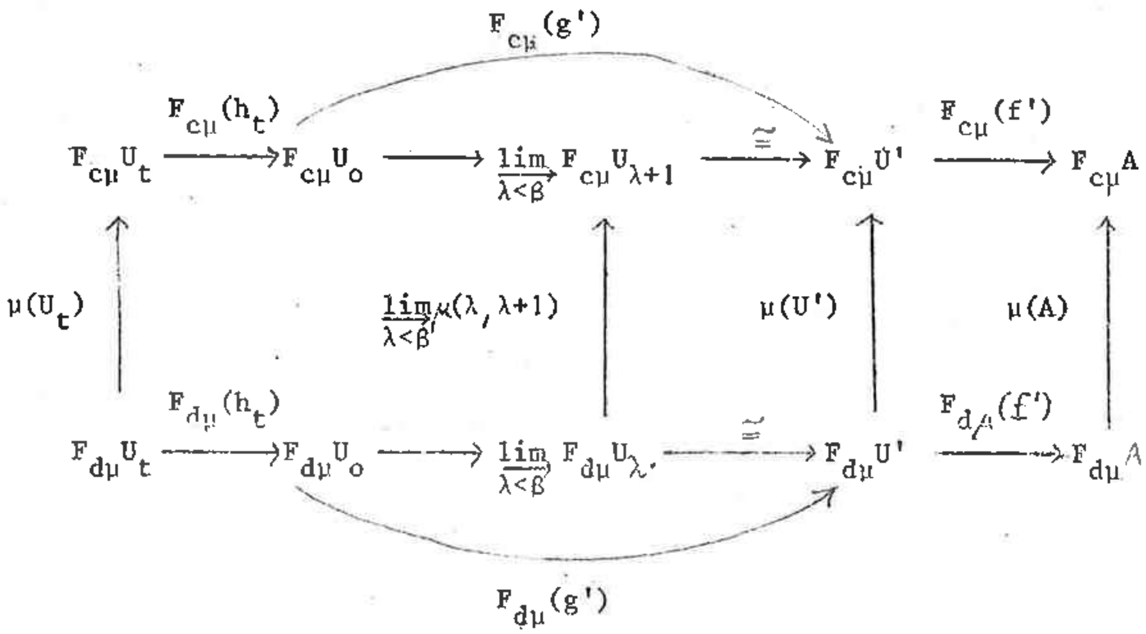


commute, where $\alpha_i^{\rho+1} = \alpha_i^{\lambda-1}\alpha_{\lambda-1}^{\rho+1}$. Note that for $\lambda-1 > 0$ the left side of the diagram commutes by induction hypothesis whereas for $\lambda = 1$ this can be established using $\pi(F_{d\mu}U_t) \leq \gamma > \mathrm{card}(T)$ in the same way as for the middle square. Since $\mathrm{card}(M) < \gamma$ and $\underline{D}_{A,f}$ is γ-filtered, there is an object $(U \xrightarrow{g_\lambda} U_\lambda \xrightarrow{f_\lambda} A)$ in $\underline{D}_{A,f}$ together with a morphism $\alpha_\lambda^i : (f_i, g_i) \to (f_\lambda, g_\lambda)$ for every $\mu \in M$ such that $\alpha_\lambda^i \circ \alpha_i^{\lambda-1} : (f_{\lambda-1}, g_{\lambda-1}) \to (f_\lambda, g_\lambda)$ is independant of $i = i(\mu)$. Hence we can define $\Phi(\lambda) = (f_\lambda, g_\lambda)$, $\Phi(\lambda-1<\lambda) = \alpha_\lambda^i \circ \alpha_i^{\lambda-1}$ and $\mu(\lambda-1,\lambda) = F_{c\mu}(\alpha_\lambda^i) \circ \mu(\lambda-1,i)$ for $\mu \in M$. With this one easily sees that $\Phi : \underline{I}_\lambda \to \underline{D}_{A,f}$ is an extension onto $\underline{I}_\lambda$ and that $\mu(\lambda-1,\lambda)$ satisfies (***).

3.16 We now construct a factorization of $f : U \to A$ into a morphism

$g' : U \to U'$ and a bialgebra morphism $f' : U' \to A$ such that the properties stated in 3.12 hold. Let $(U \xrightarrow{g'} U \xrightarrow{f'} A)$ be the colimit of $\Phi : \underline{I}_\beta \to \underline{D}_{A,f}$. Then, as mentioned in the middle of 3.13, we have $U' = \varinjlim_{\lambda<\beta} U_\lambda$ and $f' = \varinjlim_{\lambda<\beta} f_\lambda$ whereas $g' : U \to U'$ is the canonical morphism into the colimit.

For every $\mu \in M$ the functors $F_{d\mu}$ and $F_{c\mu}$ preserve β-filtered colimits, in particular $\varinjlim_{\lambda<\beta} F_{d\mu} U_\lambda \xrightarrow{\cong} F_{d\mu} \varinjlim_{\lambda<\beta} U_\lambda$ and $\varinjlim_{\lambda<\beta} F_{c\mu} U_\lambda \xrightarrow{\cong} F_{c\mu} \varinjlim_{\lambda<\beta} U_\lambda$. Passing to the colimit with 3.14 (**) and (***) yields an unique morphism $\mu(U') : F_{d\mu} U' \to F_{c\mu} U'$ such that the diagram



commutes. This shows that U' together with the morphisms $\mu(U')$, $\mu \in M$, is a bilagebra and that $f' : U' \to A$ and $g'h_t : U_t \to U'$ are bialgebra morphisms for every $t \in T$. This completes the proof of the assertion in 3.12.

3.17 For a bialgebra (A,M) let $\underline{D}_{(A,M)}$ be the category of bialgebras over (A,M) whose underlying object in $\underline{A}$ is γ-presentable. Recall that for every $A \in \underline{A}$ the category $\underline{D}_A = \underline{A}(\gamma)/A$ of γ-presentable objects over A is γ-filtered (even γ-cocomplete) and that the colimit of $\underline{D}_A \to \underline{A}, (U \xrightarrow{f} A) \rightsquigarrow U$, is A (cf. 2.8). From this and 3.12 it readily follows for a bialgebra (A,M) that the forgetful functor

(*) $\underline{D}_{(A,M)} \longrightarrow \underline{D}_A$, $\{(U,M) \xrightarrow{f} (A,M)\} \rightsquigarrow (U \xrightarrow{f} A)$

is cofinal and that $\underline{D}_{(A,M)}$ is γ-filtered (but in general not γ-cocomplete). In particular (A,M) is the colimit of

(**) $\underline{D}_{(A,M)} \longrightarrow \mathrm{Bialg}(\underline{A})$, $\{(U,M) \xrightarrow{f} (A,M)\} \rightsquigarrow (U,M)$

and $\underline{D}_{(A,M)}$ has β-wellordered colimits which are preserved by the functors (*) and (**).

3.18 We now return to the general case and drop the assumption $R = \emptyset$ which was made at the beginning of the proof in 3.12. For a bialgebra (A,M,R) let $\underline{D}_{(A,M,R)}$ be the category of bialgebras over (A,M,R) whose underlying object in $\underline{A}$ is γ-presentable. Clearly the forgetful functor

(3.19) $\underline{D}_{(A,M,R)} \longrightarrow \underline{D}_{(A,M)}$, $\{(U,M,R) \xrightarrow{f} (A,M,R)\} \rightsquigarrow \{(U,M) \xrightarrow{f} (A,M)\}$

is a full embedding. We will show below in 3.2o that it is cofinal. From this and 3.17 it follows that $\underline{D}_{(A,M,R)}$ is also γ-filtered and that (A,M,R) is the colimit of

$$\underline{D}_{(A,M,R)} \longrightarrow \mathrm{Bialg}(\underline{A}),\ \{(U,M,R) \xrightarrow{f} (A,M,R)\} \rightsquigarrow (U,M,R)$$

If (X,M,R) is γ-presentable in $\mathrm{Bialg}(\underline{A})$, then this implies that the identity of (X,M,R) admits a factorization $(X,M,R) \to (U,M,R) \xrightarrow{f} (X,M,R)$ with $f \in \underline{D}_{(X,M,R)}$. Hence X is a retract of U, in particular X is also γ-presentable. Conversely, if X is γ-presentable in $\underline{A}$, then by 3.5 (X,M,R) is γ-presentable in $\mathrm{Bialg}(\underline{A})$. This proves the second assertion of 3.8. Moreover this shows that the category $\underline{D}_{(A,M,R)}$ is the category of γ-presentable objects over (A,M,R) in $\mathrm{Bialg}(\underline{A})$ which completes the proof of 3.9.

3.2o For the cofinality of the functor 3.19 it suffices to show that a pre-bialgebra morphism $f : (U,M) \to (A,M,R)$ with $\pi(U) \leq \gamma$ factors through a bialgebra morphism $f' : (U',M,R) \to (A,M,R)$ such that $\pi(U') \leq \gamma$. The construction of f' is similar to 3.12 - 3.16. Let $\underline{D}_{(A,M),f}$ be the category whose objects are factorizations $i = \{(U,M) \xrightarrow{g_i} (U_i,M) \xrightarrow{f_i} (A,M)\}$ of f with $\pi(U_i) \leq \gamma$ and whose morphisms $i \to j$ are morphisms $\alpha^i_j : (U_i,M) \to (U_j,M)$ in P-Bialg($\underline{A}$) with the properties $f_i = f_j \alpha^i_j$ and $g_j = \alpha^i_j g_i$. Since $\underline{D}_{(A,M)}$ is γ-filtered and has β-wellordered colimits, it easily follows that the functor

$$\underline{D}_{(A,M),f} \longrightarrow \underline{D}_{(A,M)}, \quad \{g_i, f_i\} \rightsquigarrow \{f_i : (U_i,M) \to (A,M)\}$$

is cofinal and that $\underline{D}_{(A,M),f}$ is also γ-filtered and has β-well= ordered colimits.

Recall that $\underline{I}^-_\lambda$ (resp. $\underline{I}_\lambda$) denotes the wellordered set of all ordinals $\rho < \alpha$ (resp. $\rho \leq \lambda$). By means of transfinite inductions we construct a functor

$$\Omega : \underline{I}^-_\beta \longrightarrow \underline{D}_{(A,M),f}$$

with

$$\Omega(0) = \{(U,M) \xrightarrow{id} (U,M) \xrightarrow{f} (A,M)\}$$

such that the factorization $\varinjlim \Omega = \{(U,M) \xrightarrow{g'} (U',m) \xrightarrow{f'} (A,M)\}$ of f has the required properties. We write $\Omega(\rho) = \{(U,M) \xrightarrow{g_\rho} (U_\rho,M) \xrightarrow{f_\rho} (A,M)\}$ for $\rho \in \underline{I}^-_\beta$ and $\Omega(\rho<\tau) = \alpha^\rho_\tau$ for $\rho < \tau$ in $\underline{I}^-_\beta$.

We define $\Omega(0) = \{id_U, f\}$. Assume Ω has been constructed for all $\rho < \lambda$, i.e. there is a functor $\Omega : \underline{I}^-_\lambda \to \underline{D}_{(A,M),f}$ with $\Omega(0) = \{id_U, f\}$. $\Omega(0) = \{id_U, f\}$. If λ is a limit ordinal we extend Ω to $\underline{I}_\lambda$ by defining $\Omega(\lambda)$ as an appropriate upper bound of the image of $\Omega : \underline{I}^-_\lambda \to \underline{D}_{(A,M),f}$ (the details are as above in 3.15). Now let λ be a successor ordinal. For every relation $r \in R$ we have $\pi(F_{dr}U_{\lambda-1}) \leq \gamma$. Since $F_{cr}A = \varinjlim_{i \in \underline{D}_{(A,M)f}} F_{cr}U_i$ is a γ-filtered colimit and the morphisms

$r(A,M) : F_{dr}A \rightrightarrows F_{cr}A$ coincide, there is a factorization $\{(U,M) \xrightarrow{g_i} (U_i,M) \xrightarrow{f_i} (A,M)\}$ and a morphism

$$\alpha_i^{\lambda-1} : (f_{\lambda-1}, g_{\lambda-1}) \longrightarrow (f_i, g_i)$$

in $\underline{D}_{(A,M),f}$ depending on r such that the morphisms

$$r(U_{\lambda-1},M) : F_{dr}U_{\lambda-1} \rightrightarrows F_{cr}U_{\lambda-1}$$

become equal when composed with $F_{cr}\alpha_i^{\lambda-1} : F_{cr}U_{\lambda-1} \longrightarrow F_{cr}U_i$. Since card(R) $< \gamma$ and $\underline{D}_{(A,M),f}$ is γ-filtered there is a factorization

$$\{(U,M) \xrightarrow{g_\lambda} (U_\lambda,M) \xrightarrow{f_\lambda} (A,M)\}$$

together with morphisms $\alpha_\lambda^i : (f_i,g_i) \longrightarrow (f_\lambda,g_\lambda)$ in $\underline{D}_{(A,M),f}$ such that $\alpha_\lambda^{\lambda-1} = \alpha_\lambda^i \circ \alpha_i^{\lambda-1} : (f_{\lambda-1},g_{\lambda-1}) \longrightarrow (f_\lambda,g_\lambda)$ is independant of r. Thus we can define $\Omega(\lambda) = (g_\lambda,f_\lambda)$ and $\Omega(\lambda-1 < \lambda) = \alpha_\lambda^{\lambda-1}$ and it is clear that $\Omega : \underline{I}_\lambda \longrightarrow \underline{D}_{(A,M),f}$ is a functor. This shows that there is a functor $\Omega : \underline{I}_\beta^- \to \underline{D}_{(A,M),f}$ with $\Omega(0) = \{id_U, f\}$. By 3.17 and the cofinality of the forgetful functor

$$\underline{D}_{(A,M),f} \longrightarrow \underline{D}_{(A,M)},\ (f_i,g_i) \leadsto f_i$$

the colimit of Ω exists and can be computed pointwise, i.e.

$$\varinjlim \Omega = \{(U,M) \xrightarrow{g'} (\varinjlim_{\lambda<\beta} U_\lambda, M) \xrightarrow{\varinjlim f_\lambda} (A,M)\}$$

where g' denotes the canonical morphism into the colimit. From the construction of $\alpha_{\lambda+1}^\lambda : (f_\lambda,g_\lambda) \longrightarrow (f_{\lambda+1},g_{\lambda+1})$ and the diagram

$$\begin{array}{ccccccc}
F_{cr}U_\lambda & \xrightarrow{F_{cr}(\alpha^\lambda_{\lambda+1})} & F_{cr}U_{\lambda+1} & \xrightarrow{\text{can.}} & \varinjlim_{\lambda<\beta} F_{cr}U_\lambda & \xrightarrow{\cong} & F_{cr}\varinjlim_{\lambda<\beta} U_\lambda \\
{\scriptstyle r(U_\lambda,M)}\upuparrows & & \upuparrows{\scriptstyle r(U_{\lambda+1},M)} & & \upuparrows{\scriptstyle \varinjlim_{\lambda<\beta} r(U_\lambda,M)} & & \upuparrows{\scriptstyle r(\varinjlim_{\lambda<\beta} U_\lambda,M)} \\
F_{dr}U_\lambda & \xrightarrow{F_{dr}(\alpha^\lambda_{\lambda+1})} & F_{dr}U_{\lambda+1} & \xrightarrow{\text{can.}} & \varinjlim_{\lambda<\beta} F_{dr}U_\lambda & \xrightarrow{\cong} & F_{dr}\varinjlim_{\lambda<\beta} U_\lambda
\end{array}$$

it follows for every $r \in R$ that the two components of the morphism pair $r(\varinjlim_{\lambda<\beta} U_\lambda, M)$ coincide. Hence $(\varinjlim_{\lambda<\beta} U_\lambda, M)$ is a bialgebra.

3.21 Definition A sub-bialgebra of a bialgebra (A,M,R) is a bialgebra (U,M,R) together with a bialgebra morphism $f : (U,M,R) \to (A,M,R)$ whose underlying morphism in $\underline{A}$ is a monomorphism.
Clearly $f : (U,M,R) \longrightarrow (A,M,R)$ is then also a monomorphism in Bialg($\underline{A}$). However the forgetful functor V : Bialg($\underline{A}$) $\to \underline{A}$ does not preserve monomorphisms in general (for an exception see 3.4 a)).
The question arises whether there is an assertion analogous to 3.8 for α-generated objects. This is not so. The reason for this asymmetry lies in the fact that the underlying functor V : Bialg($\underline{A}$) $\to \underline{A}$ and the functors $F \in \mathbb{F}_c$ need not preserve monomorphisms. We give below in 3.22 a version of 3.8 for γ-generated objects correcting this deficiency by additional assumptions. From the point of view of applications 3.22 is useful in either of the following situations: 1) $\underline{A}$ is locally γ-noetherian, i.e. every γ-generated object is γ-presentable, cf. [13] 9.19. or 2) every $F \in \mathbb{F}_c$ preserves finite limits, e.g. in the algebraic case 3.2' II).

3.22 Theorem Let $\underline{A}$ be a locally presentable category with a data M , R and $\mathbb{F}$ for bialgebras (3.1). Assume there is a regular cardinal β such that

1) every $F \in \mathbb{F}$ preserves β-filtered colimits
2) every β-wellordered colimit of monomorphisms in $\underline{A}$ is again a monomorphism.

Let $\gamma > \beta$ be any regular cardinal such that

3) card(M) $< \gamma$ and card(R) $< \gamma$
4) $\underline{A}$ is locally γ-noetherian and if $U \in \underline{A}$ is γ-presentable, then FU is γ-presentable for every $F \in \mathbb{F}_d$ (cf. 3.6, 3.7 for $\gamma = \bar{\alpha}$).

Instead of 4) one can assume

4)' A is locally γ-generated (cf. [13] 9.1,) and every $F \in \mathbb{F}_c$ preserves finite limits; moreover if $U \in A$ is γ-generated, then FU is γ-generated for every $F \in \mathbb{F}_d$ (in this case the assumption $\mathrm{card}(R) < \gamma$ is redundant).

Then the following hold.

a) If (A,M,R) is a bialgebra and $U \in A$ is a γ-generated subobject of A, then there is a sub-bialgebra $(U',M,R) \hookrightarrow (A,M,R)$ such that U' contains U and U' is also γ-generated.

b) A bialgebra (X,M,R) is γ-generated in Bialg(A) iff X is γ-generated in A.

c) A bialgebra (A,M,R) is the γ-filtered colimit in Bialg(A) of its γ-generated sub-bialgebras.

d) If A is locally γ-noetherian, then every γ-generated bialgebra is γ-presentable in Bialg(A); in particular if Bialg(A) is cocomplete (cf. 3.24 a), b) and 3.27 below), then Bialg(A) is locally γ-noetherian.

3.23 Remarks

a) Note that γ has to be strictly bigger than β, hence $\gamma \geq \aleph_1$. If the codomain of every $F \in \mathbb{F}_d$ is locally presentable, then by 3.7 resp. 5.1 there is always a cardinal $\gamma > \beta$ such that the above conditions 3) and 4) hold (resp. the second half of 4)'). The point is of course to choose γ as small as possible. The most useful situation seems $\gamma = \aleph_1$ and $\beta = \aleph_0$. This happens in any of the following cases.

I Every $F \in \mathbb{F}$ preserves filtered colimits, $\mathrm{card}(M) \leq \aleph_0 \geq \mathrm{card}(R)$, A is locally finitely noetherian (resp. A is locally finitely generated), every $F \in \mathbb{F}_c$ takes finitely generated objects into countably presentable objects (resp. into countably generated objects and every $F \in \mathbb{F}_c$ preserves finite limits), cf. Corollary to 3.7.

II Every $F \in \mathbb{F}$ preserves filtered colimits and every countable colimit of monomorphisms in $\underline{A}$ is again a monomorphism, $\mathrm{card}(M) \leq \aleph_0 \geq \mathrm{card}(R)$, $\underline{A}$ is locally $\aleph_1$-noetherian (resp. $\underline{A}$ is locally $\aleph_1$-generated), every $F \in \mathbb{F}_d$ takes countably generated objects into countably presentable objects (resp. into countably generated objects, and every $F \in \mathbb{F}_c$ preserves finite limits), cf. Corollary to 3.7.

III Every $F \in \mathbb{F}$ preserves filtered colimits and every countable colimit of monomorphisms in $\underline{A}$ is again a monomorphism, $\mathrm{card}(M) \leq \aleph_0 \geq \mathrm{card}(R)$, $\underline{A}$ is locally $\aleph_1$-noetherian (resp. $\underline{A}$ is locally $\aleph_1$-generated).

Every $F \in \mathbb{F}_d$ has a right adjoint which preserves countably filtered colimits (resp. every $F \in \mathbb{F}_d$ has a right adjoint which preserves monomorphic countably filtered colimits, and every $F \in \mathbb{F}_c$ preserves finite limits), cf. 3.6.

b) As before in 3.8 the existence of arbitrary colimits in $\underline{A}$ is not needed for 3.22 (cf. 3.11 a), b)).

Proof of 3.22 The proof is the same as for 3.8 with the following obvious modifications. First for 3.12 - 3.16:

In 3.12 the morphisms $f : U \longrightarrow A$ and $h_t : U_t \longrightarrow U$, $t \in T$, are monomorphisms and $\varepsilon(U) \leq \gamma \geq \varepsilon(U_t)$. In 3.13 the category $\underline{D}_A$ consists of all γ-generated subobjects of A and likewise $\underline{D}_{A,f}$ consists of all γ-generated subobjects of A containing $f : U \longrightarrow A$ (clearly both categories are γ-filtered, [13] 9.1 - 9.3).

With this the proof (3.14 - 3.16) of 3.12 goes through without change because either by assumption 4) in 3.22) one has $\pi(F_{d\mu}U) \leq \gamma$ for every $\mu \in M$ and every $U \in \underline{A}$ with $\varepsilon(U) \leq \gamma$ or by 4)' in 3.22 one has $\varepsilon(F_{d\mu}U) \leq \gamma$ for every $U \in \underline{A}$ with $\varepsilon(U) \leq \gamma$ and the transition morphisms in $F_{c\mu}A = \varinjlim F_{c\mu}U_i$ are monomorphic for every $\mu \in M$. Note that by assumption 2) in 3.22 the induced morphism $\varinjlim_{\lambda<\beta} f_\lambda : \varinjlim_{\lambda<\beta} U_\lambda \longrightarrow A$ is again a monomorphism.

Second for 3.17 - 3.2o:

In 3.17 and 3.18 the categories $\underline{D}_{(A,M)}$ and $D_{(A,M,R)}$ consist of all sub-bialgebras of (A,M) (resp. sub-bialgebras of (A,M,R)) whose underlying object in $\underline{A}$ is γ-generated. In 3.2o the underlying morphism of $f : (U,M) \longrightarrow (A,M,R)$ in A is a monomorphism, and the category $\underline{D}_{(A,M),f}$ consists of all sub-prebialgebras of (A,M,R) which contain $f : (U,M) \longrightarrow (A,M,R)$ and whose underlying object in $\underline{A}$ is γ-generated.

With this the arguments in 3.17 - 3.2o go through without change. Note that as above by assumption 2) in 3.22 the induced morphism $\varinjlim_{\lambda<\beta} f_\lambda : \varinjlim_{\lambda<\beta}(U_\lambda,M) \longrightarrow (A,M)$ in 3.2o is again a monomorphism. Also note that in the presence of the assumption 1), 2), 3) and 4)' the cofinality argument in 3.2o is redundant because by 4)' a sub-prebialgebra of a bialgebra satisfies the relations automatically (whence the assumption card(R) < γ is not needed). Moreover in 3.19 a bialgebra (X,M,R) is γ-generated because of the remark following 3.5.

With these modifications it follows from 3.18 that the assertions a), b) and c) in 3.22 hold. As for d) it suffices to show that a γ-generated bialgebra (X,M,R) is γ-presentable in Bialg($\underline{A}$) . By 3.22 b) X is γ-generated in $\underline{A}$ and hence also γ-presentable because $\underline{A}$ is locally γ-noetherian. By 3.5 and assumption 4) in 3.22 (X,M,R) is γ-presentable in Bialg($\underline{A}$) .

We now investigate the completeness and cocompleteness of Bialg($\underline{A}$). Basically this occurs when the given data M , R and $\mathbb{F}$ for bialgebras (3.1) has one of the following properties: 1) every $F \in \mathbb{F}_c$ preserves limits (algebraic case, cf. 3.2 II), 2) every $F \in \mathbb{F}_d$ preserves colimits (coalgebraic case, cf. 3.2 II), and 3) the data M , R and $\mathbb{F}$ can be decomposed into one of type 1) and one of type 2) (roughly speaking every operation is algebraic or coalgebraic and every relation is algebraic or coalgebraic or a distributive law between an algebraic and coalgebraic operation).

3.24 Theorem Let $\underline{A}$ be a locally presentable category with a data M , R and $\mathbb{F}$ for bialgebras (cf. 3.1). Assume there is a regular cardinal β such that every $F \in \mathbb{F}$ preserves β-filtered colimits. Let $\gamma > \beta$ be any regular cardinal such that

1) $\mathrm{card}(M) < \gamma > \mathrm{card}(R)$ and $\underline{A}$ is locally γ-presentable,
2) if $U \in \underline{A}$ is γ-presentable, then FU is γ-presentable for every $F \in \mathbb{F}_d$ (cf. 3.6, 3.7 for $\gamma = \bar{\alpha}$).

Then the following hold.

a) If every $F \in \mathbb{F}_d$ preserves colimits, then $\mathrm{Bialg}(\underline{A})$ is locally γ-presentable and the forgetful functor $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ is cotripleable. The right adjoint $cF : \underline{A} \longrightarrow \mathrm{Bialg}(\underline{A})$ of V preserves γ-filtered colimits (cF = cofree functor).

b) If every $F \in \mathbb{F}_c$ preserves limits, then $\mathrm{Bialg}(\underline{A})$ is locally $\sup(\beta, \pi(\underline{A}))$-presentable and the forgetful functor $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ is tripleable and preserves β-filtered colimits. (The left adjoint $F : \underline{A} \longrightarrow \mathrm{Bialg}(\underline{A})$ of V is the free functor).

Remark Note the asymmetry between $\sup(\beta, \pi(\underline{A}))$ and γ in a) and b). For the locally γ-noetherian case see 3.22 d). For conditions guaranteeing $\beta = \aleph_0$ and $\gamma = \aleph_1$ see the remark following 3.3.

3.25 Corollary Let $\underline{A}$ be a Grothendieck category (resp. a topos) with a data M , R and $\mathbb{F}$ for bialgebras. If every $F \in \mathbb{F}_d$ preserves colimits and every $F \in \mathbb{F}_c$ finite limits, then $\mathrm{Bialg}(\underline{A})$ is again a Grothendieck category (resp. a topos). This follows from 3.24 a), 4.11 and 3.3.

Proof a) It follows from 3.8 and 3.4 b) that $\mathrm{Bialg}(\underline{A})$ is locally γ-presentable. By the special adjoint functor theorem the forgetful functor $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ has a right adjoint $cF : \underline{A} \longrightarrow \mathrm{Bialg}(\underline{A})$, whence by 3.4 b) V is cotripleable. For every γ-presentable object $(U,M,R) \in \mathrm{Bialg}(\underline{A})$ the functors $[(U,M,R), cF-]$ and $[U,-]$ are equivalent by adjointness, and by 3.8 U is γ-presentable. For a γ-filtered colimit $X = \varinjlim_\nu X_\nu$ the canonical morphism $\varphi : \varinjlim_\nu cFX_\nu \longrightarrow cF(\varinjlim_\nu X_\nu)$ gives rise to a commutative diagram

$$\begin{array}{ccc} [(U,M,R), \varinjlim_\nu cFX_\nu] & \xrightarrow{[(U,M,R),\varphi]} [(U,M,R), cF(\varinjlim_\nu X_\nu)] \xrightarrow{\cong} & [U, \varinjlim_\nu X_\nu] \\ \downarrow \cong & & \cong \downarrow \\ \varinjlim_\nu [(U,M,R), cFX_\nu] & \xrightarrow{\cong} & \varinjlim_\nu [U,X_\nu] \end{array}$$

Hence $[(U,M,R), \varphi]$ is a bijection for every γ-presentable object $(U,M,R) \in \mathrm{Bialg}(\underline{A})$. Since these objects forma set of (dense) generators in $\mathrm{Bialg}(\underline{A})$ (cf. 3.9), it follows that φ is an isomorphism. Thus $cF : \underline{A} \longrightarrow \mathrm{Bialg}(\underline{A})$ preserves γ-filtered colimits.

b) By 3.4 a), c) $\mathrm{Bialg}(\underline{A})$ has limits and β-filtered colimits and $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ preserves and reflects them. In order to show that V has a left adjoint, we verify the solution set condition. For every object $U \in \underline{A}$ there is a regular cardinal δ such that U is δ-presentable. By 3.7 there is a regular cardinal γ such that $\delta \leq \gamma > \beta$ and the conditions a) and b) of 3.8 hold for γ . Since the category $\underline{A}(\gamma)$ of γ-presentable objects in $\underline{A}$ is small, it follows from 3.8 and 3.1 that the same holds for $\underline{Y}(\gamma)$ (see 3.9), where $\underline{Y} = \mathrm{Bialg}(\underline{A})$. It then follows from 3.8 that a set of representatives of $\underline{Y}(\gamma)$ - i.e. a skeleton - is a solution set for U . Hence $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ has a left adjoint F and is tripleable by 3.4 a). The composite $V \circ F : \underline{A} \longrightarrow \underline{A}$ preserves β-filtered colimits and it therefore follows from Gabriel-Ulmer [13] 1o.3 that $\mathrm{Bialg}(\underline{A})$ is locally $\sup(\beta, \pi(\underline{A}))$-presentable.

3.26 It is well known that the category of commutative (resp. cocommutative) Hopfalgebras over a commutative ring Λ can be viewed as the category of cogroup (resp. group) objects in the category of commutative Λ-algebras (resp. cocommutative Λ-coalgebras). Similarly the category of commutative (resp. cocommutative) Λ-bialgebras can be viewed as the category of comonoid (resp. monoid) objects in the category of commutative Λ-algebras (resp. cocommutative Λ-coalgebras). In both cases this rests on the fact that in the category of commutative Λ-algebras (resp. cocommutative Λ-coalgebras) the categorical coproduct (resp. product) is the tensor product lifted from $\underline{\mathrm{Mod}}_\Lambda$. Thus theorem 3.24 can be applied twice - first a) and then b) or vice versa- and it follows that any of the above categories is locally $\aleph_1$-presentable. However the category of arbitrary Λ-bialgebras (resp. Λ-Hopfalgebras) cannot be expressed this way because the tensor product lifted to the category of Λ-algebras or Λ-coalgebras is not the categorical coproduct or product. The following is motivated to rectify this, at least in part.

3.27 Definition Let M , R and $\mathbb{F}$ be a data for bialgebras in $\underline{A}$ (3.1). A decomposition of M , R and $\mathbb{F}$ into an algebraic and coalgebraic part consists of a data $\bar{M}$, $\bar{R}$ and $\bar{\mathbb{F}}$ in $\underline{A}$ and a data $\bar{\bar{M}}$, $\bar{\bar{R}}$ and $\bar{\bar{\mathbb{F}}}$ in $\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A})$ with the following properties:

1) $\mathrm{Bialg}_{M,R}(\underline{A}) \cong \mathrm{Bialg}_{\bar{\bar{M}},\bar{\bar{R}}}(\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}))$,
2) every $\bar{F} \in \bar{\mathbb{F}}_c$ preserves limits,
3) every $\bar{\bar{F}} \in \bar{\bar{\mathbb{F}}}_d$ preserves colimits.

Likewise a decomposition of M , R and $\mathbb{F}$ into a coalgebraic and algebraic part consists of a data $\bar{M}$, $\bar{R}$ and $\bar{\mathbb{F}}$ in $\underline{A}$ and a data $\bar{\bar{M}}$, $\bar{\bar{R}}$ and $\bar{\bar{\mathbb{F}}}$ in $\mathrm{Bialg}_{\bar{M},\bar{R}}(A)$ with the properties

1) $\mathrm{Bialg}_{M,R}(\underline{A}) \cong \mathrm{Bialg}_{\bar{\bar{M}},\bar{\bar{R}}}(\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}))$
2) every $\bar{F} \in \bar{\mathbb{F}}_d$ preserves colimits,
3) every $\bar{\bar{F}} \in \bar{\bar{\mathbb{F}}}_c$ preserves limits.

For example to express the category Λ-Bialg of arbitrary Λ-bialgebras in this way let $\underline{A} = \underline{Mod}_\Lambda$ and choose $\bar{M}$ to consist of a multiplication $\mu : id_{\underline{A}} \otimes id_{\underline{A}} \longrightarrow id_{\underline{A}}$ and a unit $\mu : const_\Lambda \longrightarrow id_{\underline{A}}$ and $\bar{R}$ of the associative and unitary laws. Then $\underline{B} = Bialg_{\bar{M},\bar{R}}(\underline{A})$ is obviously the category of Λ-algebras and the tensor product lifts from $\underline{Mod}_\Lambda$ to $\underline{B}$. Let $\bar{\bar{M}}$ in $\underline{B}$ consist of a comultiplication $\Delta : id_{\underline{B}} \longrightarrow id_{\underline{B}} \otimes_\Lambda id_{\underline{B}}$ and a counit $\varepsilon : id_{\underline{B}} \longrightarrow const_\Lambda$ and let $\bar{\bar{R}}$ consist likewise of the coassociative and counitary laws. With this one readily checks that $Bialg_{\bar{\bar{M}},\bar{\bar{R}}}(\underline{B})$ is canonically isomorphic with Λ-Bialg$(\underline{A})$, cf. 4.4 for details. Unfortunately it doesn't seem possible to express the category of arbitrary Λ-Hopfalgebras in a similar way. While the antipode can be viewed as a morphism $s : id_{\underline{B}} \longrightarrow id_{\underline{B}}^{opp}$ I don't know how to express the relations involving s in $\underline{B}$. One would have to show that for a Λ-bialgebra $(M,\mu,u,\Delta,\varepsilon)$ the composites
$M \xrightarrow{\Delta} M \otimes_\Lambda M \xrightarrow{s \otimes id} M \otimes_\Lambda M \xrightarrow{\mu} M$ and
$M \xrightarrow{\Delta} M \otimes_\Lambda M \xrightarrow{id \otimes s} M \otimes_\Lambda M \xrightarrow{\mu} M$ which are defined in $\underline{Mod}_\Lambda$ are multiplicative or antimultiplicative whithout using that they coincide with $M \xrightarrow{\varepsilon} \Lambda \xrightarrow{u} M$.

3.28 Theorem Let $\underline{A}$ be a locally presentable category. Let M, R and $\mathbb{F}$ be a data for bialgebras in $\underline{A}$ which admits a decomposition into an algebraic part $\bar{M}$, $\bar{R}$, $\bar{\mathbb{F}}$ and a coalgebraic part $\bar{\bar{M}}$, $\bar{\bar{R}}$, $\bar{\bar{\mathbb{F}}}$ (cf. 3.27). Assume there is a regular cardinal β such that every $\bar{F} \in \bar{\mathbb{F}}$ and every $\bar{\bar{F}} \in \bar{\bar{\mathbb{F}}}$ preserve β-filtered colimits. Let $\gamma > \beta$ be any regular cardinal such that

1) $card(\bar{M}) < \gamma$, $card(\bar{\bar{M}}) < \gamma$, $card(\bar{R}) < \gamma$, $card(\bar{\bar{R}}) < \gamma$ and $\underline{A}$ is locally γ-presentable,
2) if $U \in \underline{A}$ and $(X,\bar{M},\bar{R}) \in Bialg_{\bar{M},\bar{R}}(\underline{A})$ are γ-presentable, then $\bar{F}U$ and $\bar{\bar{F}}(X,\bar{M},\bar{R})$ are γ-presentable for every $\bar{F} \in \bar{\mathbb{F}}_d$ and $\bar{\bar{F}} \in \bar{\bar{\mathbb{F}}}_d$ (cf. 3.6, 3.7 for $\gamma = \tilde{\alpha}$).

Then $Bialg_{M,R}(\underline{A}) \cong Bialg_{\bar{\bar{M}},\bar{\bar{R}}}(Bialg_{\bar{M},\bar{R}}(\underline{A}))$ is locally γ-presentable

<u>and the underlying functors</u>

$$\mathrm{Bialg}_{M,R}(\underline{A}) \longrightarrow \mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}) \quad \text{and} \quad \mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}) \longrightarrow \underline{A}$$

<u>are cotripleable and tripleable respectively.</u>

<u>Proof</u> By 3.24 b) the underlying functor $\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A}) \longrightarrow \underline{A}$ is tripleable and $\mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A})$ is locally γ-presentable. Likewise by 3.24 a) the underlying functor $\mathrm{Bialg}_{M,R}(\underline{A}) \longrightarrow \mathrm{Bialg}_{\bar{M},\bar{R}}(\underline{A})$ is cotripleable and $\mathrm{Bialg}_{M,R}(\underline{A})$ is locally γ-presentable.

3.29 <u>Remark</u> In the same way one considers morphism between algebraic theories and the corresponding algebraic functors (cf. Lawvere [21]), one can study morphisms between data for bialgebras. For a given data M, R in $\underline{A}$ and a subset $M' \subset M$ there is an obvious relative forgetful functor

$$V_{rel} : \text{P-}\mathrm{Bialg}_M(\underline{A}) \longrightarrow \text{P-}\mathrm{Bialg}_{M'}(\underline{A}) \ , \ (A,\mu(A))_{\mu \in M} \rightsquigarrow (A,\mu(A))_{\mu \in M'} \ .$$

Let R' be a set of relations on $\text{P-}\mathrm{Bialg}_{M'}(\underline{A})$ which hold in $\mathrm{Bialg}_{M,R}(\underline{A})$, i.e. $V_{rel}(A,M,R) \in \mathrm{Bialg}_{M',R'}(\underline{A})$ for every $(A,M,R) \in \mathrm{Bialg}_{M,R}(\underline{A})$. Then there is also an induced forgetful functor

$$V_{rel} : \mathrm{Bialg}_{M,R}(\underline{A}) \longrightarrow \mathrm{Bialg}_{M',R'}(\underline{A})$$

One can easily generalize the results of this chapter - in particular 3.8, 3.22, 3.24, 3.28 - to this situation. But in general it is difficult to find a data M'', R'' in $\mathrm{Bialg}_{M'R'}(\underline{A})$ - hopefully simpler than M, R - such that $\mathrm{Bialg}_{M,R}(\underline{A}) \cong \mathrm{Bialg}_{M'',R''}(\mathrm{Bialg}_{M',R'}(\underline{A}))$; (see 3.26, 3.27 for cases like $\mathrm{Bialg}(\underline{A}) \cong \mathrm{Coalg}(\mathrm{Alg}(\underline{A}))$.

§ 4 Examples of bialgebras in locally presentable categories

In this section we give a first series of examples of bialgebras and apply the main results of § 3. A second series can be found in § 6. In the following we discuss universal algebras (4.1), universal coalgebras (4.2), coalgebras over a commutative ring Λ (4.3), Λ-bialgebras and Λ-Hopfalgebras and generalizations (4.4 - 4.7), comodules over a Λ-coalgebra (4.8), bimodules over a Λ-bialgebra (4.9), coalgebras over a cotriple (4.10 - 4.12), algebras over a triple (4.13), données de recollement and descent data (4.14 - 4.18) and more generally sections and cartesian closed sections with respect to a fibration or cofibration (4.19 - 4.26). Although some of these cases are dual to each other as far as the data for bialgebras is concerned, the assertions resulting from 3.7, 3.8, 3.9, 3.22, 3.24 and 3.28 are not and can be quite different. We always assume the base category $\underline{A}$ to be locally presentable although, as for 3.8, 3.9 and 3.22 the existence of arbitrary colimits in $\underline{A}$ is not needed. We mostly leave the generalization by means of 3.11 to the reader.

We use the following notation for a data (3.1) of bialgebras M, R, $\mathbb{F}$: For an operation $\mu \in M$ and a relation $r \in R$ we write $\mu : F_{d\mu} \dashrightarrow F_{c\mu}$ and $r : F_{dr} \rightrightarrows F_{cr}$ respectively. A data will often be given by first specifying the set $\mathbb{F}$ of support functors and then indicating the operations and relations in this form.

4.1 <u>universal algebra.</u>

Let $\underline{A}$ be a category with finite products. Let Θ be a finitary algebraic theory in the sense of Lawvere [21] (or Birkhoff), e.g. groups, rings, algebras... . Let M be a set of defining operations and R a set of defining relations for Θ in the usual sense. For $\mu \in M$ let $F_{c\mu} = id_{\underline{A}}$ and let $F_{d\mu} : \underline{A} \longrightarrow \underline{A}$ be the functor $A \rightsquigarrow \prod_{n_\mu} A$ which assigns to an object its n_μ-fold product, where n_μ is the arity of μ . A pre-bialgebra (A,M) is an object $A \in \underline{A}$ together with a morphism $\prod_{n_\mu} A \longrightarrow A$ for every $\mu \in M$. For a relation $r \in R$ let $F_{cr} = id_{\underline{A}}$ and let $F_{dr} : \underline{A} \longrightarrow \underline{A}$ be the functor $A \rightsquigarrow \prod_{n_r} A$, where n_r denotes the arity of r . The functor F_{dr} is also denoted with $id_{\underline{A}}^{n_r}$. Since relations are built up of operations and projections, for every pre-bialgebra (A,M) and every relation $r \in R$ there is a morphism pair $r(A,M) : \prod_{n_r} A \rightrightarrows A$ which is natural in (A,M) - i.e. with respect to pre-bialgebra morphisms. Note that $\mathbb{F}_c = \{id_{\underline{A}}\}$ and $\mathbb{F}_d = \{id_{\underline{A}}^{\,0} , id_{\underline{A}} , id_{\underline{A}}^{\,2} \ldots\}$. It is straight forward that Bialg($\underline{A}$) is isomorphic with the category Θ-Alg($\underline{A}$) of Θ-algebras in $\underline{A}$ (i.e. the category of product preserving functors $\Theta \rightarrow \underline{A}$) .

Assume that $\underline{A}$ is locally α-presentable. Let β be the least regular cardinal such that β-filtered colimits commute with finite products, whence $\beta \leq \alpha$ by [13] 7.12. By an obvious cofinality argument for every $n \geq 0$ the functor $\underline{A} \longrightarrow \underline{A}$, $A \rightsquigarrow \prod_n A$ preserves β-filtered colimits, moreover it is right adjoint to $\underline{A} \longrightarrow \underline{A}$, $A \rightsquigarrow \coprod_n A$. Thus by 3.24 b) and 3.7 (remark) Θ-Alg($\underline{A}$) is locally α-presentable and the forgetful functor $V : \Theta\text{-Alg}(\underline{A}) \longrightarrow \underline{A}$ is tripleable and preserves β-filtered colimits (cf. also [13] 11.4).

Let γ be a regular cardinal such that

1) $\beta < \gamma \geq \alpha$, 2) card(M) $< \gamma >$ card(R) and 3) if $A \in \underline{A}$ is γ-presentable, then so is $\prod_n A$ for every finite $n \geq 0$ (cf. 3.7 remarks). Then by 3.8 a Θ-algebra (X,M,R) is γ-presentable in Θ-Alg($\underline{A}$) iff X is γ-presentable in $\underline{A}$.

Likewise, if γ is a regular cardinal such that

1) $\beta < \gamma \geq \alpha$, 2) card(M) $< \gamma$ and 3) if $A \in \underline{A}$ is γ-generated, then so is $\prod_n A$ for every finite $n \geq 0$, then a Θ-algebra (A,M,R) is γ-generated in Θ-Alg($\underline{A}$) iff A is γ-generated in $\underline{A}$ (cf. 3.22). If in addition $\underline{A}$ is locally γ-noetherian, then so is Θ-Alg($\underline{A}$) .

The generalizations to non-finitary theories in the sense of Linton [23] with rank or to partial operations are obvious generalizations and left to the reader (see also 6.14). The above can be generalized to categories $\underline{A}$ which are dual to a locally presentable category. By means of 4.2 below and $\Theta\text{-Alg}(\underline{A}) \cong (\Theta\text{-Coalg}(\underline{A}^{o}))^{o}$ it follows that the category of Θ-algebras in the dual of a locally presentable category is itself the dual of a locally presentable category. In particular if $\underline{A} = \underline{\text{Comp}}$ (compact spaces) or $\underline{A}$ is any Grothendieck AB $5)^{*}$ category with cogenerators, then $\Theta\text{-Alg}(\underline{A})$ is the dual of a locally presentable category.

4.2 universal coalgebra

Let $\underline{A}$ be a category with finite coproducts. Let Θ be a finitary algebraic theory and let M and R be sets of defining operations and relations as above. For $\mu \in M$ let $F_{d\mu} = id_{\underline{A}}$ and let $F_{c\mu} : \underline{A} \longrightarrow \underline{A}$ be the functor $A \rightsquigarrow \coprod_{n} A$. A pre-bialgebra (A,M) is an object $A \in \underline{A}$ together with a morphism $A \longrightarrow \coprod_{n_\mu} A$ for every $\mu \in M$. Likewise for a relation $r \in R$ let $F_{dr} = id_{\underline{A}}$ and let $F_{cr} : \underline{A} \longrightarrow \underline{A}$, $A \rightsquigarrow \coprod_{n_r} A$. As above there is for every pre-bialgebra (A,M) a morphism pair $r(A,M) : A \rightrightarrows \coprod_{n_r} A$ and the category $\text{Bialg}(\underline{A})$ is isomorphic with the category $\Theta\text{-Coalg}(\underline{A})$ of Θ-coalgebras in $\underline{A}$. Note that $\mathbb{F}_d = \{id_{\underline{A}}\}$ and $\mathbb{F}_c = \{id_{\underline{A}}^{(0)}, id_{\underline{A}}^{(1)}, id_{\underline{A}}^{(2)},\}$, where $id_{\underline{A}}^{(n)}$ denotes the functor $A \rightsquigarrow \coprod_{n} A$. If $\underline{A}$ has finite products, then $\underline{A} \longrightarrow \underline{A}$, $A \rightsquigarrow \coprod_{n} A$ is left adjoint to $A \rightsquigarrow \prod_{n} A$.

Assume that $\underline{A}$ is locally presentable and let

$$\gamma \geq \sup \{\aleph_1, \pi(\underline{A}), \text{card}(M)^{+}, \text{card}(R)^{+}\} .$$

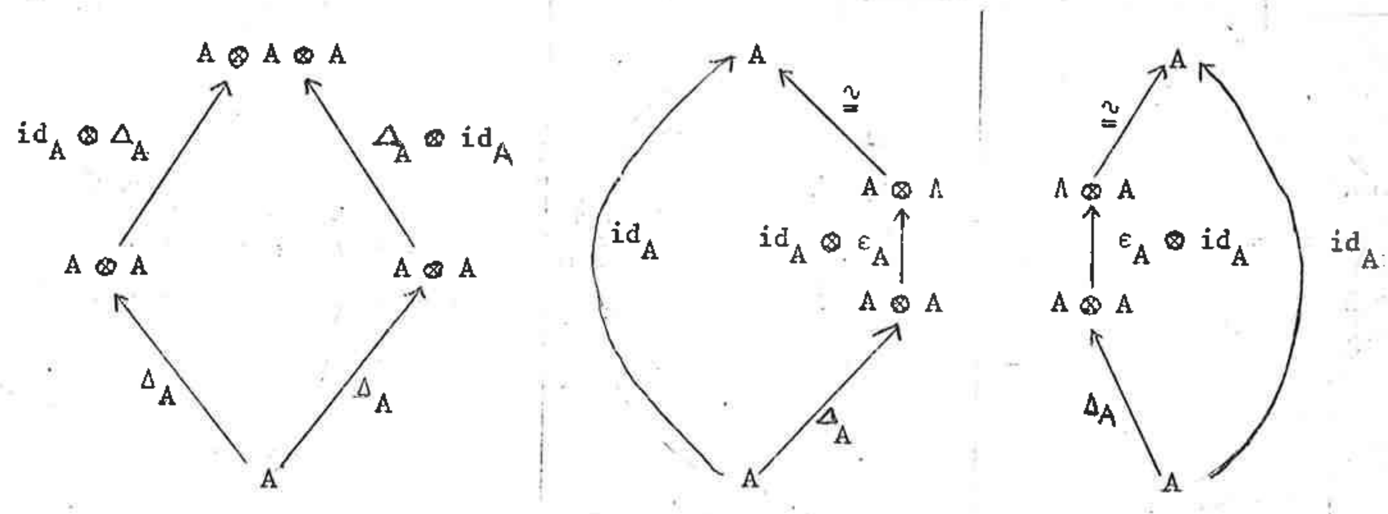
(Recall that δ^{+} denotes the least regular cardinal $> \delta$.) Since $\underline{A}(\gamma)$ is closed in $\underline{A}$ under finite coproducts (cf. 2.8), it follows from 3.24 a) that the category $\Theta\text{-Coalg}(\underline{A})$ is locally γ-presentable and the underlying functor $V : \Theta\text{-Coalg}(\underline{A}) \longrightarrow \underline{A}$ is cotripleable and its right adjoint $cF : \underline{A} \longrightarrow \Theta\text{-Coalg}(\underline{A})$ preserves γ-filtered colimits. Moreover by 3.8 a Θ-coalgebra (X,M,R) is γ-presentable in $\Theta\text{-Coalg}(\underline{A})$

iff X is γ-presentable in $\underline{A}$, in particular a morphism $U \dashrightarrow (A,M,R)$ with $\pi(U) \leq \gamma$ admits a decomposition into a morphism $U \longrightarrow U'$ and a Θ-coalgebra morphism $(U',M,R) \longrightarrow (A,M,R)$ such that $\pi(U') \leq \gamma$. Likewise if $\underline{A}$ is locally γ-noetherian and if in $\underline{A}$ β-filtered colimits of monomorphisms are monomorphic for some $\beta < \gamma$, then by 3.22 d) Θ-Coalg($\underline{A}$) is locally γ-noetherian. In addition every γ-generated subobject of a Θ-coalgebra is contained in a Θ-subcoalgebra whose underlying object is also γ-generated. (Note if $\underline{A}$ is not locally γ-noetherian, then the latter need not hold, in particular a Θ-coalgebra (X,M,R) need not be γ-generated in Θ-Coalg($\underline{A}$) if X is γ-generated in $\underline{A}$, and conversely.)

The generalizations to non-finitary theories in the sense of Linton [23] with rank or to partial co-operations are obvious and left to the reader (see also 6.14 - 6.16). The above can be generalized to categories $\underline{A}$ which are dual to a locally presentable category. This is done in some way as in 4.1 by means of $\Theta\text{-Coalg}(\underline{A}) \cong (\Theta\text{-Alg}(\underline{A}^o))^o$.

4.3 Coalgebras over a commutative ring.

Let $\underline{A} = \underline{Mod}_\Lambda$ be the category of Λ-modules over a commutative ring Λ. Let $\mathbb{F} = \{const_\Lambda, id, id \otimes id, id \otimes id \otimes id\}$, where id is the identity functor of $\underline{Mod}_\Lambda$ and $const_\Lambda : \underline{Mod}_\Lambda \longrightarrow \underline{Mod}_\Lambda$ is the constant functor $A \rightsquigarrow \Lambda$. The tensor product is taken over Λ. Let $M = \{\Delta, \varepsilon\}$, where $\Delta : id \dashrightarrow id \otimes id$ and $\varepsilon : id \dashrightarrow const_\Lambda$ are operations called comultiplication and counit. A pre-bialgebra is a Λ-module A together with homomorphisms $\Delta_A : A \longrightarrow A \otimes A$ and $\varepsilon_A : A \longrightarrow \Lambda$. Let $R = \{r_1, r_2, r_3\}$, where $r_1 : id \Rrightarrow id \otimes id \otimes id$ and $r_2 : id \Rrightarrow id$, $r_3 : id \Rrightarrow id$ are relations, called coassociative and counitary laws, which for a pre-bialgebra $(A, \Delta_A, \varepsilon_A)$ are given by the diagrams



It is straight forward that r_1, r_2 and r_3 are relations and that Bialg($\underline{\text{Mod}}_\Lambda$) is the category Λ-Coalg of Λ-coalgebras. Note that $\mathbb{F}_d = \{id\}$. Recall that $\underline{\text{Mod}}_\Lambda$ is locally γ-noetherian (i.e. every γ-generated module is γ-presentable) for some $\gamma \geq \aleph_0$ iff every ideal $I \subset \Lambda$ is γ-generated. If Λ has this property, it is called γ-noetherian; in particular for $\gamma = \aleph_0$ the notion $\aleph_0$-noetherian coincides with noetherian in the usual sense. Clearly if Λ is noetherian, then it is γ-noetherian for any $\gamma \geq \aleph_0$.

By 3.24 a) the category Λ-Coalg is locally $\aleph_1$-presentable and by 3.8 for $\gamma \geq \aleph_1$ a coalgebra $(X, \Delta_X, \varepsilon_X)$ is γ-presentable in Λ-Coalg iff its underlying module X is γ-presentable in $\underline{\text{Mod}}_\Lambda$. In particular a Λ-homomorphism $U \longrightarrow (A, \Delta_A, \varepsilon_A)$ with $\pi(U) \leq \gamma$ admits a decomposition into a Λ-homomorphism $U \longrightarrow U'$ and a coalgebra morphism $(U', \Delta_{U'}, \varepsilon_{U'}) \longrightarrow (A, \Delta_A, \varepsilon_A)$ such that $\pi(U') \leq \gamma$.

Likewise, if Λ is γ-noetherian for some $\gamma \geq \aleph_1$, then by 3.22 Λ-Coalg is locally γ-noetherian and a γ-generated Λ-submodule of a coalgebra is contained in a subcoalgebra whose underlying Λ-module is γ-generated. Moreover a coalgebra is γ-generated in Λ-Coalg iff its underlying module is γ-generated in $\underline{\text{Mod}}_\Lambda$. (Note that these assertions need not hold if Λ is not γ-noetherian.)

The same results hold for the category of cocommutative Λ-coalgebras.

For by adding to the above data of bialgebras a relation expressing the cocommutativity of $\Delta : id \dashrightarrow id \otimes id$ one obtains instead the category of cocommutative Λ-coalgebras.

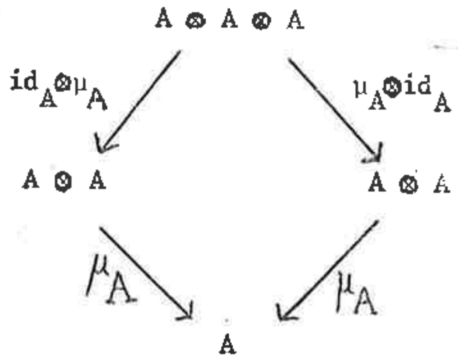
The above improves the results of M. Barr [1] considerably. He showed that for $\delta \geq \sup(\mathrm{card}(\Lambda)^+, \aleph_1)$ every δ-generated submodule of a Λ-coalgebra is contained in a subcoalgebra whose underlying module is also δ-generated; in particular the coalgebras whose underlying module is $\sup(\mathrm{card}(\Lambda)^+), \aleph_1)$-generated, form a set of generators in Λ-Coalg. As shown above these problems have something to do with the (minimal) number of generators for ideals $I \subset \Lambda$ and not with the cardinality of Λ. The latter enters his argument for a different reason. A submodule of a coalgebra which is closed under the comultiplication need not be a subcoalgebra because it need not be coassociative. If however the submodule is pure, then the coassociativity carries over. Therefore he considered only pure submodules and embedded the given submodule of the coalgebra into a pure submodule. In this way the cardinality of Λ comes in and the "generated" subcoalgebra can become much bigger than necessary.

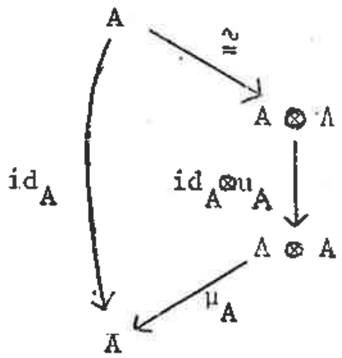
As for Fox's [8] generalization of Barr's results see 4.7 below.

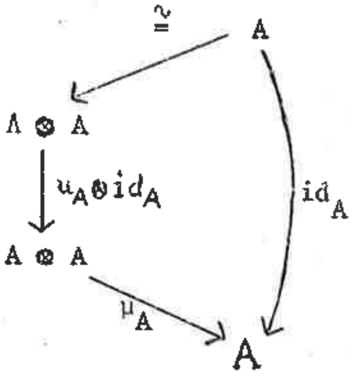
4.4 <u>Bialgebras, Hopfalgebras over a commutative ring, generalizations to Props and locally presentable categories.</u>

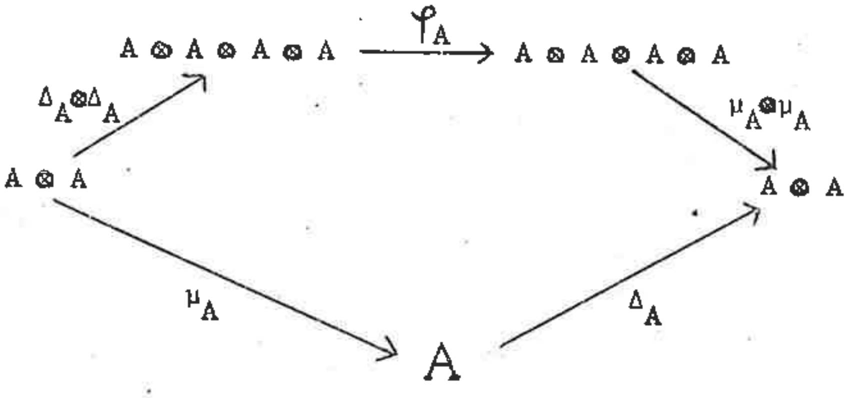
Let $\underline{A} = \underline{Mod}_\Lambda$ be the category of modules over a commutative ring Λ. The data $M, R, \mathbb{F}$ for Λ-bialgebras is as follows. Let $\mathbb{F} = \{const_\Lambda, id, id \otimes id, id \otimes id \otimes id\}$ be as above for coalgebras (4.3). Let $M = \{\Delta, \varepsilon, \mu, u\}$ be operations, where $\Delta : id \dashrightarrow id \otimes id$, $\varepsilon : id \dashrightarrow const_\Lambda$, $\mu : id \otimes id \dashrightarrow id$ and $u : const_\Lambda \dashrightarrow id$ are operations called comultiplication, counit, multiplication and unit respectively. Thus a pre-bialgebra is a Λ-module A together with homomorphisms $\Delta_A : A \longrightarrow A \otimes A$, $\varepsilon_A : A \longrightarrow \Lambda$, $\mu_A : A \otimes A \longrightarrow A$ and $u_A : \Lambda \longrightarrow A$. Let $R = \{r_1, r_2, \ldots, r_{10}\}$, where

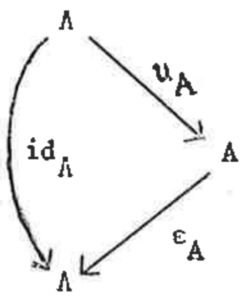
$r_1 : id \dashrightarrow id \otimes id \otimes id$, $r_2 : id \dashrightarrow id$, $r_3 : id \dashrightarrow id$ are as above in 4.3 and the relations $r_4 : id \otimes id \otimes id \dashrightarrow id$, $r_5 : id \dashrightarrow id$, $r_6 : id \dashrightarrow id$, $r_7 : id \otimes id \dashrightarrow id \otimes id$, $r_8 : const_\Lambda \dashrightarrow const_\Lambda$, $r_9 : const_\Lambda \dashrightarrow id \otimes id$, $r_{10} : id \otimes id \dashrightarrow const_\Lambda$ are given for a pre-bialgebra $(A, \Delta_A, \varepsilon_A, \mu_A, u_A)$ by the diagrams

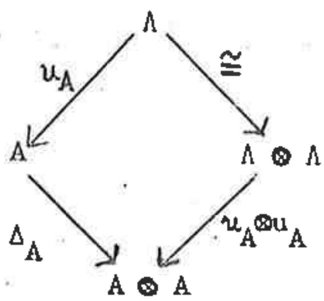


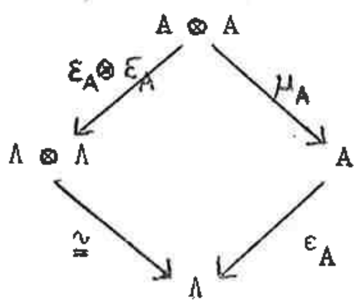




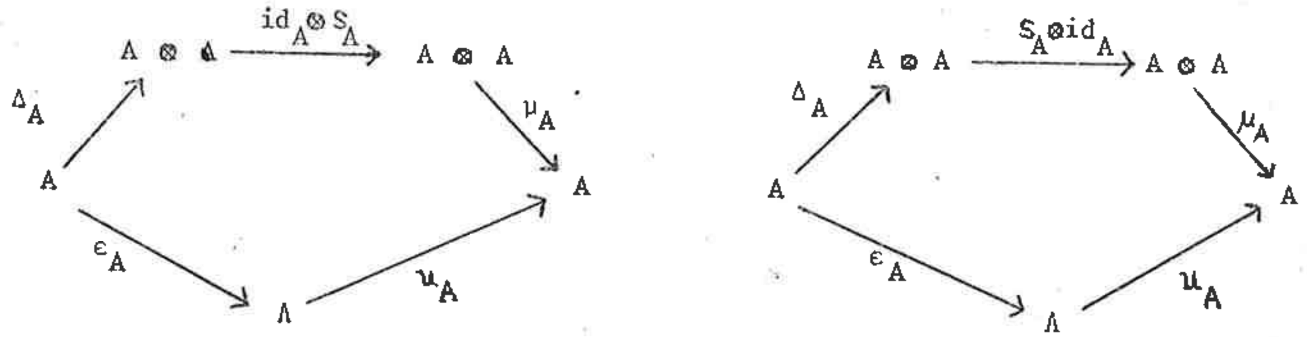








where $\rho_A : A \otimes A \otimes A \otimes A \longrightarrow A \otimes A \otimes A \otimes A$ is the homomorphism interchanging the two inner factors. It is straight forward that $r_1, r_2, \dots, r_{10}$ are relations on P-Bialg($\underline{Mod}_\Lambda$) and that Bialg($\underline{Mod}_\Lambda$) is the category Λ-Bialg of Λ-bialgebras. In order to express the category Λ-Hopf of Λ-Hopfalgebras as bialgebras in $\underline{Mod}_\Lambda$ one adds to M an operation $S : id \dashrightarrow id$ (= the antipode) and to R two relations $r_{11} : id \rightrightarrows id$, $r_{12} : id \rightrightarrows id$ which for a pre-bialgebra $(A, \Delta_A, \varepsilon_A, \mu_A, u_A, S_A)$ are given by the diagrams



Likewise by adding relations expressing the commutativity of μ or the cocommutativity of Δ or both one can obtain the categories of commutative Λ-bialgebras and Λ-Hopfalgebras, cocommutative Λ-bialgebras and Λ-Hopfalgebras and bicommutative Λ-bialgebras and Λ-Hopfalgebras. Note that $\mathbb{F}_d = \mathbb{F}$ and that $\pi(A \otimes A) \le \pi(A)$ and likewise $\varepsilon(A \otimes A) \le \varepsilon(A)$ for every $A \in \underline{Mod}_\Lambda$.

4.5 Thus for $\gamma \ge \aleph_1$ it follows from 3.8 that a Λ-bialgebra (X,M,R) is γ-presentable in Λ-Bialg iff its underlying module X is γ-presentable in $\underline{Mod}_\Lambda$. Moreover a Λ-homomorphism $U \to (A,M,R)$ with $\pi(U) \le \gamma$ admits a decomposition into a Λ-homomorphism $U \to U'$ and a Λ-bialgebra morphism $(U',M,R) \to (A,M,R)$ such that $\pi(U') \le \gamma$; in particular the Λ-bialgebras whose underlying module is $\aleph_1$-presentable form a set of dense generators in Λ-Bialg (cf. 3.8 and [13] 3.1).

If in addition Λ is γ-noetherian for some $\gamma \ge \aleph_1$ (cf. 4.3), then by 3.22 a γ-generated submodule of a bialgebra is contained in a

subbialgebra whose underlying module is also γ-generated. Moreover a Λ-bialgebra is γ-generated in Λ-Bialg iff its underlying module is γ-generated in Mod_Λ .
The same assertions hold for the categories of commutative Λ-bialgebras and Λ-Hopfalgebras, cocommutative Λ-bialgebras and Λ-Hopfalgebras and bicommutative Λ-bialgebras and Λ-Hopfalgebras.

4.6 With the exception of arbitrary Λ-Hopfalgebras all of the above categories are locally $\aleph_1$-presentable. In addition the various relative forgetful functors have left adjoints resp. right adjoints. If Λ is γ-noetherian for some $\gamma \geq \aleph_1$, then the above categories are also locally $\aleph_1$-noetherian.

The data of bialgebras for these categories admit a decomposition into algebraic and coalgebraic parts, cf. 3.27. Thus the first assertion follows from 3.28 and the last from 3.22 d) while the one concerning adjoints is a consequence of either 2.9 or the special adjoint functor theorem. For more details see 3.26 and the discussions following 3.27.

4.7 Generalizations Let $\underline{P}$ be a prop in the sense of Mac Lane [24] Section 24, and assume that it can be defined by a countable number of operations and relations (see M. Barr [ ] p. 605/606 for a discussion). It is clear that the tensor product preserving functors $\underline{P} \to \mathrm{Mod}_\Lambda$ can be expressed as bialgebras and therefore the assertions in 4.5 carry over to this situation. Likewise if the prop $\underline{P}$ is algebraic or coalgebraic ([1] 6.1) or admits a decomposition as in 3.2 , then the category of tensor product preserving functors $\underline{P} \to \mathrm{Mod}_\Lambda$ is locally $\aleph_1$-presentable, and if Λ is γ-noetherian for $\gamma \geq \aleph_1$, it is locally γ-noetherian, etc.
More generally let $\underline{A}$ be a category equipped with a bifunctor $\otimes : \underline{A} \times \underline{A} \to \underline{A}$ which is coherently associative, symmetric and unitary. Then for an arbitrary prop $\underline{P}$ one can express tensor product preser-

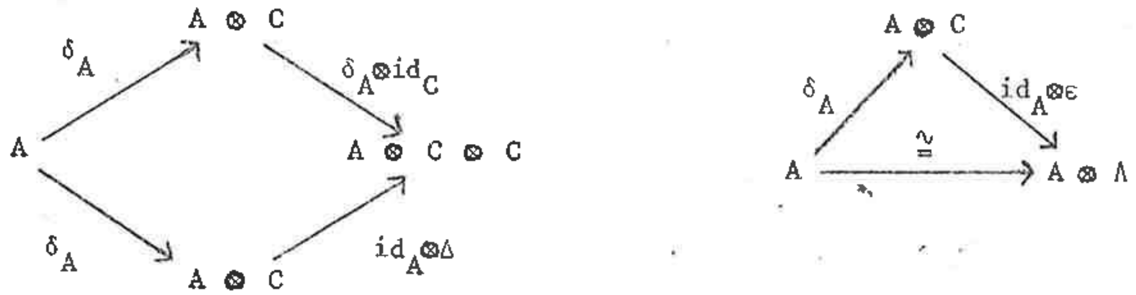
ving functors as bialgebras as above. If $\underline{A}$ is locally presentable and $\otimes$ preserves α-filtered colimits in both variables for some $\alpha \geq \aleph_0$, then 3.8 (resp. 3.7) and 3.22 (resp. 5.1) apply. Moreover if the prop $\underline{P}$ is of algebraic or coalgebraic type (cf. [1] 6.1) or admits a decomposition like in 3.27, then the category of tensor product preserving functors $\underline{P} \to \underline{A}$ is again locally presentable etc. (see 3.28 and 3.22 d)). In particular this applies to the coalgebraic situation considered by Fox [8]. We leave it to the reader to specify the minimal cardinals in 3.7 - 3.28 for tensor product preserving functors $\underline{P} \to \underline{A}$ (note that the case $\alpha = \aleph_0$ is particularly simple and useful).

While props give rise to data of bialgebras, the converse is not true, not even for $\underline{A} = \underline{Mod}_\Lambda$ and $\mathbb{F} = \{const_\Lambda, id, id \otimes id, id \otimes id \otimes id\}$. For instance, as M. Barr pointed out to me, Lie algebras over Λ cannot be expressed as tensor product preserving functors $\underline{P} \to \underline{Mod}_\Lambda$ for some prop $\underline{P}$ because the Jacoby identity involves addition of structure morphisms. However they can easily be described as bialgebras, the Jacoby identity is given by the relation $A \otimes A \otimes A \overset{f_A}{\underset{g_A}{\rightrightarrows}} A$, where $f(x,y,z) = 0$ and $g(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y]$.(Note that f_A and g_A are obviously natural with respect to Λ-homomorphisms preserving the bracket). The notion of bialgebras allows more flexibility as far as relations are concerned. It is also more natural and its simplicity should be compared with the technical problems involved with a prop $\underline{P}$ and the coherence aparatus for $\otimes$ and $\underline{P}$.

I should add that these prop problems prompted me to look for something simpler. When I met M.Barr and T. Fox in the fall of 1975 I had ten "different" proofs for the same theorem (namely 3.8); one for Σ-cocontinuous functors, one for coalgebras over a cotriple, one for descent data, one for Λ-coalgebras, one for comodules over a coalgebra, one for Λ-bialgebras,... . On the other hand Fox [8] had a proof for a tensored locally presentable category and a coalgebraic prop, but no reasonable size estimates for the generators he constructed. In order to obtain that and also to cover the case of non-coalgebraic props I had to look for an "eleventh proof" of 3.8 considering interlocking operations and relations which turned to be very technical and extremely laborious. Fox [6] had got around this problem in the same way as Barr [1] by using purity (see 4.3 above). The use of purity however makes good size estimates impossible and thus something else had to be found. In this way I was led to the notion of pre-bialgebras and bialgebras as defined in 3.1, the above mentioned example of Lie algebras served as a guide. The unification of the eleven proofs of 3.8 was a somewhat "unexpected fringe benefit".

4.8 Comodules over a Λ-coalgebra.

Let $\underline{A} = \underline{\mathrm{Mod}}_\Lambda$ be the category of modules over a commutative ring Λ and let C be a Λ-coalgebra with comultiplication $\Delta : C \longrightarrow C \otimes C$ and counit $\varepsilon : C \longrightarrow \Lambda$ (cf. 4.3). Recall that a right C-comodule is a Λ-module A together with a Λ-homomorphism $\delta_A : A \longrightarrow A \otimes C$ such that the diagrams



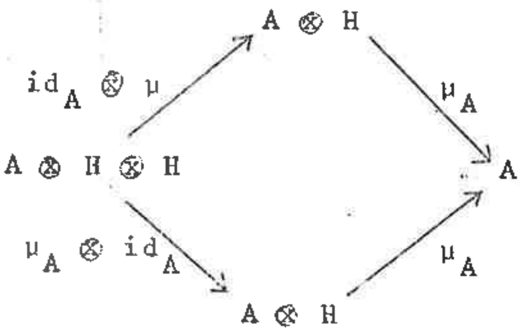
commute. The tensor product is over Λ. A right C-comodule morphism $(A,\delta_A) \longrightarrow (A',\delta_{A'})$ is a Λ-homomorphism $f : A \longrightarrow A'$ with the property $\delta_{A'} \circ f = (f \otimes \mathrm{id}_C) \circ \delta_A$. The category of right C-comodules is denoted with $\underline{\mathrm{Comod}}_C$ (cf. Demazure-Gabriel [4] p. 174, Sweedler [28] 30/31). To express $\underline{\mathrm{Comod}}_C$ as bialgebras in $\underline{\mathrm{Mod}}_\Lambda$ let $\mathbb{F} = \{\mathrm{id}, \otimes C, \otimes C \otimes C\}$ and $M = \{\delta\}$, where id is the identity of $\underline{\mathrm{Mod}}_\Lambda$ and δ an operation $id \dashrightarrow \otimes C$. Let $R = \{r_1, r_2\}$, where $r_1 : id \Rrightarrow \otimes C \otimes C$ and $r_2 : id \Rrightarrow id$ are given by the above diagrams. Clearly r_1 and r_2 are relations on P-Bialg($\underline{Mod}_\Lambda$) and $\underline{Comod}_C$ = Bialg($\underline{Mod}_\Lambda$) holds. Note that $\mathbb{F}_d = \{id\}$.

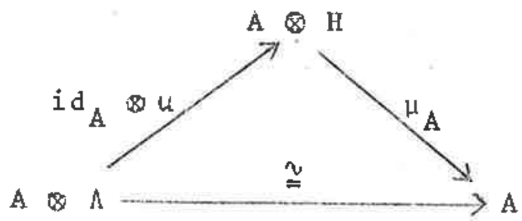
Thus by 3.22 and 3.8 the category $\underline{Comod}_C$ is locally $\aleph_1$-presentable and for $\gamma \geq \aleph_1$ a comodule (X, δ_X) is γ-presentable in $\underline{Comod}_C$ if X is γ-presentable in $\underline{Mod}_\Lambda$, in particular a Λ-homomorphism $U \longrightarrow (A, \delta_A)$ with $\pi(U) \leq \gamma$ factors into a Λ-homomorphism $U \longrightarrow U'$ and a comodule morphism $(U', \delta_{U'}) \longrightarrow (A, \delta_A)$ such that $\pi(U') \leq \gamma$. Likewise if Λ is γ-noetherian for some $\gamma \geq \aleph_1$ (cf. 4.3), then by 3.22 $\underline{Comod}_C$ is locally γ-noetherian and a comodule is γ-generated in $\underline{Comod}_C$ iff its underlying module is γ-generated in $\underline{Mod}_\Lambda$. In addition a γ-generated Λ-submodule of a comodule is contained in a subcomodule whose underlying module is γ-generated. The last assertion was first proved by Wischnewsky [36] under the additional assumption that $\gamma > \text{card}(\Lambda)$. Following Barr [1] he used purity arguments which in general make the "generated" subcomodule bigger than necessary.
If C is Λ-flat one can easily show that $\underline{Comod}_C$ is a locally $\aleph_1$-presentable Grothendieck category and that for $\alpha \geq \aleph_1$ a comodule is α-generated iff its underlying module is, etc. (cf. 3.25, 3.22, see also [36])

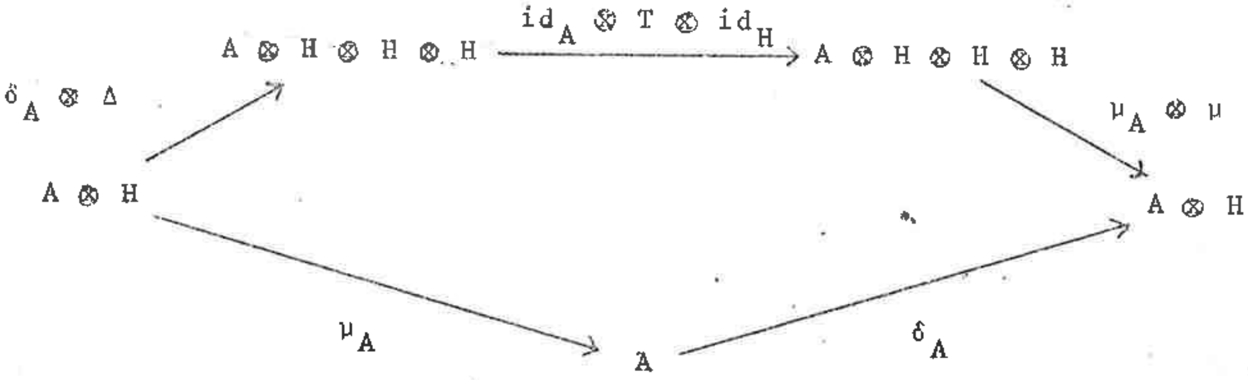
4.9 Bimodules over a Λ-bialgebra

Let $\underline{Mod}_\Lambda$ be as above and let H be a Λ-bialgebra with multiplication $\mu : H \otimes H \longrightarrow H$, unit $u : \Lambda \longrightarrow H$, comultiplication $\Delta : H \longrightarrow H \otimes H$ and counit $\varepsilon : H \longrightarrow \Lambda$ (cf. 4.4). Recall that a bimodule over H is a Λ-module A together with Λ-homomorphisms $\mu_A : A \otimes H \longrightarrow A$ and $\delta_A : A \longrightarrow A \otimes H$ such that 1) μ_A defines a right H-module structure on A with H being viewed as a Λ-algebra 2) δ_A defines a right H-comodule structure on A with H being viewed as a Λ-coalgebra 3) δ_A is H-linear, where the right H-structure on $A \otimes H$ is given by $\Delta : H \longrightarrow H \otimes H$, i.e. if $\Delta(g) = \Sigma\, g_i' \otimes g_i''$, then

$(m \otimes h)g = \Sigma\, mg_i' \otimes hg_i''$ (cf. Sweedler [28] 4.1). For instance, if $X \in \underline{Mod}_\Lambda$ then $(X \otimes H, id \otimes \mu, id \otimes \Delta)$ is a H-bimodule. A morphism between H-bimodules is a Λ-homomorphism which is compatible with both structures. Let $\underline{Bimod}_H$ denote the category of bimodules over H. To express $\underline{Bimod}_H$ as bialgebras in $\underline{Mod}_\Lambda$ let $\mathbb{F} = \{id, \otimes H, \otimes H \otimes H\}$ and let $M = \{\delta, \mu\}$ consist of operations $\delta : id \dashrightarrow \otimes H$ and $\mu : \otimes H \dashrightarrow id$, where id is the identity functor of $\underline{Mod}_\Lambda$. Let $R = \{r_1, r_2, r_3, r_4, r_5\}$ consist of relations $r_1 : id \Rrightarrow \otimes H \otimes H$, $r_2 : id \Rrightarrow id$, $r_3 : \otimes H \otimes H \Rrightarrow id$, $r_4 : id \Rrightarrow id$, $r_5 : \otimes H \Rrightarrow \otimes H$, where r_1 and r_2 are as above in 4.3 and r_3, r_4, r_5 are given for a pre-bialgebra (A, δ_A, μ_A) by the diagrams







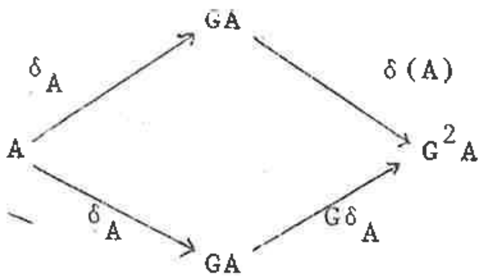
with $T : H \otimes H \longrightarrow H \otimes H$ being the twist homomorphism $h \otimes h' \rightsquigarrow h' \otimes h$. One easily checks that $r_1, \ldots, r_5$ are relations on P-Bialg($\underline{Mod}_\Lambda$) and that Bialg($\underline{Mod}_\Lambda$) = $\underline{Bimod}_H$. Note that $\mathbb{F}_d = \mathbb{F}$ and that every functor in $\mathbb{F}_d$ is colimit preserving. Moreover for every $A \in \underline{Mod}_\Lambda$ it follows from $[A \otimes H, -] \cong [A, [H, -]]$ and $[A \otimes H \otimes H, -] \cong [A \otimes H, [H, -]]$ that $\pi(A \otimes H) \le \sup(\pi(A), \pi(H)) \ge \pi(A \otimes H \otimes H)$ and likewise

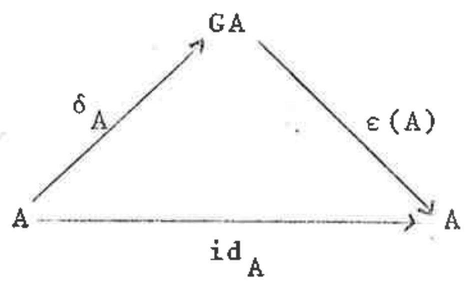
$\varepsilon(A \otimes H) \leq \sup(\varepsilon(A), \varepsilon(H)) \geq \varepsilon(A \otimes H \otimes H)$.

Thus by 3.24 a) $\underline{Bimod}_H$ is locally $\sup(\aleph_1, \pi(H))$-presentable and for $\gamma \geq \sup(\aleph_1, \pi(H))$ it follows from 3.8 that a bimodule (X, δ_X, μ_X) is γ-presentable in $\underline{Bimod}_H$ iff X is γ-presentable in $\underline{Mod}_\Lambda$; in particular a Λ-homomorphism $U \longrightarrow (A, \delta_A, \mu_A)$ with $\pi(U) \leq \gamma$ factors into a Λ-homomorphism $U \longrightarrow U'$ and a bimodule morphism $(U', \delta_{U'}, \mu_{U'}) \longrightarrow (A, \delta_A, \mu_A)$ such that $\pi(U') \leq \gamma$.
Likewise if Λ is γ-noetherian (cf. 4.3) for some $\gamma \geq \sup(\aleph_1, \pi(H))$, then $\underline{Bimod}_H$ is locally γ-noetherian and a bimodule is γ-generated in $\underline{Bimod}_H$ iff its underlying module is γ-generated in $\underline{Mod}_\Lambda$. In addition a γ-generated Λ-submodule of a bimodule is contained in a sub-bimodule whose underlying Λ-module is γ-generated.
Actually $\underline{Bimod}_H$ is locally $\aleph_1$-presentable and locally δ-noetherian where δ is the least regular cardinal $\geq \aleph_1$ such that every right ideal of H is δ-generated (in the category of right H-modules).
This follows from 3.28 resp. 3.28 and 3.24 a) because there is a decomposition $\underline{Bimod}_H \cong \underline{Comod}_H(\underline{Mod}_H)$ in the sense of 3.27. In more detail the algebraic part of $M = \{\delta, \mu\}$ and $R = \{r_1, r_2, r_3, r_4, r_5\}$ is $M' = \{\mu\}$ and $R' = \{r_3, r_4\}$ whence $Bialg_{M', R'}(\underline{Mod}_\Lambda) \cong \underline{Mod}_H$. There is a functor $\otimes H : \underline{Mod}_H \longrightarrow \underline{Mod}_H$, $A \rightsquigarrow A \otimes_\Lambda H$ (see 3) above) together with natural transformations $\otimes \varepsilon : \otimes H \longrightarrow id_{\underline{Mod}_H}$ and $\otimes \Delta : \otimes H \longrightarrow \otimes H \otimes H$, where ε is the counit of H and Δ the comultiplication. (The verification that $\otimes \Delta$ is well defined is somewhat laborious but straight forward.) With this one can define the coalgebraic part of M and R as $M'' = \{\delta : id \dashrightarrow \otimes H\}$ and $R'' = \{r_1 : id \rightrightarrows \otimes H \otimes H,\ r_2 : id \rightrightarrows id\}$, where id is the identity of $\underline{Mod}_H$ and r_1 and r_2 are defined exactly as in 4.8. It is now routine to show that $Bialg_{M'', R''}(\underline{Mod}_H) \cong \underline{Bimod}_H$. Note that if H is flat over Λ, then $\underline{Bimod}_H$ is a locally $\aleph_1$-presentable Grothendieck category and for $\alpha \geq \sup(\aleph_1, \varepsilon(H))$ a bimodule is α-generated iff its underlying module is, etc. (cf. 3.25, 3.22).

4.1o Coalgebras over a cotriple.

Recall that a cotriple $\mathbb{G} = (G,\delta,\varepsilon)$ in a category $\underline{A}$ consists of a functor $G : \underline{A} \longrightarrow \underline{A}$ and natural transformations $\delta : G \longrightarrow G^2$ (= comultiplication), $\varepsilon : G \longrightarrow id_{\underline{A}}$ (=counit) satisfying $G\delta \cdot \delta = \delta G \cdot \delta$ (coassociative law) and $G\varepsilon \cdot \delta = id_G = \varepsilon G \cdot \delta$ (counitary law). A $\mathbb{G}$-coalgebra in $\underline{A}$ is a pair (A,ξ), where $\xi : A \longrightarrow GA$ is a morphism satisfying $\varepsilon(A) \circ \xi = id_A$ and $G\xi \circ \xi = \delta(A) \circ \xi$. A morphism $(A,\xi) \longrightarrow (A',\xi')$ of $\mathbb{G}$-coalgebras is a morphism $f : A \longrightarrow A'$ satisfying $\xi' \circ f = Gf \circ \xi$. The category of all $\mathbb{G}$-coalgebras is denoted with $\underline{A}_{\mathbb{G}}$. The underlying functor $\underline{A}_{\mathbb{G}} \longrightarrow \underline{A}$, $(A,\xi) \rightsquigarrow A$ is left adjoint to the cofree functor $\underline{A} \longrightarrow \underline{A}_{\mathbb{G}}$, $A \rightsquigarrow (GA,\delta(A))$. Given a cotriple $\mathbb{G} = (G,\delta,\varepsilon)$ in $\underline{A}$ it is easy to describe $\underline{A}_{\mathbb{G}}$ in terms of bialgebras. Let $\mathbb{F} = \{id_{\underline{A}}, G, G^2\}$ and let $M = \{\delta\}$ be an operation $\delta : id_{\underline{A}} \dashrightarrow G$. Thus a pre-bialgebra is an object $A \in \underline{A}$ together with a morphism $\delta_A : A \longrightarrow GA$. Let $R = \{r_1, r_2\}$ be the relations $r_1 : id_{\underline{A}} \rightrightarrows G^2$ and $r_2 : id_{\underline{A}} \rightrightarrows id_{\underline{A}}$ which for a pre-bialgebra (A,δ_A) are given by the diagrams





Clearly r_1 and r_2 are relations on P-Bialg($\underline{A}$) and $\underline{A}_{\mathbb{G}} \cong$ Bialg($\underline{A}$) holds. Note that $\mathbb{F}_d = \{id_{\underline{A}}\}$.

Assume $\underline{A}$ is locally presentable and that G has rank (cf. 2.1) and let $\gamma \geq \sup(\aleph_1, \pi(\underline{A}), \pi(G))$. Then by 3.24 a) $\underline{A}_{\mathbb{G}}$ is locally $\sup(\aleph_1, \pi(\underline{A}), \pi(G))$-presentable and by 3.8 a coalgebra (X,δ_X) is γ-presentable in $\underline{A}_{\mathbb{G}}$ iff X is γ-presentable in $\underline{A}$; in particular a morphism $U \longrightarrow (A,\delta_A)$ with $\pi(U) \leq \gamma$ admits a decomposition into a morphism $U \longrightarrow U'$ and a coalgebra morphism $(U',\delta_{U'}) \longrightarrow (A,\delta_A)$ such that $\pi(U') \leq \gamma$. Likewise if $\underline{A}$ is locally γ-noetherian for some

$\gamma \geq \sup(\aleph_1, \pi(\underline{A}), \pi(G))$ and if in $\underline{A}$ β-filtered colimits of monomorphisms are monomorphic for some $\beta < \gamma$, then $\underline{A}_{\mathbb{G}}$ is locally γ-noetherian and a coalgebra (X, δ_X) is γ-generated in $\underline{A}_{\mathbb{G}}$ iff X is γ-generated in $\underline{A}$. Also a γ-generated subobject U of a coalgebra (A, δ_A) is contained in a subcoalgebra $(U', \delta_{U'})$ such that U' is γ-generated.

4.11 Corollary Let $\mathbb{G} = (G, \delta, \varepsilon)$ be a cotriple in a topos $\underline{A}$ (resp. Grothendieck category). Equivalent are

(i) $\underline{A}_{\mathbb{G}}$ is a topos (resp. Grothendieck category) and the left adjoint $\underline{A}_{\mathbb{G}} \longrightarrow \underline{A}$, $(A, \delta_A) \rightsquigarrow A$, preserves finite limits.

(ii) $G : \underline{A} \longrightarrow \underline{A}$ preserves finite limits and has rank.

Moreover iff i) holds, then $\underline{A}_{\mathbb{G}}$ is a locally $\sup(\aleph_1, \pi(\underline{A}), \pi(G))$-presentable topos (resp. Grothendieck category) and for $\gamma \geq \sup(\aleph_1, \pi(\underline{A}), \pi(G))$ a coalgebra (X, δ_X) is γ-generated in $\underline{A}_{\mathbb{G}}$ iff X is γ-generated in $\underline{A}$, etc. (see 3.25 and 3.22 for $\beta = \aleph_0$).

Proof (i) $\Longrightarrow$ (ii) The first assertion is trivial and the second follows from 2.9.

(ii) $\Longrightarrow$ (i) By 4.10 $\underline{A}_{\mathbb{G}}$ is locally presentable. The underlying functor $\underline{A}_{\mathbb{G}} \longrightarrow \underline{A}$ preserves and creates colimits. The same holds with respect to finite limits because G is finite limit preserving. This implies that $\underline{A}_{\mathbb{G}}$ is a Grothendieck category provided $\underline{A}$ is. Likewise if $\underline{A}$ is a topos, one readily checks with this that $\underline{A}_{\mathbb{G}}$ satisfies the conditions [13] 12.13 a) - d) (=Giraud's axioms) and hence is a topos. The last assertion follows from 3.25 and 3.22 for $\beta = \aleph_0$.

4.12 Remarks a) If G does not preserve finite limits, then $\underline{A}_{\mathbb{G}}$ need not be a Grothendieck category (resp. topos), if $\underline{A}$ is. For instance, let $\underline{U} \hookrightarrow \underline{Ab.Gr.}$ be the inclusion of the full subcategory consisting of all finite p-groups for some prime p. Let $\underline{A} = [\underline{U}, \underline{Ab.Gr.}]_+$ be the category of additive functors and let $\underline{X} \subset \underline{A}$ be the full subcategory of all cocontinuous functors. By 6.16 below the inclusion $\underline{X} \subset \underline{A}$ has a right adjoint and the resulting cotriple on $\underline{A}$ has obviously the property $\underline{A}_{\mathbb{G}} \cong \underline{X}$. By 6.25 c) below $\underline{A}_{\mathbb{G}}$ is isomorphic with the category of p-adic complete abelian groups which is not a Grothendieck category (e.g. the colimit of the system $\mathbb{Z}/p\mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p\cdot} \dots$ is zero because in the category of all abelian groups it is the Prüfer group $\mathbb{Z}(p^\infty)$ whose completion is zero.)

b) Corollary 4.11 may sound like the well known theorem "If $\mathbb{G}$ is a left exact cotriple in an elementary topos $\mathcal{E}$, then $\mathcal{E}_{\mathbb{G}}$ is again an elementary topos" but in fact in has little to do with it. The main ingredient in 4.11 is the existence of generators in $\underline{A}_{\mathbb{G}}$ which is not contained in the assertion concerning elementary topoi. Also in the latter there is no rank assumption on the cotriple which is necessary for the existence of generators.

4.13 Algebras over a triple. Let $\mathbb{T} = (T,\mu,u)$ be a triple in a category $\underline{A}$ and let $\underline{A}^{\mathbb{T}}$ denote the category of $\mathbb{T}$-algebras in $\underline{A}$, cf. [13] § 1o. The description of $\underline{A}^{\mathbb{T}}$ as bialgebras in $\underline{A}$ is dual to 4.1o, i.e. if $\mathbb{F} = \{id_{\underline{A}}, T, T^2\}$, $M = \{\mu : T \dashrightarrow id_{\underline{A}}\}$ and $R = \{r_1 : T^2 \rightrightarrows id_{\underline{A}},\ r_2 : id_{\underline{A}} \rightrightarrows id_{\underline{A}}\}$ are dual to the data for bialgebras in 4.1o, then $\underline{A}^{\mathbb{T}} = \mathrm{Bialg}(\underline{A})$. Note that $\mathbb{F}_c = \{id_{\underline{A}}\}$ and $\mathbb{F}_d = \{id_{\underline{A}}, T, T^2\}$. Assume $\underline{A}$ is locally presentable and T has rank (2.1). Then by [ ] § 1o $\underline{A}^{\mathbb{T}}$ is locally $\sup(\pi(\underline{A}),\pi(T))$-presentable. Let $\gamma \geq \aleph_1$ be a regular cardinal such that $\gamma \geq \pi(\underline{A})$, $\gamma > \pi(T)$ and that $\pi(U) \leq \gamma$ implies $\pi(TU) \leq \gamma$ for $U \in \underline{A}$. (Note that by 3.7 such cardinals exist.) Thus by 3.8 a $\mathbb{T}$-algebra (X,μ_X) is γ-presentable in $\underline{A}^{\mathbb{T}}$ iff X is γ-presentable in $\underline{A}$. Likewise if $\gamma \geq \aleph_1$ is a regular cardinal such that $\gamma \geq \varepsilon(\underline{A})$, $\gamma > \varepsilon(T)$ and that $\varepsilon(U) \leq \gamma$ implies $\varepsilon(TU) \leq \gamma$ for $U \in \underline{A}$ (cf. 5.1), then by 3.22 a $\mathbb{T}$-algebra (X,μ_X) is γ-generated in $\underline{A}^{\mathbb{T}}$ iff X is γ-generated in $\underline{A}$. If in addition $\underline{A}$ is locally γ-noetherian, then so is $\underline{A}^{\mathbb{T}}$. (Note that for $\beta \geq \sup(\pi(\underline{A}),\pi(T))$ a morphism $U \longrightarrow (A,\mu_A)$ with $\pi(U) \leq \beta$ obviously factors into a morphism $U \longrightarrow V$ and a $\mathbb{T}$-algebra morphism $(V,\mu_V) \longrightarrow (A,\mu_A)$ such that (V,μ_V) is β-presentable in $\underline{A}^{\mathbb{T}}$, namely the one given by the free $\mathbb{T}$-algebra on U, but V need not be β-presentable in $\underline{A}$).

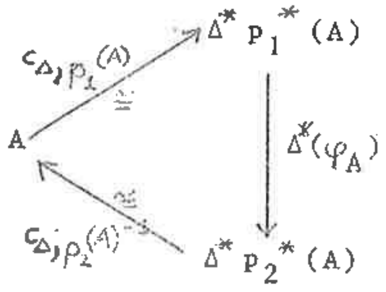
4.14 Descent data and données de recollements.

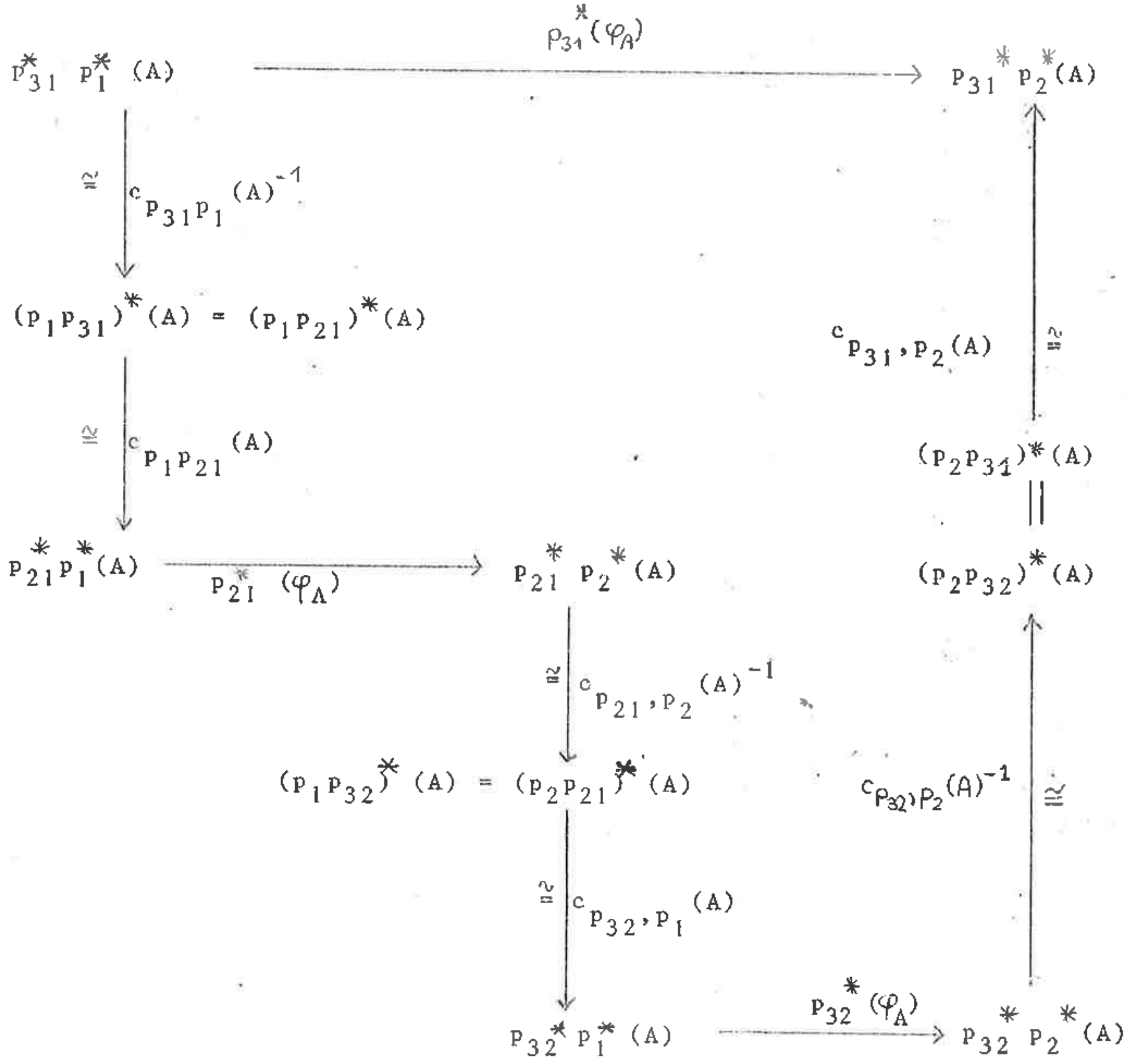
We follow Grothendieck [16] but limit ourselve to descent data. The case of données de recollement is almost identical (but simpler) and the obvious modifications are left to the reader. It should be noted that the following is a special case of cartesian closed sections below in 4.19. Let $\mathcal{F}$ be a fibration with base $\underline{C}$, i.e. for each $X \in \underline{C}$ there is a category $\underline{\mathcal{F}}_X$ (= the fibre over X) and for each morphism $f : X \to Y$ a functor $f^* : \underline{\mathcal{F}}_Y \longrightarrow \underline{\mathcal{F}}_X$ (= the inverse image of f) and for each composite $X \xrightarrow{f} Y \xrightarrow{g} C$ a natural equivalence $c_{f,g} : (gf)^* \to f^* g^*$ subject to the usual compatibility conditions (see [16] Def. 1.1 or [14]). Let $\alpha : S_o \to S$ be a morphism in $\underline{C}$ and assume that the fibre products $S_o \underset{S}{\times} S_o$ and $S_o \underset{S}{\times} S_o \underset{S}{\times} S_o$ exist. Let $S_1 = S_o \underset{S}{\times} S_o$ and let $p_i : S_1 \to S_o$ denote the projection on the i-th factor, i = 1,2. Likewise let $S_2 = S_o \underset{S}{\times} S_o \underset{S}{\times} S_o$ and let $p_{ij} : S_2 \to S_1$ denote the partial projection on the i-th and j-th factor, where (i,j) = (3,1), (3,2), (2,1). Clearly $p_1 p_{31} = p_1 p_{21}$, $p_2 p_{31} = p_2 p_{32}$ and $p_2 p_{21} = p_1 p_{32}$ hold and these morphisms together with the diagonal $\Delta : S_o \longrightarrow S_1$ give rise to a diagram

$$\underline{\mathcal{F}}_S \xrightarrow{\alpha^*} \underline{\mathcal{F}}_{S_o} \begin{array}{c} \xleftarrow{\Delta^*} \\ \xrightarrow{p_1^*} \\ \xrightarrow[p_2^*]{} \end{array} \underline{\mathcal{F}}_{S_1} \begin{array}{c} \xrightarrow{p_{31}^*} \\ \xrightarrow{p_{32}^*} \\ \xrightarrow[p_{21}^*]{} \end{array} \underline{\mathcal{F}}_{S_2}$$

and natural equivalences $c_{\Delta,p_1} : id \xrightarrow{\cong} \Delta^* p_1^*$, $c_{\Delta,p_2} : id \xrightarrow{\cong} \Delta^* p_2^*$,
$c_{p_{31},p_1} : (p_1 p_{31})^* \xrightarrow{\cong} p_{31}^* p_1^*$, $c_{p_{21},p_1} : (p_1 p_{21})^* \xrightarrow{\cong} p_{21}^* p_1^*$,
$c_{p_{31},p_2} : (p_2 p_3)^* \xrightarrow{\cong} p_{31}^* p_2^*$, $c_{p_{32},p_2} : (p_2 p_{32})^* \xrightarrow{\cong} p_{32}^* p_2^*$,
$c_{p_{21},p_2} : (p_2 p_{21})^* \xrightarrow{\cong} p_{21}^* p_2^*$ and $c_{p_{31},p_1} : (p_1 p_{31})^* \xrightarrow{\cong} p_{31}^* p_1^*$.
Recall that a descent datum on an object $A \in \underline{\mathcal{F}}_S$ is an isomorphism

$\varphi_A : p_1^* A \xrightarrow{\cong} p_2^* A$ with the properties $\Delta^*(\varphi_A) = id_A$ and $p_{31}^*(\varphi_A) = p_{32}^*(\varphi_A) \circ p_{21}^*(\varphi_A)$ modulo equivalence, i.e. the diagrams





commute. In the following we mean by a descent datum also a pair (A,φ_A) satisfying the above conditions. A morphism $(A,\varphi_A) \longrightarrow (A',\varphi_{A'})$ between descent data is a morphism $\xi : A \to A'$ in $\underline{\mathcal{F}}_{S_o}$ with the property $p_2^*(\xi) \circ \varphi_A = \varphi_{A'} \circ p_1^*(\xi)$. The resulting category of descent data is denoted with $\mathrm{Desc}(\underline{\mathcal{F}}_{S_o})$. To express descent data as bialgebras in $\underline{\mathcal{F}}_{S_o}$ let $\mathbb{F} = \{p_1^*, p_2^*, \mathrm{id}_{\underline{\mathcal{F}}_{S_o}}, \Delta^* p_2^*, p_{31}^* p_1^*, p_{31}^* p_2^*\}$ and let $M = \{\varphi, \bar{\varphi}\}$ consist of operations $\varphi : p_1^* \dashrightarrow p_2^*$ and $\bar{\varphi} : p_2 \dashrightarrow p_1$. Likewise let $R = \{r_1, r_2, r_3, r_4\}$ consist of relations $r_1 : p_1^* \rightrightarrows p_1^*$, $r_2 : p_2^* \rightrightarrows p_2^*$, $r_3 : \mathrm{id}_{\underline{\mathcal{F}}_{S_o}} \rightrightarrows \mathrm{id}_{\underline{\mathcal{F}}_{S_o}}$ and $r_4 : p_{31}^* p_1^* \rightrightarrows p_{31}^* p_2^*$ which for a pre-bialgebra $(A, \varphi_A, \bar{\varphi}_A)$ are given by

$$p_1^* A \underset{\bar{\varphi}_A \circ \varphi_A}{\overset{\mathrm{id}_{p_1^* A}}{\rightrightarrows}} p_1^* A \qquad\qquad p_2^* A \underset{\varphi_A \circ \bar{\varphi}_A}{\overset{\mathrm{id}_{p_2^* A}}{\rightrightarrows}} p_2^* A$$

and the two diagrams above. With this it is immediate that $\mathrm{Desc}(\underline{\mathcal{F}}_{S_o}) = \mathrm{Bialg}(\underline{\mathcal{F}}_{S_o})$. Note that $\mathbb{F}_d = \{p_1^*, p_2^*, \mathrm{id}_{\underline{\mathcal{F}}_{S_o}}, p_{31}^* p_1^*\}$ and $\mathbb{F}_c = \{p_1^*, p_2^*, \mathrm{id}_{\underline{\mathcal{F}}_{S_o}}, p_{31}^* p_2^*\}$. Thus by 3.3 $\mathrm{Desc}(\underline{\mathcal{F}}_{S_o})$ has colimits (resp. limits) and the forgetful functor $\mathrm{Desc}(\underline{\mathcal{F}}_{S_o}) \longrightarrow \underline{\mathcal{F}}_{S_o}$, $(A,\varphi_A) \rightsquigarrow A$ preserves them provided $\underline{\mathcal{F}}_{S_o}$ has colimits (resp. limits) and the inverse image functors p_1^*, p_2^* and p_{31}^* preserve them. Likewise if $\underline{\mathcal{F}}_{S_o}$ has γ-filtered colimits for some $\gamma \geq \aleph_o$ and the above functors preserve them, then $\mathrm{Desc}(\underline{\mathcal{F}}_{S_o})$ has γ-filtered colimits and the forgetful functor preserves them.

4.15 <u>Assume that $\underline{\mathcal{F}}_{S_o}$, $\underline{\mathcal{F}}_{S_1}$ and $\underline{\mathcal{F}}_{S_2}$ are locally presentable and that the inverse image functors p_1^*, p_2^* and p_{31}^* have rank</u> (2.1). <u>Let</u> $\gamma > \beta$ <u>be cardinals such that</u>

1) <u>$\underline{\mathcal{F}}_{S_o}$ is locally γ-presentable and</u>

2) <u>the functors p_1^*, p_2^* and p_{31}^* preserve β-filtered colimits and take γ-presentable objects into γ-presentable objects</u> (the existence of such γ's follows from 3.7, see also 3.6).

Then by 3.8 for every descent datum (A,φ_A) and every morphism $f : U \to A$ in $\mathfrak{F}_{S_o}$ with $\pi(U) \leq \gamma$ there is a decomposition of f into a morphism $U \to U'$ and a morphism $(U',\varphi_{U'}) \to (A,\varphi_A)$ of descent data such that $\pi(U') \leq \gamma$. Moreover a descent datum (X,φ_X) is γ-presentable in $\mathrm{Desc}(\mathfrak{F}_{S_o})$ iff X is γ-presentable in $\mathfrak{F}_{S_o}$. If in addition the inverse image functors p_1^*, p_2^* and p_{31}^* preserve colimits (resp. limits), then by 3.24 $\mathrm{Desc}(\mathfrak{F}_{S_o})$ is locally γ-presentable (resp. β-presentable) and the forgetful functor $\mathrm{Desc}(\mathfrak{F}_{S_o}) \to \mathfrak{F}_{S_o}$, $(A,\varphi_A) \rightsquigarrow A$ is cotripleable (resp. tripleable). In particular the canonical functor (cf. [16] 1.4)

$$\Phi : \mathfrak{F}_S \to \mathrm{Desc}(\mathfrak{F}_{S_o}),\ Y \rightsquigarrow (\alpha^*(Y), c_{p_2,\alpha}(Y) \circ c_{p_1,\alpha}(Y)^{-1})$$

is an equivalence iff $\alpha^* : \mathfrak{F}_S \to \mathfrak{F}_{S_o}$ is cotripleable (resp. tripleable). The relationship between descent and tripleability (resp. cotripleability) was first noticed by J. Beck and J. Benabou.

4.16 If $\mathfrak{F}_{S_o}$, $\mathfrak{F}_{S_1}$ and $\mathfrak{F}_{S_2}$ are Grothendieck categories (resp. topoi) and the inverse image functors p_1^*, p_2^* and p_{31}^* preserve colimits and finite limits, then $\mathrm{Desc}(\mathfrak{F}_{S_0})$ is again a Grothendieck category (resp. topos). This follows from 3.25.

4.17 The version of 4.15 for generated objects is as follows. Assume that $\mathfrak{F}_{S_0}$, $\mathfrak{F}_{S_1}$ and $\mathfrak{F}_{S_2}$ are locally presentable and that p_1^*, p_2^* and p_{31}^* have rank (2.1). Let $\gamma > \beta$ be cardinals such that

1) $\mathfrak{F}_{S_0}$ is locally γ-noetherian (resp. $\mathfrak{F}_{S_0}$ is locally γ-generated)

2) every β-well ordered colimit of monomorphisms in $\mathfrak{F}_{S_0}$ is again monomorphic

3) the functors P_1^*, P_2^* and P_{31}^* preserve β-filtered colimits and take γ-presentable objects into γ-presentable objects (resp. they preserve β-filtered colimits and finite limits and take γ-generated objects into γ-generated objects, cf. 5.1)

Then by 3.22 for every descent datum (A, φ_A) and every γ-generated subobject U of A there is a γ-generated subobject $U' \subset A$ containing U and a descent datum $(U', \varphi_{U'})$ such that the inclusion $U' \hookrightarrow A$ is a morphism of descent data. Moreover a descent datum (X, φ_X) is γ-generated in $\mathrm{Des}(\underline{\mathcal{F}}_{S_0})$ iff X is γ-generated in $\underline{\mathcal{F}}_{S_0}$, etc.

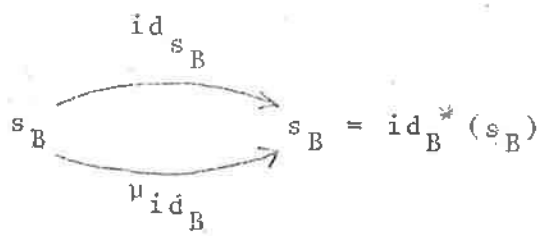
4.18 A possible application of the above is the following. If descent data are effective on small objects in $\underline{\mathcal{F}}_{S_0}$, then they are effective on all objects. In more detail let $\Phi : \underline{\mathcal{F}}_S \to \mathrm{Desc}(\underline{\mathcal{F}}_{S_0})$ be the canonical functor defined in 4.15. Recall that $\alpha : S_0 \to S$ is called of $\mathcal{F}$-descent type (resp. of strict $\mathcal{F}$-descent type) if Φ is full and faithful (resp. an equivalence), cf. [16] Def. 1.7. In addition to the assumptions made for the first half of 4.15 we assume that $\underline{\mathcal{F}}_S$ has γ-filtered colimits and that $\alpha^* : \underline{\mathcal{F}}_S \to \underline{\mathcal{F}}_{S_0}$ preserves γ-filtered colimits. Then $\alpha : S_0 \to S$ is of strict $\mathcal{F}$-descent type provided it is of $\mathcal{F}$-descent type and every descent data (U, φ_U) with U γ-presentable in $\underline{\mathcal{F}}_{S_0}$ is effective (i.e. in the image of Φ). This follows from 4.15 and 3.9 which imply that every descent datum (A, φ_A) in $\mathrm{Desc}(\underline{\mathcal{F}}_{S_0})$ is the γ-filtered colimit of descent data (U_i, φ_{U_i}) with $\pi(U_i) \leq \gamma$; whence if $\Phi(Y_i) = (U_i, \varphi_{U_i})$, then $\Phi(\varinjlim_i Y_i) \cong \varinjlim_i \Phi(Y_i) = \varinjlim_i (U_i, \varphi_{U_i}) = (A, \varphi_A)$.
The smallest cardinals γ and β which are possible for this (and 4.15) are $\aleph_1$ and $\aleph_0$. Thus, if $\underline{\mathcal{F}}_{S_0}$ is locally countably presentable (or finitely presentable) and $\underline{\mathcal{F}}_S$ has countably filtered colimits and the inverse image functors α^*, p_1^*, p_2^* and p_{31}^* preserve filtered colimits and take countably presentable objects into countably presentable objects, then every descent datum is effective provided descent data are effective on countably presentable objects and $\alpha^* : \underline{\mathcal{F}}_S \longrightarrow \underline{\mathcal{F}}_{S_0}$ is of $\mathcal{F}$-descent type.

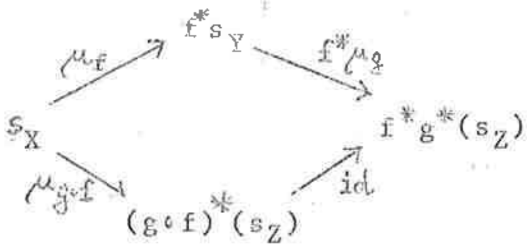
4.19 Sections and cartesian closed sections with respect to a fibration.
The following is based on, or rather prompted by exposé I in SGA 4 by

A. Grothendieck and J.L. Verdier [17] (mainly p. 138-179). At my talk [34] M. Tierney suggested to compare the notion of a bialgebra (3.1) with the notion of a section (resp. cartesian closed section) for a fibration, and theorem 3.8 with theorem I 9.25 in [17]. (Both apply to descent data and récollements in the context of Grothendieck categories or topoi and yield the existence of generators.) In order to facilitate the comparison we essentially use the notion and notation for a fibration $p : \underline{E} \longrightarrow \underline{B}$ as defined in [17] p. 160, although it differs from the one used in 4.14 above (for an exposé on the different ways to look at fibrations see Giraud [14] or SGA 1 exposé VI). Let $p : \underline{E} \to \underline{B}$ be a fibration with small base $\underline{B}$. For an object $B \in \underline{B}$ the fibre $p^{-1}(B)$ is denoted with $\underline{E}_B$ and the inverse image functor for a morphism $f : A \to B$ in $\underline{B}$ with $f^* : \underline{E}_B \to \underline{E}_A$. Let $\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$ denote the category of sections with respect to $p : \underline{E} \to \underline{B}$, i.e. the full subcategory of $[\underline{B},\underline{E}]$ consisting of all functors $s : \underline{B} \to \underline{E}$ with the property $ps = id_{\underline{B}}$, cf. [17] p. 161. Likewise let $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ be the full subcategory of $\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$ of all cartesian closed sections, i.e. all sections $s : \underline{B} \to \underline{E}$ such that for every morphism $f : B \to A$ the canonical morphism $s(B) \to f^*(A)$ is an isomorphism. The main theorems of section I.9 in [17] concern the existence of generators in $\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$ resp. $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ and implicite size estimates in terms of "filtrations cardinales". Without loss of generality one can assume that the objects of $\underline{B}$ form a set whose cardinality is the same as that of a skeleton of $\underline{B}$; this can always be achieved by pulling back the fibration $p : \underline{E} \longrightarrow \underline{B}$ along a full inclusion $\overset{\circ}{\underline{B}} \xrightarrow{\cong} \underline{B}$ (for skeleton $\overset{\circ}{\underline{B}}$ of $\underline{B}$ see [26]...). In order to express sections and cartesian closed sections as bialgebras let $\underline{A} = \coprod_{B \in \underline{B}} \underline{E}_B$ and let $\mathbb{F} = \{f^* \circ p_Y \mid (X \xrightarrow{f} Y) \in \mathrm{Mor}\,\underline{B}\}$ consist of all composites $\prod_{B \in \underline{B}} \underline{E}_B \xrightarrow[p_Y]{} \underline{E}_Y \xrightarrow{f^*} \underline{E}_X$, where f runs through all morphisms of $\underline{B}$ and p_Y denotes the canonical projection onto $\underline{E}_Y$. Let $M = \{\mu_f \mid f \in \mathrm{Mor}\,\underline{B}\}$, where $f : X \to Y$ runs through all morphisms of $\underline{B}$ and μ_f is an operation $p_X \dashrightarrow f^* p_Y$. Thus a pre-bialgebra consists of a family $(s_B)_{B \in \underline{B}}$ of objects in $\underline{E}$ together with a family $\{\mu_f : s_X \longrightarrow f^* s_Y\}_{(X \xrightarrow{f} Y) \in \text{Mor } \underline{B}}$ of morphisms. Let

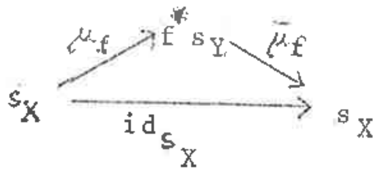
$$R = \{r_{id_B} \mid B \in \underline{B}\} \cup \{r_{f,g} \mid f, g \in \text{Mor } \underline{B} \text{ and } g \circ f \text{ defined}\}$$

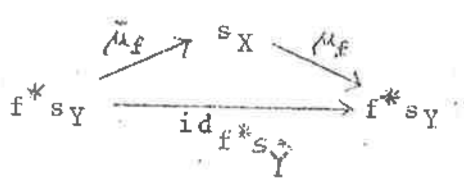
consist of relations $r_{id_B} : p_B \dashrightarrow p_B$ for every $B \in \underline{B}$ and $r_{f,g} : p_X \dashrightarrow g^* f^* p_Z$ for every composite $X \xrightarrow{f} Y \xrightarrow{g} Z$ in B , which for a pre-bialgebra $(s_B, \mu_f)_{B \in \underline{B}, f \in \text{Mor } \underline{B}}$ are given by the diagrams





With this it is straight forward that $\text{Bialg}(\underline{A}) = \text{Hom}_{\underline{B}}(\underline{B}, \underline{E})$. Clearly $\mathbb{F}_c = \mathbb{F}$, $\mathbb{F}_d = \{p_B \mid B \in \underline{B}\}$ and every projection p_B preserves all (existing) colimits and limits. In order to obtain cartesian closed sections one adds to M for every morphism $f : X \longrightarrow Y$ an operation $\bar{\mu}_f : f^* p_Y \dashrightarrow p_X$ and to R two relations which for a pre-bialgebra $(s_B, \mu_f, \bar{\mu}_f)_{B,f}$ are given by the diagrams





With this $\text{Bialg}(\underline{A}) = \text{Homcart}_{\underline{B}}(\underline{B}, \underline{E})$. Note however that in this case $\mathbb{F} = \mathbb{F}_c = \mathbb{F}_d$. Thus the functors in $\mathbb{F}_d$ preserve only those colimits (resp. limits) which are preserved by all inverse image functors f^*, $f \in \text{Mor } \underline{B}$. The above shows that sections and cartesian closed sections are special cases of bialgebras. The converse, i.e. that for a given data M , R and $\mathbb{F}$ of bialgebras in a category $\underline{X}$ there is a fibration $\underline{E} \longrightarrow \underline{B}$ such that either $\text{Hom}_{\underline{B}}(\underline{B}, \underline{E}) = \text{Bialg}(\underline{X})$ or $\text{Homcart}_{\underline{B}}(\underline{B}, \underline{E}) = \text{Bialg}(\underline{X})$, is very unlikely.

Such a hypothetical fibration would have to have weird properties (cf. discussion below in ...)

4.2o Assume that the fibre $\underline{E}_B$ is locally presentable for every $B \in \underline{B}$ and that the inverse image functor $f^* : \underline{E}_Y \to \underline{E}_X$ has rank (2.1) for every morphism $f : X \to Y$ in $\underline{B}$. Let $\gamma \geq \aleph_1$ be any regular cardinal such that 1) $\gamma \geq \pi(\underline{E}_B)$ for every $B \in \underline{B}$ 2) $\gamma > \pi(f^*)$ for every $f : X \to Y$ and 3) $\gamma > \operatorname{Mor} \underline{B}$. Let δ be the least regular cardinal $\delta \geq \aleph_1$ satisfying 1) - 3). Then in $\underline{A} = \prod_{B \in B} \underline{E}_B$ an object $(s_B)_{B \in \underline{B}}$ is γ-presentable iff s_B is γ-presentable in $\underline{E}_B$ for every $B \in \underline{B}$, in particular $\underline{A}$ is locally δ-presentable and the projections p_B take γ-presentable objects into γ-presentable objects and likewise for γ-generated objects. Moreover for every $f : X \to Y$ the functor $f^* p_Y : \prod_{B \in B} \underline{E}_B \to \underline{E}_X$ preserves δ-filtered colimits.

4.21 Assume 4.2o. Then by 3.8 <u>for every section</u> $s \in \operatorname{Hom}_{\underline{B}}(\underline{B}, \underline{E})$, <u>for every family</u> $(t_B)_{B \in B}$ <u>of objects and every family</u> $(f_B : t_B \to sB)_{B \in \underline{B}}$ <u>of morphisms such that</u> $\pi(t_B) \leq \gamma$ <u>in</u> $\underline{E}_B$ <u>for every</u> $B \in \underline{B}$ <u>there is a section</u> $t' \in \operatorname{Hom}_{\underline{B}}(\underline{B}, \underline{E})$ <u>together with a natural transformation</u> $\varphi : t' \to s$ <u>such that</u> $t'B$ <u>is</u> γ-<u>presentable in</u> $\underline{E}_B$ <u>and</u> $f_B : t_B \to sB$ <u>admits a decomposition</u> $t_B \to t'B \xrightarrow{\varphi(B)} sB$ <u>for every</u> $B \in \underline{B}$. <u>Moreover a section</u> $s : \underline{B} \to \underline{E}$ <u>is</u> γ-<u>presentable in</u> $\operatorname{Hom}_{\underline{B}}(\underline{B}, \underline{E})$ <u>iff</u> sB <u>is</u> γ-<u>presentable in</u> $\underline{E}_B$ <u>for every</u> $B \in \underline{B}$.

<u>Assume in addition to</u> 4.2o <u>that there is a regular cardinal</u> $\beta < \gamma$ <u>such that for every</u> $B \in \underline{B}$ β-<u>well ordered colimits of monomorphisms in</u> $\underline{E}_B$ <u>are monomorphic and that either</u> $\underline{E}_B$ <u>is locally</u> γ-<u>noetherian or all inverse image functors</u> f^*, <u>where</u> $f \in \operatorname{Mor} \underline{B}$, <u>preserve finite limits. Then for every section</u> $s : \underline{B} \to \underline{E}$ <u>and every family</u> $(f_B : t_B \hookrightarrow sB)_{B \in \underline{B}}$ <u>of</u> γ-<u>generated subobjects there is a subsection</u> $\varphi : t' \hookrightarrow s$ <u>such that</u> $t'B$ <u>contains</u> t_B <u>and</u> $t'B$ <u>is</u> γ-<u>generated in</u> $\underline{E}_B$ <u>for every</u> $B \in \underline{B}$. <u>Moreover a section</u> $s : \underline{B} \to \underline{E}$ <u>is</u> γ-<u>generated in</u> $\operatorname{Hom}_{\underline{B}}(\underline{B}, \underline{E})$ <u>iff</u> sB <u>is</u> γ-<u>generated in</u> $\underline{E}_B$ <u>for every</u> $B \in \underline{B}$.

4.22 Assume 4.2o. Then by 3.24 a) $\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$ is locally δ-presentable and the functor

$$\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E}) \dashrightarrow \prod_{B\in \underline{B}} \underline{E}_B \ , \quad s \rightsquigarrow (sB)_{B\in \underline{B}}$$

is cotripleable. Its right adjoint preserves δ-filtered colimits. Assume in addition to 4.2o that every inverse image functor f^*, where $f \in \mathrm{Mor}\ \underline{B}$, preserves finite limits and that every fibre $\underline{E}_B$, where $B \in \underline{B}$, is a Grothendieck category (resp. a topos). Then by 3.25 $\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$ is also a Grothendieck category (resp. a topos).

4.23 Assume 4.2o and that for every $f \in \mathrm{Mor}\ \underline{B}$ the inverse image functor f^* preserves limits. Let δ' be the least regular cardinal such that every f^* preserves δ'-filtered colimits. Then by 3.24 b) $\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E})$ is locally δ'-presentable and the functor

$$\mathrm{Hom}_{\underline{B}}(\underline{B},\underline{E}) \longrightarrow \prod_{B\in \underline{B}} \underline{E}_B, \quad s \rightsquigarrow (sB)_{B\in \underline{B}}$$

is tripleable and preserves δ'-filtered colimits. (Note that in contrast to 4.22 the case $\delta' = \aleph_0$ is possible, e.g. if $\mathrm{Mor}\ \underline{B}$ is finite, every fibre $\underline{E}_B$, $B \in \underline{B}$, is locally finitely presentable and f^* preserves filtered colimits for every $f \in \mathrm{Mor}\ \underline{B}$.)

4.24 The situation for $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ is different because $\mathbb{F}_d$ consists of all composites $\prod_{B\in \underline{B}} \underline{E}_B \xrightarrow{p_Y} \underline{E}_Y \xrightarrow{f^*} \underline{E}_X$. In general these functors neither take γ-presentable objects into γ-presentable objects nor do they preserve colimits. Therefore additional conditions are needed to guarantee the validity of 4.21 - 4.23 for $\mathrm{Homcart}_{\underline{B}}(\underline{B},\underline{E})$. They are as follows.

For the first half of 4.21 one has to assume in addition to 4.20 that for every $f \in \mathrm{Mor}\ \underline{B}$ the inverse image functor f^* takes γ-presentable objects into γ-presentable objects, and for the second half of 4.21 that every f^* takes γ-generated objects into γ-generated objects.

For 4.22 (both the first and second half) one has to assume in addition to 4.2o that for every $f \in \text{Mor}\ \underline{B}$ the inverse image functor f^* preserves colimits and takes γ-presentable objects into γ-presentable objects. (Note that by the special adjoint functor theorem every f^* has a right adjoint f_*. Thus it follows from $[X, f_* -] \cong [f^* X, -]$ that the functor f^* takes γ-presentable objects into γ-presentable objects iff f_* preserves γ-filtered colimits for every $f \in \text{Mor}\ \underline{B}$.)

For 4.23 no additional assumptions to 4.2o are needed. The functor f^* may not take δ'-presentable objects into δ'-presentable objects with δ' as in 4.23, but by 3.7 there is a regular cardinal $\tilde{\delta} \geq \delta'$ such that every f^* takes $\tilde{\delta}$-presentable objects into $\tilde{\delta}$-presentable objects. Thus by 3.24 b) $\text{Homcart}_{\underline{B}}(\underline{B}, \underline{E})$ is locally δ'-presentable with δ' as in 4.23 and $\text{Homcart}_{\underline{B}}(\underline{B}, \underline{E}) \longrightarrow \prod_{B \in \underline{B}} \underline{E}_B$, $s \rightsquigarrow (sB)_{B \in \underline{B}}$ is tripleable and preserves δ'-filtered colimits.

4.25 Remark For the first half of 4.21 (in particular the existence of γ-presentable generators in $\text{Hom}_{\underline{B}}(\underline{B}, \underline{E})$) the assumptions 4.2o are not fully used, in particular the existence of arbitrary colimits in the fibres $\underline{E}_B$, $B \in \underline{B}$, is not needed. Instead of 4.2o it suffices to assume that there are regular cardinals $\gamma > \beta \geq \aleph_0$ such that the following conditions hold

1) $\text{card}(\text{Mor}\ \underline{B}) < \gamma$

2) for every $B \in \underline{B}$ the fibre $\underline{E}_B$ has β-filtered colimits and for every $f \in \text{Mor}\ \underline{B}$ the inverse image functor f^* preserves β-filtered colimits

3) for every $B \in \underline{B}$ and every $A \in \underline{E}_B$ the category $\underline{E}_B(\gamma)/A$ of γ-presentable objects over A is γ-filtered and A is the colimit of $\underline{E}_B(\gamma)/A \longrightarrow \underline{A}$, $(U \longrightarrow A) \rightsquigarrow U$.

This follows from 3.11 a), b) and 4.19.

Likewise the first half of 4.21 holds also for $\text{Homcart}_{\underline{B}}(\underline{B}, \underline{E})$ provided in addition to the above conditions 1) - 3) the following is satisfied.

4) for every $f \in \operatorname{Mor} \underline{B}$ the inverse image functor f^* takes γ-presentable objects into γ-presentable objects.

4.26 Comparison with SGA 4 I.9. The main theorem I. 9.25 asserts the following. Let $p : \underline{E} \longrightarrow \underline{B}$ be a fibration with a small base $\underline{B}$ and assume that for every morphism f in $\underline{B}$ the inverse image functor f^* has rank (2.1). Then both $\operatorname{Hom}_{\underline{B}}(\underline{B},\underline{E})$ and $\operatorname{Homcart}_{\underline{B}}(\underline{B},\underline{E})$ admit a set of strict generators provided the following four conditions are satisfied a) every fibre $\underline{E}_B$, $B \in \underline{B}$, has a set of strict*) generators b) every fibre $\underline{E}_B$, $B \in \underline{B}$, has colimits and pullbacks c) for every $B \in \underline{B}$ the kernel functor $\operatorname{Mor}^2 \underline{E}_B \longrightarrow \underline{E}_B$, $(u,v) \rightsquigarrow \ker(u,v)$, has rank (2.1), where $\operatorname{Mor}^2 \underline{E}_B$ denotes the category of morphism pairs in $\underline{E}_B$ with common domain and codomain d) in every fibre $\underline{E}_B$, $B \in \underline{B}$, the pullback of a strict epimorphism*) is again a strict epimorphism.
The conditions a) - d) imply that for every $B \in \underline{B}$ the fibre $\underline{E}_B$ is locally presentable. (This is because by [17] I. 9.11 every object in $\underline{E}_B$ is presentable.) The converse is not true. A locally presentable category satisfies a), b) and c) but not d) in general; e.g. the category Cat of small categories does not satisfy d). Grothendieck and Verdier do not give explicite size estimates for the generators and the ones which result from their proof are not very effective. For instance if the fibres are locally finitely or locally $\aleph_1$-presentable and the inverse image functors preserve filtered colimits and if the set of morphisms of $\underline{B}$ is countable, then their proof yields that all sections $s \in \operatorname{Hom}_{\underline{B}}(\underline{B},\underline{E})$ with $\pi(sB) < (2^{\aleph_0})^+$ for every $B \in \underline{B}$ form a small generating (even dense) subcategory. (Recall that $(2^{\aleph_0})^+$ denotes the least regular cardinal $> 2^{\aleph_0}$). In contrast it follows from 4.21 above

*) An epimorphism $f : A \longrightarrow B$ is called strict if it is the cokernel of its kernel pair $A \times_B A \rightrightarrows A$. In Gabriel-Ulmer [13] § 1 strict epimorphisms are called regular.

that already all sections $s \in Hom_{\underline{B}}(\underline{B},\underline{E})$ such that sB is countably presentable for every $B \in \underline{B}$ form a small generating (even dense) subcategory.

The proof of Grothendieck and Verdier (cf. I. 9.22 - I. 9.26) for the existence of generators in $Homcart_{\underline{B}}(\underline{B},\underline{E})$ is not correct. The error is on p. 173 in [17], where they claim that the indicated composed morphism $f^*(X(\beta)_i) \longrightarrow f^*(X(\beta)) \xrightarrow{X(f)^{-1}} X(\alpha)$ factors through a canonical morphism $X(\alpha)_j \longrightarrow \varinjlim_{j \in I}(X_j) = X(\alpha)$ for some j, assuming that I is c^+-filtered (= I est grand devant c) and $X(\beta)_i$ is c^+-presentable (= c-accessible). This need not be so because in general f^* does not take c^+-presentable objects into c^+-presentable objects! As a matter of fact with c as in [17] p. 173 the cardinal $\pi(f^*(\beta)_i)$ can be arbitrary large although f^* has rank. (As a guidance for this phenomenon we mention the filtered colimit preserving forgetful functor $\underline{Mod}_\Lambda \to \underline{Sets}$ for some ring Λ, which takes finitely presentable objects into $card(\Lambda)^+$-presentable objects, cf. also 3.5 - 3.7.) As a consequence of this error the lemmata 9.21.16 (ii) and 9.21.19 are incorrect and the "filtrations" of $Homcart_{\underline{B}}(\underline{B},\underline{E})$ given in I. 9.22- I. 9.26 are not "filtrations cardinales" as claimed.

§ 5 Purity

In this section we generalize some aspects of purity to locally presentable categories. This will be crucial for the next section but seems of independant interest and we therefore state it seperately. Recall that over a ring Λ a submodule $i : U \hookrightarrow A$ of a left Λ-module $A \in {}_{\Lambda}\underline{\text{Mod}}$ is called pure iff for every right Λ-module $B \in \underline{\text{Mod}}_{\Lambda}$ the induced map $B \otimes_{\Lambda} i : X \otimes_{\Lambda} U \dashrightarrow X \otimes_{\Lambda} A$ is a monomorphism. Clearly $i : U \dashrightarrow A$ is already pure if $B \otimes_{\Lambda} i$ is a monomorphism for every finitely presentable module B because every module is a filtered colimit of finitely presentable ones. (Actually one can test purity with finitely presentable cyclic modules, but this is not relevant in the following. The important thing is that purity can be tested with a set of modules.) Among the various characterizations of purity the following is instructive for our purposes. A monomorphism $i : U \dashrightarrow A$ is pure iff it is a filtered colimit of splitting monomorphisms. Therefore any functor $T : {}_{\Lambda}\underline{\text{Mod}} \dashrightarrow \underline{X}$ which preserves filtered colimits takes pure monomorphisms into monomorphisms. The proviso is that in $\underline{X}$ a filtered colimit of splitting monomorphisms is again a monomorphism. Note that the class of all filtered colimit preserving T ($\underline{X}$ variable) contains a subset with which purity can be tested. Fakir [F] used the above characterization to define purity in locally presentable categories. We introduce here a weaker notion of purity.

Now let $\underline{A}$ be an arbitrary category and let $(T_V : \underline{A} \dashrightarrow \underline{X}_V)_{V \in M}$ be a family of functors. A monomorphism $i : U \dashrightarrow A$ in $\underline{A}$ is called pure with respect to $(T_V)_{V \in M}$ if $T_V i : T_V U \dashrightarrow T_V A$ is a monomorphism for every $V \in M$. Given a subobject Y of $A \in \underline{A}$ we are concerned with the problem of constructing a pure subobject Y' of A which contains Y and is not much bigger than Y. For locally presentable categories and a set M we give a construction and size estimates which are the best possible in the cases envisaged here. It should be noted that the existence of arbitrary colimits in $\underline{A}$ and $\underline{X}_V$ is not needed, the "minimal" conditions on $\underline{A}$, $\underline{X}_V$ and T_V can be found in 5.3c) and 5.6 b).

We begin with some preparation. Recall that $\varepsilon(A)$ denotes the least regular cardinal γ such that A is γ-generated, i.e. the functor $[A,-] : \underline{A} \to \underline{Sets}$ preserves monomorphic γ-filtered colimits (cf. 2.2).

5.1 <u>Lemma</u> <u>Let $\underline{A}$ be a locally α-generated category (cf. 2.3) and $T : \underline{A} \to \underline{X}$ a functor which preserves monomorphic α-filtered colimits. Let $\bar{\alpha} \geq \alpha$ be any regular cardinal such that</u>

1) <u>if</u> $W \in \underline{A}$ <u>and</u> $\varepsilon(W) \leq \alpha$, <u>then</u> $\varepsilon(TW) \leq \bar{\alpha}$

2) <u>if</u> $\rho < \alpha$ <u>and</u> $\beta < \bar{\alpha}$, <u>then</u> $\beta^{\rho} < \bar{\alpha}$

<u>Then if</u> $U \in \underline{A}$ <u>is $\bar{\alpha}$-generated, so is</u> TU.

<u>Remark</u> Note that if $\alpha = \aleph_0$ or $\bar{\alpha} = (2^{\gamma})^+$ for some $\gamma^+ \geq \alpha$, then the "akward" condition 2) is automatically satisfied. (Recall that γ^+ denotes the least regular cardinal $> \gamma$.)

<u>Proof</u> The case $\alpha = \bar{\alpha}$ is trivial and we assume $\bar{\alpha} > \alpha$. Let $U \in \underline{A}$ be an $\bar{\alpha}$-generated object. By 2.7 there is a family $(W_\iota)_{\iota \in I}$ of α-generated objects $W_\iota \in \underline{A}$ and a proper epimorphism $\varphi: \coprod_{\iota \in I} W_\iota \to U$ such that $\text{card}(I) < \bar{\alpha}$. Let R be the set of all subsets J of I with $\text{card}(J) < \alpha$ ordered by inclusion. Clearly R is α-filtered and it follows from condition 2) that $\text{card}(R) < \bar{\alpha}$. Let U_J denote the image of the composite $\coprod_{\iota \in J} W_\iota \to \coprod_{\iota \in I} W_\iota \xrightarrow{\varphi} U$, where the first morphism is given by the inclusion $J \subset I$. Then by [ ] 6.7 d) U_J is again α-generated. Hence by condition 1) TU_J is $\bar{\alpha}$-generated. By [13] 6.7 a) the canonical morphism $\psi: \varinjlim_{J \in R} U_J \to U$ is monomorphic. Since $\varphi: \coprod_{\iota \in I} W_\iota \to U$ factors through $\psi : \varinjlim_{J \in R} U_J \to U$ and φ is a proper epimorphism, it follows that ψ is an isomorphism. Summarizing we

obtain

$$\varepsilon(TU) = \varepsilon(T \varinjlim_{J\in R} U_J) = \varepsilon(\varinjlim_{J\in R} TU_J) \leq \bar{\alpha}$$

because by [ ]6.2 an $\bar{\alpha}$-colimit of $\bar{\alpha}$-generated objects is again $\bar{\alpha}$-generated. This completes the proof. (We note the similarity with the proof of 3.7)). Concluding we remark that the existence of colimits in $\underline{A}$ is not needed for the above argument. We have only used that an $\bar{\alpha}$-generated object U is an α-filtered $\bar{\alpha}$-colimit of α-generated subobjects.

Recall that a locally δ-presentable category is called locally δ-noetherian if every δ-generated object is δ-presentable. By [13] 13.3 every locally presentable category is locally δ-noetherian for sufficiently large δ.

5.2 Theorem Let $(T_V : \underline{A} \to \underline{X}_V)_{V\in M}$ be a family of functors, where M is a set and $\underline{A}$ and $\underline{X}_V$, $V \in M$, are locally presentable categories. Assume there is a regular cardinal α such that every T_V preserves monomorphic α-filtered colimits and that in $\underline{A}$ and $\underline{X}_V$ α-filtered colimits of monomorphisms are monomorphic for every $V\in M$. Let δ be any regular cardinal such that

1) $\mathrm{card}(M) < \delta > \alpha$

2) $\underline{A}$ is locally δ-generated and $\underline{X}_V$ is locally δ-noetherian for every $V \in M$

3) if $U \in \underline{A}$ and $\varepsilon(U) \leq \delta$, then $\varepsilon(T_V U) \leq \delta$ for every $V \in M$ (cf. 5.1)

Then every δ-generated subobject Y of $A \in \underline{A}$ is contained in a pure subobject Y' of A which is also δ-generated.

5.3 Remarks

a) Note that by 5.1 and [13] 13.3 there is always a regular cardinal δ satisfying 1) - 3) and it can be choosen so as to exceed any given cardinal. The point is of course to choose δ as small as possible.

b) Note that γ has to be strictly bigger than α, whence $\gamma \geq \aleph_1$. For instance for left modules over a ring Λ – i.e. $\underline{A} = {}_{\Lambda}\underline{\mathrm{Mod}}$, $\underline{X}_V = \underline{\mathrm{Ab}}.\underline{\mathrm{Gr}}.$, $T_V = X_V \otimes_{\Lambda}$ for a finitely presentable right Λ-module X_V and M = set of equivalence classes of finitely presentable right Λ-modules – one has $\alpha = \aleph_o$ and $\mathrm{card}(M) = \mathrm{card}(\Lambda)$ if Λ is infinite and $\mathrm{card}(M) = \aleph_o$ if Λ is finite. Clearly for $\delta > \mathrm{card}(\Lambda)$ every δ-generated module U is δ-presentable and $\mathrm{card}(U) < \delta$, whence $\mathrm{card}(X \otimes_{\Lambda} U) < \delta$ for every finitely presentable X. Thus for $\delta > (\mathrm{card}(\Lambda), \aleph_o)$ every δ-generated submodule Y of A is contained in a δ-generated pure submodule Y' of A (cf. Barr [1]).

c) From the proof of 5.2 below it will be obvious that not all assumptions on $\underline{A}$ and $\underline{X}_V$ are needed, in particular the existence of arbitrary colimits in $\underline{A}$ and $\underline{X}_V$ is redundant. Besides conditions 1) and 3) only the following properties of $\underline{A}, \underline{X}_V$ and T_V are used : $\underline{A}$ has α-filtered colimits and every $T_V : \underline{A} \longrightarrow \underline{X}_V$, $V \in M$, preserves them. In $\underline{A}$ and in every $\underline{X}_V$, $V \in M$, an α-filtered colimit of monomorphisms is again a monomorphism. Every object $A \in \underline{A}$ is a δ-filtered colimit of δ-generated subobjects. In $\underline{X}_V$, $V \in M$, every δ-generated object is δ-presentable and every morphism admits a factorization into a proper epimorphism and a monomorphism.

Proof of 5.2 Let $i : Y \longrightarrow A$ be a monomorphism in $\underline{A}$, where $\varepsilon(Y) \leq \delta$. Then $T_V(Y)$ is δ-generated and hence δ-presentable for every $V \in M$. Let $A = \varinjlim_{\mu} Y_{\mu}$ be the colimit presentation of A as the δ-filtered colimit of its δ-generated subobjects Y_{μ} and let $i_{\mu} : Y_{\mu} \longrightarrow A$ denote the inclusion (cf. 2.8). Clearly $i : Y \longrightarrow A$ belongs to this system and we write $Y = Y_o$ and $i = i_o$. Since $T_V(Y_o)$ is δ-generated, so is $\mathrm{im}(T_V(i_o)$ for every $V \in M$, cf. [ ] 6.7 d). Thus $\mathrm{im}(T_V(i_o))$ is δ-presentable and from $\varinjlim_{\mu} T_V(Y_{\mu}) \xrightarrow{\cong} T_V(A)$ it follows that there is a δ-generated subobject Y_{μ} together with a morphism

$\xi_\mu : im(T_V(i_o)) \longrightarrow T_V(Y_\mu)$ – depending on T_V and we therefore write $\xi_V : im(T_V(i_o)) \longrightarrow T_V(Y_V)$ instead – such that the diagram

$$\begin{array}{ccc} im(T_V(i_o)) & \xrightarrow{j_V^o} & T_V(A) \\ {\scriptstyle \xi_V}\downarrow & & \downarrow{\scriptstyle id} \\ T_V(Y_V) & \xrightarrow{T_V(i_V)} & T_V(A) \end{array}$$

commutes, where j_V^o and $i_V : Y_V \longrightarrow A$ denote the canonical inclusions. (Note that $T_V(i_V)$ need not be a monomorphism). Let $i'_V : Y'_V \longrightarrow A$ be a δ-generated subobject containing $i_o : Y_o \longrightarrow A$ and $i_V : Y_V \longrightarrow A$. The inclusions $u : Y_o \longrightarrow Y'_V$ and $w : Y_V \longrightarrow Y'_V$ give rise to a pair of morphisms

$$\begin{array}{c} T_V(Y_o) \xrightarrow{p_V^o} im(T_V(i_o)) \xrightarrow{\xi_V} T_V(Y_V) \xrightarrow{T_V(w)} T_V(Y'_V) \\ T_V(Y_o) \xrightarrow{\qquad\qquad\qquad T_V(u) \qquad\qquad\qquad} T_V(Y'_V) \end{array}$$

-where p_V^o denotes the canonical projection – which become equal when composed with $T_V(i'_V) : T_V(Y'_V) \longrightarrow T_V(A)$. Since $T_V(Y_o)$ is δ-presentable and $T_V(A) = \varinjlim_\mu T_V(Y_\mu)$, there is a δ-generated subobject $i''_V : Y'' \longrightarrow A$ containing $i'_V : Y'_V \longrightarrow A$ such that the above pair becomes already equal when composed with $T_V(z) : T_V(Y'_V) \longrightarrow T_V(Y''_V)$, where $z : Y'_V \longrightarrow Y''_V$ denotes the inclusion. Since $card(M) < \delta$, there is a δ-generated subobject $i_1 : Y_1 \longrightarrow A$, containing Y''_V for every $V \in M$, together with a morphism $\xi_V^o : im(T_V(i_o)) \longrightarrow T_V(Y_1)$ for every $V \in M$ – namely the obvious composite $im(T(i_o)) \xrightarrow{\xi_V} T_V(Y_V) \xrightarrow{T_V(c)} T_V(Y_1)$ – such that the diagram

$$\begin{array}{ccccc}
T_V(Y_o) & \xrightarrow{\;p_V^o\;} & im(T_V(i_o)) & \xrightarrow[\subset]{\;j_V^o\;} & T_V(A) \\
{\scriptstyle T_V(u_o)}\downarrow & \swarrow{\scriptstyle \xi_V^o} & \downarrow\cap & & \downarrow{\scriptstyle id} \\
T_V(Y_1) & \xrightarrow{\;p_V^1\;} & im(T_V(i_1)) & \xrightarrow[\subset]{\;j_V^1\;} & T_V(A)
\end{array}$$

commutes, where $u_o : Y_o \longrightarrow Y_1$ denotes the inclusion. We now proceed by transfinite induction. If λ is a successor ordinal, then Y_λ is constructed from $Y_{\lambda-1}$ as above and so are the morphisms $\xi_V^{\lambda-1} : im(T_V(i_{\lambda-1})) \longrightarrow T_V(Y_\lambda)$ for every $V \in M$. If $\lambda < \alpha$ is a limit ordinal, then let Y_λ be any δ-generated subobject of A containing every Y_ρ for $\rho<\alpha$. We claim that $Y' = \varinjlim_{\lambda<\alpha} Y_\lambda$ is a δ-generated pure subobject of A containing $Y_o = Y$. The latter is obvious because $\varinjlim_{\lambda<\alpha} i_\lambda : \varinjlim_{\lambda<\alpha} Y_\lambda \longrightarrow A$ is a monomorphism. Since $\alpha<\delta$ the object Y' is δ-generated by 2.8 . The purity of the inclusion $i' : Y' \longrightarrow A$ results from the induced diagram

$$\begin{array}{ccccc}
\varinjlim_{\lambda<\alpha} T_V(Y_\lambda) & \xrightarrow{\;\varinjlim p_V^\lambda\;} & \varinjlim_{\lambda<\alpha} im\,(T_V(i_\lambda)) & \xrightarrow{\;\varinjlim j_V^\lambda\;} & T_V(A) \\
{\scriptstyle \varinjlim_{\lambda<\alpha} T_V(i_\lambda)}\downarrow\cong & & \Big|\,{\scriptstyle q} & & \downarrow{\scriptstyle id} \\
T_V(Y') & \xrightarrow{\;p_V'\;} & im\,(T_V(i')) & \xrightarrow{\;j_V'\;} & T_V(A)
\end{array}$$
$$T_V(Y') \xrightarrow{\;T_V(i')\;} T_V(A)$$

in which $\varinjlim j_V^\lambda$ is a monomorphism for every $V \in M$. Hence q is monomorphic. Since p_V' is a proper epimorphism, so is q and thus q is an isomorphism. Moreover $\varinjlim_{\lambda<\alpha} p_V^\lambda$ is an isomorphism, it's inverse is $\varinjlim_{\lambda<\alpha} \xi_V^\lambda$. Hence $T_V(i')$ is a monomorphism for every $V \in M$, i.e. $i' : Y' \longrightarrow A$ is pure which completes the proof.

5.4 <u>Definition</u> Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor. A monomorphism

$i : Y \dashrightarrow A$ is called T-pure if $T(B,i) : T(B,Y) \longrightarrow T(B,A)$ is a monomorphism for every $B \in \underline{B}$.

Clearly T-purity is equivalent with purity as defined above for $\{T(B,-) : \underline{A} \dashrightarrow \underline{C}\}_{B \in \underline{B}}$.

5.5. Corollary Let $T : \underline{B} \times \underline{A} \dashrightarrow \underline{C}$ be a bifunctor, where $\underline{A}$, $\underline{B}$ and $\underline{C}$ are locally presentable categories. Assume there is a regular cardinal α such that $T(-,-)$ preserves α-filtered colimits in both variables and such that in both $\underline{A}$ and $\underline{C}$ α-filtered colimits of monomorphisms are again monomorphisms. Let δ be any regular cardinal such that

1) $\delta > \alpha$ and the set M of equivalence classes of α-presentable objects in $\underline{B}$ has cardinality $< \delta$

2) $\underline{B}$ is locally α-presentable, $\underline{C}$ is locally δ-noetherian and $\underline{A}$ is locally δ-generated

3) if $V \in \underline{B}$ is α-presentable and $U \in \underline{A}$ δ-generated, then $T(V,U)$ is δ-generated.

Then every δ-generated subobject Y of $A \in \underline{A}$ is contained in a δ-generated T-pure subobject Y' of A.

Proof Since $T(-,-)$ preserves α-filtered colimits in the first variable and every $B \in \underline{B}$ is an α-filtered colimit of α-presentable objects, a monomorphism $i : Y \longrightarrow A$ is T-pure iff $T(V,i)$ is a monomorphism for every α-presentable object $V \in \underline{B}$. The assertion now follows from 5.2.

5.6 Remarks

a) Note that by 5.1 and [13] 13.3 there are always cardinals α and δ such that $\underline{A}$, $\underline{B}$, $\underline{C}$ and T satisfy the conditions in 5.5 (The only exception is that $T(-,-)$ preserves α-filtered colimits in both variables which has to be required separately). The point is of course to choose δ as small as possible.

b) As in 5.2 (cf. 5.3 c)) the assumptions on $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ are not fully used and 5.5 can be generalized considerably, in particular the existence of arbitrary colimits in $\underline{A}$, $\underline{B}$ and $\underline{C}$ is not needed. The following conditions suffice to establish 5.5. There is a set M of objects in $\underline{B}$ such that T-purity can be tested with the functors $T(V,-) : \underline{A} \to \underline{C}$ with V running through M . Putting $T_V = T(V,-)$ and $\underline{X}_V = \underline{C}$ for $V \in M$, then all the conditions listed in 5.3 c) hold, i.e. $\underline{A}$ has α-filtered colimits and ... .

c) The notion of T-purity was independantly introduced by T. Fox [8]. For a locally presentable category $\underline{X}$ and a coherently symmetric, associative and unitary tensor product $\otimes : \underline{X} \times \underline{X} \to \underline{X}$ with rank he proved that every γ-generated subobject in $\underline{X}$ is contained in a γ'-generated pure subobject for some γ' . He gives no size estimate for γ' and the case of purity over non-commutative rings is excluded. The present versions of 5.2 and 5.5 represent a slight (but useful) improvement over the original statement in [13]. The proofs of Fox [8] and the one given here have little in common. While our proof often gives the best possible upper bound for γ', the one resulting from his proof is much too large to be useful in practice. Following Barr [1], Fox [8] used 5.5 to prove that the category of coalgebras in a locally presentable category $\underline{A}$ with respect to some tensor product $\otimes : \underline{A} \times \underline{A} \to \underline{A}$ and a co-algebraic Prop has generators (cf. 4.7). We use 5.5 in the next section to prove that the category of Σ-cocontinuous functors $\underline{U} \to \underline{A}$ has generators when Σ is a proper class.

d) Fakir [7] defined the notion of an α-algebraically closed mono-morphism in locally α-presentable categories. He showed that a monomorphism is α-algebraically closed iff it is an α-filtered colimit of splitting monomorphisms. From this the relationship with purity becomes evident and it is clear that the test functors T_V in 5.2 (resp. $T(V,-)$ in 5.5) preserve α-algebraically closed monomorphisms, in particular the latter are pure. The converse need not hold and obviously depends on the family $(T_V)_{V \in M}$ of test-functors. It might be interesting to investigate (and characterize) pure monomorphisms with respect to functors different from tensor product type functors, e.g. (co)homology functors $(H_n)_{n \in \mathbb{N}}, (H^n)_{n \in N}, (Tor_n)_{n \in N}, (Ext^n)_{n \in N}$ etc. (i.e. functors which preserve α-filtered colimits for some α). Note that for any of these sets of test functors theorem 5.2 applies and the size estimates for δ can be effectively handled.

§ 6 Local presentability of $\underline{A}_{\Sigma,T}$, $Cc_{\Sigma}[\underline{U},\underline{X}]$ and $Adj(\underline{A},\underline{B})$; examples

This section is a continuation of § 4. We give further examples of bialgebras - in particular Σ-cocontinuous and Σ-continuous functors, pairs of adjoint functors etc. - and apply the results of § 3 and § 5. Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor and let $(\sigma : d\sigma \longrightarrow r\sigma)_{\sigma \in \Sigma}$ be a class of morphisms in $\underline{B}$. Let $\underline{A}_{\Sigma,T}$ be the full subcategory of $\underline{A}$ consisting of all objects $X \in \underline{A}$ such that $T(\sigma,X)$ is an isomorphism for every $\sigma \in \Sigma$. The bifunctor $T(-,-)$ and the class Σ give rise to a data for bialgebras in $\underline{A}$ such that $Bialg(\underline{A}) = \underline{A}_{\Sigma,T}$ and the forgetful functor $Bialg(\underline{A}) \longrightarrow \underline{A}$ is the inclusion $\underline{A}_{\Sigma,T} \xrightarrow{\subset} \underline{A}$ (cf. 6.1). The main result 6.12 (resp. 6.15) concerns conditions on $T : \underline{B} \times \underline{A} \longrightarrow C$ and on a class Σ which guarantee that $\underline{A}_{\Sigma,T}$ is locally δ-presentable (resp. locally δ-noetherian) for some specified cardinal δ depending on T and Σ. If T is the bifunctor $\otimes : [\underline{U}^{o},\underline{Sets}] \times [\underline{U},\underline{X}] \longrightarrow \underline{X}$ (resp. $T = [-,-]$) as defined in 2.1o, then $\underline{A} = [\underline{U},\underline{X}]$ and $\underline{A}_{\Sigma,T}$ consists exactly of all Σ-cocontinuous (resp. Σ-continuous) functors on $\underline{U}$ with values in $\underline{X}$ (cf. 6.14, 6.15). By choosing Σ accordingly one can obtain colimit (resp. limit) preserving functors $\underline{U} \longrightarrow \underline{X}$ or cosheaves (resp. sheaves) with respect to a Grothendieck topology on $\underline{U}$ and values in $\underline{X}$ (cf. 6.16 - 6.17). Moreover the category of pairs of adjoint functors between locally presentable categories is equivalent with a category of Σ-cocontinuous functors (cf. 6.18 - 6.2o).
Another example for T is the tensor product $\otimes_{\Lambda}$ over some ring Λ. If $\Sigma = \{I \hookrightarrow \Lambda\}_{I \in \mathcal{F}}$ is the set of all inclusions for a family $\mathcal{F}$ of right ideals in Λ, then ${}_{\Lambda}\underline{Mod}_{\Sigma,\otimes_{\Lambda}}$ consists exactly of all left Λ-modules X which are uniquely divisible by $\mathcal{F}$, i.e. for which multiplication $I \otimes_{\Lambda} X \longrightarrow X$ is an isomorphism for every $I \in \mathcal{F}$. For instance, if $\underline{A}$ is a Grothendieck category and $\Lambda = [U,U]$ the endomorphism ring of a generator $U \in \underline{A}$, then the functor $Cocont[\underline{A},\underline{Ab.Gr.}] \longrightarrow {}_{\Lambda}\underline{Mod}$, $t \mapsto tU$

induces an equivalence between cocontinuous functors $t : \underline{A} \longrightarrow \underline{Ab.Gr.}$ and uniquely $\mathcal{F}$-divisible left A-modules, where $\mathcal{F}$ is the Gabriel filter on A associated with $\underline{A}$ (cf. 6.25b)). Cocontinuous functors can have unexpected features, e.g. the category of cocontinuous functors from abelian p-groups to abelian groups is equivalent with the category of p-adic complete abelian groups. Similar assertions hold in more general situations (cf. 6.25c)).

6.1 Lemma <u>Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor and $(\sigma : d\sigma \longrightarrow r\sigma)_{\sigma \in \Sigma}$ a class of morphisms in $\underline{B}$. Then there is a data $M, R, \mathbb{F}$ for bialgebras in $\underline{A}$ (cf. 3.1) such that $\mathrm{Bialg}(\underline{A}) = \underline{A}_{\Sigma,T}$ and the forgetful functor $V : \mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}$ is the inclusion $\underline{A}_{\Sigma,T} \subset \underline{A}$. The class $\mathbb{F}$ consists of all functors $T(d\sigma,-) : \underline{A} \longrightarrow \underline{C}$ and $T(r\sigma,-) : \underline{A} \longrightarrow \underline{C}$, where σ runs through Σ and $\mathbb{F} = \mathbb{F}_d = \mathbb{F}_c$ holds. Moreover if Σ is a set, then so are M, R and $\mathbb{F}$.</u>

<u>Proof</u> Let $\mathbb{F}$ be as above. For M and R we limit ourselves to an intuitive description. A pre-bialgebra is an object $X \in \underline{A}$ together with a morphism $\sigma(X) : T(r\sigma,X) \longrightarrow T(d\sigma,X)$ for every $\sigma \in \Sigma$. Note that the forgetful functor $\text{P-Bialg}(\underline{A}) \longrightarrow \underline{A}$ need neither be an embedding nor full. The relations on a pre-bialgebra $(X,\sigma(X))_{\sigma \in \Sigma}$ express that the composites

$$T(d\sigma,X) \xrightarrow{T(\sigma,X)} T(r\sigma,X) \xrightarrow{\sigma(X)} T(d\sigma,X) \quad \text{and} \quad T(r\sigma,X) \xrightarrow{\sigma(X)} T(d\sigma,X) \xrightarrow{T(\sigma,X)} T(r\sigma,X)$$

are the identities of $T(d\sigma,X)$ and $T(r\sigma,X)$ respectively for every $\sigma \in \Sigma$. In other words $T(\sigma,X)$ is an isomorphism and $\sigma(X)$ its inverse. Hence a bialgebra is an object $X \in \underline{A}$ together with an isomorphism $\sigma(X) : T(r\sigma,X) \xrightarrow{\sim} T(d\sigma,X)$ whose inverse is $T(\sigma,X)$. Therefore the map $\mathrm{Bialg}(\underline{A}) \longrightarrow \underline{A}_{\Sigma,T}$, $(X,\sigma(X))_{\sigma \in \Sigma} \rightsquigarrow X$ is bijective on objects and it can be made into a functor Φ by mapping a bialgebra morphism $f : (X,\sigma(X))_{\sigma \in \Sigma} \longrightarrow (X',\sigma(X'))_{\sigma \in \Sigma}$ onto $f : X \longrightarrow X'$. Then Φ is obviously an isomorphism. So we can identify $\underline{A}_{\Sigma,T}$ with $\mathrm{Bialg}(\underline{A})$

and the forgetful functor $\text{Bialg}(\underline{A}) \longrightarrow \underline{A}, (X, \sigma(X))_{\sigma \in \Sigma} \rightsquigarrow X$ becomes the inclusion $\underline{A}_{\Sigma,T} \subset \underline{A}$. The other assertions in 6.1 are obvious.

6.2 Theorem Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor, where $\underline{A}$ is a locally presentable category. Let Σ be a set of morphisms in $\underline{B}$. Assume there is a regular cardinal α such that $T(d\sigma,-) : \underline{A} \longrightarrow \underline{C}$ and $T(r\sigma,-)$ preserve α-filtered colimits for every $\sigma \in \Sigma$. Let $\gamma > \alpha$ be any regular cardinal such that

a) $\text{card}(\Sigma) < \gamma$ and $\underline{A}$ is locally γ-presentable

b) if $U \in \underline{A}$ is γ-presentable, then so are $T(d\sigma,U)$ and $T(r\sigma,U)$ for every $\sigma \in \Sigma$ (cf. 3.7 for $\gamma = \bar{\alpha}$).

Then every morphism $f : U \longrightarrow A$ with the properties $A \in \underline{A}_{\Sigma,T}$ and $\pi(U) \leq \gamma$ admits a factorization $U \longrightarrow U' \longrightarrow A$ such that $\pi(U') \leq \gamma$ and $U' \in \underline{A}_{\Sigma,T}$. Moreover an object $X \in \underline{A}_{\Sigma,T}$ is γ-presentable in $\underline{A}_{\Sigma,T}$ iff it is γ-presentable in $\underline{A}$.

Proof The assertions follow directly from 6.1 and 3.8. It should be noted however that a direct proof can be given following the pattern in 1... . This proof is simpler because it involves only a one-step construction in contrast to the two-step construction in 3.8.

6.3 Remarks

a) Note that γ has to be strictly bigger than* α. Moreover if every object in $\underline{C}$ is presentable, then by 3.7 there is always a cardinal γ satisfying the conditions a) and b). The point is of course to choose γ as small as possible (cf. remark following 3.8).

b) The theorem also holds when $\underline{A}$ is not locally presentable, but merely satisfies conditions a) and b) in 3.11. In either case $\underline{A}_{\Sigma,T}$ need not be locally presentable, however it is equivalent with the category of γ-flat functors on the category $\underline{A}_{\Sigma,T}(\gamma)$

consisting of all γ-presentable objects in $\underline{A}_{\Sigma,T}$ (cf. 3.11 and 3.9).

6.4 <u>Definition</u> <u>Let</u> $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ <u>be a bifunctor and</u> Σ <u>a set of morphisms in</u> $\underline{B}$. <u>Assume</u> $\underline{A}$ <u>and</u> $\underline{C}$ <u>are locally presentable. Then</u> $\mathrm{rank}_\Sigma(T)$ <u>denotes the least cardinal</u> $\delta \geq \pi(A)$ <u>such that for every</u> $\sigma \in \Sigma$ <u>and every</u> $\pi(A)$-<u>presentable object</u> $U \in \underline{A}$ <u>the objects</u> $T(d\sigma,U)$ <u>and</u> $T(r\sigma,U)$ <u>are</u> δ-<u>presentable. For a set</u> M <u>of objects in</u> $\underline{A}$ $\mathrm{rank}_M(T)$ <u>is defined likewise.</u>

If the functors $T(d\sigma,-)$ and $T(r\sigma,-)$ preserve colimits for every $\sigma \in \Sigma$, then by the special adjoint functor theorem they have right adjoints $S(d\sigma,-)$ and $S(r\sigma,-)$ and the latter have rank (2.9, 2.1). Since by adjointness $[T(d\sigma,U),-] \cong [U,S(d\sigma,-)]$ and $[T(r\sigma,U),-] \cong [U,S(r\sigma,-)]$, it is not difficult to see, that $\mathrm{rank}_\Sigma(T)$ is the least regular cardinal δ such that $\delta \geq \pi(\underline{A})$ and $\pi(S(d\sigma,-)) \leq \delta \geq \pi(S(r\sigma,-))$ for every $\sigma \in \Sigma$. With this it is not hard to check the following.

1) Let Λ be a commutative ring and $\underline{A}$ a Grothendieck category. Let T be the bifunctor $\otimes_\Lambda : \underline{\mathrm{Mod}}_\Lambda \times {}_\Lambda\underline{A} \longrightarrow {}_\Lambda\underline{A}$, where ${}_\Lambda\underline{A}$ is the category of Λ-objects in $\underline{A}$. Then $\mathrm{rank}_\Sigma(\otimes_\Lambda) \leq \sup^*_{\sigma\in\Sigma}(\pi(\underline{A}),\pi(d\sigma),\pi(r\sigma))$, where $\sup^*(\ )$ denotes the least regular cardinal $\geq \sup(\ )$. Likewise if Λ is not commutative and T is the bifunctor $\otimes_\Lambda : \mathrm{Mod}_\Lambda \times {}_\Lambda\underline{A} \longrightarrow \underline{A}$, then $\mathrm{rank}_\Sigma(\otimes_\Lambda) \leq \sup^*_{\sigma\in\Sigma}(\pi(\underline{A}),\pi(d\sigma),\pi(r\sigma),\mathrm{card}(\Lambda)^+)$, where $\mathrm{card}(\Lambda)^+$ denotes the least regular cardinal $> \mathrm{card}(\Lambda)$.

2) Let T be the bifunctor $\otimes : [\underline{U}^o,\underline{\mathrm{Sets}}] \times [\underline{U},\underline{X}] \longrightarrow \underline{X}$ as defined in 2.1o, where $\underline{X}$ is a locally presentable category. Then $\mathrm{rank}_\Sigma(\otimes)$ is the least regular $\delta \geq \pi(X)$ such that $\mathrm{card}(d\sigma(U)) < \delta > \mathrm{card}(r\sigma(U))$ for every $\sigma \in \Sigma$ and every object $U \in \underline{U}$. To see that note that the right adjoint $S(d\sigma,-) : \underline{X} \longrightarrow [\underline{U},\underline{X}]$ assigns to an object X the functor $U \rightsquigarrow \prod_{d\sigma(U)} X$ (= $d\sigma(U)$-fold product of X); and likewise for $S(r\sigma,-)$.

6.5 Corollary Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor, where $\underline{A}$ and $\underline{C}$ are locally presentable categories. Let Σ be a set of morphisms in $\underline{B}$. Assume that $T(d\sigma,-)$ and $T(r\sigma,-)$ preserve colimits for every $\sigma \in \Sigma$ (resp. limits and α-filtered colimits for some α). Then $\underline{A}_{\Sigma,T}$ is locally presentable and the inclusion $\underline{A}_{\Sigma,T} \hookrightarrow \underline{A}$ has a right adjoint (resp. left adjoint). Moreover if

$$\gamma = \sup(\pi(\underline{A}), \aleph_1, \operatorname{card}(\Sigma)^+, \operatorname{rank}_\Sigma(T)) \ (\text{resp. } \gamma' = \sup(\pi(\underline{A}), \alpha))$$

then $\underline{A}_{\Sigma,T}$ is locally γ-presentable (resp. γ'-presentable) and the right adjoint $\underline{A} \longrightarrow \underline{A}_{\Sigma,T}$ (resp. the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$) preserves γ-filtered colimits (resp. γ'-filtered colimits). In the first case (i.e. $T(d\sigma,-)$ and $T(r\sigma,-)$ cocontinuous), the assertions in 6.2 hold for γ as defined here.

Proof The corollary is a consequence of 3.24, 6.1, 6.2 and 6.4.

6.6 Remark The second case ($T(d\sigma,-)$ and $T(r\sigma,-)$ continuous) can also be obtained from [13] 8.6 b). The proofs for 6.5 and [13] 8.6 b) are entirely different.

6.7 The analoguous assertion to 4.8 and 6.2 for γ-generated objects requires stronger hypotheses. They are listed in the following

Theorem Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor, where $\underline{A}$ and $\underline{C}$ are locally presentable categories. Let Σ be a set of morphisms in $\underline{B}$ and assume there is a regular cardinal α such that every α-filtered colimit of monomorphisms in $\underline{A}$ is a monomorphism and such that $T(d\sigma,-)$ and $T(r\sigma,-)$ preserve α-filtered colimits for every $\sigma \in \Sigma$. Let $\gamma > \alpha$ be any regular cardinal such that

a) $\operatorname{card}(\Sigma) < \gamma$ and $\underline{A}$ is locally γ-generated

b) if $U \in \underline{A}$ is γ-generated, then so are $T(d\sigma,U)$ and $T(r\sigma,U)$ for every $\sigma \in \Sigma$.

c1) $T(d\sigma,-)$ and $T(r\sigma,-)$ preserve finite limits for every $\sigma \in \Sigma$.

Instead of c 1) one can assume

c2) in $\underline{A}$ and $\underline{C}$ every γ-generated object is γ-presentable.

Then the following assertions hold.

I If $A \in \underline{A}_{\Sigma,T}$ and $U \in \underline{A}$ is a γ-generated subobject of A, then there is a subobject U' of A containing U such that $U' \in \underline{A}_{\Sigma,T}$ and U' is γ-generated in $\underline{A}$.

II An object $X \in \underline{A}_{\Sigma,T}$ is γ-generated in $\underline{A}_{\Sigma,T}$ iff it is γ-generated in $\underline{A}$.

III An object $X \in \underline{A}_{\Sigma,T}$ is the γ-filtered colimit of its γ-generated subobjects in $\underline{A}_{\Sigma,T}$.

IV In the presence of c2) every γ-generated object in $\underline{A}_{\Sigma,T}$ is γ-presentable in $\underline{A}_{\Sigma,T}$.

Proof The theorem is an immediate consequence of 3.22 and 6.1 .

6.8 Remarks

a) Assume that the conditions in 6.7 are satisfied except for c1) and c2) and that instead the following holds.

c3) In $\underline{A}$ every object is the γ-filtered colimit of its T-pure γ-generated subobjects (cf. 5.4, 5.5, 5.6 b)).

Then assertion I) can be strengthened as follows.

I' If $A \in \underline{A}_{\Sigma,T}$ and $U \in \underline{A}$ is a γ-generated subobject of A, then there is a T-pure γ-generated subobject U' of A containing U such that $U' \in \underline{A}_{\Sigma,T}$.

This follows from 6.1 and the proof of 3.22. Instead of using in 3.22 the presentation of an object as the γ-filtered colimit of its γ-generated subobjects one considers the cofinal subsystem of all T-pure γ-generated subobjects; for pre-bialgebras and sub-pre-bialgebras whose underlying objects in $\underline{A}$ are γ-generated one proceeds likewise.

Note however that a), b) c3) do not imply II, III, IV because the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ need not preserve monomorphisms.

b) As above in 6.2 not all assumptions on $\underline{A}$ and $\underline{C}$ are needed for I - IV and one can get by as in 6.3 b), Note that there is always a cardinal $\gamma > \alpha$ such that 6.7 a), b), c 2) hold. The point is of course to choose γ as small as possible, cf. also 3.23.

In order to deal with the situation when Σ is not a set - which is necessary in order to consider functors on a small category $\underline{U}$ which preserve all existing colimits in $\underline{U}$ - we have to use purity with respect to a bifunctor $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$. We assume in the following that $\underline{A}$, $\underline{B}$ and $\underline{C}$ are locally presentable, although the existence of arbitrary colimits is not needed for 6.12 (cf. 6.3b), 6.8b)).

6.9 Definition Let Σ be a class of morphisms in a category $\underline{B}$. Assume that in $\underline{B}$ every morphism $\beta : B \longrightarrow B'$ admits a factorization into a proper epimorphism $\beta'' : B \longrightarrow \mathrm{im}\,\beta$ and a monomorphism $\beta' : \mathrm{im}\,\beta \longrightarrow B'$. Then $\mathfrak{M}_\Sigma$ denotes the class of those subobjects of $r\sigma$ which are of the form $\sigma' : \mathrm{im}\,\sigma \longrightarrow r\sigma$ for some $\sigma \in \Sigma$.

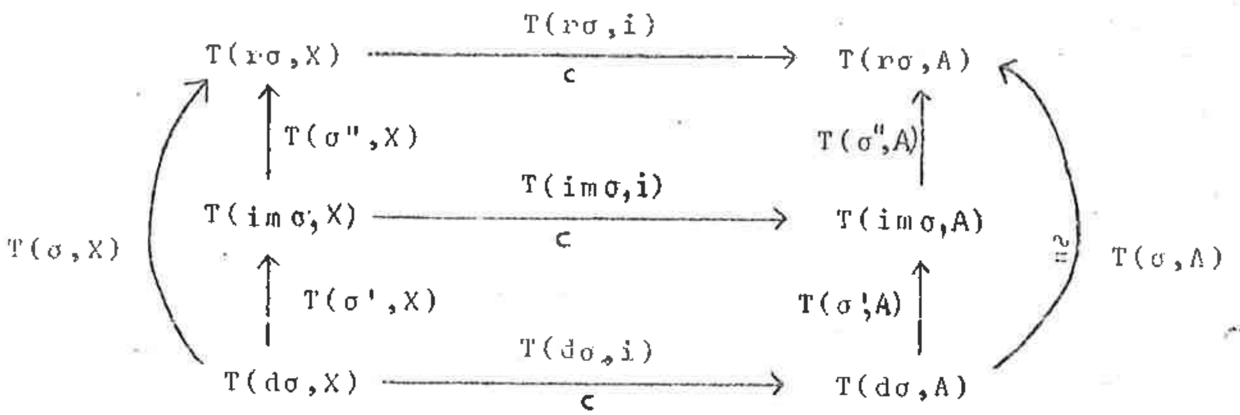
Conditions on $\underline{B}$ which guarantee the existence of such factorizations can be found in [13] 1.5, 1.6. Clearly they hold in every locally presentable category. Note that $\mathfrak{M}_\Sigma$ is a set provided the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set and $\underline{B}$ is well powered.

6.10 Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ be a bifunctor and Σ a class of morphisms in $\underline{B}$. If $T(-,-)$ preserves proper epimorphisms in the first variable (resp. takes proper epimorphisms into proper monomorphisms in case T is contravariant in the first variable), then it follows easily from the above that $A \in \underline{A}_{\Sigma,T}$ implies $A \in \underline{A}_{\mathfrak{M}_\Sigma,T}$, i.e. $\underline{A}_{\Sigma,T} \subset \underline{A}_{\mathfrak{M}_\Sigma,T}$. The converse is "unfortunately" not true, but the following shows that $\underline{A}_{\Sigma,T}$ is closed in $\underline{A}_{\mathfrak{M}_\Sigma,T}$ under T-pure subobjects.

6.11 Lemma Assume that $\underline{A}$, $\underline{B}$ and $\underline{C}$ are locally presentable and that

$T(-,-)$ preserves regular epimorphisms and well ordered colimits in the first variable (resp. takes them into regular monomorphisms and well ordered limits in case T is contravariant in the first variable). Let $A \in \underline{A}_{\Sigma,T}$ and let $i : X \rightarrowtail A$ be a T-pure monomorphism in $\underline{A}$. Then $X \in \underline{A}_{\Sigma,T}$ iff $X \in \underline{A}_{\mathfrak{M}_\Sigma,T}$.

Proof We limit ourselves to the first case because the second one is dual. By [13].6.6 b), 1.5 a morphism in a locally presentable category is a proper epimorphism iff it is a well ordered colimit of regular epimorphisms. Hence $T(-,-)$ preserves proper epimorphisms in the first variable. The assertion now results from the commutative diagram



observing that $T(\sigma',X)$ and $T(\sigma',A)$ are proper epimorphisms, that $T(\sigma,A)$ is an isomorphism and that $T(d\sigma,i)$ is a monomorphism.

6.12 Theorem Let $\underline{A}$, $\underline{B}$ and $\underline{C}$ be locally presentable categories. Let $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ (resp. $T : \underline{B}^o \times \underline{A} \longrightarrow \underline{C}$) be a bifunctor which preserves colimits in both variables (resp. limits and for every $B \in \underline{B}$ the functors $T(B,-) : \underline{A} \to \underline{C}$ preserves β-filtered colimits for some β depending on B). Let Σ be a class of morphisms in $\underline{B}$ such that the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set. Then the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ has a right adjoint (resp. left adjoint). In first case (but not in the second) $\underline{A}_{\Sigma,T}$ is locally presentable. In more detail let $\delta \geq \aleph_1$ be any regular cardinal such that

1) *$\underline{A}$ and $\underline{C}$ are locally δ-noetherian*

2) *there is a regular cardinal $\alpha < \delta$ such that in $\underline{A}$ and $\underline{C}$ α-filtered colimits of monomorphisms are again monomorphic*

3) $\delta \geq \sup \{\mathrm{card}(M)^{+}, \mathrm{rank}_{\Sigma}(T), \mathrm{rank}_{M}(T), \mathrm{card}(\mathfrak{M}_{\Sigma})^{+}\}$

where M is a set of objects in $\underline{B}$ with which T-purity can be tested (e.g. $\pi(\underline{B})$-presentable objects in $\underline{B}$, cf. 5.6 b) and 5.5; for $\mathfrak{M}_{\Sigma}$ and $\mathrm{rank}_{\mathfrak{M}_{\Sigma}}(T)$, $\mathrm{rank}_{M}(T)$ see 6.9 and 6.4 respectively). *Then $\underline{A}_{\Sigma,T}$ is locally δ-noetherian and the right adjoint $\underline{A} \dashrightarrow \underline{A}_{\Sigma,T}$ preserves δ-filtered colimits. Moreover an object $X \in \underline{A}_{\Sigma,T}$ is δ-generated in $\underline{A}_{\Sigma,T}$ iff it is in $\underline{A}$, and every morphism $f : U \longrightarrow A$ with $A \in \underline{A}_{\Sigma,T}$ and U δ-generated in $\underline{A}$ factors through a monomorphism $U' \hookrightarrow A$ in $\underline{A}$ such that $U' \in \underline{A}_{\Sigma,T}$ and U' is δ-generated.*

6.13 Remark The existence of a left adjoint $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ in the second case (i.e. $T : \underline{B}^{o} \times \underline{A} \longrightarrow \underline{C}$) can also be obtained from the main result of Freyd-Kelly [10]. One shows that there is a class Ω of morphisms in $\underline{A}$ such that $\underline{A}_{\Sigma,T} = \underline{A}_{\Omega,[-,-]}$ and the codomains of Ω form a set. Also the proof given below can easily be extended to locally bounded categories in the sense of Freyd-Kelly [10]. An example for $\underline{A}$ and Σ such that $\underline{A}_{\Sigma,[-,-]}$ is not locally presentable can be found in [13] 8.15.

Proof of 6.12 We first settle the case $T : \underline{B}^{o} \times \underline{A} \longrightarrow \underline{C}$ which is much simpler because the results of § 5 about purity are not needed. Since $T(d\sigma,-)$ and $T(r\sigma,-)$ are continuous for every $\sigma \in \Sigma$, the category $\underline{A}_{\Sigma,T}$ is complete and the inclusion $\underline{A}_{\Sigma,T} \longrightarrow \underline{A}$ preserves limits. In addition every monomorphism in $\underline{A}$ is trivially T-pure. For the existence of a left adjoint $\underline{A} \longrightarrow \underline{A}_{\Sigma,T}$ it suffices to verify the solution set condition (c.f. Freyd [9]). This means that for every object $X \in \underline{A}$ there is a small subcategory $\underline{M}_{X}$ of $\underline{A}_{\Sigma,T}$ such that every morphism $f : X \dashrightarrow A$ with $A \in \underline{A}_{\Sigma,T}$ admits a factorization

$X \dashrightarrow X' \longrightarrow A$ with $X' \in \underline{M}_X$. By 2.8 there is a cardinal α_X such that X is α_X-generated. Since $\mathfrak{M}_\Sigma$ is a set, there is a regular cardinal β such that $\pi(\underline{A}) \leq \beta \geq \alpha_X$ and $T(r\sigma,-)$ and $T(\mathrm{im}\ \sigma,-)$ preserve β-filtered colimits for every $\sigma \in \Sigma$. By 5.1 there is a cardinal $\gamma > \beta$ such that $\varepsilon(T(r\sigma,U)) \leq \gamma \geq \varepsilon(T(\mathrm{im}\ \sigma,U))$ for every $\sigma \in \Sigma$ and every γ-generated object $U \in \underline{A}$. Then $\underline{M}_X = \underline{\tilde{A}}(\gamma) \cap \underline{A}_{\Sigma,T}$ is a "solution set", where $\underline{\tilde{A}}(\gamma)$ denotes the full small subcategory of all γ-generated objects in $\underline{A}$ (cf. 2.8). To see that let $f : X \longrightarrow A$ be a morphism with $A \in \underline{A}_{\Sigma,T}$ as above. Then by [13] 6.7 a) the image of f is also γ-generated and by 6.11 A is also in $\underline{A}_{\mathfrak{M}_\Sigma,T}$. So 6.7 a), b), c1) can be applied to $\mathfrak{M}_\Sigma$ and the inclusion $\mathrm{im}\ f \longrightarrow A$. Therefore the latter admits a factorization $\mathrm{im}\ f \xrightarrow{\subseteq} X' \xrightarrow{\subseteq} A$ such that $X' \in \underline{A}_{\mathfrak{M}_\Sigma,T}$ and X' is γ-generated in $\underline{A}$. By 6.11 $X' \in \underline{A}_{\Sigma,T}$ which shows that $\underline{M}_X$ is a "solution set" for X.

As for the first case (i.e. $T : \underline{B} \times \underline{A} \dashrightarrow \underline{C}$) the inclusion $A_{\Sigma,T} \dashrightarrow \underline{A}$ preserves colimits and $\underline{A}_{\Sigma,T}$ is cocomplete. Thus by the special adjoint functor theorem there is a right adjoint $\underline{A} \dashrightarrow \underline{A}_{\Sigma,T}$ provided $\underline{A}_{\Sigma,T}$ has generators. To establish that let δ be any regular cardinal with the properties 1), 2) and 3) stated in 6.12. We show that $\underline{\tilde{A}}(\delta) \cap \underline{A}_{\Sigma,T}$ is a small generating subcategory of $\underline{A}_{\Sigma,T}$. The δ-generated objects in $\underline{A}$ obviously form a small generating subcategory. Therefore it suffices to show that every morphism $f : X \dashrightarrow A$ with $A \in \underline{A}_{\Sigma,T}$ and $\varepsilon(X) \leq \delta$ admits a factorization $X \rightarrow X' \rightarrow A$ such that $X' \in \underline{A}_{\Sigma,T}$ and X' is δ-generated in $\underline{A}$. This is done in the same pattern as above. First by 6.11 $A \in \underline{A}_{\Sigma,T}$ implies $A \in \underline{A}_{\mathfrak{M}_\Sigma,T}$ and by [13] 6.7 d) $\mathrm{im}\ f$ is δ-generated. In order to apply 6.8 a) to the inclusion $\mathrm{im}\ f \xrightarrow{\subseteq} A$ with respect to $\gamma = \delta$ and $\mathfrak{M}_\Sigma$ (not Σ), it suffices to verify 6.7 b) and c3); the other assumptions in 6.8 follow trivially from those in 6.12. As for c3) we use 5.2 and the fact that in $\underline{A}$ every object is the δ-filtered colimit of its δ-generated subobjects. In 5.2 let $\underline{X}_V = \underline{C}$ and $T_V = T(V,-)$ for every $V \in M$.

Then the hypothesis in 5.2 follow trivially from those in 6.12 except for condition 3) in 5.2. The latter and condition 6.7 b) express the following. For every $V \in M_\gamma$, every $\sigma \in \Sigma'$ and every δ-generated object $U \in \underline{U}$ the inequalities

$$\varepsilon(T(V,U)) \leq \delta \quad \text{and} \quad \varepsilon(T(r\sigma,U)) \leq \delta \geq \varepsilon(T(\text{im } \sigma,U)$$

hold. To verify them first recall that in $\underline{A}$ and $\underline{C}$ the notions δ-generated and δ-presentable coincide by assumption, i.e. $\tilde{\underline{A}}(\delta) = \underline{A}(\delta)$ and $\tilde{\underline{C}}(\delta) = \underline{C}(\delta)$, cf. 2.8. By the special adjoint functor theorem the functors $T(V,-)$, $T(r\sigma,-)$ and $T(\text{im } \sigma,-)$ have right adjoints for every $V \in M$ and $\sigma \in \Sigma$ which we denote with $S(V,-)$, $S(r\sigma,-)$ and $S(\text{im } \sigma,-)$ respectively. By 2.9 the latter have rank (2.1) and as mentioned in 6.4 the inequalities $\text{rank}_M(T) \leq \delta$ and $\text{rank}_{\mathfrak{M}_\Sigma}(T) \leq \delta$ imply that the functors $S(V,-)$, $S(r\sigma,-)$ and $S(\text{im } \sigma,-)$ preserve δ-filtered colimits for every $V \in M$ and $\sigma \in \Sigma$. Hence for every δ-generated object $U \in \underline{A}$ the adjunction isomorphisms

$$[T(V,U),-] \cong [U,S(V,-)], \; [T(r\sigma,U),-] \cong [U,S(r\sigma,-)], \; [T(\text{im } \sigma,U),-] \cong [U,S(\text{im } \sigma,-)]$$

yield the desired inequalities $\varepsilon(T(V,U)) \leq \delta$ and $\varepsilon(T(r\sigma,U)) \leq \delta \geq \varepsilon(T(\text{im } \sigma,U))$. With this the assumptions in 6.8 are verified for $\gamma = \delta$ and $\mathfrak{M}_\Sigma$. Thus the inclusion $\text{im } f \longrightarrow A$ admits a factorization $\text{im } f \longrightarrow X' \longrightarrow A$ such that X' is a T-pure subobject of A which belongs to $A_{\mathfrak{M}_\Sigma, T}$ and is δ-generated in $\underline{A}$. Then 6.11 implies $X' \in \underline{A}_{\Sigma,T}$ which shows that $\tilde{\underline{A}}(\delta) \cap \underline{A}_{\Sigma,T}$ is a small generating subcategory of $\underline{A}_{\Sigma,T}$. Since the inclusion $A_{\Sigma,T} \longrightarrow \underline{A}$ preserves colimits and the objects of $\underline{A}(\delta) \cap \underline{A}_{\Sigma,T}$ are δ-presentable in $\underline{A}$, they are a *fortiori* δ-presentable in $\underline{A}_{\Sigma,T}$, whence $\underline{A}_{\Sigma,T}$ is locally δ-presentable. The last assertion in 6.12 is obviously part of the above construction of generators in $\underline{A}_{\Sigma,T}$. Since $\tilde{\underline{A}}(\delta) = \underline{A}(\delta)$ an object $X \in \underline{A}_{\Sigma,T}$ which is δ-generated in $\underline{A}$ is likewise a fortiori δ-generated in $\underline{A}_{\Sigma,T}$. For the converse let $A \in \underline{A}_{\Sigma,T}$ be any object. Then it follows from the above that A is the δ-filtered colimit in $\underline{A}$ of subobjects $X_i \subset A$ which are δ-presentable in $\underline{A}$ and belong to $\underline{A}_{\Sigma,T}$. Thus the X_i's are a fortiori δ-presentable in $\underline{A}_{\Sigma,T}$ and $A = \varinjlim_i X_i$ holds in $\underline{A}_{\Sigma,T}$. If $A \in \underline{A}_{\Sigma,T}$ is δ-generated in $\underline{A}$, then the identity of A admits a factorization $A \to X_i \hookrightarrow A$, whence $X_i \xrightarrow{\cong} A$ for some i. Thus A is δ-presentable in $\underline{A}_{\Sigma,T}$. Summarizing we obtain that an object $A \in \underline{A}_{\Sigma,T}$ is δ-generated in $\underline{A}_{\Sigma,T}$ iff it is in $\underline{A}$ and that $\underline{A}_{\Sigma,T}$ is locally δ-noetherian. With this one can show as in 3.24 a) that the right adjoint $\underline{A} \to \underline{A}_{\Sigma,T}$ preserves δ-filtered colimits which completes the proof of 6.12.

6.14 We now apply 6.2 - 6.12 to the bifunctor $\otimes : [\underline{U}^o, \underline{Sets}] \times [\underline{U}, \underline{X}] \to \underline{X}$ as defined in 2.1o, where $\underline{U}$ is a small category and $\underline{X}$ is cocomplete. We do not apply them to the bifunctor symbolic hom $[-,-]$ (cf. 2.1o) because the resulting statements for Σ-continuous functors are, except for size estimates, contained in [13] § 8. Also it is straight forward to deduce the corresponding size estimates for Σ-continuous functors from 6.2 and 6.7 a), b), c)).

Let Σ be a class of morphisms in $[\underline{U}^o, \underline{Sets}]$. Then by 2.1o a functor $t : \underline{U} \to \underline{X}$ is Σ-cocontinuous iff $\sigma \otimes t$ is an isomorphism for every $\sigma \in \Sigma$, in other words $[\underline{U}, \underline{X}]_{\Sigma,\otimes}$ coincides with the full subcategory $Cc_\Sigma[\underline{U}, \underline{X}]$ of $[\underline{U}, \underline{X}]$ consisting of all Σ-cocontinuous functors $\underline{U} \to \underline{X}$. With card(Ob $\underline{U}$) and card(Mor $\underline{U}$) we denote the cardinality of the set of objects and the set of morphisms of a <u>skeleton</u> of $\underline{U}$ respectively (cf. Schubert [26] p. 17o). Recall that if Σ is a set and $\underline{X}$ locally presentable, then $\mathrm{rank}_\Sigma(\otimes)$ exists and is the least regular cardinal $\delta \geq \pi(\underline{X})$ such that

$\mathrm{card}(d\sigma(U)) < \delta > \mathrm{card}(r\sigma(U))$ for every $U \in \underline{U}$ and every $\sigma \in \Sigma$, cf. 6.4 and 6.4 2). It might be instructive to show directly how this condition on δ implies $\pi(d\sigma \otimes t) \leq \delta \geq \pi(r\sigma \otimes t)$ for every $\sigma \in \Sigma$ and every finitely presentable functor $t \in [\underline{U}, \underline{X}]$. Since $d\sigma \otimes$ and $r\sigma \otimes$ are cocontinuous, it suffices to verify this when t belongs to a set of regular δ-presentable generators. By [13] 7.2 h) the generalized representable functors $X \otimes [U,-] : \underline{U} \longrightarrow \underline{X}$, $U \mapsto \coprod_{[U,U]} X$ form a set of regular (even dense) generators, where U is running through Ob $\underline{U}$ and X through $\mathrm{Ob}(\underline{X}(\delta))$ (note $\delta \geq \pi(\underline{X})$) . Since

$$d\sigma \otimes (X \otimes [U,-]) \cong X \otimes d\sigma(U) = \coprod_{d\sigma(U)} X$$

and likewise $r\sigma \otimes (X \otimes [U,-]) \cong \coprod_{r\sigma(U)} X$, the conditions $\mathrm{card}(d\sigma(U)) < \delta > \mathrm{card}(r\sigma(U))$ obviously imply

$$\pi\big(d\sigma \otimes (X \otimes [U,-])\big) \leq \delta \geq \pi\big(r\sigma \otimes (X \otimes [U,-])\big)$$

for every $\sigma \in \Sigma$.

6.15 Corollary Let $\underline{U}$ be a small category and let Σ be a class of morphisms in $[\underline{U}^o, \underline{Sets}]$ such that the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set. Let $\underline{X}$ be a locally presentable category. Then $Cc_\Sigma[\underline{U},\underline{X}]$ is locally presentable. In more detail, let

$$\delta = \sup^*\{\aleph_1, \pi(\underline{X}), \sup_{\substack{\sigma\in\Sigma \\ U\in\underline{U}}}(\mathrm{card}(d\sigma(U))^+, \mathrm{card}(r\sigma(U))^+), \mathrm{card}(\Sigma)^+\}$$

if Σ is a set, resp. let

$$\delta' = \sup\{\aleph_1, \pi(\underline{X}), \sup^*_{\sigma\in\Sigma} \varepsilon(r\sigma), \mathrm{card}(\mathcal{M}_\Sigma)^+, \mathrm{card}(\mathrm{Mor}\ \underline{U})^+\}$$

if Σ is a class. (In the latter case it is assumed in addition that $\underline{X}$ is locally δ'-noetherian and that there is a cardinal $\alpha < \delta'$ such that in $\underline{X}$ α-filtered colimits of monomorphisms are monomorphic.) Then $Cc_\Sigma[\underline{U},\underline{X}]$ is locally δ-presentable (resp. locally δ'-noetherian). Moreover a Σ-cocontinuous functor $t : \underline{U} \longrightarrow \underline{X}$ is δ-presentable (resp. δ'-generated) in $Cc_\Sigma[\underline{U},\underline{X}]$ iff it is δ-presentable (resp. δ'-generated) in $[\underline{U},\underline{X}]$. In particular every morphism (resp. monomorphism) $t \longrightarrow s$ with $s \in Cc_\Sigma[\underline{U},\underline{X}]$ and t δ-presentable (resp. δ'-generated) in $[\underline{U},\underline{X}]$ factors through a morphism (resp. monomorphism) $t' \longrightarrow s$ such that $t' \in Cc_{\Sigma}[\underline{U},\underline{X}]$ and t' is δ-presentable (resp. δ'-generated).

Proof If Σ is a set the assertion follows from 6.14, 6.5 and 6.4 ($\gamma=\delta$). If Σ is a class we apply 6.11, 6.12 and 6.4 and revert δ' to δ. For this we have to verify the conditions 1)-3) in 6.12. The first two conditions are obvious. As for 3) note that purity with respect to $\otimes$ can be tested in $[\underline{U}^o,\underline{Sets}]$ with finitely presentable functors; hence we choose $M = Ob([\underline{U}^o,\underline{Sets}](\aleph_o))$. By [13] 7.6 a functor $r : \underline{U}^o \longrightarrow \underline{Sets}$ is finitely presentable iff there is a cokernel diagram

$$\coprod_{i=1}^{n} [-,U_i'] \rightrightarrows \coprod_{j=1}^{m} [-,U_j] \longrightarrow r$$

in other words, a finitely presentable functor can be described by a finite set of morphisms in $\underline{U}$. Since the set of finite subsets of $Mor(\underline{U})$ has the same cardinality as $Mor(\underline{U})$, this shows that $card(M) \leq card\, Mor(\underline{U})$; whence $card(M)^+ \leq \delta$. Since $\varepsilon(r\sigma) \leq \delta$ for every $\sigma \in \Sigma$, there is an epimorphism $\coprod_{\iota \in I_\sigma} [-,U_\iota] \longrightarrow r\sigma$ in $[\underline{U}^o,\underline{Sets}]$ such that $card(I_\sigma) < \delta$. From $card(Mor\underline{U}) < \delta$ it therefore follows that

$$\delta > card(\coprod_{i \in I_\sigma} [U,U_i]) \geq card(r\sigma(U)) \geq card(im\ \sigma(U))$$

for every $U \in \underline{U}$ and every $\sigma \in \Sigma$. Hence $rank\, \mathcal{M}_{\Sigma}(\otimes) \leq \delta$ by 6.14 (resp. 6.4). In the same way one shows $rank_M(\otimes) \leq \delta$. With this conditions 1) - 3) in 6.12 are verified which completes the proof of 6.14 when Σ is a class.

6.16 Colimit preserving functors. Let $\underline{U}$ be a small category and let $(U^k = \varinjlim_{\nu_k} U_{\nu_k})_{k \in K}$ be a class of small colimits in $\underline{U}$. Every $k \in K$ gives rise to a canonical morphism $\sigma_k : \varinjlim_{\nu_k} [-,U_{\nu_k}] \longrightarrow [-,U^k]$ in $[\underline{U}^o,\underline{Sets}]$. Let $\Sigma_K = \{\sigma_k | k \in K\}$ and let $\underline{X}$ be a cocomplete category. Then for every functor t and every $k \in K$ there is a canonical morphism $u_k : \varinjlim_{\nu_k} tU_{\nu_k} \longrightarrow tU^k$. By adjointness σ_k and u_k give rise to a commutative diagram

$$\begin{array}{ccc} [[-,U^k]\otimes t,\ X] & \xrightarrow{\ \cong\ } & [tU^k,\ X] \\ \Big\downarrow [\sigma_k \otimes t,\ X] & & \Big\downarrow [u_k, X] \\ [\varinjlim_{\nu_k}[-,U_{\nu_k}]\otimes t,\ X] & \xrightarrow{\ \cong\ } & [\varinjlim_{\nu_k} tU_{\nu_k},\ X] \end{array}$$

for every $X \in \underline{X}$. Thus $\sigma_k \otimes t$ is an isomorphism iff u_k is and the category $Cc_{\Sigma_K}[\underline{U},\underline{X}]$ coincides with the category $Cc_K[\underline{U},\underline{X}]$ of all functors $\underline{U} \longrightarrow \underline{X}$ which preserve the colimits in K . In order to apply 6.15 the codomains of Σ have to form a set. In order to obtain this let $\overset{\circ}{\underline{U}}$ be a skeleton of $\underline{U}$ and $\Phi : \underline{U} \longrightarrow \overset{\circ}{\underline{U}}$ an inverse to the inclusion $I : \overset{\circ}{\underline{U}} \rightarrow \underline{U}$ (cf. Schubert [26] 16. 3.4). The resulting class $\overset{\circ}{K}$ of colimits $(I\circ\Phi)U^k = \varinjlim_{\nu_k}(I\circ\Phi)U_{\nu_k}$ in $\underline{U}$ has the property $Cc_K[\underline{U},\underline{X}] = Cc_{\overset{\circ}{K}}[\underline{U},\underline{X}]$ and the codomains of $\overset{\circ}{\Sigma} = \{\sigma_k \mid k \in \overset{\circ}{K}\}$ form a set (two colimits in K are considered equal if their index categories, their diagrams and their canonical morphisms coincide). Therefore we can assume without loss of generality that $K = \overset{\circ}{K}$.

If K is a set of colimits in $\underline{U}$ and $\underline{X}$ locally presentable, then by 6.15 $Cc_K[\underline{U},\underline{X}]$ is locally δ-presentable for

$$\delta = \sup{}^*\{\aleph_1,\ \pi(\underline{X}),\ \sup_{\substack{k\in K \\ \mathcal{U}\in Ob\,\underline{\mathcal{U}}}}(\text{card}\ \varinjlim_{\nu_k}[U,U_{\nu_k}],\ \text{card}[U,U^k]),\ \text{card}(K)^+\}$$

and a K-cocontinuous functor $t : \underline{U} \longrightarrow \underline{X}$ is δ-presentable in $Cc_K[\underline{U},\underline{X}]$ iff it is δ-presentable in $[\underline{U},\underline{X}]$, etc. (see 6.15). Likewise if K is a class of colimits in $\underline{U}$ and $\underline{X}$ is a locally δ-noetherian category for some regular cardinal

$$\delta \geq \sup{}^*\{\aleph_1,\ \pi(\underline{X}),\ \text{card}(\mathfrak{M}_{\Sigma_K})^+,\ \text{card}(\text{Mor}\ \underline{U})^+\}$$

and if in addition α-filtered colimits of monomorphisms are monomorphic in $\underline{X}$ for some $\alpha < \delta$, then $Cc_K[\underline{U},\underline{X}]$ is locally δ-noetherian etc. (see 6.15). <u>In particular</u> $\pi(Cc_K[\underline{U},\underline{X}])$ <u>is bounded by</u> $\sup\{\aleph_1, \pi(\underline{X}), \mathrm{card}(2^{\mathrm{Mor}\,\underline{U}})^+\}$.

The passage from K to $\mathring{K}$ is essential for the above size estimates of $\pi(Cc_K[\underline{U},\underline{X}])$. Also given $\underline{U}$ and K one may find $\underline{U}'$ and K' such that $Cc_K[\underline{U},\underline{X}] \cong Cc_{K'}[\underline{U}',\underline{X}]$ and the latter gives a better size estimate for δ . For instance, let $\underline{U}$ = p-<u>Ab</u>.<u>Gr</u>. and $\underline{X}$ = <u>Ab</u>.<u>Gr</u>. be the category of abelian p-groups and abelian groups respectively and let K be the class of all colimits in $\underline{U}$. Let $\underline{U}' \subset \underline{U}$ be the full subcategory of all finite p-groups and let K' be the class of finite colimits in $\underline{U}'$. Then $Cc_K[\underline{U},\underline{X}] \xrightarrow{\cong} Cc_{K'}[\underline{U}',\underline{X}]$, $t \rightsquigarrow t|_{\underline{U}'}$ is an equivalence and $\mathrm{card}(\mathring{K}') = \mathrm{card}(\mathfrak{M}_{\Sigma_{\mathring{K}'}}) = \aleph_0 = \mathrm{card}(\mathrm{Mor}\,\underline{U}')$ holds. Thus by the above the category of cocontinuous functors p-<u>Ab</u>.<u>Gr</u>. $\to$ <u>Ab</u>.<u>Gr</u>. is locally $\aleph_1$-noetherian. This cannot be improved. If this category were locally finitely generated, then by [13] 7.12 a countable colimit of monomorphisms would be again a monomorphism. But this need be so. To show that we use the equivalence p-$\widehat{\text{Ab.Gr}}$. $\to Cc_K$[p-<u>Ab</u>.<u>Gr</u>. <u>Ab</u>.<u>Gr</u>.] , $X \rightsquigarrow \otimes_{\mathbb{Z}} X$ of 6.25 c) below, where p-$\widehat{\text{Ab.Gr}}$. denotes the category of p-adic complete abelian groups. Then the colimit of $\mathbb{Z}/p\mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z}/p^3\mathbb{Z} \to \ldots$ in <u>Ab</u>.<u>Gr</u>. is the Prüfer group $\mathbb{Z}(p^\infty)$ whose completion is zero, whence the colimit in p-$\widehat{\text{Ab.Gr}}$. is zero. This shows in particular that the colimit of the vertical non-zero monomorphisms

$$\begin{array}{ccccccc}
\otimes(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{id} & \otimes(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{id} & \otimes(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{id} & \ldots \\
\downarrow id & & \downarrow \otimes p\cdot & & \downarrow \otimes p^2\cdot & & \\
\otimes(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\otimes p\cdot} & \otimes(\mathbb{Z}/p^2\mathbb{Z}) & \xrightarrow{\otimes p\cdot} & \otimes(\mathbb{Z}/p^3\mathbb{Z}) & \xrightarrow{\otimes p\cdot} & \ldots
\end{array}$$

in Cc_K[p-<u>Ab</u>.<u>Gr</u>., <u>Ab</u>.<u>Gr</u>.] is zero. (Note that 6.25 c) was used to show that $\otimes p\cdot$ is a monomorphism in Cc_K[p-<u>Ab</u>.<u>Gr</u>., <u>Ab</u>.<u>Gr</u>.] although $\otimes p\cdot^n$ is obviously not pointwise a monomorphism, eg. $(\mathbb{Z}/p^n\mathbb{Z})\otimes p\cdot^n = 0$.)

<u>Remarks</u> a) The problem of whether the inclusion $Cc_K[\underline{U},\underline{X}] \hookrightarrow [\underline{U},\underline{X}]$ has

a right adjoint has been around for quite a while. Partial results were stated without proof in Freyd [9] p. 118/119 and Freyd-Kelly [10] p. 170. Recently G.M. Kelly has communicated to me a simple proof for $\underline{X} = \underline{Sets}$ which makes use of the explicit description of colimits in $\underline{Sets}$ in an elegant way.

b) By [13] 7.9 a category $\underline{A}$ is locally α-presentable iff it is equivalent with the category of α-continuous functors $\underline{A}(\alpha)^{o} \longrightarrow \underline{Sets}$. The question may arise whether locally presentable categories can also be characterized as categories of set-valued K-cocontinuous functors (or more generally as Σ-cocontinuous functors for some class Σ as in 6.15). The answer is negative. Any category of the form $Cc_{\Sigma}[\underline{U},\underline{Sets}]$ has a small cogenerating subcategory (even codense [ ] 3.1) because the category $\underline{Sets}$ has one ([13] 4.15) and the inclusion $Cc_{\Sigma}[\underline{U},\underline{Sets}] \xrightarrow{\subset} [\underline{U},\underline{Sets}]$ has a right adjoint. This shows that categories of the form $Cc_{\Sigma}[\underline{U},\underline{Sets}]$ constitute only a very small subclass of the class of locally presentable categories.

6.17 Cosheaves. Let $\underline{U}$ be a small category with a pretopology τ, i.e. with each $U \in \underline{U}$ there is associated a set $J(U)$ of subfunctors of $[-,U] : \underline{U}^{o} \longrightarrow \underline{Sets}$ – called covering cribles – such that $id_{[-,U]} \in J(U)$ and for every natural transformation $\varphi : [-,U'] \longrightarrow [-,U]$ and every $R \in J(U)$ the inverse image $\varphi^{-1}(R)$ belongs to $J(U')$. Recall that a functor $t : \underline{U} \longrightarrow \underline{X}$ is called a τ-cosheaf on $\underline{U}$ with values in $\underline{X}$ if for every triple $U \in \underline{U}$, $R \in J(U)$ and $X \in \underline{X}$ the inclusion $\sigma : R \longrightarrow [-,U]$ induces a bijection

$$[\sigma,[t-,X]] : [[-,U],[t-,X]] \longrightarrow [R,[t-,X]], \psi \rightsquigarrow \psi\sigma$$

or, what is equivalent by 2.10 – assuming $\underline{X}$ has colimits – the morphism $\sigma \otimes t : R \otimes t \longrightarrow [-,U] \otimes t$ is an isomorphism for every σ, cf. Borel-Moore [3], Gray [15], Kultze [20]. The full subcategory of $[\underline{U},\underline{X}]$ consisting of all τ-cosheaves is denoted with $Csh_{\tau}[\underline{U},\underline{X}]$. Let

Σ_τ be the set of all inclusions $R \hookrightarrow [-,U]$, where $R \in J(U)$ and U runs through a skeleton of $\underline{U}$. Then $Cc_{\Sigma_\tau}[\underline{U},\underline{X}] = Csh_\tau[\underline{U},\underline{X}]$. <u>If</u> $\underline{X}$ <u>is locally presentable, then</u> $Csh_\tau[\underline{U},\underline{X}]$ <u>is locally</u> δ<u>-presentable for</u> $\delta = \sup^*\{\aleph_1, \pi(\underline{X}), \mathrm{card}(\Sigma_\tau)^+, \sup_{U,U'\in\underline{U}}(\mathrm{card}[U,U'])\}$, etc. (see 6.15). <u>Likewise if</u> $\underline{X}$ <u>is locally</u> δ<u>-noetherian and</u> $\delta \geq \sup\{\aleph_1, \pi(\underline{X}), \mathrm{card}(\Sigma_\tau)^+, \mathrm{card}(\mathrm{Mor}\,\underline{U})^+\}$ <u>and if in addition</u> α<u>-filtered colimits of monomorphisms are monomorphic in</u> $\underline{X}$ <u>for some</u> $\alpha < \delta$, <u>then</u> $Csh_\tau[\underline{U},\underline{X}]$ <u>is locally</u> δ<u>-noetherian</u>, etc. (see 6.15).

<u>Remark</u> Let τ be a Grothendieck topology on $\underline{U}$. Let τ_0 be a pre-subtopology of τ – i.e. $J_0(U) \subset J(U)$ for $U \in \underline{U}$ – which generates τ (cf. [??] 2o. 1.6). Then one can show that $Csh_\tau[\underline{U},\underline{X}] = Csh_{\tau_0}[\underline{U},\underline{X}]$ and thus in the above estimate for $\pi(Csh_\tau[\underline{U},\underline{X}])$ one can therefore replace $\mathrm{card}(\Sigma_\tau)$ by $\mathrm{card}(\Sigma_{\tau_0})$ which can be much smaller. To see that the cosheaves on $\underline{U}$ with respect to τ_0 and τ coincide first note that every τ-cosheaf is a τ_0-cosheaf. For the converse let $\bar{\Sigma}_{\tau_0}$ be the closure of Σ_{τ_0} (cf. 2.1o). Then by 2.1o every τ_0-cosheaf is $\bar{\Sigma}_{\tau_0}$-cocontinuous. Moreover by [13] 12.5 Σ_τ is contained in $\bar{\Sigma}_{\tau_0}$. Hence every $\bar{\Sigma}_{\tau_0}$-cocontinuous functor $\underline{U} \to \underline{X}$ is a τ-cosheaf.

6.18 <u>Adjoint functors.</u> Let $\underline{A}$ and $\underline{B}$ be categories and let $Adj(\underline{A},\underline{B})$ be the full subcategory of $[\underline{A},\underline{B}]$ consisting of all functors $\underline{A} \to \underline{B}$ admitting a right adjoint. Then $Adj(\underline{A},\underline{B})$ is equivalent with the category whose objects are pairs $\underline{A} \xrightarrow{T} \underline{B} \xrightarrow{S} \underline{A}$ of adjoint functors (T = left adjoint) and whose morphisms are pairs $(\varphi: T \to T', \psi: S' \to S)$ of natural transformations subject to the usual compatibility condition. The equivalence is given by the forgetful functor $(T,S) \rightsquigarrow T$. Our aim is to show that $Adj(\underline{A},\underline{B})$ is locally presentable if $\underline{A}$ and $\underline{B}$ are, and to give an estimate of $\pi(Adj(\underline{A},\underline{B}))$ in terms of $\underline{A}$ and $\underline{B}$. This is done by identifying $Adj(\underline{A},\underline{B})$ with a category of Σ-cocontinuous functors $\underline{U} \to \underline{B}$, where

$\underline{U}$ is a small generating subcategory of $\underline{A}$.

We start out somewhat more generally. *Let* $\underline{U}$ *be a small category and let* Σ *be a class of morphisms in* $[\underline{U}^o, \underline{Sets}]$ *such that the codomains* $\{r\sigma \mid \sigma \in \Sigma\}$ *form a set.* Recall that $C_\Sigma[\underline{U}^o, \underline{Sets}]$ denotes the category of all Σ-continuous functors $\underline{U}^o \longrightarrow \underline{Sets}$ and that $C_\Sigma[\underline{U}^o, \underline{Sets}]$ is locally presentable if Σ is a set (cf. 2.1o, 2.11). In either case the inclusion $I : C_\Sigma[\underline{U}^o, \underline{Sets}] \longrightarrow [\underline{U}^o, \underline{Sets}]$ has a left adjoint $L : [\underline{U}^o, \underline{Sets}] \longrightarrow C_\Sigma[\underline{U}^o, \underline{Sets}]$ by 2.1o. Let $\underline{A} = C_\Sigma[\underline{U}^o, \underline{Sets}]$ and let $\underline{X}$ be a category with colimits. Then by 2.1o every functor $t : \underline{U} \longrightarrow \underline{X}$ gives rise to an adjoint pair

$$\otimes\, t : [\underline{U}^o, \underline{Sets}] \longrightarrow \underline{X} \quad \text{and} \quad \underline{X} \longrightarrow [\underline{U}^o, \underline{Sets}],\ X \rightsquigarrow [t-, X] \ .$$

Clearly t is Σ-cocontinuous iff $[t-, X] \in C_\Sigma[\underline{U}^o, \text{Sets}]$ for every $X \in \underline{X}$. Hence $t \rightsquigarrow (\otimes\, t) \cdot I$ induces a functor $\Phi : Cc_\Sigma[\underline{U}, \underline{X}] \longrightarrow Adj(\underline{A}, \underline{X})$. On the other hand the Yoneda embedding $Y : \underline{U} \longrightarrow [\underline{U}^o, \underline{Sets}]$ and the left adjoint $L : [\underline{U}^o, \underline{Sets}] \longrightarrow C_\Sigma[\underline{U}^o, \underline{Sets}]$ give rise to a functor $\Psi : Adj(\underline{A}, \underline{X}) \longrightarrow [\underline{U}, \underline{X}]$, $T \rightsquigarrow TLY$.

6.19 *Lemma* *The functor*

$$\Phi : Cc_\Sigma[\underline{U}, \underline{X}] \longrightarrow Adj(\underline{A}, \underline{X}),\ t \rightsquigarrow (\otimes\, t) \cdot I$$

is an equivalence and its inverse is Ψ. *If the representable functors* $\underline{U}^o \longrightarrow \underline{Sets}$ *are* Σ-*continuous, then* $\Psi(T) = TLY$ *is equivalent with the composite* $\underline{U} \xrightarrow{Y} C_\Sigma[\underline{U}^o, \underline{Sets}] \xrightarrow{T} \underline{X}$ *for every* $T \in Adj(\underline{A}, \underline{X})$.

6.2o *Corollary* *Let* $\underline{A} = C_\Sigma[\underline{U}^o, \underline{Sets}]$ *with* $\underline{U}$ *and* Σ *as above and let* $\underline{X}$ *be a locally presentable category. Then* $Adj(\underline{A}, \underline{X})$ *is locally presentable. In particular the category of adjoint pairs between two locally presentable categories is itself locally presentable.*
In more detail, if Σ *is a set, then* $Adj(\underline{A}, \underline{X})$ *is locally* δ-*presentable for* $\delta = \sup^*_{\sigma \in \Sigma}\{\aleph_1, \pi(\underline{X}), \sup(\text{card } d\sigma(U)^+, \text{card } r\sigma(U)^+), \text{card}(\Sigma)^+\}$, *and a left adjoint* $T : \underline{A} \longrightarrow \underline{X}$ *is* δ-*presentable in* $Adj(\underline{A}, \underline{X})$ *iff* $TLY : \underline{U} \longrightarrow \underline{X}$ *is* δ-*presentable in* $[\underline{U}, \underline{X}]$. *In addition every natural transformation* $H \longrightarrow T$ *with* $T \in Adj(\underline{A}, \underline{B})$ *and* HLY δ-*presentable in*

$[\underline{U},\underline{X}]$ admits a factorization $H \longrightarrow H' \longrightarrow T$ such that H' has a right adjoint and $H'LY$ is δ-presentable in $[\underline{U},\underline{X}]$, cf. 6.15.
Likewise if Σ is a class and $\underline{X}$ is locally δ-noetherian for $\delta > \sup\{\aleph_1, \pi(\underline{X}), \operatorname{card}(\mathcal{M}_\Sigma)^+, \operatorname{card}(\operatorname{Mor} \underline{U})^+\}$ and if in addition α-filtered colimits of monomorphisms are again monomorphic in $\underline{X}$ for some $\alpha < \delta$, then $\operatorname{Adj}(\underline{A},\underline{X})$ is locally δ-noetherian, etc. (see 6.15).

Remark If $\underline{A}$ is a locally presentable category, then the above estimate for δ resp. $\pi(\operatorname{Adj}(\underline{A},\underline{X}))$ depends on the presentation $\underline{A} \simeq C_\Sigma[\underline{U}^o,\underline{Sets}]$, cf. 2.11. The point is of course to choose a presentation in such a way that $\sup\limits_{U\in\underline{U}}(\operatorname{card}(\Sigma)^+, \sup\limits_{\sigma\in\Sigma}\{\operatorname{card} d\sigma(U)^+, \operatorname{card} r\sigma(U)^+)\}$ is as small as possible. If the representable functors $\underline{U}^o \longrightarrow \underline{Sets}$ are Σ-continuous – which is often the case – then a left adjoint $T : \underline{A} \longrightarrow \underline{X}$ is δ-presentable in $\operatorname{Adj}(\underline{A},\underline{X})$ iff its "restriction" on $\underline{U}$ is δ-presentable in $[\underline{U},\underline{X}]$.
Proof of the lemma If the representable functors $[-,U]$, $U \in \underline{U}$, are Σ-continuous, then the assertion follows from 6.18 and the well known fact (due to Kan[19]) that the Kan extension $[\underline{U},\underline{X}] \longrightarrow \operatorname{Adj}([\underline{U}^o,\underline{Sets}], \underline{X})$, $t \leadsto t \otimes$ is an equivalence (cf. [13] § 2).
So we basically have to deal with the (technical) complication that the representable functors need not be Σ-continuous. Let $T : \underline{A} \longrightarrow \underline{X}$ be a functor with a right adjoint. The Σ-cocontinuity of the functor $t = TLY$ results from the diagram

$$\begin{array}{ccccccc}
[r\sigma,[t-,X]] & = & [r\sigma,[TLY-,X]] & \simeq & [r\sigma,[Y-,ISX]] & \simeq & [r\sigma,ISX] \\
\Big\downarrow{\scriptstyle [\sigma,[t-,X]]} & & & & & & \Big\downarrow{\scriptstyle [\sigma,ISX]} \\
[d\sigma,[t-,X]] & = & [d\sigma,[TLY-,X]] & \simeq & [d\sigma,[Y-,ISX]] & \simeq & [d\sigma,ISX]
\end{array}$$

where $\sigma \in \Sigma$, $X \in \underline{X}$, in which $[\sigma,ISX]$ is a bijection because SX is Σ-continuous. We show that there are natural isomorphisms $(\Psi\Phi)(t) \simeq t$ and $(\Phi\Psi)(T) \simeq T$. Recall that the closure $\bar{\Sigma}$ of Σ consists of all morphisms $\bar{\sigma}$ in $[\underline{U}^o,\underline{Sets}]$ such that $[\bar{\sigma},F]$ is a bijection for every $F \in \underline{C}_\Sigma[\underline{U}^o, \underline{Sets}]$ and that a functor $t : \underline{U} \to \underline{X}$ is Σ-cocontinuous iff it is $\bar{\Sigma}$-cocontinuous (cf. 2.1o). Then for every $U \in \underline{U}$ the canonical morphism $\tau_U : [-,U] \to IL[-,U]$ belongs to $\bar{\Sigma}$ because for every $F \in C_\Sigma[\underline{U}^o, \underline{Sets}]$ the map

$$[\tau_U, F] : [[-,U], F] \longrightarrow [IL[-,U], F]$$

is bijective. (Note $[IL[-,U], F] = [L[-,U], F] \cong [[-,U], F]$.) Hence for every $U \in \underline{U}$ and every Σ-cocontinuous functor $t : \underline{U} \to \underline{X}$ the morphism $\tau_U \otimes t : [-,U] \otimes t \to IL[-,U] \otimes t$ is an isomorphism. Since $IL[-,U] \otimes t = ((\otimes t) \cdot I \cdot L \cdot Y)(U)$ and the composite

$$tU \xrightarrow{\cong} [-,U] \otimes t \xrightarrow[\cong]{\tau_U \otimes t} IL[-,U] \otimes t$$

is natural in U, we obtain $t \cong (\otimes t) \cdot I \cdot L \cdot Y = (\Psi \Phi)(t)$. Second if $T : \underline{A} \to \underline{X}$ has a right adjoint S, then so does $TL : [\underline{U}^o, \underline{Sets}] \to \underline{X}$, namely IS. If $t = TLY$, then by 2.1o the right adjoint of $\otimes t$ is the functor $X \to [\underline{U}^o, \underline{Sets}], X \rightsquigarrow [TLY-, X]$. By adjointness the latter is isomorphic with IS. Hence $TL \cong \otimes t$ which implies $(\Phi \cdot \Psi)(T) = \Phi(TLY) = \Phi(t) = (\otimes t) \cdot I \cong TLI \cong T$.

6.21 Generalizations to topological and additive categories. In view of the work of Wischnewsky [35], Ertel-Schubert [6], Wyler [37] and others, the assertions in 6.15, 6.16, 6.17 and 6.2o can be generalized to the situation, where $\underline{X}$ is replaced by a topological category over a locally presentable category. Note that in 6.14, 6.18 and 6.19 it was only assumed that $\underline{X}$ has colimits. In more detail let $\underline{U}$ be a small category and Σ a class of morphisms in $[\underline{U}^o, \underline{Sets}]$ such that the codomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set. Moreover let $F : \tilde{\underline{X}} \to \underline{X}$ be an initial structure functor, where $\underline{X}$ is locally presentable, cf. Hoffmann [13], Wyler [38], Wischnewsky [35]. Then by Wischnewsky [35] 2.13, 2.22, 2.23

$$Cc_\Sigma[\underline{U}, \tilde{\underline{X}}] \longrightarrow Cc_\Sigma[\underline{U}, \underline{X}], \quad t \rightsquigarrow F \cdot t$$

is again an initial structure functor and by 6.15 $Cc_{\Sigma}[\underline{U},\underline{X}]$ is locally presentable. Hence all of Wischnewsky's assertions in [36] 2.13-2.24 and elsewhere apply, in particular $Cc_{\Sigma}[\underline{U},\tilde{\underline{X}}]$ has limits, dense generators and the inclusion $Cc_{\Sigma}[\underline{U},\tilde{\underline{X}}] \xrightarrow{c} [\underline{U},\tilde{\underline{X}}]$ has a right adjoint, etc. In particular the functor $\mathrm{Adj}(C_{\Sigma}[\underline{U}^{o},\underline{Sets}],\tilde{\underline{X}}) \longrightarrow [\underline{U},\tilde{\underline{X}}]$, $T \rightsquigarrow TLY$, is full and faithful and has a right adjoint.

The assertions in 2.1o, 2.11, 6.14 - 6.2o can also be formulated in the additive case. For this assume that the categories $\underline{A},\underline{B}\ldots$, $\underline{U},\underline{X},\ldots$ are additive (or preadditive) and that all functors are additive. If the category $\underline{Sets}$ of sets is replaced by the category $\underline{Ab}.\underline{Gr}.$ of abelian groups and if $[\underline{U},\underline{X}]$, $[\underline{U},\underline{Ab}.\underline{Gr}.]$ etc. denote the categories of additive functors, then there is an additive bifunctor $\otimes : [\underline{U}^{o},\underline{Ab}.\underline{Gr}.] \times [\underline{U},\underline{X}] \longrightarrow \underline{X}$ with the same properties as in 2.1o, 2.11 and 6.14. (Note that in 6.14 the additive generalized representable functors are composites of the form $X \otimes [U,-] : \underline{U} \longrightarrow \underline{Ab}.\underline{Gr}. \dashrightarrow \underline{X}$, where $X\otimes$ is the left adjoint of $[X,-] : \underline{X} \dashrightarrow \underline{Ab}.\underline{Gr}.$) . With these modifications all assertions in 6.14 - 6.2o hold also in the additive case. If there is danger of confusion we denote the category of additive functors $\underline{U} \dashrightarrow \underline{X}$ with $[\underline{U},\underline{X}]_{+}$ in order to distinguish it from the category $[\underline{U},\underline{X}]$ of all functors $\underline{U} \longrightarrow \underline{X}$.

6.22 Closure properties of $\mathrm{Adj}(\underline{A},\underline{B})$. Whereas $\mathrm{Adj}(\underline{A},\underline{B})$ is locally presentable provided $\underline{A}$ and $\underline{B}$ are (6.2o), there is no corresponding assertion for topoi or Grothendieck categories. Likewise if $\underline{X}$ is a topos or a Grothendieck category, then $Cc_{\Sigma}[\underline{U},\underline{X}]$ need not be so, not even when Σ is given by a Grothendieck topology on $\underline{U}$ (cf 6.25 c)). The following definition is "designed to rectify" this, at least in the additive case. It is motivated by Lazard's [22] characterization of flat modules as filtered colimits of finitely generated free modules.

6.23 Definition A class Σ of morphisms in $[\underline{U}^{o},\underline{Sets}]$, $\underline{U}$ small, is called flat if the dodomains $\{r\sigma \mid \sigma \in \Sigma\}$ form a set and $d\sigma$ and

$r\sigma$ are filtered colimits of representable functors for every $\sigma \in \Sigma$. A category $\underline{A}$ is called flat if there is a small category $\underline{U}$ and a flat class Σ of morphisms in $[\underline{U}^{o}, \underline{Sets}]$ such that $\underline{A} \cong C_{\Sigma}[\underline{U}^{o}, \underline{Sets}]$. In the additive case (6.21) flat classes and flat additive categories are defined likewise.

This is obviously somewhat an ad hoc definition and it raises many questions. We limit ourselves to the following.

6.24 Corollary Let $\underline{A}$ be a flat category and let Σ be a flat class of morphisms in $[\underline{U}^{o}, \underline{Sets}]$, where $\underline{U}$ is a small category. If $\underline{X}$ is a topos (resp. a Grothendieck category), then so are $Cc_{\Sigma}[\underline{U}, \underline{X}]$ and $Adj(\underline{A}, \underline{X})$.

Likewise, if $\underline{A}$ and Σ are flat additive, then $Adj(\underline{A}, \underline{X})$ and $Cc_{\Sigma}[\underline{U}, \underline{X}]_{+}$ are Grothendieck categories, provided $\underline{X}$ is.

Proof We limit ourselves to the non-additive case, the proof for the additive case is similar. Let $\underline{X}$ be a topos (resp. a Grothendieck category) and let Σ be a flat class in $[\underline{U}^{o}, \underline{Sets}]$. Clearly $Cc_{\Sigma}[\underline{U}, \underline{X}]$ is closed in $[\underline{U}, \underline{X}]$ under colimits. Since $d\sigma$ and $r\sigma$ are filtered colimits of representable functors for every $\sigma \in \Sigma$ and $\underline{X}$ is a topos (resp. a Grothendieck category), one readily sees that the functors $d\sigma \otimes : [\underline{U}, \underline{X}] \longrightarrow \underline{X}$ and $r\sigma \otimes : [\underline{U}, \underline{X}] \longrightarrow \underline{X}$ preserve finite limits. Hence $Cc_{\Sigma}[\underline{U}, \underline{X}]$ is closed in $[\underline{U}, \underline{X}]$ under finite limits and by 6.15 it is locally presentable. If $\underline{X}$ is a Grothendieck category, then so is $[\underline{U}, \underline{X}]$ and therefore, by the above, the same holds for $Cc_{\Sigma}[\underline{U}, \underline{X}]$. On the other hand, if $\underline{X}$ is a topos, then so is $[\underline{U}, \underline{X}]$, and it follows from the above and Giraud's characterization of topoi (cf [13] 12.13 a) - d)) that $Cc_{\Sigma}[\underline{U}, \underline{X}]$ is again a topos.

6.25 Examples of categories $Adj(\underline{A}, \underline{X})$.

6.25 a) Let $\underline{U}$ be a small category with a pretopology τ (resp. with a class K of colimits). Then by 6.19 the category $Adj(Sh_{\tau}[\underline{U}^{o}, \underline{Sets}], \underline{X})$

of adjoint functors between the category of set valued sheaves on the site (U,τ) and a cocomplete category $\underline{X}$ is equivalent with the category $Csh_\tau[\underline{U},\underline{X}]$ of τ-cosheaves on $\underline{U}$ with values in $\underline{X}$. Likewise the category $Adj(C_K[\underline{U}^o,\underline{Sets}],\underline{X})$ of adjoint functors between the category of K-limits preserving functors $\underline{U}^o \longrightarrow \underline{Sets}$ and a cocomplete category $\underline{X}$ is equivalent with the category of K-cocontinuous functors $\underline{U} \longrightarrow \underline{X}$.

6.25 b) Grothendieck categories. We give a description of $Adj(\underline{A},\underline{X})$ for Grothendieck categories $\underline{A}$ and $\underline{X}$ in terms of those objects in $\underline{X}$ which are uniquely divisible by all covering right ideals of the endomorphism ring of a generator in $\underline{A}$. We start out somewhat more generally.

Let Λ be a ring and $\mathcal{F}$ a set of right ideals in Λ. Let $\underline{X}$ be a Grothendieck category and ${}_\Lambda\underline{X}$ the category of left Λ-objects in $\underline{X}$. An object $X \in {}_\Lambda X$ is called uniquely divisible by $\mathcal{F}$ if for every $I \in \mathcal{F}$ the evaluation $I \otimes_\Lambda X \longrightarrow X$ is an isomorphism. Let $\mathcal{F}_\Lambda\underline{X}$ denote the full subcategory of ${}_\Lambda\underline{X}$ consisting of all uniquely $\mathcal{F}$-divisible objects. Dually a module $Y \in \underline{Mod}_\Lambda$ is called $\mathcal{F}$-closed (cf. Gabriel [11], Stenström [37] p. 37) if for every $I \in \mathcal{F}$ the restriction $[\Lambda,Y] \longrightarrow [I,Y]$ is an isomorphism. Let $(\underline{Mod}_\Lambda)_{\mathcal{F}}$ denote the full subcategory of $\underline{Mod}_\Lambda$ consisting of all $\mathcal{F}$-closed modules. By 6.2 the inclusion $(\underline{Mod}_\Lambda)_{\mathcal{F}} \xrightarrow{\subset} \underline{Mod}_\Lambda$ has a left adjoint $\mathcal{F}$-loc : $\underline{Mod}_\Lambda \longrightarrow (\underline{Mod}_\Lambda)_{\mathcal{F}}$ called localization at $\mathcal{F}$. In particular $(\underline{Mod}_\Lambda)_{\mathcal{F}}$ is locally δ-presentable for $\delta = \sup^*_{I \in \mathcal{F}} \pi(I)$. In general $\mathcal{F}$-loc is not exact unless $\mathcal{F}$ is a pretopology, cf. 6.17 and [30] 22. Let Σ be the set of all inclusions $I \subset \Lambda$ for $I \in \mathcal{F}$. If $\{\Lambda\}$ denotes the full subcategory of $\underline{Mod}_\Lambda$ whose only object is Λ, then there are canonical isomorphisms $[\{\Lambda\}^{opp},\underline{Ab}.\underline{Gr}.] \cong \underline{Mod}_\Lambda$, $C_\Sigma[\{\Lambda\}^{opp},\underline{Ab}.\underline{Gr}.] \cong (\underline{Mod}_\Lambda)_{\mathcal{F}}$, $[\{\Lambda\},\underline{X}] \cong {}_\Lambda\underline{X}$ and $Cc_\Sigma[\{\Lambda\},\underline{X}] \cong \mathcal{F}_\Lambda\underline{X}$. Together with the functor Ψ from 6.18 they give rise to a diagram

$$\begin{array}{ccc} \mathrm{Adj}\left(C_{\Sigma}[\{\Lambda\}^{opp},\underline{Ab}.\underline{Gr}.],\underline{X}\right) & \xrightarrow[\cong]{\Psi} & Cc_{\Sigma}[\{\Lambda\},\underline{X}] \\ \uparrow \cong & & \downarrow \cong \\ \mathrm{Adj}\left((\underline{Mod}_{\Lambda})_{\mathcal{F}},\underline{X}\right) & \overset{\Psi'}{\underset{\cong}{- - - \to}} & \mathcal{F}_{\Lambda}\underline{X} \end{array}$$

and one readily checks by means of 6.19 that the composite Ψ' is the functor $T \rightsquigarrow (T\cdot\mathcal{F}\text{-loc})(\Lambda)$ and that its inverse assigns to an object $X \in {}_{\Lambda}\underline{X}$ the restriction of $\otimes_{\Lambda}X$ onto $(\underline{Mod}_{\Lambda})_{\mathcal{F}}$. From 6.2o it follows that $\mathrm{Adj}\left((\underline{Mod}_{\Lambda})_{\mathcal{F}},\underline{X}\right)$ <u>is locally δ-presentable for</u> $\delta = \sup\{\aleph_1, \pi(\underline{X}), \mathrm{card}(\Lambda)^+, \mathrm{card}(\mathcal{F})^+\}$, <u>and a functor</u> $T : (\underline{Mod}_{\Lambda})_{\mathcal{F}} \longrightarrow \underline{X}$ <u>admitting a right adjoint is δ-presentable in</u> $\mathrm{Adj}\left((\underline{Mod}_{\Lambda})_{\mathcal{F}},\underline{X}\right)$ <u>iff</u> $(T\cdot\mathcal{F}\text{-loc})(\Lambda)$ <u>is δ-presentable in</u> ${}_{\Lambda}\underline{X}$, etc. see 6.15.

Now let $\underline{A}$ be a Grothendieck category. Let $U \in \underline{A}$ be a generator and $\Lambda = [U,U]$ its endomorphism ring. Let $\mathcal{F}$ be the filter of all right ideals $I \subset \Lambda$ which cover U in the sense that $U = \bigcup_{\gamma \in I} \mathrm{im}\,\gamma$, where $\mathrm{im}\,\gamma$ denotes the image of $\gamma : U \longrightarrow U$. Then it follows from Gabriel-Popescu [12] (see also [30] (6)) that the functor $\underline{A} \longrightarrow (\underline{Mod}_{\Lambda})_{\mathcal{F}}$, $A \rightsquigarrow [-,A]$ is an equivalence. This together with the above yields that <u>the functor</u>

$$\mathrm{Adj}(\underline{A},\underline{X}) \longrightarrow \mathcal{F}_{\Lambda}\underline{X}, \quad T \rightsquigarrow TU$$

<u>is an equivalence for every Grothendieck category</u> $\underline{X}$. <u>In addition</u> $\mathrm{Adj}(\underline{A},\underline{X})$ <u>is locally δ-presentable for</u> $\delta = \sup\{\aleph_1, \pi(\underline{X}), \mathrm{card}(\mathcal{F})^+, \mathrm{card}(\Lambda)^+\}$, etc (see 6.15).

6.25 c) In the above case $\mathrm{Adj}(\underline{A},\underline{X})$ was described in terms of divisible objects in $\underline{X}$. In the following we give a rather special example of a Grothendieck category $\underline{A}$ such that $\mathrm{Adj}(\underline{A},\underline{X})$ can be described in terms of complete objects in $\underline{X}$. The details are somewhat involved and have nothing to do with what has been done above. Instead they center around the condition of Mittag-Leffler.

Let R be a commutative ring and $\mathfrak{a} \subset R$ an ideal. Let $\mathfrak{a}$-$\underline{\text{Mod}}_R$ be the full subcategory of $\underline{\text{Mod}}_R$ consisting of all modules A such that every cyclic submodule (a) is a quotient of $R/\mathfrak{a}^n$ for some $n \geq 1$ depending on $a \in A$. In analogy to the category of abelian p-groups we call $\mathfrak{a}$-$\underline{\text{Mod}}_R$ the category of $\mathfrak{a}$-modules. Clearly $\mathfrak{a}$-$\underline{\text{Mod}}_R$ is a Grothendieck category with $\{R/\mathfrak{a}, R/\mathfrak{a}^2, \ldots\}$ as a set of generators, and thus by the special adjoint functor theorem every cocontinuous functor $\mathfrak{a}\text{-}\underline{\text{Mod}}_R \longrightarrow \underline{X}$ has a right adjoint. In particular the right adjoint of the inclusion $I : \mathfrak{a}\text{-}\underline{\text{Mod}}_R \longrightarrow \underline{\text{Mod}}_R$ assigns to an R-module the largest $\mathfrak{a}$-submodule. Let $\underline{X}$ be a Grothendieck category and $\underline{X}_R$ the category of R-objects in $\underline{X}$. *An object $X \in \underline{X}_R$ is called $\mathfrak{a}$-adic complete if the canonical morphism* $X \longrightarrow \varprojlim_{\nu} X/\mathfrak{a}^{\nu}X$ *is an isomorphism, where $\mathfrak{a}^{\nu}X$ is the image of the evaluation morphism* $\mathfrak{a}^{\nu} \otimes_R X \longrightarrow X$ *and the transition morphisms* $X/\mathfrak{a}^{\nu+1}X \longrightarrow X/\mathfrak{a}^{\nu}X$ *are given by the inclusions* $\mathfrak{a}^{\nu+1} \subset \mathfrak{a}^{\nu}$. Let $\widehat{\mathfrak{a}\text{-}\underline{X}}_R$ denote the full subcategory of $\underline{X}_R$ of all $\mathfrak{a}$-adic complete objects. Note that even in general the inclusion $\widehat{\mathfrak{a}\text{-}\underline{X}}_R \longrightarrow \underline{X}_R$ need not have a left adjoint.

Then the functors

$$\Omega : \text{Adj}(\mathfrak{a}\text{-}\underline{\text{Mod}}_R, \underline{X}) \longrightarrow \widehat{\mathfrak{a}\text{-}\underline{X}}_R, \quad T \rightsquigarrow \varprojlim_{\nu} T(R/\mathfrak{a}^{\nu})$$

and

$$\Psi : \widehat{\mathfrak{a}\text{-}\underline{X}}_R \longrightarrow \text{Adj}(\mathfrak{a}\text{-}\underline{\text{Mod}}_R, \underline{X}), \quad X \rightsquigarrow (\otimes_R X) \circ I$$

are well defined and inverse equivalences provided either $\mathfrak{a}$ is finitely generated and $R/\mathfrak{a}^{\nu}$ is artinian for $\nu \geq 1$ or $\mathfrak{a}$ is a principal ideal generated by a non zero divisor. Moreover the inclusion $\widehat{\mathfrak{a}\text{-}\underline{X}}_R \longrightarrow \underline{X}_R$ *has a left adjoint, namely* $X \rightsquigarrow \varprojlim_{\nu} X/\mathfrak{a}^{\nu}$, *and* $\text{Adj}(\mathfrak{a}\text{-}\underline{\text{Mod}}_R, \underline{X})$ *is locally* $\sup(\aleph_1, \pi(\underline{X}))$*-presentable (resp. locally* $\sup(\aleph_1, \varepsilon(\underline{X}))$*-generated)*. Note that if R is noetherian, then $R/\mathfrak{a}^{\nu}$ is artinian for $\gamma \geq 1$ iff the associated prime ideals of $\mathfrak{a}$ are maximal.

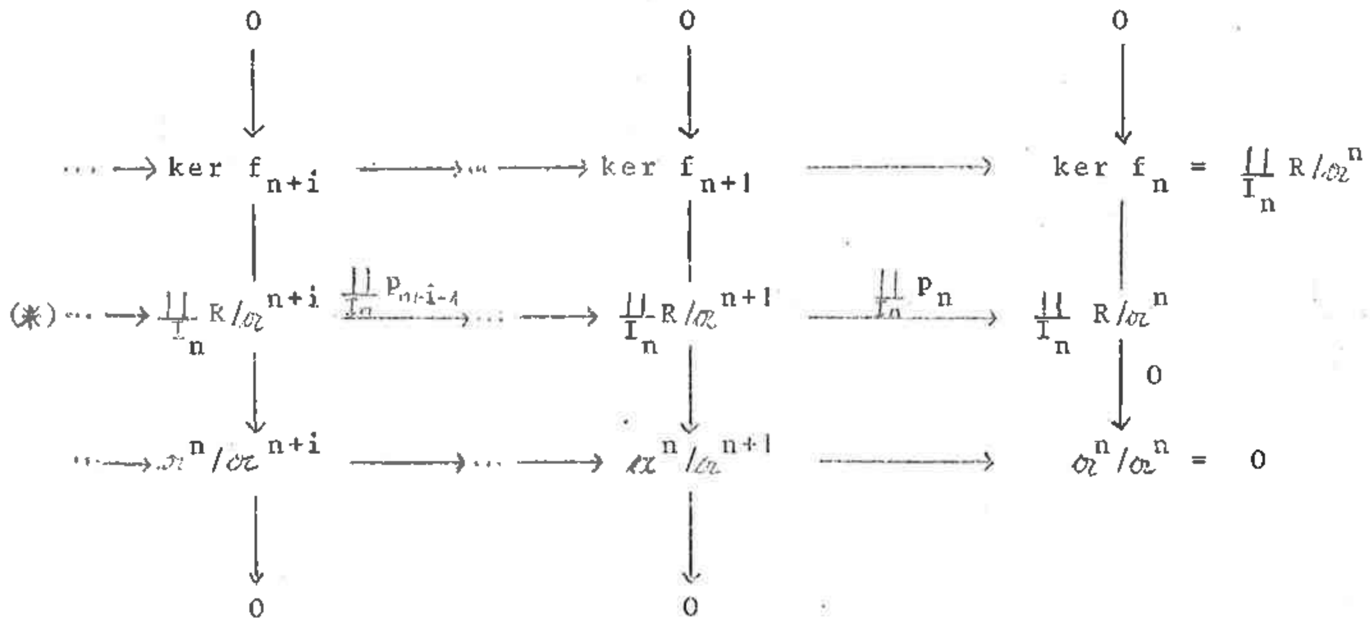
Proof We limit ourselves to the case $\underline{X} = \underline{Ab}.\underline{Gr}.$ and give an outline for the modifications in the general case at the end. Note that $\underline{X}_R = \underline{Mod}_R$ and $\widehat{\mathfrak{a}}-\underline{X}_R = \widehat{\mathfrak{a}}-\underline{Mod}_R$.

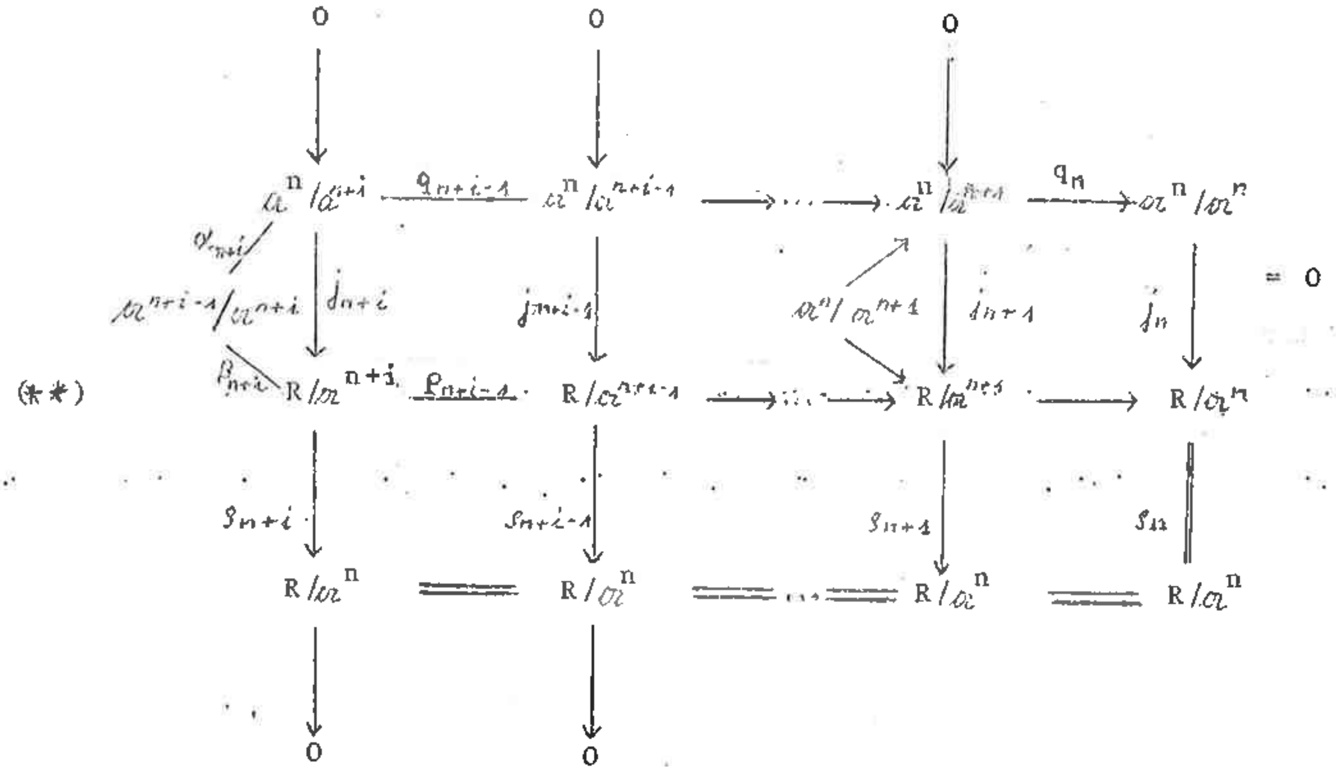
We first show that Ψ and Ω are well defined. For Ψ this is obvious because the inclusion $I : \mathfrak{a}\text{-}\underline{Mod}_R \longrightarrow \underline{Mod}_R$ preserves colimits. As for Ω let $T : \mathfrak{a}\text{-}\underline{Mod}_R \longrightarrow \underline{Ab}.\underline{Gr}.$ be a functor with a right adjoint. Since T is additive, for every $\mathfrak{a}$-module A the map $R \longrightarrow [TA, TA]$, $r \rightsquigarrow Tr$, makes TA into a R-module. This gives rise to a factorization of T through the forgetful functor $V : \underline{Mod}_R \longrightarrow \underline{Ab}.\underline{Gr}.$, and thus $\Omega(T) = \varprojlim_\nu T(R/\mathfrak{a}^\nu)$ is a R-module which is obviously functorial in T. It will be shown below that $\varprojlim_\nu TR/\mathfrak{a}^\nu$ is $\mathfrak{a}$-adic complete.

If $X \in \widehat{\mathfrak{a}}\text{-}\underline{Mod}_R$, then there are canonical isomorphisms $X \xrightarrow{\sim} \varprojlim_\nu X/\mathfrak{a}^\nu X \xrightarrow{\cong} \varprojlim_\nu (R/\mathfrak{a}^\nu \otimes_R X)$, whence $\Omega\Psi(X) \cong X$ is a natural equivalence in X. The converse - i.e. $\Psi\Omega(T) \cong T$ for every T admitting a right adjoint - is more involved.

Since every R-module is in a canonical way a colimit of copies of $R \oplus R$ (cf [23] 1.5 b)), it follows that every $X \in \mathfrak{a}\text{-}\underline{Mod}_R$ is a colimit of copies of $R/\mathfrak{a}^n \oplus R/\mathfrak{a}^m$ for $n, m = 1, 2\ldots$ Hence two colimit preserving functors F and F' on $\mathfrak{a}\text{-}\underline{Mod}_R$ are isomorphic iff for every $n \geq 1$ there is an isomorphism $F(R/\mathfrak{a}^n) \cong F'(R/\mathfrak{a}^n)$ which is natural in $R/\mathfrak{a}^n$. We will show that for every cocontinuous functor $T : \mathfrak{a}\text{-}\underline{Mod}_R \longrightarrow \underline{Ab}.\underline{Gr}.$ and every $n \geq 1$ there is an isomorphism $\xi_n : T(R/\mathfrak{a}^n) \xrightarrow{\cong} R/\mathfrak{a}^n \otimes_R \varprojlim_\nu TR/\mathfrak{a}^\nu$ which is natural in $R/\mathfrak{a}^n$ and T and such that the canonical projection $\varprojlim_\nu T(R/\mathfrak{a}^\nu) \longrightarrow T(R/\mathfrak{a}^n)$ is the composite of the canonical morphism $\varprojlim_\nu T(R/\mathfrak{a}^\nu) \longrightarrow R/\mathfrak{a}^n \otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu)$ with ξ_n^{-1}. The latter implies that $\varprojlim_\nu T(R/\mathfrak{a}^\nu)$ is $\mathfrak{a}$-adic complete. Assume $\mathfrak{a}$ is finitely generated and let $(a_k)_{k \in I_n}$ be a set of generators of $\mathfrak{a}^n$. Let $f : \coprod_{I_n} R \longrightarrow R$ be the homomorphism whose restriction onto the k-th summand is multiplication with a_k. For every $\nu \geq n$ f induces a morphism

$f_{\nu} : \coprod_{I_n} R/\mathfrak{a}^{\nu} \longrightarrow R/\mathfrak{a}^{\nu}$. The exact sequence $\coprod_{I_n} R \xrightarrow{f} R \longrightarrow R/\mathfrak{a}^{n} \longrightarrow 0$ and the filtration $\dots \mathfrak{a}^{\nu+1} \subset \mathfrak{a}^{\nu} \subset \dots \subset \mathfrak{a}^{n} \subset R$ give rise to commutative diagrams





where p_{n+i-1} , q_{n+i-1} , α_{n+i} , β_{n+i} , j_{n+i} , and ρ_{n+i} denote the obvious canonical morphisms. Let $\mathrm{im}(T(\ker f_{n+i}))$ denote the image of $T(\ker f_{n+i}) \longrightarrow T(\coprod_{I_n} R/\mathfrak{a}^{n+i})$. It suffices to show that the induced sequences in the inverse limit

$$(***) \qquad 0 \longrightarrow \varprojlim_i \mathrm{im}(T(\ker f_{n+i})) \longrightarrow \varprojlim_i T(\coprod_{I_n} R/\mathfrak{a}^{n+i}) \longrightarrow \varprojlim_i T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \longrightarrow 0$$

$$(****) \qquad \varprojlim_i T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \longrightarrow \varprojlim_i T(R/\mathfrak{a}^{n+i}) \longrightarrow R/\mathfrak{a}^n \longrightarrow 0$$

are exact. For then the composite of (***) and (****)

$$\varprojlim_i T(\coprod_{I_n} R/\mathfrak{a}^{n+i}) \xrightarrow{\varprojlim_i T(f_{n+i})} \varprojlim_i T(R/\mathfrak{a}^{n+i}) \longrightarrow R/\mathfrak{a}^n \longrightarrow 0$$

is also exact and from the commutative diagram with exact rows

$$\begin{array}{ccccc}
(\coprod_{I_n} R)\otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu) & \xleftarrow{\ \cong\ } & \coprod_{I_n}\varprojlim_i T(R/\mathfrak{a}^{n+i}) & \xrightarrow{\ \sim\ } & \varprojlim_i T(\coprod_{I_n} R/\mathfrak{a}^{n+i}) \\
\downarrow{\scriptstyle f\otimes_R id} & & \downarrow{\scriptstyle (\alpha_\kappa\cdot)_{\kappa\in I_n}} & \swarrow{\scriptstyle \varprojlim_i Tf_{n+i}} & \\
R\otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu) & \xleftarrow{\ \cong\ } & \varprojlim_i T(R/\mathfrak{a}^{n+i}) & & \\
\downarrow & & \downarrow & & \\
R/\mathfrak{a}^n \otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu) & & T(R/\mathfrak{a}^n) & & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

it follows that there is an isomorphism

$\xi_n : T(R/\mathfrak{a}^n) \xrightarrow{\ \cong\ } R/\mathfrak{a}^n \otimes_R \varprojlim_\nu T(R/\mathfrak{a}^\nu)$ which is natural in T .

One readily checks that ξ_n has the two properties mentioned above, as for the naturality in $R/\mathfrak{a}^n$ note that a homomorphism $g : R/\mathfrak{a}^n \longrightarrow R/\mathfrak{a}^m$ can be decomposed into $R/\mathfrak{a}^n \xrightarrow{r\cdot} R/\mathfrak{a}^n \xrightarrow{can} R/\mathfrak{a}^m$ for some $r \in R$ if $n \geq m$ resp. into $R/\mathfrak{a}^m \xrightarrow{can} R/\mathfrak{a}^n \xrightarrow{r\cdot} R/\mathfrak{a}^n$ if $m \geq n$, where can denotes the canonical projection. This completes the proof modulo the exactness of (***) and (****).

As for the exactness of (***) note that by the first assumption on $\mathfrak{a}$ $\ker f_{n+i}$ is a submodule of the artinian module $\coprod_{I_n} R/\mathfrak{a}^{n+i}$. Hence the system $(\ker f_{n+i})_{i\in\mathbb{N}}$ satisfies the condition of Mittag-Leffler. Since T preserves epimorphisms the system $(T(\ker f_{n+i}))_{i\in\mathbb{N}}$ also satisfies Mittag-Leffler and thus the same holds for its image $(\operatorname{im} T(\ker f_{n+i}))_{i\in\mathbb{N}}$. Therefore applying the right exact functor T to the diagram (*) and passing to the limit yields the exactness of (***). On the other hand, if $\mathfrak{a} = (a)$ and $a \in R$ is not a zero divisor, one can chose $I_n = \{a^n\}$. Then $\ker f_{n+i} = \ker\left((R/a^{n+i}R) \xrightarrow{a^n\cdot} R/a^{n+i}R\right) \cong R/\mathfrak{a}^n$ and the morphism $\ker f_{n+i} \longrightarrow \ker f_{n+i-1}$ induced by $p_{n+i} : (R/a^{n+i}R) \longrightarrow (R/a^{n+i-1}R)$ can be identified with $R/a^nR \xrightarrow{a\cdot} R/a^nR$. Since $R/a^nR \xrightarrow{a^\nu\cdot} R/a^nR$ is zero for $\nu \geq n$, the system $(\ker f_{n+i})_{i\in\mathbb{N}}$ satisfies the condition of Mittag-Leffler trivially and one proceeds as in the first case.

For the exactness of (****) it suffices to show that the transition morphisms of the systems

$$\cdots \longrightarrow \ker(Tj_{n+i}) \longrightarrow \ker(Tj_{n+i-1}) \longrightarrow \cdots \longrightarrow \ker(Tj_n) = 0$$

$$\cdots \longrightarrow \ker(T\rho_{n+i}) \longrightarrow \ker(T\rho_{n+i-1}) \longrightarrow \cdots \longrightarrow \ker(T\rho_n) = T(R/\mathfrak{a}^n)$$

induced by diagram (**) are epimorphisms. For the latter this is obvious because by (**) it is an epimorphic image of the system

$$\longrightarrow T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \xrightarrow{Tq_{n+i-1}} T(\mathfrak{a}^n/\mathfrak{a}^{n+i-1}) \longrightarrow \cdots T(\mathfrak{a}^n/\mathfrak{a}^n)$$

whose transition morphism are epimorphic. (Note that T is right exact.) For the former this requires some diagram chasing on the diagram (cf. (**))

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \ker Tj_{n+i} & \longrightarrow & \ker (Tj_{n+i-1}) & \longrightarrow & \cdots \\
 & & \downarrow \cap & & \downarrow \cap & & \\
\cdots & \longrightarrow & T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) & \xrightarrow{T(q_{n+i-1})} & T(\mathfrak{a}^n/\mathfrak{a}^{n+i-1}) & \longrightarrow & \cdots \\
T(\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i}) & \nearrow_{T(\alpha_{n+i})} \searrow^{T(\beta_{n+i})} & \downarrow Tj_{n+i} & & \downarrow T(j_{n+i-1}) & & \\
 & & T(R/\mathfrak{a}^{n+i}) & \xrightarrow{T(p_{n+i-1})} & T(R/\mathfrak{a}^{n+i-1}) & \longrightarrow & \cdots
\end{array}$$

For $x \in \ker T(j_{n+i-1})$ there is an element $\bar{y} \in T(\mathfrak{a}^n/\mathfrak{a}^{n+i})$ which is mapped onto x by the epimorphism $T(q_{n+i-1})$. The aim is to find an $y \in \ker T(j_{n+i})$ which is also mapped onto x under $T(q_{n+i-1})$. Since $T(j_{n+i-1})x = 0$, the image $\bar{\bar{y}} = T(j_{n+i})\bar{y}$ is in the kernel of $T(p_{n+i-1}) : T(R/\mathfrak{a}^{n+i}) \longrightarrow T(R/\mathfrak{a}^{n+i-1})$. Since

$$\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i} \xrightarrow{\beta_{n+i}} R/\mathfrak{a}^{n+i} \xrightarrow{p_{n+i-1}} R/\mathfrak{a}^{n+i-1} \longrightarrow 0$$

is exact and T is right exact, there is an element $z \in T(\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i})$ which is mapped by $T(\beta_{n+i})$ onto $\bar{\bar{y}}$. On the other hand the composite

$$T(\mathfrak{a}^{n+i-1}/\mathfrak{a}^{n+i}) \xrightarrow{T(\alpha_{n+i})} T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \xrightarrow{T(q_{n+i-1})} T(\mathfrak{a}^n/\mathfrak{a}^{n+i-1})$$

is zero, whence the image of $\bar{y} - T(\alpha_{n+i})(z)$ under $T(q_{n+i-1}) : T(\mathfrak{a}^n/\mathfrak{a}^{n+i}) \longrightarrow T(\mathfrak{a}^n/\mathfrak{a}^{n+i-1})$ is also x. But $T(j_{n+i})(\bar{y} - T(\alpha_{n+i})(z)) = \bar{\bar{y}} - T(\beta_{n+i})(z) = \bar{\bar{y}} - \bar{\bar{y}} = 0$ which shows that $y = \bar{y} - T(\alpha_{n+i})(z)$ is in $\ker T(j_{n+i})$. Hence $\ker T(j_{n+i}) \longrightarrow \ker T(j_{n+i-1})$ is surjective which completes the proof of the exactness of (***). Note that for the exactness of (****) none of the conditions on $\mathfrak{a}$ was used.

The generalization to Grothendieck categories is straight forward and requires only the exactness of (***) and (****). The diagram chasing for (***) can be done in any abelian category and by Roos [25]..

the functor $\varprojlim$ preserves the exactness of sequences $0 \longrightarrow (A_i')_{i \in N} \longrightarrow (A_i)_{i \in N} \longrightarrow (A_i'')_{i \in N} \longrightarrow 0$ in Grothendieck categories provided the transition morphisms of $(A_i')_{i \in \mathbb{N}}$ are epimorphic. Thus the sequence (****) is exact without any condition on $\mathfrak{a}$. The above result of Roos also implies that Mittag-Leffler holds in Grothendieck categories so that the proof for the exactness of (***) goes through without change.

It remains to show that the inclusion $\widehat{\mathfrak{a}\text{-}\underline{X}_R} \longrightarrow \underline{X}_R$ has a left adjoint and that $\mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X})$ is locally $\sup(\aleph_1, \pi(\underline{X}))$-presentable. For $\delta = \sup(\aleph_1, \pi(\underline{X}))$ the inclusion $I : \widehat{\mathfrak{a}\text{-}\underline{X}_R} \longrightarrow \underline{X}_R$ preserves δ-filtered colimits and in $\underline{X}_R$ δ-filtered colimits commute with δ-limits. To see the former let $X = \varinjlim_\mu X_\mu$ be a δ-filtered colimit in $\underline{X}_R$ of $\mathfrak{a}$-adic complete objects X_μ. Then the composite

$$\varinjlim_\mu X_\mu \cong \varinjlim_\mu(\varprojlim_i X_\mu/\mathfrak{a}^i X_\mu) \cong \varprojlim_i(\varinjlim_\mu X_\mu/\mathfrak{a}^i X_\mu) \cong \varprojlim_i(\varinjlim_\mu X_\mu/\mathfrak{a}^i \varinjlim_\mu X_\mu)$$

is the canonical map from $\varinjlim_\mu X_\mu$ to its $\mathfrak{a}$-adic completion, whence $\varinjlim_\mu X_\mu$ is $\mathfrak{a}$-adic complete and the inclusion $I : \widehat{\mathfrak{a}\text{-}\underline{X}_R} \longrightarrow \underline{X}_R$ preserves δ-filtered colimits. (Note that this holds for any ideal $\mathfrak{a} \in R$.) The functor $\underline{X}_R \longrightarrow \underline{X}_R$, $X \rightsquigarrow \varprojlim_i X/\mathfrak{a}^i X$ has its value in $\widehat{\mathfrak{a}\text{-}\underline{X}_R}$ because

$$\Omega(\otimes_R X) = \varprojlim_i X/\mathfrak{a}^i X$$

is, as shown above, $\mathfrak{a}$-adic complete. Thus by the universal property of $\varprojlim_i X/\mathfrak{a}^i X$ the functor $L : \underline{X}_R \longrightarrow \widehat{\mathfrak{a}\text{-}\underline{X}_R}$, $X \rightsquigarrow \varprojlim_i /\mathfrak{a}^i X$ is left adjoint to the inclusion $I : \widehat{\mathfrak{a}\text{-}\underline{X}_R} \longrightarrow \underline{X}_R$. Since $\pi(\underline{X}) = \pi(\underline{X}_R)$ it follows from $[LU,-] \cong [U, I-]$, where $U \in \underline{X}_R$ and $\pi(U) \leq \delta$, that $\widehat{\mathfrak{a}\text{-}\underline{X}_R}$ is locally δ-presentable. Thus the same holds for $\mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X})$ because it is equivalent with $\widehat{\mathfrak{a}\text{-}\underline{X}_R}$. In the same way one can show that $\mathrm{Adj}(\mathfrak{a}\text{-}\underline{\mathrm{Mod}}_R, \underline{X})$ is locally $\sup(\aleph_1, \pi(\underline{X}))$-generated.

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List of Symbols

a) non alphabetical

$[X,Y]$	set of morphisms $X \longrightarrow Y$	
$[\underline{A},\underline{B}]$	category of functors $\underline{A} \longrightarrow \underline{B}$	
$\underline{A}(\alpha)$	full subcategory of a category $\underline{A}$ consisting of all α-presentable objects	2.8
$\tilde{\underline{A}}(\alpha)$	full subcategory of a category $\underline{A}$ consisting of all α-generated objects.	2.8
$\bar{\Sigma}$	closure of a class Σ of morphisms	
$\underline{A}_{\Sigma,T}$	full subcategory of a category $\underline{A}$ consisting of all objects $X \in \underline{A}$ such that $T(\sigma,X)$ is an isomorphism for every $\sigma \in \Sigma$, where $T : \underline{B} \times \underline{A} \longrightarrow \underline{C}$ is a given bifunctor and Σ a class of morphisms in $\underline{B}$	6.
$\underline{A}_{\mathbb{G}}$	category of all $\mathbb{G}$-coalgebras in $\underline{A}$ for some cotriple $\mathbb{G}$	4.1o
$\underline{A}^{\mathbb{T}}$	category of all $\mathbb{T}$-algebras in $\underline{A}$ for some triple $\mathbb{T}$	4.13
${}_{\Lambda}\underline{A}$	category of left Λ-objects in $\underline{A}$	6.4

b) alphabetical

$Adj(\underline{A},\underline{B})$	category of all functors $\underline{A} \longrightarrow \underline{B}$ admitting a right adjoint	6.18
$\mathfrak{M}$-$\underline{Mod}_R$	full subcategory of $\underline{Mod}_R$ consisting of all R-modules A such that every cyclic submodule (a) is a quotient of $R/\mathfrak{M}^n$ for some $n \geq 1$ depending on a	6.25
$Bialg_{M,R}(\underline{A})$	category of bialgebras in $\underline{A}$ with respect to operations M and relations R	3.1
Λ-$\underline{Bialg}$	category of bialgebras over a commutative ring Λ	4.4
$\underline{Bimod}_H$	category of bimodules over a bialgebra H	4.9

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