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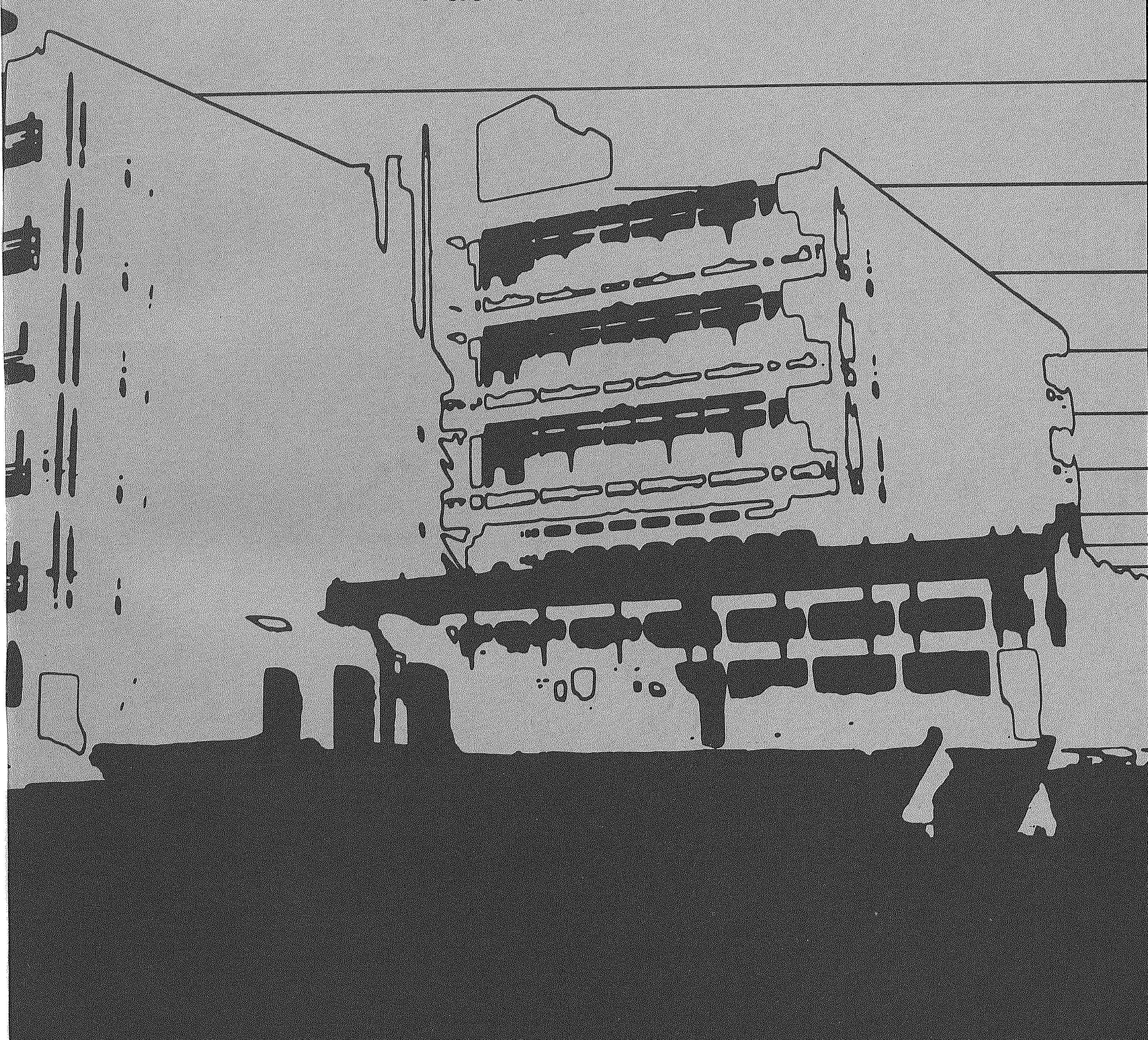

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EILENBERG-MOORE ALGEBRAS REVISITED

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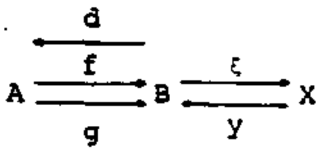
Recently, there has been again a growing interest in monadic and premonadic categories. The Eilenberg-Moore algebras, together with the theorems of Beck and Paré, give us the means to decide, whether a category or, more precisely, a functor is *algebraic*, i.e. *monadic*. If the functor $V : A \to Set$, Set the category of sets, is only premonadic, A is equivalent to a full subcategory of the category of the Eilenberg-Moore algebras via the comparison functor. If, in this case, one is able to compute the Eilenberg-Moore algebras explicitly as universal algebras with operations and equations, one has determined the "*algebraic component*" of the mathematical theory described by $V : A \to Set$ (cp.e.g.[7]).

Hence, it seems to be appropriate to give a review of the Eilenberg-Moore construction. This review is thought as a guideline for anybody investigating the algebraic character of a mathematical theory given by a functor $V : A \to X$ with a left adjoint. Consequently, the starting point for introducing the Eilenberg-Moore algebras in paragraph 1 is an elementary exercise from algebra, more precisely from semigroups. In paragraph 2 the universal property of the category of Eilenberg-Moore algebras is proved, not as general as in [6], but of sufficient generality for most applications. The last section contains the Theorem of Beck (cp. [1],[8]), a Theorem of Tholen characterizing premonadic categories (cp. [9]) and a sufficient condition for the comparison functor to have a left adjoint (see [9]).

§ 1 The Category of Eilenberg-Moore Algebras

Let us begin with a paradigmatic exercise. Consider a semigroup B, a set X and a surjective set mapping $\xi : B \to X$. We will try to answer the following question: "*Under which condition is there a semigroup structure on* X, *s.th.* ξ *is a semigroup homomorphism?*"

To get an idea how to solve this, let us assume there is such a structure on X and form the kernel pair of ξ in the category of semigroups, $f,g : A \to B$, $A := \{(b_0,b_1) \mid b_i \in B,\ i = 0,1,\ \text{and}\ \xi(b_0) = \xi(b_1)\}$ with the induced semigroup structure, and $f(b_0,b_1) := b_0$, $g(b_0,b_1) := b_1$. Besides, take a set mapping $y : X \to B$, right inverse to ξ, $\xi y = id_X$, which picks a representative out of every equivalence class $\xi^{-1}(x)$, $x \in X$. Of course, f,g is also a kernel pair in the category of sets and we have $\xi y \xi = \xi = \xi id_B$, hence, there is a unique set mapping $d : B \to A$ with $fd = y\xi$ and $gd = id_B$. All in all, we have the following diagram:

(SC) 

Surprisingly, the existence of such mappings fulfilling the equations $\xi f = \xi g$, $\xi y = id_X$, $fd = y\xi$, and $gd = id_B$, also turns out to be sufficient for the existence of a (unique) semigroup structure on X making ξ a homomorphism. To show this, one has no choice but to define $x_o x_1 := \xi(y(x_o)y(x_1))$, $x_o, x_1 \in X$, as the product on X (which, incidentally, shows the uniqueness!). With this definition one gets for $b_o, b_1 \in B$

$$\xi(b_o b_1) = \xi(gd(b_o)gd(b_1)) = \xi g(d(b_o)d(b_1))$$

$$= \xi f(d(b_o)d(b_1)) = \xi(fd(b_o)fd(b_1))$$

$$= \xi(y(\xi(b_o))y(\xi(b_1))) = \xi(b_o)\xi(b_1) .$$

Hence, ξ is a surjective mapping preserving products. But this, at once, yields the missing property for the multiplication on X, namely the associative law and we are finished. ξ is, by the way, the coequalizer of f and g in the category of semigroups. For, take any semigroup homomorphism $h : B \to C$ with $hf = hg$, then $(hy)\xi = hfd = hgd = h$ and hy is uniquely determined by this equation, because $h'\xi = h$, $h' : X \to C$, implies $hy = h'\xi y = h'$! It remains to show that hy is a homomorphism. For $x_o, x_1 \in X$ one has

$$hy(x_o x_1) = hy\xi(y(x_o)y(x_1)) = h(y(x_o)y(x_1))$$

$$= hy(x_o)hy(x_1) .$$

This exercise and its solution is not only typical for semigroups, but the same result holds for any type of equationally defined universal algebras. The only reason semigroups were considered here, was, because in semigroups there is only one operation and only one equation, the associative law, which have to be verified. Thus, the situation described by the diagram (SC) and the equations between the mappings seems to be typical for algebraic structures and, therefore, a special name is introduced for it. Even though our examples are categories of algebras over sets, we introduce this new notion for an arbitrary functor.

(1.1) <u>DEFINITION</u>. If $V : A \to X$ is a (covariant) functor, a diagram

$$(SC)\qquad V(A)\ \underset{\underset{d}{\longleftarrow}}{\overset{V(f)}{\underset{V(g)}{\rightrightarrows}}}\ V(B)\ \underset{y}{\overset{\xi}{\rightleftarrows}}\ X$$

in X **, is called a** *split* V-*coequalizer*, **iff the following equations hold:**

$$\xi V(f) = \xi V(g)\ , \quad \xi y = X\ ,$$

$$V(f)d = y\xi\ , \quad V(g)d = V(B)\ .$$

If $V = X$ is the identity functor, we simply speak of a *split coequalizer in* X .

Just as in our exercise, it can also be shown in the general situation that, in a split V-coequalizer, ξ is a coequalizer of $V(f)$ and $V(g)$ in X.

Now, if this notion is really important for algebraic structures, it should be possible to represent any equationally defined (universal) algebra as a split V-coequalizer in a canonical way, i.e. find for such an algebra A_o morphisms f,g and mappings y,d, such that the set $V(A_o)$, underlying A_o, appears as the X in such a diagram (SC). This is, indeed, true and we will, again, as a typical example, consider only the case of semigroups.

Let $V : \mathit{SemiGrp} \to \mathit{Set}$ be the usual forgetful functor from the category *SemiGrp* of semigroups to the category *Set* of sets, assigning the underlying set to any semigroup and the underlying set mapping to any homomorphism. The free semigroup $F(X)$ generated by a set X, also induces a functor $F : \mathit{Set} \to \mathit{SemiGrp}$, the *left adjoint* of V, uniquely determined (up to isomorphism) by the universal property

(UM I)

$$\begin{array}{ccc} X & \xrightarrow{\eta(X)} & V(F(X)) \\ & \searrow^{f} & \Big\downarrow \exists ! V(\varphi) \\ & & V(S) \end{array}$$

that to any set mapping $f : X \to V(S)$, $S \in \mathit{SemiGrp}$, there is exactly one homomorphism $\varphi : F(X) \to S$ with $V(\varphi)\eta(X) = f$, $\eta(X)$ denoting the canonical embedding. There is also, for every semigroup S, the canonical representation of S as a quotient of $F(V(S))$, $\varepsilon(S) : F(V(S)) \to S$, which is also characterized by a universal property:

$$\text{(UN II)} \qquad \begin{array}{ccc} F(V(S)) & \xrightarrow{\varepsilon(S)} & S \\ \exists!\, F(f) \uparrow & \nearrow_{\varphi} & \\ F(Y) & & \end{array}$$

To any morphism φ there is a unique mapping f with $\varphi = \varepsilon(S)F(f)$.

Actually, $\eta : \mathit{Set} \to V \circ F$, $\varepsilon : F \circ V \to \mathit{SemiGrp}$ are natural transformations (Here and in the sequel, the identity functor on a category X is denoted by X , too!), fulfilling the following equations:

$$\text{(ADJ)} \quad (V \circ \varepsilon)(\eta \circ V) = V \ , \quad (\varepsilon \circ F)(F \circ \eta) = F \ .$$

It is well-known that the equations (ADJ) are equivalent to (UN I) as well as to (UN II) and this situation is not particular for semigroups, but, if for any two functors $V : A \to X$, $F : X \to A$, these three equivalent conditions are fulfilled, one says that F is *left adjoint* to V or V *right adjoint* to F or, that $(V, F, \varepsilon, \eta)$ is **an *adjunction*.** Now, returning to our problem at hand, **whether** any semigroup can be canonically represented **by a split** V-coequalizer, one gets the

(1.2) <u>PROPOSITION</u>. *For any semigroup* S *the diagram*

$$V \circ F \circ V \circ F \circ V(S) \;\underset{V \circ \varepsilon \circ F \circ V(S)}{\overset{\overset{\eta \circ V \circ F \circ V(S)}{\longleftarrow}}{\underset{\longrightarrow}{\overset{V \circ F \circ V \circ \varepsilon(S)}{\longrightarrow}}}}\; V \circ F \circ V(S) \;\underset{\eta \circ V(S)}{\overset{V \circ \varepsilon(S)}{\rightleftarrows}}\; V(S)$$

is a split V-*coequalizer.*

Proof: The proof is routine and the easiest way to do it is by using the equations (ADJ). The first of these equations yields $\xi y = X$ in (1.1) for $\xi := V\circ\varepsilon(S)$, $y := \eta\circ V(S)$. Putting $f := F\circ V\circ\varepsilon(S)$, $g := \varepsilon\circ F\circ V(S)$, $d := \eta\circ V\circ F\circ V(S)$, one gets the first equation in (1.1) by applying V to the trivial identity

$$\varepsilon\circ\varepsilon = \varepsilon(F\circ V\circ\varepsilon) = \varepsilon(\varepsilon\circ F\circ V) .$$

The third equation in (1.1) results from

$$(V\circ F\circ V\circ\varepsilon)(\eta\circ V\circ F\circ V) = ((V\circ F\circ V)\circ\varepsilon)((\eta\circ V)\circ F\circ V)$$
$$= \eta\circ V\circ\varepsilon = \eta\circ(V\circ\varepsilon) = (\eta\circ V)(V\circ\varepsilon) ,$$

while the last equation in (1.1) is just the first equation in (ADJ) for the object $F\circ V(S)$.

Now, this representation of the semigroup S by a split V-equalizer is not only canonical but gives rise to an interesting observation. Writing $T := V\circ F$, $\mu := V\circ\varepsilon\circ F$, as abbreviations and $X := V(S)$, $\xi := V\circ\varepsilon(S)$, the diagram takes the form

$$\textbf{(t-ALG)} \qquad T^2(X) \underset{\mu(X)}{\overset{\eta\circ T(X)}{\underset{T(\xi)}{\leftleftarrows\!\!\rightrightarrows}}} T(X) \underset{\eta(X)}{\overset{\xi}{\rightleftarrows}} X$$

and the following equations hold: $\xi T(\xi) = \xi\mu(X)$, $\xi\eta(X) = X$, $\mu(X)\eta\circ T(X) = T(X)$, $T(\xi)\eta\circ T(X) = \eta(X)\xi$. And it is obvious from the above that these conditions induce a semigroup structure on X, s.th. ξ becomes a homomorphism. But (t-ALG) is *not* a split V-coequalizer, provided we forget for the moment where T comes from.

T together with μ and η is an example of a so-called monad:

(1.3) <u>DEFINITION</u>. $t = (T, \mu, \eta)$ is called a *monad* (cp. [3], where the name triple is used instead), iff $T : X \to X$, X a category, is a (covariant) endofunctor and $\mu : T^2 \to T$, $\eta : X \to T$ are natural transformations, s.th. the following equations hold:

(MON)
$$\mu(\eta \circ T) = \mu(T \circ \eta) = T ,$$
$$\mu(T \circ \mu) = \mu(\mu \circ T) .$$

It is clear from the above that any adjunction induces a monad. Eilenberg and Moore in [3] and Kleisli in [4] proved the converse, i.e. any monad is induced by an adjunction. The interesting thing from our point of view is that Eilenberg and Moore used exactly diagrams (t-ALG) for their construction.

On the other hand, it has been already pointed out that any equationally defined, universal algebra can be represented as a diagram as in (1.2) and, hence, also as a diagram (t-ALG). It will turn out that this is no coincidence, because the construction of Eilenberg and Moore in [3] is actually an excellent method for representing categories of universal algebras. We will now describe the Eilenberg-Moore construction.

(1.4) <u>DEFINITION</u>. Let $t = (T,\mu,\eta)$, $T : \mathcal{X} \to \mathcal{X}$, be a monad. A t-*algebra* is a diagram of the form (t-ALG) together with the equations

$$\xi T(\xi) = \xi\mu(X) \ , \quad \xi\eta(X) = X \ ,$$

for $X \in \mathrm{OB}\,\mathcal{X}$. $\mathrm{Ob}\,\mathcal{X}$ for a category $\mathcal{X}$ denotes the class of objects; objects will invariably denoted by capital letters, thus sometimes one simply writes $X \in \mathcal{X}$ instead of $X \in \mathrm{Ob}\,\mathcal{X}$.

The other two equations we had in our example are automatically fulfilled. Often, a t-algebra will be simply denoted by $\xi : T(X) \to X$. A *morphism from the* t-*algebra* $\xi_o : T(X_o) \to X_o$ *to the* t-*algebra* $\xi_1 : T(X_1) \to X_1$ is a morphism $f : X_o \to X_1$, s.th.

$$\begin{array}{ccc} T(X_o) & \xrightarrow{\ \xi_o\ } & X_o \\ {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle f} \\ T(X_1) & \xrightarrow{\ \xi_1\ } & X_1 \end{array}$$

commutes. For the sake of brevity this morphism will be denoted by $\overline{f}$ in the following; $\overline{f} : (T(X_o) \xrightarrow{\xi_o} X_o) \to (T(X_1) \xrightarrow{\xi_1} X_1)$. The t-algebras with these morphisms form the *category* $\mathcal{X}^t$ *of the Eilenberg-Moore algebras of* t. There is a canonical *"forgetful"* functor $V^t : \mathcal{X}^t \to \mathcal{X}$, mapping a t-algebra $\xi : T(X) \to X$ to X and a morphism $\overline{f}$ of t-algebras to the underlying morphism f in $\mathcal{X}$. Obviously V^t is faithful and one gets the

(1.5) <u>THEOREM OF EILENBERG-MOORE</u> (cp. [3]). *For any monad* t *on* $\mathcal{X}$ V^t *has a left adjoint* $F^t : \mathcal{X} \to \mathcal{X}^t$. *The unit of this adjunction is* $\eta : \mathcal{X} \to V^t \circ F^t$ *and the counit* $\varepsilon^t : F^t \circ V^t \to \mathcal{X}^t$ *is given by* $V^t(\varepsilon^t(T(X) \xrightarrow{\xi} X)) = \xi$. *This adjunction induces the monad* t.

Proof: It follows immediately from (1.3) that $\mu(X) : T^2(X) \to T(X)$ is a t-algebra for any object $X \in \mathcal{X}$. One defines $F^t(X) := T^2(X) \xrightarrow{\mu(X)} T(X)$. We prove that F^t is a left adjoint by veryfying (UN I):

$$\begin{array}{ccc} X & \xrightarrow{\eta(X)} & V^t(F^t(X)) = T(X) \\ & \searrow^{f} & \downarrow V^t(\bar{\varphi}) \\ & & V^t(T(X') \xrightarrow{\xi'} X') = X' \end{array}$$

If a $\bar{\varphi}$ exists making this commutative, then it is induced by a morphism $\varphi : T(X) \to X'$, which implies $\varphi\eta(X) = f$, $\varphi\mu(X) = \xi' T(\varphi)$, hence $\varphi = \xi' T(f)$, showing the uniqueness of $\bar{\varphi}$, because V^t is faithful. Now, defining $\varphi := \xi' T(f)$, a straightforward computation proves that φ induces a t-algebra morphism $\bar{\varphi}$ and

$$V^t(\bar{\varphi})\eta(X) = \varphi\eta(X) = \xi' T(f)\eta(X) = \xi'\eta(X')f = f .$$

Hence, F^t is a left adjoint of V^t and, as in every adjunction, the counit ε^t is uniquely determined by the equation $V^t(\varepsilon^t(T(X) \xrightarrow{\xi} X))\eta(X) = X$. Luckily, $\xi : T(X) \to X$ induces a morphism $\bar{\xi} : F^t \circ V^t(T(X) \xrightarrow{\xi} X) \to (T(X) \xrightarrow{\xi} X)$ and $\xi\eta(X) = X$, thus $\bar{\xi}$ is the counit.

If $t = (T,\mu,\eta)$, then $V^t \circ \varepsilon^t \circ F^t = \mu$ follows from the definition of ε^t , which proves the last assertion.

§ 2 The Universal Property of the Eilenberg-Moore Algebras

In this paragraph it will be shown that the Eilenberg-Moore construction $V^t : X^t \to X$ for a monad t is uniquely - up to isomorphisms- characterized by a universal property (cp. [3], 2.2, [5], [6]), which proves very useful in applications.

Putting aside our (hopefully existing!) set-theoretic scruples we will in the sequel define the category of monads and of adjunctions over a fixed category. The reason for this is not so much mathematical, but in this way the results can be formulated more elegantly. Besides, this (dubious) constructions have no influence whatsoever on the proofs and results.

(2.1) DEFINITION (cp. [5],[6]). Let X be a fixed category in the following. We define the category $\mathcal{Mon}(X)$ by taking as objects the monads over X and by defining the morphisms as follows: If (T,μ,η) , (T',μ',η') are monads over X , a morphism $\bar{\alpha} : (T,\mu,\eta) \to (T',\mu',\eta')$ is given by a natural transformation $\alpha : T' \to T$ fulfilling the equations

(i) $\quad \alpha\eta' = \eta$,

(ii) $\quad \alpha\mu' = \mu(\alpha\circ\alpha) = \mu(\alpha\circ T)(T'\circ\alpha)$.

If $\overline{\beta} : (T',\mu',\eta') \to (T'',\mu'',\eta'')$ is a second morphism, we have $\beta : T'' \to T'$, $\alpha\beta : T'' \to T$ and

$$(\alpha\beta)\eta'' = \alpha(\beta\eta'') = \alpha\eta' = \eta \ ,$$

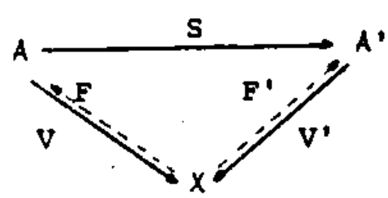
$$(\alpha\beta)\mu'' = \alpha\mu'(\beta\circ\beta) = \mu(\alpha\circ\alpha)(\beta\circ\beta) = \mu((\alpha\beta)\circ(\alpha\beta)) \ .$$

Hence, one defines

$$\overline{\beta}\,\overline{\alpha} := \overline{\alpha\beta} \ ,$$

which obviously yields a category $Mon(X)$, *the category of monads over* X .

(2.2) <u>DEFINITION</u>. The adjunctions (V,F,ε,η) , $V : A \to X$ are taken as the objects of the *category* $Ad(X)$ *of adjunctions over* X . A morphism in this category $S : (V,F,\varepsilon,\eta) \to (V',F',\varepsilon',\eta')$ is just a functor $S : A \to A'$ with $V'\circ S = V$.



Composition of the morphisms S is just the usual composition "$\circ$" of functors. Note that S only forms a commutative triangle with V,V' *not* with the left adjoints.

(2.3) <u>LEMMA</u>. *Let* $(V,F,\varepsilon,\eta),(V',F',\varepsilon',\eta') \in Ad(X)$.
Then the following statements hold:

(a) *For any morphism* $S:(V,F,\varepsilon,\eta) \to (V',F',\varepsilon',\eta')$,
$S:A \to A'$, *there is a unique natural transformation* $\alpha : F' \to S \circ F$ *with* $(V' \circ \alpha)\eta' = \eta$.

(b) *If* $S:(V,F,\varepsilon,\eta) \to (V',F',\varepsilon',\eta')$ *is a morphism in* $Ad(X)$ *and* $\alpha : F' \to S \circ F$ *a natural transformation, then the following equations are equivalent:*

(i) $(V' \circ \alpha)\eta' = \eta$,

(ii) $\alpha = (\varepsilon' \circ S \circ F)(F' \circ \eta)$,

(iii) $V' \circ \varepsilon' \circ S = (V \circ \varepsilon)(V' \circ \alpha \circ V)$,

(iv) $\varepsilon' \circ S = (S \circ \varepsilon)(\alpha \circ V)$.

Proof: (a): α is uniquely defined pointwise by the following diagram (cp. (UN I)):

$$\begin{array}{ccc} X & \xrightarrow{\eta'(X)} & V' \circ F'(X) \\ & \searrow^{\eta(X)} & \Big\downarrow V'(\alpha(X)) \\ & & V \circ F(X) = V' \circ S \circ F(X) \end{array} .$$

It is elementary to verify that this yields a natural transformation. The asserted equation is fulfilled by definition.

(b): (i)$\Longleftrightarrow$(ii): Starting with (ii) one has:

$$(V' \circ [(\varepsilon' \circ S \circ F)(F' \circ \eta)])\eta' = (V' \circ \varepsilon' \circ S \circ F)(V' \circ F' \circ \eta)\eta'$$

$$= (V' \circ \varepsilon' \circ S \circ F)(V' \circ F' \circ \eta)(\eta' \circ X)$$

$$= (V'\circ\varepsilon'\circ S\circ F)(\eta'\circ\eta)$$

$$= (V'\circ\varepsilon'\circ S\circ F)((\eta'X)\circ((V\circ F)\eta))$$

$$= (V'\circ\varepsilon'\circ S\circ F)(\eta'\circ V\circ F)\eta$$

$$= (V'\circ\varepsilon'\circ S\circ F)(\eta'\circ V'\circ S\circ F)\eta = \eta .$$

One realizes that this equation holds without any reference to α or to (i) resp. (ii). Hence, continuing the chain of equations by (i) yields

$$(V'\circ[(\varepsilon'\circ S\circ F)(F'\circ\eta)])\eta' = (V'\circ\alpha)\eta' .$$

This implies (ii), because of the universal property of η'.

(ii) $\Longrightarrow$ (iii):

$$(V\circ\varepsilon)(V'\circ\alpha\circ V) = (V\circ\varepsilon)(V'\circ\varepsilon'\circ S\circ F\circ V)(V'\circ F'\circ\eta\circ V) ;$$

now

$$(S\circ\varepsilon)(\varepsilon'\circ S\circ F\circ V) = \varepsilon'\circ(S\circ\varepsilon) = (\varepsilon'\circ S)\circ\varepsilon$$

$$= [(\varepsilon'\circ S)(F'\circ V'\circ S)]\circ[A\varepsilon] = (\varepsilon'\circ S)(F'\circ V'\circ S\circ\varepsilon) ,$$

which implies

$$(V\circ\varepsilon)(V'\circ\alpha\circ V) = (V'\circ\varepsilon'\circ S)(V'\circ F'\circ V'\circ S\circ\varepsilon)(V'\circ F'\circ\eta\circ V)$$

$$= (V'\circ\varepsilon'\circ S) .$$

(iii) $\Longrightarrow$ (ii):

$$(V'\circ((\varepsilon'\circ S\circ F)(F'\circ\eta)))\eta' = (V'\circ\varepsilon'\circ S\circ F)(V'\circ F'\circ\eta)\eta' .$$

But we have

$$(\alpha\circ V\circ F)(F'\circ\eta) = \alpha\circ\eta = ((S\circ F)\alpha)\circ(\eta X)$$

$$= (S\circ F\circ\eta)\alpha .$$

Hence, we get, by inserting (iii) into the right side of the first equation:

$$[V'\circ((\varepsilon'\circ S\circ F)(F'\circ\eta))]\eta' = (V\circ\varepsilon\circ F)(V'\circ\alpha\circ V\circ F)(V'\circ F'\circ\eta)\eta'$$

$$= (V\circ\varepsilon\circ F)(V'\circ S\circ F\circ\eta)(V'\circ\alpha)\eta'$$

$$= (V'\circ\alpha)\eta' \ ,$$

i.e. (ii), because of the universal property of η'.

(iii)$\Longleftrightarrow$(iv) is obvious.

(2.4) <u>DEFINITION</u>. The functor $\mathrm{Mon} : \mathit{Ad}(X) \to \mathit{Mon}(X)$ is defined by:

$$\mathrm{Mon}(V,F,\varepsilon,\eta) := (V\circ F, V\circ\varepsilon\circ F, \eta) \ ,$$

for $(V,F,\varepsilon,\eta) \in \mathit{Ad}(X)$, and, for a morphism $S : (V,F,\varepsilon,\eta) \to (V',F',\varepsilon',\eta')$ in $\mathit{Ad}(X)$, by:

$$\mathrm{Mon}(S) := \overline{V'\circ\alpha}$$

with the unique $\alpha : F' \to S\circ F$ in (2.3).

One has: $V'\circ\alpha = V'\circ F' \to V'\circ S\circ F$, $V'\circ S\circ F = V\circ F$, and the **defining** equations (i),(ii) in (2.1) are fulfilled, **because** of (2.3),(b),(i), and the following:

$$V'\circ\alpha = (V'\circ\varepsilon'\circ S\circ F)(V'\circ F'\circ\eta) \qquad \text{(cp. (2.3),(b),(ii))}$$

$$= (V'\circ S\circ\varepsilon\circ F)(V'\circ\alpha\circ V\circ F)(V'\circ F'\circ\eta) \qquad \text{(cp. (2.3),(b),(iv))}$$

$$= (V\circ\varepsilon\circ F)(V'\circ\alpha\circ V\circ F)(V'\circ F'\circ V'\circ\alpha)(V'\circ F'\circ\eta')$$

(cp. (2.3),(b),(i)).

Multiplying this equation from the right with $V'\circ\varepsilon'\circ F'$, one gets

$$(V'\circ\alpha)(V'\circ\varepsilon'\circ F') = (V\circ\varepsilon\circ F)(V'\circ\alpha\circ V\circ F)(V'\circ F'\circ V'\circ\alpha) ,$$

i.e. (ii) in (2.1). Hence,

$$\mathrm{Mon}(S) : \mathrm{Mon}(V,F,\varepsilon,\eta) \to \mathrm{Mon}(V',F',\varepsilon',\eta')$$

is a morphism in $\mathit{Mon}(X)$.

(2.5) <u>LEMMA</u>. Mon : $\mathit{Ad}(X) \to \mathit{Mon}(X)$ *is a functor.*

Proof: Let $S : (V,F,\varepsilon,\eta) \to (V',F',\varepsilon',\eta')$, $S' : (V',F',\varepsilon',\eta') \to (V'',F'',\varepsilon'',\eta'')$ be two morphisms in $\mathit{Ad}(X)$. Let $\alpha : F' \to S\circ F$ be induced by S and $\alpha' : F'' \to S'\circ F'$ be induced by S' via (2.3). Then $(S'\circ\alpha)\alpha' : F'' \to (S'\circ S)\circ F$ is a natural transformation, and

$$(V''\circ[(S'\circ\alpha)\alpha'])\eta'' = (V''\circ S'\circ\alpha)(V''\circ\alpha')\eta''$$

$$= (V'\circ\alpha)\eta' = \eta ,$$

because of (2.3),(b),(i).

Hence, in view of (2.3),(a), $(S'\circ\alpha)\alpha'$ is the natural transformation induced by $S'\circ S$, i.e.

$$\mathrm{Mon}(S'\circ S) = \overline{V''\circ((S'\circ\alpha)\alpha')}$$

$$= \overline{(V''\circ S'\circ\alpha)(V''\circ\alpha')} = \overline{(V'\circ\alpha)(V''\circ\alpha')}$$

$$= \overline{(V''\circ\alpha')}\ \overline{(V'\circ\alpha)} = \mathrm{Mon}(S')\ \mathrm{Mon}(S) .$$

(2.6) <u>DEFINITION</u>. If $t=(T,\mu,\eta)\in Mon(X)$, denote the adjunction given by the category X^t of Eilenberg-Moore algebras and the forgetful functor $V^t : X^t \to X$ with left adjoint $F^t : X \to X^t$ by

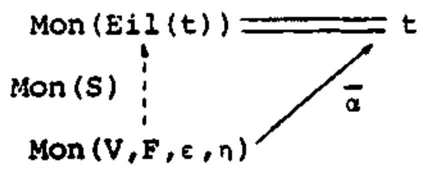
$$\mathrm{Eil}(t) := (V^t, F^t, \varepsilon^t, \eta^t) .$$

(2.7) <u>PROPOSITION</u> (cp. [5],[6]). Eil *induces a full embedding*

$$\mathrm{Eil} : Mon(X) \to Ad(X)$$

with left adjoint $\mathrm{Mon} : Ad(X) \to Mon(X)$, *i.e. represents* $Mon(X)$ *as a full, reflective subcategory of* $Ad(X)$.

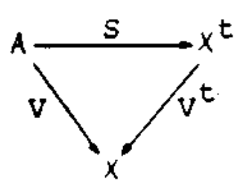
Proof: In order to show that Eil induces a functor, we prove that $\mathrm{Eil}(t)$, $t\in \mathrm{Ob}\,Mon(X)$, satisfies the universal property (UN II)



,

i.e. one shows the unique existence of a morphism $S : (V,F,\varepsilon,\eta) \to \mathrm{Eil}(t)$ with $\mathrm{Mon}(S) = \bar{\alpha}$ for a given $\bar{\alpha}$ in $Mon(X)$.

At first the uniqueness of S is proved. Assume you have such an S , i.e. a functor



with $V^t \circ S = V$, $Mon(S) = \overline{\alpha}$, $\alpha : T \to V \circ F$, then $Mon(S) = \overline{V^t \circ \alpha_S} = \overline{\alpha}$, where $\alpha_S : F^t \to S \circ F$ is the natural transformation induced by S (cp. (2.3),(a)). Now, by (2.3),(b),(iii), we have

$$(*) \qquad V^t \circ \varepsilon^t \circ S = (V \circ \varepsilon)(V^t \circ \alpha_S \circ V) = (V \circ \varepsilon)(\alpha \circ V) .$$

For $A \in Ob\,A$, $V^t(S(A)) = V(A)$ implies that $S(A)$ must be of the form

$$S(A) = (T(V(A)) \xrightarrow{\xi_A} V(A))$$

with a suitable ξ_A. Remembering now the value of $\varepsilon^t(T(X) \xrightarrow{\xi} X)$, (*) yields

$$V^t(\varepsilon^t(S(A))) = \xi_A = V(\varepsilon(A))\alpha(V(A)) : T(V(A)) \to V(A) .$$

Next, the existence of S is shown. It is obvious now, how one has to define S: For $A \in Ob\,A$ put

$$S(A) := (V(\varepsilon(A))\ \alpha(V(A)) : T(V(A)) \to V(A)) .$$

$S(A) \in Ob\,X^t$, because, $\overline{\alpha}$ being a morphism in $Mon(X)$, we get from (2.1),(i),

$$(V \circ \varepsilon)(\alpha \circ V)(\eta^t \circ V) = (V \circ \varepsilon)((\alpha\eta^t) \circ V) = (V \circ \varepsilon)(\eta \circ V) = V ,$$

and

$$(V \circ \varepsilon)(\alpha \circ V)(\mu^t \circ V) = (V \circ \varepsilon)((\alpha\mu^t) \circ V) \qquad \text{(cp. (2.1),(ii}$$

$$= (V \circ \varepsilon)(\mu \circ V)(\alpha \circ \alpha \circ V) = (V \circ \varepsilon)(V \circ \varepsilon \circ F \circ V)(\alpha \circ \alpha \circ V) .$$

On the other hand,

$$(V \circ \varepsilon)(\alpha \circ V)(T \circ V \circ \varepsilon)(T \circ \alpha \circ V)$$

$$= (V \circ \varepsilon)(\alpha \circ V \circ \varepsilon)(T \circ \alpha \circ V)$$

$$= (V\circ\varepsilon)[((V\circ F)\alpha)\circ((V\circ\varepsilon)(V\circ F\circ V))](T\circ\alpha\circ V)$$

$$= (V\circ\varepsilon)(V\circ F\circ V\circ\varepsilon)(\alpha\circ V\circ F\circ V)(T\circ\alpha\circ V)$$

$$= (V\circ\varepsilon)(V\circ F\circ V\circ\varepsilon)(\alpha\circ\alpha\circ V) \ .$$

Hence,

$$(V\circ\varepsilon)(\alpha\circ V)(\mu^t\circ V) = (V\circ\varepsilon)(\alpha\circ V)T\circ[(V\circ\varepsilon)(\alpha\circ V)] \ ,$$

because

$$\varepsilon(\varepsilon\circ F\circ V) = \varepsilon\circ\varepsilon = (\varepsilon(F\circ V))\circ(A\varepsilon) = \varepsilon(F\circ V\circ\varepsilon)$$

holds.

If $u : A \to B$ is a morphism in A, we have the following commutative diagram

$$\begin{array}{ccc}
T(V(A)) & \xrightarrow{\ T\circ V(u)\ } & T(V(B)) \\
{\scriptstyle(\alpha\circ V)(A)}\downarrow & & \downarrow{\scriptstyle(\alpha\circ V)(B)} \\
V\circ F\circ V(A) & \xrightarrow{\ V\circ F\circ V(u)\ } & V\circ F\circ V(B) \\
{\scriptstyle V\circ\varepsilon(A)}\downarrow & & \downarrow{\scriptstyle V\circ\varepsilon(B)} \\
V(A) & \xrightarrow{\ V(u)\ } & V(B)
\end{array}$$

proving $\overline{u} : S(A) \to S(B)$ to be a morphism, i.e. we extend S to morphisms by

$$S(u) := \overline{u} \ .$$

It is routine to check that this definition makes S a functor.

Obviously $V = V^t\circ S$ holds and, because of (2.3),(b),(ii),

$$\alpha_S = (\varepsilon^t\circ S\circ F)(F^t\circ\eta)$$

for the natural transformation $\alpha_S : F^t \to S \circ F$ induced by S (cp. (2.3)). Thus

$$V^t \circ \alpha_S = (V^t \circ \varepsilon^t \circ S \circ F)(V^t \circ F^t \circ \eta)$$

and

$$V^t \circ \varepsilon^t \circ S = (V \circ \varepsilon)(\alpha \circ V) ,$$

because of the definition of S, and, hence,

$$V^t \circ \alpha_S = (V \circ \varepsilon \circ F)(\alpha \circ V \circ F)(T \circ \eta)$$

$$= (V \circ \varepsilon \circ F)(\alpha \circ \eta) = (V \circ \varepsilon \circ F)(((V \circ F)\alpha) \circ (\eta X))$$

$$= (V \circ \varepsilon \circ F)(V \circ F \circ \eta)\alpha = \alpha ,$$

which implies

$$\mathrm{Mon}(S) = \overline{V^t \circ \alpha_S} = \bar{\alpha} .$$

In the uniqueness proof for S we have apparently forgotten the morphisms $u : A \to B$, because we have proved only that the values $S(A)$, $A \in \mathrm{Ob}\,\mathcal{A}$, are uniquely determined. But, because of $V^t \circ S = V$, S is uniquely determined on the morphisms u, too. Let S' be a second functor fulfilling $S' : (V,F,\varepsilon,\eta) \to \mathrm{Eil}(t)$ and $\mathrm{Mon}(S') = \bar{\alpha}$. Then, for a morphism $u : A \to B$, we have $S(A) = S'(A)$, $S(B) = S'(B)$, because of the first part of the proof of (2.7). But then $V(u) = V^t(S(u)) = V^t(S'(u))$ implying $S(u) = S'(u)$, because V^t is faithful.

Summing up, we get that Eil induces a right-adjoint to $\mathrm{Mon} : \mathit{Ad}(X) \to \mathit{Mon}(X)$. As the counit is an identity, therefore, in particular, an isomorphism,

$\mathrm{Eil} : \mathit{Mon}(X) \to \mathit{Ad}(X)$ is full and faithful. Eil is also injective on objects, for, let $\mathrm{Eil}(t) = \mathrm{Eil}(t')$, $t, t' \in \mathrm{Ob}\,\mathit{Mon}(X)$, then

$$(V^t, F^t, \varepsilon^t, \eta^t) = (V^{t'}, F^{t'}, \varepsilon^{t'}, \eta^{t'}) ,$$

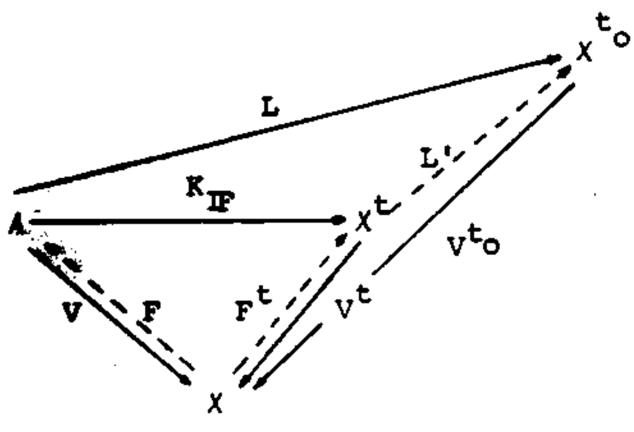
which implies $t = t'$ (according to the construction of Eil(t) !). Hence, Eil is a full embedding.

(2.8) <u>COROLLARY</u>. *The unit of the adjunction in* (2.7), $K_{\mathbb{F}} : \mathbb{F} \to \mathrm{Eil} \circ \mathrm{Mon}(\mathbb{F})$ *for a* $\mathbb{F} = (V, F, \varepsilon, \eta) \in \mathrm{Ob}\,\mathit{Ad}(X)$, $V : A \to X$, *is given by*

$$K_{\mathbb{F}}(A) = (V \circ F \circ V(A) \xrightarrow{(V \circ \varepsilon)(A)} V(A))$$

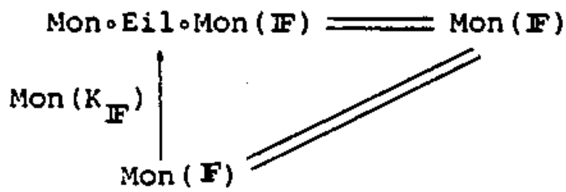
for $A \in \mathrm{Ob}\,A$. $K_{\mathbb{F}}$ *can obviously be canonically and uniquely extended to the whole of* A, $K_{\mathbb{F}} : A \to X^{\mathrm{Mon}(\mathbb{F})}$.

The universal property of the unit $K_{\mathbb{F}}$ *can alternatively be described by the following diagram:*



If $t := \mathrm{Mon}(\mathbb{F})$, $t_o \in \mathrm{Ob}\,\mathit{Mon}(X)$ *arbitrary, then, for any functor* $L : A \to X^{t_o}$ *with* $V^{t_o} \circ L = V$, *there is a unique functor* $L' : X^t \to X^{t_o}$ *with* $V^{t_o} \circ L' = V^t$ *and* $L = L' \circ K_{\mathbb{F}}$.

Proof: As in any adjunction with known counit, the unit, which we call $K_{\mathbb{F}}$, is uniquely determined as the morphism making the following diagram commutative for every $\mathbb{F} \in \mathrm{Ob}\, Ad(X)$:



Looking at the proof of (2.7), one sees that, for $A \in \mathrm{Ob}\, A$,

$$K_{\mathbb{F}}(A) = (V \circ F \circ V(A) \xrightarrow{V \circ \varepsilon(A)} V(A)) \ ,$$

because in the present situation we have $\bar{\alpha} = \mathrm{Mon}(\mathbb{F})$, $\alpha = V \circ F : V \circ F \to V \circ F$.

The rest of the assertion is just a rewriting of the universal property.

An interesting and useful result is the following

(2.9) PROPOSITION. *Let* $\mathbb{F} = (V, F, \varepsilon, \eta) \in \mathrm{Ob}\, Ad(X)$, $V : A \to X$ ***and put*** $t := \mathrm{Mon}(\mathbb{F})$. *If* $S : \mathbb{F} \to \mathrm{Eil} \circ \mathrm{Mon}(\mathbb{F})$ *is a* ***morphism*** *in* $Ad(X)$ *with* $S \circ F = F^{t}$, *then*

$$S = K_{\mathbb{F}} .$$

This means that the comparison functor $K_{\mathbb{F}}$ *is, among all functors* S *making the diagram in* (2.2) *commutative, uniquely determined by the property that it also commutes with the left adjoints.*

Proof: We have, because of $\eta^t = \eta$,

$$\alpha_S = S \circ F = F^t$$

according to (2.3),(a). Hence

$$(*) \qquad \varepsilon^t \circ S = S \circ \varepsilon$$

holds, because of (2.3),(b),(iv). As $V^t \circ S = V$ and $S(A) \in X^t$, $A \in \mathrm{Ob}\,A$, $S(A)$ must be of the form

$$S(A) = (T(V(A)) \xrightarrow{\xi_A} V(A))$$

with a certain ξ_A and $T = V \circ F$. (*) now yields (cp. the definition of $\varepsilon^t(T(X) \xrightarrow{\xi} X)$ in (1.5)):

$$\varepsilon^t(S(A)) = \overline{\xi}_A = S(\varepsilon(A))$$

and $V^t \circ S(\varepsilon(A)) = V(\varepsilon(A))$, i.e. $S(\varepsilon(A)) = \overline{V(\varepsilon(A))}$, which implies $\overline{\xi}_A = \overline{V(\varepsilon(A))}$, resp. $\xi_A = V \circ \varepsilon(A)$ or $S(A) = K_{\mathbb{F}}(A)$. The equality $S = K_{\mathbb{F}}$ on the morphisms follows as in the proof of (2.7) from the faithfulness of V.

(2.10) <u>DEFINITION</u>. If $\mathbb{F} = (V, F, \varepsilon, \eta) \in \mathrm{Ob}\,Ad(X)$, $V : A \to X$, then $\mathbb{F}$ is called *monadic*, iff $K_{\mathbb{F}}$ is an isomorphism, *weakly monadic*, iff $K_{\mathbb{F}}$ is an equivalence, *premonadic*, iff $K_{\mathbb{F}}$ is full and faithful (cp. [9]).

$K_{\mathbb{F}}$ is called the (canonical) *comparison functor* between A and the Eilenberg-Moore category X^t.

Often, one simply calls A monadic, resp. weakly or premonadic instead of V. But this can lead to serious misunderstandings, if it is not clear beyond any doubt, which functor V is meant. For instance, the usual forgetful functor $V : \mathcal{Ban}_1 \to \mathcal{Set}$, $\mathcal{Ban}_1$ the category of (real or complex) Banach spaces and linear contractions, is not premonadic, while the unit ball functor $O : \mathcal{Ban}_1 \to \mathcal{Set}$, $O(B) := \{x \mid \|x\| \le 1\}$, for $B \in \mathcal{Ban}_1$, is premonadic. There exists an analogous example for the category *Top* of topological spaces.

However, it will in general not lead to misunderstandings, if one calls $V : A \to X$ *monadic* (resp. *weakly monadic*, *premonadic*) instead of (V,F,ε,η). This is widely done in the literature and we will also do so occasionally.

§ 3 Monadic Categories

In § 1 we have shown that, for any semigroup S, the diagram (t-ALG) obtained by re-writing the result of (1.2) is a t-algebra for the monad t induced by the usual adjunction of semigroups, $V : \mathcal{SemiGrp} \to \mathcal{Set}$, $F : \mathcal{Set} \to \mathcal{SemiGrp}$. In the light of (2.8) (t-ALG), for $S \in \mathcal{SemiGrp}$, is just $K_{\mathbb{F}}(S)$, i.e. our paradigmatic exercise on semigroups would have led us also, necessarily, to the comparison functor $K_{\mathbb{F}} : \mathcal{SemiGrp} \to \mathcal{Set}^{\mathbf{t}}$. It is straightforward to verify that $K_{\mathbb{F}}$ is an isomorphism and a completely analogous proof leads to the same result for any category of equationally defined universal algebras. The reason, why we do not give these proofs here ,is, because it is by far easier to give them with the Theorem of Beck (3.2).

Thus, all categories of equationally defined universal algebras are monadic (with respect to their usual forgetful functors!) and the Eilenberg-Moore construction appears as an elegant way to describe the notion of "*algebraic*" theory or category. But actually it is of far greater importance. The Theorem of Beck (cp. [1],[8]), which will be proved in this paragraph, gives a handy criterion to recognize a category as monadic, and its range goes beyond the scope of universal algebras. Thus, it is possible by this theorem, to classify other categories as algebraic as, for instance, the category *Comp* of compact Hausdorff spaces with the usual forgetful functor. As interesting, perhaps even more interesting, is another theorem, proved by Tholen in [9], giving several necessary and sufficient conditions for a functor to be premonadic. "Premonadic" means, in most interesting cases, that the category is equivalent to a full subcategory of its Eilenberg-Moore category (cp. [9],(10.6)). But, because of the universal property (2.8), the Eilenberg-Moore category can be considered as the "*algebraic hull*" or the "*algebraic component*" of the theory described by the category in question. Thus, if one succeeds to give an explicit description of the Eilenberg-Moore algebra for a concrete premonadic functor $V: A \to Set$, one knows the algebraic structure of the mathematical theory described by V (cp. e.g. [7]).

For formulating the Theorem of Beck we need the following notion:

(3.1) <u>DEFINITION</u>. A functor $V: A \to X$ is said to *create uniquely split V-coequalizers*, iff, for any split V-coequalizer as in (1.1), the following conditions are fulfilled:

(a) There is a unique morphism $c: B \to C$ in A with $V(c) = \xi$.

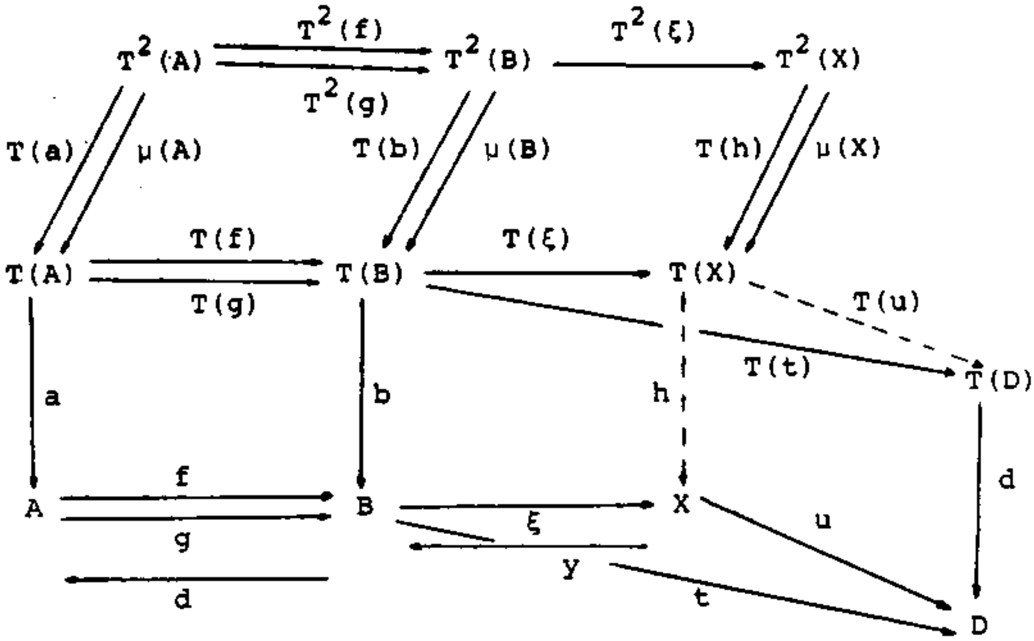
(b) $c = \mathrm{coeq}(f,g)$, i.e. c is a coequalizer of f and g in A.

(3.2) <u>THEOREM OF BECK</u> (cp. [1],[8]). *Let* $(V\ F,\varepsilon,\eta)$ *be an adjunction,* $V: A \to X$, *then the following are equivalent:*

(i) (V,F,ε,η) *is monadic.*

(ii) V *creates uniquely split V-coequalizers.*

***Proof*:** **(i)**$\Rightarrow$(ii): We may assume $(V,F,\varepsilon,\eta) = (V^t,F^t,\varepsilon^t,\eta^t)$ for a monad $t = (T,\mu,\eta)$. Take a split V^t-coequalizer as in (1.1), then one gets the following diagram:



where $V^t(T(A) \overset{a}{\to} A) = A$, $V^t(\overline{f}) = f$, $V^t(\overline{g}) = g$, etc. First, we have to lift ξ uniquely. Now, obviously, $T(\xi)$ is a split coequalizer of $T(f)$ and $T(g)$ in X and $\xi bT(f) = \xi bT(g)$, i.e. there is a unique $h : T(X) \to X$ with $hT(\xi) = \xi b$. Besides, $h : T(X) \to X$ is an Eilenberg-Moore algebra:

$$h\mu(X)T^2(\xi) = hT(\xi)\mu(B) = \xi b\mu(B) = \xi bT(b)$$

$$= hT(\xi)T(b) = hT(\xi b) = hT(hT(\xi)) = hT(h)T^2(\xi) ,$$

hence, $h\mu(X) = hT(h)$, $T^2(\xi)$ having a right inverse. Also, one has

$$h\eta(X)\xi = hT(\xi)\eta(B) = \xi b\eta(B) = \xi ,$$

i.e. $h\eta(X) = X$ and our assertion is proved. Besides, $\overline{\xi} : (T(B) \overset{b}{\to} B) \to (T(X) \overset{h}{\to} X)$ is a morphism in X^t with $V^t(\overline{\xi}) = \xi$. This morphism is uniquely determined by $V^t(\overline{\xi}) = \xi$, because, if there is a $\overline{u}_o : (T(B) \overset{b}{\to} B) \to (T(C) \overset{c}{\to} C)$ with $V^t(\overline{u}_o) = \xi$, then $C = X$, $T(C) = T(X)$ and

$cT(\xi) = \xi b = hT(\xi)$, which yields $c = h$.

Now, let $\overline{t} : (T(B) \xrightarrow{b} B) \to (T(D) \xrightarrow{d} D)$ be in X^t with $\overline{t}\,\overline{f} = \overline{t}\,\overline{g}$. Then, one gets $tf = tg$ implying the unique existence of a $u : X \to D$ with $t = u\xi$. This leads to

$$dT(u)T(\xi) = dT(u\xi) = dT(t) = tb = u\xi b = uhT(\xi)$$

and $dT(u) = uh$, i.e. $\overline{u} : (T(X) \xrightarrow{h} X) \to (T(D) \xrightarrow{d} D)$ is a morphism with $\overline{t} = \overline{u}\,\overline{\xi}$ and $\overline{u}$ is obviously determined uniquely by $\overline{t}$, because $\overline{\xi}$ is an epimorphism in X^t. Hence, (ii) is proved.

(ii)$\Rightarrow$(i): To prove this implication, we will define a functor $L : X^t \to A$, inverse to the comparison functor $K_{\mathbb{F}} : A \to X^t$. We know already (actually, in (1.2) we only showed it for semigroups, but the proof carries over verbatim!), that, for any $A \in \mathrm{Ob}\, A$,

$$V\circ F\circ V\circ F\circ V(A) \underset{\underset{\eta\circ V\circ F\circ V(A)}{\longleftarrow}}{\overset{V\circ\varepsilon\circ F\circ V(A)}{\underset{V\circ F\circ V\circ\varepsilon(A)}{\rightrightarrows}}} V\circ F\circ V(A) \overset{V\circ\varepsilon(A)}{\underset{\eta\circ V(A)}{\rightleftarrows}} V(A)$$

is a split V-coequalizer. Hence, there exists a unique morphism $c_A : F\circ V(A) \to A$ with $V(c_A) = V(\varepsilon(A))$ and, besides, $c_A = \mathrm{coeq}(\varepsilon\circ F\circ V(A), F\circ V\circ\varepsilon(A))$. Thus $c_A = \varepsilon(A)$ is a coequalizer and, in particular, an epimorphism, hence V is faithful.

Now, put $t := \mathrm{Mon}(\mathbb{F})$, $\mathbb{F} := (V,F,\varepsilon,\eta)$ and take any Eilenberg-Moore algebra $\xi : T(X) \to X$. ξ is a split V-coequalizer, because we have

$$T^2(X) \; \underset{\underset{\eta\circ T(X)}{\longleftarrow}}{\overset{\mu(X) = V(\varepsilon\circ F(X))}{\underset{T(\xi) = V(F(\xi))}{\rightrightarrows}}} \; T(X) \; \underset{\eta(X)}{\overset{\xi}{\rightleftarrows}} \; X ,$$

with $\xi\eta(X) = X$, $\xi\mu(X) = \xi T(\xi)$. Besides, we have $V(\varepsilon\circ F(X))\eta\circ T(X) = \mu(X)\eta\circ T(X) = T(X)$ and $V(F(\xi))\eta\circ T(X) = \eta(X)\xi$, because η is a natural transformation.

Hence, (ii) implies the unique existence of a $c_\xi : F(X) \to L_X$ in A with $V(c_\xi) = \xi$ and $c_\xi = \mathrm{coeq}(\varepsilon\circ F(X), F(\xi))$. Therefore, for $\xi : T(X) \to X$ in X^t, one defines

$$L(T(X) \xrightarrow{\xi} X) := L_X .$$

Now,

$$\begin{array}{ccc} V(L_X) & \xrightarrow{\eta\circ V(L_X)} & V\circ F\circ V(L_X) \\ & \searrow\!\!\!\!\searrow & \downarrow V(c_\xi) \\ & & V(L_X) \end{array}$$

commutes, which implies

$$c_\xi = \varepsilon(L_X) = \varepsilon(L(T(X) \xrightarrow{\xi} X)) .$$

If $\bar{u} : (T(X) \xrightarrow{\xi} X) \to (T(X') \xrightarrow{\xi'} X')$ is a morphism in X^t, i.e. the diagram

$$(*) \quad \begin{array}{ccc} V\circ F\circ V(L_X) & \xrightarrow{V\circ F(u)} & V\circ F\circ V(L_Y) \\ V\circ\varepsilon(L_X) \downarrow & & \downarrow V\circ\varepsilon(L_Y) \\ V(L_X) & \xrightarrow{u} & V(L_Y) \end{array}$$

commutes, then

$$\varepsilon(L_Y)F(u)F(\xi) = \varepsilon(L_Y)F(u)\varepsilon\circ F(X)$$

follows, because V is faithful, and the above equation holds for the V-images of both sides.

Now, $\varepsilon(L_X) = c_\xi = \mathrm{coeq}(F(\xi), \varepsilon\circ F(X))$, hence there is a unique

$$w : L_X \to L_Y$$

with

$$w\,\varepsilon(L_X) = \varepsilon(L_Y)F(u) .$$

This yields

$$V(w)V\circ\varepsilon(L_X) = uV\circ\varepsilon(L_X) ,$$

resp.

$$V(w) = u .$$

Define now $L(\bar{u}) := w$; w is obviously uniquely determined by u resp. $\bar{u}$. This extends L to a functor

$$L : X^t \to A .$$

Taking now, for $A \in \mathrm{Ob}\,A$, the Eilenberg-Moore algebra

$$K_{\mathbb{F}}(A) = (T(V(A)) \xrightarrow{V(\varepsilon(A))} V(A)) ,$$

one concludes

$$c_{V(\varepsilon(A))} = \varepsilon(A) : F\circ V(A) \to A ,$$

i.e.

$$L(K_{\mathbb{F}}(A)) = A .$$

If $v : A \to B$ is a morphism in $\mathcal{A}$, we have $K_{\mathbb{F}}(v) : K_{\mathbb{F}}(A) \to K_{\mathbb{F}}(B)$, $K_{\mathbb{F}}(v) = \overline{V(v)}$, i.e. we have to replace u in diagram (*) by $V(v)$. As V is faithful, $V(v) = V(w)$ implies $v = w$ and we get

$$L(K_{\mathbb{F}}(v)) = v\ ,$$

i.e.

$$L \circ K_{\mathbb{F}} = \mathcal{A}\ .$$

Conversely, for $(T(X) \xrightarrow{\xi} X) \in \mathrm{Ob}\, \mathcal{X}^t$, one gets

$$\begin{aligned} K_{\mathbb{F}}(L(T(X) \xrightarrow{\xi} X)) &= K_{\mathbb{F}}(L_X) \\ &= V \circ F \circ V(L_X) \xrightarrow{V \circ \varepsilon(L_X)} V(L_X) \\ &= (T(X) \xrightarrow{\xi} X)\ . \end{aligned}$$

Let $\overline{u} : (T(X) \xrightarrow{\xi} X) \to (T(Y) \xrightarrow{\zeta} Y)$ be a morphism in $\mathcal{X}^t$, i.e. a commutative diagram

$$\begin{array}{ccccc} V \circ F \circ V(L_X) = & T(X) & \xrightarrow{T(u)} & T(Y) & = V \circ F \circ V(L_Y) \\ V \circ \varepsilon(L_X) \big\downarrow & \big\downarrow \xi & & \big\downarrow \zeta & \big\downarrow V \circ \varepsilon(L_Y) \\ V(L_X) = & X & \xrightarrow{\quad u \quad} & Y & = V(L_Y) \end{array}$$

One has $L(\overline{u}) = w$ with the unique $w : L_X \to L_Y$ with $V(w) = u$ and $w\varepsilon(L_X) = \varepsilon(L_Y)F(u)$.

Now, $K_{\mathbb{F}}(w)$ is given by the diagram

$$\begin{array}{ccc} V\circ F\circ V(L_X) & \xrightarrow{V\circ F\circ V(w)} & V\circ F\circ V(L_Y) \\ {\scriptstyle V\circ\varepsilon(L_X)}\downarrow & & \downarrow{\scriptstyle V\circ\varepsilon(L_Y)} \\ V(L_X) & \xrightarrow{V(w)=u} & V(L_Y) \end{array} ,$$

i.e. the same diagram as above. Hence,

$$K_{\mathbb{F}}\circ L(\overline{u}) = \overline{u}$$

holds, i.e.

$$K_{\mathbb{F}}\circ L = X^t .$$

Thus (i) has been proved.

Paré proved in [8] another version of Beck's Theorem by showing that (V,F,ε,η) is monadic, iff V creates uniquely *absolute coequalizers*, where a coequalizer is called absolute, iff it is preserved as a colimit by any functor.

Beck's Theorem is very suitable for proving concrete functors to be monadic, e.g. the usual forgetful functor from any category of equationally defined universal algebras to *Set* . We demonstrate this by restricting ourselves to the paradigmatic case of non-abelian groups.

(3.3) PROPOSITION. $V : \mathcal{G}r p \to \mathcal{S}et$, *the usual forgetful functor from the category of non-abelian groups to sets is monadic.*

Proof: Let

$$V(A) \underset{V(g)}{\overset{V(f)}{\rightrightarrows}} V(B) \underset{y}{\overset{\xi}{\rightleftarrows}} X, \qquad V(A) \xleftarrow{\;d\;} V(B)$$

be a split V-coequalizer, $\xi y = X$, $\xi V(f) = \xi V(g)$, $V(f)d = y\xi$, $V(g)d = V(B)$. For $x_1, x_2 \in X$ define

(*) $\quad x_1 \cdot x_2 := \xi(y(x_1)y(x_2))$.

For the sake of simplicity, we will write f resp. g instead of $V(f)$ resp. $V(g)$. For $b_o, b_1 \in B$ we have

$$\xi(b_o b_1) = \xi(g(d(b_o))g(d(b_1))) = \xi g(d(b_o)d(b_1))$$
$$= \xi f(d(b_o)d(b_1)) = \xi(fd(b_o)fd(b_1))$$
$$= \xi(y(\xi(b_o))y(\xi(b_1))) = \xi(b_o)\xi(b_1) \;,$$

i.e. ξ is surjective and preserves the product. Hence, the product (*) in X is associative and X with this product is a semigroup.

If e_B is the identity element of B, put

$$e := \xi(e_B) \,.$$

For every $x \in X$ this yields

$$ex = \xi(e_B)\xi(y(x)) = \xi(e_B y(x)) = \xi(y(x)) = x$$

and, likewise, $xe = x$.

Thus, e is the identiy element of X and X is a monoid.

For $x \in X$, define

$$x^{-1} := \xi((y(x))^{-1}) ,$$

then

$$x \cdot x^{-1} = x\xi((y(x))^{-1}) = \xi(y(x))\xi(y(x)^{-1}) =$$
$$= \xi(y(x)y(x)^{-1}) = \xi(e_B) = e ;$$

likewise

$$x^{-1} \cdot x = e .$$

Hence, X is a group and ξ a group homomorphism.

As the construction is obviously unique, the lifting of ξ to a group homomorphism is unique. Besides, ξ is a coequalizer of f and g in *Grp* . Take a group homomorphism $\varphi : B \to C$ with $\varphi g = \varphi f$. As ξ is a coequalizer of V(f) and V(g) in *Set* , there is a set mapping $h : X \to C$ with $\varphi = h\xi$. Now ξ , as a surjective group homomorphism, is V-final, hence, h is a group homomorphism. Thus V creates uniquely split V-coequalizers which implies V to be monadic because of (3.2).

(3.4) REMARK. Mutatis mutandis the proof of (3.3) carries over to any category of equationally defined universal algebras over *Set* : These categories are all monadic!

(3.5) <u>PROPOSITION</u> (cp. [8]). *Let Comp denote the category of compact Hausdorff spaces and* $V : Comp \to Set$ *the usual forgetful functor. Then* V *is monadic.*

Proof: Let

$$\begin{array}{c} V(Y) \underset{V(g)}{\overset{V(f)}{\rightrightarrows}} V(Z) \underset{y}{\overset{\xi}{\rightleftarrows}} X \\ \underset{d}{\longleftarrow} \end{array}$$

be a split V-coequalizer, $\xi y = X$, $\xi V(f) = \xi V(g)$, $V(f)d = y\xi$, $V(g)d = V(Z)$. Again, we write in the following simply f for $V(f)$, g for $V(g)$. For a subset $A \subset Z$ of a topological space, let $\overline{A}$ denote the closure of A in Z.

For $A \subset X$ define

$$\overline{A} := \xi(\overline{y(A)}) .$$

Then, for $A, B \subset X$,

$$\begin{aligned} \overline{\overline{A}} &= \overline{(\overline{A})} = \overline{\xi(\overline{y(A)})} = \xi(\overline{y(\xi(\overline{y(A)}))}) \\ &= \xi(\overline{fd(\overline{y(A)})}) = \xi f(\overline{d(\overline{y(A)})}) = \xi g(\overline{d(\overline{y(A)})}) \\ &= \xi(\overline{gd(\overline{y(A)})}) = \xi(\overline{\overline{y(A)}}) = \xi(\overline{y(A)}) = \overline{A} \end{aligned}$$

and

$$\begin{aligned} \overline{A \cup B} &= \xi(\overline{y(A \cup B)}) = \xi(\overline{y(A) \cup y(B)}) \\ &= \xi(\overline{y(A)} \cup \overline{y(B)}) = \xi(\overline{y(A)}) \cup \xi(\overline{y(B)}) = \overline{A} \cup \overline{B} \end{aligned}$$

Hence, $A \mapsto \overline{A}$ is a Kuratowski-closure-operator on X, making it a topological space. ξ becomes a continuous mapping, because for $B \subset Z$:

$$\overline{\xi(B)} = \xi(\overline{y(\xi(B))}) = \xi(\overline{fd(B)}) = \xi f(\overline{d(B)})$$

$$= \xi g(\overline{d(B)}) = \xi(\overline{gd(B)}) = \xi(\overline{B}) .$$

ξ is even a closed mapping, for, take $B \subset Z$ with $B = \overline{B}$, then

$$\xi(B) = \xi(\overline{B}) = \overline{\xi(B)}$$

is closed.

As a surjective mapping from a compact space has a quasicompact range, X is quasicompact. This lifting of ξ is obviously unique, if we assume for the moment that X is even Hausdorff. Because, if ξ is continuous, then, for every $B \subset Z$, we have necessarily $\xi(\overline{B}) = \overline{\xi(B)}$. Hence, for $A \subset X$, because of $A = \xi(y(A))$,

$$\xi(\overline{y(A)}) = \overline{A}$$

holds.

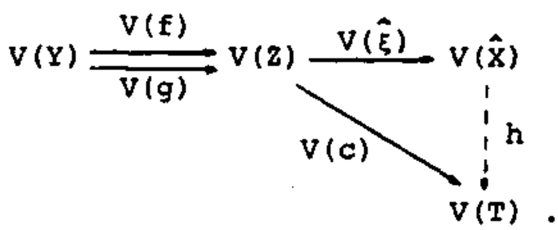
It remains to show X to be Hausdorff. Take $x_o, x_1 \in X$, $x_o \neq x_1$. $\{y(x_i)\}$ is closed in Z, Z being Hausdorff, therefore $\{x_i\} = \xi(\{y(x_i)\})$ is closed, $i = 0,1$ and $\{x_o\} \cap \{x_1\} = \emptyset$.
This implies that $\xi^{-1}(\{x_i\})$ is closed in Z and $\xi^{-1}(\{x_o\}) \cap \xi^{-1}(\{x_1\}) = \emptyset$. Hence, there are open $O_i \subset Z$, $i = 0,1$, $O_o \cap O_1 = \emptyset$ and $\xi^{-1}(\{x_i\}) \subset O_i$. Thus, $C(O_i) := Z \setminus O_i$, $i = 0,1$, is closed and $C(O_o) \cup C(O_1) = Z$, i.e. $\xi(C(O_i))$, $i = 0,1$, is closed in X and $\xi(C(O_o)) \cup \xi(C(O_1)) = X$.

We have $x_i \notin \xi(C(O_i))$, because, if $x_i \in \xi(C(O_i))$, then there is a $z_i \in C(O_i)$ with $x_i = \xi(z_i)$, i.e. $z_i \in \xi^{-1}(\{x_i\}) \subset O_i$, which is a contradiction. Hence, we have $x_i \in C(\xi(C(O_i)))$, $i = 0,1$, where this set is open and

$$C(\xi(C(O_0))) \cap C(\xi(C(O_1))) = \emptyset ,$$

i.e. X is Hausdorff and therefore compact.

The lifted $\hat{\xi}: Z \to \hat{X}$, $\hat{X}$ with the above topology, is also a coequalizer of f and g in *Comp* : If $c: Z \to T$ is in *Comp* with $cf = cg$, we have



As ξ is a coequalizer of $V(f), V(g)$ in *Set*, there is a unique $h: V(\hat{X}) \to V(T)$ in *Set* with

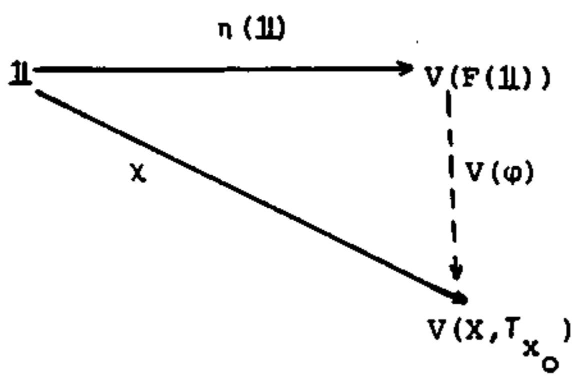
$$hV(\hat{\xi}) = V(c) .$$

Now, $\hat{\xi}$, as a surjective continuous mapping between compact spaces is a quotient mapping, i.e. V-final. Hence, there is a unique continuous mapping $\hat{h}: \hat{X} \to T$ with $V(\hat{h}) = h$, $\hat{h}\hat{\xi} = c$.

Note, that the lifting $\hat{\xi}$ of ξ is uniquely determined by ξ alone, because any such lifting $\tilde{\xi}$ necessarily has the property $\tilde{\xi}(\bar{B}) = \overline{\tilde{\xi}(B)}$ for any $B \subset Z$. Hence, necessarily, for any $A \subset X$, $\xi(\overline{y(A)}) = \overline{\xi(y(A))} = \bar{A}$.

(3.6) <u>REMARK.</u> Let Top^* denote the category of topological spaces and continuous, closed mappings, and let $V : Top^* \to Set$ be the usual forgetful functor. Then, as in (3.5) one shows that V creates uniquely split V-coequalizers. This notwithstanding that V is *not* monadic, because it does not have a left adjoint.

Assume $F: Set \to Top^*$ to be a left adjoint of V. For $\mathbb{1} := \{o\} \in \mathrm{Ob}Set$, one sees at once that the closure $\overline{\eta_1(\mathbb{1})}$ is equal to $F(\mathbb{1})$. Now, take a set X with a cardinality greater than $V(F(\mathbb{1}))$, $|X| > |V(F(\mathbb{1}))|$, and supply it with the topology $T_{x_o} := \{\emptyset, \{x_o\}, X\}$ for some fixed $x_o \in X$. Define $\chi: \mathbb{1} \to V(X, T_{x_o})$ by $\chi(o) := x_o$. Then there is a unique continuous and closed mapping $\varphi: F(\mathbb{1}) \to (X, T_{x_o})$ making the diagram



commutative. As $\varphi(\eta_1(\mathbb{1})) = \{x_o\}$, we get $\varphi(F(\mathbb{1})) = \overline{\{x_o\}} = X$, i.e. φ is surjective implying $|V(F(1))| \geq |X|$, which is a contradiction.

Similar reasonings apply, if one defines the categories Top_1^* and Top_4^* analogously, Top_1^* as the category of T_1-spaces and closed, continuous mappings, Top_4^* as the category of T_4-spaces (normal spaces) and closed, continuous mappings.

The following result of Tholen is very well suited for recognizing concretely given functors as premonadic. This is important in all those cases, where one knows that a certain functor $V : A \to X$ is not monadic and wants to "embed" the category A into a category of algebras, namely the category of Eilenberg-Moore algebras.

(3.6) <u>THEOREM</u> (cp. [9],(10.1)). *For an adjunction* (V,F,ε,η) , $V : A \to X$, *the following statements are equivalent:*

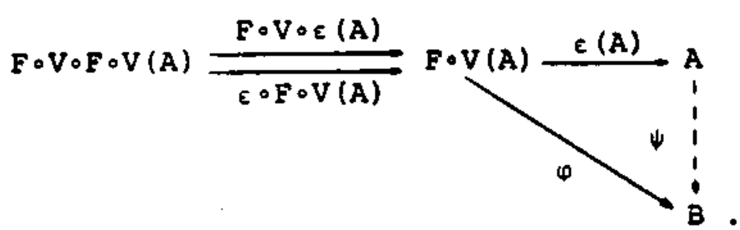
(i) (V,F,ε,η) *is premonadic.*

(ii) ε *is pointwise a coequalizer.*

(iii) ε *is pointwise a regular epimorphism.*

(iv) V *is faithful and* ε *is pointwise* V-*final.*

Proof: **(i)⇒(ii)**: We show that for any $A \in \mathrm{Ob}\, A$, $\varepsilon(A)$ is a coequalizer in the following diagram:



Let φ be such that $\varphi\varepsilon\circ F\circ V(A) = \varphi F\circ V\circ\varepsilon(A)$. Now, by applying V to this diagram and by taking into account $V = V^t\circ K$, where $K = K_{\mathbb{F}}$ denotes the comparison functor, one gets

$$V^t(K\circ F\circ V\circ F\circ V(A)) \underset{\underset{\eta(V\circ F\circ V(A))}{\longleftarrow}}{\overset{V^t(K\circ F\circ V\circ\varepsilon(A))}{\underset{V^t(K\circ\varepsilon\circ F\circ V(A))}{\rightrightarrows}}} V^t(K\circ F\circ V(A) \underset{\eta\circ V(A)}{\overset{V^t(K\circ\varepsilon(A))}{\rightleftarrows}} V^t(K(A)$$

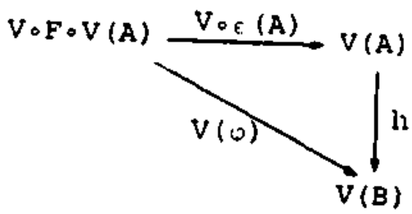
Now this is a split V^t-coequalizer, because this is just a rewriting of (1.2) with V^t and K. As V^t uniquely creates split V^t-coequalizers

$$\begin{array}{ccccc} K\circ F\circ V\circ F\circ V(A) & \underset{K\circ\varepsilon\circ F\circ V(A)}{\overset{K\circ F\circ V\circ\varepsilon(A)}{\rightrightarrows}} & K\circ F\circ V(A) & \xrightarrow{K\circ\varepsilon(A)} & K(A) \\ & & & \searrow^{K(\varphi)} & \downarrow u \\ & & & & K(B) \end{array}$$

is a coequalizer diagram, hence there is a unique u with $K(\varphi) = uK\circ\varepsilon(A)$. This induces a unique $\psi : A \to B$ with $\varphi = \psi\varepsilon(A)$, because K is full and faithful.

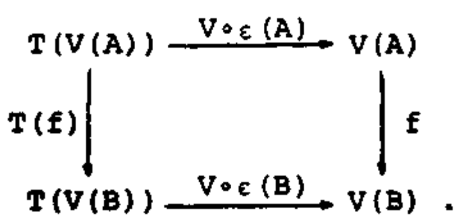
(ii)⇒(iii): is trivial. But one should remember the **definition** of a *regular epimorphism*: If $f : A \to B$ and $g : A \to C$, then $g \leq f$ is used as a notation for the following property: For any two morphisms $u_o, u_1 : D \to A$ $fu_o = fu_1$ implies $gu_o = gu_1$. An *epimorphism* $e : A \to B$ is called *regular*, iff, for any $g : A \to C$, $g \leq e$ implies the existence of a $g' : B \to C$ with $g = g'e$.

(iii)$\Rightarrow$(iv): As ε is pointwise an epimorphism, V is faithful. If



is a commutative diagram, then, for any $u_o, u_1 : C \to F\circ V(A)$, $\varepsilon(A)u_o = \varepsilon(A)u_1$ implies $V(\varphi u_o) = V(\varphi u_1)$, i.e. $\varphi u_o = \varphi u_1$. Hence, $\varphi \leq \varepsilon(A)$ and there is a unique $\varphi' : A \to B$ with $\varphi = \varphi'\varepsilon(A)$. But $V\circ\varepsilon(A)$ is an epimorphism (cp. (ADJ)), which implies $h = V(\varphi')$. Thus, $\varepsilon(A)$ is V-final.

(iv)$\Rightarrow$(i): As V is faithful, so is K. For $A, B \in \mathrm{Ob}\,\mathcal{A}$ let $\bar{f} : K(A) \to K(B)$ be a morphism, i.e. one has a commutative diagram



As $\varepsilon(A)$ is V-final, there is a unique $u : A \to B$ with $V(u) = f$ and $u\varepsilon(A) = \varepsilon(B)F(f) = \varepsilon(B)F\circ V(u)$, hence $\bar{f} = K(u)$ and K is full.

In many applications, the comparison functor $K_{\mathbb{F}}$ has a left adjoint for a premonadic adjunction. This was proved by Dubuc in [2] and improved by Tholen in [9].

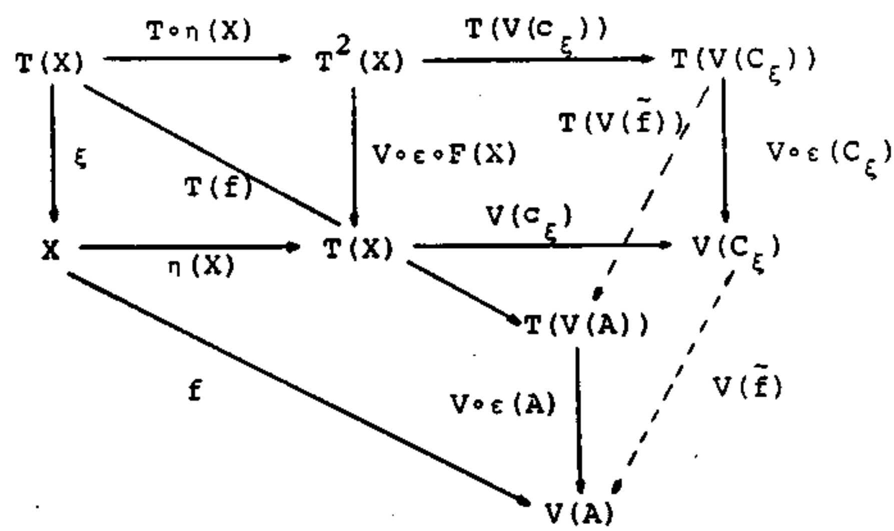
We give a direct proof for the sufficiency of a condition, often fulfilled in applications.

(3.7) <u>PROPOSITION</u>. *Let* $\mathbb{F} = (V, F, \varepsilon, \eta)$ *be a premonadic adjunction,* $V: A \to X$*, and let* A *have coequalizers. Then* $K_{\mathbb{F}}: A \to X^t$, $t = \mathrm{Mon}(\mathbb{F})$*, has a left adjoint.*

Proof: Take an object $(T(X) \overset{\xi}{\to} X) \in \mathrm{Ob}\, X^t$. Select some fixed coequalizer as shown in the diagram

$$F(T(X)) \underset{\varepsilon \circ F(X)}{\overset{F(\xi)}{\rightrightarrows}} F(X) \xrightarrow{c_\xi} C_\xi$$

in A. Now, $K_{\mathbb{F}}(C_\xi) = (V \circ F \circ V(C_\xi) \xrightarrow{V \circ \varepsilon(C_\xi)} V(C_\xi))$ is in X^t and we get the following diagram:



The two horizontal diagrams together are a morphism in X^t, because

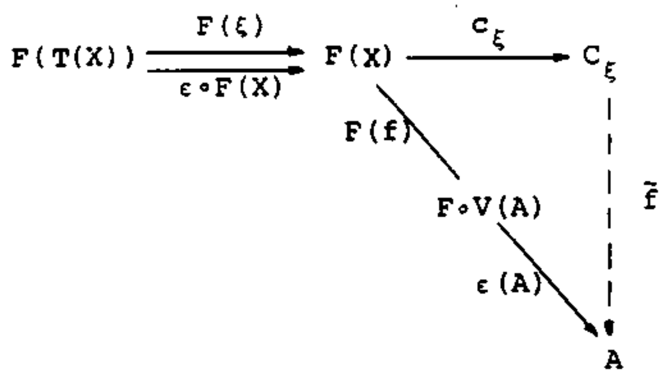
$$V\circ\varepsilon(C_\xi)(T(V(c_\xi))T\circ\eta(X)) = V(c_\xi)V\circ\varepsilon\circ F(X)T\circ\eta(X)$$

$$= V(c_\xi)\mu(X)T\circ\eta(X) = V(c_\xi)\mu(X)\eta\circ T(X)$$

$$= V(c_\xi)T(\xi)\eta\circ T(X) = (V(c_\xi)\eta(X))\xi$$

holds.

This morphism is even universal, for, given any morphism $\bar{f}: (T(X) \xrightarrow{\xi} X) \to K_{\mathbb{F}}(A)$ in X^t, we get a unique $\tilde{f}: C_\xi \to A$, s.th.



commutes, because

$$(\varepsilon(A)F(f))F(\xi) = \varepsilon(A)F(f\xi) = \varepsilon(A)F\circ V\circ\varepsilon(A)F\circ T(f)$$

$$= \varepsilon(A)\varepsilon\circ F\circ V(A)F(T(f)) = (\varepsilon(A)F(f))\varepsilon\circ F(X) .$$

Hence,

$$V(\tilde{f})V(c_\xi)\eta(X) = V\circ\varepsilon(A)V\circ F(f)\eta(X)$$

$$= V\circ\varepsilon(A)\eta\circ V(A)f = f$$

holds.

Now, let $g: C_\xi \to A$ be a morphism in A, s.th. $K_{\mathbb{F}}(g)\,\overline{V(c_\xi)\eta(X)} = \bar{f}$, then

$$V(g)V(c_\xi)\eta(X) = f = V(\tilde{f})V(c_\xi)\eta(X)$$

holds, i.e.

$$V(gc_\xi)\eta(X) = V(\tilde{f}c_\xi)\eta(X) ,$$

resp. $g c_\xi = \tilde{f} c_\xi$, i.e. $g = \tilde{f}$, which means the uniqueness of $\tilde{f}$.

Thus $R(T(X) \xrightarrow{\xi} X) := C_\xi$ defines a functor $R : X^t \to A$, left adjoint to $K_{\mathbb{F}}$ with unit
$\overline{V(c_\xi)\eta(X)} : (T(X) \xrightarrow{\xi} X) \to K_{\mathbb{F}} \circ R(T(X) \xrightarrow{\xi} X)$.

(3.8) <u>EXAMPLES</u>. Let us consider two easy examples to illustrate these results.

(1) In the category *Ab* of abelian groups, consider the full subcategory *Torfree* of torsion-free abelian groups and denote the usual forgetful functors by $V : \mathit{Torfree} \to \mathit{Set}$, $W : \mathit{Ab} \to \mathit{Set}$ and the embedding by $E : \mathit{Torfree} \to \mathit{Ab}$. V is *not monadic*, for, consider the following diagram in *Set* for some $n \in \mathbb{N}$, $n > 1$,

$$A \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{Z} \underset{y}{\overset{\xi}{\rightleftarrows}} \mathbb{Z}/_{n\mathbb{Z}}, \qquad \mathbb{Z} \xrightarrow{d} A$$

where ξ is the canonical projection in *Ab*, $\mathbb{Z}$ the **abelian group** with the usual addition. A is the **torsion-free** abelian subgroup of $\mathbb{Z}^2$ defined by $A := \{(k,l) \mid k \equiv l \bmod n\}$, $f(k,l) := k$, $g(k,l) = l$. The set mapping y is defined by denoting by $y(k + n\mathbb{Z})$ the unique number in $(k + n\mathbb{Z}) \cap \{0,1,\dots,n-1\}$.
With $\delta(k) := y(\xi(k))$, $k \in \mathbb{Z}$, we put $d(k) := (\delta(k),k)$, $k \in \mathbb{Z}$, and have a split V-coequalizer, which can obviously not be lifted. On the other hand,

$\varepsilon(A) : F\circ V(A) \to A$, $A \in \mathrm{Ob}\,\mathit{Torfree}$, is a coequalizer, thus V is premonadic because of (3.6),(ii), and the comparison functor $K_{\mathbb{F}}$ has a left adjoint because of (3.7). Actually, the category of Eilenberg-Moore algebras for V is *Ab* and $K_{\mathbb{F}} = E$. The left adjoint $R: \mathit{Ab} \longrightarrow \mathit{Torfree}$ is given by $R(A) := A/_{\mathrm{Tor}(A)}$ for $A \in \mathrm{Ob}\,\mathit{Ab}$, if Tor(A) denotes the subgroup of torsion elements of A.

(2) Let Ban_1 denote the category of (real or complex) Banach spaces and linear contractions. $O : \mathit{Ban}_1 \to \mathit{Set}$ is the so-called unit ball functor defined by $O(B) := \{x \mid x \in B \text{ and } \|x\| \le 1\}$. If, for a set X, $l_1(X)$ denotes the l_1-space generated by X,

$$l_1(X) := \{f \mid f : X \to \mathbb{K} \quad \|f\| := \sum_{x \in X} |f(x)| < \infty\}, \quad \mathbb{K} = \mathbb{R}, \mathbb{C},$$

then l_1 is a left adjoint of O. A basis of $l_1(X)$ is given by the Dirac functionals δ_x, $x \in X$, $\delta_x(x') := 0$ for $x \neq x'$ and $\delta_x(x) = 1$. It is not difficult to see that $\varepsilon(B) : l_1\circ O(B) \to B$, $B \in \mathit{Ban}_1$, is a coequalizer. Hence, due, to (3.6), O is *premonadic* and the comparison functor has a left adjoint because of (3.7). It is less simple to show that O is not monadic and to explicitly compute the category of Eilenberg-Moore algebras, which has been done in [7].

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