A concise course in complex analysis and Riemann surfaces

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Preface

During their first year at the University of Chicago, graduate students in mathematics take classes in algebra, analysis, and geometry, one of each every quarter. The analysis classes typically cover real analysis and measure theory, functional analysis, and complex analysis. This book grew out of the author's notes for the complex analysis class which he taught during the Spring quarter of 2007 and 2008. The course covered elementary aspects of complex analysis such as the Cauchy integral theorem, the residue theorem, Laurent series, and the Riemann mapping theorem with Riemann surface theory. Needless to say, all of these topics have been covered in excellent textbooks as well as classic treatise. This book does not try to compete with the works of the old masters such as Ahlfors [**1**], Hurwitz–Courant [**20**], Titchmarsh [**39**], Ahlfors–Sario [**2**], Nevanlinna [**34**], Weyl [**41**]. Rather, it is intended as a fairly detailed yet fast paced guide through those parts of the theory of one complex variable that seem most useful in other parts of mathematics. There is no question that complex analysis is a corner stone of the analysis education at every university and each area of mathematics requires at least some knowledge of it. However, many mathematicians never take more than an introductory class in complex variables that often appears awkward and slightly outmoded. Often, this is due to the omission of Riemann surfaces and the assumption of a computational, rather than geometric point of view. Therefore, the authors has tried to emphasize the very intuitive geometric underpinnings of elementary complex analysis which naturally lead to Riemann surface theory. As for the latter, today it is either not taught at all or sometimes given a very algebraic slant which does not appeal to more analytically minded students. This book intends to develop the subject of Riemann surfaces as a natural continuation of the elementary theory without which the latter would indeed seem artificial and antiquated. At the same time, we do not overly emphasize the algebraic aspect such as elliptic curves. The author feels that those students who wish to pursue this direction will be able to do so quite easily after mastering the material in this book. Because of such omissions as well as the reasonably short length of the book it is to be considered "intermediate".

Partly because of the fact that the Chicago first year curriculum covers topology and geometry this book assumes knowledge of basic notions such as homotopy, the fundamental group, differential forms, co-homology and homology, and from algebra we require knowledge of the notions of groups and fields, and some familiarity with the resultant of two polynomials (but the latter is needed only for the definition of the Riemann surfaces of an algebraic germ). However, only the most basic knowledge of these concepts is assumed and we collect the few facts that we do need in Chapter 13.

Let us now describe the contents of the individual chapters in more detail. Chapter 1 introduces the concept of differentiability over $\mathbb{C}$, the calculus of $\partial_z, \partial_{\bar{z}}$, Möbius (or fractional linear) transformations and some applications of these transformations to hyperbolic geometry. In particular, we prove the Gauss-Bonnet theorem in that case. Next, we develop integration and Cauchy's theorem in various guises, then apply this to the study of analyticity, and harmonicity, the logarithm and the winding number. We conclude the chapter with some brief comments about co-homology and the fundamental group.

Chapter 2 refines the Cauchy formula by extending it to zero homologous cycles, i.e., those cycles which do not wind around any point outside of the domain of holomorphy. We then classify isolated singularities, prove the Laurent expansion and the residue theorems with applications. After that, Chapter 2 studies analytic continuation and presents the monodromy theorem. Then, we turn to convergence of analytic functions and normal families with application to the Mittag-Leffler and Weierstrass theorems in the entire plane, as well as the Riemann mapping theorem. The chapter concludes with Runge's theorem.

In Chapter 3 we study the Dirichlet problem on the unit disk. This means that we solve the boundary value problem for the Laplacian on the disk via the Poisson kernel. We present the usual L^p based Hardy classes of harmonic functions on the disk, and discuss the question of representing them via their boundary data both in the L^p and the almost every sense. We then sketch the more subtle theory of homolomorphic functions in the Hardy class, or equivalently of the boundedness properties of the conjugate harmonic functions (with the F.& M. Riesz theorem and the notion of inner and outer functions being the most relevant here).

The theory of Riemann surfaces begins with Chapter 4. This chapter covers the basic definitions of such surfaces and the analytic functions on them. Elementary results such as the Riemann-Hurwitz formula for the branch points are discussed and several examples of surfaces and analytic functions defined on them are presented. In particular, we show how to define Riemann surfaces via discontinuous group actions and give examples of this procedure. The chapter closes with a discussion of tori and some aspects of the classical theory of meromorphic functions on these tori (doubly periodic or elliptic functions).

Chapter 5 presents another way in which Riemann surfaces arise naturally, namely via analytic continuation. Historically, the desire to resolve unnatural issues related to "multi-valued functions" (most importantly for algebraic functions) lead Riemann to introduce his surfaces. Even though the underlying ideas leading from a so-called analytic germ to its Riemann surface are very geometric and intuitive (and closely related to covering spaces in topology), their rigorous rendition requires some patience as ideas such as "analytic germ", "branch point", "(un)ramified Riemann surface of an analytic germ" etc., need to be defined precisely. This typically proceeds via some factorization procedure of a larger object (i.e., equivalence classes of sets which are indistinguishable from the point of view of the particular object we wish to construct). The chapter also develops some basic aspects of algebraic functions and their Riemann surfaces. At this point the reader will need to be familiar with the resultant of two polynomials. In particular, we will see that *every* (!) compact Riemann surface is obtained through analytic continuation of some algebraic germ. This uses the machinery of Chapter 5 together with a potential theoretic result that guarantees the existence of a non-constant meromorphic function on every Riemann surface, which we prove in Chapter 7.

Chapter 6 introduces differential forms on Riemann surfaces and their integrals. Needless to say, the only really important class are the 1-forms and we define harmonic, holomorphic and meromorphic forms and the residues in the latter case. Furthermore, the Hodge $*$ operator appears naturally. We then present some examples that lead up to the Hodge decomposition in the next chapter. This refers to the fact that every 1-form can be decomposed additively into three components: a closed, co-closed, and a harmonic form (the latter being characterized as being simultaneously closed and co-closed). In this book, we follow the classical L^2-based derivation of this theorem. Thus, via Hilbert space methods one first derives this decomposition with L^2-valued forms and then uses Weyl's regularity lemma (weakly harmonic functions are smoothly harmonic) to upgrade to smooth forms.

The proof of the Hodge theorem is presented in Chapter 7. This chapter includes a theorem on the existence of meromorphic differentials and functions on a general Riemann surface. In particular, we derive the striking fact that every Riemann surface carries a non-constant meromorphic function which is needed to complete the result on compact surfaces being algebraic in Chapter 5.

Chapter 8 presents the well-known Riemann-Roch theorem which computes the dimension of certain spaces of meromorphic differentials from properties of the so-called divisor and the genus of the underlying compact Riemann surface. Before proving the theorem, there are a number of prerequisites to be dealt with, such as the Riemann period relations and the definition of a divisor.

The remaining Chapters 9, 10 and 11 are devoted to the proof of the uniformization theorem. This theorem states that the only simply connected Riemann surfaces (up to isomorphisms) are $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{C}P^1$. For the compact case, we deduce this from the Riemann-Roch theorem. But for the other two cases we use methods of potential theory which are motivated by the proof of the Riemann mapping theorem which is based on the existence of a Green function. It turns out that such a function only exists for the hyperbolic surfaces (such as $\mathbb{D}$) but not for the parabolic case (such as $\mathbb{C}$) or the compact case. Via the Perron method, we prove the existence of a Green function for hyperbolic surfaces, thus establishing the conformal equivalence with the disk. For the parabolic case, a suitable substitute for the Green function needs to be found. We discuss this in detail for the simply connected case, and also sketch some aspects of the non-simply connected cases.

CHAPTER 1

From i to z: the basics of complex analysis

1. The field of complex numbers

The field $\mathbb{C}$ of complex numbers is obtained by adjoining i to the field $\mathbb{R}$ of reals. The defining property of i is $i^2+1=0$ and complex numbers $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ are added component-wise and multiplied according to the rule

$$z_1 \cdot z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

which follows from $i^2+1=0$ and the distributional law. The *conjugate* of $z = x+iy$ is $\bar{z} = x - iy$ and we have $|z|^2 := z\bar{z} = x^2+y^2$. Therefore every $z \neq 0$ has a multiplicative inverse given by $\frac{1}{z} := \bar{z}|z|^{-2}$ and $\mathbb{C}$ becomes a field. Since complex numbers z can be represented as points or vectors in $\mathbb{R}^2$ in the Cartesian way, we can also assign polar coordinates (r,θ) to them. By definition, $r = |z|$ and $z = r(\cos\theta + i\sin\theta)$. The addition theorems for cosine and sine imply that

$$z_1 \cdot z_2 = |z_1||z_2|(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

which reveals the remarkable fact that complex numbers are multiplied by *multiplying their lengths and adding their angles.* In particular, $|z_1z_2| = |z_1||z_2|$. This shows that power series behave as in the real case with respect to convergence, i.e.,

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{converges on } |z|<R \text{ and diverges for every } |z|>R$$
$$R^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

where the sense of convergence is relative to the length metric $|\cdot|$ on complex numbers which is the same as the Euclidean distance on $\mathbb{R}^2$ (the reader should verify the triangle inequality); the formula for R of course follows from comparison with the geometric series. Note that the convergence is absolute on the disk $|z|<R$ and uniform on every compact subset of that disk. Moreover, the series diverges for *every* $|z|>R$ as can be seen by the comparison test. We can also write $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, provided this limit exists. The first example that comes to mind here is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z|<1.$$

Another example is of course

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1.1}$$

which converges absolutely and uniformly on every compact subset of $\mathbb{C}$. Expanding $(z_1+z_2)^n$ via the binomial theorem shows that $E(z_1+z_2) = E(z_1)E(z_2)$. Recall the definition of the Euler constant e: consider the ordinary differential equation (ODE) $\dot{y} = y$ with $y(0) = 1$ which has a unique solution $y(t)$ for all $t \in \mathbb{R}$. Then set $e := y(1)$. Let us solve our ODE iteratively by the Picard method. Thus,

$$\begin{aligned} y(t) &= 1 + \int_0^t y(s)\,ds = 1 + t + \int_0^t (t-s)y(s)\,ds = \dots \\ &= \sum_{j=0}^n \frac{t^j}{j!} + \int_0^t (t-s)^n y(s)\,ds. \end{aligned}$$

The integral on the right vanishes as $n \to \infty$ and we obtain

$$y(t) = \sum_{j=0}^\infty \frac{t^j}{j!}$$

which in particular yields the usual series expansion for e. Also, by the group property of flows,

$$y(t_2)y(t_1) = y(t_1+t_2)$$

which proves that $y(t) = e^t$ for every rational t and motivates why we *define*

$$e^t := \sum_{j=0}^\infty \frac{t^j}{j!} \qquad \forall\, t \in \mathbb{R}.$$

Hence, our series $E(z)$ above is used as the *definition* of e^z for all $z \in \mathbb{C}$. We have the group property $e^{z_1+z_2} = e^{z_1}e^{z_2}$, and by comparison with the power series of cos and sin on $\mathbb{R}$, we arrive at the famous Euler formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

for all $\theta \in \mathbb{R}$. This in particular shows that $z = re^{i\theta}$ where (r,θ) are the polar coordinates of z. This in turn implies that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

for every $n \geq 1$ (de Moivre's formula). Now suppose that $z = re^{i\theta}$ with $r > 0$. Then by the preceding,

$$z = e^{\log r + i\theta} \quad \text{or} \quad \log z = \log r + i\theta.$$

Note that the logarithm is not well-defined since θ and $\theta + 2\pi n$ for any $n \in \mathbb{Z}$ both have the property that exponentiating leads to z. Similarly,

$$\left(r^{\frac{1}{n}} e^{i\frac{\theta}{n}} e^{\frac{2\pi i k}{n}}\right)^n = z \quad \forall\, 1 \leq k \leq n$$

which shows that there are n different possibilities for $\sqrt[n]{z}$. Later on we shall see how these functions become single-valued on their natural Riemann surfaces. Let us merely mention at this point that the complex exponential is most naturally viewed as the covering map

$$\begin{cases} \mathbb{C} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\} \\ z \mapsto e^z \end{cases}$$

showing that $\mathbb{C}$ is the universal cover of $\mathbb{C}^*$.

But for now, we of course wish to differentiate functions defined on some open set $\Omega \subset \mathbb{C}$. There are two relevant notions of derivative here and we will need to understand how they relate to each other.

2. Differentiability and conformality

The first is the crucial linearization idea from multivariable calculus and the second copies the idea of difference quotients from calculus. In what follows we shall either use U or Ω to denote planar regions, i.e., open and connected subsets of $\mathbb{R}^2$. Also, we will identify $z = x + iy$ with the real pair (x, y) and will typically write a complex-valued function as $f(z) = u(z) + iv(z) = (u, v)(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$.

DEFINITION 1.1. *Let* $f : \Omega \to \mathbb{C}$.
(a) We say that $f \in C^1(\Omega)$ *iff there exists* $df \in C(\Omega, M_2(\mathbb{R}))$, *a* 2×2 *matrix-valued function, such that*

$$f(z + h) = f(z) + df(z)(h) + o(|h|) \qquad h \in \mathbb{R}^2,\ |h| \to 0$$

where $df(z)(h)$ *means the matrix* $df(z)$ *acting on the vector* h.
(b) We say that f *is* holomorphic *on* Ω *if*

$$f'(z) := \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for all $z \in \Omega$ *and is continuous on* Ω. *We denote this by* $f \in \mathcal{H}(\Omega)$. *A function* $f \in \mathcal{H}(\mathbb{C})$ *is called* entire.

Note that (b) is equivalent to the existence of a function $f' \in C(\Omega)$ so that

$$f(z + h) = f(z) + f'(z)h + o(|h|) \qquad |h| \to 0$$

where $f'(z)h$ is the product between the complex numbers $f'(z)$ and h. Hence, we conclude that the holomorphic functions are precisely those functions in $C^1(\Omega)$ in the sense of (a) for which the differential $df(z)$ acts as linear map via multiplication by a complex number. Obvious examples of holomorphic maps are the powers $f(z) = z^n$ for all $n \in \mathbb{Z}$ (if n is negative, then we exclude $z = 0$). They satisfy $f'(z) = nz^{n-1}$ by the binomial theorem. Also, since we can do algebra in $\mathbb{C}$ the same way we did over $\mathbb{R}$ it follows that the basic differentiation rules like the sum, product, quotient, and chain rules continue to hold for holomorphic functions. Let us demonstrate this for the chain rule: if $f \in \mathcal{H}(\Omega)$, $g \in \mathcal{H}(\Omega')$ and $f : \Omega \to \Omega'$, then we know from the C^1-chain rule that

$$(f \circ g)(z + h) = (f \circ g)(z) + Df(g(z))Dg(z)h + o(|h|) \qquad |h| \to 0.$$

From (b) above we infer that $Df(g(z))$ and $Dg(z)$ act as multiplication by the complex numbers $f'(g(z))$ and $g'(z)$, respectively. Thus, we see that $f \circ g \in \mathcal{H}(\Omega)$ and $(f \circ g)' = f'(g)g'$. We leave the product and quotient rules to the reader.

It is clear that all polynomials are holomorphic functions. In fact, we can generalize this to all power series within their disk of convergence. Let us make this more precise.

DEFINITION 1.2. *We say that* $f : \Omega \to \mathbb{C}$ *is* analytic *(or* $f \in \mathcal{A}(\Omega)$*) if* f *is represented by a convergent power series expansion on a neighborhood around every point of* Ω.

LEMMA 1.3. $\mathcal{A}(\Omega) \subset \mathcal{H}(\Omega)$

PROOF. Suppose $z_0 \in \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \forall\, z \text{ so that } |z - z_0| < r(z_0)$$

where $r(z_0) > 0$. As in real calculus, one checks that differentiation can be interchanged with summation and

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1} \qquad \forall\, |z - z_0| < r(z_0).$$

In fact, one can differentiate any number of times and

$$f^{(k)}(z) = \sum_{n=0}^{\infty} (n)_k\, a_n (z - z_0)^{n-k} \qquad \forall\, |z - z_0| < r(z_0)$$

where $(n)_k = n(n-1)\dots(n-k+1)$. This proves also that $a_n = \frac{f^{(n)}(z_0)}{n!}$ for all $n \geq 0$. $\square$

We note that with e^z defined as above, $(e^z)' = e^z$ from the series representation (1.1). It is a remarkable fact of basic complex analysis that one has equality in Lemma 1.3, i.e., $\mathcal{A}(\Omega) = \mathcal{H}(\Omega)$. In order to establish this equality, we need to be able to integrate; see the section about integration below.

Recall that $f = u + iv = (u, v)$ belongs to $C^1(\Omega)$ iff the partials u_x, u_y, v_x, v_y exist and are continuous on Ω. If $f \in \mathcal{H}(\Omega)$, then by letting w approach z along the x or y–directions, respectively,

$$f'(z) = u_x + i v_x = -i u_y + v_y$$

so that

$$u_x = v_y, \qquad u_y = -v_x.$$

These relations are known as the *Cauchy–Riemann equations*. They are equivalent to the property that

$$df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \rho A \quad \text{for some} \quad \rho \geq 0, \quad A \in SO(2, \mathbb{R}).$$

In other words, at each point where a holomorphic function f has a nonvanishing derivative, its differential df is a conformal matrix: it preserves angles and the orientation between vectors. Conversely, if $f \in C^1(\Omega)$ has the property that df is proportional to a rotation everywhere on Ω, then $f \in \mathcal{H}(\Omega)$. Let us summarize these observations.

THEOREM 1.4. *A complex-valued function $f \in C^1(\Omega)$ is holomorphic iff the Cauchy-Riemann system holds in Ω. This is equivalent to df being the composition of a rotation and a dilation (possibly by zero) at every point in Ω.*

PROOF. We already saw that the Cauchy-Riemann system is necessary. Conversely, since $f \in C^1(\Omega)$, we can write:

$$u(x + \xi, y + \eta) = u(x, y) + u_x(x, y)\xi + u_y(x, y)\eta + o(|(\xi, \eta)|)$$
$$v(x + \xi, y + \eta) = v(x, y) + v_x(x, y)\xi + v_y(x, y)\eta + o(|(\xi, \eta)|).$$

Using that $u_x = v_y$ and $u_y = -v_x$ we obtain, with $\zeta = \xi + i\eta$,

$$f(z + \zeta) - f(z) = (u_x + i v_x)(z)(\xi + i\eta) + o(|\zeta|)$$

which of course proves that $f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z)$ as desired. The second part was already discussed above. $\square$

The following notion is of central importance for all of complex analysis:

Definition 1.5. *A function $f \in C^1(\Omega)$ is called* conformal *if and only if $df \neq 0$ in Ω and df preserves the angle and orientation at each point.*

Thus, the holomorphic functions are precisely those C^1 functions which are conformal at all points at which $df \neq 0$. Note that $f(z) = \bar{z}$ belongs to $C^1(\mathbb{C})$ but is not holomorphic since it reverses orientations. Also note that $f(z) = z^2$ doubles angles at $z = 0$ (in the sense that curves crossing at 0 at angle α get mapped onto curves intersecting at 0 at angle 2α), so conformality is lost there.

A particularly convenient — as well as insightful — way of distinguishing holomorphic functions from C^1 functions is given by the $\partial_z, \partial_{\bar{z}}$ calculus. Assume that $f \in C^1(\Omega)$. Then the real-linear map $df(z)$ can be written as the sum of a complex-linear (meaning that $T(zv) = zT(v)$) and a complex anti-linear transformation (meaning that $T(zv) = \bar{z}T(v)$); see Lemma 6.2 below. In other words, there exist complex numbers $w_1(z), w_2(z)$ such that

$$df(z) = w_1(z)\, dz + w_2(z)\, d\bar{z}$$

where dz is simply the identity map and $d\bar{z}$ the reflection about the real axis followed by multiplication by the complex numbers w_1 and w_2, respectively. We used here that all complex linear transformations on $\mathbb{R}^2$ are given by multiplication by a complex number, whereas the complex anti-linear ones become complex linear by composing them with a reflection. To find w_1 and w_2 simply observe that

$$\begin{aligned} df(x) &= \partial_x f\, dx + \partial_y f\, dy = \partial_x f\, \frac{1}{2}(dz + d\bar{z}) + \partial_y f\, \frac{1}{2i}(dz - d\bar{z}) \\ &= \frac{1}{2}(\partial_x f - i\partial_y f)\, dz + \frac{1}{2}(\partial_x f + i\partial_y f)\, d\bar{z} \\ &=: \partial_z f\, dz + \partial_{\bar{z}} f\, d\bar{z}. \end{aligned}$$

In other words, $f \in \mathcal{H}(\Omega)$ iff $f \in C^1(\Omega)$ and $\partial_{\bar{z}} f = 0$ in Ω.
One can immediately check that $\partial_{\bar{z}} f = 0$ is the same as the Cauchy-Riemann system. As an application of this formalism we record the following crucial fact: for any $f \in \mathcal{H}(\Omega)$,

$$d(f(z)\, dz) = \partial_z f\, dz \wedge dz + \partial_{\bar{z}} f\, d\bar{z} \wedge dz = 0$$

which means that $f(z)\, dz$ is a closed differential form. This property is *equivalent* to the homotopy invariance of the Cauchy integral that we will encounter below. We leave it to the reader to verify the chain rules

$$\begin{aligned} \partial_z(g \circ f) &= \big[(\partial_w g) \circ f\big]\, \partial_z f + \big[(\partial_{\bar{w}} g) \circ f\big]\, \partial_z \bar{f} \\ \partial_{\bar{z}}(g \circ f) &= \big[(\partial_w g) \circ f\big]\, \partial_{\bar{z}} f + \big[(\partial_{\bar{w}} g) \circ f\big]\, \partial_{\bar{z}} \bar{f} \end{aligned} \tag{1.2}$$

as well as the representation of the Laplacean $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. These ideas will be of particular importance once we discuss differential forms on Riemann surfaces.

To continue our introductory chapter, we next turn to the simple but important idea of extending the notion of analyticity to functions that take the value ∞. In a similar vein, we can make sense of functions being analytic at $z = \infty$. To start with, we define the one-point compactification of $\mathbb{C}$, which we denote by $\mathbb{C}_\infty$, with the usual basis of the topology; the neighborhoods of ∞ are the complements of all compact sets. It is intuitively clear that $\mathbb{C}_\infty \simeq S^2$ in the homeomorphic sense. Somewhat deeper as well as much more relevant for complex analysis is the fact that $\mathbb{C} \simeq S^2 \setminus \{p\}$ in the sense of *conformal equivalence* where $p \in S^2$ is arbitrary. This is done via the well-known stereographic projection; see the homework and Chapter 4 below as well as Figure 1.1. If the circle in that figure is the unit circle, $N = (0,1)$, and $X = (x,y)$, then $Z = \frac{x}{1-y}$ as the reader will easily verify using similarity of triangles. This identifies the stereographic projection as the map

$$\Phi : S^2 \setminus \{(0,0,1)\} \to \mathbb{C},\ X = (x_1,x_2,x_3) \mapsto \frac{x_1 + ix_2}{1 - x_3}$$

The stereographic projection preserves angles as well as circles; see Problem 1.4. We will see in Chapter 4 that

$$\mathbb{C}_\infty \simeq S^2 \simeq \mathbb{C}P^1$$

in the sense of conformal equivalences, and each of these Riemann surfaces are called the *Riemann sphere*. Without going into details about the exact definition of a Riemann

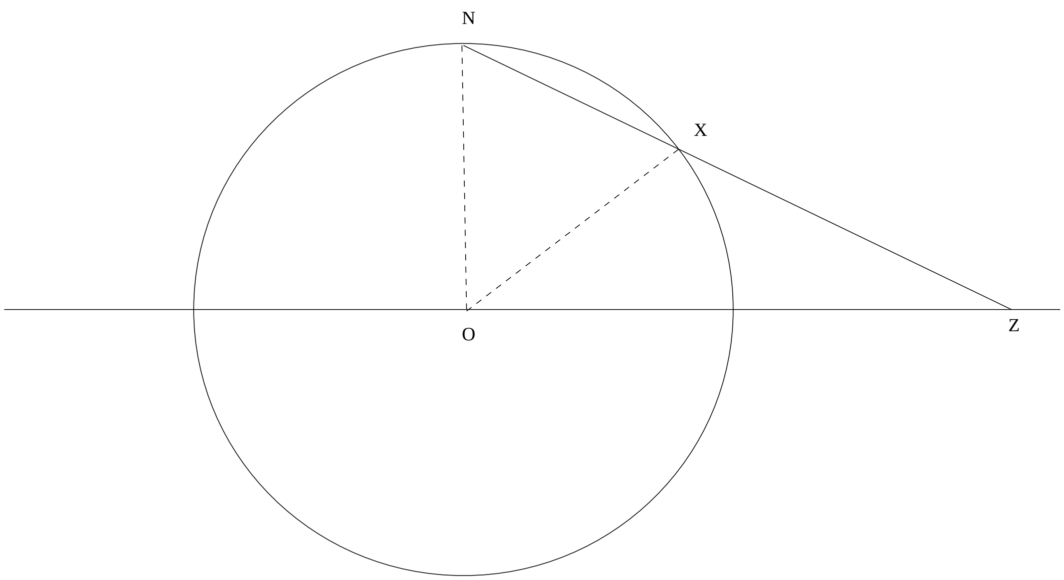

FIGURE 1.1. Stereographic projection

surface, we mention in passing that $\mathbb{C}_\infty$ is covered by two charts, namely $(\mathbb{C}, z)$ and $(\mathbb{C}_\infty \setminus \{0\}, z^{-1})$, both of which are homeomorphisms onto $\mathbb{C}$. On the overlap region $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, the change of charts is given by the map $z \mapsto z^{-1}$, which is of course a conformal equivalence.

It is now clear how to extend the domain and range of holomorphic maps to

$$f : \mathbb{C} \to \mathbb{C}_\infty, \quad f : \mathbb{C}_\infty \to \mathbb{C}, \quad f : \mathbb{C}_\infty \to \mathbb{C}_\infty. \tag{1.3}$$

First, we need to require that f is continuous in each case. This is needed in order to ensure that we can localize f to charts. Second, we require f to be holomorphic relative to the respective charts. For example, if $f(z_0) = \infty$ for some $z_0 \in \mathbb{C}$, then we say that f is holomorphic close to z_0 if and only if $\frac{1}{f(z)}$ is holomorphic around z_0. To make sense of f being analytic at $z = \infty$ with values in $\mathbb{C}$, we simply require that $f(\frac{1}{z})$ is holomorphic around $z = 0$. For the final example in (1.3), if $f(\infty) = \infty$, then f is analytic around $z = \infty$ if and only if $1/f(1/z)$ is analytic around $z = 0$. We remark that $z \mapsto \frac{1}{z}$ is conformal as a map from $C_\infty \to C_\infty$; this is a tautology in view of our choice of chart at $z = \infty$. On the other hand, if we interpret C_∞ as the Riemann sphere then one needs to use here that stereographic projection is a conformal map.

We shall see later in this chapter that the holomorphic maps $f : \mathbb{C}_\infty \to \mathbb{C}$ are constants (indeed, such a map would have to be entire and bounded and therefore constant by Liouville's theorem, see Corollary 1.19 below). On the other hand, the maps $f : \mathbb{C} \to \mathbb{C}_\infty$ are precisely the *meromorphic ones* which we shall encounter in the next chapter. Finally, the holomorphic maps $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ are precisely the rational functions $\frac{P(z)}{Q(z)}$ where P, Q are polynomials. To see this[1], one simply argues that any such f is necessarily meromorphic with only finitely many poles in $\mathbb{C}$ and possibly a pole at $z = \infty$.

3. Möbius transforms

If we now accept that the holomorphic, and thus conformal, maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$ are precisely the rational ones, it is clear how to identify the *conformal automorphisms* (or *automorphisms*) amongst these maps. Indeed, in that case necessarily P and Q both have to be linear which immediately leads to the following definition. Based on the argument of the previous paragraph (which the reader for now can ignore if desired), the lemma identifies all automorphisms of $\mathbb{C}_\infty$.

LEMMA 1.6. *Every $A \in GL(2,\mathbb{C})$ defines a transformation*

$$T_A(z) := \frac{az+b}{cz+d}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which is holomorphic as a map from $\mathbb{C}_\infty \to \mathbb{C}_\infty$. It is called a fractional linear *or* Möbius transformation. *The map $A \mapsto T_A$ only depends on the equivalence class of A under the relation $A \sim B$ iff $A = \lambda B$, $\lambda \in \mathbb{C}^*$. In other words, the family of all Möbius transformations is the same as*

$$PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\{\pm \mathrm{Id}\} \tag{1.4}$$

We have $T_A \circ T_B = T_{A\circ B}$ and $T_A^{-1} = T_{A^{-1}}$. In particular, every Möbius transform is an automorphism of $\mathbb{C}_\infty$.

PROOF. It is clear that each T_A is a holomorphic map $\mathbb{C}_\infty \to \mathbb{C}_\infty$. The composition law $T_A \circ T_B = T_{A\circ B}$ and $T_A^{-1} = T_{A^{-1}}$ are simple computations that we leave to the reader. In particular, T_A has a conformal inverse and is thus an automorphism of $\mathbb{C}_\infty$. If $T_A = T_{\widetilde{A}}$ where $A, \widetilde{A} \in SL(2,\mathbb{C})$, then

$$T_A'(z) = \frac{ad-bc}{(cz+d)^2} = T_{\widetilde{A}}'(z) = \frac{\widetilde{a}\widetilde{d} - \widetilde{b}\widetilde{c}}{(\widetilde{c}z + \widetilde{d})^2}$$

and thus $cz + d = \pm(\widetilde{c}z + \widetilde{d})$ under the assumption that

$$ad - bc = \widetilde{a}\widetilde{d} - \widetilde{b}\widetilde{c} = 1$$

Hence, A and $\widetilde{A}$ are the same matrices in $SL(2,\mathbb{C})$ possibly up to a choice of sign, which establishes (1.4). $\square$

[1]The reader should not be alarmed in case he or she does not follow these arguments – they will become clear once this chapter and the next one have been read.

Fractional linear transformations enjoy many important properties which can be checked separately for each of the following four elementary transformations. In particular, Lemma 1.7 proves that the group $PSL(2, \mathbb{C})$ has four generators.

LEMMA 1.7. *Every Möbius transformation is the composition of four elementary maps:*

- *translations* $z \mapsto z + z_0$
- *dilations* $z \mapsto \lambda z$, $\lambda > 0$
- *rotations* $z \mapsto e^{i\theta} z$, $\theta \in \mathbb{R}$
- *inversion* $z \mapsto \frac{1}{z}$

PROOF. If $c = 0$, then $T_A(z) = \frac{a}{d}z + \frac{b}{d}$. If $c \neq 0$, then

$$T_A(z) = \frac{bc - ad}{c^2} \frac{1}{z + \frac{d}{c}} + \frac{a}{c}$$

and we are done. □

The reader will have no difficulty verifying that $z \mapsto \frac{z-1}{z+1}$ take the right half-plane on the disk $\mathbb{D} := \{|z| < 1\}$. In particular, $i\mathbb{R}$ gets mapped on the unit circle. Similarly, $z \mapsto \frac{2z-1}{2-z}$ takes $\mathbb{D}$ onto itself with the boundary going onto the boundary. If we include all lines into the family of circles (they are circles passing through ∞) then these examples can serve to motivate the following lemma.

LEMMA 1.8. *Fractional linear transformations take circles onto circles.*

PROOF. In view of the previous lemma, the only case requiring an argument is the inversion. Thus, let $|z - z_0| = r$ be a circle and set $w = \frac{1}{z}$. Then

$$\begin{aligned} 0 &= |z|^2 - 2\mathrm{Re}\,(\bar{z}z_0) + |z_0|^2 - r^2 \\ &= \frac{1}{|w|^2} - 2\frac{\mathrm{Re}\,(wz_0)}{|w|^2} + |z_0|^2 - r^2 \end{aligned}$$

If $|z_0| = r$, then one obtains the equation of a line in w. Note that this is precisely the case when the circle passes through the origin. Otherwise, we obtain the equation

$$0 = \left| w - \frac{\bar{z}_0}{|z_0|^2 - r^2} \right|^2 - \frac{r^2}{(|z_0|^2 - r^2)^2}$$

which is a circle. A line is given by an equation

$$2\mathrm{Re}\,(z\bar{z}_0) = a$$

which transforms into $2\mathrm{Re}\,(z_0 w) = a|w|^2$. If $a = 0$, then we simply obtain another line through the origin. Otherwise, we obtain the equation $|w - z_0/a|^2 = |z_0/a|^2$ which is a circle.

An alternative argument uses the fact that stereographic projection preserves circles, see homework problem #4. To use it, note that the inversion $z \mapsto \frac{1}{z}$ corresponds to a rotation of the Riemann sphere about the x_1 axis (the real axis of the plane). Since such a rotation preserves circles, a fractional linear transformation does, too. □

Since $Tz = \frac{az+b}{cz+d} = z$ is a quadratic equation[2] for any Möbius transform T, we see that T can have at most two fixed points unless it is the identity.

[2]Strictly speaking, this is a quadratic equation provided $c \neq 0$; if $c = 0$ one obtains a linear equation with a fixed point in $\mathbb{C}$ and another one at $z = \infty$.

It is also clear that every Möbius transform has at least one fixed point. The map $Tz = z+1$ has exactly one fixed point, namely $z = \infty$, whereas $Tz = \frac{1}{z}$ has two, $z = \pm 1$.

Lemma 1.9. *A fractional linear transformation is determined completely by its action on three distinct points. Moreover, given $z_1, z_2, z_3 \in \mathbb{C}_\infty$ distinct, there exists a unique fractional linear transformation T with $Tz_1 = 0$, $Tz_2 = 1$, $Tz_3 = \infty$.*

Proof. For the first statement, suppose that S, T are Möbius transformations that agree at three distinct points. Then $S^{-1} \circ T$ has three fixed points and is thus the identity. For the second statement, let

$$Tz := \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}$$

in case $z_1, z_2, z_3 \in \mathbb{C}$. If any one of these points is ∞, then we obtain the correct formula by passing to the limit here. □

Definition 1.10. *The cross ratio of four points $z_0, z_1, z_2, z_3 \in \mathbb{C}_\infty$ is defined as*

$$[z_0 : z_1 : z_2 : z_3] := \frac{z_0 - z_1}{z_0 - z_3} \frac{z_2 - z_3}{z_2 - z_1}$$

This concept is most relevant for its relation to Möbius transformations.

Lemma 1.11. *The cross ratio of any four distinct points is preserved under Möbius transformations. Moreover, four distinct points lie on a circle iff their cross ratio is real.*

Proof. Let z_1, z_2, z_3 be distinct and let $Tz_j = w_j$ for T a Möbius transformation and $1 \le j \le 3$. Then for all $z \in \mathbb{C}$,

$$[w : w_1 : w_2 : w_3] = [z : z_1 : z_2 : z_3] \;\text{ provided } w = Tz$$

This follows from the fact that the cross ratio on the left-hand side defines a Möbius transformation $S_1 w$ with the property that $S_1 w_1 = 0, S_1 w_2 = 1, S_1 w_3 = \infty$, whereas the right-hand side defines a transformation S_0 with $S_0 z_1 = 0, S_0 z_2 = 1, S_0 z_3 = \infty$. Hence $S_1^{-1} \circ S_0 = T$ as claimed. The second statement is an immediate consequence of the first and the fact that for any three distinct points $z_1, z_2, z_3 \in \mathbb{R}$, a fourth point z_0 has a real-valued cross ratio with these three iff $z_0 \in \mathbb{R}$. □

We can now define what it means for two points to be symmetric relative to a circle (or line — recall that this is included in the former).

Definition 1.12. *Let $z_1, z_2, z_3 \in \Gamma$ where $\Gamma \subset \mathbb{C}_\infty$ is a circle. We say that z and z^* are symmetric relative to Γ iff*

$$\overline{[z : z_1 : z_2 : z_3]} = [z^* : z_1 : z_2 : z_3]$$

Obviously, if $\Gamma = \mathbb{R}$, then $z^* = \bar{z}$. In other words, if Γ is a line, then z^* is the reflection of z across that line. If Γ is a circle of finite radius, then we can reduce matters to this case by an inversion.

Lemma 1.13. *Let $\Gamma = \{|z - z_0| = r\}$. Then for any $z \in \mathbb{C}_\infty$,*

$$z^* = \frac{r^2}{\bar{z} - \bar{z}_0}$$

PROOF. It suffices to consider the unit circle. Then

$$\overline{[z; z_1; z_2 : z_3]} = [\bar{z} : z_1^{-1} : z_2^{-1} : z_3^{-1}] = [1/\bar{z} : z_1 : z_2 : z_3]$$

In other words, $z^* = \frac{1}{\bar{z}}$. The general case now follows from this via a translation and dilation. □

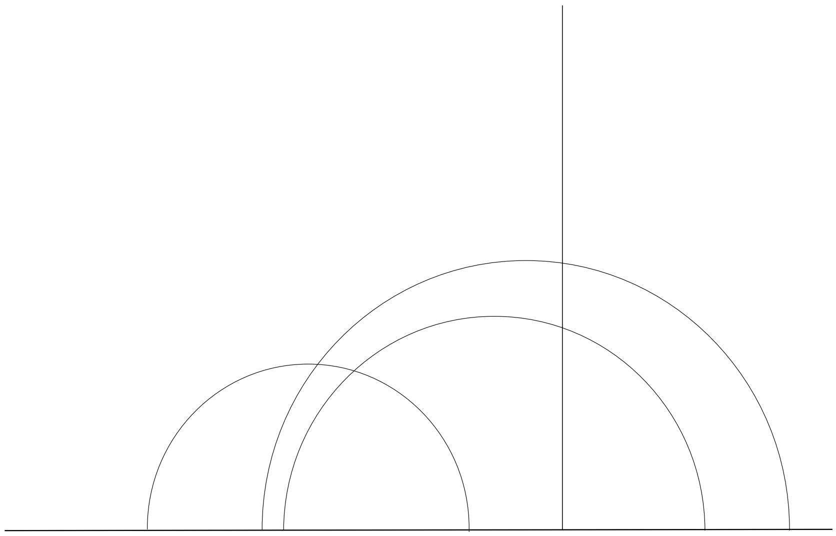

FIGURE 1.2. Geodesics in the hyperbolic plane

Möbius transformations are important for several reasons. We already observed that they are precisely the automorphisms of the Riemann sphere (though to see that every automorphism is a Möbius transformation requires material from this entire chapter as well as the next). In the 19th century there was much excitement surrounding non-Euclidean geometry and there is an important connection between Möbius transforms and hyperbolic geometry: the isometries of the hyperbolic plane $\mathbb{H}$ are precisely those Möbius transforms which preserve it. Let us be more precise. Consider the upper half-plane model of the hyperbolic plane given by

$$\mathbb{H} = \{z \in \mathbb{C} \,:\, \operatorname{Im} z > 0\}, \quad ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{d\bar{z}dz}{(\operatorname{Im} z)^2}$$

It is not hard to see that the subgroup of $PSL(2, \mathbb{C})$ which preserves the upper half-plane is precisely $PSL(2, \mathbb{R})$. Indeed, $z \mapsto \frac{az+b}{cz+d}$ preserves $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ if and only if $a, b, c, d \in \lambda\mathbb{R}$ for some $\lambda \in \mathbb{C}^*$. In other words, the stabilizer of $\mathbb{R}$ (as a set) is $PGL(2, \mathbb{R})$ which contains $PSL(2, \mathbb{R})$ as an index two subgroup. The latter preserves the upper half plane, whereas those matrices with negative determinant interchange the upper with the lower half-plane. It is easy to check (see the home work problems) that $PSL(2, \mathbb{R})$ operates transitively on $\mathbb{H}$ and preserves the metric: for the latter, one simply computes

$$w = \frac{az + b}{cz + d} \Longrightarrow \frac{d\bar{w}\, dw}{(\operatorname{Im} w)^2} = \frac{d\bar{z}\, dz}{(\operatorname{Im} z)^2}$$

In particular, the geodesics are preserved under $PSL(2, \mathbb{R})$. Since the metric does not depend on x it follows that all vertical lines are geodesics. We leave it to the reader to verify that for all $z_0 \in \mathbb{H}$

$$\{T'(z_0) \mid T \in \mathrm{Stab}(z_0)\} = SO(2, \mathbb{R})$$

which means that the stabilizer subgroup in $PSL(2, \mathbb{R})$ at any point z_0 in the upper half plane acts on the tangent space at z_0 by arbitrary rotations. Therefore, the geodesics of $\mathbb{H}$ are precisely all circles which intersect the real line at a right angle (with the vertical lines being counted as circles of infinite radius). From this it is clear that the hyperbolic plane satisfies all axioms of Euclidean geometry with the exception of the parallel axiom: there are many "lines" (i.e., geodesics) passing through a point which is not on a fixed geodesic that do not intersect that geodesic. Let us now prove the famous Gauss-Bonnet theorem which describes the hyperbolic area of a triangle whose three sides are geodesics (those are called geodesic triangles). This is of course a special case of a much more general statement about integrating the Gaussian curvature over a geodesic triangle on a general surface. The reader should prove the analogous statement for spherical triangles.

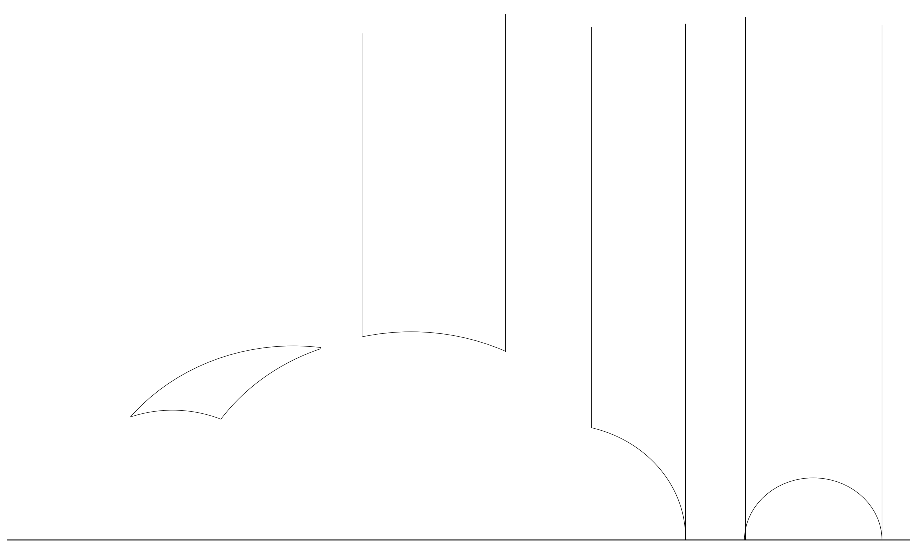

FIGURE 1.3. Geodesic triangles

THEOREM 1.14. *Let T be a geodesic triangle with angles $\alpha_1, \alpha_2, \alpha_3$. Then* $\mathrm{Area}(T) = \pi - (\alpha_1 + \alpha_2 + \alpha_3)$.

PROOF. There are four essentially distinct types of geodesic triangles, depending on how many of its vertices lie on $\mathbb{R}_\infty$. Up to equivalences via transformations in $PSL(2, \mathbb{R})$ (which are isometries and therefore also preserve the area) we see that it suffices to consider precisely those cases described in Figure 1.3. Let us start with the case in which exactly two vertices belong to $\mathbb{R}_\infty$ as shown in that figure (the second triangle from the right). Without loss of generality one vertex coincides with 1, the other with ∞, and the circular arc lies on the unit circle with the projection of the second finite vertex onto the real axis being x_0. Then

$$\begin{aligned}\text{Area}(T) &= \int_{x_0}^{1}\int_{y(x)}^{\infty}\frac{dxdy}{y^2} = \int_{x_0}^{1}\frac{dx}{\sqrt{1-x^2}}\\ &= \int_{\alpha_0}^{0}\frac{d\cos\phi}{\sqrt{1-\cos^2(\phi)}} = \alpha_0 = \pi - \alpha_1\end{aligned}$$

as desired since the other two angles are zero. By additivity of the area we can deal with the other two cases in which at least one vertex is real. We leave the case where no vertex lies on the (extended) real axis to the reader, the idea is to use Figure 1.4. □

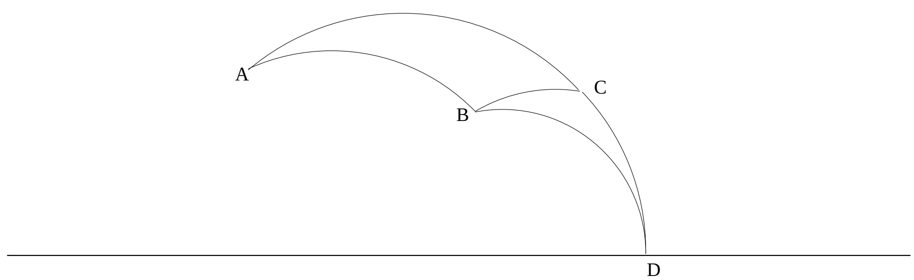

FIGURE 1.4. The case of no real vertex

We leave it to the reader to generalize the Gauss-Bonnet theorem to geodesic polygons. Many interesting questions about Möbius transformations remain, for example how to characterize those that correspond to rotations of the sphere, or how to determine all finite subgroups of $PSL(2,\mathbb{C})$. For some answers see Problem 4.9 as well as [**22**]. A whole topic onto itself are the Fuchsian and Kleinian groups, see for example [**25**]. These groups are of crucial importance for the uniformization theory of Riemann surfaces in the non-simply connected case.

4. Integration

We now develop our complex calculus further. The following definition defines the complex integral canonically in the sense that it is the only definition which preserves the fundamental theorem of calculus for holomorphic functions.

DEFINITION 1.15. *For any C^1-curve $\gamma : [0,1] \to \Omega$ and any compex-valued $f \in C(\Omega)$ we define*

$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt$$

If γ is a closed curve ($\gamma(0) = \gamma(1)$) then we also write $\oint_\gamma f(z)\,dz$ for this integral.

We remark that the integral is $\mathbb{C}$-valued, and that $f(\gamma(t))\gamma'(t)$ is understood as multiplication of complex numbers. From the chain rule, we deduce the fundamental fact that the line integrals of this definition do not depend on any particular C^1 parametrization of the curve as long as the orientation is preserved (hence, there is no loss in assuming that γ is parametrized by $0 \le t \le 1$). Again from the chain rule, we immediately obtain the following: if $f \in \mathcal{H}(\Omega)$, then

$$\int_\gamma f'(z)\,dz = \int_0^1 f'(\gamma(t))\gamma'(t)\,dt = \int_0^1 \frac{d}{dt} f(\gamma(t))\,dt = f(\gamma(1)) - f(\gamma(0))$$

for any γ as in the definition. In particular,

$$\oint_\gamma f'(z)\,dz = 0 \qquad \forall \text{ closed curves } \gamma \text{ in } \Omega$$

On the other hand, let us compute with $\gamma_r(t) := re^{it}$, $r > 0$,

$$\oint_{\gamma_r} z^n\,dz = \int_0^{2\pi} r^n e^{int}\, rie^{it}\,dt = \begin{cases} 0 & n \ne -1 \\ 2\pi i & n = -1 \end{cases} \tag{1.5}$$

In $\Omega = \mathbb{C}^*$, the function $f(z) = z^n$ has the primitive $F_n(z) = \frac{z^{n+1}}{n+1}$ provided $n \ne -1$. This explains why we obtain 0 for all $n \ne -1$. On the other hand, if $n = -1$ we realize from our calculation that $\frac{1}{z}$ *does not* have a (holomorphic) primitive in $\mathbb{C}^*$. This issue merits further investigation (for example, we need to answer the question whether $\frac{1}{z}$ has a *local primitive* in $\mathbb{C}^*$ — this is indeed the case and this primitive is a branch of $\log z$). Before doing so, however, we record Cauchy's famous theorem in its homotopy version. Figure 1.5 shows two curves, namely γ_1 and γ_2, which are homotopic within the

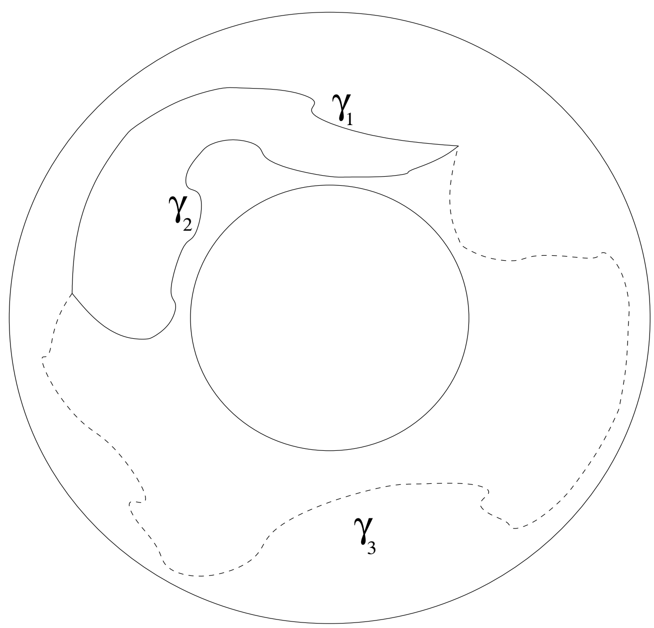

FIGURE 1.5. Homotopy

annular region they lie in. The dashed curve is not homotopic to either of them within the annulus.

THEOREM 1.16. *Let $\gamma_0, \gamma_1 : [0,1] \to \Omega$ be C^1 curves*[3] *with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ (the fixed endpoint case) or $\gamma_0(0) = \gamma_0(1)$, $\gamma_1(0) = \gamma_1(1)$ (the closed case). Assume that they are C^1-homotopic*[4] *in the following sense: there exists a continuous map $H : [0,1]^2 \to \Omega$ with $H(t,0) = \gamma_0(t)$, $H(t,1) = \gamma_1(t)$ and such that $H(\cdot, s)$ is a C^1 curve for each $0 \le s \le 1$. Moreover, in the fixed endpoint case we assume that H freezes the endpoints, whereas in the closed case we assume that each curve from the homotopy is closed. Then*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$$

for all $f \in \mathcal{H}(\Omega)$. In particular, if γ is a closed curve in Ω which is homotopic to a point, then

$$\oint_{\gamma} f(z)\, dz = 0$$

PROOF. We first note the important fact that $f(z)\, dz$ is a closed form. Indeed,

$$d(f(z)\, dz) = \partial_z f(z)\, dz \wedge dz + \partial_{\bar{z}} f(z)\, d\bar{z} \wedge dz = 0$$

by the Cauchy-Riemann equation $\partial_{\bar{z}} f = 0$. Thus, Cauchy's theorem is a special case of the homotopy invariance of the integral over closed forms which in turn follows from Stokes's theorem. Let us briefly recall the details: since a closed form is locally exact, we first note that

$$\oint_{\eta} f(z)\, dz = 0$$

for all closed curves η which fall into sufficiently small disks, say. But then we can triangulate the homotopy so that

$$\int_{\gamma_0} f(z)\, dz - \int_{\gamma_1} f(z)\, dz = \sum_j \oint_{\eta_j} f(z)\, dz = 0$$

where the sum is over a finite collection of small loops which constitute the triangulation of the homotopy H. The more classically minded reader may prefer to use Green's formula (which of course follows from the Stokes theorem): provided $U \subset \Omega$ is a sufficiently small neighborhood which is diffeomorphic to a disk, say, one can write

$$\begin{aligned}\oint_{\partial U} f(z)\, dz &= \oint_{\partial U} u\, dx - v\, dy + i(u\, dy + v\, dx) \\ &= \iint_U (-u_y - v_x)\, dxdy + i \iint_U (-v_y + u_x)\, dxdy = 0\end{aligned}$$

where the final equality sign follows from the Cauchy–Riemann equations. □

This theorem is typically applied to very simple configurations, such as two circles which are homotopic to each other in the region of holomorphy of some function f. As an example, we now derive the following fundamental fact of complex analysis which is intimately tied up with the $n = -1$ case of (1.5).

[3] This can be relaxed to piece-wise C^1, which means that we can write the curve as a finite concatenation of C^1 curves. The same comment applies to the homotopy.

[4] In light of commonly used terminology it is probably best to refer to this as *homotopic through C^1 curves* but for simplicity, we shall continue to abuse terminology and use C^1 homotopic.

Proposition 1.17. *Let $\overline{D(z_0, r)} \subset \Omega$ and $f \in \mathcal{H}(\Omega)$. Then*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{where } \gamma(t) = z_0 + re^{it} \tag{1.6}$$

for all $z \in D(z_0, r)$.

Proof. Fix any $z \in D(z_0, r)$ and apply Theorem 1.16 to the region $U_\varepsilon := D(z_0, r) \setminus D(z, \varepsilon)$ where $\varepsilon > 0$ is small. We use here that the two boundary circles of U_ε are homotopic to each other relative to the region Ω. Then

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{\partial U_\varepsilon} \frac{f(\zeta)}{z - \zeta}\, d\zeta = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(\zeta)}{z - \zeta}\, d\zeta \\ &\quad - \frac{1}{2\pi i} \int_{\partial D(z,\varepsilon)} \frac{f(\zeta) - f(z)}{z - \zeta}\, d\zeta - \frac{f(z)}{2\pi i} \int_{\partial D(z,\varepsilon)} \frac{1}{z - \zeta}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(\zeta)}{z - \zeta}\, d\zeta + O(\varepsilon) - f(z) \quad \text{as } \varepsilon \to 0 \end{aligned}$$

where we used the $n = -1$ case of (1.5) to pass to the third term of the last line. □

We can now derive the astonishing fact that holomorphic functions are in fact analytic. This is done by noting that the integrand in (1.6) is analytic relative to z.

Corollary 1.18. *$\mathcal{A}(\Omega) = \mathcal{H}(\Omega)$. In fact, every $f \in \mathcal{H}(\Omega)$ is represented by a convergent power series on $D(z_0, r)$ where $r = \operatorname{dist}(z_0, \partial\Omega)$.*

Proof. We have already observed that analytic functions are holomorphic. For the converse, we use the previous proposition to conclude that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z_0 - (z - z_0)}\, d\zeta \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^\infty \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n d\zeta \\ &= \sum_{n=0}^\infty \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\ (z - z_0)^n \end{aligned}$$

where the interchange of summation and integration is justified due to uniform and absolute convergence of the series. Thus, we obtain that f is analytic and, moreover,

$$f(z) = \sum_{n=0}^\infty \frac{f^{(n)}}{n!} (z - z_0)^n$$

converges on $|z - z_0| < \operatorname{dist}(z_0, \partial\Omega)$ with

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \tag{1.7}$$

for any $n \geq 0$. □

In contrast to power series over $\mathbb{R}$, over $\mathbb{C}$ there is an explanation for the radius of convergence: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ has finite and positive radius of convergence R iff $f \notin \mathcal{H}(\Omega)$ for every Ω which compactly contains $D(z_0, R)$. We immediately obtain a number of corollaries from this.

Corollary 1.19. *(a) Cauchy's estimates: Let $f \in \mathcal{H}(\Omega)$ with $|f(z)| \le M$ on Ω. Then*

$$|f^{(n)}(z)| \le \frac{Mn!}{\operatorname{dist}(z, \partial\Omega)^n}$$

for every $n \ge 0$ and all $z \in \Omega$.

(b) Liouville's theorem: If $f \in \mathcal{H}(\mathbb{C}) \cap L^\infty(\mathbb{C})$, then f is constant. More generally, if $|f(z)| \le C(1 + |z|^N)$ for all $z \in \mathbb{C}$, for some fixed integer $N \ge 0$ and a finite constant C, then f is polynomial of degree at most N.

Proof. (a) follows by putting absolute values inside (1.7). For (b) apply (a) to $\Omega = D(0, R)$ and let $R \to \infty$. This shows that $f^{(k)} \equiv 0$ for all $k > N$. $\square$

Part (b) has a famous consequence, namely the *fundamental theorem of algebra*.

Proposition 1.20. *Every $P \in \mathbb{C}[z]$ of positive degree has a complex zero; in fact it has exactly as many zeros over $\mathbb{C}$ (counted with multiplicity) as its degree.*

Proof. Suppose $P(z) \in \mathbb{C}[z]$ is a polynomial of positive degree and without zero in $\mathbb{C}$. Then $f(z) := \frac{1}{P(z)} \in \mathcal{H}(\mathbb{C})$ and since $|P(z)| \to \infty$ as $|z| \to \infty$, f is evidently bounded. Hence $f =$ const and so $P =$ const contrary to the assumption of positive degree. So $P(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Factoring out $z - z_0$ we conclude inductively that P has exactly $\deg(P)$ many complex zeros as desired. $\square$

Next, we show how Theorem 1.16 allows us to define local primitives. In particular, we can clarify the characterization of the logarithm as the local primitive of $\frac{1}{z}$.

Proposition 1.21. *Let Ω be simply connected. Then for every $f \in \mathcal{H}(\Omega)$ so that $f \neq 0$ everywhere on Ω there exists $g \in \mathcal{H}(\Omega)$ with $e^{g(z)} = f(z)$. Thus, for any $n \ge 1$ there exists $f_n \in \mathcal{H}(\Omega)$ with $(f_n(z))^n = f(z)$ for all $z \in \Omega$. In particular, if $\Omega \subset \mathbb{C}^*$ is simply connected, then there exists $g \in \mathcal{H}(\Omega)$ with $e^{g(z)} = z$ everywhere on Ω. Such a g is called* a branch of $\log z$. *Similarly, there exist holomorphic branches of any $\sqrt[n]{z}$ on Ω, $n \ge 1$.*

Proof. If $e^g = f$, then $g' = \frac{f'}{f}$ in Ω. So fix any $z_0 \in \Omega$ and define

$$g(z) := \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta$$

where the integration path joins z_0 to z and consists a finite number of line segments (say). We claim that $g(z)$ does not depend on the choice of path. First note that $\frac{f'}{f} \in \mathcal{H}(\Omega)$ due to analyticity and nonvanishing of f. Second, by the simple connectivity of Ω, any two curves with coinciding initial and terminal points are homotopic to each other via a piece-wise C^1 homotopy. Thus, Theorem 1.16 yields the desired equality of the integrals. It is now an easy matter to check that $g'(z) = \frac{f'(z)}{f(z)}$. Indeed,

$$\frac{g(z+h) - g(z)}{h} = \int_0^1 \frac{f'(z+th)}{f(z+th)}\, dt \to \frac{f'(z)}{f(z)} \quad \text{as } h \to 0$$

So $g \in \mathcal{H}(\Omega)$ and $(fe^{-g})' \equiv 0$ shows that $e^g = cf$ where c is some constant different from zero and therefore $c = e^k$ for some $k \in \mathbb{C}$. Hence, $e^{g(z)-k} = f(z)$ for all $z \in \Omega$ and we are done. $\square$

Throughout, for any disk D, the punctured disk D^* denotes D with its center removed. The first part of the following result is known as the *uniqueness theorem*. The name derives from the fact that two functions $f, g \in \mathcal{H}(\Omega)$ are identical if $\{z \in \Omega : f(z) = g(z)\}$ has an accumulation point in Ω. The second part establishes a *normal form* for nonconstant analytic functions locally around any point, see (1.8). Amongst other things, this yields the *open mapping theorem* for analytic functions.

COROLLARY 1.22. *Let $f \in \mathcal{H}(\Omega)$. Then the following are equivalent:*

- $f \equiv 0$
- *for some $z_0 \in \Omega$, $f^{(n)}(z_0) = 0$ for all $n \geq 0$*
- *the set $\{z \in \Omega \mid f(z) = 0\}$ has an accumulation point in Ω*

Assume that f is not constant. Then at every point $z_0 \in \Omega$ there exist a positive integer n and a holomorphic function h locally at z_0 such that

$$f(z) = f(z_0) + [(z - z_0)h(z)]^n, \qquad h(z_0) \neq 0 \tag{1.8}$$

In particular, there are disks $D(z_0, \rho), D(f(z_0), r)$ with the property that every $w \in D(f(z_0), r)^*$ has precisely n pre-images under f in $D(z_0, \rho)^*$. If $f'(z_0) \neq 0$, then f is a local C^∞ diffeomorphism. Finally, every nonconstant holomorphic map is an open map (i.e., it takes open sets to open sets).

PROOF. Let $z_n \to z_0 \in \Omega$ as $n \to \infty$, where $f(z_n) = 0$ for all $n \geq 1$. Suppose $f^{(m)}(z_0) \neq 0$ for some $m \geq 0$. Then

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = a_N (z - z_0)^N (1 + O(z - z_0)) \quad \text{as } z \to z_0$$

locally around z_0 where $N \geq 0$ is minimal with the property that $a_N \neq 0$. But then it is clear that f does not vanish on some disk $D(z_0, r)^*$, contrary to assumption. Thus, $f^{(n)}(z_0) = 0$ for all $n \geq 0$ and thus $f \equiv 0$ locally around z_0. Since Ω is connected, it then follows that $f \equiv 0$ on Ω. This settles the equivalencies. If f' does not vanish identically, let us first assume that $f'(z_0) \neq 0$. We claim that *locally* around z_0, the map $f(z)$ is a C^∞ diffeomorphism from a neighborhood of z_0 onto a neighborhood of $f(z_0)$ and, moreover, that the inverse map to f is also holomorphic. Indeed, in view of Theorem 1.4, the differential *df* is invertible at z_0. Hence, by the usual inverse function theorem we obtain the statement about diffeomorphisms. Furthermore, since *df* is conformal locally around z_0, its inverse is, too and so f^{-1} is conformal and thus holomorphic. If $f'(z_0) = 0$, then there exists some positive integer n with $f^{(n)}(z_0) \neq 0$. But then

$$f(z) = f(z_0) + (z - z_0)^n g(z)$$

with $g \in \mathcal{H}(\Omega)$ satisfying $g(z_0) \neq 0$. By Proposition 1.21 we can write $g(z) = (h(z))^n$ for some $h \in \mathcal{H}(U)$ where U is a neighborhood of z_0 and $h(z_0) \neq 0$, whence (1.8). Figure 1.6 shows that case of $n = 8$. The dots symbolize the eight pre-images of some point. Finally, by the preceding analysis of the $n = 1$ case we conclude that $(z - z_0)g(z)$ is a local diffeomorphism which implies that f has the stated n-to-one mapping property. The openness is now also evident. $\square$

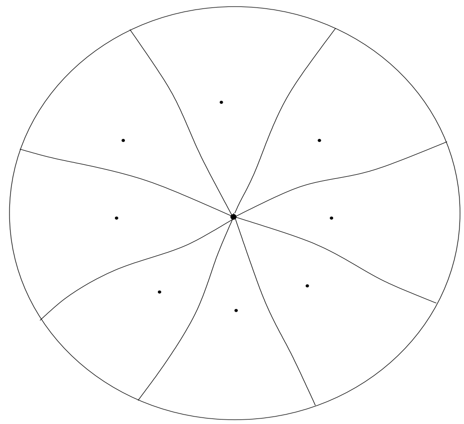

FIGURE 1.6. A branch point

We remark that any point $z_0 \in \Omega$ for which $n \geq 2$ is called a *branch point*. The branch points are precisely the zeros of f' in Ω and therefore form a discrete subset of Ω. The open mapping part of Corollary 1.22 has an important implication known as the *maximum principle*.

COROLLARY 1.23. *Let $f \in \mathcal{H}(\Omega)$. If there exists $z_0 \in \Omega$ with $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$, then $f = \text{const}$.*

PROOF. If f is not constant, then $f(\Omega)$ is open contradicting that $f(z_0) \in \partial f(\Omega)$, which is required by $|f(z)| \leq |f(z_0)|$ on Ω. $\square$

The maximum principle has numerous important applications as well as variants and extensions. In Problem 1.11, we present the simple but powerful Schwarz lemma as an application, whereas for such extensions as the three lines and circle theorems, as well as the Phragmen-Lindelöf theorems we refer the reader to the classical literature, see [**29**] and [**39**], as well as [**37**] (in fact, a version of the Phragmen-Lindelöf principle is discussed in Problem 3.6).

To conclude this chapter, we present Morera's theorem (a kind of converse to Cauchy's theorem) and (conjugate) harmonic functions. The latter is of central importance tp complex analysis and Riemann surfaces. We begin with Morera's theorem.

THEOREM 1.24. *Let $f \in C(\Omega)$ and suppose $\mathcal{T}$ is a collection of triangles in Ω which contains all sufficiently small triangles[5] in Ω. If*

$$\oint_{\partial T} f(z)\, dz = 0 \qquad \forall\, T \in \mathcal{T}$$

then $f \in \mathcal{H}(\Omega)$.

[5]This means that every point in Ω has a neighborhood in Ω so that all triangles which lie inside that neighborhood belong to $\mathcal{T}$

PROOF. The idea is simply to find a local holomorphic primitive of f. Thus, assume $D(0,r) \subset \Omega$ is a small disk and set

$$F(z) := \int_0^z f(\zeta)\, d\zeta = z \int_0^1 f(tz)\, dt$$

for all $|z| < r$. Then by our assumption, for $|z| < r$ and h small,

$$\frac{F(z+h) - F(z)}{h} = \int_0^1 f(z+ht)\, dt \to f(z)$$

as $h \to 0$. This shows that $F \in \mathcal{H}(D(0,r))$ and therefore also $F' = f \in \mathcal{H}(D(0,r))$. Hence $f \in \mathcal{H}(\Omega)$ as desired. $\square$

Next, we introduce harmonic functions.

5. Harmonic functions

DEFINITION 1.25. *A function $u : \Omega \to \mathbb{C}$ is called harmonic iff $u \in C^2(\Omega)$ and $\Delta u = 0$.*

Typically, harmonic functions are taken to be real-valued but there is no need to make this restriction in general. The following result explains the ubiquity of harmonic functions in complex analysis.

PROPOSITION 1.26. *If $f \in \mathcal{H}(\Omega)$, then $\mathrm{Re}\,(u), \mathrm{Im}\,(v)$ are harmonic in Ω.*

PROOF. First, $u := \mathrm{Re}\,(f), v := \mathrm{Im}\,(f) \in C^\infty(\Omega)$ by analyticity of f. Second, by the Cauchy–Riemann equations,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0, \qquad v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$$

as claimed. $\square$

This motivates the following definition.

DEFINITION 1.27. *Let u be harmonic on Ω and real–valued. We say that v is the harmonic conjugate of u iff v is harmonic and real–valued on Ω and $u + iv \in \mathcal{H}(\Omega)$.*

Let us first note that a harmonic conjugate, if it exists, is unique up to constants: indeed, if not, then we would have a real-valued harmonic function v on Ω so that $iv \in \mathcal{H}(\Omega)$. But from the Cauchy–Riemann equations we would then conclude that $\nabla v = 0$ or $v = \text{const}$ by connectedness of Ω.

This definition of course presents us with the question whether every harmonic function on a region of $\mathbb{R}^2$ has a harmonic conjugate function. The classical example for the failure of this is $u(z) = \log|z|$ on $\mathbb{C}^*$; the unique harmonic conjugate v with $v(1) = 0$ would have to be the polar angle which is not defined on $\mathbb{C}^*$. However, in view of Proposition 1.21 it is defined on harmonic on every simply connected subdomain of $\mathbb{C}^*$. As the following proposition explains, this is a general fact.

PROPOSITION 1.28. *Let Ω be simply connected and u real-valued and harmonic on Ω. Then $u = \mathrm{Re}\,(f)$ for some $f \in \mathcal{H}(\Omega)$ and f is unique up to an additive imaginary constant.*

PROOF. We already established the uniqueness property. To obtain existence, we need to solve the Cauchy–Riemann system. In other words, we need to find a potential v to the vector field $(-u_y, u_x)$ on Ω, i.e., $\nabla v = (-u_y, u_x)$. If v exists, then it is $C^2(\Omega)$ and

$$\Delta v = -u_{yx} + u_{xy} = 0$$

hence v is harmonic. Define

$$v(z) := \int_{z_0}^{z} -u_y\, dx + u_x\, dy$$

where the line integral is along a curve connecting z_0 to z which consists of finitely many line segments, say. If γ is a closed curve of this type in Ω, then by Green's theorem,

$$\oint_\gamma -u_y\, dx + u_x\, dy = \iint_U (u_{yy} + u_{xx})\, dxdy = 0$$

where $\partial U = \gamma$ (this requires Ω to be simply connected). So the line integral defining v does not depend on the choice of curve and v is therefore well-defined on Ω. Furthermore, as usual one can check that $\nabla v = (-u_y, u_x)$ as desired. A quick but less self-contained proof is as follows: the differential form

$$\omega := -u_y\, dx + u_x\, dy$$

is closed since $d\omega = \Delta u\, dx \wedge dy = 0$. Hence, it is locally exact and by simple connectivity of Ω, exact on all of Ω. In other words, $\omega = dv$ for some smooth function v on Ω as desired. $\square$

From this, we can easily draw several conclusions about harmonic functions. We begin with the important observation that a conformal change of coordinates preserves harmonic functions.

COROLLARY 1.29. *Let u be harmonic in Ω and $f : \Omega_0 \to \Omega$ holomorphic. Then $u \circ f$ is harmonic in Ω_0.*

PROOF. Locally around every point of Ω there is v so that $u+iv$ is holomorphic. Since the composition of holomorphic functions is again holomorphic, the statement follows. There is of course a direct way of checking this: since $\Delta = 4\partial_z\partial_{\bar z}$ one has from the chain rule (1.2)

$$\partial_z(u \circ f) = (\partial_w u) \circ f\, \partial_z f + (\partial_{\bar w} u) \circ f\, \overline{\partial_{\bar z} f} = (\partial_w u) \circ f\, f'$$

and thus furthermore

$$\Delta(u \circ f) = 4\partial_{\bar z}\partial_z(u \circ f) = 4(\partial_{\bar w}\partial_w u) \circ f |f'|^2 = |f'|^2\, (\Delta u) \circ f$$

whence the result. $\square$

Next, we describe the well-known mean value and maximum properties of harmonic functions. We can motivate them in two ways: first, they are obvious for the one-dimensional case since then harmonic functions on an interval are simply the linear ones; second, in the discrete setting (i.e., on the lattice $\mathbb{Z}^2$ and similarly on any higher-dimensional lattice), the harmonic functions $u : \mathbb{Z}^2 \to \mathbb{R}$ are characterized by

$$u(n) = \frac{1}{4} \sum_{|n-m|=1} u(m)$$

where the sum is over the four nearest neighbors (thus, $|\cdot|$ is the $\ell^1(\mathbb{Z}^2)$ metric). The reader will easily verify that this implies

$$u(n) = \frac{1}{8}\sum_{|n-m|=2} u(m) = \frac{1}{16}\sum_{|n-m|=3} u(m)$$

and so forth. In other words, the mean value property over the nearest neighbors extends to larger ℓ^1 balls.

COROLLARY 1.30. *Let u be harmonic on Ω. Then $u \in C^\infty(\Omega)$, u satisfies the mean-value property*

$$u(z_0) = \int_0^1 u(z_0 + re^{2\pi it})\,dt \qquad \forall\, r < \operatorname{dist}(z_0, \partial\Omega) \tag{1.9}$$

and u obeys the maximum principle: if u attains a local maximum or minimum in Ω, then u is constant. In particular, if Ω is bounded and $u \in C(\bar{\Omega})$, then

$$\min_{\partial\Omega} u \le u(z) \le \max_{\partial\Omega} u \qquad \forall\, z \in \Omega$$

where equality can be attained only if u is constant.

PROOF. Let $U \subset \Omega$ be simply connected, say a disk. By Proposition 1.28, $u = \operatorname{Re}(f)$ where $f \in \mathcal{H}(U)$. Since $f \in C^\infty(U)$, so is u. Moreover,

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz = \int_0^1 f(z_0 + re^{2\pi it})\,dt$$

Passing to the real part proves (1.9). For the maximum principle, suppose that u attains a local extremum on some disk in Ω. Then it follows from (1.9) that u has to be constant on that disk. Since any two points in Ω are contained in a simply connected subdomain of Ω, we conclude from the existence of conjugate harmonic functions on simply connected domains as well as the uniqueness theorem for analytic functions that u is globally constant. ☐

It is not too surprising that both these properties by themselves, i.e., the mean value property as well as the maximum property, already characterize harmonic functions.

6. The winding number

Let us apply the procedure of the proof of Proposition 1.28 to $u(z) = \log|z|$ on $\mathbb{C}^*$. Then, with $\gamma_r(t) = re^{it}$,

$$\oint_{\gamma_r} -u_y\,dx + u_x\,dy = \int_0^{2\pi} (\sin^2(t) + \cos^2(t))\,dt = 2\pi.$$

This is essentially the same calculation as (1.5) with $n = -1$. Indeed, on the one hand, the differential form

$$\omega = -\frac{y}{r^2}\,dx + \frac{x}{r^2}\,dy$$

pulls back to any circle as the form $d\theta$ — this of course explains the appearance of 2π. On the other hand, the local primitive of $\frac{1}{z}$ is (any branch of) $\log z$. So integrating over a loop that encircles the origin once, we create a jump by 2π. On the one hand, this property shows that $\log|z|$ does not have a conjugate harmonic function on $\mathbb{C}^*$ and on the other, it motivates the following definition.

LEMMA 1.31. *Let $\gamma : [0,1] \to \mathbb{C}$ be a closed curve. Then for any $z_0 \in \mathbb{C} \setminus \gamma([0,1])$ the integral*

$$n(\gamma; z_0) := \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0}$$

is an integer. It is called the index *or* winding number *of γ relative to z_0. It is constant on each component of $\mathbb{C} \setminus \gamma([0,1])$ and vanishes on the unbounded component.*

PROOF. Let

$$g(t) := \int_0^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds.$$

The integrand equals $\frac{d}{ds}\log(\gamma(s) - z_0)$ for an arbitrary branch of log. In fact,

$$\frac{d}{dt}\Big(e^{-g(t)}(\gamma(t) - z_0)\Big) = -e^{-g(t)}\gamma'(t) + e^{-g(t)}\gamma'(t) = 0$$

which implies that

$$e^{-g(1)}(\gamma(1) - z_0) = e^{-g(0)}(\gamma(0) - z_0) = e^{-g(0)}(\gamma(1) - z_0)$$

and thus $e^{g(0)-g(1)} = 1$; in other words, $g(0) - g(1) \in 2\pi i\mathbb{Z}$ as claimed. To establish the constancy on the components, observe that

$$\frac{d}{dz_0} \oint_\gamma \frac{dz}{z - z_0} = \oint_\gamma \frac{dz}{(z - z_0)^2} = -\oint_\gamma \frac{d}{dz}\Big[\frac{1}{z - z_0}\Big]\, dz = 0.$$

Finally, on the unbounded component we can let $z_0 \to \infty$ to see that the index vanishes. $\square$

This carries over to cycles of the form $c = \sum_{j=1}^J n_j \gamma_j$ where each γ_j is a closed curve and $n_j \in \mathbb{Z}$. If $n_j < 0$, then $n_j\gamma_j$ means that we take $|n_j|$ copies of γ_j with the opposite orientation. The index of a cycle c relative to a point z_0 not on the cycle is simply

$$n(c; z_0) := \sum_{j=1}^J n_j\, n(\gamma_j; z_0).$$

Observe that

$$n(c; z_0) = \frac{1}{2\pi i}\oint_c d\log(\zeta - z_0) = \frac{1}{2\pi}\oint_c d\theta_{z_0}$$

where θ_{z_0} is the argument relative to the point z_0. The real part of $\log(\zeta - z_0)$ does not contribute since c is made up of closed curves. The differential form (we set $z_0 = 0$)

$$d\theta_0 = -\frac{y}{r^2}\, dx + \frac{x}{r^2}\, dy$$

is closed but not exact. In fact, it is essentially the only form with this property in the domain $\mathbb{C}^* = \mathbb{R}^2 \setminus \{0\}$. To understand this, note that a closed form ω on a domain Ω is exact if and only if

$$\oint_c \omega = 0 \qquad \forall \text{ closed curves } c \subset \Omega. \tag{1.10}$$

Indeed, it is clearly necessary; for the sufficiency set

$$f(z) := \int_{z_0}^z \omega$$

where the integral is along an arbitrary path in Ω connecting z_0 to z. It is well-defined due to the vanishing condition (1.10) and satisfies $df = \omega$. Now let ω be an arbitrary closed form on $\mathbb{C}^*$ and set

$$\widetilde{\omega} := \omega - \frac{\lambda}{2\pi} d\theta_0, \quad \lambda := \oint_{[|z|=1]} \omega.$$

Then $d\widetilde{\omega} = 0$ and (1.10) holds due to the homotopy invariance of integrals of closed forms (Stokes's theorem). Finally, this implies that the map $\omega \mapsto \lambda$ is one-to-one on the space

$$\mathcal{H}^1(\mathbb{C}^*) := \frac{\text{closed forms}}{\text{exact forms}},$$

so we have established the well-known fact that $\mathcal{H}^1(\mathbb{C}^*) \simeq \mathbb{R}$. Incidentally, it also follows that

$$[c] \mapsto n(c; 0), \qquad [c] \in \pi_1(\mathbb{C}^*) \tag{1.11}$$

is an isomorphism of the fundamental group onto $\mathbb{Z}$ (for this, the cycles c have to be rooted at some fixed base-point), see Problem 1.16.

Let us repeat this analysis on the space $X := \mathbb{R}^2 \setminus \{z_j\}_{j=1}^k$ where $z_j \in \mathbb{C}$ are distinct and $k \geq 2$. As before, let ω on X be a closed form and set

$$\widetilde{\omega} := \omega - \sum_{j=1}^k \frac{\lambda_j}{2\pi} d\theta_{z_j}, \quad \lambda_j = \oint_{[|z-z_j|=\varepsilon_j]} \omega$$

where $\varepsilon_j > 0$ is so small that the disks $D(z_j, \varepsilon_j)$ are all disjoint. Then we again conclude

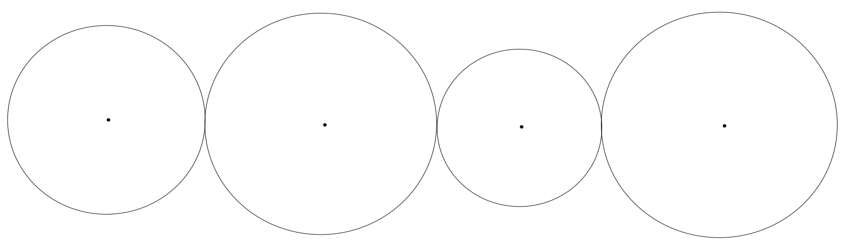

FIGURE 1.7. Bouquet of circles

that (1.10) holds and thus that $\widetilde{\omega}$ is exact. Since the map

$$\omega \mapsto \{\lambda_j\}_{j=1}^k$$

is a linear map from all closed forms on X onto $\mathbb{R}^k$ with kernel equal to the exact forms, we have recovered the well-known fact

$$\mathcal{H}^1(X) \simeq \mathbb{R}^k.$$

We note that any closed curve in X is homotopic to a "bouquet of circles", see Figure 1.7; more formally, up to homotopy, it can be written as a word

$$a_{i_1}^{\nu_1} a_{i_2}^{\nu_2} a_{i_3}^{\nu_3} \dots a_{i_m}^{\nu_m}$$

where $i_\ell \in \{1, 2, \dots, k\}$, $\nu_\ell \in \mathbb{Z}$, and a_ℓ are circles around z_ℓ with a fixed orientation. It is an exercise in algebraic topology to prove from this that $\pi_1(X) = \langle a_1, \dots, a_m \rangle$, the free group with m generators (use van Kampen's theorem). Finally, the map

$$[c] \mapsto \{n(c; z_j)\}_{j=1}^m, \quad \pi_1(X) \to \mathbb{Z}^m$$

is a surjective homomorphism, but is not one-to-one; the kernel consists of all curves with winding number zero around each point. See Figure 1.8 for an example with $k = 2$.

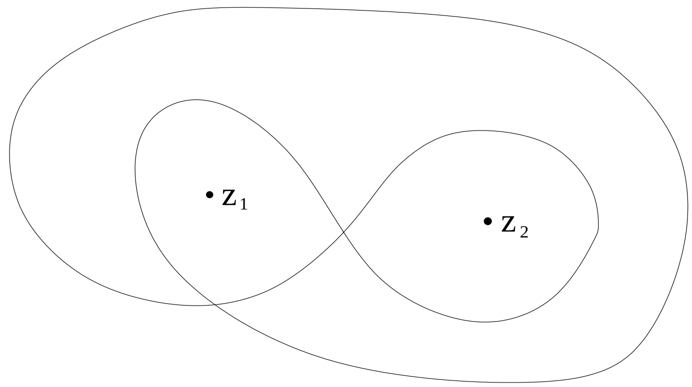

FIGURE 1.8. Zero homologous but not homotopic to a point

7. Problems

PROBLEM 1.1. *(a) Let $a, b \in \mathbb{C}$ and $k > 0$. Describe the set of points $z \in \mathbb{C}$ which satisfy*

$$|z - a| + |z - b| \leq k.$$

(b) Let $|a| < 1$, $a \in \mathbb{C}$. The plane $\{z \in \mathbb{C}\}$ is divided into three subsets according to whether

$$w = \frac{z - a}{1 - \bar{a}z}$$

satisfies $|w| < 1$, $|w| = 1$, or $|w| > 1$. Describe these sets (in terms of z).

PROBLEM 1.2. *Let $P(z) = \sum_{j=0}^{n} a_j\, z^j$ be a polynomial of degree $n \geq 1$ with all roots inside the unit circle $|z| < 1$. Define $P^*(z) = z^n \bar{P}(z^{-1})$ where $\bar{P}(z) = \sum_{j=0}^{n} \bar{a}_j\, z^j$. Show that all roots of*

$$P(z) + P^*(z) = 0$$

lie on the unit circle $|z| = 1$. Do the same for $P(z) + e^{i\theta} P^*(z) = 0$, with $\theta \in \mathbb{R}$ arbitrary.

PROBLEM 1.3. *Suppose $p_0 > p_1 > p_2 > \cdots > p_n > 0$. Prove that all zeros of the polynomial $P(z) = \sum_{j=0}^{n} p_j\, z^j$ lie in $\{|z| > 1\}$.*

PROBLEM 1.4. *Let $\Phi : S^2 \to \mathbb{C}_\infty$ be the stereographic projection $(x_1, x_2, x_3) \mapsto \frac{x_1 + ix_2}{1 - x_3}$. (a) Give a detailed proof that Φ is conformal. (b) Define a metric $d(z, w)$ on $\mathbb{C}_\infty$ as the Euclidean distance of $\Phi^{-1}(z)$ and $\Phi^{-1}(w)$ in $\mathbb{R}^3$. Find a formula for $d(z, w)$. In particular, find $d(z, \infty)$. (c) Show that circles on S^2 go to circles or lines in $\mathbb{C}$ under Φ.*

PROBLEM 1.5. *Find a Möbius transformation that takes $|z - i| < 1$ onto $|z - 2| < 3$. Do the same for $|z + i| < 2$ onto $x + y \geq 2$. Is there a Möbius transformation that takes*

$$\{|z - i| < 1\} \cap \{|z - 1| < 1\}$$

onto the first quadrant? How about $\{|z - 2i| < 2\} \cap \{|z - 1| < 1\}$ and $\{|z - \sqrt{3}| < 2\} \cap \{|z + \sqrt{3}| < 2\}$ onto the first quadrant?

PROBLEM 1.6. *Let* $\{z_j\}_{j=1}^n \subset \mathbb{C}$ *be distinct points and* $m_j > 0$ *for* $1 \le j \le n$. *Assume* $\sum_{j=1}^n m_j = 1$ *and define* $z = \sum_{j=1}^n m_j z_j$. *Prove that every line* ℓ *through* z separates *the points* $\{z_j\}_{j=1}^n$ *unless all of them are co-linear. Here "separates" means that there are points from* $\{z_j\}_{j=1}^n$ *on both sides of the line* ℓ *(without being on* ℓ*).*

PROBLEM 1.7. *(a) Suppose* $\{z_j\}_{j=1}^\infty \subset \{z \in \mathbb{C} : \operatorname{Re} z \ge 0\}$ *is a given sequence. True or false: if both* $\sum_{j=1}^\infty z_j$ *and* $\sum_{j=1}^\infty z_j^2$ *converge, then* $\sum_{j=1}^\infty |z_j|^2$ *also converges.*

(b) True or false: there are sequences of complex numbers $\{z_j\}_{j=1}^\infty$ *such that for* each *integer* $k \ge 1$ *the infinite series* $\sum_{j=1}^\infty z_j^k$ *converges, but fails to converge absolutely.*

PROBLEM 1.8. *Find the holomorphic function* $f(z) = f(x + iy)$ *with real part*

$$\frac{x(1 + x^2 + y^2)}{1 + 2x^2 - 2y^2 + (x^2 + y^2)^2}$$

and so that $f(0) = 0$.

PROBLEM 1.9. *Discuss the mapping properties of* $z \mapsto w = \frac{1}{2}(z + z^{-1})$ *on* $|z| < 1$. *Is it one-to-one there? What is the image of* $|z| < 1$ *in the* w*-plane? What happens on* $|z| = 1$ *and* $|z| > 1$*? What is the image of the circles* $|z| = r < 1$, *and of the half rays* $\operatorname{Arg} z = \theta$ *emanating from zero?*

PROBLEM 1.10. *Let* $T(z) = \frac{az+b}{cz+d}$ *be a Möbius transformation.*

(a) Show that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ *iff we can choose* $a, b, c, d \in \mathbb{R}$. *(b) Find all* T *such that* $T(\mathbb{T}) = \mathbb{T}$, *where* $\mathbb{T} = \{|z| = 1\}$ *is the unit circle. (c) Find all* T *for which* $T(\mathbb{D}) = \mathbb{D}$, *where* $\mathbb{D} = \{|z| < 1\}$ *is the unit disk.*

PROBLEM 1.11. *Let* $f \in \mathcal{H}(\mathbb{D})$ *with* $|f(z)| < 1$ *for all* $z \in \mathbb{D}$.

a) If $f(0) = 0$, *show that* $|f(z)| \le |z|$ *on* $\mathbb{D}$ *and* $|f'(0)| \le 1$. *If* $|f(z)| = |z|$ *for some* $z \ne 0$, *or if* $|f'(0)| = 1$, *then* f *is a rotation.*

b) Without any assumption on $f(0)$, *prove that*

$$\Big|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)}\Big| \le \frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|} \quad \forall\, z_1, z_2 \in \mathbb{D} \tag{1.12}$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2} \quad \forall\, z \in \mathbb{D}. \tag{1.13}$$

Show that equality in (1.12) *for some pair* $z_1 \ne z_2$ *or in* (1.13) *for some* $z \in \mathbb{D}$ *implies that* $f(z)$ *is a fractional linear transformation.*

PROBLEM 1.12. *a) Let* $f \in \mathcal{H}(\Omega)$ *be one-to-one. Show that necessarily* $f'(z) \ne 0$ *everywhere in* Ω, *that* $f(\Omega)$ *is open (do you need one-to-one for this? If not, what do you need?), and that* $f^{-1} : f(\Omega) \to \Omega$ *is also holomorphic. Such a map is called a* bi-holomorphic *map between the open sets* Ω *and* $f(\Omega)$. *If* $f(\Omega) = \Omega$, *then* f *is also called an* automorphism.

b) Determine all automorphisms of $\mathbb{D}$, $\mathbb{H}$, *and* $\mathbb{C}$.

PROBLEM 1.13. *Endow* $\mathbb{H}$ *with the Riemannian metric*

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) = \frac{1}{(\operatorname{Im} z)^2} dz d\bar{z}$$

and $\mathbb{D}$ with the metric

$$ds^2 = \frac{4}{(1-|z|^2)^2} dz d\bar{z}$$

These Riemannian manifolds, which turn out to be isometric, are known as hyperbolic space*. By definition, for any two Riemannian manifolds M, N a map $f : M \to N$ is called an* isometry *if it is one-to-one, onto, and preserves the metric.*

(a) The distance between any two points z_1, z_2 in hyperbolic space (on either $\mathbb{D}$ or $\mathbb{H}$) is defined as

$$d(z_1, z_2) = \inf_{\gamma} \int_0^1 \|\gamma'(t)\| \, dt$$

where the infimum is taken over all curves joining z_1 and z_2 and the length of $\dot{\gamma}$ is determined by the hyperbolic metric ds. Show that any holomorphic $f : \mathbb{D} \to \mathbb{D}$ or holomorphic $f : \mathbb{H} \to \mathbb{H}$ satisfies

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2)$$

for all z_1, z_2 in hyperbolic space.

(b) Determine all orientation preserving isometries of $\mathbb{H}$ to itself, $\mathbb{D}$ to itself, as well as from $\mathbb{H}$ to $\mathbb{D}$.

(c) Determine all geodesics of hyperbolic space as well as its scalar curvature (we are using the terminology of Riemannian geometry).

Problem 1.14. *Let $f \in \mathcal{H}(\mathbb{D})$ be such that $\operatorname{Re} f(z) > 0$ for all $z \in \mathbb{D}$, and $f(0) = a > 0$. Prove that $|f'(0)| \leq 2a$. Is this inequality sharp? If so, which functions attain it?*

Problem 1.15. *Prove Goursat's theorem: if f is complex differentiable in Ω (but without assuming that f' is continuous), then $f \in \mathcal{H}(\Omega)$.*

Problem 1.16. *Show that two closed loops in $\mathbb{C}^*$ are homotopic if and only if they have the same winding number around the origin. This proves that the map* (1.11) *is an isomorphism.*

CHAPTER 2

From z to the Riemann mapping theorem: some finer points of basic complex analysis

1. The winding number version of Cauchy's theorem

We now use the notion of winding number from the previous section to formulate a more general version of Cauchy's formula. We say that $z \in c$ where c is a cycle iff z lies on one of the curves that make up the cycle. In general, we write c both for the cycle as well as the points on it. In what follows, we shall call a cycle c in a region Ω a 0*–homologous cycle in* Ω *or relative to* Ω, iff $n(c; z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$. From the discussion at the end of the previous section we know that such a cycle is not necessarily homotopic to a point (via a homotopy inside Ω, of course), see Figure 1.8. On the other hand, it is clear that a cycle homotopic to a point is also homologous to zero.

THEOREM 2.1. *Let c be a 0–homologous cycle in Ω. Then for any $f \in \mathcal{H}(\Omega)$,*

$$n(c; z_0)\, f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0}\, dz \tag{2.1}$$

for all $z_0 \in \Omega \setminus c$. Conversely, if (2.1) holds for all $f \in \mathcal{H}(\Omega)$ and a fixed $z_0 \in \Omega \setminus c$, then c is a 0–homologous cycle in Ω.

PROOF. Define

$$\phi(z, w) := \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \in \Omega \\ \\ f'(z) & \text{if } z = w \in \Omega \end{cases}$$

Then by analyticity of f, $\phi(z, w)$ is analytic in z and jointly continuous (this is clear for $z \neq w$ and for z close to w Taylor expand in z around w). The set

$$\Omega' := \{z \in \mathbb{C} \setminus c \mid n(\gamma; z) = 0\}$$

is open, $\Omega' \cup \Omega = \mathbb{C}$, and very importantly, $\partial\Omega \subset \Omega'$. This property is due to c being zero homologous as well as the winding number being constant on all components of $\mathbb{C} \setminus c$. The function

$$g(z) := \begin{cases} \oint_c \phi(z, w)\, dw & \text{if } z \in \Omega \\ \\ \oint \frac{f(w)}{w-z}\, dw & \text{if } z \in \Omega' \end{cases}$$

is therefore well–defined and $g \in \mathcal{H}(\mathbb{C})$; for the former, note that for any $z \in \Omega$,

$$\oint_c \phi(z, w)\, dw = \oint_c \frac{f(w) - f(z)}{w - z}\, dw = \oint_c \frac{f(w)}{w - z}\, dw - 2\pi i f(z) n(c; z) \tag{2.2}$$

with $n(c; z) = 0$ for all $z \in \Omega' \cap \Omega$. The analyticity of g on Ω' is clear, whereas on Ω it follows from Fubini's and Morera's theorems. Finally, since $g(z) \to 0$ as $|z| \to \infty$, we see that $g \equiv 0$ on $\mathbb{C}$. The theorem now follows from (2.2).

For the converse, fix any $z_1 \in \mathbb{C} \setminus \Omega$ and apply (2.1) to $f(z) = \frac{1}{z-z_1}$. Then $f \in \mathcal{H}(\Omega)$ and therefore

$$\begin{aligned} n(c; z_0) f(z_0) &= \frac{1}{2\pi i} \oint_c \frac{1}{(z-z_0)(z-z_1)}\, dz \\ &= f(z_0) \frac{1}{2\pi i} \oint_c \Big[\frac{1}{z-z_0} - \frac{1}{z-z_1} \Big]\, dz \\ &= f(z_0) n(c; z_0) - n(c; z_1) f(z_0) \end{aligned}$$

whence $n(c; z_1) = 0$ as claimed. □

We can now derive the following more general version of Cauchy's Theorem (cf. Theorem 1.16). As the reader will easily verify, it is equivalent to Theorem 2.1.

Corollary 2.2. *With c and Ω as in Theorem 2.1,*

$$\oint_c f(z)\, dz = 0 \tag{2.3}$$

for all $f \in \mathcal{H}(\Omega)$. In particular, if Ω is simply connected, then (2.3) *holds for* all *cycles in Ω and $f \in \mathcal{H}(\Omega)$.*

Proof. Apply the previous theorem to $h(z) = (z - z_0) f(z)$ where $z_0 \in \Omega \setminus c$. As for the second statement, it uses the fact that $\mathbb{C} \setminus \Omega$ is connected if Ω is simply connected (and conversely). But then $n(\gamma; z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$ by Lemma 1.31, and we are done. □

The final formulation of Cauchy's theorem is the homotopy invariance. We say that two C^1–cycles c_1 and c_2 are *homotopic* iff each closed curve from c_1 (counted with multiplicity) is C^1–homotopic to exactly one closed curve from c_2.

Theorem 2.3. *Let c_1 and c_2 be two cycles in Ω that are C^1–homotopic. Then*

$$\oint_{c_1} f(z)\, dz = \oint_{c_2} f(z)\, dz$$

for all $f \in \mathcal{H}(\Omega)$. In particular, if c is homotopic to a sum of points, then $\oint_c f(z)\, dz = 0$ for all $f \in \mathcal{H}(\Omega)$.

Proof. By summation, it suffices to consider closed curves instead of cycles. For the case of closed curves one can apply Theorem 1.16 and we are done. □

This is a most important statement, as it implies, for example, that the winding number is homotopy invariant (a fact that we deduced from the homotopy invariance of integrals of closed forms before); in particular, if a cycle $c \subset \Omega$ is 0–homologous relative to Ω, then any cycle homotopic to c relative to Ω is also 0–homologous. As already noted before the converse of this is false, see Figure 1.8.

We remark that Theorem 2.3 can be proven with continuous curves instead of C^1. For this, one needs to define the integral along a continuous curve via analytic continuation of primitives. In that case, Theorem 2.3 becomes a corollary of the monodromy theorem; see Theorem 2.16 below.

2. Isolated singularities and residues

We now consider isolated singularities of holomorphic functions.

Definition 2.4. *Suppose $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ where $z_0 \in \Omega$. Then z_0 is called an* isolated singularity *of f. We say that z_0 is* removable *if f can be assigned a complex value at z_0 that renders f holomorphic on Ω. We say that z_0 is* a pole *of f provided $f(z) \to \infty$ as $z \to z_0$. Otherwise, z_0 is called an* essential singularity.

An example of a pole at $z = 0$ is exhibited by $\frac{1}{z}$, whereas an example of an essential singularity at zero is given by $e^{\frac{1}{z}}$. Indeed, simply consider the behavior of the latter function as $z \to 0$ along the imaginary and real axes, respectively.

We will now give some criteria which allow one to characterize these different types of isolated singularities. As usual, D^* denotes a disk D with the center removed.

Proposition 2.5. *Suppose $f \in \mathcal{H}(\Omega \setminus \{z_0\})$. Then there is the following mutually exclusive trichotomy:*

- *z_0 is removable iff $\lim_{z \to z_0}(z - z_0)f(z) = 0$*
- *z_0 is a pole iff there exists a positive integer $n \geq 1$ and $h \in \mathcal{H}(\Omega)$ with $h(z_0) \neq 0$ such that $f(z) = \frac{h(z)}{(z-z_0)^n}$*
- *z_0 is essential iff for every $\varepsilon > 0$, the set $f(D(z_0, \varepsilon)^*)$ is dense in $\mathbb{C}$*

Proof. Suppose $\lim_{z \to z_0}(z - z_0)f(z) = 0$. Then $g(z) := (z - z_0)f(z_0) \in C(\Omega)$ and from Morera's theorem it follows that $g \in \mathcal{H}(\Omega)$. To apply Morera's theorem, distinguish the cases where z_0 lies outside the triangle, on the boundary of the triangle, or in the interior of the triangle.
Suppose that z_0 is a pole. Then by the previous criterion, $g(z) := \frac{1}{f(z)}$ has a removable singularity at z_0 (in fact, $g(z) \to 0$ as $z \to z_0$). Hence, for some positive integer n there is the representation $g(z) = (z - z_0)^n \widetilde{g}(z)$ where $\widetilde{g} \in \mathcal{H}(\Omega)$ and $\widetilde{g}(z_0) \neq 0$. This implies that $f(z) = \frac{h(z)}{(z-z_0)^n}$ where $h(z_0) \neq 0$ and $h \in \mathcal{H}(\Omega)$. Conversely, suppose that $f(z)$ has this form. Then $f(z) \to \infty$ as $z \to z_0$ (which is equivalent to $|f(z)| \to \infty$ as $z \to z_0$) and z_0 is a pole of f.
Finally, suppose $f(D(z_0, \varepsilon)) \cap D(w_0, \delta) = \emptyset$ for some $\varepsilon > 0$ and $w_0 \in \mathbb{C}$, $\delta > 0$. Then $\frac{1}{f(z)-w_0} \in \mathcal{H}(D(z_0, \varepsilon))$ has a removable singularity at z_0 which then further implies that $f(z)$ has a removable singularity or a pole at z_0. In the converse direction, the density of the sets $f(D(z_0, \varepsilon))$ for every $\varepsilon > 0$ clearly precludes a removable singularity of pole at z_0. $\square$

Let n be the integer arising in the previous characterization of a pole; then we say that the *order of the pole* at z_0 is n. We also remark that the characterization of essential singularities in Proposition 2.5 is referred to as *Casorati-Weierstrass theorem*. The *great Picard theorem* in fact states that for every $\varepsilon > 0$ such a function necessarily assumes every value – with one possible exception – infinitely often on $D(z_0, \varepsilon)^*$.

Definition 2.6. *We say that f is a* meromorphic *function on Ω iff there exists a discrete set $\mathcal{P} \subset \Omega$ such that $f \in \mathcal{H}(\Omega \setminus \mathcal{P})$ and such that each point in $\mathcal{P}$ is a pole of f. We denote the field of meromorphic functions by $\mathcal{M}(\Omega)$.*

A standard and very useful tool in the study of isolated singularities are the *Laurent series.*

PROPOSITION 2.7. *Suppose that* $f \in \mathcal{H}(\mathcal{A})$ *where*

$$\mathcal{A} = \{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}, \quad 0 \le r_1 < r_2 \le \infty$$

is an annulus. Then there exist unique $a_n \in \mathbb{C}$ *such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{2.4}$$

where the series converges absolutely on $\mathcal{A}$ *and uniformly on compact subsets of* $\mathcal{A}$. *Furthermore,*

$$a_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \tag{2.5}$$

for all $n \in \mathbb{Z}$ *and any* $r_1 < r < r_2$. *The series* (2.4) *is called the* Laurent *series of* f *around* z_0*, and*

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

is its principal part.

PROOF. Fix $z \in \Omega$. Let c be the cycle defined in Ω by

$$c = -\gamma_{r_1'} + \gamma_{r_2'} - \eta_\varepsilon$$

where $\gamma_r(t) := z_0 + re^{2\pi it}$, $\eta_\varepsilon(t) := z + \varepsilon e^{2\pi it}$, and $r_1 < r_1' < |z - z_0| < r_2' < r_2$ and ε is small. Then $n(c; w) = 0$ for all $w \in \mathbb{C} \setminus \mathcal{A}$ and $n(c; z) = 0$. Hence, by the Cauchy formula of Theorem 2.1,

$$\frac{1}{2\pi i} \oint_c \frac{f(w)}{w - z}\, dw = 0 \Longrightarrow f(z) = \frac{1}{2\pi i} \oint_{\gamma_{r_2'}} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \oint_{\gamma_{r_1'}} \frac{f(w)}{w - z}\, dw. \tag{2.6}$$

Now proceed as in the proof of Corollary 1.18 with $\gamma_{r_2'}$ contributing the a_n, $n \ge 0$ as in (2.5), and the inner curve $\gamma_{r_1'}$ contributing a_n with $n < 0$ as in (2.5). Indeed, we simply expand:

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}} \qquad \text{if } |w - z_0| > |z - z_0|$$

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = -\sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^{n+1}} \qquad \text{if } |w - z_0| < |z - z_0|$$

Inserting these expansions into (2.6) and interchanging summation and integration yields the desired representation (the interchange being justified by the uniform convergence of these series on the integration curves). The absolute and uniform convergence of the resulting series on compact sets follow as well. Note that these formulas, as well as the uniqueness, follow from our previous calculation (1.5) (divide the Laurent series by $(z - z_0)^\ell$ and integrate). $\square$

Suppose now that $r_1 = 0$ so that z_0 becomes an isolated singularity. Amongst all Laurent coefficients, a_{-1} is the most important due to its invariance properties (this will only become clear in the context of differential forms on Riemann surfaces). It is called *the residue* of f at z_0 and denoted by $\mathrm{res}(f; z_0)$. It is easy to read off from the Laurent series which kind of isolated singularity we are dealing with:

COROLLARY 2.8. *Suppose z_0 is an isolated singularity of f and suppose*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent expansion of f around z_0 convergent on $0 < |z - z_0| < \delta$ for some $\delta > 0$. Then z_0 is removable iff $a_n = 0$ for all $n < 0$. It has a pole iff there exists an integer $n_0 < 0$ such that $a_n = 0$ for all $n < n_0$ but $a_{n_0} \neq 0$ (and n_0 is the order of the pole). Otherwise, z_0 is an essential singularity.

PROOF. Simply apply Proposition 2.5 to the Laurent series of f at z_0. □

Let us now clarify why holomorphic maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$ are necessarily rational. Recall that this fact was already mentioned in Chapter 1 in connection with fractional linear transformations.

LEMMA 2.9. *The analytic maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$ which are not identically equal to ∞ are precisely the rational functions, i.e., all maps of the form $\frac{P}{Q}$ with P, Q polynomials over $\mathbb{C}$ and $Q \not\equiv 0$.*

PROOF. All rational maps are analytic from the extended plane to itself. For the converse, suppose $f(z) \in \mathbb{C}$ for all $z \in \mathbb{C}_\infty$. Then f is entire and bounded and thus constant. We can therefore assume that $f(z_0) = \infty$ for some $z_0 \in \mathbb{C}$ (consider $f(1/z)$ if necessary). By continuity of f the point z_0 cannot be an essential singularity of f, cf. Proposition 2.5. In other words, z_0 is a removable singularity or a pole. By the uniqueness theorem, the poles cannot accumulate in $\mathbb{C}_\infty$. Since the latter is compact, there can thus only be finitely many poles. Hence, after subtracting the principal part of the Laurent series of f around each pole in $\mathbb{C}$ from f, we obtain an entire function which grows at most like a polynomial. By Liouville's theorem, see Corollary 1.19, such a function must be a polynomial and we are done. □

A most useful result of elementary complex analysis is the *residue theorem*.

THEOREM 2.10. *Suppose $f \in \mathcal{H}(\Omega \setminus \{z_j\}_{j=1}^J)$. If c is a 0–homologous cycle in Ω which does not pass through any of the z_j, then*

$$\frac{1}{2\pi i} \oint_c f(z)\, dz = \sum_{j=1}^{J} n(c; z_j) \operatorname{res}(f; z_j). \tag{2.7}$$

PROOF. Let $\nu_j := n(c; z_j)$ and define a new cycle

$$c' := c - \sum_{j=1}^{J} \nu_j \gamma_j, \qquad \gamma_j(t) := z_j + \varepsilon e^{2\pi i t}$$

where $\varepsilon > 0$ is small. Then $n(c'; w) = 0$ for all $w \in \mathbb{C} \setminus \Omega$ and $n(c'; z_j) = 0$ for all $1 \le j \le J$. The residue formula (2.7) now follows from Theorem 2.1 applied to $\Omega \setminus \{z_j\}_{j=1}^J$. □

The residue theorem can be used to evaluate definite integrals; see Problem 2.3. We will apply it now to derive the *argument principle*. To motivate this principle, consider $f(z) = z^n$ with $n = 0$. If $\gamma_r(t) = re^{2\pi i t}$ is the circle of radius r around 0, then $f \circ \gamma_r$ has winding number n around 0. Hence, that winding number counts how many zeros of f there are inside of γ_r. If $n < 0$, then we obtain the order of the pole at 0 with a negative sign.

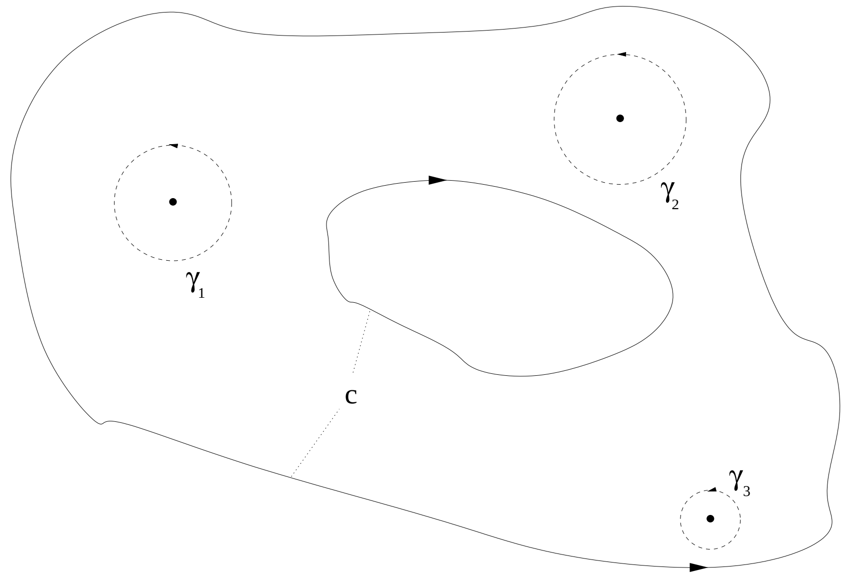

FIGURE 2.1. An example of the cycle in the residue theorem

PROPOSITION 2.11. *Let c be a 0–homologous cycle relative to Ω. If $f \in \mathcal{M}(\Omega)$ is such that no zero or pole of f lie on c, then*

$$n(f \circ c\,;0) = \sum_{z\in\Omega: f(z)=0} n(c;z) - \sum_{\zeta\in\Omega: f(\zeta)=\infty} n(c;\zeta) \tag{2.8}$$

where zeros and poles are counted with multiplicity. In other words, the winding number — or increase in the argument — of f along c counts zeros minus poles with multiplicity and weighted by the winding number of c around the respective points.

PROOF. We first point out the sum on the right-hand side of (2.8) only has finitely many nonzero terms; indeed, zeros and poles can only cluster at the boundary where the winding number necessarily vanishes. By definition,

$$n(f \circ c\,;0) = \frac{1}{2\pi i}\oint_{f\circ c} \frac{dw}{w} = \frac{1}{2\pi i}\oint_{c} \frac{f'(z)}{f(z)}\,dz. \tag{2.9}$$

If $f(z) = (z - z_0)^n g(z)$ with $n \neq 0$, $g(z_0) \neq 0$, and $g \in \mathcal{H}(\Omega)$, then

$$\operatorname{res}\Big(\frac{f'}{f}\,;z_0\Big) = n$$

and the proposition follows by applying the residue theorem to (2.9). □

It is clear that the argument principle gives another (direct) proof of the fundamental theorem of algebra. Combining the homotopy invariance of the winding number (see Theorem 2.3) with the argument principle yields *Rouche's theorem*.

PROPOSITION 2.12. *Let c be a 0-homologous cycle in Ω such that*

$$\{z \in \mathbb{C} \setminus c \,:\, n(c;z) = 1\} = \Omega_0$$

has the property

$$\{z \in \mathbb{C} \setminus c \,:\, n(c; z) = 0\} = \mathbb{C} \setminus (\Omega_0 \cup c).$$

Let $f, g \in \mathcal{H}(\Omega)$ *and suppose that* $|g| < |f|$ *on* c. *Then*

$$\#\{z \in \Omega_0 \mid f(z) = 0\} = \#\{z \in \Omega_0 \mid (f+g)(z) = 0\}$$

where the zeros are counted with multiplicity.

Proof. The function $(f + sg) \circ c$, $0 \le s \le 1$ is a homotopy between the cycles $f \circ c$ and $(f + g) \circ c$ relative to $\mathbb{C}^*$ with the property that Proposition 2.11 applies to each s–slice (note that in particular, $f \neq 0$ on c). Consequently,

$$\begin{aligned} n((f+sg)\circ c; 0) &= \sum_{z \in \Omega : (f+sg)(z)=0} n(c; z) = \sum_{z \in \Omega_0 : (f+sg)(z)=0} 1 \\ &= \#\{z \in \Omega_0 \mid (f+sg)(z) = 0\} \end{aligned}$$

does not depend on s and Rouche's theorem follows. $\square$

Rouche's theorem allows for yet another proof of the fundamental theorem of algebra: If $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, then set $f(z) := z^n$ and $g(z) := a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$. On $|z| = R$ with R very large, $|f| > |g|$ and Rouche's theorem applies.

3. Analytic continuation

Many special functions, such as the Gamma and zeta functions, are defined by integral or series representations in subdomains of the complex plane (such as a half-plane). The question then arises whether these functions can be *analytically continued* outside of this domain. Historically, this question turned out to be of fundamental importance to complex analysis with many ramifications to other areas of mathematics. In fact, Riemann surfaces appeared as the natural domains of analytic functions obtained by analytic continuation of roots of algebraic equations.

In this chapter, we discuss the most elementary aspects of this theory and we begin with analytic continuation along curves. First, we define a chain of disks along a continuous curve. Next, we will put analytic functions on the disks which are naturally continuations of one another.

Definition 2.13. *Suppose* $\gamma : [0, 1] \to \Omega$ *is a continuous curve inside a region* Ω. *We say that* $D_j = D(\gamma(t_j), r_j) \subset \Omega$, $0 \le j \le J$, *is a* chain of disks along γ in Ω *iff* $0 = t_0 < t_1 < t_2 < \ldots < t_N = 1$ *and* $\gamma([t_j, t_{j+1}]) \subset D_j \cap D_{j+1}$ *for all* $0 \le j \le N - 1$.

For any γ and Ω as in this definition there exists a chain of disks along γ in Ω, by uniform continuity of γ. Next, we analytically continue along such a chain.

Definition 2.14. *Let* $\gamma : [0, 1] \to \Omega$ *be a continuous curve inside* Ω. *Suppose* $f \in \mathcal{H}(U)$ *and* $g \in \mathcal{H}(V)$ *where* $U \subset \Omega$ *and* $V \subset \Omega$ *are neighborhoods of* $p := \gamma(0)$ *and* $q := \gamma(1)$, *respectively. Then we say that* g *is an* analytic continuation of f along γ *iff there exists a chain of disks* $D_j := D(\gamma(t_j), r_j)$ *along* γ *in* Ω *where* $0 \le j \le J$, *and* f_j *in* $\mathcal{H}(D_j)$ *such that* $f_j = f_{j+1}$ *on* $D_j \cap D_{j+1}$ *and* $f_0 = f$ *and* $f_J = g$ *locally around* p *and* q, *respectively.*

In what follows, the only relevant information about f and g is their definition locally at p and q, respectively, and not their domains of definition. This is equivalent to saying that we identify f and g with their Taylor series around p and q, respectively.

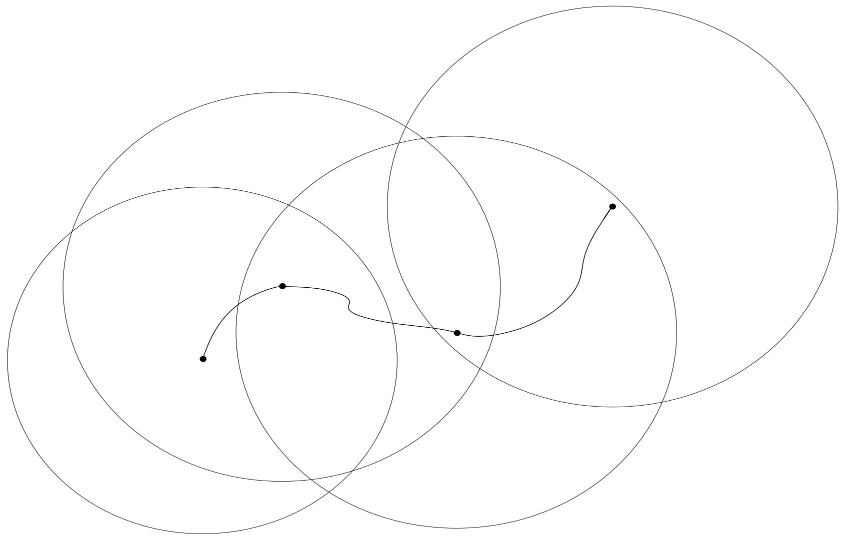

FIGURE 2.2. A chain of disks

As expected, the analytic continuation of f along γ is unique whenever it exists. In particular, it does not depend on the chain of disks along γ, but only on γ itself. This follows from the uniqueness theorem, see Corollary 1.22 above.

LEMMA 2.15. *The analytic continuation g of f along γ as in Definition 2.14 only depends on f and γ, but not on the specific choice of the chain of circles. In particular, it is unique.*

PROOF. Suppose that D_j and $\widetilde{D}_k$ are two different chains of disks along γ with underlying partitions $\{t_j\}_{j=1}^J$ and $\{s_k\}_{k=1}^K$, respectively. Denote the chain of analytic functions defined on these disks by f_j and g_k. Then we claim that for any j, k with $t_{j-1} \leq s_k \leq t_j$,

$$f_j = g_k \quad \text{on} \quad D_j \cap \widetilde{D}_k.$$

Applying this claim to the end point of γ yields the desired uniqueness. To prove the claim, one uses induction on $j + k$ and the uniqueness theorem. As an exercise, supply the details. ☐

We have already encountered a special case of this: suppose that $f \in \mathcal{H}(\Omega)$. Then locally around every point in Ω there exists an anti-derivative (or primitive). Any such primitive can be analytically continued along an arbitrary C^1-curve $\gamma : [0, 1] \to \Omega$ by integration:

$$F(z) := \int_\gamma f(\zeta)\, d\zeta$$

where $\gamma(1) = z$ and $\gamma(0) = z_0$ is kept fixed. This procedure, however, does not necessarily lead to a "global" primitive $F \in \mathcal{H}(\Omega)$. The standard example $\Omega = \mathbb{C}^*$ and $f(z) = \frac{1}{z}$ shows otherwise. On the other hand, it is clear from Theorem 2.3 that we *do obtain a global* F if Ω is simply connected. This holds in general for analytic continuations and is known as the *monodromy theorem*.

THEOREM 2.16. *Suppose γ_0 and γ_1 are two homotopic curves (relative to some region $\Omega \subset \mathbb{C}$) with the same initial point p and end point q. Let U be a neighborhood of p and assume further that $f \in \mathcal{H}(U)$ can be analytically continued along every curve of the homotopy. Then the analytic continuations of f along γ_j, $j = 0, 1$ agree locally around q.*

PROOF. Let $H : [0,1]^2 \to \Omega$ be the homotopy between γ_0 and γ_1 which fixes the initial and endpoints. Thus $H = H(t,s)$ where $\gamma_0(t) = H(t,0)$ and $\gamma_1(t) = H(t,1)$, respectively. Denote the continuation of f along $H(\cdot, s)$ by g_s. We need to prove that the Taylor series of g_s around q does not depend on s. It suffices to prove this locally in s. The idea is of course to change s so little that essentially the same chain of disks can be used. The details are as follows: let $\gamma_s(t) := H(t,s)$, fix any $s_0 \in [0,1]$ and suppose $\{D_j\}_{j=1}^J$ is a chain of circles along γ_{s_0} with underlying partition $0 = t_0 < t_1 < \ldots < t_N = 1$ and functions f_j on D_j defining the analytic continuation of f along γ_{s_0}. We claim the following: let $D_j(s)$ denote the largest disk centered at $\gamma_s(t_j)$ which is contained in D_j. There exists $\varepsilon > 0$ such that for all $s \in [0,1]$, $|s - s_0| < \varepsilon$, the $D_j(s)$ form a chain of disks along γ_s. In that case, we can use the same f_j, which proves that for all $|s - s_0| < \varepsilon$, the g_s agree with g_{s_0} locally around q. It remains to prove the claim. For this, we use the uniform continuity of the homotopy H to conclude that there exists $\varepsilon > 0$ so that for all $|s - s_0| < \varepsilon$, each disk $D_j(s)$ contains the ε-neighborhood of $\gamma_s([t_{j-1}, t_j])$ for each $1 \le j \le J$. This of course guarantees that $\{D_j(s)\}_{j=1}^J$ is a chain of disks along γ_s inside Ω as desired. □

In particular, since any two curves with the same initial and end points are homotopic in a simply connected region, we conclude that under the assumption of simple connectivity, analytic continuations are always unique. This of course implies all previous results of this nature (the existence of the logarithm etc.). Any reader familiar with universal covers should be reminded here of the homeomorphism between a simply connected manifold and its universal cover. Making this connection between the monodromy theorem and the universal cover requires the notion of a *Riemann surface* to which we turn in Chapter 4.

An instructive example of how analytic continuation is performed "in practice", such as in the context of special functions, is furnished by the Gamma function $\Gamma(z)$. If $\mathrm{Re}\,(z) > 0$, then

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt$$

is holomorphic. This follows from Fubini's and Morera's theorems. One checks via integration by parts that $\Gamma(n+1) = n!$ and the functional equation

$$\Gamma(z+1) = z\Gamma(z) \qquad \forall\, \mathrm{Re}\,(z) > 0.$$

Since the left-hand side is defined for all $\mathrm{Re}\,(z) > -1$, we set

$$\Gamma(z) := \frac{\Gamma(z+1)}{z}, \quad \forall\, \mathrm{Re}\,(z) > -1$$

Note that $z = 0$ is a pole of first order. Iterating this identity yields, with $k \ge 0$ an arbitrary integer,

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)(z+2)\ldots(z+k)} \qquad \forall\, \mathrm{Re}\,(z) > -k-1.$$

This allows one to analytically continue Γ as a meromorphic function to all of $\mathbb{C}$; it has simple poles at $\{n \in \mathbb{Z} : n \le 0\}$ with residues $\frac{(-1)^n}{n!}$. For details as well as other examples we refer the reader to Problems 2.10 and 2.13.

Returning to general properties, let us mention that any domain Ω carries analytic functions which cannot be continued beyond any portion of the boundary $\partial\Omega$. Simply let $\{z_n\}_{n=1}^\infty$ be dense in $\partial\Omega$ and define

$$f(z) = \sum_{n=1}^\infty 2^{-n} \frac{1}{z - z_n}$$

which is analytic on Ω. It follows from Corollary 1.18 that a power series with finite and positive radius of convergence cannot be analytically continued across its entire circle of convergence. A natural class of power series with $R = 1$ and which cannot be continued across *any portion* of $|z| = 1$ are the gap series, see Problem 2.11.

4. Convergence and normal families

The next topic we address is that of convergence and compactness of sequences of holomorphic functions. The concept of complex differentiability is so rigid that it survives under uniform limits – this is no surprise as Morera's theorem characterizes it by means of the vanishing of integrals.

LEMMA 2.17. *Suppose $\{f_n\}_{n=1}^\infty \subset \mathcal{H}(\Omega)$ converges uniformly on compact subsets of Ω to a function f. Then $f \in \mathcal{H}(\Omega)$ and $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of Ω for each $k \ge 1$. Furthermore, suppose that*

$$\sup_{n\ge1} \#\{z \in \Omega \mid f_n(z) = w\} \le N < \infty.$$

Then either $f \equiv w$ in Ω or

$$\#\{z \in \Omega \mid f(z) = w\} \le N.$$

The cardinalities here include multiplicity.

PROOF. The first assertion is immediate from Morera's theorem. The second one follows from Cauchy's formula

$$f_n^{(k)}(z_0) = \frac{k!}{2\pi i} \oint_{|z-z_0|=r} \frac{f_n(z)}{(z-z_0)^{k+1}}\, dz \tag{2.10}$$

whereas the third is a consequence of Rouche's theorem: assume that $f \not\equiv w$ and let

$$\{z \in \Omega \mid f(z) = w\} \supseteq \{z_j\}_{j=1}^J$$

with $J < \infty$. Since the set on the left-hand side is discrete in Ω, there exist $\delta, \varepsilon > 0$ small so that $|f(z) - w| > \delta$ on each circle $|z - z_j| = \varepsilon$, $1 \le j \le J$, cf. (1.8). Now let n_0 be so large that $|f - f_n| < \delta$ on $|z - z_j| = \varepsilon$ for all $n \ge n_0$ and each $1 \le j \le J$. By Rouche's theorem, it follows that f has as many zeros counted with multiplicity as each f_n with $n \ge n_0$ inside these disks and we are done. □

The following proposition shows that Lemma 2.17 applies to any *bounded* family $\{f_n\}_{n=1}^\infty \subset \mathcal{H}(\Omega)$, or at least subsequences thereof. This is*Montel's normal family theorem*; one says that $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{H}(\Omega)$ is a *normal family* provided for each compact

$K \subset \Omega$ one has

$$\sup_{z\in K}\sup_{\alpha\in\mathcal{A}} |f_\alpha(z)| < \infty \tag{2.11}$$

This is equivalent to *local uniform boundedness*, i.e., each point in Ω has a neighborhood K for which (2.11) holds.

PROPOSITION 2.18. *Suppose $\mathcal{F} = \{f_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{H}(\Omega)$ is a normal family. Then there exists a sequence f_n in $\mathcal{F}$ that converges uniformly on compact subsets of Ω.*

PROOF. By (2.10), we see that $f_\alpha^{(k)}$ with $\alpha \in \mathcal{A}$ is uniformly bounded on compact subsets of Ω for all $k \geq 0$. By the cases $k = 0$ and $k = 1$, we see that $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is in particular equi-continuous and bounded. By the Arzela-Ascoli theorem and a diagonal subsequence argument we thus construct a subsequence converging uniformly on all compact subsets of Ω as desired. $\square$

5. The Mittag-Leffler and Weierstrass theorems

Let us now use this machinery to answer some fundamental and classical questions in complex analysis:

- Can we find $f \in \mathcal{M}(\mathbb{C})$ so that f has poles exactly at a prescribed sequence $\{z_n\}$ that does not cluster in $\mathbb{C}$, and such that f has prescribed principal parts at these poles (this refers to fixing the entire portion of the Laurent series with negative powers at each pole)?
- Can we find $f \in \mathcal{H}(\mathbb{C})$ such that f has zeros exactly at a given sequence $\{z_n\}$ that does not cluster with prescribed orders $\nu_n \geq 1$?

In both cases the answer is "yes", as we can easily see now.

THEOREM 2.19 (Mittag-Leffler). *Given $\{z_n\}_{n=1}^N \subset \mathbb{C}$, with $|z_n| \to \infty$ if $N = \infty$, and polynomials P_n with positive degrees, there exists $f \in \mathcal{M}(\mathbb{C})$ so that f has poles exactly at z_n and*

$$f(z) - P_n\Big(\frac{1}{z - z_n}\Big)$$

is analytic around z_n for each $1 \leq n \leq N$.

PROOF. If N is finite, there is nothing to do: simply define

$$f(z) := \sum_{n=1}^N P_n\Big(\frac{1}{z - z_n}\Big).$$

If $N = \infty$, then we need to guarantee convergence of this series on compact sets by making at most a holomorphic error. Let $D_n := \{|z| < |z_n|/2\}$ and $T_n(z)$ be the Taylor polynomial of $P_n\Big(\frac{1}{z-z_n}\Big)$ of sufficiently high degree so that

$$\sup_{z\in D_n} \Big| P_n\Big(\frac{1}{z - z_n}\Big) - T_n(z) \Big| < 2^{-n}.$$

Then

$$\sum_{n=1}^\infty \Big[P_n\Big(\frac{1}{z - z_n}\Big) - T_n(z) \Big]$$

converges on compact subsets of $\mathbb{C} \setminus \{z_n\}_{n=1}^\infty$ and thus defines a holomorphic function there. Moreover, the z_n are isolated singularities; in fact, they are poles with $P_n\left(\frac{1}{z-z_n}\right)$ as principal parts. $\square$

As an example of this procedure, let $f(z) = \frac{\pi^2}{\sin^2(\pi z)}$. Then $f \in \mathcal{M}(\mathbb{C})$ with poles $z_n = n \in \mathbb{Z}$ and principal part $h_n(z) = (z-n)^{-2}$. Clearly,

$$\sum_{n \in \mathbb{Z}} h_n(z)$$

converges uniformly on $\mathbb{C} \setminus \mathbb{Z}$ to a function $s(z)$ holomorphic there. Moreover, s and $g := f - s$ are both 1–periodic. In addition, $g \in \mathcal{H}(\mathbb{C})$. Finally, in the strip $0 \le \operatorname{Re} z \le 1$ we see that both f and s are uniformly bounded; in fact, they both tend to zero as $|\operatorname{Im} z| \to \infty$. Hence, $g \in \mathcal{H}(\mathbb{C}) \cap L^\infty(\mathbb{C})$ is bounded and in fact vanishes identically. In conclusion,

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

Setting $z = \frac{1}{2}$ shows that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \quad \text{and thus} \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

where the second series is obtained from the first by splitting into even and odd n. As another example, consider $f(z) = \pi \cot(\pi z)$. It has simple poles at each $n \in \mathbb{Z}$ with principal parts $h_n(z) = \frac{1}{z-n}$. In this case we do require the T_n from the proof of Theorem 2.19:

$$\begin{aligned} s(z) &:= \frac{1}{z} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{1}{z-n} + \frac{1}{n}\right] = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{1}{z-n} + \frac{1}{z+n}\right] \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \end{aligned}$$

By inspection, $g := f - s$ is 1-periodic and analytic on $\mathbb{C}$. A simple estimate of both f and s reveals that g is bounded. Hence $g = \text{const}$, and expanding around $z = 0$ shows that in fact $g \equiv 0$. We have thus obtained the *partial fraction decomposition*

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{2.12}$$

Let us now turn to the second question, the construction of an entire function with prescribed zeros.

Definition 2.20. *Given $\{z_n\}_{n=1}^\infty \subset \mathbb{C}^*$, we say that $\prod_{n=1}^\infty z_n$ converges, iff*

$$P_N := \prod_{n=1}^{N} z_n \to P_\infty \in \mathbb{C}^*.$$

We say that this product converges absolutely iff $\sum_{n=1}^\infty |1 - z_n| < \infty$. We shall also allow $\prod_{n=1}^\infty z_n$ with all but finitely many $z_n \neq 0$. In the case that some $z_n = 0$, this product *is defined to be* 0 *provided the infinite product with all* $z_n = 0$ *removed converges in the previous sense.*

Here is an elementary lemma whose proof we leave to the reader. $\operatorname{Log} z$ denotes the principal branch of the logarithm, i.e.,

$$\operatorname{Log} z := \log|z| + i\operatorname{Arg} z, \qquad \operatorname{Arg} z \in [0, 2\pi).$$

LEMMA 2.21. *Let* $\{z_n\}_{n=1}^\infty \subset \mathbb{C}^*$. *Then* $\prod_{n=1}^\infty z_n$ *converges (absolutely) iff* $\sum_{n=1}^\infty \operatorname{Log} z_n$ *converges (absolutely). The notion of* uniform convergence *of* $\prod z_n$ *relative to some complex parameter is also reduced to the same question for the series* $\sum \log z_n$.

We can now easily answer the question concerning entire functions with prescribed zeros.

THEOREM 2.22 (Weierstrass). *Let* $\{z_n\}_n \subset \mathbb{C}$ *be a sequence* $\mathcal{Z}$ *(finite or infinite) that does not accumulate in* $\mathbb{C}$. *Then there exists an entire function* f *that vanishes exactly at* z_n *to the order which equals the multiplicity of* z_n *in* $\mathcal{Z}$.

PROOF. We set

$$f(z) := z^\nu \dot{\prod_n} \Big[\Big(1 - \frac{z}{z_n}\Big) \exp\Big(\sum_{\ell=1}^{m_n} \frac{1}{\ell}\Big(\frac{z}{z_n}\Big)^\ell\Big)\Big] \tag{2.13}$$

where ν is the number of times 0 appears in $\mathcal{Z}$, whereas $\dot{\prod}$ is the product with all $z_n = 0$ deleted. The $m_n \geq 0$ are integers chosen so that

$$\Big|\operatorname{Log}\Big(1 - \frac{z}{z_n}\Big) + \sum_{\ell=1}^{m_n} \frac{1}{\ell}\Big(\frac{z}{z_n}\Big)^\ell\Big| < 2^{-n}$$

on $|z| < \frac{1}{2}|z_n|$. Given any $R > 0$, all but finitely many z_n satisfy $|z_n| \geq 2R$. By our construction, the (tail of the) infinite product converges absolutely and uniformly on every disk to an analytic function. In particular, the zeros of f are precisely those of the factors, and we are done. □

As in the case of the Mittag–Leffler theorem, one typically applies the Weierstrass theorem to give entire functions. Here is an example: let $f(z) = \sin(\pi z)$ with zero set $\mathcal{Z} = \mathbb{Z}$. The zeros are simple. In view of (2.13), we define

$$g(z) := z \dot{\prod_{n\in\mathbb{Z}}} \Big(1 - \frac{z}{n}\Big) e^{\frac{z}{n}} = z \prod_{n=1}^\infty \Big(1 - \frac{z^2}{n^2}\Big).$$

There exists an entire function h such that $f(z) = g(z)e^{h(z)}$. In other words:

$$f(z) = \sin(\pi z) = z e^{h(z)} \prod_{n=1}^\infty \Big(1 - \frac{z^2}{n^2}\Big)$$

$$\frac{f'(z)}{f(z)} = \pi \cot(\pi z) = h'(z) + \frac{1}{z} + \sum_{n=1}^\infty \frac{2z}{z^2 - n^2}$$

By (2.12), $h' \equiv 0$ or $h = \text{const}$. Expanding everything around $z = 0$ we conclude that $e^h = \pi$ and we have shown that

$$\sin(\pi z) = \pi z \prod_{n=1}^\infty \Big(1 - \frac{z^2}{n^2}\Big)$$

Setting $z = \frac{1}{2}$ in particular yields the *Wallis formula*

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n+1)(2n-1)}$$

We remark that the expression in brackets appearing in (2.13) is called the *canonical factor* and denoted by $E_{m_n}(z/z_n)$. In other words,

$$E_k(z) = (1-z)e^{z+\frac{1}{2}z^2+\ldots+\frac{1}{k}z^k} \qquad k \geq 0.$$

It is natural to ask under what circumstances the numbers m_n in the proof of Theorem 2.22 remain bounded. By the proof of Weierstrass' theorem, this questions is tied up with the problem of analyzing the distribution of zeros of entire functions. More precisely, we will need to control the number of zeros in the disk $D(0,R)$ as $R \to \infty$. As evidenced by the Jensen formula (3.11), this number is related to the growth of the entire function at infinity.

DEFINITION 2.23. *An entire function is of finite order provided*

$$|f(z)| \leq Ae^{B|z|^\rho} \qquad \forall\, z \in \mathbb{C}$$

for some constants A, B and $\rho \geq 0$. The infimum of all possible ρ is called the order *of f.*

This class of entire functions satisfies the more precise *Hadamard factorization theorem*:

THEOREM 2.24. *Let $f \in \mathcal{H}(\mathbb{C})$ with order at most $\rho \geq 0$. Let $k = \lfloor \rho \rfloor$. Then*

$$f(z) = e^{P(z)} z^\nu \prod_{\zeta \in \mathbb{C}^*: f(\zeta)=0} E_k(z/\zeta)$$

where P is a polynomial of degree at most k and $\nu \geq 0$ is the order of vanishing of f at $z = 0$.

PROOF. We note that we can control the number of zeros of such functions that fall into a disk of radius r. Indeed, from the Jensen formula (3.11), assuming $f(z) \neq 0$ if $|z - z_0| = r$ and $f(z_0) \neq 0$,

$$\int_0^1 \log|f(z_0 + re(\theta))|\, d\theta - \log|f(z_0)| = \sum_{|z-z_0|<r, f(z)=0} \log(r/|z-z_0|)$$

one concludes that

$$\#\{|z| < r \;:\; f(z) = 0\} \leq Cr^{\rho+\varepsilon} \qquad \forall\, r \geq 1$$

for any $\varepsilon > 0$. Considering shells $\{2^j < |z| < 2^{j+1}\}$ with arbitrary $j \geq 0$, one infers from this estimate that

$$\sum_{z \in \mathbb{C}^*,\, f(z)=0} |z|^{-b} < \infty$$

provided $b > \rho$. But this of course implies that the canonical factor E_k suffices in the Weierstrass theorem where $k = \lfloor \rho \rfloor$ (the largest integer which is $\leq \rho$). FINISH THIS ; is the degree of P = k or k+1? $\square$

As an application of this theorem, we leave it to the reader to check that an entire function with non-integral order must have infinitely many zeros. For further material on entire functions of finite order we refer the reader to [**39**], [**29**], or [**37**].

To conclude the discussion of the Mittag-Leffler and Weierstrass theorems let us note the following: given arbitrary disjoint sequences $\{z_n\}$ and $\{\zeta_m\}$ with no accumulation point in $\mathbb{C}$, there exists a function $f \in \mathcal{M}(\mathbb{C})$ with poles precisely at the $\{z_n\}$ and zeros at the $\{\zeta_m\}$. Moreover, the principal parts at the poles can be arbitrarily prescribed. To see this, apply the Mittag-Leffler theorem which yields a function $f_0 \in \mathcal{M}(\mathbb{C})$ with the prescribed poles and principal parts. Then, divide out the zeros of f_0 by means of the Weierstrass function and then multiply the resulting function with another entire function which has the prescribed zeros. In other words, there is no obstruction involving zeros and poles. However, if we consider the compactified plane $\mathbb{C}_\infty$, then there is an important obstruction: first, we know that each (nonconstant) $f \in \mathcal{M}(\mathbb{C}_\infty)$ is a rational function. It is easy to see from this representation that

$$\#\{z \in \mathbb{C}_\infty \,:\, f(z) = 0\} = \#\{z \in \mathbb{C}_\infty \,:\, f(z) = \infty\}$$

where the zeros and poles are counted according to multiplicity (including a possible one at ∞) and the cardinality here is finite. A better way of arriving at the same conclusion is given by the argument principle:

$$n(f \circ \gamma_R; 0) = \#\{z \in D(0,R) \,:\, f(z) = 0\} - \#\{z \in D(0,R) \,:\, f(z) = \infty\}$$

where γ_R is a circle of radius R centered at the origin so that no zero or pole of f lies on γ_R. Taking the limit $R \to \infty$ now yields the desired conclusion. We leave it to the reader to verify that this is the only obstruction to the existence of a meromorphic function with the given number of zeros and poles. Of interest is also that in fact

$$\#\{z \in \mathbb{C}_\infty \mid f(z) = w\} = \text{const}$$

(counted with multiplicity) independently of $w \in \mathbb{C}_\infty$ in that case. We shall see later that this is a general fact about analytic functions on *compact* Riemann surfaces (the constancy of the *valency* or degree). We remark that both the Mittag–Leffler and Weierstrass theorems remain valid on regions $\Omega \subset \mathbb{C}$, see Conway [**7**], for example.

6. The Riemann mapping theorem

We now present the famous and fundamental *Riemann mapping theorem.* Later, it will become part of the much wider *uniformization theory* of Riemann surfaces.

THEOREM 2.25. *Let $\Omega \subset \mathbb{C}$ be simply connected and $\Omega \neq \mathbb{C}$. Then there exists a conformal homeomorphism $f : \Omega \to \mathbb{D}$ onto the unit disk $\mathbb{D}$.*

PROOF. We first find such a map *into* $\mathbb{D}$. Then we will "maximize" all such f to select the desired homeomorphism. We may assume that $0 \notin \Omega$. By Proposition 1.21 there exists a branch of $\sqrt{\cdot}$ on Ω which we denote by ρ. Let $\widetilde{\Omega} := \rho(\Omega)$. Then ρ is one-to-one and if $w \in \widetilde{\Omega}$, then $-w \notin \widetilde{\Omega}$. Indeed, otherwise $\rho(z_1) = w = -\rho(z_2)$ with $z_1, z_2 \in \Omega$ would imply that $z_1 = z_2$ or $w = -w = 0$ contrary to $0 \notin \Omega$. Since $\widetilde{\Omega}$ is open, we deduce that

$$\widetilde{\Omega} \cap D(w_0, \delta) = \emptyset$$

for some $w_0 \in \mathbb{C}$ and $\delta > 0$. Now define $f(z) := \frac{\delta}{\rho(z)-w_0}$ and observe that f is one-to-one and into $\mathbb{D}$. Henceforth, we assume that $\Omega \subset \mathbb{D}$ and also that $0 \in \Omega$ (scale and translate). Define

$$\mathcal{F} := \{f : \Omega \to \mathbb{D} \mid f \in \mathcal{H}(\Omega) \text{ is one-to-one and } f(0) = 0, f'(0) > 0\}.$$

Note in particular that $f'(0)$ is assumed to be real-valued. Then $\mathcal{F} \neq \emptyset$ (since $\mathrm{id}_\Omega \in \mathcal{F}$) and $\mathcal{F}$ is a normal family; see Proposition 2.18. We claim that

$$s_0 := \sup_{f \in \mathcal{F}} f'(0) > 0$$

is attained by some $f \in \mathcal{F}$. Indeed, let $f_n'(0) \to s_0$ with $f_n \in \mathcal{F}$ and $f_n \to f_\infty \in \mathcal{H}(\Omega)$ uniformly on compact subsets of Ω. Then $f_\infty(0) = 0$, $f_\infty'(0) > 0$ and, by the maximum principle and the open mapping theorem, $f_\infty : \Omega \to \mathbb{D}$ since this map is clearly not constant. Finally, from Lemma 2.17 we infer that f_∞ is also one-to-one and thus $f_\infty \in \mathcal{F}$ as claimed.

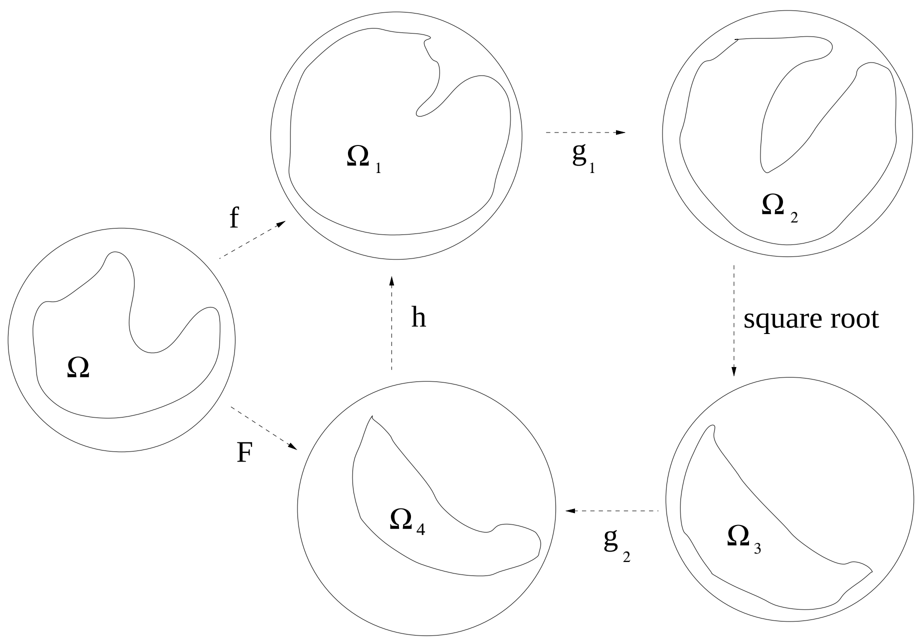

FIGURE 2.3. The final step in the Riemann mapping theorem

It remains to prove that f_∞ is onto $\mathbb{D}$. Suppose not, and let $w_0 \in \mathbb{D} \setminus f_\infty(\Omega) =: \Omega_1$. Pick $g_1 \in \mathrm{Aut}(\mathbb{D})$ (a Möbius transform) such that $g_1(w_0) = 0$ and let $\Omega_2 := g_1(\Omega_1)$, which is simply connected. It therefore admits a branch of the square root, denoted by $\sqrt{\cdot}$. Let $g_2 \in \mathrm{Aut}(\mathbb{D})$ take $\sqrt{g_1(0)}$ onto 0. By construction,

$$F := g_2 \circ \sqrt{\cdot} \circ g_1 \circ f_\infty$$

satisfies $e^{i\theta} F \in \mathcal{F}$ for suitable θ. The inverse of $g_2 \circ \sqrt{\cdot} \circ g_1$ exists and equals the analytic function

$$h(z) := g_1^{-1}((g_2^{-1}(z))^2) : \mathbb{D} \to \mathbb{D}$$

which takes 0 to 0 and is not an automorphism of $\mathbb{D}$. Hence, by the Schwarz lemma, $|h'(0)| < 1$. Since $h \circ F = f_\infty$, we have $h'(0)F'(0) = f'_\infty(0)$ which yields the desired contradiction. $\square$

We refer to f as in the theorem as a "Riemann map". It is clear that f becomes unique once we ask for $f(z_0) = 0$, $f'(z_0) > 0$ for any $z_0 \in \Omega$ (this can always be achieved). It is clear that $\Omega = \mathbb{C}$ does not admit such a map (constancy of bounded entire functions). Next, we address the important issue of the boundary behavior of the Riemann map.

DEFINITION 2.26. *We say that $z_0 \in \partial\Omega$ is* regular *provided for all $0 < r < r_0(z_0)$*

$$\Omega \cap \{z \in \mathbb{C} \mid |z - z_0| = r\} = \{z_0 + re^{i\theta} \mid \theta_1(r) < \theta < \theta_2(r)\}$$

for some $\theta_1(r) < \theta_2(r)$ which are continuous in r. In other words, $\Omega \cap \partial D(z_0, r)$ is an arc for all small $r > 0$. We say that Ω is regular provided all points of $\partial\Omega$ are regular.

This notion of regularity only applies to the Riemann mapping theorem (later we shall encounter another — potential theoretic — notion of regularity at the boundary). An example of a regular domain Ω is a manifold with C^1–boundary and corners, see below.

THEOREM 2.27. *Suppose Ω is bounded, simply connected, and regular. Then any conformal homeomorphism as in Theorem 2.25 extends to a homeomorphism $\bar{\Omega} \to \bar{\mathbb{D}}$.*

PROOF. Let $f : \Omega \to \mathbb{D}$ be a Riemann map. We first show that $\lim_{z \to z_0} f(z)$ exists for all $z_0 \in \partial\Omega$, the limit being taken from within Ω. Suppose this fails for some $z_0 \in \partial\Omega$. Then there exist sequences $\{z_n\}_{n=1}^\infty$ and $\{\zeta_n\}_{n=1}^\infty$ in Ω converging to z_0 and such that

$$f(z_n) \to w_1, \qquad f(\zeta_n) \to w_2$$

as $n \to \infty$. Here $w_1 \neq w_2 \in \partial\mathbb{D}$. Let γ_1 be a continuous curve that connects the points $\{f(z_n)\}_{n=1}^\infty$ in this order and let γ_2 do the same with $\{f(\zeta_n)\}_{n=1}^\infty$. Denote $\eta_j := f^{-1} \circ \gamma_j$ for $j = 1, 2$. Then η_j are continuous curves both converging to z_0. Let

$$z_r \in \partial D(z_0, r) \cap \eta_1, \qquad \zeta_r \in \partial D(z_0, r) \cap \eta_2$$

where we identify the curves with their set of points. By regularity of z_0 there exists an arc $c_r \subset \Omega \cap \partial D(z_0, r)$ with

$$f(z_r) - f(\zeta_r) = \int_{c_r} f'(z)\, dz$$

which further implies that

$$\begin{aligned} |f(z_r) - f(\zeta_r)|^2 &\le \Big| \int_{c_r} f'(z)\, dz \Big|^2 \le \Big(\int_{\alpha(r)}^{\beta(r)} |f'(re^{i\theta})|\, r\, d\theta \Big)^2 \\ &\le 2\pi r \int_{\theta_1(r)}^{\theta_2(r)} |f'(re^{i\theta})|^2\, r\, d\theta, \end{aligned}$$

using Definition 2.26 and the Cauchy-Schwarz inequality. Dividing by r and integrating over $0 < r < r_0(z_0)$ implies that

$$\int_0^{r_0(z_0)} |f(z_r) - f(\zeta_r)|^2 \frac{dr}{r} \le \iint_\Omega |f'(z)|^2\, dz = \text{area}\,(\mathbb{D}) < \infty$$

contradicting that $f(z_r) \to w_1$ and $f(\zeta_r) \to w_2$ as $r \to 0$ where $w_1 \neq w_2$. Hence,

$$\lim_{z \to z_0} f(z)$$

does exist and defines a continuous extension $F : \bar{\Omega} \to \bar{\mathbb{D}}$ of f. The continuity part here is implicit in the preceding; indeed, if it fails, then there would have to exist some $z_0 \in \partial\Omega$ and a sequence $z_n \in \partial\Omega$ so that $F(z_n) \not\to F(z_0)$ as $n \to \infty$. Since we can find $z_n' \in \Omega$ which are arbitrarily close to z_n this would then imply that $f(z_n') \not\to F(z_0)$ even though $z_n' \to z_0$. This contradicts the previous step and the continuity holds. Next, apply the same argument to $f^{-1} : \mathbb{D} \to \Omega$. This can be done since obviously $\mathbb{D}$ is regular in the sense of Definition 2.26 and moreover, since any sequence $z_n \in \Omega$ converging to $z_0 \in \partial\Omega$ can be connected by a continuous curve inside Ω — indeed, use the continuity of $\theta_1(r), \theta_2(r)$ in Definition 2.26. Therefore, f^{-1} extends to a continuous map $G : \bar{\mathbb{D}} \to \bar{\Omega}$. Finally, it is evident that $F \circ G = \mathrm{Id}_{\bar{\mathbb{D}}}$ and $G \circ F = \mathrm{Id}_{\bar{\Omega}}$ as desired. □

The same statement applies to unbounded Ω. In that case, we regard Ω as a region in $\mathbb{C}_\infty$ and call ∞ regular provided 0 is regular for $\Omega^{-1} := \{z^{-1} \mid z \in \Omega\}$. The following obvious result gives some examples of regular domains.

LEMMA 2.28. *Any region $\Omega \subset \mathbb{C}$ so that $\bar{\Omega}$ is a C^1–manifold with boundary and corners is regular. This means that for every $z_0 \in \Omega$ there exists a C^1–diffeomorphism ϕ of a neighborhood U of z_0 onto a disk $D(0, r_1)$ for some $r_1 = r_1(z_0) > 0$ and such that*

$$\phi(\Omega \cap U) = \{re^{i\theta} \mid 0 < r < r_1,\ 0 < \theta < \theta_1 < 2\pi\}$$

A particularly important example of such regions are polygons. In that case the conformal maps are given by the Schwartz-Christoffel formulas, see [**37**]. For example, the map

$$z \mapsto \int_0^z \frac{d\zeta}{\zeta^{\frac{1}{2}}(\zeta - 1)^{\frac{1}{2}}}, \quad \operatorname{Im} z > 0$$

takes the upper half-plane onto the half-strip

$$\{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0,\ 0 < \operatorname{Im} z < \pi\}$$

where 0 and 1 get mapped to the two finite vertices of the half-strip. In a similar spirit,

$$z \mapsto \int_0^z \frac{d\zeta}{\zeta^{\frac{1}{2}}(\zeta - 1)^{\frac{1}{2}}(\zeta - 2)^{\frac{1}{2}}}, \quad \operatorname{Im} z > 0$$

takes the upper half-plane onto a rectangle with 0, 1, and 2 being mapped onto three of the vertices and the fourth vertex being the image of ∞. The square roots $(z-a)^{\frac{1}{2}}$ here are defined to be positive when $z > a$ and to take the upper half-plane into itself.

7. Runge's theorem

We close this chapter with Runge's theorem. It addresses the question as to whether any $f \in \mathcal{H}(\Omega)$ can be approximated on compact sets by a polynomial. Again, there is a topological obstruction: $f(z) = \frac{1}{z}$ cannot be approximated on $1 \leq |z| \leq 2$ by polynomials – otherwise, $\oint_{|z|=1} \frac{dz}{z} = 0$, which is false. However, on simply connected domains this can be done. On general domains, it can be done by *rational functions*.

THEOREM 2.29. *Let $K \subset \mathbb{C}$ be compact. Any function holomorphic on a neighborhood of K can be approximated uniformly on K by rational functions all of whose poles belong to $\mathbb{C} \setminus K$. In particular, if $\mathbb{C} \setminus K$ is connected, then the approximation is uniform by polynomials.*

PROOF. Let $f \in \mathcal{H}(U)$ where $U \supset K$ is open. Let $\varepsilon > 0$ be sufficiently small that all squares with side-length ε which intersect K belong (together with their interior) entirely to U. Now define a cycle c_K by tracing out the boundaries of all square in an ε-grid with the property that they intersect both K and $U \setminus K$. We can assume that there are no degenerate configurations (squares that intersect ∂K but not K). The cycle is made up

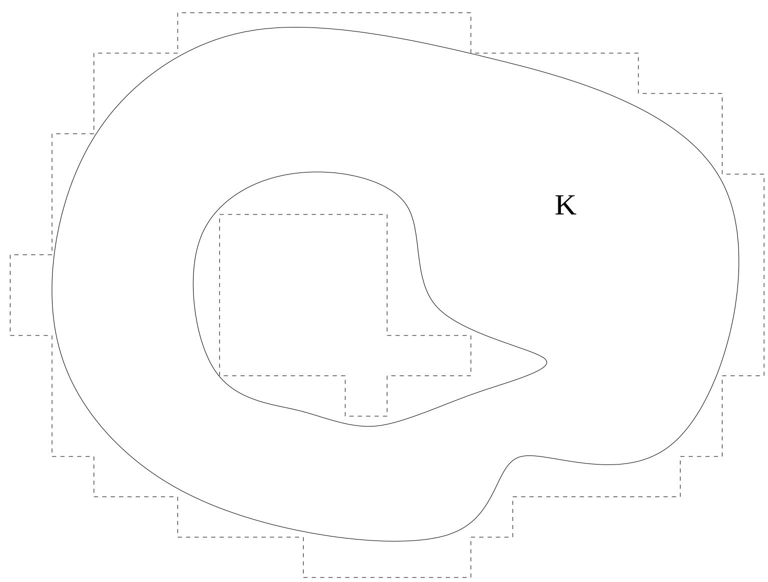

FIGURE 2.4. The cycle in Runge's theorem

of finitely many curves which consist of finitely many line segments. Moreover, equipped with the natural orientation, c_K is 0-homologous relative to U, and the winding number $n(c; z) = 1$ for all $z \in K$. It follows from Cauchy's theorem that

$$f(z) = \frac{1}{2\pi i} \oint_c \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \forall\, z \in K.$$

The integral on the right–hand side can be approximated by a Riemann sum uniformly on K and any such Riemann sum defines a rational function with poles on c and thus in $\mathbb{C} \setminus K$. To finish the theorem, we need to show that if $\mathbb{C} \setminus K$ is connected, then the poles can be "pushed to ∞". In other words, we need to prove that $f(z,\zeta) := \frac{1}{z-\zeta}$ can be approximated uniformly by polynomials if $\zeta \notin K$. To this end, let $\phi \in C(K)^*$ (a bounded linear functional on $C(K)$) that vanishes on all polynomials. We remark that the polynomials are in general not dense in $C(K)$ (see Proposition 2.18) so that ϕ does not need to vanish. We claim, however, that $\phi(f(\cdot,\zeta)) = 0$ for all $\zeta \in \mathbb{C} \setminus K$. If

$|\zeta| > \sup_{z\in K}|z|$, then this follows from

$$f(z,\zeta) = -\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n \qquad \forall\, z\in K.$$

Next, observe that $\phi(f(\cdot,\zeta))$ is analytic in ζ. Since $\mathbb{C}\setminus K$ is connected, the claim is proved. By the Hahn–Banach theorem,

$$f(\cdot,\zeta)\in\overline{\operatorname{span}\{p(z)\mid p\in\mathbb{C}[z]\}} =: L$$

where the closure is with respect to $C(K)$. Indeed, assume this fails. Then

$$\phi(p+tf(\cdot,\zeta)) := t \qquad \forall\, p\in L,\ \forall\, t\in\mathbb{C}$$

defines a bounded linear functional on the span of L and $f(\cdot,\zeta)$ which vanishes on L and does not vanish on $f(\cdot,\zeta)$. Extend it as a bounded functional to $C(K)$ (without increasing its norm which is $(\operatorname{dist}(f(\cdot,\zeta),L))^{-1}$) using the Hahn-Banach theorem. ☐

Runge's theorem has many deep consequences, such as the local Mittag-Leffler theorem, which we do not discuss here; for one application of Runge's theorem see Problem 2.12.

8. Problems

Problem 2.1. *(a) Suppose $f\in\mathcal{H}(\mathbb{D})$ satisfies $|f(z)|\le M$ for all $z\in\mathbb{D}$. Assume further that $f(z)$ vanishes at the points $\{z_j\}_{j=1}^N$ where $1\le N\le\infty$. Prove that*

$$|f(z)|\le M\Big|\prod_{j=1}^{m}\frac{z-z_j}{1-\bar z_j z}\Big| \qquad \forall\, z\in\mathbb{D}$$

for any $1\le m\le N$ (or, if $N=\infty$, then $1\le m<N$).

(b) If $N=\infty$ and $f\not\equiv 0$, then show that

$$\sum_{j=1}^{\infty}(1-|z_j|)<\infty.$$

Problem 2.2. *Let $z_1,z_2,\ldots,z_n\in\mathbb{C}$ be distinct points. Suppose γ is a closed (large) circle that contains these points in its interior and let f be analytic on a disk containing γ. Then the unique polynomial $P(z)$ of degree $n-1$ which satisfies $P(z_j)=f(z_j)$ for all $1\le j\le n$ is given by*

$$P(z)=\frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\omega(\zeta)}\frac{\omega(\zeta)-\omega(z)}{\zeta-z}\,d\zeta$$

provided $\omega(z)$ is a suitably chosen polynomial. Find ω and prove this formula.

Problem 2.3. *In this exercise you are asked to use the residue theorem, Theorem 2.10 to evaluate several integrals.*

First, compute the value of the following definite integrals.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = ?, \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = ?, \quad \int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^4} = ?$$

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = ?, \quad \int_0^{\infty} \frac{1-\cos x}{x^2}\,dx = ?$$

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = ? \quad a > 1, \quad \int_0^{\infty} \frac{\log x}{1+x^2} = ?, \quad \int_0^{\infty} \frac{(\log x)^3}{1+x^2}\,dx = ?$$

Second, prove that

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t}\,dt = \frac{\pi}{\sin \pi a}, \qquad 0 < a < 1$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx = e^{-\pi\xi^2} \qquad \forall\, \xi \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} e^{-2\pi i x\xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a}\,dx = \frac{2\sinh 2\pi a\xi}{\sinh 2\pi\xi} \qquad \forall\, \xi \in \mathbb{R},\ 0 < a < 1$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \pi\frac{e^{-a}}{a}, \quad \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = \pi e^{-a}, \quad a > 0$$

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2}\,dx = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}, \qquad \forall\, \xi \in \mathbb{R}$$

as well as

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot \pi$$

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{\frac{3}{2}}}, \qquad a > 1$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \qquad a, b \in \mathbb{R},\ |a| > b$$

and finally, show that

$$\int_0^{2\pi} \log|1-ae^{i\theta}|\,d\theta = 0, \quad \int_0^{\infty} \frac{\log x}{x^2+a^2}\,dx = \frac{\pi}{2a}\log a, \qquad a > 0$$

$$\int_0^1 \log(\sin \pi x)\,dx = -\log 2$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}, \qquad \textit{Hint: use } f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$$

$$\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}.$$

PROBLEM 2.4. *a) Prove that*

$$\int_0^{\frac{\pi}{2}} \frac{x\,d\theta}{x^2+\sin^2\theta} = \frac{\pi}{2\sqrt{1+x^2}} \qquad \forall\, x > 0.$$

b) Prove that

$$\int_0^{2\pi} \frac{(1+2\cos\theta)^n\cos(n\theta)}{1-r-2r\cos\theta}\,d\theta = \frac{2\pi}{\sqrt{1-2r-3r^2}}\Big(\frac{1-r-\sqrt{1-2r-3r^2}}{2r^2}\Big)^n$$

for any $-1 < r < \frac{1}{3}$, $n = 0, 1, 2, \ldots$.

PROBLEM 2.5. *a) Let* Ω *be an open set in* $\bar{\mathbb{H}}$ *and denote* $\Omega_0 = \Omega \cap \mathbb{H}$. *Suppose* $f \in \mathcal{H}(\Omega_0) \cap C(\Omega)$ *with* $\operatorname{Im} f(z) = 0$ *for all* $z \in \Omega \cap \partial\mathbb{H}$. *Define*

$$F(z) := \begin{cases} f(z) & z \in \Omega \\ \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}$$

where $\Omega^- = \{z \,:\, \bar{z} \in \Omega\}$. *Prove that* $F \in \mathcal{H}(\Omega \cup \Omega^-)$.

b) Suppose $f \in \mathcal{H}(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ *so that* $|f(z)| = 1$ *on* $|z| = 1$. *If* f *does not vanish anywhere in* $\mathbb{D}$, *then prove that* f *is constant.*

PROBLEM 2.6. *(a) Is there a bi-holomorphic map between the punctured disk* $\{0 < |z| < 1\}$ *and the annulus* $\{\frac{1}{2} < |z| < 1\}$*? If "yes", then find it; if "no", then prove that it cannot exist.*

(b) Prove that $\mathbb{C}$ *is not conformally equivalent to any proper subdomain of itself.*

PROBLEM 2.7. *Let* $f(z) = \sinh(\pi z)$,

$$\Omega_0 = \Big\{z \in \mathbb{C} \,:\, \operatorname{Re} z > 0,\ -\frac{1}{2} < \operatorname{Im} z < \frac{1}{2}\Big\},$$

and $\Omega_1 = -i\mathbb{H}$ *(the right half-plane). Prove that* $f : \Omega_0 \to \Omega_1$ *is one-to-one, onto, and bi-holomorphic (use the argument principle).*

PROBLEM 2.8. *Let* $\lambda > 1$. *Show that the equation* $\lambda - e^{-z} - z = 0$ *has a unique zero in the closed right half-plane* $\operatorname{Re} z \geq 0$.

PROBLEM 2.9. *(a) Give the partial fraction expansion of* $r(z) = \frac{z^2+1}{(z^2+z+1)(z-1)^2}$.

(b) Let $f(z) = \frac{1}{z(z-1)(z-2)}$. *Find the Laurent expansion of* f *on each of the following three annuli:*

$$\mathcal{A}_1 = \{0 < |z| < 1\}, \quad \mathcal{A}_2 = \{1 < |z| < 2\}, \quad \mathcal{A}_3 = \{2 < |z| < \infty\}$$

PROBLEM 2.10. *This exercise introduces and discusses some basic properties of the Gamma function* $\Gamma(z)$, *which is of fundamental importance in mathematics:*

(a) Show that

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt \tag{2.14}$$

defines an analytic function in the half-plane $\operatorname{Re} z > 0$. *Also, verify the functional equation* $\Gamma(z+1) = z\Gamma(z)$ *for all* $\operatorname{Re} z > 0$ *as well as the identity* $\Gamma(n+1) = n!$ *for all integers* $n \geq 0$.

(b) Using the functional equation, show that there exists a unique meromorphic function on $\mathbb{C}$ *which agrees with* $\Gamma(z)$ *on the right half-plane. Denoting this globally defined function again by* Γ, *prove that it has poles exactly at the nonpositive integers* $-n$ *with* $n \geq 0$. *Moreover, show that these poles are simple with residues* $\operatorname{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}$ *for all* $n \geq 0$.

(c) With $\Gamma(z)$ as in (a), verify the identity

$$\Gamma(z) = \int_1^\infty e^{-t} t^{z-1}\, dt + \sum_{n=0}^\infty \frac{(-1)^n}{n!(z+n)} \tag{2.15}$$

for all $\operatorname{Re} z > 0$. Now repeat part (b) using (2.15) instead of the functional equation.

(d) Verify that

$$\int_0^\infty \frac{v^{a-1}}{1+v}\, dv = \frac{\pi}{\sin \pi a}, \qquad \forall\, 0 < \operatorname{Re} a < 1. \tag{2.16}$$

Now apply this to establish that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{2.17}$$

as an identity between meromorphic functions defined on $\mathbb{C}$. In particular, we see that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Find an expression for $|\Gamma(\frac{1}{2} + it)|$ with $t \in \mathbb{R}$.
To pass from (2.16) to (2.17), use the identity

$$\Gamma(1-x) = y^{1-x} \int_0^\infty e^{-uy} u^{-x}\, du, \qquad \forall\, y > 0$$

which holds for any $0 < x < 1$.

(e) Check that

$$\int_0^1 (1-t)^{\alpha-1} t^{\beta-1}\, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \forall\, \operatorname{Re}(\alpha) > 0,\ \operatorname{Re}(\beta) > 0.$$

(f) Prove that

$$\int_0^\infty t^{z-1} \cos t\, dt = \Gamma(z)\cos(\pi z/2), \qquad \forall\, 0 < \operatorname{Re} z < 1$$
$$\int_0^\infty t^{z-1} \sin t\, dt = \Gamma(z)\sin(\pi z/2), \qquad \forall\, -1 < \operatorname{Re} z < 1.$$

Deduce from this that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}, \qquad \int_0^\infty \frac{\sin x}{x^{3/2}}\, dx = \sqrt{2\pi}.$$

(g) Let γ be a version of Hankel's loop contour. This refers to a smooth curve $\gamma = \gamma(t) : \mathbb{R} \to \mathbb{C} \setminus (-\infty, 0]$ which approaches $(-\infty, 0]$ from above as $t \to \infty$ and from below as $t \to -\infty$. Moreover, it encircles $w = 0$ once in a positive sense. An example

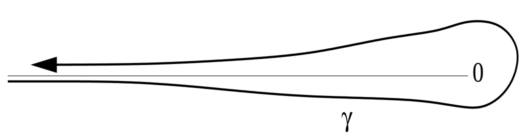

FIGURE 2.5. A Hankel contour γ

of such a γ would be with $\varepsilon : \mathbb{R} \to (0,1)$, $\varepsilon(t) \to 0$ as $|t| \to \infty$, $\gamma(t) = t - i\varepsilon(t)$ for $-\infty < t < -1$ and $\gamma(t) = -t + i\varepsilon(t)$ for $1 < t < \infty$ as well as a circular arc $\gamma(t)$ for $-1 \le t \le 1$ encircling $w = 0$ in a positive sense and joining the point $-1 - i\varepsilon(-1)$ to the point $-1 + i\varepsilon(1)$.

Now prove that for all $z \in \mathbb{C}$

$$\frac{1}{2\pi i}\int_\gamma e^w w^{-z}\, dw = \frac{1}{\Gamma(z)} \tag{2.18}$$

as an identity between entire functions. On the left-hand side $w^{-z} = e^{-z\,\mathrm{Log}\, w}$ *where* $\mathrm{Log}\, w$ *is the principal branch of the logarithm.*

PROBLEM 2.11. *Suppose that* $f(z) = \sum_{n=0}^\infty a_n\, z^{2^n}$ *has radius of convergence* $R = 1$. *Prove that* f *cannot be analytically continued to any disk centered at any point* z_0 *with* $|z_0| = 1$ *(assume that you can analytically continue to a neighborhood of* $z = 1$ *and substitute* $z = aw^2 + bw^3$ *where* $0 < a < 1$ *and* $a + b = 1$*).*
Generalize to other gap series $\sum_{k=0}^\infty a_k z^{n_k}$ *where* $n_{k+1} > \lambda n_k$ *for all* $k \geq 1$ *with* $\lambda > 1$ *fixed.*

PROBLEM 2.12. *Suppose* $K \subsetneq \partial\mathbb{D}$, $K \neq \emptyset$. *Show that for any* $\varepsilon > 0$ *there exists a polynomial* P *with* $P(0) = 1$ *and* $|P(z)| < \varepsilon$ *on* K *(hint: use Runge's theorem).*

PROBLEM 2.13. *This exercise continues our investigation of the important Gamma function.*

(a) Prove that there is some constant $A \in \mathbb{C}$ *such that*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^1 \Big(1 - (1-t)^{z-1}\Big)\frac{dt}{t} + A \qquad \forall\, \mathrm{Re}\, z > 0. \tag{2.19}$$

Deduce from (2.19) *that*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^\infty \Big(\frac{1}{n+1} - \frac{1}{n+z}\Big) + A \tag{2.20}$$

as an identity between meromorphic functions on $\mathbb{C}$.

(b) Derive the following product expansion from (2.20)*:*

$$\frac{1}{\Gamma(z)} = e^{\gamma z}\, z \prod_{n=1}^\infty \Big(1 + \frac{z}{n}\Big)e^{-\frac{z}{n}} \tag{2.21}$$

as an identity between entire functions. Here γ *is the Euler constant*

$$\gamma = \lim_{N\to\infty}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{N} - \log N\Big)$$

Alternatively, derive (2.21) *from Hadamard's theorem, see Theorem 2.24.*

(c) Prove Gauss' formula*: if* $z \in \mathbb{C} \setminus \{n \,:\, n \leq 0\}$, *then*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\ldots(z+n)}$$

CHAPTER 3

Harmonic functions on $\mathbb{D}$

1. The Poisson kernel

There is a close connection between Fourier series and analytic (harmonic) functions on the disc $\mathbb{D}$. Heuristically speaking, a Fourier series can be viewed as the "boundary values" of a Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

Let us use this observation to derive a solution formula for the following fundamental problem: *Given a function f on the boundary of $\mathbb{D}$ find a harmonic function u on $\mathbb{D}$ which attains these boundary values.*

Notice that this so-called *Dirichlet* problem is formulated too vaguely. In fact, much of this as well as the following two chapters is devoted to a proper interpretation of what we mean by *attaining* the boundary values and what kind of boundedness properties we wish u to satisfy on all of $\mathbb{D}$.

But for the moment, let us proceed heuristically. Starting with the Fourier series $f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e(n\theta)$ with $e(\theta) := e^{2\pi i\theta}$, we observe that one harmonic extension to the interior is given by

$$u(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)z^n = \sum_{n\in\mathbb{Z}} \hat{f}(n)r^n e(n\theta), \qquad z = re(\theta)$$

This is singular at $z = 0$, though, in case $\hat{f}(n) \neq 0$ for one $n < 0$. Since both z^n and $\overline{z^n}$ are (complex) harmonic, we can avoid the singularity by defining

$$u(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n + \sum_{n=-\infty}^{-1} \hat{f}(n)\bar{z}^{|n|} \tag{3.1}$$

which at least formally is a solution of our Dirichlet problem.

Inserting $z = re(\theta)$ and $\hat{f}(n) = \int_0^1 e(-n\varphi)f(\varphi)\,d\varphi$ into (3.1) yields

$$u(re(\theta)) = \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} r^{|n|}e(n(\theta-\varphi))f(\varphi)\,d\varphi =: \int_0^1 P_r(\theta-\varphi)\,f(\varphi)\,d\varphi$$

where the *Poisson kernel*

$$P_r(\theta) := \sum_{n\in\mathbb{Z}} r^{|n|}e(n\theta) = \frac{1-r^2}{1-2r\cos(2\pi\theta)+r^2}$$

via explicit summation. We start the rigorous theory by stating some properties of P_r.

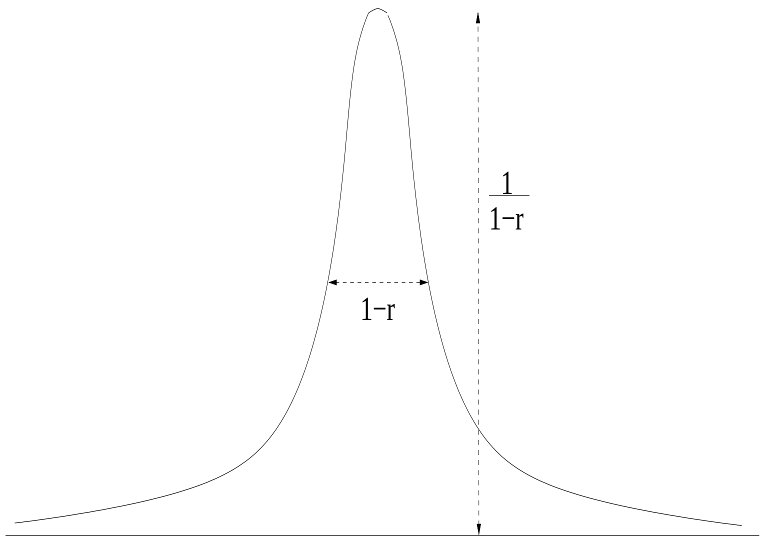

FIGURE 3.1. The poisson kernel

LEMMA 3.1. *The function $z = re(\theta) = P_r(\theta)$ is a positive harmonic function on $\mathbb{D}$. It satisfies $\int_0^1 P_r(\theta)\,d\theta = 1$ and for any (complex) Borel measure μ on $\mathbb{T}$,*

$$z = re(\theta) \mapsto (P_r * \mu)(\theta)$$

defines a harmonic function on $\mathbb{D}$.

PROOF. These properties are all either evident from the explicit form of the kernel or via the defining series. □

The behavior of the Poisson kernel close to the boundary can be captured by means of the following notion.

DEFINITION 3.2. *A sequence $\{\Phi_n\}_{n=1}^{\infty} \subset L^\infty(\mathbb{T})$ is called* an approximate identity *provided*

A1) $\int_0^1 \Phi_n(\theta)\,d\theta = 1$ *for all n*
A2) $\sup_n \int_0^1 |\Phi_n(\theta)|\,d\theta < \infty$
A3) *for all $\delta > 0$ one has $\int_{|x|>\delta} |\Phi_n(\theta)|\,d\theta \to 0$ as $n \to \infty$.*

The same definition applies, with obvious modifications, to families of the form $\{\Phi_t\}_{0<t<1}$ (with $n \to \infty$ replaced by $t \to 1-$).

A standard example is the box kernel

$$\left\{\frac{1}{2\varepsilon}\chi_{[-\varepsilon,\varepsilon]}\right\}_{0<\varepsilon<\frac{1}{2}}$$

in the limit $\varepsilon \to 0$. Another example is the Fejer kernel from Fourier series. The relevant example for our purposes is of course the Poisson kernel $\{P_r\}_{0<r<1}$, and we leave it to the reader to check that it satisfies A1)–A3). The significance of approximate identities lies with their reproducing properties (as their name suggests).

LEMMA 3.3. *For any approximate identity $\{\Phi_n\}_{n=1}^{\infty}$ one has*

(1) *If* $f \in C(\mathbb{T})$, *then* $\|\Phi_n * f - f\|_\infty \to 0$ *as* $n \to \infty$
(2) *If* $f \in L^p(\mathbb{T})$ *where* $1 \le p < \infty$, *then* $\|\Phi_n * f - f\|_p \to 0$ *as* $n \to \infty$.

These statements carry over to approximate identities Φ_t, $0 < t < 1$ *simply by replacing* $n \to \infty$ *with* $t \to 1$.

PROOF. Since $\mathbb{T}$ is compact, f is uniformly continuous. Given $\varepsilon > 0$, let $\delta > 0$ be such that

$$\sup_x \sup_{|y|<\delta} |f(x-y) - f(x)| < \varepsilon$$

Then, by A1)–A3),

$$\begin{aligned}
|(\Phi_n * f)(x) - f(x)| &= \Big| \int_{\mathbb{T}} (f(x-y) - f(x))\Phi_n(y)\,dy \Big| \\
&\le \sup_{x\in\mathbb{T}} \sup_{|y|<\delta} |f(x-y) - f(x)| \int_{\mathbb{T}} |\Phi_n(t)|\,dt + \int_{|y|\ge\delta} |\Phi_n(y)| 2\|f\|_\infty\,dy \\
&< C\varepsilon
\end{aligned}$$

provided n is large. For the second part, fix $f \in L^p$. Let $g \in C(\mathbb{T})$ with $\|f - g\|_p < \varepsilon$. Then

$$\begin{aligned}
\|\Phi_n * f - f\|_p &\le \|\Phi_n * (f-g)\|_p + \|f - g\|_p + \|\Phi_n * g - g\|_p \\
&\le \left(\sup_n \|\Phi_n\|_1 + 1\right) \|f - g\|_p + \|\Phi_n * g - g\|_\infty
\end{aligned}$$

where we have used Young's inequality ($\|f_1 * f_2\|_p \le \|f_1\|_1 \|f_2\|_p$) to obtain the first term on the right-hand side. Using A2), the assumption on g, as well as the first part finishes the proof. □

An immediate consequence is the following simple and fundamental result.

THEOREM 3.4. *Let* $f \in C(\mathbb{T})$. *The unique harmonic function* u *on* $\mathbb{D}$, *with* $u \in C(\overline{\mathbb{D}})$ *and* $u = f$ *on* $\mathbb{T}$ *is given by* $u(z) = (P_r * f)(\theta)$, $z = re(\theta)$.

PROOF. Uniqueness follows from the maximum principle. For the existence, we observed before that $u(z) := (P_r * f)(\theta)$ with $|z| < 1$ is harmonic on $\mathbb{D}$. By Lemma 3.3, $\|u(re(\cdot) - f\|_\infty \to 0$ as $r \to 1-$. This implies that we can extend u continuously to $\bar{\mathbb{D}}$ by setting it equal to f on $\mathbb{T}$. □

2. Hardy classes of harmonic functions

Next, we wish to reverse this process and understand which classes of harmonic functions on $\mathbb{D}$ assume boundary values on $\mathbb{T}$. Moreover, we need to clarify which boundary values arise here and what we mean by "assume". Particularly important classes known as the "little" Hardy spaces are as follows:

DEFINITION 3.5. *For any* $1 \le p \le \infty$ *define*

$$h^p(\mathbb{D}) := \left\{ u : \mathbb{D} \to \mathbb{C} \text{ harmonic} \;\Big|\; \sup_{0<r<1} \int_0^1 |u(re(\theta))|^p\,d\theta < \infty \right\}$$

with norm

$$|||u|||_p := \sup_{0<r<1} \|u(re(\cdot))\|_{L^p(\mathbb{T})}$$

By the mean value property, any positive harmonic function belongs to the space $h^1(\mathbb{D})$. Amongst those, the most important example is $P_r(\theta) \in h^1(\mathbb{D})$. Observe that this function has boundary values $P_r \to \delta_0$ (the Dirac mass at $\theta = 0$) as $t \to 1-$ where the convergence is in the sense of distributions. In what follows, $\mathcal{M}(\mathbb{T})$ denotes the complex-valued Borel measures and $\mathcal{M}^+(\mathbb{T}) \subset \mathcal{M}(\mathbb{T})$ the positive Borel measures.

THEOREM 3.6. *There is a one-to-one correspondence between $h^1(\mathbb{D})$ and $\mathcal{M}(\mathbb{T})$ given by $\mu \in \mathcal{M}(\mathbb{T}) \mapsto F_r(\theta) := (P_r * \mu)(\theta)$. Under this correspondence, any $\mu \in \mathcal{M}^+(\mathbb{T})$ relates uniquely to a positive harmonic function. Furthermore,*

$$\|\mu\| = \sup_{0<r<1} \|F_r\|_1 = \lim_{r\to 1} \|F_r\|_1 \tag{3.2}$$

and the following properties hold:

(1) *μ is absolutely continuous with respect to Lebesgue measure ($\mu \ll d\theta$) if and only if $\{F_r\}$ converges in $L^1(\mathbb{T})$. If so, then $d\mu = f\,d\theta$ where $f = L^1$-limit of F_r.*

(2) *The following are equivalent for $1 < p \le \infty$: $d\mu = f d\theta$ with $f \in L^p(\mathbb{T})$*

$\iff \{F_r\}_{0<r<1}$ *is L^p- bounded*

$\iff \{F_r\}$ *converges in L^p if $1 < p < \infty$ and in weak-$*$ sense in L^∞ if*

$p = \infty$ *as* $r \to 1$

(3) *f is continuous $\Leftrightarrow$ F extends to a continuous function on $\overline{\mathbb{D}}$ $\Leftrightarrow$ F_r converges uniformly as $r \to 1-$.*

This theorem identifies $h^1(\mathbb{D})$ with $\mathcal{M}(\mathbb{T})$, and $h^p(\mathbb{D})$ with $L^p(\mathbb{T})$ for $1 < p \le \infty$. Moreover, $h^\infty(\mathbb{D})$ contains the subclass of harmonic functions that can be extended continuously onto $\overline{\mathbb{D}}$; this subclass is the same as $C(\mathbb{T})$. Before proving the theorem we present two simple lemmas. In what follows we use the notation $F_r(\theta) := F\big(re(\theta)\big)$.

LEMMA 3.7.

(1) *If $F \in C(\overline{\mathbb{D}})$ and $\triangle F = 0$ in $\mathbb{D}$, then $F_r = P_r * F_1$ for any $0 \le r < 1$.*
(2) *If $\triangle F = 0$ in $\mathbb{D}$, then $F_{rs} = P_r * F_s$ for any $0 \le r, s < 1$.*
(3) *As a function of $r \in (0,1)$ the norms $\|F_r\|_p$ are non-decreasing for any $1 \le p \le \infty$.*

PROOF. 1.) is a restatement of Theorem 3.4. For 2.), rescale the disc $s\mathbb{D}$ to $\mathbb{D}$ and apply the first property. Finally, by Young's inequality

$$\|F_{rs}\|_p \le \|P_r\|_1 \|F_s\|_p = \|F_s\|_p$$

as claimed. □

LEMMA 3.8. *Let $F \in h^1(\mathbb{D})$. Then there exists a unique measure $\mu \in \mathcal{M}(\mathbb{T})$ such that $F_r = P_r * \mu$.*

PROOF. Since the unit ball of $\mathcal{M}(\mathbb{T})$ is weak-$*$ compact there exists a subsequence $r_j \to 1$ with $F_{r_j} \to \mu$ in weak-$*$ sense to some $\mu \in \mathcal{M}(\mathbb{T})$. Then, for any $0 < r < 1$,

$$P_r * \mu = \lim_{j\to\infty} (F_{r_j} * P_r) = \lim_{j\to\infty} F_{rr_j} = F_r$$

by Lemma 3.7. Let $f \in C(\mathbb{T})$. Then, again by Lemma 3.7,

$$\langle F_r, f\rangle = \langle P_r * \mu, f\rangle = \langle \mu, P_r * f\rangle \to \langle \mu, f\rangle$$

as $r \to 1$. This shows that, in the weak-$*$ sense,

$$\mu = \lim_{r\to 1} F_r \tag{3.3}$$

which implies uniqueness of μ. $\square$

Proof of Theorem 3.6. If $\mu \in \mathcal{M}(\mathbb{T})$, then $P_r * \mu \in h^1(\mathbb{D})$. Conversely, given $F \in h^1(\mathbb{D})$ then by Lemma 3.8 there is a unique μ so that $F_r = P_r * \mu$. This gives the one-to-one correspondence. Moreover, (3.3) and Lemma 3.7 show that

$$\|\mu\| \le \limsup_{r\to 1} \|F_r\|_1 = \sup_{0<r<1} \|F_r\|_1 = \lim_{r\to 1} \|F_r\|_1 \,.$$

Since one also has

$$\sup_{0<r<1} \|F_r\|_1 \le \sup_{0<r<1} \|P_r\|_1 \|\mu\| = \|\mu\| \,,$$

(3.2) follows. If $f \in L^1(\mathbb{T})$ and $d\mu = f d\theta$, then Lemma 3.3 shows that $F_r \to f$ in $L^1(\mathbb{T})$. Conversely, if $F_r \to f$ in the sense of $L^1(\mathbb{T})$, then because of (3.3) necessarily $d\mu = f d\theta$ which proves the first part. The other parts are equally easy and we skip the details—simply invoke Lemma 3.3, part 2.) for $1 < p < \infty$ and Lemma23.3 part 1.) if $p = \infty$. $\square$

In passing we remark the following: an important role is played by the kernel $Q_r(\theta)$ which is the *harmonic conjugate* of $P_r(\theta)$. Recall that this means that $P_r(\theta) + iQ_r(\theta)$ is analytic in $z = re(\theta)$ and $Q_0 = 0$. In this case it is easy to find $Q_r(\theta)$ since

$$P_r(\theta) = \mathrm{Re}\left(\frac{1+z}{1-z}\right)$$

and therefore

$$Q_r(\theta) = \mathrm{Im}\left(\frac{1+z}{1-z}\right) = \frac{2r\sin(2\pi\theta)}{1 - 2r\cos(2\pi\theta) + r^2}$$

Observe that $\{Q_r\}_{0<r<1}$ is *not* an approximate identity, since $Q_1(\theta) = \cot(\pi\theta)$ which is not the density of a measure – it behaves like $\frac{1}{\pi\theta}$ close to $\theta = 0$. The Hilbert transform is the map which is formally defined as follows:

$$f \mapsto u_f \mapsto \widetilde{u}_f \mapsto \widetilde{u}_f\big|_{\mathbb{T}}$$

where u_f denotes the harmonic extension to $\mathbb{D}$ and $\widetilde{u}_f$ its harmonic conjugate. From the preceding, Q_1 is the kernel of the Hilbert transform. It is a very important object, especially for the role it played in the development of function theory. Similarly famously, the Dirichlet kernel in Fourier series is not an approximate identity and the many efforts in understanding its mapping properties have been of enormous importance in analysis. But we will not pursue these topics any further here.

3. Almost everywhere convergence to the boundary data

Instead, we turn to the issue of almost everywhere convergence of $P_r * f$ to f as $r \to 1$. The main idea here is to mimic the proof of the Lebesgue differentiation theorem. In particular, we need the Hardy-Littlewood maximal function Mf, which is defined as follows:

$$Mf(x) = \sup_{x\in I\subset\mathbb{T}} \frac{1}{|I|}\int_I |f(y)|\,dy$$

where $I \subset \mathbb{T}$ is an (open) interval and $|I|$ is the length of I. The most basic facts concerning this (sublinear) operator are contained in the following result.

PROPOSITION 3.9. *M is bounded from L^1 to weak L^1, i.e.,*

$$\mathrm{mes}[x \in \mathbb{T} | Mf(x) > \lambda] \le \frac{3}{\lambda} \|f\|_1$$

for all $\lambda > 0$. For any $1 < p \le \infty$, M is bounded on L^p.

PROOF. Fix some $\lambda > 0$ and any compact

$$K \subset \{x \mid Mf(x) > \lambda\} \tag{3.4}$$

There exists a finite cover $\{I_j\}_{j=1}^N$ of $\mathbb{T}$ by open arcs I_j such that

$$\int_{I_j} |f(y)|\, dy > \lambda |I_j| \tag{3.5}$$

for each j. We now pass to a more convenient sub-cover (this is known as Wiener's covering lemma): Select an arc of maximal length from $\{I_j\}$; call it J_1. Observe that any I_j such that $I_j \cap J_1 \neq \emptyset$ satisfies $I_j \subset 3 \cdot J_1$ where $3 \cdot J_1$ is the arc with the same center as J_1 and three times the length (if $3 \cdot J_1$ has length larger than 1, then set $3 \cdot J_1 = \mathbb{T}$). Now remove all arcs from $\{I_j\}_{j=1}^N$ that intersect J_1. Let J_2 be one of the remaining ones with maximal length. Continuing in this fashion we obtain arcs $\{J_\ell\}_{\ell=1}^L$ which are pair-wise disjoint and so that

$$\bigcup_{j=1}^N I_j \subset \bigcup_{\ell=1}^L 3 \cdot J_\ell$$

In view of (3.4) and (3.5) therefore,

$$\begin{aligned} \mathrm{mes}(K) \le \mathrm{mes}\left(\bigcup_{\ell=1}^L 3 \cdot J_\ell\right) &\le 3 \sum_{\ell=1}^L \mathrm{mes}(J_\ell) \\ &\le \frac{3}{\lambda} \sum_{\ell=1}^L \int_{J_\ell} |f(y)|\, dy \le \frac{3}{\lambda} \|f\|_1 \end{aligned}$$

as claimed. To prove the L^p statement, one interpolates the weak L^1 bound with the trivial L^∞ bound

$$\|Mf\|_\infty \le \|f\|_\infty$$

by means of Marcinkiewicz's interpolation theorem. □

We now introduce a class of approximate identities which can be reduced to the box kernels. The importance of this idea is that it allows us to dominate the maximal function associated with an approximate identity by the Hardy-Littlewood maximal function, see Lemma 3.11 below.

DEFINITION 3.10. *Let $\{\Phi_n\}_{n=1}^\infty$ be an approximate identity as in Definition 3.2. We say that it is* radially bounded *if there exist functions $\{\Psi_n\}_{n=1}^\infty$ on $\mathbb{T}$ so that the following additional property holds:*

A4) *$|\Phi_n| \le \Psi_n$, Ψ_n is even and decreasing, i.e., $\Psi_n(x) \le \Psi_n(y)$ for $0 \le y \le x \le \frac{1}{2}$, for all $n \ge 1$. Finally, we require that $\sup_n \|\Psi_n\|_1 < \infty$.*

Now for the domination lemma.

LEMMA 3.11. *If* $\{\Phi_n\}_{n=1}^\infty$ *satisfies* A4), *then for any* $f \in L^1(\mathbb{T})$ *one has*

$$\sup_n |(\Phi_n * f)(x)| \le Mf(x) \ \sup_n \|\Psi_n\|_1$$

for all $x \in \mathbb{T}$.

PROOF. It clearly suffices to show the following statement: let $K : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}^+ \cup \{0\}$ be even and decreasing. Then for any $f \in L^1(\mathbb{T})$

$$|(K * f)(x)| \le \|K\|_1 Mf(x) \tag{3.6}$$

Indeed, assume that (3.6) holds. Then

$$\sup_n |(\Phi_n * f)(x)| \le \sup_n (\Psi_n * |f|)(x) \le \sup_n \|\Psi_n\|_1 Mf(x)$$

and the lemma follows. The idea behind (3.6) is to show that K can be written as an average of box kernels, i.e., for some positive measure μ

$$K(x) = \int_0^{\frac{1}{2}} \chi_{[-y,y]}(x) \, d\mu(y) \tag{3.7}$$

We leave it to the reader to check that

$$d\mu = -dK + K\left(\frac{1}{2}\right)\delta_{\frac{1}{2}}$$

is a suitable choice. Notice that (3.7) implies that

$$\int_0^1 K(x)\, dx = \int_0^{\frac{1}{2}} 2y \, d\mu(y)$$

Moreover, by (3.7),

$$\begin{aligned} |(K * f)(x)| &= \Big| \int_0^{\frac{1}{2}} \big(\frac{1}{2y}\chi_{[-y,y]} * f\big)(x)\, 2y \, d\mu(y) \Big| \le \int_0^{\frac{1}{2}} Mf(x) 2y \, d\mu(y) \\ &= Mf(x)\|K\|_1 \end{aligned}$$

which is (3.6). □

Finally, we can properly address the question of whether $P_r * f \to f$ in the almost everywhere sense for $f \in L^1(\mathbb{T})$. The idea is as follows: the pointwise convergence is clear from Lemma 3.3 for continuous f. This suggests approximating $f \in L^1$ by a sequence of continuous ones, say $\{g_n\}_{n=1}^\infty$, in the L^1 norm. Evidently, we encounter an interchange of limits here, namely $r \to 1$ and $n \to \infty$. As always in such a situation, we require some form of uniform control. The needed uniform control is precisely furnished by the Hardy–Littlewood maximal function.

THEOREM 3.12. *If* $\{\Phi_n\}_{n=1}^\infty$ *satisfies A1)–A4), then for any* $f \in L^1(\mathbb{T})$ *one has* $\Phi_n * f \to f$ *almost everywhere as* $n \to \infty$.

Proof. Pick $\varepsilon > 0$ and let $g \in C(\mathbb{T})$ with $\|f-g\|_1 < \varepsilon$. By Lemma 3.3, with $h = f - g$ one has, with $|\cdot|$ being Lebesgue measure,

$$\begin{aligned}
&\Big|\Big[x \in \mathbb{T} \;\Big|\; \limsup_{n\longrightarrow\infty} |(\Phi_n * f)(x) - f(x)| > \sqrt{\varepsilon}\Big]\Big| \\
\le &\Big|\Big[x \in \mathbb{T} \;\Big|\; \limsup_{n\to\infty} |(\Phi_n * h)(x)| > \sqrt{\varepsilon}/2\Big]\Big| + \Big|\Big[x \in \mathbb{T} \;\Big|\; |h(x)| > \sqrt{\varepsilon}/2\Big]\Big| \\
\le &\Big|\Big[x \in \mathbb{T} \;\Big|\; \sup_n |(\Phi_n * h)(x)| > \sqrt{\varepsilon}/2\Big]\Big| + \Big|\Big[x \in \mathbb{T} \;\Big|\; |h(x)| > \sqrt{\varepsilon}/2\Big]\Big| \\
\le &\Big|\Big[x \in \mathbb{T} \;\Big|\; CMh(x) > \sqrt{\varepsilon}/2\Big]\Big| + \Big|\Big[x \in \mathbb{T} \;\Big|\; |h(x)| > \sqrt{\varepsilon}/2\Big]\Big| \\
\le &C\sqrt{\varepsilon}
\end{aligned}$$

To pass to the final inequality we used Proposition 3.9 as well as Markov's inequality (recall $\|h\|_1 < \varepsilon$). $\square$

As a corollary we not only obtain the classical Lebesgue differentiation theorem, but also almost everywhere convergence of of the Poisson integrals $P_r * f \to f$ for any $f \in L^1(\mathbb{T})$ as $r \to 1-$. In view of Theorem 3.6 we of course would like to know whether a similar statement holds for measures instead of L^1 functions. It turns out, see Problem 3.2 below, that $P_r * \mu \to f$ almost everywhere where f is the density of the absolutely continuous part of μ in the Lebesgue decomposition. A most important example here is P_r itself! Indeed, its boundary measure is δ_0 and the almost everywhere limit is identically zero. Hence, in the almost everywhere limit we lose a lot of information namely the singular part of the boundary measure. An amazing fact, known as the F. & M. Riesz theorem, states that there is no such loss in the class $h^1(\mathbb{D}) \cap \mathcal{H}(\mathbb{D})$. Indeed, any such function is the Poisson integral of an L^1 function rather than a measure. Another way of expressing this fact is as follows: if $\mu \in \mathcal{M}(\mathbb{T})$ satisfies $\hat{\mu}(n) = 0$ for all $n < 0$, then μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$. For this important result we refer to the reader to the literature on Hardy spaces $H^p(\mathbb{D}) := h^p(\mathbb{D}) \cap \mathcal{H}(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$, see for example [**17**] and [**35**].

4. Problems

Problem 3.1. *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ have radius of convergence $R = 1$. Problems (a)–(c) further explore the connection between the behavior of the series and the function f at the boundary. They are completely elementary but a bit tricky.*

(a) Suppose $a_n \in \mathbb{R}$ for all $n \ge 0$ and $s_n = a_0 + a_1 + \ldots + a_n \to \infty$ as $n \to \infty$. Prove that $f(z)$ cannot be analytically continued to any neighborhood of $z = 1$. Is it meaningful to call $z = 1$ a pole of f? Does the same conclusion hold if all a_n are real and $|s_n| \to \infty$ as $n \to \infty$?

(b) Suppose $\sum_{n=0}^\infty a_n = s$. Show that then $f(z) \to s$ as $z \to 1$ inside the region $z \in K_\alpha \cap \mathbb{D}$, $0 < \alpha < \pi$ arbitrary but fixed. Here K_α is a cone with tip at $z = 1$, symmetric about the x-axis, opening angle α, and with $(-\infty, 1) \subset K_\alpha$ (this type of convergence $z \to 1$ is called "non-tangential convergence"). Note that $z = 1$ can be replaced by any $z \in \partial\mathbb{D}$.

(c) Now assume that $na_n \to 0$ as $n \to \infty$. If $f(z) \to s$ as $z \to 1$ non-tangentially, then prove that $\sum_{n=0}^\infty a_n = s$. Note again that $z = 1$ can be replaced by any $z \in \partial\mathbb{D}$.

Problem 3.2. *It is natural to ask whether there is an analogue of Theorem 3.12 for measures $\mu \in \mathcal{M}(\mathbb{T})$. Prove the following:*

(a) If $\mu \in \mathcal{M}(\mathbb{T})$ is singular with respect to Lebesgue measure ($\mu \perp d\theta$), then for a.e. $x \in \mathbb{T}$ (with respect to Lebesgue measure)

$$\frac{\mu([x-\varepsilon, x+\varepsilon])}{\varepsilon} \to 0 \text{ as } \epsilon \to 0$$

(b) Let $\{\Phi_n\}_{n=1}^{\infty}$ satisfy A1)–A4) from Chapter 3, and assume that the kernels $\{\Psi_n\}_{n=1}^{\infty}$ from Definition 3.10 also satisfy

$$\sup_{\delta<|\theta|<\frac{1}{2}} |\Psi_n(\theta)| \to 0 \text{ as } n \to \infty \tag{3.8}$$

for all $\delta > 0$. Under these assumptions show that for any $\mu \in \mathcal{M}(\mathbb{T})$

$$\Phi_n * \mu \to f \text{ a.e. as } n \to \infty$$

where $d\mu = f d\theta + d\nu_s$ is the Lebesgue decomposition, i.e., $f \in L^1(\mathbb{T})$ and $d\nu_s \perp d\theta$.

PROBLEM 3.3. *(a) Let $0 \le r_1 < r_2 \le \infty$ and suppose that u is a real-valued harmonic function on the annulus $\mathcal{A} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Prove that there exists some unique $k \in \mathbb{R}$ and $f \in \mathcal{H}(\mathcal{A})$ such that*

$$u(z) = k \log|z| + \operatorname{Re} f(z) \qquad \forall\, z \in \mathcal{A}$$

Next, assume that $r_1 = 0$. Prove that if u is bounded on $\mathcal{A}$, then $k = 0$ and u extends to a harmonic function throughout $|z| < r_2$.

(b) Suppose $\Omega \subset \mathbb{C}$ is open and simply connected. Let $z_0 \in \Omega$ and suppose that $u \in \Omega \setminus \{z_0\} \to \mathbb{R}$ is harmonic such that

$$u(z) - \log|z - z_0|$$

remains bounded as $z \to z_0$. Show that there exists $f \in \mathcal{H}(\Omega)$ such that $f(z_0) = 0$, $u(z) = \log|f(z)|$, and f is one-to-one one some disk around z_0.

PROBLEM 3.4. *Suppose that u, v are harmonic in Ω so that ∇u and ∇v never vanish in Ω (we call this* non-degenerate*). If $f = u + iv$ is conformal (i.e., $f \in \mathcal{H}(\Omega)$), then we know that the level curves $u = \text{const}$ and $v = \text{const}$ in Ω are perpendicular to each other (why?). This exercise addresses the converse:*

(a) Suppose v, w are harmonic and non-degenerate in Ω such that the level curves of v and w coincide in Ω. How are v and w related?

(b) Suppose u, v are harmonic and non-degenerate in Ω, and assume their level curves are perpendicular throughout Ω. Furthermore, assume that $|\nabla u(z_0)| = |\nabla v(z_0)|$ at one point *$z_0 \in \Omega$. Prove that either $u + iv$ or $u - iv$ is conformal in Ω.*

PROBLEM 3.5. *We say that $u : \Omega \to [-\infty, \infty)$ is* subharmonic *($u \in \mathfrak{sh}(\Omega)$) provided it is continuous and it satisfies the* sub mean value property *(SMVP): for every $z_0 \in \Omega$ and any $0 \le r < \operatorname{dist}(z_0, \partial\Omega)$,*

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \tag{3.9}$$

In addition, we require that $u \not\equiv -\infty$. Establish the following ten properties:

(i) Maximum principle: if Ω is bounded and $u \in \mathfrak{sh}(\Omega)$, then

$$\sup_{\zeta \in \partial\Omega} \limsup_{\substack{z \to \zeta \\ z \in \Omega}} u(z) \le M < \infty \Longrightarrow u(z) \le M \quad \forall\, z \in \Omega$$

with equality being attained on the right-hand side for some $z \in \Omega$ iff $u = \text{const}$.

(ii) Let $u \in \mathfrak{sh}(\Omega)$ on Ω and suppose h is harmonic on some open disk K compactly contained in Ω and $h \in C(\bar{K})$. Further, assume that $u \leq h$ on ∂K. Show that $u \leq h$ on K. Further if $u = h$ at some point in K, then $u = h$ on K (this explains the name sub-harmonic).

(iii) If $u_1, \ldots, u_N \in \mathfrak{sh}(\Omega)$, then so is $\max(u_1, \ldots, u_N)$ and $\sum_{j=1}^N c_j u_j$ with $c_j \geq 0$.

(iv) If h is harmonic on Ω and $\phi : \mathbb{R} \to \mathbb{R}$ is convex, then $\phi \circ h \in \mathfrak{sh}(\Omega)$. If $v \in \mathfrak{sh}(\Omega)$, and $\phi : [-\infty, \infty) \to \mathbb{R}$ is continuous, non-decreasing, and convex, then $\phi \circ v \in \mathfrak{sh}(\Omega)$.

(v) This is a converse of (ii): suppose $u : \Omega \to [-\infty, \infty)$ is continuous so that for every disk K with $\bar{K} \subset \Omega$ the harmonic majorization property *holds: if $h \in C(\bar{K})$ is harmonic on K and satisfies $u \leq h$ on ∂K then $u \leq h$ on K. Prove that u is subharmonic.*

(vi) Prove that subharmonic functions are characterized already by the local SMVP*: for every $z_0 \in \Omega$ there exists $0 < \rho(z_0) \leq \operatorname{dist}(z_0, \partial\Omega)$ such that (3.9) holds for every $0 < r < \rho(z_0)$.*

(vii) Suppose $u \in \mathfrak{sh}(\Omega)$. Prove that for any $z_0 \in \Omega$ and any $0 < r_1 < r_2 < \operatorname{dist}(z_0, \partial\Omega)$,

$$-\infty < \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r_1 e^{i\theta})\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r_2 e^{i\theta})\, d\theta$$

and

$$\lim_{r \to 0+} \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta = u(z_0)$$

as well as

$$\int_0^{2\pi} |u(z_0 + re^{i\theta})|\, d\theta < \infty$$

for any $0 < r < \operatorname{dist}(z_0, \partial\Omega)$.

(viii) Suppose $u \in C^2(\Omega)$. Then $u \in \mathfrak{sh}(\Omega)$ iff $\Delta u \geq 0$ in Ω.

(ix) Suppose $f \in \mathcal{H}(\Omega)$, $f \not\equiv 0$. Prove that the following functions are in $\mathfrak{sh}(\Omega)$: $\log|f(z)|$, $|f(z)|^\alpha$ for any $\alpha > 0$, $e^{\beta|f(z)|}$ for any $\beta > 0$.

(x) Show that in $C(\Omega)$ the harmonic functions are precisely those that satisfy the mean value property. Use this to prove that the limit of any sequence $\{u_n\}_{n=1}^\infty$ of harmonic functions on Ω which converges uniformly on every compact subset of Ω is again harmonic.

PROBLEM 3.6. *(a) Let $\lambda \geq 1$ and let $\mathcal{S}$ be the sector*

$$\mathcal{S} := \left\{ re^{i\theta} \,\middle|\, 0 < r < \infty,\ |\theta| < \frac{\pi}{2\lambda} \right\}$$

Let $u \in \mathfrak{sh}(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ satisfy $u \leq M$ on $\partial\mathcal{S}$ and $u(z) < |z|^\rho$ in $\mathcal{S}$ where $\rho < \lambda$. Prove that $u \leq M$ on $\mathcal{S}$.

(b) Let $u \in \mathfrak{sh}(\Omega)$ where Ω is a bounded domain. Further, suppose $E := \{z_n\}_{n=1}^\infty \subset \partial\Omega$ has the property that

$$\limsup_{z \to \partial\Omega \setminus E} u(z) \leq M$$

Prove that $u \leq M$ in Ω.

PROBLEM 3.7. *Let u be subharmonic on a domain $\Omega \subset \mathbb{C}$.*

(a) Prove that

$$\langle u, \Delta\phi \rangle \geq 0 \quad \forall\, \phi \in \mathbb{C}^\infty_{\rm comp}(\Omega),\ \phi \geq 0$$

where $\langle \cdot, \cdot \rangle$ denotes the standard $L^2(\Omega)$ pairing, and deduce from it that there exists a unique positive Borel measure (called the Riesz measure) on Ω such that

$$\langle u, \Delta\phi \rangle = \iint_\Omega \phi(x)\mu(dx)$$

for all $\phi \in \mathbb{C}^\infty_{\text{comp}}(\Omega)$ (from this identity, $\mu(K) < \infty$ for all compact $K \subset \Omega$). In other words, even if a subharmonic function is not C^2 its distributional Laplacean is no worse than a measure. Find μ for $u = \log |f|$ where $f \in \mathcal{H}(\Omega)$.

(b) Show that with μ as in (a) and for any $\Omega_1 \subset \Omega$ compactly contained,

$$u(z) = \iint_{\Omega_1} \log |z - \zeta| \, \mu(d\zeta) + h(z) \tag{3.10}$$

where h is harmonic on Ω_1. This is "Riesz's representation of subharmonic functions". Interpret its meaning for $u = \log |f|$ with $f \in \mathcal{H}(\Omega)$. Show that, conversely, any nonnegative Borel measure μ which is finite on compact sets of Ω defines a subharmonic function u via (3.10) *(with $h = 0$) provided the integral on the right-hand side is continuous with values in $[-\infty, \infty)$. Give an example of a μ where u is not continuous. But show that* (3.10) *is always upper semicontinuous (usc), i.e,*

$$u(z_0) \geq \limsup_{z \to z_0} u(z)$$

for all $z_0 \in \Omega$. Check that upper semicontinuous functions always attain their supremum on compact sets. In fact, the theory of subharmonic functions which we have developed so far applies to the wider class of usc functions satisfying the SMVP (try to see this) basically unchanged.

(c) With u and μ as in (a), show that

$$\int_0^1 u(z + re(\theta)) \, d\theta - u(z) = \int_0^r \frac{\mu(D(z,t))}{t} \, dt \tag{3.11}$$

for all $D(z, r) \subset \Omega$ (this is "Jensen's formula"). In other words, μ measures the extent to which the mean value property fails and really is a sub mean value *property. Now find an estimate for $\mu(K)$ where $K \subset \Omega$ is compact in terms of the* pointwise size *of u. Finally, write* (3.11) *down explicitly for $u = \log |f|$ with $f \in \mathcal{H}(\Omega)$.*

PROBLEM 3.8. *This exercise introduces the important Harnack inequality and principle for harmonic functions.*

(a) Let $P_r(\phi) = \frac{1-r^2}{1-2r\cos\phi + r^2}$ be the Poisson kernel. Show that for any $0 < r < 1$

$$\frac{1-r}{1+r} \leq P_r(\phi) \leq \frac{1+r}{1-r} \tag{3.12}$$

and deduce from this that for any nonnegative *harmonic function u on $\mathbb{D}$ one has*

$$\sup_{|z| \leq r} u(z) \leq C(r) \inf_{|z| \leq r} u(z)$$

where $C(r) < \infty$ for $0 < r < 1$. What is the optimal constant $C(r)$? Now show that for any Ω and K compactly contained in Ω one has the inequality

$$\sup_{z \in K} u(z) \leq C(K, \Omega) \inf_{z \in K} u(z)$$

for nonnegative harmonic functions u on Ω. Now prove that if $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic, and bounded from one side (thus, $u \le M$ in Ω for some finite constant M or $u \ge M$), then u is constant.

(b) Suppose $u_1 \le u_2 \le u_3 \le \dots$ are harmonic functions in Ω. Let $u = \sup_n u_n$. Then either $u \equiv \infty$ or u is harmonic in Ω.

PROBLEM 3.9. *Let $u \in \mathfrak{sh}(\mathbb{D})$. Show that the following two properties are equivalent:*

(i) u has a harmonic majorant on $\mathbb{D}$, i.e., there exists $h : \mathbb{D} \to \mathbb{R}$ harmonic such that $u \le h$ on $\mathbb{D}$.

(ii) $\sup_{0<r<1} \int_0^1 u(re(\theta))\, d\theta < \infty$ where $e(\theta) = e^{2\pi i\theta}$.

We say that h_0 is a least harmonic majorant *of u iff h_0 is a harmonic majorant of u on $\mathbb{D}$ and if $h \ge h_0$ for every other harmonic majorant h of u.*

Prove that if u has a harmonic majorant on $\mathbb{D}$, then it has a least harmonic majorant. Given an example of a $u \in \mathfrak{sh}(\mathbb{D})$ that has no harmonic majorant.

PROBLEM 3.10. *Let $f \in \mathcal{H}(\mathbb{D})$, $f \not\equiv 0$. Then prove that the following two properties are equivalent (here $\log^+ x = \max(0, \log x)$):*

(i) $\log^+ |f|$ has a harmonic majorant in $\mathbb{D}$.

(ii) $f = \frac{g}{h}$ where $g, h \in \mathcal{H}(\mathbb{D})$ with $|g| \le 1$, $0 < |h| \le 1$ in $\mathbb{D}$.

PROBLEM 3.11. *You should compare this to Problem 2.1.*

(a) Suppose $\mathcal{Z} = \{z_n\}_{n=0}^\infty \subset \mathbb{D} \setminus \{0\}$ satisfies

$$\sum_{n=0}^{\infty} (1 - |z_n|) < \infty$$

Prove that

$$B(z) = \prod_{n=0}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$$

converges uniformly on every $D(0, r)$ with $0 < r < 1$ to a holomorphic function $B \in \mathcal{H}(\mathbb{D})$ with $|B(z)| \le 1$ for all $|z| < 1$. It vanishes exactly at the z_n (with the order of the zero being equal to the multiplicity of z_n in $\mathcal{Z}$).

(b) We know that $\lim_{r \to 1-} B(re^{i\theta})$ exists for almost every θ (after all, $B \in h^\infty(\mathbb{D})$ so Chapter 3 applies). Denote these boundary values by $B(e^{i\theta})$. Prove that $|B(e^{i\theta})| = 1$ for almost every θ.

CHAPTER 4

Riemann surfaces: definitions, examples, basic properties

1. The basic definitions

The following definition introduces the concept of a Riemann surface. Needless to say, this concept arose naturally in an attempt to understand algebraic functions (which are the analytic continuations of the roots of a polynomial equation $P(z, w) = 0$ relative to w, say) as well as other "multi-valued" analytic functions. We will discuss this important construction in the following chapter, but for now start with the abstract definition.

Definition 4.1. *A* Riemann surface *is a two–dimensional, connected, Hausdorff topological manifold M with a countable base for the topology and with conformal transition maps between charts. I.e., there exists a familiy of open set $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covering M and homeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha$ where $V_\alpha \subset \mathbb{R}^2$ is some open set so that*

$$\phi_\alpha \circ \phi_\beta^{-1} \ : \ \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is biholomorphic (in other words, a conformal homeomorphism). We refer to each (U_α, ϕ_α) as a chart *and to $\mathcal{A}$ as an* atlas *of M.*

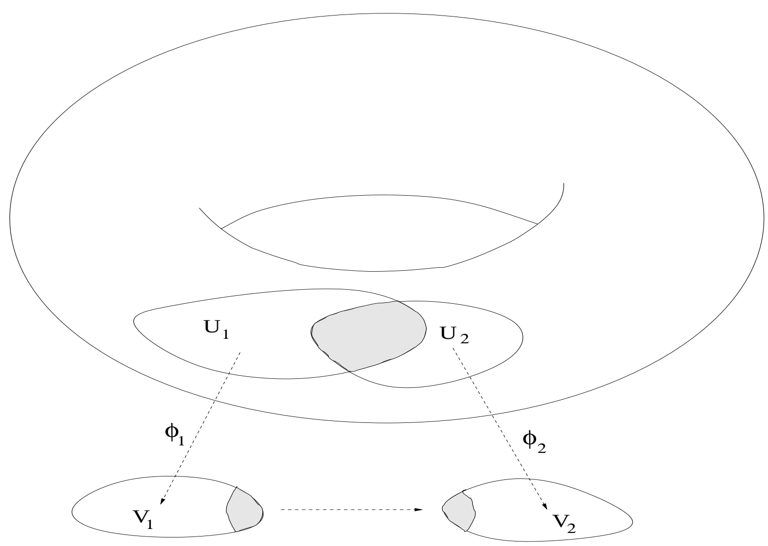

Figure 4.1. Charts and analytic transition maps

The countability axiom can be dispensed with as it can be shown to follow from the other axioms (this is called Rado's theorem, see [**15**]), but in all applications it is easy to check directly. Two atlases $\mathcal{A}_1, \mathcal{A}_2$ of M are called equivalent iff $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas of M. An equivalence class of atlases of M is called *conformal structure* and a *maximal atlas* of

M is the union of all atlases in a conformal structure. We shall often write (U, z) for a chart indicative of the fact that $p \mapsto z(p)$ takes U into the complex z–plane. Moreover, a *parametric disk* is a set $D \subset U$ where (U, z) is chart with $z(D)$ a disk in $\mathbb{C}$. We shall always assume that $\overline{z(D)} \subset z(U)$ is compact. By a parametric disk D *centered at* $p \in M$ we mean that (U, z) is a chart with $p \in U$, $z(p) = 0$, and $D = z^{-1}(D(0, r))$ for some $r > 0$.

We say that the Riemann surface M is an *extension* of the Riemann surface N iff $N \subset M$ as an open subset and if the conformal structure of M restricted to N is exactly the conformal structure that N carried to begin with.

DEFINITION 4.2. *A continuous map $f : M \to N$ between Riemann surfaces is said to be* analytic *iff it is analytic in charts. I.e., if $p \in M$ is arbitrary and $p \in U_\alpha$, $f(p) \in V_\beta$ where (U_α, z_α) is a chart of M and (V_β, w_β) is a chart of N, respectively, then $w_\beta \circ f \circ z_\alpha^{-1}$ is analytic where it is defined. We say that f is a* conformal isomorphism *iff f is an analytic homeomorphism. If $N = \mathbb{C}$ then one says that f is holomorphic, if $N = \mathbb{C}_\infty$, it is called meromorphic.*

It is clear that the meromorphic functions on a Riemann surface form a field. One refers to this field as the *function field*[1] of a surface M.

2. Examples

In this section we discuss a number of examples of Riemann surfaces, some of which will play an important role in the development of the theory. We begin with an obvious class of examples which serve to illustrate that complex analysis as we have developed it so far in this book, is really a special case of general Riemann surfaces (and the "local case").

1) *Any open region* $\Omega \subset \mathbb{C}$: Here, a single chart suffices, namely (Ω, z) with z being the identity on Ω. The associated conformal structure consists of all (U, ϕ) with $U \subset \Omega$ open and $\phi : U \to \mathbb{C}$ biholomorphic. Notice that an alternative, non-equivalent conformal structure is $(\Omega, \bar{z})$.

2) *The Riemann sphere $S^2 \subset \mathbb{R}^3$, which can be described in three, conformally equivalent, ways*: S^2, $\mathbb{C}_\infty$, $\mathbb{C}P^1$.

2a) We define a conformal structure on S^2 via two charts

$$(S^2 \setminus (0,0,1), \phi_+), \quad (S^2 \setminus (0,0,-1), \phi_-)$$

where $\phi_\pm$ are the stereographic projections

$$\phi_+(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}, \quad \phi_-(x_1, x_2, x_3) = \frac{x_1 - ix_2}{1 + x_3}$$

from the north, and south pole, respectively, see Figure 1.1. If $p = (x_1, x_2, x_3) \in S^2$ with $x_3 \neq \pm 1$, then

$$\phi_+(p)\phi_-(p) = 1$$

This shows that the transition map between the two charts is $z \mapsto \frac{1}{z}$ from $\mathbb{C}^* \to C^*$.

2b) The one-point compactification of $\mathbb{C}$ denoted by $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The neighborhood base of ∞ in $\mathbb{C}_\infty$ is given by the complements of all compact sets of $\mathbb{C}$. Again

[1] At least when M is compact this is commonly used terminology from algebraic geometry.

there are two charts, namely

$$(\mathbb{C}, z), \quad (\mathbb{C}_\infty \setminus \{0\}, \frac{1}{z})$$

in the obvious sense. The transition map is again given by $z \mapsto \frac{1}{z}$.

2c) The one-dimensional complex projective space

$$\mathbb{C}P^1 := \big\{[z:w] \mid (z,w) \in \mathbb{C}^2 \setminus \{(0,0)\}\big\} \big/ \sim$$

where the equivalence relation is $(z_1, w_1) \sim (z_2, w_2)$ iff $z_2 = \lambda z_1, w_2 = \lambda w_1$ for some $\lambda \in \mathbb{C}^*$. Our charts are (U_1, ϕ_1) and (U_2, ϕ_2) where

$$U_1 := \{[z:w] \in \mathbb{C}P^1 \mid w \neq 0\}, \quad \phi_1([z:w]) = \frac{z}{w}$$
$$U_2 := \{[z:w] \in \mathbb{C}P^1 \mid z \neq 0\}, \quad \phi_2([z:w]) = \frac{w}{z}$$

Here, too, the transition map is $z \mapsto \frac{1}{z}$.

3) *Any smooth, orientable two-dimensional submanifold of* $\mathbb{R}^3$ *is a Riemann surface*

4) *Any polyhedral surface* $S \subset \mathbb{R}^3$ *is a Riemann surface*: Such an S is defined to be a compact toplogical manifold which can be written as the finite union of *faces* $\{f_i\}$, *edges* $\{e_j\}$, and *vertices* $\{v_k\}$. Any f_i is assumed to be an open subset of a plane in $\mathbb{R}^3$, an edge is an open line segment and a vertex a point in $\mathbb{R}^3$ with the obvious relations between them (the boundaries of faces in $\mathbb{R}^3$ are finite unions of edges and vertices and the endpoints of the edges are vertices; an edge is where two faces meet etc.). To

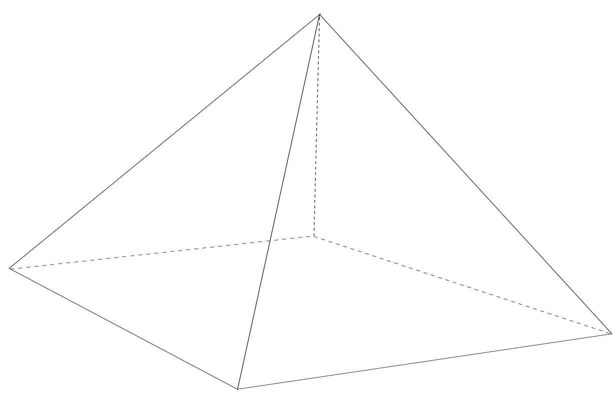

FIGURE 4.2. Polyhedra are Riemann surfaces

define a conformal structure on such a polyhedral surface, proceed as follows: each f_i defines a chart (f_i, ϕ_i) where ϕ_i is Euclidean motion (affine isometry) that takes f_i into $\mathbb{C} = \mathbb{R}^2 =\subset \mathbb{R}^3$ where we identify $\mathbb{R}^2$ with the (x_1, x_2)–plane of $\mathbb{R}^3$, say. Each edge e_j defines a chart as follows: let f_{i_1} and f_{i_2} be the two unique faces that meet in e_j. Then $(f_{i_1} \cup f_{i_2} \cup e_j, \phi_j)$ is a chart where ϕ_j maps that folds $f_{i_1} \cup f_{i_2} \cup e_j$ at the edge so that it becomes straight (piece of a plane) and then maps that plane isometrically into $\mathbb{R}^2$. Finally, at a vertex v_k we define a chart as follows: for example, suppose three faces meet at v_k, say $f_{i_1}, f_{i_2}, f_{i_3}$ with respective angles α_1, α_2, and α_3. Let $\gamma > 0$ be defined so that

$$\gamma \sum \alpha_i = 2\pi \tag{4.1}$$

and let the chart map these faces with their edges meeting at v into the plane in such a way that angles get dilated by γ. It is easy to see that this defines a conformal structure (for example, at a vertex, the transition maps are z^γ where γ is as in (4.1)).

5) *Covers and universal covers, and their quotients*

6) *Surfaces defined as smooth (projective) algebraic curves*: Let $P(z,w)$ be an irreducible polynomial such that $dP \neq 0$ on

$$S := \{(z,w) \in \mathbb{C}^2 \mid P(z,w) = 0\}$$

In other words, $(\partial_z P, \partial_w P)(z,w) \neq (0,0)$ when $P(z,w) = 0$ (such P are called *nonsingular*). Then $S \subset \mathbb{C}^2$ is a Riemann surface imbedded in $\mathbb{C}^2$, called an affine *algebraic curve*. To defined the complex structure on S, one can use either z or w as local co-ordinates depending on whether $\partial_z P \neq 0$ or $\partial_w P \neq 0$ on that neighborhood. The irreducibility of P implies that S is connected. By construction, any function of the form $\frac{f(z)}{g(w)}$ where f, g are meromorphic on $\mathbb{C}$ and g not identically zero, is a meromorphic function on S (this of course raises the question as to what all meromorphic functions on S are). To compactify S, we pass to the homogenized version of P: thus, let $\nu \geq 1$ be the minimal integer for which

$$u^\nu P(z/u, w/u) =: Q(z,w,u) \tag{4.2}$$

has no negative powers of u. Then

$$\widetilde{S} := \{[z : w : u] \in \mathbb{C}P^2 \mid Q(z,w,u) = 0\} \tag{4.3}$$

is well-defined and $\widetilde{S}$ is called a *smooth projective algebraic curve*, whereas $S = \widetilde{S} \cap \{[z : w : 1] \,:\, (z,w) \in \mathbb{C}^2\}$ is called the *affine part* of $\widetilde{S}$. Assuming that Q is nonsingular, i.e., $dQ \neq 0$ on $\widetilde{S}$, it follows just as before that $\widetilde{S}$ is a Riemann surface which is compact as a closed subset of the compact space $\mathbb{C}P^2$. To be more precise, we use the three charts

$$\{[1 : w : u] \mid (w,u) \in \mathbb{C}^2\},\ \{[z : w : 1] \mid (z,w) \in \mathbb{C}^2\},\ \{[z : 1 : u] \mid (z,u) \in \mathbb{C}^2\}$$

to cover $\mathbb{C}P^2$. If follows from Euler's relation $dQ(z,w,u)(z,w,u) = \nu Q(z,w,u)$ that on $\widetilde{S}$ one has $dQ(z,w,u)(z,w,u) = 0$. Consequently, if we set any one of the coordinates equal to 1, then the polynomial in the two remaining variables is nonsingular in the affine sense from above. This allows us to define complex structures on $\widetilde{S}$ over each of the three projective charts which are of course compatible with each other. The meromorphic functions $\frac{f(z)}{g(w)}$ from above extend to meromorphic functions on $\widetilde{S}$ provided f, g are rational (in this case, too, we would like to characterize the meromorphic functions on $\widetilde{S}$ – in other words, the function field on $\widetilde{S}$). Being compact, $\widetilde{S}$ has finite genus. An example of (the affine part of) a curve of genus $g \geq 1$ is given by

$$w^2 - \prod_{j=1}^{N} (z - z_j) = 0 \tag{4.4}$$

where $\{z_j\}_{j=1}^N \subset \mathbb{C}$ are distinct and $N = 2g$ or $N = 2g - 1$. For any $z_0 \in \mathbb{C} \setminus \{z_j\}_{j=1}^N$ one has local coordinates

$$w(z) = \pm \sqrt{\prod_{j=1}^{N} (z - z_j)}$$

where the two signs correspond precisely to the two "sheets" locally near z_0; note that the square root is analytic. Near any z_ℓ, $1 \leq \ell \leq N$ one sets $z := z_\ell + \zeta^2$ so that

$$w(\zeta) = \zeta \sqrt{\prod_{j=1,\, j\neq \ell}^{N} (z_\ell - z_j + \zeta^2)}$$

where the ambiguity of the choice of sign can be absorbed into ζ (the square root is again analytic). It is clear that the transition maps between the charts are holomorphic. The reader will easily verify that the projective version of (4.4) with $N \geq 2$, i.e.,

$$Q(z, w, u) := w^2 u^{N-2} - \prod_{j=1}^{N} (z - u z_j)$$

is nonsingular for $N = 2, 3$ but singular when $N \geq 4$. Indeed, if $N \geq 4$ one has $dQ = 0$ "at the point at infinity" of $\widetilde{S}$ (which means when $u = 0$). Returning to the affine curve, one refers to the z_j as branch points, since the natural projection $w \mapsto z$, which is a covering map on $\mathbb{C}_\infty \setminus \{z_j\}_{j=1}^N$, ceases to be a covering map near each z_j. Rather, one refers to this case as a *branched covering map* and an algebraic curve is a *branched cover* of the Riemann sphere. The specific algebraic curve we just considered is called *elliptic curve* if $g = 1$ or hyper-elliptic curve if $g > 1$. A very remarkable fact of the theory of Riemann surfaces is that any compact Riemann surface is the same as (i.e., conformally equivalent to) an algebraic curve defined by some irreducible polynomial $P \in \mathbb{C}[z, w]$, see Chapter 5. For this, we no longer assume that P is nonsingular and therefore proceed differently with our construction of the Riemann surface of P – it is defined via all possible analytic continuations of a locally defined analytic solution $w = w(z)$ of $P(z, w(z)) = 0$. The difference in this procedure is that we do not immediately seek to imbed this Riemann surface into $\mathbb{C}P^2$. In fact, in general this cannot be done when Q is singular (but is a result of Riemann surface theory that every compact Riemann surface can be imbedded into some $\mathbb{C}P^d$, in fact $d = 3$ suffices).

7) *Riemann surfaces defined via analytic continuation of an analytic germ*:

3. Functions on Riemann surfaces

We already defined analytic functions between Riemann surfaces and also introduced the concept of a conformal isomorphism. Note that any conformal isomorphism has an analytic inverse. In Example 1) above, the Riemann surfaces with with conformal structures induced by (Ω, z) and $(\Omega, \bar{z})$, respectively, do not have equivalent conformal structures but are conformally isomorphic (via $z \mapsto \bar{z}$). As the reader may have guessed, examples 3), 4), and 5) are isomorphic (we shall drop "conformal" when it is clearly implied). As usual,

$$PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm \mathrm{Id}\}$$

THEOREM 4.3. *The Riemann surfaces $S^2, \mathbb{C}_\infty$, and $\mathbb{C}P^1$ are conformally isomorphic. Furthermore, the group of automorphisms of these surfaces is $PSL(2, \mathbb{C})$.*

PROOF. we leave it to the reader to write down the explicit isomorphisms between these surfaces. As for the automorphism group, each

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C})$$

defines an automorphism of $\mathbb{C}P^1$ via

$$[z : w] \mapsto [az + bw : cz + dw]$$

Note that $\pm A$ define the same map (a Möbius transform). On the other hand, if f is an automorphism of $\mathbb{C}_\infty$, then composing with a Möbius transformation we may assume that $f(\infty) = \infty$. Hence, restricting f to $\mathbb{C}$ yields a map from $\mathrm{Aut}(\mathbb{C})$ which is of the form (see Problem 1.12) $f(z) = az + b$ and we are done. $\square$

We now state the important uniqueness and open mapping theorems for analytic functions on Riemann surfaces.

THEOREM 4.4 (Uniqueness theorem). *Let $f, g : M \to N$ be analytic. Then either $f \equiv g$ or $\{p \in M \mid f(p) = g(p)\}$ is discrete in M.*

PROOF. Define

$$A := \{p \in M \mid \text{locally at } p,\ f \text{ and } g \text{ are identically equal}\}$$
$$B := \{p \in M \mid \text{locally at } p,\ f \text{ and } g \text{ agree only on a discrete set}\}$$

It is clear that both A and B are open subsets of M. We claim that $M = A \cup B$ which then finishes the proof since M is connected. If $p \in M$ is such that $f(p) \neq g(p)$ then $p \in B$. If, on the other hand, $f(p) = g(p)$, then we see via the usual uniqueness theorem in charts that $\{f = g\}$ not discrete implies that $f = g$ locally around p. $\square$

As an obvious corollary, note that for any analytic $f : M \to N$ each "level set" $\{f \in M \mid f(p) = q\}$ with $q \in N$ fixed, is either discrete or all of M (and thus $f = \text{const}$). In particular, if M is compact and f not constant, then $\{p \in M \mid f(p) = q\}$ is *finite*.

THEOREM 4.5 (Open mapping theorem). *Let $f : M \to N$ be analytic. If f is not constant, then $f(M)$ is an open subset of N. More generally, f takes open subsets of M to open subsets of N.*

PROOF. By the uniqueness theorem, if f is locally constant around any point, then f is globally constant. Hence we can apply the usual open mapping theorem in every chart to conclude that $f(M) \subset N$ is open. $\square$

COROLLARY 4.6. *Let M be compact and $f : M \to N$ analytic and nonconstant. Then f is onto and N is compact.*

PROOF. Since $f(M)$ is both closed (since compact and N Hausdorff), and open by Theorem 4.5, it follows that $f(M) = N$ as claimed. $\square$

It is customary to introduce the following terminology.

DEFINITION 4.7. *The* holomorphic *functions on a Riemann surface M are defined as all analytic $f : M \to \mathbb{C}$. They are denoted by $\mathcal{H}(M)$. The* meromorphic *functions on M are defined as all analytic $f : M \to \mathbb{C}_\infty$. They are denoted by $\mathcal{M}(M)$.*

In view of the preceding the following statements are immediate.

COROLLARY 4.8. *Let M be a Riemann surface. The the following properties hold:*
i) if M is compact, then every holomorphic function on M is constant.
ii) Every meromorphic function on a compact Riemann surface is onto $\mathbb{C}_\infty$.
iii) If f is a nonconstant holomorphic function on a Riemann surface, then $|f|$ attains neither a local maximum nor a positive local minimum on M.

To illustrate what we have accomplished so far, let us give a "topological proof" of Liouville's theorem: thus assume that $f \in \mathcal{H}(\mathbb{C}) \cap L^\infty(\mathbb{C})$. Then $f(1/z)$ has a removable singularity at $z = 0$. In other words, $f \in \mathcal{H}(\mathbb{C}_\infty)$ and thus constant. The analytical ingredient in this proof consists of the uniqueness and open mapping theorems as well as the removability theorem: the first two are reduced to the same properties in charts which then require power series expansions. But instead of using expansions that converge on all of $\mathbb{C}$ and Cauchy's estimate we used connectivity to pass from a local property to a global one.

It is a good exercise at this point to verify the following: the meromorphic functions on $\Omega \subset \mathbb{C}$ in the sense of standard complex analysis coincide exactly with $\mathcal{M}(\Omega) \setminus \{\infty\}$ in the sense of Definition 4.7 (we need to discard the function which is constant equal to infinity). In particular,

$$\mathcal{M}(\mathbb{C}_\infty) = \Big\{ \frac{P}{Q} \mid P, Q \in \mathbb{C}[z],\ Q \not\equiv 0 \Big\} \cup \{\infty\}$$

In other words, the meromorphic functions on $\mathbb{C}_\infty$ up to the const $= \infty$ are exactly the rational functions. Note that we may prescribe the location of the finitely many zeros and poles of $f \in \mathcal{M}(\mathbb{C}P^1)$ arbitrarily provided the combined order of the zeros exactly equals the combined order of the poles and provided the set of zeros is distinct from the set of poles (construct the corresponding rational function).

4. Degree and genus

DEFINITION 4.9. *Let $f : M \to N$ be analytic and nonconstant. Then the* valency *of f at $p \in M$, denoted by $\nu_f(p)$, is defined to be the unique positive integer with the property that in charts (U, ϕ) around p (with $f(p) = 0$) and (V, ψ) around $f(p)$ (with $\psi(f(p)) = 0$) we have $(\psi \circ f \circ \phi^{-1})(z) = (zh(z))^n$ where $h(0) \neq 0$. If M is compact, then the degree of f at $q \in N$ is defined as*

$$\deg_f(q) := \sum_{p: f(p)=q} \nu_f(p)$$

which is a positive integer.

Locally around any point $p \in M$ with valency $\nu_f(p) = n \geq 1$ the map f is n-to-one; in fact, every point q' close but not equal to $q = f(p)$ has exactly n pre-images close to p.

Let $f = \frac{P}{Q}$ be a nonconstant rational function on $\mathbb{C}_\infty$ represented by a reduced fraction (i.e., P and Q are relatively prime). Then for every $q \in \mathbb{C}_\infty$, the reader will easily verify that $\deg_f(q) = \max(\deg(Q), \deg(P))$ where the degree of P, Q is in the sense of polynomials. It is a general fact that $\deg_f(q)$ does not depend on $q \in N$.

LEMMA 4.10. *Let $f : M \to N$ be analytic and nonconstant with M compact. Then $\deg_f(q)$ does not depend on q. It is called* the degree *of f and denoted by* $\deg(f)$. *The isomorphisms from M to N are precisely those nonconstant analytic maps f on M with* $\deg(f) = 1$.

PROOF. Recall that f is necessarily onto N. We shall prove that $\deg_f(q)$ is locally constant. Let $f(p) = q$ and suppose that $\nu_f(p) = 1$. As remarked before, f is then an isomorphism from a neighborhood of p onto a neighborhood of q. If, on the other hand, $n = \nu_f(p) > 1$, then each q' close but not equal to q has exactly n preimages $\{p'_j\}_{j=1}^n$ and

$\nu_f(p'_j) = 1$ at each $1 \le j \le n$. This proves that $\deg_f(q)$ is locally constant and therefore globally constant by connectivity of N. The statement concerning isomorphisms is evident. $\square$

We remark that this notion of degree coincides with the usual one from differentiable manifolds, see Chapter 13. Let us now prove the Riemann–Hurwitz formula for *branched covers*. If necessary, the reader should review the Euler characteristic and the genus on compact surfaces, see Chapter 13. This simply refers to an analytic nonconstant map $f : M \to N$ from a compact Riemann surface M onto another compact surface N.

THEOREM 4.11 (Riemann–Hurwitz). *Let $f : M \to N$ be an analytic nonconstant map between compact Riemann surfaces. Define the* total branching number *to be*

$$B := \sum_{p\in M} (\nu_f(p) - 1)$$

Then

$$g_M - 1 = \deg(f)(g_N - 1) + \frac{1}{2}B \tag{4.5}$$

where g_M and g_N are the genera of M and N, respectively. In particular, B is always an even nonnegative integer.

PROOF. Denote by $\mathcal{B}$ all $p \in M$ with $\nu_f(p) > 1$ (the branch points). Let $\mathcal{T}$ be a triangulation of N such that all $f(p)$, $p \in \mathcal{B}$ are vertices of $\mathcal{T}$. Lift $\mathcal{T}$ to a triangulation $\widetilde{\mathcal{T}}$ on M. If $\mathcal{T}$ has V vertices, E edges and F faces, then $\mathcal{T}$ has $nV - B$ vertices, nE edges, and nF faces where $n = \deg(f)$. Therefore, by the Euler–Poincaré formula (13.1),

$$\begin{aligned} 2(1 - g_N) &= V - E + F \\ 2(1 - g_M) &= nV - B - nE - nF = 2n(1 - g_N) - B \end{aligned}$$

as claimed. $\square$

5. Riemann surfaces as quotients

Many Riemann surfaces M are generated as quotients of other surfaces N modulo an equivalence relation, i.e., $M = N/\sim$. A common way of defining the equivalence relation is via the action of a subgroup $G \subset \mathrm{Aut}(N)$. Then $q_1 \sim q_2$ in N iff there exists some $g \in G$ with $gq_1 = q_2$. Let us state a theorem to this effect where $N = \mathbb{C}_\infty$. Examples will follow immediately after the theorem. First, let us recall the notion of a *covering map*.

DEFINITION 4.12. *Let X, Y be topological spaces and $f : Y \to X$ is called a* covering map *if and only if every $x \in X$ has a neighborhood U in X so that $f^{-1}(U) = \bigcup_j V_j$ with open and disjoint $V_j \subset Y$ so that $f : V_j \to U$ is a homeomorphism for each j.*

In particular, a covering map is a local homeomorphism. For example $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map, as is $z^n : \mathbb{C}^* \to C^*$ for each $n \ge 1$. Note that if $n \ge 2$, then the latter example does not extend to a covering map $\mathbb{C} \to \mathbb{C}$; rather, we encounter a branch point at zero and this extension is then referred to as a *branched cover*.

THEOREM 4.13. *Let $\Omega \subset \mathbb{C}_\infty$ and $G \subset \mathrm{Aut}(\mathbb{C}_\infty)$ with the property that*

- *$g(\Omega) \subset \Omega$ for all $g \in G$*
- *for all $g \in G$, $g \ne \mathrm{id}$, all fixed points of g in $\mathbb{C}_\infty$ lie outside of Ω*

• *let $K \subset \Omega$ be compact. Then the cardinality of $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is finite.* *Under these assumptions, the natural projection $\pi : \Omega \to \Omega/G$ is a covering map which turns Ω/G canonically into a Riemann surface.*

Proof. By definition, the topology on Ω/G is the coarsest one that makes π continuous. In this case, π is also open; indeed, for every open $A \subset \Omega$,

$$\pi^{-1}(\pi(A)) = \bigcup_{g\in G} g(A)$$

is open since $g(A)$ is open. Next, let us verify that the topology is Hausdorff. Suppose $\pi(z_1) \neq \pi(z_2)$ and define for all $n \geq 1$,

$$A_n := \left\{ z \in \Omega \,\Big|\, |z - z_1| < \frac{r}{n} \right\} \subset \Omega$$
$$B_n := \left\{ z \in \Omega \,\Big|\, |z - z_2| < \frac{r}{n} \right\} \subset \Omega$$

where $r > 0$ is sufficiently small. Define $K := \overline{A}_1 \cup \overline{B}_1$ and suppose that $A_n \cap B_n \neq \emptyset$ for all $n \geq 1$. Then for some $a_n \in A_n$ and $g_n \in G$ we have

$$g_n(a_n) \in B_n \qquad \forall\, n \geq 1$$

Since in particular $g_n(K) \cap K \neq \emptyset$, we see that there are only finitely many possibilities for g_n and one of them therefore occurs infinitely often. Let us say that $g_n = g \in G$ for infinitely many n. Passing to the limit $n \to \infty$ implies that $g(z_1) = z_2$ or $\pi(z_1) = \pi(z_2)$, a contradiction. For all $z \in \Omega$ we can find a small pre-compact open neighborhood of z denoted by $K_z \subset \Omega$, so that

$$g(\overline{K_z}) \cap \overline{K_z} = \emptyset \qquad \forall\, g \in G,\ g \neq \mathrm{id} \tag{4.6}$$

where we are using all three assumptions. Then $\pi : K_z \to K_z$ is the identity and therefore we can use the K_z as charts. Note that the transition maps are given by $g \in \mathrm{Aut}(\mathbb{C}_\infty)$ (which are Möbius transformations) and are therefore holomorphic. Finally, $\pi^{-1}(K_z) = \bigcup_{g\in G} g^{-1}(K_z)$ with pair wise disjoint open $g^{-1}(K_z)$. The disjointness follows from (4.6) and we are done. $\square$

We remark that any group G as in the theorem is necessarily *discrete* in the topological sense. First, this is meaningful as $G \subset \mathrm{Aut}(\mathbb{C}) = PSL(2, \mathbb{C})$ with the latter carrying a natural topology; second, if G is not discrete, then the third requirement in the theorem will fail (since we can find group elements in G as close to the identity as we wish). There are many natural examples to which this theorem applies (in what follows, we use $\langle g_1, g_2, \ldots, g_k\rangle$ to denote the group generated by these k elements) and as usual, $\mathbb{H}$ is the upper half-plane.

1) The punctured plane and disk: $\mathbb{C}/\langle z \mapsto z + 1\rangle \simeq \mathbb{C}^*$ where the isomorphism is given by the exponential map $e^{2\pi i z}$. Here $\Omega = \mathbb{C}$, and $G = \langle z \mapsto z + 1\rangle$. Similarly, $\mathbb{H}/\langle z \mapsto z + 1\rangle \simeq \mathbb{D}^*$.

2) The tori: Let $\omega_1, \omega_2 \in \mathbb{C}^*$ be linearly independent over $\mathbb{R}$. Then

$$\mathbb{C}/\langle z \mapsto z + \omega_1, z \mapsto z + \omega_2\rangle$$

is a Riemann surface. It is the same as $\mathbb{C}/\Lambda$ with the lattice

$$\Lambda = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\} \tag{4.7}$$

In Figure 4.4 the lattice is generated by any distinct pair of vectors from $\{\omega_1, \omega_2, \omega_3\}$

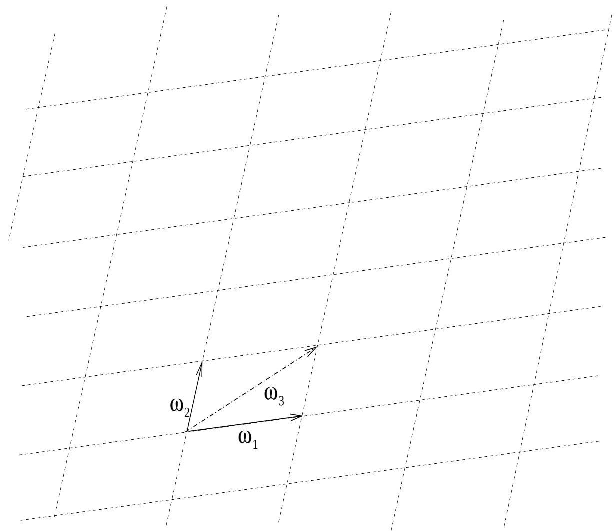

FIGURE 4.3. A lattice in $\mathbb{C}$

which proves that a pair of generators is not unique. Furthermore, up to conformal equivalence we may always assume that $\omega_1 = 1$ and $\omega_2 = \tau$ where $\mathrm{Im}\,(\tau) > 0$. A special case is given by

$$\mathbb{C}^*/\langle z \mapsto \lambda z\rangle \simeq \mathbb{C}/\langle z \mapsto z+1, z \mapsto z + \frac{1}{2\pi i}\log\lambda\rangle$$

where $\lambda > 1$. The same exponential map as in 1) induces the isomorphism here. A very important question is to determine the so-called moduli space of tori, which is defined to be the space of all conformal equivalence classes of tori. For this, see Problem 4.3.

3) The annuli: consider $\mathbb{H}/\langle z \mapsto \lambda z\rangle$ with $\lambda > 1$. Then $\log z$ maps this onto

$$\{w \in \mathbb{C} \;:\; 0 < \mathrm{Im}\, w < \pi i,\; 0 \le \mathrm{Re}\, w \le \log\lambda\}$$

with the sides $\mathrm{Re}\, w = 0, \mathrm{Re}\, w = \log\lambda$ identified. Next, send this via the conformal map

$$w \mapsto \exp\big(2\pi i \frac{w}{\log\lambda}\big)$$

onto the annulus $\Delta_r := \{r < |z| < 1\}$ where $\log r = -\frac{2\pi^2}{\log\lambda}$. We leave it to the reader to check that no two Δ_r are conformally equivalent (hence, the moduli space of tori $\{z \in \mathbb{C}^* \;:\; r_1 < |z| < r_2\}$ with $0 < r_1 < r_2$ is the same as all $\frac{r_2}{r_1}$, i.e., $(0,\infty)$.

This list of examples is important for a number of reasons. First, we remark that we have exhausted all possible examples with $\Omega = \mathbb{C}$. Indeed, we leave it to the reader to verify that all nontrivial discrete subgroups of $\mathrm{Aut}(\mathbb{C})$ *that have no fixed point* are either $\langle z \mapsto z+\omega\rangle$ with $\omega \ne 0$, or $\langle z \mapsto z+\omega_1, z \mapsto z+\omega_2\rangle$ with $\omega_1 \ne 0, \omega_2/\omega_1 \notin \mathbb{R}$, see Problem 4.1. Second, $\mathbb{C}^*, \mathbb{D}^*, \Delta_r$ and $\mathbb{C}/\Lambda$ where Λ is a lattice, is a complete list of Riemann surfaces (up to conformal equivalence, of course) with nontrivial, abelian fundamental group. This latter property is of course not so easy to see, cf. Chapter 11 as well as [**16**], Chapter IV.6.

6. Elliptic functions

For the remainder of this section, we let

$$M = \mathbb{C}/\langle z \mapsto z + \omega_1, z \mapsto z + \omega_2 \rangle$$

be the torus of Example 2). As usual, we refer to the group relative to which we are factoring as the lattice Λ. We first remark that $\omega_1' := a\omega_1 + b\omega_2$, $\omega_2' := c\omega_1 + d\omega_2$ is another generator of the same lattice iff $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$. Thus, in Figure 4.4 we can pass to other generators as well as other fundamental regions. The latter here refers to any closed connected set $P \subset \mathbb{C}$ with the property that

i) every point in z is congruent modulo the lattice to some point of P

ii) no pair of points from the interior of P are congruent

In Figure 4.4 the parallelogram spanned by ω_1, ω_2 is one such fundamental region, whereas the parallelogram spanned by ω_1, ω_3 is another. An important, and in some ways canonical choice of such a region is given by the *Dirichlet polygon*, see Problem 4.2. Let us now turn to the study of meromorphic functions on the torus M. By definition

$$\mathcal{M}(M) = \{f \in \mathcal{M}(\mathbb{C}) \mid f = f(\cdot + \omega_1) = f(\cdot + \omega_2)\}$$

where we ignore the function constant equal ∞. These are called *doubly periodic* or *elliptic functions*. First, since M is compact the only holomorphic functions are the constants. Next, we claim that any nonconstant function $f \in \mathcal{M}(M)$ satisfies $\deg(f) \geq 2$. Indeed, suppose $\deg(f) = 1$. Then, in the notation of Riemann–Hurwitz, $B = 0$ and therefore $1 = g_M = g_{S^2} = 0$, a contradiction. The reader should check that we can arrive at the same conclusion by verifying that

$$\oint_{\partial P} \frac{f'(z)}{f(z)}\, dz = 0$$

which implies that the sum of the residues inside P is zero (here P is a fundamental domain so that f has neither zeros no poles on its boundary). Notice also, from Riemann–Hurwitz, that any elliptic function f with $\deg(f) = 2$ satisfies $B = 4$ and therefore has exactly four branch points each with valency 2. An interesting question concerns the existence of elliptic functions of minimal degree, viz. $\deg(f) = 2$. We shall now present the classical Weierstrass function $\wp$ which is of this type. It is an important fact that all elliptic functions can be expressed in terms of this one function, see Proposition 4.16.

Proposition 4.14. *For any $n \geq 3$, the series*

$$f(z) = \sum_{w \in \Lambda} (z + w)^{-n}$$

defines a function $f \in \mathcal{M}(M)$ with $\deg(f) = n$. Furthermore, the Weierstrass function

$$\wp(z) := \frac{1}{z^2} + \sum_{w \in \Lambda^*} \big[(z + w)^{-2} - w^{-2}\big],$$

is an elliptic function of degree two with Λ as its group of periods. Here Λ is as in (4.7) *and $\Lambda^* := \Lambda \setminus \{0\}$.*

Proof. If $n \geq 3$, then we claim that

$$f(z) = \sum_{w \in \Lambda} (z + w)^{-n}$$

converges absolutely and uniformly on every compact set $K \subset \mathbb{C}\setminus\Lambda$. Indeed, there exists $C > 0$ such that

$$C^{-1}(|x| + |y|) \le |x\omega_1 + y\omega_2| \le C(|x| + |y|)$$

for all $x, y \in \mathbb{R}$. Hence, when $z \in \{x\omega_1 + y\omega_2 \mid 0 \le x, y \le 1\}$, then

$$|z + (k_1\omega_1 + k_2\omega_2)| \ge C^{-1}(|k_1| + |k_2|) - |z| \ge (2C)^{-1}(|k_1| + |k_2|)$$

provided $|k_1| + |k_2|$ is sufficiently large. Since

$$\sum_{|k_1|+|k_2|>0} |k_1\omega_1 + k_2\omega_2|^{-n} < \infty$$

as long as $n > 2$, we conclude $f \in \mathcal{H}(\mathbb{C} \setminus \Lambda)$. Since $f(z) = f(z + \omega_1) = f(z + \omega_2)$ for all $z \in \mathbb{C} \setminus \Lambda$, it is clear that $f \in \mathcal{M}(M)$.

For the second part, we note that

$$\left|(z + w)^{-2} - w^{-2}\right| \le \frac{|z||z + w|}{|w|^2|z + w|^2} \le \frac{C}{|w|^3}$$

provided $|w| > 2|z|$ so that the series defining $\wp$ converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$. For the periodicity of $\wp$, note that

$$\wp'(z) = -2 \sum_{w\in\Lambda} (z + w)^{-3}$$

is periodic relative to the lattice Λ. Thus, for every $w \in \Lambda$,

$$\wp(z + w) - \wp(z) = C(w) \quad \forall\, z \in \mathbb{C}$$

with some constant $C(w)$. Expanding around $z = 0$ yields $C(w) = 0$ as desired. $\square$

Another way of obtaining the function $\wp$ is as follows: let σ be defined as the Weierstrass product

$$\sigma(z) := z \prod_{\omega\in\Lambda^*} E_2(z/\omega) \tag{4.8}$$

with canonical factors E_2 as in Chapter 2. Then σ is entire with simple zeros precisely at the points of Λ. Now let

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega\in\Lambda^*} \Big[\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\Big]$$

By inspection, $\wp = -\zeta'$.

The $\wp$ function has many remarkable properties, the most basic of which is the following differential equation.

Lemma 4.15. *With $\wp$ as before, one has*

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \tag{4.9}$$

where $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$, and $e_3 = \wp((\omega_1 + \omega_2)/2)$. Furthermore, one has $e_1 + e_2 + e_3 = 0$ so that (4.9) *can be written in the form*

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$$

with constants $g_j(\omega_1, \omega_2)$, $j = 2, 3$.

PROOF. By inspection,

$$\wp'(z) = -2\sum_{z\in\Lambda}(z+w)^{-3}$$

is an odd function $\in \mathcal{M}(M)$ of degree three. Thus,

$$\wp'(\omega_1/2) = -\wp'(-\omega_1/2) = -\wp'(\omega_1/2) = 0$$

Similarly, $\wp'(\omega_2/2) = \wp'((\omega_1+\omega_2)/2) = 0$. In other words, the three points $\omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2$ are the three zeros of $\wp'$ and thus also the unique points where $\wp$ has valency two apart from $z = 0$. Denoting the right-hand side of (4.9) by $F(z)$, this implies that $\frac{\wp'(z)}{F(z)} \in \mathcal{H}(M)$ and therefore equals a constant. Considering the expansion of $\wp'(z)$ and $F(z)$, respectively, around $z = 0$ shows that the value of this constant equals 1, as claimed. The final statement follows by observing from the Laurent series around zero that

$$(\wp'(z))^2 - 4(\wp(z))^3 - g_2\wp(z)$$

for suitable g_2 is analytic and therefore constant. □

The previous proof shows that $0, \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2$ are precisely the branch points of $\wp$. We are now able to establish the following property of $\wp$.

PROPOSITION 4.16. *Every $f \in \mathcal{M}(M)$ is a rational function of $\wp$ and $\wp'$. If f is even, then it is a rational function of $\wp$ alone.*

PROOF. Suppose that f is nonconstant and even. Then for all but finitely many values of $w \in \mathbb{C}_\infty$, the equation $f(z) - w = 0$ has only simple zeros (since there are only finitely many zeros of f'). Pick two such $w \in \mathbb{C}$ and denote them by c, d. Moreover, we can ensure that the zeros of $f - c$ and $f - d$ are distinct from the branch points of $\wp$. Thus, since f is even and with $2n = \deg(f)$,

$$\{z \in M \,:\, f(z) - c = 0\} = \{a_j, -a_j\}_{j=1}^n$$
$$\{z \in M \,:\, f(z) - d = 0\} = \{b_j, -b_j\}_{j=1}^n$$

The elliptic functions

$$g(z) := \frac{f(z) - c}{f(z) - d}$$

and

$$h(z) := \prod_{j=1}^n \frac{\wp(z) - \wp(a_j)}{\wp(z) - \wp(b_j)}$$

have the same zeros and poles which are all simple. It follows that $g = \alpha h$ for some $\alpha \neq 0$. Solving this relation for f yields the desired conclusion.

If f is odd, then $f/\wp'$ is even so $f = \wp' R(\wp)$ where R is rational. Finally, if f is any elliptic function then

$$f(z) = \frac{1}{2}(f(z) + f(-z)) + \frac{1}{2}(f(z) - f(-z))$$

is a decomposition into even/odd elliptic functions whence

$$f(z) = R_1(\wp) + \wp' R_2(\wp)$$

with rational R_1, R_2, as claimed. □

For another result along these lines see Problem 4.4. It is interesting to compare the previous result for the tori to a similar one for the simply periodic functions, i.e., functions on the surface $\mathbb{C}/\langle z \mapsto z+1 \rangle \simeq \mathbb{C}^*$. These can be represented via Fourier series, i.e, infinite expansions in the basis $e^{2\pi i z}$ which plays the role of $\wp$ in this case. The reason we obtain infinite expansions rather than the finite ones in the case of tori lies with the fact that the latter are compact whereas $\mathbb{C}^*$ is not.

We conclude our discussion of elliptic functions by turning to the following natural question: to given disjoint finite sets of distinct points $\{z_j\}$ and $\{\zeta_k\}$ in M as well as positive integers n_j for z_j and ν_k for ζ_k, respectively, is there an elliptic function with precisely these zeros and poles and the given orders? We remark that for the case of $\mathbb{C}_\infty$ the answer was affirmative if and only if the constancy of the degree was not violated, i.e.,

$$\sum_j n_j = \sum_k \nu_k \tag{4.10}$$

For the tori, however, there is a new obstruction:

$$\sum_j n_j z_j - \sum_k \nu_k \zeta_k = 0 \qquad \text{mod } \Lambda \tag{4.11}$$

This follows from the residue theorem since

$$\frac{1}{2\pi i} \oint_{\partial P} z \frac{f'(z)}{f(z)}\, dz = \sum_j n_j z_j - \sum_k \nu_k \zeta_k \tag{4.12}$$

where ∂P is the boundary of a fundamental domain P as in Figure 4.4 so that no zero or pole of f lies on that boundary. Comparing parallel sides of the fundamental domain and using the periodicity shows that the left-hand side in (4.12) is of the form $n_1\omega_1 + n_2\omega_2$ with $n_1, n_2 \in \mathbb{Z}$ and thus 0 in Λ.

It is a remarkable fact that (4.10) and (4.11) are also *sufficient* for the existence of an elliptic function with precisely these given sets of zeros and poles.

THEOREM 4.17. *Suppose* (4.10) *and* (4.11) *hold. Then there exists an elliptic function which has precisely these zeros and poles with the given orders.*

PROOF. Listing the points z_j and ζ_k with their respective multiplicities, we obtain sequences z_j' and ζ_k' of the same length, say n. Shifting by a lattice element if needed, one has

$$\sum_{j=1}^n z_j' = \sum_{k=1}^n \zeta_k'$$

Define, see (4.8) for σ,

$$f(z) = \prod_{j=1}^n \frac{\sigma(z - z_j')}{\sigma(z - \zeta_j')}$$

which has the desired zeros and poles. It remains to check the periodicity which we leave to the reader. ∎

7. Problems

Problem 4.1. *Show that all nontrivial discrete subgroups of* $\mathrm{Aut}(\mathbb{C})$ that have no fixed point *are either* $\langle z \mapsto z + \omega \rangle$ *with* $\omega \neq 0$, *or* $\langle z \mapsto z + \omega_1, z \mapsto z + \omega_2 \rangle$ *with* $\omega_1 \neq 0, \omega_2/\omega_1 \notin \mathbb{R}$.

Problem 4.2. *Let* $\Lambda \subset \mathbb{C}$ *be the lattice (i.e., the discrete subgroup) generated by* ω_1, ω_2 *which are independent over* $\mathbb{R}$. *Show that the* Dirichlet polygon

$$\{z \in \mathbb{C} \, : \, |z| \leq |z - \omega| \quad \forall \, \omega \in \Lambda\}$$

is a fundamental region, cf. Chapter 4.

Problem 4.3. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{C})$ *and* $\Lambda = \langle z \mapsto z + \omega_1, z \mapsto z + \omega_2 \rangle$ *with* ω_1, ω_2 *linearly independent over* $\mathbb{R}$. *Prove that* $\Lambda = \langle z \mapsto z + \omega_1', z \mapsto z + \omega_2' \rangle$ *where*

$$\begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} = A \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

if and only if $A \in GL(2, \mathbb{Z})$ *with* $\det(A) = \pm 1$. *Now show that the moduli space of tori as defined in Section 4 of Chapter 4 is* $\mathbb{H}/PSL(2, \mathbb{Z})$. *The group* $PSL(2, \mathbb{Z})$ *is called* modular group, *see for example* [**22**] *as well as many other sources.*

Problem 4.4. *Let* $M = \mathbb{C}/\Lambda$ *where* Λ *is the lattice generated by* $\omega_1, \omega_2 \in \mathbb{C}^*$ *with* $\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0$. *As usual* $\mathcal{P}$ *denotes the Weierstrass function on* M. *Suppose that* $f \in \mathcal{M}(M)$ *has degree two. Prove that there exists* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C})$ *and* $w \in \mathbb{C}$ *such that*

$$f(z) = \frac{a\wp(z - w) + b}{c\wp(z - w) + d}$$

Problem 4.5. *Suppose* N *is a Riemann surface such that* $\bar{N}$ *is compact and is a manifold with boundary. I.e., for every* $p \in \partial N$ *there exists a neighborhood* U *of* p *in* $\bar{N}$ *and a map* $\phi : U \to \mathbb{R}^2_+$ *such that* ϕ *takes* U *homeomorphically onto* $\mathbb{D} \cap \{\mathrm{Im}\, z \geq 0\}$. *Moreover, we demand that the transition maps between such charts are conformal on* $\mathrm{Im}\, z > 0$. *Prove that then* $\bar{N} \subset M$ *where* M *is a Riemann surface. In other words,* N *can be extended to a strictly larger Riemann surface.*

Problem 4.6. *Let* M *be a compact Riemann surface and* $S \subset M$ *discrete. Suppose* $f : M \setminus S \to \mathbb{C}$ *is analytic and nonconstant. Show that the image of* $M \setminus S$ *under* f *is dense in* $\mathbb{C}$.

Problem 4.7. *Fill in the missing details in Corollary 5.20. I.e., first check that* Φ *is indeed a homeomorphism from* $\mathbb{C}/\Gamma$ *onto* E. *Second, verify the integrals* (5.11).

Problem 4.8. *Let* M, N *be compact Riemann surfaces and suppose* $\Phi : M \setminus \mathcal{S} \to N \setminus \mathcal{S}'$ *is an isomorphism where* $\mathcal{S}, \mathcal{S}'$ *are finite sets. Then* Φ *extends to an isomorphism from* $M \to N$.

Problem 4.9. *This exercise revisits fractional linear transformations.*
(a) Prove that

$$G = \left\{ \begin{bmatrix} a & \bar{b} \\ b & \bar{a} \end{bmatrix} \, : \, a, b \in \mathbb{C}, \ |a|^2 - |b|^2 = 1 \right\}$$

is a subgroup of $SL(2,\mathbb{C})$ *(it is known as* $SU(1,1)$*). Establish the group isomorphism* $G/\{\pm I\} \simeq \mathrm{Aut}(\mathbb{D})$ *in two ways: (i) by showing that each element of* G *defines a fractional linear transformation which maps* $\mathbb{D}$ *onto* $\mathbb{D}$*; and conversely, that every such fractional linear transformation arises in this way uniquely up to the signs of* a, b*. (ii) By showing that the map*

$$e^{2i\theta}\frac{z-z_0}{1-\bar{z}_0 z} \mapsto \begin{bmatrix} \frac{e^{i\theta}}{\sqrt{1-|z_0|^2}} & \frac{-z_0 e^{i\theta}}{\sqrt{1-|z_0|^2}} \\ -\frac{\bar{z}_0 e^{-i\theta}}{\sqrt{1-|z_0|^2}} & \frac{e^{-i\theta}}{\sqrt{1-|z_0|^2}} \end{bmatrix} \tag{4.13}$$

leads to an explicit isomorphism.

(b) We know from a previous problem that $\mathrm{Aut}(\mathbb{C}_\infty)$ *is the group of all fractional linear transformations, i.e.,*

$$\mathrm{Aut}(\mathbb{C}_\infty) = PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm \mathrm{Id}\}$$

and that each such transformation induces a conformal homeomorphism of S^2 *(indeed, the stereographic projection is conformal). The purpose of this exercise is to identity the subgroup* G_{rig} *of those transformations in* $\mathrm{Aut}(\mathbb{C}_\infty)$ *which are* isometries *(in other words, rigid motions) of* S^2 *(viewing* $\mathbb{C}_\infty$ *as the Riemann sphere* S^2*). Prove that*

$$G_{\mathrm{rig}} \simeq SO(3) \simeq SU(2)/\{\pm I\}$$

where

$$SU(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \,:\, a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}$$

Find the fractional linear transformation which corresponds to a rotation of S^2 *of angle* $\frac{\pi}{2}$ *about the* x_1*-axis.*

(c) Show that the quaternions can be viewed as the four-dimensional real vector space spanned by the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

and with the algebra structure being defined via the matrix products of the e_j*'s (typically, one writes* $1, i, j, k$ *instead of* e_1, e_2, e_3, e_4*). Show that in this representation the unit quaternions are nothing but* $SU(2)$ *and exhibit a homomorphism* Q *of the unit quaternions onto* $SO(3)$ *so that* $\ker(Q) = \{\pm 1\}$*.*

Which rotation does the unit quaternion $\xi_1 + \xi_2 i + \xi_3 j + \xi_4 k$ *represent (i.e., what are the axis and angle of rotation)?*

CHAPTER 5

Analytic continuation, covering surfaces, and algebraic functions

1. Analytic continuation

This chapter takes us back to the origins of Riemann surfaces as a way of "explaining" multi-valued functions arising through analytic continuation. As an example, consider the polynomial equation

$$w^4 - 2w^2 + 1 - z = 0 \tag{5.1}$$

It has a solution $(z, w) = (1, \sqrt{2})$ and near $z = 1$ this gives rise to the function $w_1(z) = \sqrt{1 + \sqrt{z}}$ which solves (5.1); the convention here is that $\sqrt{z} > 0$ if $z > 0$. However, there are other solutions, namely $(z, w) = (1, -\sqrt{2})$ leading to $w_2(z) = -\sqrt{1 + \sqrt{z}}$ as well as $(z, w) = (1, 0)$. The latter formally corresponds to the functions $w(z) = \pm\sqrt{1 - \sqrt{z}}$ which are not analytic near $z = 0$. Thus, $(1, 0)$ is referred to as a branch point and it is characterized as a point obstruction to analytic continuation. The purpose of this chapter is to put this example, as well as all such algebraic equations, on a solid foundation in the context of Riemann surfaces.

Some of the material in this chapter may seem overly "abstract" due to the cumbersome definitions we will have to work through. The reader should therefore always try to capture the simple geometric ideas underlying these notions. To begin with, we define *function elements* or *germs* and their analytic continuations. There is a natural equivalence relation on these function elements leading to the notion of a *complete analytic function.*

Definition 5.1. *Let M, N be fixed Riemann surfaces. A* function element *is a pair (f, D) where $D \subset M$ is a connected, open non-empty subset of M and $f : D \to N$ is analytic. We say that two function elements (f_1, D_1) and (f_2, D_2) are* direct analytic continuations *of each other iff*

$$D_1 \cap D_2 \neq \emptyset, \qquad f_1 = f_2 \quad on \quad D_1 \cap D_2$$

Note that by the uniqueness theorem on Riemann surfaces there is at most one f_2 that makes (f_2, D_2) a direct analytic continuation of (f_1, D_1). This relation, denoted by $\simeq$, is reflexive and symmetric but not transitive, cf. Figure 5.1. On the other hand, it gives rise to an equivalence relation, denoted by $\sim$, in the following canonical way:

Definition 5.2. *We say that two function elements (f, D) and $(g, \widetilde{D})$ are* analytic continuations *of each other iff there exist function elements (f_j, D_j), $0 \le j \le J$ such that $(f_0, D_0) = (f, D)$, $(f_J, D_J) = (g, \widetilde{D})$, and $(f_j, D_j) \simeq (f_{j+1}, D_{j+1})$ for each $0 \le j < J$.*

The complete analytic function *of (f, D) is simply the equivalence class $[(f, D)]_\sim$ of this function element under $\sim$.*

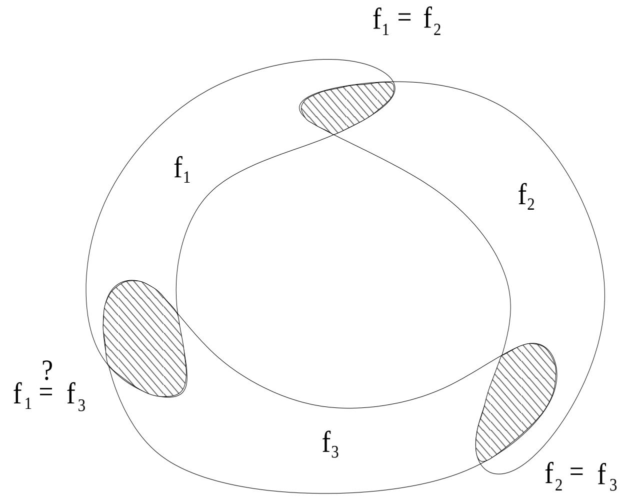

FIGURE 5.1. Failure of transitivity

Our main goal for now is to analyze the complete analytic function in more detail. But before doing so, we need to clarify the process of analytic continuation on a Riemann surface some more. In particular, we wish to prove an analogue of the monodromy theorem, Theorem 2.16, for Riemann surfaces. To begin with, we remark that the notion of analytic continuation from Definition 5.2 agrees with analytic continuation along paths.

LEMMA 5.3. *Two function elements (f_1, D_1) and (f_2, D_2) are analytic continuations of each other in the sense of Definition 5.2 iff there exists a continuous path $\gamma : [0,1] \to M$ connecting a point of D_1 to one of D_2 with the following property: there exists a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ so that for each $0 \leq i < m$, $\gamma([t_i, t_{i+1}])$ belongs to a single parametric disk of M and analytic continuation relative to each of these local coordinates in the sense of Definition 2.14 leads (necessarily uniquely) from f_1 defined around $\gamma(0)$ to f_2 defined around $\gamma(1)$.*

PROOF. This is an immediate consequence of the path-connectedness of M and the uniqueness theorem on Riemann surfaces. It there is such a path γ, then we let the (f_j, D_j) in Definition 5.2 be given by evaluating the analytically continued function along a sufficiently fine net of points on γ. Conversely, connecting points from the sets D_j by continuous paths and partitioning the path into sufficiently many intervals shows that f_2 is obtained from f_1 via continuation in local coordinates. □

As an important corollary, we can now state the monodromy theorem.

COROLLARY 5.4. *Let $H : [0,1]^2 \to M$ be a homotopy with fixed endpoints p and q. Suppose (f, D) is a function element with $p \in D$ which can be analytically continued along each path $H(\cdot, s)$ of the homotopy leading to a function element (g_s, D_s) with $q \in D_s$, $0 \leq s \leq 1$. The $g_0 = g_1$ on $D_0 \cap D_1$. If H does not fix end points but each path $H(\cdot, s)$ is a closed curve, then the following holds: suppose (f_0, D_0) is a function element with $H(0,0) \in D_0$ which can be analytically continued along $H(0,s)$ yielding function elements (f_s, D_s) with $H(0,s) \in D_s$, $0 \leq s \leq 1$. Assume further that each (f_s, D_s) can be analytically continued along $H(\cdot, s)$ yielding $(g_s, \widetilde{D}_s)$. If $f_0 = g_0$ on $D_0 \cap \widetilde{D}_0$ then $f_1 = g_1$ on $D_1 \cap \widetilde{D}_1$.*

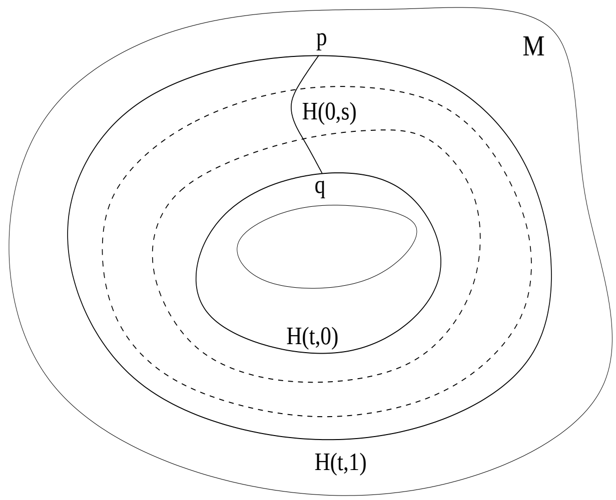

FIGURE 5.2. A homotopy of closed curves

PROOF. We begin with the fixed endpoint case. By the previous lemma, analytic continuation can be carried out in local coordinates. More precisely, we can cover each path by parametric disk and analytically continue locally in each of these disks. We can use the same disks as long as we move the parameter s by very little. But then the analytic continuation does not depend on s since it has the same property in local coordinates. For the closed curves version, we argue analogously, covering by parametric disks and changing s by very little. Figure 5.1 explains the meaning of the monodromy statement for closed curves. The short arc connecting p and q is $H(0, s)$ and we are assuming that we are allowed to analytically continue f_s along all curves $t \mapsto H(t, s)$. The point is that if we come back to the same function at p, then we have to come back to the same function at q. $\square$

The monodromy theorem of course implies that on *simply connected* Riemann surfaces M any function element (f, D) that can be analytically continued *everywhere on* M defines a global analytic function on M. Another way of arriving at the same conclusion is furnished by the following "sheaf theoretic" device that can be a very useful tool. The logic is as follows: if on a simply connected Riemann surface we can always glue function elements together locally, then this can also be done globally.

LEMMA 5.5. *Suppose M is a simply connected Riemann surface and $\{D_\alpha \subset M : \alpha \in A\}$ a collection of domains (connected, open). Assume further that $M = \cup_{\alpha \in A} D_\alpha$ and that for each $\alpha \in A$ there is a family F_α of analytic functions $f : D_\alpha \to N$ such that if $f \in F_\alpha$ and $p \in D_\alpha \cap D_\beta$ then there is some $g \in F_\beta$ so that $f = g$ near p. Then, given $\gamma \in A$ and some $f \in F_\gamma$ there exists an analytic function $F_\gamma : M \to N$ so that $F_\gamma = f$ on D_γ.*

PROOF. Let

$$\mathcal{U} := \{(p, f) \mid p \in D_\alpha,\ f \in F_\alpha,\ \alpha \in A\}/\sim$$

where $(p, f) \sim (q, g)$ iff $p = q$ and $f = g$ in a neighborhood of p. Let $[p, f]$ denote the equivalence class of (p, f). As usual, $\pi([p, f]) := p$. For each $f \in F_\alpha$, let

$$D'_{\alpha,f} := \{[p, f] \mid p \in D_\alpha\}$$

Clearly, $\pi : D'_{\alpha,f} \to D_\alpha$ is bijective. We define a topology on $\mathcal{U}$ as follows: $\Omega \subset D'_{\alpha,f}$ is open iff $\pi(\Omega) \subset D_\alpha$ is open. An arbitrary $\Omega \subset \mathcal{U}$ is open iff $\Omega \cap D'_{\alpha,f}$ is open for each $\alpha, f \in F_\alpha$. This does indeed define open sets in $\mathcal{U}$: since $\pi(D'_{\alpha,f} \cap D'_{\beta,g})$ is the union of connected components of $D_\alpha \cap D_\beta$ by the uniqueness theorem (if it is not empty), it is open in M as needed. With this topology, $\mathcal{U}$ is a Hausdorff space since M is Hausdorff (we use this if the base points differ) and because of the uniqueness theorem (which we use if the base points coincide).

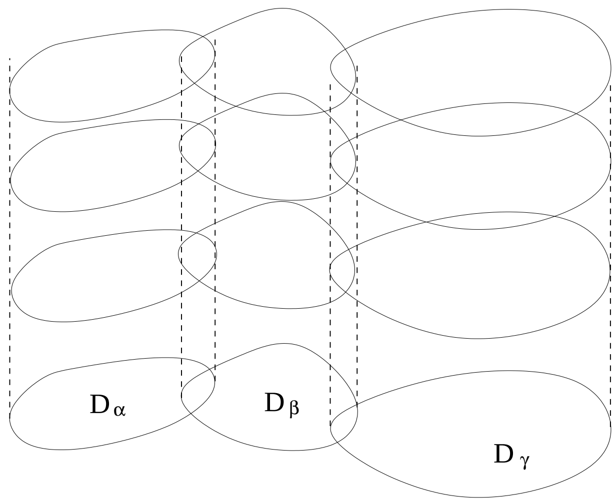

FIGURE 5.3. Local gluing and simple connectivity implies global gluing

Note that by construction, we have made the fibers indexed by the functions in F_α discrete in the topology of $\mathcal{U}$. The main point is now to realize that, if $\widetilde{M}$ is a connected component of $\mathcal{U}$, then $\pi : \widetilde{M} \to M$ is onto and in fact is a covering map. Let us check that it is onto. First, we claim that $\pi(\widetilde{M}) \subset M$ is open. Thus, let $[p, f] \in \widetilde{M}$ and pick D_α with $p \in D_\alpha$ and $f \in F_\alpha$. Clearly, $D'_{\alpha,f} \cap \widetilde{M} \neq \emptyset$ and since D_α, and thus also $D'_{\alpha,f}$, are open and connected the connected component $\widetilde{M}$ has to contain $D'_{\alpha,f}$ entirely. Therefore, $D_\alpha \subset \pi(\widetilde{M})$ as claimed. Next, we need to check that $M \setminus \pi(\widetilde{M})$ is open. Let $p \in M \setminus \pi(\widetilde{M})$ and pick D_β so that $p \in D_\beta$. If $D_\beta \cap \pi(\widetilde{M}) = \emptyset$ then we are done. Otherwise, let $q \in D_\beta \cap \pi(\widetilde{M})$ and pick D_α containing q and some $f \in F_\alpha$ with $D'_{\alpha,f} \subset \widetilde{M}$ (we are using the same "nonempty intersection implies containment" argument as above). But now we can find $g \in F_\beta$ with the property that $f = g$ on a component of $D_\alpha \cap D_\beta$. As before, this implies that $\widetilde{M}$ would have to contain $D'_{\beta,g}$ which is a contradiction. To see that $\pi : \widetilde{M} \to M$ is a covering map, one verifies that

$$\pi^{-1}(D_\alpha) = \bigcup_{f \in F_\alpha} D'_{\alpha,f}$$

The sets on the right-hand side are disjoint and in fact they are the connected components of $\pi^{-1}(D_\alpha)$. Since M is simply connected, $\widetilde{M}$ is homeomorphic to M, see Chapter 13. This property reduces to the existence of a globally defined analytic function which agrees with some $f \in F_\alpha$ on each D_α. By picking the connected component that contains any given $D'_{\alpha,f}$ one can fix the "sheet" locally on a given D_α. $\square$

The reader should try to apply this proof to the case where $M = \mathbb{C}^*$, D_α are all possible disks in $\mathbb{C}^*$, and F_α all possible branches of the logarithm on the disk D_α. What are all possible $\widetilde{M}$ here, and is π still a covering map?

Arguments as in the previous lemma are very powerful and allow one to base the theory of analytic continuations and thus the constructions of the following section entirely on the theory of covering spaces (for example, the mondromy theorem then becomes the well-known invariance of lifts under homotopies). For such an approach see [**15**]. The author feels, however, that using these "sheaves" is somewhat less constructive (due to arguments by contradiction as in the previous proof) and he has therefore chosen to follow the more direct traditional approach based on analytic continuation along curves.

2. The unramified Riemann surface of an analytic germ

Heuristically, we can regard the complete analytic function from Definition 5.2 as a analytic function F defined on a new Riemann surface $\widetilde{M}$ as follows: writing

$$[(f, D)]_\sim = \{(f_\alpha, D_\alpha) \mid \alpha \in \mathcal{A}\}$$

we regard each D_α as distinct from any other D_β (even if $D_\alpha = D_\beta$). Next, define $f = f_\alpha$ on D_α. Finally, identify $p \in D_\alpha$ with $q \in D_\beta$ iff (i) $p = q$ when considered as points in M and (ii) $f_\alpha = f_\beta$ near p. In other words, we let the functions label the points and only identify if we have local agreement. You should convince yourself that this is precisely the naive way in which we picture the Riemann surfaces of $\log z$, $\sqrt{z}$ etc.

In the following lemma, we prove that this construction does indeed give rise to a Riemann surface $\widetilde{M}$ and a function defined on it. Throughout, M, N will be fixed Riemann surfaces and any function element and complete analytic function will be defined relative to them.

LEMMA 5.6. *(a) Given a complete analytic function $\mathcal{A}$ and $p \in M$, we define an equivalence relation $\sim_p$ on function elements in $\{(f, D) \in \mathcal{A} \mid p \in D\}$ as follows:*

$$(f_0, D_0) \sim_p (f_1, D_1) \Longleftrightarrow f_0 = f_1 \quad \textit{near} \quad p$$

We define $[f, p]$ to be the equivalence class of (f, D), $p \in D$ under $\sim_p$ and call this a germ. *Then the germ $[f, p]$ uniquely determines three things: the point p, the value $f(p)$, and the complete analytic function $\mathcal{A}$.*

(b) Let $[f_0, p_0]$ be a germ and let $\mathcal{A} = \mathcal{A}(f_0, p_0)$ be the associated complete analytic function. Define

$$\mathcal{RS}(M, N, f_0, p_0) = \{[f, p] \mid p \in D,\ (f, D) \in \mathcal{A}\}$$

and endow this set with a topology as follows: the base for the topology is

$$[f, D] = \{[f, p] \mid p \in D, \}, \quad (f, D) \in \mathcal{A}$$

With this topology, $\mathcal{RS}(M, N, f_0, p_0)$ is a two-dimensional, arcwise connected, Hausdorff manifold with a countable base for the topology.

(c) On $\mathcal{RS}(M, N, f_0, p_0)$ there are two natural maps: the first is the canonical map $\pi : \mathcal{RS}(M, N, f_0, p_0) \to M$ defined by $\pi([f, p]) = p$. The second is the analytic continuation of (f_0, p_0), denoted by F, and defined as $F([f, p]) = f(p)$. The map π is a local homeomorphism and thus defines a complex structure on $\mathcal{RS}(M, N, f_0, p_0)$ which renders

π a local conformal isomorphism. Hence $\mathcal{RS} = \mathcal{RS}(M, N, f_0, p_0)$ is a Riemann surface, called the unramified Riemann surface *of the germ $[f_0, p_0]$ and F is an analytic function $\mathcal{RS} \to N$.*

Proof. (a) It is clear that the germ determines p as well as the Taylor series at p.

(b) M is arcwise connected and $\mathcal{RS}(M, N, f_0, p_0)$ is obtained by analytic continuation along curves — so it, too, is arcwise connected. If two points $[f, p]$ and $[g, q]$ in $\mathcal{RS}(M, N, f_0, p_0)$ satisfy $p \neq q$, then use that M is Hausdorff. If $p = q$, then the germs are distinct and can therefore be separated by open connected neighborhoods via the uniqueness theorem. For the countable base, use that M satisfies this and check that only countably many paths are needed to analytically continue a germ.

(c) The statements regarding π and F are clear. □

The map π in part (c) does not need to be onto. Indeed, suppose $f \in \mathcal{H}(\mathbb{D})$ cannot be analytically continued to any larger region than $\mathbb{D}$. In that case, $\mathcal{RS}(\mathbb{C}, \mathbb{C}, f, 0) = \mathbb{D}$, $\pi(z) = z$ and $F = f$. As another example, consider the function elements given by $\log z$ or $z^{\frac{1}{n}}$ with integer $n > 1$ on a neighborhood of $z = 1$. Then we cannot analytically continue into $z = 0$ which leads us to the notion of a "branch point" of the Riemann surface constructed in the previous lemma. We remark that these two classes of examples (logarithms and roots) are representative of all possible types of branch points and that in the case of the roots there is a way to adjoin the branch point to $\mathcal{RS}(M, N, f_0, p_0)$ and to make it "essentially disappear", see below.

Definition 5.7. *Let (U, ϕ) be a chart at $p_1 \in M$ with $\phi(p_1) = 0$, $\phi(U) = \mathbb{D}$. Let $(f, p) \in \mathcal{RS}(M, N, f_0, p_0)$ with $p \in U \setminus \{p_1\}$. If (f, p) can be analytically continued along every path in $U \setminus \{p_1\}$ but not into p_1 itself, then we say that $\mathcal{RS}(U, N, f, p)$ represents a* branch point *of $\mathcal{RS} = \mathcal{RS}(M, N, f_0, p_0)$ rooted at p_1. Under a* branch point *$\mathfrak{p}_1$ rooted at $p_1 \in M$ we mean an equivalence class as follows: suppose $\mathcal{RS}(U, N, f, p)$ and $\mathcal{RS}(V, N, g, q)$ each represent a branch point of $\mathcal{RS}$ rooted at p_1. We say that they are equivalent iff there is another such $\mathcal{RS}(W, N, h, r)$ with*

$$\mathcal{RS}(W, N, h, r) \subset \mathcal{RS}(U, N, f, p) \cap \mathcal{RS}(V, N, g, q)$$

The reader should convince himself or herself that $\mathcal{RS}(U, N, f, p)$ is not necessarily the same as $\pi^{-1}(U)$ (it can be smaller, see Figure 5.5). This is why we need to distinguish between $\mathfrak{p}_1$ and its root p_1 in M. We now define the *branching number* at a branch point.

Definition 5.8. *Let $p_1 \in M$ be the root of some branch point $\mathfrak{p}_1$ and pick $\mathcal{RS}(U, N, f, p)$ from the equivalence class of surfaces representing this branch point $\mathfrak{p}_1$ as explained above. Let $\phi(U) = \mathbb{D}$, $\phi(p_1) = 0$ be a chart and let $\alpha(t) = \phi^{-1}(\phi(p)e^{2\pi i t})$ be a closed loop in U around p_1. Then we let $[f_n, p]$ be the germ obtained by analytic continuation of $[f, p]$ along $\alpha^n = \alpha \circ \ldots \circ \alpha$ (n-fold composition), $n \geq 1$. We define the* branching number *at $\mathfrak{p}_1$ to be*

$$B(\mathfrak{p}_1) := \begin{cases} \infty \quad \textit{iff} \quad [f_n, p] \neq [f, p] \quad \forall\, n \geq 1 \\ \min\{n \geq 1 \mid [f_n, p] = [f, p]\} - 1 \quad \textit{otherwise} \end{cases} \tag{5.2}$$

If $B(\mathfrak{p}_1) = \infty$, then we say that $\mathfrak{p}_1$ is a logarithmic *branch point.*

Figure 5.4 is a schematic view of a branch point with branching number equal to two. We now need to check that these notions are well-defined. In what follows, we shall freely use the simple fact that the winding number characterizes equivalence classes of homotopic loops in $\mathbb{D}^*$.

LEMMA 5.9. *The branch number introduced in the previous definition is well-defined, i.e., it does not depend on the representative $\mathcal{RS}(U,N,f,p)$. Furthermore, if $\mathcal{RS}(U,N,f,p)$ is a representative of the branch point $\mathfrak{p}_1$ rooted at p_1, then the following holds: let γ be a closed loop in $U\setminus\{p_1\}$. Then $[f,p]$ is invariant under analytic continuation along γ if and only if*

$$B(\mathfrak{p}_1)+1 \quad \textit{divides} \quad n(\phi\circ\gamma;0)$$

PROOF. Suppose $\mathcal{RS}(U,N,f,p)$ and $\mathcal{RS}(V,N,g,q)$ are equivalent in the sense of Definition 5.7. Then there exists $\mathcal{RS}(W,N,h,r)$ with

$$\mathcal{RS}(W,N,h,r)\subset\mathcal{RS}(U,N,f,p)\cap\mathcal{RS}(V,N,g,q)$$

In particular, the germ $[h,r]$ is an analytic continuation of $[f,p]$ as well as $[g,q]$ along paths in $U\setminus\{p\}$ and $V\setminus\{q\}$, respectively. Let α be the closed loop from Definition 5.8 for U, β be the one for V, and γ the one for W. Since the winding number classifies homotopy classes of closed curves in the punctured disk, we see that $\alpha\sim_{U^*}\gamma$ in the sense of homotopy relative to $U^*:=U\setminus\{p_1\}$, as well as $\beta\sim_{V^*}\gamma$ relative to $V^*:=V\setminus\{p_1\}$. By the closed-curve version of the monodromy theorem, see Corollary 5.4, we conclude that $[f,p]$ and $[h,r]$ yield the same number in (5.2) and by the same token, also $[g,q]$ and $[h,r]$. It therefore follows that $B(\mathfrak{p}_1)$ is well-defined on the equivalence class defining the branch point.

For the final statement, suppose $n_0:=B(\mathfrak{p}_1)+1$ does not divide $n(\phi\circ\gamma;0)$. Then

$$n(\phi\circ\gamma;0)=kn_0+r_0,\qquad 0<r_0<n_0$$

Since a loop of winding number kn_0 brings $[f,p]$ back to itself, this implies that there exists a loop of winding number r_0 which does so, too. But this contradicts (5.2) and we are done. ∎

3. The ramified Riemann surface of an analytic germ

We now show that at each branch point $\mathfrak{p}_1$ of $\mathcal{RS}$ with $B(\mathfrak{p}_1)<\infty$ and for every representative $\mathcal{RS}(U)=\mathcal{RS}(U,N,f,p)$ of that branch point there is a chart Ψ defined globally on $\mathcal{RS}(U)$ (known as *uniformizing variable*) which maps $\mathcal{RS}(U)$ bi-holomorphically onto $\mathbb{D}^*$. The construction is very natural and is as follows:

LEMMA 5.10. *With $\mathcal{RS}(U,N,f,p)$ representing a branch point, let $\phi:U\to\mathbb{D}$ be a chart that takes $p_1\mapsto 0$. Pick a path γ that connects $[f,p]$ with an arbitrary $[g,q]\in\mathcal{RS}(U)$, and pick a branch ρ_0 of the n^{th} root $z^{\frac{1}{n}}$ locally around $z_0=\phi(p)$, $n:=B(\mathfrak{p}_1)+1$. Now continue the germ $[\rho_0,z_0]$ analytically along the path $\phi\circ\pi\circ\gamma$ to a germ $[\rho_\gamma,z]$ where $z=\phi(q)$. Define $\Psi([g,q])=\rho_\gamma(z)$. The map Ψ, once ρ_0 has been selected, is well-defined. Moreover, Ψ is analytic, and a homeomorphism onto $\mathbb{D}^*$.*

PROOF. First, we check that the choice of γ does not affect Ψ. Let $\widetilde{\gamma}$ be another path connecting $[f,p]$ with $[g,q]\in\mathcal{RS}(U)$. As usual, $\widetilde{\gamma}^-$ is the reversed path and $\gamma\widetilde{\gamma}^-$ is the composition. Analytically continuing $[f,p]$ along $\pi\circ\gamma\widetilde{\gamma}^-$ then leads back to $[f,p]$. This implies that $\phi\circ\pi\circ\gamma\widetilde{\gamma}^-$ has winding number around zero which is divisible by n (for otherwise we could obtain a smaller integer in (5.2)). Therefore, $[\rho_0,z_0]$ is invariant under analytic continuation along $\phi\circ\pi\circ\gamma\widetilde{\gamma}^-$; in other words, $\rho_\gamma(z)=\rho_\beta(z)$ as was to be shown. This also shows that Ψ is analytic since π is a local homeomorphism as well as a analytic map. Ψ is onto $\mathbb{D}^*$ by our standing assumption that analytic continuation can be performed freely in U^*. Finally, we need to check that Ψ is one-to-one. Suppose

$\Psi([g_1, q_1]) = \Psi([g_2, q_2])$. Then $\Psi([g_1, q_1])^n = \Psi([g_2, q_2])^n$ which means that $\phi(q_1) = \phi(q_2)$ and thus $q_1 = q_2$. By construction, $\Psi([g_j, q_j]) = \rho_{\gamma_j}(z)$ where $z = \phi(q_1)$. Since Ψ was

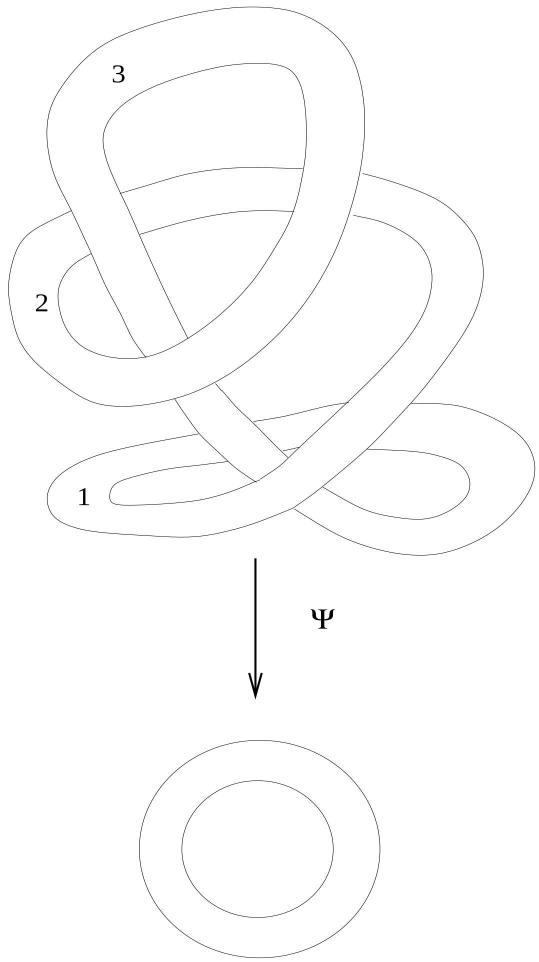

FIGURE 5.4. A schematic depiction of a uniformizing chart at a branch point with branching number 2

obtained as analytic continuation of a branch of $z^{\frac{1}{n}}$ along $\phi \circ \pi \circ \gamma$, it follows that

$$n(\phi \circ \pi \circ \gamma_1 \gamma_2^- \,; 0) = kn, \quad k \in \mathbb{Z}$$

But then $[g_1, q_1] = [g_2, q_2]$ by (5.2) and we are done. $\square$

Convince yourself that for the case of $\mathcal{RS}(\mathbb{C}, \mathbb{C}, z^{\frac{1}{n}}, 1)$ one can think of $\Psi^{-1}(z) = z^n$. Obviously, in that case $(\Psi^{-1}(z))^{\frac{1}{n}} = (z^n)^{\frac{1}{n}} = z$ for all $z \in \mathbb{D}^*$. The point of our discussion here is that locally at a branch point with finite branching number $n-1$ *any* unramified Riemann surface behaves the same as the n^{th} root. And, moreover, adjoining the branch point $\mathfrak{p}_1$ to the unramified surface yields another Riemann surface with a chart around $\mathfrak{p}_1$ that maps a neighborhood bi-holomorphically onto $\mathbb{D}$. This is very relevant from the point of view of analytically continuing the global function F into a branch point by means of the chart Ψ. This can indeed be done, at least in the algebraic case to which we now turn.

DEFINITION 5.11. *We define the* ramified Riemann surface *by adjoining all algebraic branch points. The latter are defined as being precisely those branch points with finite branching number so that F (relative to the uniformizing variable Ψ) has a removable singularity at zero, i.e., $F \circ \Psi^{-1} : \mathbb{D}^* \to N$ extends as an analytic function $\mathbb{D} \to N$.*

An example of a non-algebraic branch point with finite branching number is

$$\mathcal{R}S(\mathbb{C}^*, \mathbb{C}_\infty, \exp(z^{\frac{1}{2}}), 1)$$

Note that $z = 0$ is the root of an algebraic branch point (set $z = \zeta^2$ which yields e^ζ) whereas $z = \infty$ is not algebraic since $z = \zeta^{-2}$ with ζ near zero leads to $e^{\zeta^{-1}}$ which has an essential singularity at $\zeta = 0$.

We can state precisely how to define the ramified Riemann surface.

LEMMA 5.12. *Let $\mathcal{P}_{\mathrm{al}}$ be the set of algebraic branch points of $\mathcal{R}S(M, N, f_0, p_0)$ and define*

$$\widetilde{\mathcal{R}S} = \widetilde{\mathcal{R}S}(M, N, f_0, p_0) := \mathcal{R}S(M, N, f_0, p_0) \cup \mathcal{P}_{\mathrm{al}}$$

Then $\widetilde{\mathcal{R}S}$ is canonically a Riemann surface to which π and F have analytic continuations $\widetilde{\pi} : \widetilde{\mathcal{R}S} \to M$, and $\widetilde{F} : \widetilde{\mathcal{R}S} \to N$, respectively. We call $\widetilde{\mathcal{R}S}$ the ramified Riemann surface *(or just Riemann surface), and $\widetilde{F}$ the* complete analytic function *of the germ $[f_0, p_0]$. At each $\mathfrak{p} \in \mathcal{P}_{\mathrm{al}}$ the branching number $B(\mathfrak{p}) = \nu(\widetilde{\pi}, \mathfrak{p}) - 1$ where ν is the valency as defined earlier.*

PROOF. This is an immediate consequence of the preceding results and definitions, in particular, of Lemma 5.10. $\square$

The term "complete analytic function" was introduced previously for the collection of all function elements obtained via analytic continuation from a given one. However, from now on we will use this term exclusively in this new sense. Next, we turn to the important special case where the Riemann surface is compact.

LEMMA 5.13. *If $\widetilde{\mathcal{R}S} = \widetilde{\mathcal{R}S}(M, N, f_0, p_0)$ is compact, then M is compact. Moreover, there can only be finitely many branch points in $\mathcal{R}$; we denote the set of their projections onto M by $\mathcal{B}$ and define $\mathcal{P} := \widetilde{\pi}^{-1}(\mathcal{B})$. The map $\widetilde{\pi}_1 : \widetilde{\mathcal{R}S} \setminus \mathcal{P} \to M \setminus \mathcal{B}$ (the restriction of $\widetilde{\pi}$ to $\widetilde{\mathcal{R}S} \setminus \mathcal{P}$) is a covering map and the number of pre-images of this restricted map is constant; this is called the* number of sheets of $\widetilde{\mathcal{R}S}$ *and it equals the degree of $\widetilde{\pi}$. Finally, the following Riemann-Hurwitz type relation holds:*

$$g_{\widetilde{\mathcal{R}S}} = 1 + S(g_M - 1) + \frac{1}{2} \sum_{\mathfrak{p}} B(\mathfrak{p}) \tag{5.3}$$

where $g_{\widetilde{\mathcal{R}S}}, g_M$ are the respective genera, S is the number of sheets of $\widetilde{\mathcal{R}S}$, and the sum runs over the branch points $\mathfrak{p}$ in $\mathcal{R}S$ with $B(\mathfrak{p})$ being the respective branching numbers.

PROOF. It is clear from the compactness of $\widetilde{\mathcal{R}S}$ that there can be only finitely many branch points. Note that $\mathcal{R}S \setminus \mathcal{P} = \widetilde{\mathcal{R}S} \setminus \mathcal{P}$ and $\widetilde{\pi}_1 : \mathcal{R}S \setminus \mathcal{P} \to M \setminus \mathcal{B}$ is a local homeomorphism which is also proper; i.e., the pre-images of compact sets in $M \setminus \mathcal{B}$ are compact. This then easily implies that $\widetilde{\pi}$ restricted to $\widetilde{\mathcal{R}S} \setminus \mathcal{P}$ is a covering map.

Figure 5.5 is a schematic depiction of a three-sheeted surface with two (finite) branch points. Note that $\mathcal{R}S \neq \widetilde{\mathcal{R}S} \setminus \mathcal{P}$ in that case due to the fact that there are unbranched sheets covering the roots of branch points. It is clear that the cardinality of the fibres

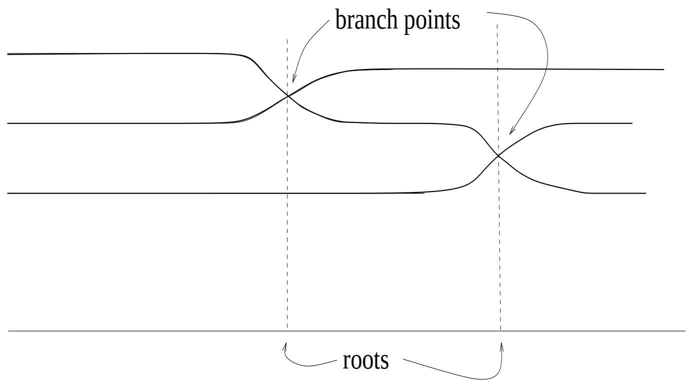

FIGURE 5.5. Three sheets

equals the degree of $\widetilde{\pi}$. Finally, the Riemann-Hurwitz relation follows from the general Riemann-Hurwitz formula above applied to $\widetilde{\pi}$. $\square$

We remark that $\widetilde{\pi}$ is what one calls a branched covering map.

4. Algebraic germs and functions

So far, our exposition has been very general in the sense that no particular kind of function element was specified to begin with. This will now change as we turn to a more systematic development of the ramified Riemann surfaces of algebraic functions. For example, consider the ramified Riemann surface $\widetilde{\mathcal{R}S}$ of $\sqrt{(z-z_1)\cdots(z-z_m)}$ where $z_j \in \mathbb{C}$ are distinct points (it is easy to see that $\widetilde{R}$ is compact, see below). In our notation, we are looking at

$$\widetilde{\mathcal{R}S}(\mathbb{C}_\infty, \mathbb{C}_\infty, \sqrt{(z-z_1)\cdots(z-z_m)}, z_0), \qquad z_0 \neq z_j \quad \forall\, 1 \leq j \leq m$$

and with one of the two branches of the square root fixed at z_0. What is the genus of $\widetilde{\mathcal{R}S}$? If m is even, then $\mathcal{R}S$ has $M = m$ branch points, if m is odd, then is has $M = m+1$ branch points (the point at ∞ is a branch point in that case). In all cases, the branching

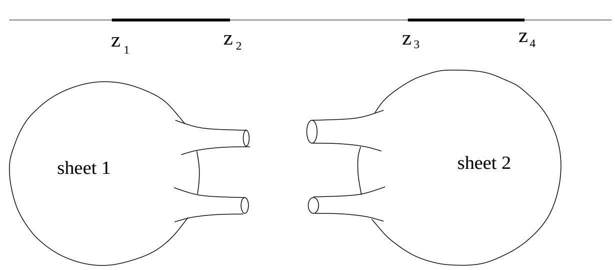

FIGURE 5.6. The case $m = 2$ with genus 1

numbers are one. The number of sheets is $S = 2$. Hence, in view of (5.3),

$$g_{\widetilde{\mathcal{RS}}} = \frac{M}{2} - 1$$

In other words, the Riemann surface is a sphere with $\frac{M}{2} - 1$ handles attached. This can also be seen directly, by cutting the Riemann sphere along $[z_1, z_2]$, $[z_3, z_4], \ldots, [z_{m-1}, z_m]$ if m is even. Then the surface consists of two copies of the Riemann sphere glued together along these edges which results in genus $\frac{m}{2} - 1$ as claimed. The reason for this cutting is of

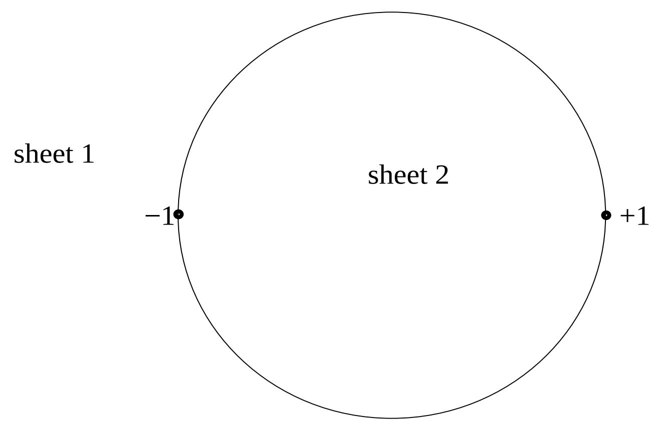

FIGURE 5.7. The Riemann surface of $\sqrt{z^2 - 1}$ via the Jukowski map as a global uniformizing map

course to allow for the transition from one sheet to the other as we continue analytically along a small loop centered at one of the z_j. Note that we skipped the cut $[z_2, z_3]$ above since analytic continuation along a loop surrounding the pair z_1, z_2 but encircling none of the other branch points does not change the sheet. If m is odd, then we also need to introduce a cut from z_m to ∞. For the case of $m = 2$ it is possible to make all of this very explicit. Indeed, setting $z = \frac{1}{2}(\zeta + \zeta^{-1})$ (this is the Jukowski map from Problem 1.9) yields

$$\sqrt{(z-1)(z+1)} = \frac{1}{2}(\zeta - \zeta^{-1}) \tag{5.4}$$

where we have made a choice of branch of the square root. Since the right-hand side of (5.4) is analytic as a map from $\mathbb{C}_\infty$ to itself, we see immediately that the Riemann surface of the left-hand side is the Riemann sphere, cf. Figure 5.7. Moreover, in the ζ plane, the two sheets of the Riemann surface of $\sqrt{z^2 - 1}$ correspond precisely to the regions $\{|\zeta| < 1\}$ and $\{|\zeta| > 1\}$, respectively. Recall that $\zeta \mapsto z$ is a conformal isomorphism from each of these regions onto $\mathbb{C} \setminus [-1, 1]$ with $\{|z| = 1\}$ mapped onto $[-1, 1]$. More precisely, $z = \pm 1$ are mapped onto ± 1, respectively, whereas each point $z \in (-1, 1)$ is represented exactly twice in the form $z = \cos(\theta)$, $0 < |\theta| < \pi$. Moreover, $z'(\zeta) = \frac{1}{2}(1 - \zeta^{-2}) = 0$ exactly at $\zeta = \pm 1$ and $\zeta''(\zeta) \neq 0$ at these points, which corresponds precisely to the simple branch points rooted at $z = \pm 1$. This example serves to illustrate the uniformizing charts from Lemma 5.10: at a branch point the ramified Riemann surface looks like every other point in the plane. Finally, note that the map $\zeta \mapsto \zeta^{-1}$ switches these two sheets and also changes the sign in (5.4) which needs to happen given the two different signs of

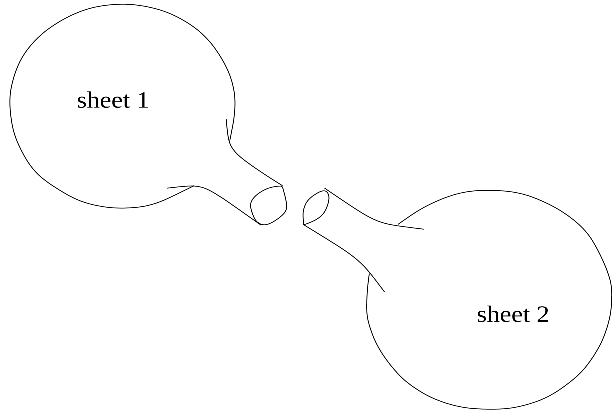

FIGURE 5.8. The two sheets form a Riemann sphere for $m = 1$

the square root. Clearly, the genus is zero, see also Figure 5.8. Another example of an algebraic function is furnished by the ramified Riemann surface generated by any root of $P(w, z) = w^3 - 3w - z = 0$ relative to the w variable. It is natural that the branch points are given precisely by the failure of the implicit function theorem. In other words, by all those pairs of (w, z) for which $P = 0, \partial_w P = 0$. This means that $w = \pm 1$ and $z = \mp 2$. Since $\partial_w^2 P \neq 0$ at these points we conclude that the branching number equals two at these points, whereas at $z = \infty$ the branching number equals three. Finally, the number of sheets is three, so that the surface looks schematically like the one in Figure 5.5 (see Figure 3.7.3 in Beardon [**3**] for a more artistic rendition of this surface). Finally, the genus $g_{\widetilde{\mathcal{R}S}} = 1$ from (5.3).

Throughout this section, $M = N = \mathbb{C}P^1$ are fixed; in particular, note that analytic functions are allowed to take the value ∞. As usual, we need to start with some definitions:

DEFINITION 5.14. *An analytic germ* $[f_0, z_0]$ *is called* algebraic *iff there is a polynomial* $P \in \mathbb{C}[w, z]$ *of positive degree so that* $P(f_0(z), z) = 0$ *identically for all* z *close to* z_0*. The complete analytic function*

$$\widetilde{F} : \widetilde{\mathcal{R}S} = \widetilde{\mathcal{R}S}(\mathbb{C}P^1, \mathbb{C}P^1, f_0, z_0) \to \mathbb{C}P^1$$

generated by $[f_0, z_0]$ *is called* an algebraic function.

The following lemma develops some of the basic properties of algebraic functions. In particular, we show that all branch points of an algebraic function are algebraic and that $\widetilde{\mathcal{R}S}$ is compact.

LEMMA 5.15. *Let* P *and* $\widetilde{R}$ *be as in the previous definition. Then one has the following properties:*

(a) $P(f(z), z) = 0$ *for all* $[f, z] \in \widetilde{\mathcal{R}S}$*. In fact, the same is true with an irreducible factor of* P *which is uniquely determined up to constant multiples.*

(b) there is the following version of the implicit function theorem: Let $P(w, z) \in \mathbb{C}[w, z]$ *satisfy* $P(w_1, z_1) = 0$, $P_w(w_1, z_1) \neq 0$*. Then there is a unique analytic germ* $[f_1, z_1]$ *with* $P(f_1(z), z) = 0$ *locally around* z_1 *and with* $f_1(z_1) = w_1$*.*

(c) If $P(w,z) = \sum_{j=0}^{n} a_j(z) w^j \in \mathbb{C}[w,z]$ *satisfies* $a_n \not\equiv 0$*, then up to finitely many* z *the polynomial* $w \mapsto P(w,z)$ *has exactly* n *simple roots.*
(d) Given an algebraic germ $[f_0, z_0]$*, there are finitely many points* $\{\zeta_j\}_{j=1}^{J} \in \mathbb{C}P^1$ *(called "critical points") such that* $[f_0, z_0]$ *can be analytically continued along every path in* $\mathbb{C}P^1 \setminus \{\zeta_1, \ldots, \zeta_J\}$*. If the unramified Riemann surface*

$$\mathcal{RS}(\mathbb{C}P^1, \mathbb{C}P^1, f_0, z_0)$$

has a branch point at $\mathfrak{p}$*, then* $\mathfrak{p}$ *has to be rooted over one of the* ζ_j*. Furthermore,* $\#\pi^{-1}(z)$ *is constant on* $\mathbb{C}P^1 \setminus \{\zeta_1, \ldots, \zeta_J\}$ *and no larger than the degree of* $P(w,z)$ *in* w*.*
(e) All branch points of the unramified Riemann surface

$$\mathcal{RS}(\mathbb{C}P^1, \mathbb{C}P^1, f_0, z_0)$$

generated by an algebraic germ are necessarily algebraic. In particular, the ramified Riemann surface of an algebraic germ is compact.

Proof. (a) Since $[f,z]$ is obtained from $[f_0, z_0]$ via analytic continuation, we conclude from the uniqueness theorem that $P(f(z), z) = 0$ for all $[f,z] \in \widetilde{\mathcal{RS}}$. To pass to an irreducible factor, we will need to use the resultant, see Chapter 13: given two relatively prime polynomials $P, Q \in \mathbb{C}[w,z]$, there exist $A, B \in \mathbb{C}[w,z]$ such that

$$A(w,z)P(w,z) + B(w,z)Q(w,z) = R(z) \in \mathbb{C}[z] \tag{5.5}$$

is a *nonzero* polynomial in z alone; it is called the *resultant* of P and Q. To conclude the proof of (a), let $P_0(w,z)$ be an irreducible factor of P for which $P_0(f_0(z), z) = 0$ near z_0. In view of (5.5) this factor is unique up to constant factors, as claimed.

(b) First, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$P(w,z) \neq 0 \quad \forall\, |z - z_1| < \delta, |w - w_1| = \varepsilon$$

Next, from the residue theorem,

$$\frac{1}{2\pi i} \oint_{|w-w_1|=\varepsilon} \frac{P_w(w,z)}{P(w,z)}\, dw = 1 \quad \forall\, |z - z_1| < \delta$$

and one infers from this that $P(w,z) = 0$ has a unique zero $w = f_1(z) \in \mathbb{D}(w_1, \varepsilon)$ for all $|z - z_1| < \delta$. Finally, write

$$f_1(z) = \frac{1}{2\pi i} \oint_{|w-w_1|=\varepsilon} w \frac{P_w(w,z)}{P(w,z)}\, dw$$

again from the residue theorem, which allows one to conclude that $f_1(z)$ is analytic in $\mathbb{D}(z_1, \delta)$. The reader familiar with the Weierstrass preparation theorem will recognize this argument from the proof of that result, see Problem 5.2.

(c) This follows from the (a) and (b) by considering the *discriminant* of P, which is defined to be the resultant of $P(w,z)$ and $P_w(w,z)$.

(d) This is a consequence of (b) and (c). The constancy of the cardinality of $\pi^{-1}(z)$ follows from the fact that this cardinality equals the degree of $\widetilde{\pi}$. It is clear that there can be no more sheets than the degree of P in w specifies (there could be fewer sheets, though, but as we shall see, only if P is reducible).

(e) Let $\mathfrak{p}$ be a branch point of $\mathcal{RS}(\mathbb{C}P^1, \mathbb{C}P^1, f_0, z_0)$. If $\mathfrak{p}$ is rooted over ∞, then consider instead $f_0(\zeta + z^{-1})$ where ζ is not the root of any branch point. So let us assume that $\mathfrak{p}$ is rooted over some point $z_1 \in \mathbb{C}$. We need to show that relative to the uniformizing chart Ψ from Lemma 5.10 the analytic function F has a removable singularity or a pole at z_1. For this it suffices to show that any solution $f(z)$ of $P(f(z), z) = 0$ can grow at most like $|z - z_1|^{-k}$as $z \to z_1$ for some $k \geq 0$ (indeed, the role of Ψ is merely to remove "multi-valued" issues but does not affect the polynomial growth as $z \to z_1$ other than by changing the power k). This follows easily from the fact that

$$\sum_{j=0}^{n} a_j(z)(f(z))^j = 0$$

Indeed, suppose $a_n(z_1) \neq 0$. Then let $|a_n(z)| > \delta > 0$ for all $|z - z_1| < r_0$ small. Fix such a z. Since

$$f(z) = -a_n(z)^{-1} \sum_{j=0}^{n-1} a_j(z)(f(z))^{j-(n-1)}$$

either $|f(z)| \leq 1$ or $|f(z)| \leq \delta^{-1} \sum_{j=0}^{n-1} |a_j(z)|$. If, on the other hand, $a_n(z_1) = 0$, then $a_n(z) = (z - z_1)^\ell b_n(z)$ where $b_n(z_1) \neq 0$. Thus $|b_n(z)| > \delta$ for all $|z - z_1| < r_0$. From this we conclude that

$$|f(z)(z - z_1)^\ell| \leq \delta^{-1} \sum_{j=0}^{n-1} |a_j(z)||z - z_1|^{\ell(n-1-j)} |f(z)(z - z_1)^\ell|^{j-(n-1)}$$

As before, this implies that $f(z)(z - z_1)^\ell$ remains bounded and we have shown that all branch points are indeed algebraic. Finally, since $\mathbb{C}P^1$ is compact and due to Lemma 5.10 it follows that that the ramified surface $\widetilde{\mathcal{R}S}$ can be covered by finitely many compact sets. This of course implies that $\widetilde{\mathcal{R}S}$ is itself compact and we are done. $\square$

Lemma 5.15 is not just of theoretical value, but also has important practical implications. Let us sketch how to "build" the ramified Riemann surface of some irreducible polynomial equation $P(w, z) = 0$ which we solve for w. Suppose the degree of P in w is $n \geq 1$. First, let $D(z) = R(P, P_w)$ be the discriminant of P (thus, the resultant of P and P_w). First, we remove all critical points $\mathcal{C} \subset \mathbb{C}_\infty$ which are defined to be all zeros of D and ∞ from $\mathbb{C}_\infty$. It is clear that for all $z_0 \in \mathbb{C}_\infty \setminus \mathcal{C}$ there are n analytic functions $w_j(z)$ defined near z_0 so that $P(w_j(z), z) = 0$ for $1 \leq j \leq n$ and such that all $w_j(z)$ are distinct. Locally at each finite critical point ζ the following happens: there is a neighborhood of ζ, say U, so that on U at least one of the zeros w_j ceases to be analytic due to the fact that $P(\cdot, \zeta) = 0$ has at least one multiple root. But in view of Lemma 5.10 we see that locally around ζ there is always a representation of the form (called Puiseux series)

$$w_j(z) = \sum_{k=0}^{\infty} a_{jk}(z - \zeta)^{\frac{k}{\nu_j}}$$

with some $\nu_j \geq 1$. If $\nu_j = 1$ then w_j is of course analytic, whereas in all other cases it is analytic as a function of η which is defined via the *uniformizing change of variables* $z = \zeta + \eta^{\nu_j}$. This is precisely the meaning of Lemma 5.10. It may seem clear from this example that the number of sheets needs to be precisely n. However, this requires some work as the reader will see below. For related examples see the problems of this chapter. For an explicit example of a Puiseux series, consider the irreducible polynomial

$$P(w, z) = w^4 - 2w^2 + 1 - z$$

and solve for w locally around $(w, z) = (1, 0)$. This leads to

$$w(z) = \sqrt{1 + \sqrt{z}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} z^{\frac{n}{2}}$$

which converges in $|z| < 1$.
Next, we turn to the following remarkable result which is in some sense a converse of what we have done so far. More precisely, we will show how to generate an algebraic equation from a nonconstant meromorphic function on an (abstract) compact Riemann surface!

PROPOSITION 5.16. *Let $z = z(p)$ be a meromorphic function of degree $n \geq 1$ on a compact Riemann surface M. If $f : M \to \mathbb{C}P^1$ is any other nonconstant meromorphic function, then f satisfies an algebraic equation*

$$f^n + \sigma_1(z) f^{n-1} + \sigma_2(z) f^{n-2} + \ldots + \sigma_{n-1}(z) f + \sigma_n(z) = 0 \tag{5.6}$$

of degree n, with rational functions $\sigma_j(z)$. In particular, if the ramified Riemann surface

$$\widetilde{\mathcal{RS}}(\mathbb{C}P^1, \mathbb{C}P^1, f_0, z_0)$$

is compact, where $[f_0, z_0]$ is any holomorphic germ, then $[f_0, z_0]$ is algebraic.

PROOF. To prove the theorem, we proceed as follows: remove from $\mathbb{C}_\infty$ the point ∞, as well as the image $z(p)$ of any branch point of the map $p \mapsto z(p)$ (recall that a branch point of an analytic function is defined as having valency strictly bigger than one). Denote these finitely many points as $\mathcal{C} := \{z_j\}_{j=0}^{J}$ and we refer to them as "critical points". If $z \in \mathbb{C}_\infty \setminus \mathcal{C}$, then let

$$\{p_1(z), \ldots, p_n(z)\}$$

be the n pre-images under $z(p)$ in arbitrary order and define

$$\sigma_j(z) := \sum_{1 \leq \nu_1 < \nu_2 < \ldots < \nu_j \leq n} f(p_{\nu_1}(z)) \cdot \ldots \cdot f(p_{\nu_j}(z))$$

with $\sigma_0 = 1$. Thus, the σ_j are the elementary symmetric functions in $f(p_1), \ldots, f(p_n)$ and they satisfy

$$\sum_{j=0}^{n} w^j \sigma_{n-j}(z) = \prod_{\ell=1}^{n} (w - f(p_\ell(z)))$$

By Lemma 5.5, each $p_j(z)$ is a holomorphic function on any simply connected subdomain of $\mathbb{C} \setminus \mathcal{C}$ (possibly after a renumbering of the branches). This implies that σ_j is meromorphic on any such domain. Furthermore, analytic continuation of σ_j along a small loop surrounding an arbitrary point of $\mathcal{C}$ takes σ_j back to itself as the different branches of p_j can only be permuted; however, σ_j is invariant under such a permutation. This implies that σ_j has isolated singularities at the points of $\mathcal{C}$ and since f is meromorphic these singularities can be at worst poles. In conclusion, σ_j is meromorphic on $\mathbb{C}_\infty$ and therefore rational as claimed. Note that (5.6) holds by construction. For the final statement, let $\tilde{\pi} : \widetilde{\mathcal{RS}}(\mathbb{C}P^1, \mathbb{C}P^1, f_0, z_0) \to \mathbb{C}P^1$ be the meromorphic function $z(p)$ from the first part. Thus, (f, p) is algebraic as desired. $\square$

Needless to say, Proposition 5.16 is a fundamental statement in the theory of algebraic functions. It raises a number of important questions: Which Riemann surfaces carry nontrivial meromorphic functions? Furthermore, what degrees can one achieve on a given surface? These questions can be restated as follows: how can we realize a given compact Riemann surface M as a branched cover of S^2, and which restrictions exist for the number of sheets of such a cover? The Riemann-Hurwitz formula provides one such obstruction, albeit a rather weak one. A much more useful tool will later be given by the Riemann-Roch theorem. Let us first observe the trivial fact that for M to admit a meromorphic function of degree one, it is necessary and sufficient that $M \simeq S^2$ in the sense of conformal isomorphism. We shall see later (in Chapter 8) that in fact every compact simply connected Riemann surface is isomorphic to S^2.

Before pursuing these matters any further, let us first derive a simple but important corollary from Proposition 5.16. It will allow us to determine the number of sheets of the Riemann surface determined by an irreducible polynomial.

Corollary 5.17. *Let $P \in \mathbb{C}[w,z]$ be irreducible and suppose $\deg_w(P) = n$. Then the ramified Riemann surface $\widetilde{R}$ generated by some germ $[f_0, z_0]$ with $P(f_0(z), z) = 0$ locally at z_0 has n sheets and satisfies*

$$\widetilde{R} = \widetilde{R}(\mathbb{C}_\infty, \mathbb{C}_\infty, f_1, z_1)$$

for every germ $[f_1, z_1]$ with $P(f_1(z), z) = 0$ for all z near z_1.

Proof. If the number of sheets were $m < n$, then by the previous result we could find a polynomial Q of degree m with the property that $Q(f_0(z), z) = 0$ for z near z_0. But then Q would necessarily need to be a factor of P which is impossible. $\square$

In other words, the ramified Riemann surface of a germ satisfying an irreducible polynomial equation contains all germs satisfying this equation.
Let us now clarify the connection between the smooth, affine or projective, algebraic curves which were introduced in the previous chapter, and the ramified Riemann surfaces of an algebraic germ which we just constructed. In particular, the reader needs to recall the notion of nonsingular polynomials as well as the homogenization of a polynomial, see (4.2) and (4.3).

Lemma 5.18. *Let $P \in \mathbb{C}[w,z]$ be an irreducible polynomial so that its homogenization Q is nonsingular. Then the smooth projective algebraic curve $\widetilde{S}$ defined by Q, see* (4.3)*, is isomorphic to the Riemann surface $\widetilde{R} := \widetilde{\mathcal{RS}}(\mathbb{C}P^1, \mathbb{C}P^1, f_0, z_0)$ of any algebraic germ $[f_0, z_0]$ defined by P. In particular, $\widetilde{\mathcal{RS}}$ can be imbedded into $\mathbb{C}P^2$.*

Proof. We will need to following fact, see Problem 4.8: *Let M, N be compact Riemann surfaces and suppose $\Phi : M \setminus \mathcal{C} \to N \setminus \mathcal{C}'$ is an isomorphism where $\mathcal{C}, \mathcal{C}'$ are finite sets. Then Φ extends to an isomorphism $\widetilde{\Phi} : M \to N$.*

Remove from $\widetilde{R}$ all points "at infinity", more precisely, all germs rooted at $z = \infty$. In other words, we consider

$$M := \widetilde{\mathcal{RS}}(\mathbb{C}, \mathbb{C}_\infty, f_0, z_0)$$

for an arbitrary germ $[f_0, z_0]$ of P with $z_0 \in \mathbb{C}$. Similarly, remove from $\widetilde{S}$ the line at infinity, i.e.,

$$N := \widetilde{S} \setminus L_\infty, \qquad L_\infty := \{[z : w : 0] \mid (z, w) \in \mathbb{C}^2\}$$

In other words, consider the affine curve S instead of the projective one $\widetilde{S}$. The reader should convince himself or herself that $\widetilde{S} \cap L_\infty$ is always finite. By the aforementioned fact concerning isomorphisms, it will suffice to show that $M \simeq N$. We will accomplish this by showing that we can identify M and N as sets and also use the same charts. First, locally at all non-critical points (z_1, w_1) (which are defined via $P(w_1, z_1) = 0$, $\partial_w P(w_1, z_1) \neq 0$) one has $n = \deg_w(P)$ branches $w_j(z)$ defined and analytic near z_1. These branches define n charts on both M and N on neighborhoods which we can identify provided they are chosen small enough (so that there is no overlapping of values); indeed, we simply map the germ $[f, z]$ on M onto the pair $(f(z), z) \in N$.

Next, consider any critical point (z_1, w_1). Since P is non-singular, one has $\partial_z P(z_1, w_1) \neq 0$ which implies that there exists an analytic function $Z_1(w)$ defined near $w = w_1$ and an analytic and nonvanishing function Q near (z_1, w_1) so that

$$P(z, w) = (z - Z_1(w))Q(z, w)$$

locally around (z_1, w_1). This follows either from Lemma 5.15 or the Weierstrass preparation theorem, see Problem 5.2. Now suppose that $\partial_w^j P(w_1, z_1) = 0$ for all $0 \leq j < \ell$ but $\partial_w^\ell P(w_1, z_1) \neq 0$ with some $\ell \geq 2$. This is equivalent to

$$\frac{d^j Z}{dw^j}(w_1) = 0 \quad \forall\, 0 \leq j < \ell, \qquad \frac{d^\ell Z}{dw^\ell}(w_1) \neq 0$$

Thus, the branching number of the branch point of M at (z_1, w_1) is $\ell - 1 \geq 1$ and a uniformizing chart as in Lemma 5.10 is given by $z = Z_1(w)$ provided $w = f(z)$ with $[f, z]$ a germ from the Riemann surface representing the branch point ($z \neq z_1$). On the other hand, by definition $z = Z_1(w)$ is a chart on N near w_1. Hence, this again allows one to identify M and N near (z_1, w_1) with the same chart and we are done. $\square$

Proposition 5.16 raises the following natural and important question: *Is M conformally equivalent to the Riemann surface associated with the algebraic equation* (5.6) *over $\mathbb{C}P^1$?*. For this to hold (5.6) needs to be irreducible . We shall see later, as an application of the Riemann-Roch theorem, that we can in fact find f to a given z such that (5.6) is irreducible. This will then allow us to obtain an affirmative answer to our question, which has a remarkable consequence: *any compact Riemann surface which carries a nonconstant meromorphic function is isomorphic to the Riemann surface associated to some irreducible polynomial $P(w, z)$.* Since we will later see that in fact every Riemann surface carries a nonconstant meromorphic function, we can now state the following truly remarkable conclusion.

Theorem 5.19. *Every compact Riemann surface M is the ramified Riemann surface of some algebraic germ.*

Proof. As already explained, the proof hinges on two facts:

- every Riemann surface carries a nonconstant meromorphic function $z : M \to \mathbb{C}P^1$
- given a nonconstant meromorphic function on M, we can find another meromorphic function f on M such that (5.6) is irreducible

We will prove these two facts in Chapters 7, see Corollary 7.14, and 8, see Problem 8.5, respectively. Assuming them for the moment, we can easily finish the argument by going back to proof of Proposition 5.16. Indeed, let $P(w, z)$ be the irreducible polynomial associated with (5.6) and let $\widetilde{\mathcal{R}S}$ be the ramified Riemann surface of $P(w,z)$. In view of Corollary 5.17, $\widetilde{\mathcal{R}S}$ has exactly $n = \deg(z)$ sheets. For any $z_0 \in \mathbb{C}P^1 \setminus \mathcal{C}$ there are n distinct pre-images $\{p_j(z_0)\}_{j=1}^n$ with the property that locally around these points $p \mapsto z$ is an isomorphism onto some neighborhood of z_0. Moreover, these neighborhoods are also isomorphic to neighborhoods of $\widetilde{\mathcal{R}S}$. To see this, simply note that each $f(p_j(z))$ is analytic close to z_0 and therefore an unbranched function element from $\widetilde{\mathcal{R}S}$. But since $\widetilde{\mathcal{R}S}$ has n sheets this implies that the germs $\{[f \circ p_j, z_0]\}_{j=1}^n$ are distinct and in fact parametrize neighborhoods on distinct sheets. This in turn defines isomorphisms between each of these neighborhoods and respective neighborhoods of the points $p_j(z_0)$ on M: simply map $[f \circ p_j, z]$ onto $p_j(z)$ for all z close to z_0. Now connect the points of $\mathcal{C}$ by a piecewise linear path γ so that $\Omega := \mathbb{C}_\infty \setminus \gamma$ is simply connected. By Lemma 5.5 these local isomorphisms extend to global ones between those sheets of $\widetilde{\mathcal{R}S}$ that cover Ω via the canonical projection $\widetilde{\pi} : \widetilde{\mathcal{R}S} \to \mathbb{C}_\infty$ as well as the respective components of M which cover Ω via the map z. It remains to check that over each critical point $\widetilde{\mathcal{R}S}$ is branched in exactly the same way as M. More precisely, one needs to verify that at each critical point $z_0 \in \mathcal{C}$ there is a one-to-one correspondence between the branch points of M and $\mathcal{R}S$, respectively, with equal branching numbers (recall that we may have unbranched sheets over a given root, see Figure 5.5). However, this is clear from the fact that the germs are given by $f \circ p_j$ and thus inherit the branching numbers from the map z: suppose analytic continuation of the germ $[f \circ p_j, z_1]$ with a fixed $z_1 = z_0 + \varepsilon e^{i\theta_0}$ close to z_0 along the loop $\rho := z_0 + \varepsilon e^{i\theta}$, $\theta_0 \le \theta \le \theta_0 + 2\pi$ yields the germ $[f \circ p_k, z_1]$ with $j \neq k$. Since we verified before that these germs are distinct, it follows that the lift of ρ to M under the branched cover z which starts at p_j must end at p_k. But this means precisely that there is the desired one-to-one correspondence between the branch points which preserves the branching number. $\square$

In other words, *any* compact Riemann surface is obtained by analytic continuation of a suitable algebraic germ! Note that the proof gives an "explicit" conformal isomorphism:

$$p \mapsto (z(p), f(p))$$

where z and f are the meromorphic functions from the proof.

As the surface of the logarithm shows, this is not true in the non-compact case. Another example is given by

$$\mathcal{R}S(\mathbb{C}^*, \mathbb{C}_\infty, \exp(z^{\frac{1}{2}}), 1)$$

It is clearly not algebraic and, in fact, is not compact since the branch point at $z = \infty$ is not part of the ramified Riemann surface. The existence of a meromorphic function on an (abstract) Riemann surface is a highly non-trivial issue; indeed, even though we can of course define such functions locally on every chart, the challenge lies with the extension of such a function beyond the chart as partitions of unity, say, would take us outside of the analytic category (by the uniqueness theorem).

As already note before, the case $n = 1$ of Proposition 5.16 means precisely that $z : M \to \mathbb{C}P^1$ is an isomorphism and we recover the result that all meromorphic functions on $\mathbb{C}P^1$ are rational. The first interesting case is $n = 2$, and any compact M of genus $g > 1$ carrying such a meromorphic function $z(p)$ is called *hyper-elliptic*, whereas the genus one case is typically referred to as *elliptic* (it will follow from the Riemann-Roch theorem that every compact surface of genus one carries a meromorphic function of degree two). Suppose now that $n = 2$ in Proposition 5.16. To proceed we quote a result that we shall prove in Chapter 7, see Corollary 7.14: *Let $p, q \in M$ be two distinct points on a Riemann surface M. Then there exists $f \in \mathcal{M}(M)$ with $f(p) = 0$ and $f(q) = 1$.*

In other words, the function field on M separates points. Apply this fact with $\{p, q\} = z^{-1}(z_0)$ where z_0 is not a critical value of z. By the preceding, f satisfies an equation of the form

$$f^2 + \sigma_1(z)f + \sigma_2(z) = 0 \tag{5.7}$$

with rational σ_1, σ_2. This equation needs to be irreducible. Indeed, otherwise f is a rational function of z which would contradict that f takes different values at p and q. Replacing f by $f + \sigma_1/2$ shows that the Riemann surface generated by (5.7) is the same as that generated by

$$f^2 + \sigma_3(z) = 0, \qquad \sigma_3 := \sigma_2 + \sigma_1^2/4$$

By irreducibility, σ_3 is not the square of a rational function. Let $\sigma_3 = \frac{P}{Q}$ where P, Q are without loss of generality, relatively prime, monic polynomials. By the proof of Theorem 5.19, we see that M is isomorphic to the Riemann surface generated by $\sqrt{\frac{P}{Q}}$. A moments reflection shows that this surface is isomorphic to the Riemann surface of $\sqrt{PQ}$, and moreover, we can assume that each linear factor in PQ appears only to the first power. In other words, we infer that every elliptic or hyper-elliptic Riemann surface is isomorphic to one of the examples which we already encountered above, viz.

$$\widetilde{\mathcal{RS}}(\mathbb{C}P^1, \mathbb{C}P^1, \sqrt{(z - z_1) \cdot \ldots \cdot (z - z_m)}, z_0)$$

where $\{z_j\}_{j=0}^m$ are all distinct. Let us now examine the elliptic surfaces more closely.

Corollary 5.20. *Suppose M is a compact surface of genus one. Then M is isomorphic to the set of zeros in $\mathbb{C}P^2$ of a cubic polynomial*

$$E_\zeta := \{[z : w : u] \in \mathbb{C}P^2 \mid Q_\zeta(z, w, u) := w^2 u - z(z - u)(z - \zeta u) = 0\} \tag{5.8}$$

where $\zeta \in \mathbb{C} \setminus \{0, 1\}$. In other words, M can be imbedded into $\mathbb{C}P^2$. For the particular case of tori $\mathbb{C}/\Gamma$ with $\Gamma := \langle z \mapsto z + \omega_1, z \mapsto z + \omega_2\rangle$ and ω_1, ω_2 independent over $\mathbb{R}$, this imbedding is given explicitly in terms of the Weierstrass function $\wp$ associated with the lattice Γ:

$$\Psi\ : z \in \mathbb{C}/\Gamma \mapsto \begin{cases} [\wp(z) : \wp'(z) : 1] & z \neq 0 \\ [0 : 1 : 0] & z = 0 \end{cases} \tag{5.9}$$

Proof. We need to refer to the aforementioned fact (see Chapter 8, Corollary 8.14) that every genus one compact surface can be realized as a two sheeted branched cover of S^2 (in other words, it carries a meromorphic function of degree two). Note that for the case of tori this is nothing but the existence of the Weierstrass $\wp$ function. By our previous discussion, M is therefore isomorphic to the Riemann surface of $\sqrt{\prod_{j=1}^m (z - z_j)}$ with either $m = 3$ or $m = 4$ and distinct z_j. However, these two cases are isomorphic so it suffices to consider $m = 3$. Composing with a Möbius transform which moves z_0, z_1 to $0, 1$, respectively, and fixed ∞, we now arrive at the the cubic polynomial

$$P_\zeta := w^2 - z(z - 1)(z - \zeta) = 0, \qquad \zeta \in \mathbb{C} \setminus \{0, 1\}$$

Not only is P_ζ non-singular but also its homogenization Q_ζ from (5.8). We leave this to the reader to check. Hence, applying Lemma 5.18, we see that M is isomorphic to both the Riemann surface of P_ζ from this chapter as well as the algebraic curve E_ζ as defined in the previous chapter.

Finally, for the torus we proved in Chapter 4 that the Weierstrass function satisfies the differential equation

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

In other words, $z \mapsto (\wp(z), \wp'(z))$ maps the torus into the set of zeros (where E stands for "elliptic curve")

$$E := \{(\zeta, \eta) \in \mathbb{C}^2_\infty \;:\; \eta^2 - 4(\zeta - e_1)(\zeta - e_2)(\zeta - e_3) = 0\}$$

We need to "projectivize" this in the usual way leading to (5.9). The choice of $[0 : 1 : 0]$ for $z = 0$ is the only possible one as can be seen from the Laurent expansion of $\wp$ and $\wp'$ at $z = 0$. The reader will easily verify that this map is a homeomorphism between the torus and the projective version of E. Since the latter is in a canonical way the Riemann surface of the irreducible polynomial $\eta^2 - 4(\zeta - e_1)(\zeta - e_2)(\zeta - e_3)$, we have obtained the desired isomorphism. Historically, the inverse to the map $\Phi : z \mapsto (\zeta, \eta) := (\wp(z), \wp'(z))$ was given by the "elliptic integral"

$$z(p) = \int_\infty^p \frac{d\zeta}{\eta} = \int_\infty^p \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}} \tag{5.10}$$

where $p \in E$ where the latter is viewed as the Riemann surface of

$$\eta^2 - 4(\zeta - e_1)(\zeta - e_2)(\zeta - e_3) =: \eta^2 - (4\zeta^3 - g_2\zeta - g_3)$$

The integration here is along any path that avoids the branch points and the branch of the square root in (5.10) is determined by analytic continuation along that path. Clearly, the integral in (5.10) is invariant under homotopies, but the path is determined only up to integral linear combinations of the homology basis a, b, see Figure 5.9. The dashed line on b means that we are entering the other sheet. The torus is shown as E is topologically equivalent to one. However, such an integral linear combination changes $z(p)$ only by

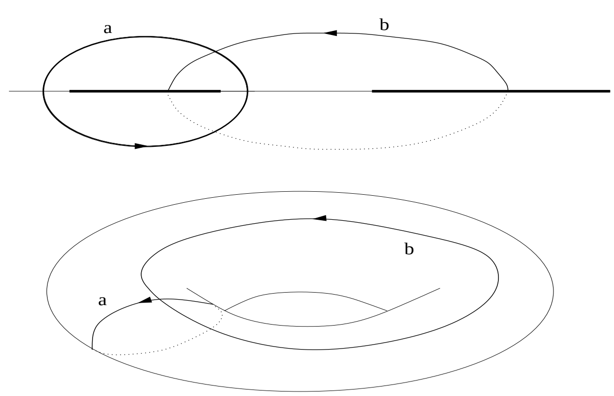

FIGURE 5.9. The homology basis for E

$$m \oint_a \frac{d\zeta}{\eta} + n \oint_b \frac{d\zeta}{\eta} = m\omega_1 + n\omega_2 \tag{5.11}$$

with $n, m \in \mathbb{Z}$. We leave the evaluation of the integrals to the reader, see Problem 4.7. Consequently, $z(p)$ is well-defined as an element of the torus $\mathbb{C}/\Gamma$. Finally, since $\Phi : z \mapsto (\wp(z), \wp'(z))$ is clearly onto, we can write a path connecting ∞ to p as $\Phi \circ \gamma$ where $\gamma : [0, 1] \to \mathbb{C}/\Gamma$ is a path connecting 0 to z_0. But then

$$z(p) = \int_0^1 \frac{d(\wp \circ \gamma)}{\wp'(\gamma)} = \int_0^1 \gamma'(t)\, dt = \gamma(1) = z_0$$

which shows that $p \mapsto z(p)$ is indeed the inverse to Φ. □

Elliptic curves have many remarkable properties, of which we just mention the following one: *the three distinct points* $\{\Psi(z_j)\}_{j=1}^3$ *are colinear iff* $z_1 + z_2 + z_3 = 0$. In other words,

$$\det \begin{bmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z_3) & \wp'(z_3) & 1 \end{bmatrix} = 0 \Leftrightarrow z_1 + z_2 + z_3 = 0$$

where the final equality is to be understood modulo Γ. For the proof, as well as how to use this fact to put a group structure on an elliptic curve (and many other properties of these curves) we refer the reader to [**11**] and Chapter 3 of [**22**].

This result of course also raises a number of questions, for example: *does any compact Riemann surface of higher genus admit an imbedding into some projective space* $\mathbb{C}P^d$? In fact, the answer is "yes" with $d = 3$. For more on these topics see Chapter 8 or [**23**].

To conclude this chapter, we illustrate the methods of algebraic functions by means of the (ramified) Riemann surface defined by the germ $[\sqrt{1 + \sqrt{z}}, 1]$. The convention regarding the square root shall be $\sqrt{x} > 0$ for $x > 0$. This germ gives rise to the unique irreducible monomial $P(w, z) = (w^2 - 1)^2 - z$. The system $P = 0, \partial_w P = 0$ has solutions

$$(w, z) \in \{(0, 1), (1, 0), (-1, 0)\}$$

Note that $\partial_w^2 P(w, z) \neq 0$ as well as $\partial_z P(z, w) \neq 0$ at each of these points. Thus,

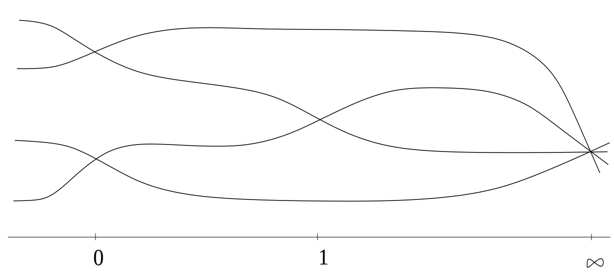

FIGURE 5.10. The sheets of $\sqrt{1 + \sqrt{z}}$

any branch point associated with them has branching number 1. Moreover, if $z = 0$ then necessarily $w = \pm 1$, whereas $z = 1$ yields $w = 0$ as well as $w = \pm\sqrt{2}$. Finally, $z = \infty$ is a branch point and analytic continuation around a large circle (more precisely, any loop encircling $z = 0, 1$ once) permutes the four sheets cyclically. For this reason, the sheets look schematically as shown in Figure 5.9 (the two sheets which branch over $z = 1$ cannot branch again over $z = 0$ since this would contradict the aforementioned cyclic permutation property at $z = \infty$). Let us now analyze the four sheets and their permutation properties more carefully. First, on the simply connected region

$$\Omega := \mathbb{C} \setminus (-\infty, 0]$$

there exist four branches $f_j(z)$, $0 \le j \le 3$ uniquely determined by the asymptotic equalities

$$f_0(x) \sim x^{\frac{1}{4}},\ f_1(x) \sim ix^{\frac{1}{4}},\ f_2(x) \sim -x^{\frac{1}{4}},\ f_3(x) \sim -ix^{\frac{1}{4}}$$

as $x \to \infty$. The enumeration here has been chosen so that analytic continuation along any circle containing $z = 0, 1$ with positive orientation induces the cyclic permutation

$$f_0 \mapsto f_1 \mapsto f_2 \mapsto f_3 \mapsto f_0$$

Next, for all $x > 1$ there are the explicit expressions

$$f_0(x) = \sqrt{1 + \sqrt{x}}, \quad f_2(x) = -\sqrt{1 + \sqrt{x}}$$
$$f_1(x) = i\sqrt{\sqrt{x} - 1}, \quad f_3(x) = -i\sqrt{\sqrt{x} - 1}$$

Analytic continuation to Ω yields

$$f_1(i0+) = f_3(i0-) = f_2(0) = -1, \quad f_1(i0-) = f_3(i0+) = f_0(0) = 1$$

Recall that by our convention, $\sqrt{z}$ is analytic on Ω with $\sqrt{x} > 0$ if $x > 0$. Analytic continuation around the loop $z(\theta) = 1+\varepsilon^2 e^{i\theta}$, $0 \le \theta \le 2\pi$, with $\varepsilon > 0$ small leaves f_0 and f_2 invariant, whereas f_1 and f_3 are interchanged. Similarly, analytic continuation around $z(\theta) = \varepsilon^2 e^{i\theta}$ (starting with $\theta > 0$ small) takes f_0 into f_3 and f_2 into f_1. This implies that the *monodromy group* of this Riemann surface is generated by the permutations (12)(03) and (13). The formal definition of the monodromy group is as follows: for any $c \in \pi_1(\mathbb{C} \setminus \{0, 1\})$ let $\mu(c) \in S_4$ (the group of permutations on four symbols) be defined as the permutation of the four sheets which is induced by analytic continuation along the closed loop c. For this we assume, as we may, that c has its base point somewhere in Ω and we pick the four germs defined on each branch over that base point for analytic continuation. Note that μ is well-defined by the monodromy theorem. The map

$$\mu : \pi_1(\mathbb{C} \setminus \{0, 1\}) \to S_4$$

is a group homomorphism and the monodromy group is the image $\mu(\pi_1)$. For example, (13)(12)(03) = (0123) which is precisely the cyclic permutation at $z = \infty$. The genus g of this Riemann surface is given by the Riemann-Hurwitz formula as

$$g - 1 = 4(-1) + \frac{1}{2}(1 + 1 + 1 + 3) = -4 + 3 = -1$$

which implies that $g = 0$.

5. Problems

PROBLEM 5.1. *Picture the unramified Riemann surfaces*

$$\mathcal{RS}(\mathbb{C}, \mathbb{C}, \log z, 1), \qquad \mathcal{RS}(\mathbb{C}, \mathbb{C}, z^{\frac{1}{n}}, 1), \tag{5.12}$$

$n \ge 2$. *Prove that they cover* $\mathbb{C}^*$. *Compute the fundamental groups* $\pi_1(\mathcal{RS})$ *of these surfaces and prove that*

$$\mathcal{RS}(\mathbb{C}, \mathbb{C}, \log z, 1) \simeq \mathbb{C}$$
$$\mathcal{RS}(\mathbb{C}, \mathbb{C}, z^{\frac{1}{n}}, 1) \simeq \mathbb{C}^*, \quad n \ge 2$$

in the sense of conformal isomorphisms. Show that each of the surfaces in (5.12) *has a branch point rooted at zero.*

PROBLEM 5.2. *Prove the Weierstrass preparation theorem: suppose* $f(z,w)$ *is analytic in both variables*[1] *and such that* $f(z,w) \neq 0$ *for all* $|w-w_0| = r_0 > 0$ *and* $|z-z_0| < r_1$. *Then there exists a polynomial*

$$P(z,w) = \sum_{j=0}^{n} a_j(z) w^j$$

with a_j *analytic on* $|z-z_0| < r_1$ *so that*

$$f(z,w) = P(z,w)g(z,w)$$

with g *analytic and nonvanishing on* $D(z_0,r_1) \times D(w_0,r_0)$. *Moreover, all solutions of* $P(z,\cdot) = 0$ *lie inside of* $D(w_0,r_0)$ *for all* $z \in D(z_0,r_1)$. *In particular, show the following: if* $\partial_w^\ell f(z_0,w_0) = 0$ *for all* $0 \le \ell < n$, *but* $\partial_w^n f(z_0,w_0) \neq 0$, *then there exist* $r_0, r_1 > 0$ *so that the previous statement applies.*

PROBLEM 5.3. *Let* $A(z)$ *be an* $n \times n$ *matrix so that each entry* $A_{ij}(z)$ *is a polynomial in* z. *Let the eigenvalues be denoted by* $\lambda_j(z)$, $1 \le j \le n$. *Prove that around each point* z_0 *at which* $\lambda_j(z_0)$ *is a simple eigenvalue,* $\lambda_j(z)$ *is an analytic function of* z. *Furthermore, if* z_1 *is a point at which* $\lambda_j(z_1)$ *has multiplicity* k, *then there is a local representation of the form*

$$\lambda_j(z) = \sum_{n=0}^{\infty} a_n (z-z_1)^{\frac{n}{\ell}}$$

with some $1 \le \ell \le k$ *(this is called a Puiseaux series). Now assume that* $A(z)$ *is Hermitian for all* $z \in \mathbb{R}$. *Prove that each* λ_j *is* analytic *on a neighborhood of* $\mathbb{R}$. *In other words, if* $z_1 \in \mathbb{R}$ *then the Puiseux series is actually a power series. Check these statements by means of the examples*

$$A(z) = \begin{bmatrix} 0 & z \\ 1 & 0 \end{bmatrix}, \qquad A(z) = \begin{bmatrix} 0 & z \\ z & 0 \end{bmatrix}$$

PROBLEM 5.4. *For each of the following algebraic functions, you are asked to understand their Riemann surfaces by answering each of the following questions: Where are the branch points on the surface (be sure to check infinity)? How many sheets does it have? How are these sheets permuted under analytic continuation along closed curves which avoid the (roots of the) branch points? What is its genus? You should also try to obtain a sketch or at least some geometric intuition of the Riemann surface.*

$$w = \sqrt[4]{\sqrt{z}-1}\,, \quad w = \sqrt[3]{2\sqrt{z}+z+1}\,, \quad w^3 - 3w - z = 0$$
$$w = \sqrt{(z-z_1)\cdot \ldots \cdot (z-z_m)}\,, \quad w = \sqrt[3]{z^2-1}$$

[1] This means that f is jointly continuous and analytic in each variable.

CHAPTER 6

Differential forms on Riemann surfaces

1. Holomorphic and meromorphic differentials

We already observed that every Riemann surface is orientable as a smooth two-dimensional manifold. Next, we state another important fact. Throughout, M and N will denote Riemann surfaces.

LEMMA 6.1. *Every tangent space T_pM is in a natural way a complex vector space. In particular, M is orientable and thus carries a volume form. Moreover, if $f : M \to N$ is a C^1 map between Riemann surfaces, then f is analytic iff $Df(p)$ is complex linear as a map $T_pM \to T_{f(p)}N$ for each $p \in M$.*

PROOF. First note that $\sphericalangle(\vec{v}, \vec{w})$ is well-defined in T_pM. Simply measure this angle in any chart — because of conformality of the transition maps this does not depend on the choice of chart. The sign of the angle is also well-defined because of the orientation on M. Now let R be a rotation in T_pM by $\frac{\pi}{2}$ in the positive sense. Then we define

$$i\vec{v} := R\vec{v}$$

It is clear that this turns each T_pM into a complex one-dimensional vector space. To fix an orientation on M, simply define (v, iv) with $v \in T_pM$, $v \neq 0$, as the positive orientation. Since $f : U \to \mathbb{R}^2$ with $f \in C^1(U)$, $U \subset \mathbb{C}$ open is holomorphic iff Df is complex linear, we see via charts that the same property lifts to the Riemann surface case. $\square$

As a smooth manifold, M carries k-forms for each $0 \leq k \leq 2$. We allow these forms to be complex valued and denote the respective spaces by

$$\Omega^0(M; \mathbb{C}), \quad \Omega^1(M; \mathbb{C}), \quad \Omega^2(M; \mathbb{C})$$

By definition, $\Omega^0(M; \mathbb{C})$ are simply C^∞ functions on M, whereas because of orientability $\Omega^2(M; \mathbb{C})$ contains a 2-form denoted by vol which never vanishes; hence, every other element in $\Omega^2(M; \mathbb{C})$ is of the form $f\mathrm{vol}$ where $f \in \Omega^0(M; \mathbb{C})$. This leaves $\Omega^1(M; \mathbb{C})$ as only really interesting object here. By definition, each $\omega \in \Omega^1(M; \mathbb{C})$ defines a *real-linear* functional ω_p on T_pM. We will be particularly interested in those that are complex linear. We start with a simple observation from linear algebra.

LEMMA 6.2. *If $T : V \to W$ is a $\mathbb{R}$-linear map between complex vector spaces, then there is a unique representation $T = T_1 + T_2$ where T_1 is complex linear and T_2 complex anti-linear. The latter property means that $T_2(\lambda\vec{v}) = \bar{\lambda}T_2(\vec{v})$.*

PROOF. Uniqueness follows since a $\mathbb{C}$-linear map which is simultaneously $\mathbb{C}$-anti linear vanishes identically. For existence, set

$$T_1 = \frac{1}{2}(T - iTi), \qquad T_2 = \frac{1}{2}(T + iTi)$$

Then $T_1 i = iT_1$ and $T_2 i = -iT_2$, $T = T_1 + T_2$, as desired. $\square$

As an application, consider the following four complex valued maps on $U \subset \mathbb{R}^2$ where U is any open set: $\pi_1, \pi_2, z, \bar{z}$ which are defined as follows

$$\pi_1(x,y) = x,\ \pi_2(x,y) = y,\ z(x,y) = x+iy,\ \bar{z} = x - iy$$

Identifying the tangent space of U with $\mathbb{R}^2$ at every point the differentials of each of these maps correspond to the following constant matrices

$$d\pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ d\pi_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ dz = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ d\bar{z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Let us now write $\omega \in \Omega^1(M;\mathbb{C})$ in local coordinates

$$\omega = a\,dx + b\,dy = \frac{1}{2}(a-ib)\,dz + \frac{1}{2}(a+ib)\,d\bar{z} = u\,dz + v\,d\bar{z}$$

Of course, this is exactly the decomposition of Lemma 6.2 in each tangent space. A very important special case if $\omega = df$ with $f \in \Omega^0(M;\mathbb{C})$ (we already encountered this in Chapter 1). Then

$$\begin{aligned} df &= \partial_z f\,dz + \partial_{\bar{z}} f\,d\bar{z} \\ \partial_z f &= \frac{1}{2}(\partial_x f - i\partial_y f), \quad \partial_{\bar{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f) \end{aligned}$$

If $f \in \mathcal{H}(M)$, then df needs to be complex linear. In other words, $\partial_{\bar{z}} f = 0$ which are precisely the Cauchy–Riemann equations. In this notation, it is easy to give a one line proof of Cauchy's theorem: Let $f \in \mathcal{H}(U)$ where $U \subset \mathbb{C}$ is a domain with piecewise C^1 boundary and $f \in C^1(\bar{U})$. Then

$$\int_{\partial U} f\,dz = \int_U d(f\,dz) = \int_U \partial_{\bar{z}} f\,d\bar{z} \wedge dz = 0$$

We leave it to the reader the verify the chain rules

$$\begin{aligned} \partial_z(g\circ f) &= (\partial_w g)\circ f\ \partial_z f + (\partial_{\bar{w}} g)\circ f\ \partial_z \bar{f} \\ \partial_{\bar{z}}(g\circ f) &= (\partial_w g)\circ f\ \partial_{\bar{z}} f + (\partial_{\bar{w}} g)\circ f\ \partial_{\bar{z}} \bar{f} \end{aligned}$$

as well as the representation of the Laplacean $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

DEFINITION 6.3. *The* holomorphic differentials *on a Riemann surface M, denoted by $\mathcal{H}\Omega^1(M)$, are precisely those $\omega \in \Omega^1(M;\mathbb{C})$ so that $\omega = u\,dz$ in arbitrary local coordinates with u holomorphic. The* meromorphic differentials, *denoted by $\mathcal{M}\Omega^1(M)$, are all $\omega \in \mathcal{H}\Omega^1(M;\mathbb{C}\setminus\mathcal{S})$ where $\mathcal{S} \subset M$ is discrete and so that in local coordinates around an arbitrary point of M one has $\omega = u\,dz$ where u is meromorphic. The points of $\mathcal{S}$ will be called* poles *of ω.*

Note that we are assuming here that those points of $\mathcal{S}$ which are removable singularities of u have been removed. Obvious examples of holomorphic and meromorphic differentials, respectively, are given by df where $f \in \mathcal{H}(M)$ or $f \in \mathcal{M}(M)$.

2. Integrating differentials and residues

Let us introduce the following terminology, with M, N Riemann surfaces as usual:

DEFINITION 6.4. *We say that $N \subset M$ is a Stokes region if $\bar{N}$ is compact and ∂N is piecewise C^1. This means that ∂N is the finite union of curves $\gamma_j : [0,1] \to M$ which are C^1 on the closed interval $[0,1]$.*

The importance of a Stokes region lies with the fact that Stokes theorem applies to it. We have the following simple but important properties:

PROPOSITION 6.5. *Suppose $\omega \in \mathcal{H}\Omega^1(M)$. Then $d\omega = 0$. Thus, $\int_{\partial N} \omega = 0$ for every Stokes region $N \subset M$. Moreover, for any closed curve γ the integral $\oint_\gamma \omega$ only depends on the homology class of γ. In particular, $\oint_\gamma \omega = \oint_\eta \omega$ if γ and η are homotopic closed curves. Finally, if c is a curve with initial point p and endpoint q, then*

$$\int_c \omega = f_1(q) - f_0(p)$$

where $df_0 = \omega$ locally around p and f_1 is obtained via analytic continuation of f_0 along c. In particular, $df_1 = \omega$ locally around q.

PROOF. Since $\omega = u\, dz$ in a chart, one has

$$d\omega = \partial_{\bar{z}} u \, d\bar{z} \wedge dz = 0$$

as claimed. The other properties are immediate consequences of this via Poincare's

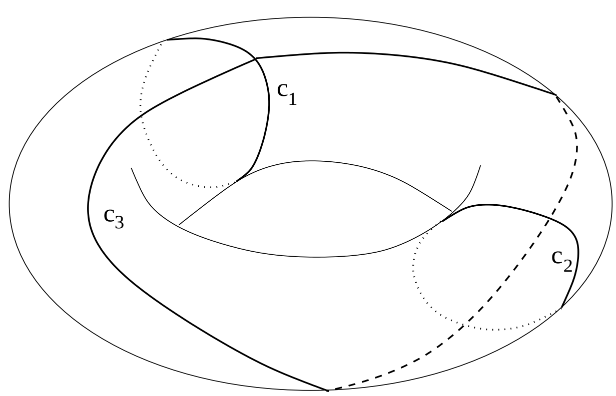

FIGURE 6.1. c_1 and c_2 are homologous to each other, but not to c_3

lemma (closed means locally exact) and Stokes' integral theorem. We skip the details. $\square$

For meromorphic differentials, we have the following facts.

PROPOSITION 6.6. *Suppose $\omega \in \mathcal{M}\Omega^1(M)$ with poles $\{p_j\}_{j=1}^J$. Then at each of these poles their order $\operatorname{ord}(\omega, p_j) \in \mathbb{Z}^-$ and residue $\operatorname{res}(\omega, p_j) \in \mathbb{C}$ are well-defined. In fact,*

$$\operatorname{res}(\omega, p_j) = \frac{1}{2\pi i} \oint_c \omega$$

where c is any small loop around p_j. Given any Stokes region $N \subset M$ so that ∂N does not contain any pole of ω, we have

$$\frac{1}{2\pi i}\oint_{\partial N} \omega = \sum_{p\in N} \operatorname{res}(\omega, p) \tag{6.1}$$

Finally, if M is compact, then

$$\sum_{p\in M} \operatorname{res}(\omega; p) = 0$$

for all $\omega \in \mathcal{M}\Omega^1(M)$. With $N \subset M$ a Stokes region and $f \in \mathcal{M}(M)$,

$$\frac{1}{2\pi i}\oint_{\partial N} \frac{df}{f} = \#\{p \in N \mid f(p) = 0\} - \#\{p \in N \mid p \text{ is a pole of } f\}$$

assuming that no zero or pole lies on ∂N.

Proof. Let $\omega = u\,dz$ with $u(p) = \sum_{n=-n_0}^{\infty} a_n\, z^n$ in local coordinates (U, z) around p_j with $z(p_j) = 0$ and $a_{n_0} \neq 0$. Note that n_0 does not depend on the choice of the chart but the coefficients do in general. However, since

$$a_{-1} = \frac{1}{2\pi i}\oint \omega$$

this coefficient does not depend on the chart and it is the residue. The "residue theorem" (6.1) follows from Stokes applies to $N' := N \setminus D_j$ where D_j are small parametric disks centered at $p_j \in N$. Finally, if M is compact, then we triangulate M in such a way that no edge of the triangulation passes through a pole (there are only finitely many of them). This follows simply by setting $\omega = \frac{df}{f}$ in (6.1). □

In the following chapter we shall prove that to a given finite sequence $\{p_j\}$ of points and complex numbers $\{c_j\}$ adding up to zero at these points we can find a meromorphic differential that has simple poles at *exactly* these points with residues equal to the c_j. This will be based on the crucial Hodge theorem to which we now turn.

3. The Hodge $*$ operator and harmonic differentials

Definition 6.7. *To every $\omega \in \Omega^1(M; \mathbb{C})$ we associate a one–form $*\omega$ defined as follows: if $\omega = u\,dz + v\,d\bar z = f\,dx + g\,dy$ in local coordinates, then*

$$*\omega := -iu\,dz + iv\,d\bar z = -g\,dx + f\,dy$$

Moreover, if $\omega, \eta \in \Omega^1_{\mathrm{comp}}(M; \mathbb{C})$ (the forms with compact support), we set

$$\langle \omega, \eta \rangle := \int_M \omega \wedge \overline{*\eta} \tag{6.2}$$

This defines an inner product on $\Omega^1_{\mathrm{comp}}(M; \mathbb{C})$. The completion of this space is denoted by $\Omega^1_2(M; \mathbb{C})$.

Some comments are in order: first, $*\omega$ is well-defined as can be seen from the change of coordinates $z = z(w)$. Then

$$\omega = u\,dz + v\,d\bar z = uz'\,dw + v\,\overline{z'}\,d\bar z$$

and $*\omega$ transforms the same way. Second, it is evident that (6.2) does not depend on coordinates and, moreover, if ω, η are supported in U where (U, z) is a chart, then with $\omega = u\,dz + v\,d\bar{z}$, $\eta = r\,dz + s\,d\bar{z}$ we obtain

$$\omega \wedge \overline{*\eta} = i(u\bar{r} + v\bar{s})\,dz \wedge d\bar{z} = 2(u\bar{r} + v\bar{s})\,dx \wedge dy$$

and in particular,

$$\langle \omega, \eta \rangle = 2 \iint_U (u\bar{r} + v\bar{s})\,dx \wedge dy$$

which is obviously a positive definite scalar product locally on U. By a partition of unity, this shows that indeed (6.2) is a scalar product on $\Omega^1_{\mathrm{comp}}(M)$ and, moreover, that the abstract completion $\Omega^1_2(M)$ consists of all 1-forms ω which in charts have measurable L^2_{loc} coefficients and which satisfy the global property $\|\omega\|_2^2 := \langle \omega, \omega \rangle < \infty$.

Let us state some easy properties of the Hodge-$*$ operator.

LEMMA 6.8. *For any $\omega, \eta \in \Omega^1_2(M)$ we have*

$$*\overline{\omega} = \overline{*\omega}, \quad **\,\omega = -\omega, \quad \langle *\,\omega, *\eta \rangle = \langle \omega, \eta \rangle$$

PROOF. The first two identities follow immediately from the representation in local coordinates, whereas the third is a consequence of the first two. □

We now come to the very important topic of harmonic functions and forms. Recall that the class of harmonic functions is invariant under conformal changes of coordinates, see Corollary 1.29.

DEFINITION 6.9. *We say that $f \in \Omega^0(M; \mathbb{C})$ is harmonic iff f is harmonic in every chart. We say that $\omega \in \Omega^1(M; \mathbb{C})$ is harmonic iff $d\omega = d{*}\omega = 0$, i.e., iff ω is both closed and co-closed. We denote the harmonic forms on M by $\mathfrak{h}(M; \mathbb{R})$ if they are real-valued and by $\mathfrak{h}(M; \mathbb{C})$ if they are complex-valued.*

Let us state some basic properties of harmonic functions, mainly the important maximum principle.

LEMMA 6.10. *Suppose $f \in \Omega^0(M; \mathbb{C})$ is harmonic with respect to some atlas. Then it is harmonic with respect the any equivalent atlas and therefore, also with respect to the conformal structure. Moreover the* maximum principle *holds: if such an f is real-valued and the open connected set $U \subset M$ has compact closure in M, then*

$$\min_{\partial U} f \le f(p) \le \max_{\partial U} f \qquad \forall\, p \in U$$

with equality being attained at some $p \in U$ iff $f = \mathrm{const}$. In particular, if M is compact, then f is constant.

PROOF. Under the conformal change of coordinates $w = w(z)$ we have

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = |w'(z)|^2 \frac{\partial^2 f}{\partial w \partial \bar{w}}$$

Thus, harmonicity is preserved under conformal changes of coordinates as claimed. For the maximum principle, we note that $f : U \to \mathbb{R}$ is harmonic in a chart and attains a local maximum in that chart, then it is constant on the chart by the maximum principle for harmonic functions on open sets of $\mathbb{C}$. But then f would have to be constant on all of U by connectedness and the fact that harmonic functions in the plane that are constant on some open subset of a planar domain have to be constant on the entire domain. Hence we have shown that f cannot attain a local maximum on U. Finally, if M is compact, then by taking $U = M$ we are done. □

For harmonic forms we have the following simple properties.

LEMMA 6.11. *Let $\omega \in \mathfrak{h}(M;\mathbb{R})$ (or $\mathfrak{h}(M;\mathbb{C})$). Then, locally around every point of M, $\omega = df$ where f is real-valued (or complex-valued) and harmonic. If M is simply connected, then $\omega = df$ where f is a harmonic function on all of M. Conversely, if f is a harmonic function on M, then df is a harmonic 1–form. If u is a harmonic (and either real- or complex-valued) function, then the complex linear part of du is a holomorphic differential. In other words, in local coordinates, $\omega = \partial_z u\, dz \in \mathcal{H}\Omega^1(M)$.*

PROOF. Since ω is locally exact, we have $\omega = df$ locally with f being either real- or complex-valued depending on whether ω is real- or complex-valued. Then ω is co-closed iff $d{*}df = 0$. In local coordinates, this is the same as

$$d(-f_y\, dx + f_x\, dy) = (f_{xx} + f_{yy})\, dx \wedge dy = 0$$

which is the same as f being harmonic. This also proves the converse. If M is simply connected, then f is a global primitive of ω. For the final statement, note that

$$\partial_{\bar z}\partial_z u = 0$$

since u is harmonic. □

Here is a useful characterization of harmonic differentials. We will omit the field $\mathbb{R}$ or $\mathbb{C}$ from our notation if this choice makes no difference.

LEMMA 6.12. *Let $\omega \in \Omega^1(M)$ and suppose that $\omega = a\, dx + b\, dy$ in some chart (U, z). Then ω is harmonic iff $f := a - ib$ is holomorphic on $z(U)$.*

PROOF. Since

$$d\omega = (-a_y + b_x)\, dx \wedge dy, \quad d{*}\omega = (b_y + a_x)\, dx \wedge dy$$

we see that ω is harmonic iff $a, -b$ satisfy the Cauchy–Riemann system on $z(U)$ which is equivalent to $a - ib$ being holomorphic on $z(U)$. □

Next, we make the following observation linking holomorphic and harmonic differentials.

LEMMA 6.13. *Let $\omega \in \Omega^1(M;\mathbb{C})$. Then*

(1) *ω is harmonic iff $\omega = \alpha + \overline{\beta}$ where $\alpha, \beta \in \mathcal{H}\Omega^1(M)$*
(2) *$\omega \in \mathcal{H}\Omega^1(M)$ iff $d\omega = 0$ and ${*}\omega = -i\omega$ iff $\omega = \alpha + i{*}\alpha$ where $\alpha \in \mathfrak{h}(M;\mathbb{R})$*

In particular, every holomorphic differential is harmonic and the only real-valued holomorphic differential is zero.

PROOF. Write $\omega = u\, dz + v\, d\bar z$ in local coordinates. For (1), observe that

$$d\omega = (-\partial_{\bar z} u + \partial_z v)\, dz \wedge d\bar z$$
$$d{*}\omega = i(\partial_{\bar z} u + \partial_z v)\, dz \wedge d\bar z$$

both vanish identically iff $\partial_{\bar z} u = 0$ and $\partial_z v = 0$. In other words, iff $\alpha = u\, dz$ and $\beta = \bar v\, dz$ are both holomorphic differentials.

For (2), ω is holomorphic iff $v = 0$ and $\partial_{\bar z} u = 0$ iff $d\omega = 0$ and ${*}\omega = -i\omega$. If $\omega = \alpha + i{*}\alpha$ with α harmonic, then $d\omega = 0$ and ${*}\omega = {*}\alpha - i\alpha = -i\omega$. For the converse, set $\alpha = \frac{1}{2}(\omega + \bar\omega)$. Then ${*}\alpha = \frac{i}{2}(-\omega + \bar\omega)$, $\alpha + i{*}\alpha = \omega$, and $\alpha \in \mathfrak{h}(M;\mathbb{R})$ as desired.

Finally, it is clear from (1) that every holomorphic differential is also harmonic. On the other hand, if $\omega \in \mathcal{H}\Omega^1(M)$ and real-valued, then we can write

$$\omega = \alpha + i{*}\alpha = \overline{\omega} = \alpha - i{*}\alpha$$

or ${*}\alpha = 0$ which is the same as $\alpha = 0$. $\square$

In the simply connected compact case it turns out that there are no non zero harmonic or holomorphic differentials.

COROLLARY 6.14. *If M is compact and simply connected, then*

$$\mathfrak{h}(M;\mathbb{R}) = \mathfrak{h}(M;\mathbb{C}) = \mathcal{H}\Omega^1(M) = \{0\}$$

PROOF. Any harmonic 1–form ω can be written globally on M as $\omega = df$ with f harmonic, see Lemma 5.5. But then $f = \text{const}$ by the maximum principle and so $\omega = 0$. Consequently, the only harmonic 1–form is also zero. $\square$

The obvious example for this corollary is of course $M = \mathbb{C}P^1$. Let us now consider some examples to which Corollary 6.14 does not apply. In the case of $M \subset \mathbb{C}$ simply connected we have in view of Lemmas 6.11–6.13,

$$\begin{aligned} \mathcal{H}\Omega^1(M) &= \{df \mid f \in \mathcal{H}(M)\} \\ \mathfrak{h}(M;\mathbb{C}) &= \{df + \overline{dg} \mid f, g \in \mathcal{H}(M)\} \\ \mathfrak{h}(M;\mathbb{R}) &= \{a\,dx + b\,dy \mid a = \mathrm{Re}\,(f),\ b = -\mathrm{Im}\,(f),\ f \in \mathcal{H}(M)\} \\ &= \{df + \overline{df} \mid f \in \mathcal{H}(M)\} \end{aligned} \tag{6.3}$$

In these examples harmonic (or holomorphic) 1–forms are globally differentials of harmonic (or holomorphic) functions.

For a non-simply connected example, take $M = \{r_1 < |z| < r_2\}$ with $0 \le r_1 < r_2 \le \infty$. In these cases, a closed form ω is exact iff

$$\oint_{\gamma_r} \omega = 0, \qquad \gamma_r(t) = re^{2\pi i t}$$

for one (and thus every) $\gamma_r \subset M$ (or any closed curve in M that winds around 0). This implies that every closed ω can be written uniquely as

$$\omega = k\,d\theta + df, \quad k = \frac{1}{2\pi}\oint_{\gamma_r} \omega, \quad f \in C^\infty(M)$$

and, with θ being any branch of the polar angle,

$$d\theta := -\frac{y}{r^2}\,dx + \frac{x}{r^2}\,dy, \quad r^2 = x^2 + y^2$$

We remark that $d\theta \in \mathfrak{h}(M;\mathbb{R})$ since (any branch of) the polar angle is harmonic. This is of course is in agreement with the de Rham chomology fact $H^1(\mathbb{R}^2 \setminus \{0\}) \simeq \mathbb{R}$. The conclusion is that

$$\mathfrak{h}(M;\mathbb{R}) = \{df + k\,d\theta \mid f \text{ harmonic and } \mathbb{R}\text{-valued on } M,\ k \in \mathbb{R}\}$$

$$\mathfrak{h}(M;\mathbb{C}) = \{df + k\,d\theta \mid f \text{ harmonic and } \mathbb{C}\text{-valued on } M,\ k \in \mathbb{C}\}$$

$$\mathcal{H}\Omega^1(M) = \left\{f_z\,dz + ik\frac{dz}{z} \mid f \text{ harmonic, } \mathbb{R}\text{-valued on } M,\ k \in \mathbb{R}\right\} \tag{6.4}$$

$$= \left\{dg + \kappa\frac{dz}{z} \mid g \in \mathcal{H}(M),\ \kappa \in \mathbb{C}\right\} \tag{6.5}$$

The representation (6.4) follows from Lemma 6.13, whereas for (6.5) we note that $\omega \in \mathcal{H}\Omega^1(M)$ is exact iff

$$\oint_{\gamma_r} \omega = 0$$

Hence, (6.5) follows by setting

$$\kappa = \frac{1}{2\pi i}\oint_{\gamma_r} \omega$$

which then allows us to define $g(z) := \int_1^z \omega$ along an arbitrary curve. To reconcile (6.5) with (6.4), we first observe that

$$\oint_{\gamma_r} f_z\,dz = -\oint_{\gamma_r} f_{\bar z}\,d\bar z = -\overline{\oint_{\gamma_r} f_z\,dz} \in i\mathbb{R}$$

since f is real-valued. Conversely, if $a \in \mathbb{R}$, then there exists f real-valued and harmonic on M such that

$$\oint_{\gamma_r} f_z\,dz = 2\pi i a$$

Indeed, simply set $f(z) = a\log|z|$ for which $f_z(z)\,dz = a\frac{\bar z}{r^2}\,dz$. This explains why it suffices to add $ik\frac{dz}{z}$ with $k \in \mathbb{R}$ in (6.4).
As a final example, let $M = \mathbb{C}/\langle 1,\tau\rangle$ where $\langle 1,\tau\rangle \subset \mathrm{Aut}(\mathbb{C})$ is the group generated by $z \mapsto z+1$, $z \mapsto z+\tau$ and $\operatorname{Im}\tau > 0$. Then any $\omega \in \mathfrak{h}(M;\mathbb{R})$ lifts to the universal cover of M which is $\mathbb{C}$. Thus, we can write $\omega = a\,dx + b\,dy$ where $a - ib$ is an analytic function on M and thus constant. Hence,

$$\dim_{\mathbb{R}} \mathfrak{h}(M;\mathbb{R}) = \dim_{\mathbb{C}} \mathfrak{h}(M;\mathbb{C}) = 2 = 2\dim_{\mathbb{C}} \mathcal{H}\Omega^1(M)$$

Any reader familiar with Hodge's theorem will recognize the statement here that $H^1(M) \simeq \mathbb{R}^{2g}$ where M is a compact surface of genus g.

4. Statement and examples of the Hodge decomposition

In Chapter 7 we shall prove the following version of Hodge's theorem:

$$\Omega^1_2(M;\mathbb{R}) = E \oplus *E \oplus \mathfrak{h}_2(M;\mathbb{R}) \tag{6.6}$$

Here $\Omega^1_2(M;\mathbb{R})$ are the square integrable, real-valued, one forms from Definition 6.7,

$$\mathfrak{h}_2(M;\mathbb{R}) := \mathfrak{h} \cap \Omega^1_2(M;\mathbb{R}),$$

and

$$E := \overline{\{df \mid d \in \Omega^0_{\mathrm{comp}}(M;\mathbb{R})\}}, \quad *E := \overline{\{*df \mid d \in \Omega^0_{\mathrm{comp}}(M;\mathbb{R})\}}$$

where the closure is meant in the sense of $\Omega^1_2(M)$. Figure 6.2 describes the subspaces appearing in Hodge's theorem (the reason for the (co)closed planes will become clear later, see Lemma 7.2). Let us first clarify that $E \perp *E$: thus, let $f, g \in \Omega^0_{\mathrm{comp}}(M;\mathbb{R})$ and compute

$$\int_M df \wedge \overline{**dg} = -\int_M df \wedge dg = -\int_M d(f dg) = 0$$

by Stokes. Hence,

$$\Omega^1_2(M;\mathbb{R}) = E \oplus *E \oplus (E^\perp \cap (*E)^\perp)$$

and the main issue then becomes equating the intersection at the end with $\mathfrak{h}_2(M;\mathbb{R})$. This is nontrivial, since we will need to prove that all forms in $E^\perp \cap (*E)^\perp$ are *smooth*

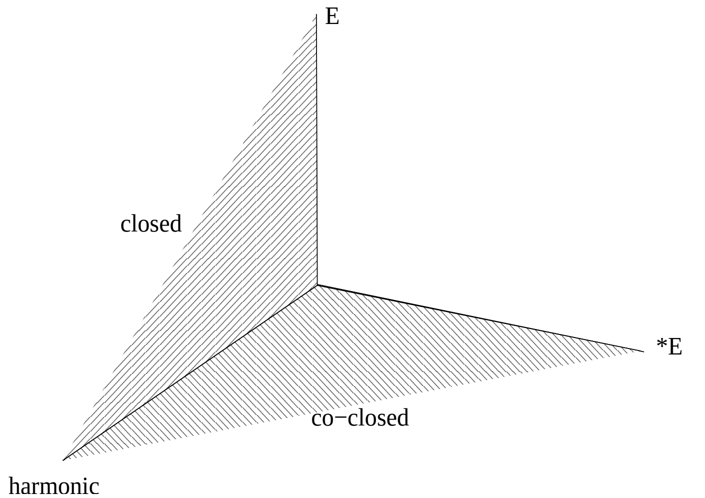

FIGURE 6.2. A schematic view of the Hodge decomposition

while at first sight they only have L^2 coefficients. It will turn out that it is easy to see from the definition that $E^\perp \cap (*E)^\perp$ are *weakly harmonic*[1] so that the issue then is to prove that weakly harmonic forms are strongly harmonic. This of course is the content of the basic Weyl's lemma which is an example of elliptic regularity theory.

The purpose of this section is to establish the Hodge decomposition (6.6) on four manifolds: $\mathbb{C}$, S^2, $\mathbb{C}/\mathbb{Z}^2$, and $\mathbb{D}$. In each case we shall determine the space of harmonic differentials and exhibit the role they play in (6.6). Heuristically speaking, the space $\mathfrak{h}_2$ is nonzero due to either a nontrivial cohomology or a "large boundary" of M. The former is revealed by the torus $\mathbb{C}/\mathbb{Z}^2$, whereas the latter arises in the case of $\mathbb{D}$. More technically speaking, in each of the four examples we will prove the Hodge decomposition by solving the Poisson equation and the harmonic forms arise either because of an integrability condition or due to a boundary condition.

Example 1: $M = \mathbb{C}$

Pick any $\omega \in \mathfrak{h}_2(\mathbb{C})$. In view of (6.3), $\omega = a\,dx + b\,dy$ with a, b being harmonic and L^2 bounded:

$$\iint_{\mathbb{R}^2} (|a|^2 + |b|^2)\,dxdy < \infty$$

We claim that necessarily $a = b = 0$. Indeed, from the mean-value theorem

$$\begin{aligned} |a(z)|^2 &= \Big|\frac{1}{|D(z,r)|} \iint_{D(z,r)} a(\zeta)\,d\xi d\eta\Big|^2 \\ &\le \frac{1}{|D(z,r)|} \iint_{D(z,r)} |a(\zeta)|^2\,d\xi d\eta \le \frac{\|a\|_2^2}{|D(z,r)|} \to 0 \end{aligned}$$

as $r \to \infty$. So $\mathfrak{h}(M) = \{0\}$ in that case. Note that while $\mathfrak{h}(\mathbb{C}; \mathbb{R})$ is a huge space (since there are many entire functions by the Weierstrass theorem), the L^2 condition only leaves the zero form. This can be thought of the fact that the boundary of $\mathbb{C}$ is tiny, and in fact it consists only of the point at infinity in S^2. Indeed, heuristically the previous argument

[1] This will be made precise in the next chapter.

can be thought of as follows: any form in $\mathfrak{h}_2$ can be continued to a harmonic form on S^2 which then has to vanish.

Hence, by (6.6) every L^2 form ω is the sum of an exact and a co-exact form (more precisely, up to L^2 closure). Let us understand this first for smooth, compactly supported ω. Thus, let $\omega = a\,dx + b\,dy$ with $a, b \in C^\infty_{\mathrm{comp}}(\mathbb{R}^2)$. Then we seek $f, g \in C^\infty(\mathbb{R}^2)$ with

$$\omega = df + {*dg} \tag{6.7}$$

Since $\mathbb{C}$ is simply connected, this is equivalent to writing $\omega = \alpha + \beta$ where $d\alpha = 0$ and $d{*\beta} = 0$. This in turn shows that (6.7) is equivalent to writing a smooth, compactly supported vector field as the sum of a divergence-free field and a curl-free field. To find f, g, we apply d and $d*$ to (6.7) which yields

$$\Delta f = a_x + b_y, \quad \Delta g = -a_y + b_x \tag{6.8}$$

We therefore need to solve the Poisson equation $\Delta f = h$ with $h \in C^\infty_{\mathrm{comp}}(\mathbb{R}^2)$. A solution to this equation is not unique; indeed, we can add linear polynomials to f. On the other hand, solutions that decay at infinity are necessarily unique from the maximum principle. To obtain existence, we invoke the *fundamental solution* of the Laplacian on $\mathbb{R}^2$, which is $\Gamma(z) = \frac{1}{2\pi}\log|z|$. This means that $\Delta\Gamma = \delta_0$ in the sense of distributions and we therefore expect to find a solution via $f = \Gamma * h$. We now derive the solution to Poisson's equation in the smooth setting; much weaker conditions suffice but we do not wish to dwell on that for now.

LEMMA 6.15. *Let $h \in C^\infty_{\mathrm{comp}}(\mathbb{R}^2)$. Then the function*

$$f(z) := \frac{1}{2\pi}\iint_{\mathbb{R}^2} h(\zeta)\log|z-\zeta|\,d\xi d\eta = \frac{1}{2\pi}\iint_{\mathbb{R}^2} h(z-\zeta)\log|\zeta|\,d\xi d\eta \tag{6.9}$$

(with $\zeta = \xi + i\eta$) satisfies $f \in C^\infty$ and solves $\Delta f = h$. Moreover,

$$f(z) = \frac{1}{\pi}\langle h\rangle \log|z| + O(1/|z|) \quad \text{as } |z| \to \infty \tag{6.10}$$

where $\langle h\rangle := \int_{\mathbb{R}^2} h\,d\xi d\eta$ is the mean of h. In fact, f is the unique solution which is of the form

$$f(z) = k\log|z| + o(1) \quad \text{as } |z| \to \infty \tag{6.11}$$

for some constant $k \in \mathbb{R}$.

PROOF. Differentiating under the integral sign yields

$$\begin{aligned}\Delta f(z) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} \Delta_z h(z-\zeta)\log|\zeta|\,d\xi d\eta \\ &= \frac{1}{2\pi}\lim_{\varepsilon\to 0}\iint_{|z-\zeta|>\varepsilon} \Delta_\zeta h(z-\zeta)\log|\zeta|\,d\xi d\eta\end{aligned}$$

Now apply Green's identity (which follows from Stokes theorem on manifolds)

$$\int_\Omega (v\Delta u - u\Delta v)\,d\xi d\eta = \int_{\partial\Omega}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})\,d\sigma$$

to conclude that (and using that $\log|\zeta|$ is harmonic away from zero)

$$\begin{aligned}
&\iint_{|z-\zeta|>\varepsilon} \log|\zeta|\, \Delta_\zeta h(z-\zeta)\, d\xi d\eta \\
&\iint_{|z-\zeta|>\varepsilon} \Big[\log|\zeta| \Delta_\zeta h(z-\zeta) - h(z-\zeta)\Delta_\zeta \log|\zeta|\Big]\, d\xi d\eta \\
&= \int_{|z-\zeta|=\varepsilon} \Big[\log|\zeta| \frac{\partial}{\partial n_\zeta} h(z-\zeta) - h(z-\zeta)\frac{\partial}{\partial n_\zeta}\log|\zeta|\Big]\, d\sigma
\end{aligned}$$

where n is the outward pointing norm vector relative to the region $|z-\zeta|>\varepsilon$. Thus,

$$\frac{\partial}{\partial n_\zeta}\log|\zeta| = -\frac{1}{|\zeta|}$$

In conclusion, letting $\varepsilon \to 0$ yields $\Delta f = h$ as desired. We remark that in dimensions $d \geq 3$ the fundamental solutions of Δ are $c_d|x-y|^{2-d}$ with a dimensional constant c_d by essentially the same proof. In $d=1$ a natural choice is $x_+ := \max(x,0)$ or anything obtained from this by adding a linear function. Inspection of our solution formula (6.9) now establishes (6.10); indeed, simply expand the logarithm

$$\log|z-\zeta| = \log|z| + \log|1-\zeta/z|$$

To establish the uniqueness, one uses that the only harmonic function of the form (6.11) vanishes identically (use the mean value property). In particular, the solution f of $\Delta f = h$ and h as above decays at infinity if and only if $\langle h\rangle = 0$. □

Returning to our discussion of Hodge's decomposition, recall that h is given by the right-hand sides of (6.8). Evidently, in that case $\langle h\rangle = 0$ so that (6.9) yields smooth functions f, g decaying like $1/|z|$ at infinity and which solve (6.8). It remains to check that indeed

$$\omega = df + {*dg} = (f_x - g_y)dx + (f_y + g_x)dy$$

To this end we simply observe that

$$\begin{aligned}
(f_x - g_y)(z) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} \Delta a(\zeta)\, \log|z-\zeta|\, d\xi d\eta = a(z) \\
(f_y + g_x)(z) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} \Delta b(\zeta)\, \log|z-\zeta|\, d\xi d\eta = b(z)
\end{aligned}$$

To obtain the final two equality signs no calculations are necessary; in fact, since a vanishes at infinity, the only decaying solution to $\Delta f = \Delta a$ is $f = a$ and the same holds for b. To summarize: we have shown that every compactly supported $\omega \in \Omega^1(\mathbb{C};\mathbb{C})$ is the sum of an exact and a co-exact smooth 1–form and each of these summands typically decay only like $|z|^{-2}$ (as differentials of functions decaying like $|z|^{-1}$). It is important to note that this rate of decay is square integrable at infinity relative to Lebesgue measure in the plane; if this were not so, then one could not derive the Hodge decomposition in the L^2 setting. In particular, we see here this is crucial that we are dealing with forms and not functions, as the harmonic functions themselves only decay at the rate $|z|^{-1}$ which is not square integrable in the plane.

It is important to now that our proof extends to the L^2 setting; that is, given $a, b \in L^2$ there exist $f, g \in H^1(\mathbb{R}^2)$ (which is the Sobolev space of L^2 functions with an L^2 weak derivative) so that $a\,dx + b\,dy = df + {*dg}$ as an equality between L^2 functions. As usual, this can be obtain from the smooth, compactly supported case which we just discussed since those functions are dense in $H^1(\mathbb{R}^2)$. The required uniform control required to pass to the limit is given by the L^2 boundedness of the double Riesz transforms R_{ij}. We leave the the details of this to the reader, see Problem 6.1.

Example 2: $M = S^2$ or M is any compact, simply connected Riemann surface.

Since we already observed that $\mathfrak{h}(M) = \{0\}$ in this case, we see that trivially $\mathfrak{h}_2(M) = \{0\}$ so that Hodge's decomposition (6.6) again reduces to

$$\Omega^1_2(M) = E \oplus *E$$

Indeed, it is easy to prove that every $\omega \in \Omega^1(M)$ is of the form

$$\omega = df + *dg, \quad f, g \in C^\infty(M)$$

Applying d to this yields $d\omega = \Delta_M g\,\mathrm{vol}$ where vol is a suitably normalized volume form on M and Δ_M is the Laplace–Beltrami operator on M. Recall that the Laplace Beltrami operator Δ_M on any compact orientable manifold M has discrete spectrum and $L^2(M)$ has an orthonormal basis consisting of (smooth) eigenfunctions Y_n of Δ_M with eigenvalues $\{\lambda_n\}_{n=0}^\infty$, $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots$. The lowest eigenvalue λ_0 is simple, i.e., Y_0 is necessarily a constant as follows from the maximum principle. Finally, from Weyl's law the eigenvalue λ_n grow at some power rate in n. Hence, continuing in this degree of generality, one concludes that $\Delta_M f = h$ with $f \in C^\infty(M)$, say, has a solution iff $\langle h\rangle = \int_M h\,dv = 0$ (v is the volume form on M). In that case the solution is given by

$$f = \sum_{n\ge1} \lambda_n^{-2}\langle h, Y_n\rangle Y_n$$

which converges rapidly since h is smooth. Returning to $d\omega = \Delta_M g\,\mathrm{vol}$, one sees from Stokes theorem that $\int_M d\omega = 0$ so that the integrability condition holds. This yields a smooth solution g whence $\omega - *dg$ is a closed form on M. Since M is simply connected, it is also exact and thus $\omega = df + *dg$ for some smooth f. Note that the simple connectivity of M entered only at the final step. In the following example, we shall see how the harmonic functions precisely eliminate the obstruction to the exactness of closed forms in the genus one case.

Example 3: $M = \mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2 = \mathbb{C}/\langle z \mapsto z+1, z \mapsto z+i\rangle$

In view of our previous discussion of the harmonic forms in this case, (6.6) reduces the following: any $\omega = a\,dx + b\,dy$ with smooth, $\mathbb{Z}^2$–periodic functions a, b can be written as

$$a\,dx + b\,dy = df + *dg + c_1\,dx + c_2\,dy \tag{6.12}$$

where f, g are smooth, $\mathbb{Z}^2$–periodic functions and suitable constants c_1, c_2. In fact, it will turn out that

$$c_1 = \int_0^1\int_0^1 a(x,y)\,dxdy, \quad c_2 = \int_0^1\int_0^1 b(x,y)\,dxdy$$

As in the discussion of the whole plane, finding f, g reduces to a suitable Poisson equation. Hence, let us first understand how to solve $\Delta f = h$ on $\mathbb{T}^2$ with smooth h. Integrating over $\mathbb{T}^2$ shows that the vanishing condition $\int_0^1\int_0^1 h(x,y)\,dxdy = 0$ is necessary. It is also sufficient for solvability; indeed, any such smooth h has a convergent Fourier expansion

$$h(x,y) = \sum_{n_1,n_2} \hat h(n_1,n_2)e(xn_1 + yn_2)$$

where $\hat{h}(0,0)=0$ (and with $e(x):=e^{2\pi i x}$). The solution to $\Delta f = h$ is therefore given by

$$f(x,y) = -\sum_{n_1,n_2} \frac{\hat{h}(n_1,n_2)}{4\pi^2(n_1^2+n_2^2)} e(xn_1+yn_2)$$

which is again smooth. We write this schematically as $f = \Delta^{-1}h$. As in the case of $M=\mathbb{C}$, solving (6.12) reduces to solving (6.8) on $\mathbb{T}^2$. Notice that our vanishing condition is automatically satisfied and we therefore obtain smooth solutions f, g. In order to conclude that $\omega = df + {*dg}$ it remains to check that $a = \Delta^{-1}\Delta a$ and the same for b. This is true iff a and b have vanishing means and confirms our choice of c_1, c_2 above. Here we see an example where $\mathfrak{h}(M) = \mathfrak{h}_2(M)$ plays a *topological* role. This is typical of the *compact* case but not of the non-compact case.

Example 3: $M = \mathbb{D}$

Our fourth example is the disk $\mathbb{D}$ (which is the same as the upper half-plane or any other simply connected true subdomain of $\mathbb{C}$). In this case there is not only an abundance of harmonic and holomorphic one-forms, but also of *square integrable* ones. First, we remark that

$$E = \{df \mid f \in H_0^1(\mathbb{D})\}$$

where $H_0^1(\mathbb{D})$ is the usual Sobolev space with vanishing trace on $\partial\mathbb{D}$ (see [**12**], for example). Second, let us reformulate (6.6) as an equivalent fact for vector fields $\vec{v} = (v_1, v_2) \in L^2(\mathbb{D})$ rather than forms: there exist $f, g \in H_0^1(\mathbb{D})$, as well as $\vec{\omega} = (\omega_1, \omega_2)$ smooth and both divergence–free and curl–free, and with $\omega_1, \omega_2 \in L^2(\mathbb{D})$ so that

$$\vec{v} = \vec{\nabla} f + \vec{\nabla}^{\perp} g + \vec{\omega}$$

where $\vec{\nabla}^{\perp} g := (-g_y, g_x)$. To find f, we need to solve

$$\Delta f = \operatorname{div} \vec{v} \in H^{-1}(\mathbb{D}), \quad f \in H_0^1(\mathbb{D})$$

whereas for g, we need to solve

$$\Delta g = \operatorname{div}^{\perp} \vec{v} \in H^{-1}(\mathbb{D}), \quad g \in H_0^1(\mathbb{D})$$

where $\operatorname{div}^{\perp}\vec{v} = -\partial_y v_1 + \partial_x v_2$. This can be done uniquely with $f, g \in H_0^1(\mathbb{D})$ via the usual machinery of weak solutions for elliptic equations, see the chapter on elliptic equations in [**12**]. For the uniqueness, suppose that $\Delta g = 0$ and $g \in H_0^1(\mathbb{D})$. Then $\Delta g \in H^{-1}(\mathbb{D}) = H_0^1(\mathbb{D})^*$ and

$$0 = \langle -\Delta g, g\rangle = \int_{\mathbb{D}} |\nabla g|^2 \, dxdy$$

which implies that g is constant and therefore zero. This shows that any $\omega \in E$ which is also harmonic is zero. Notice the importance of the "boundary condition" in this regard which was built into the space E (coming from the compact support condition). Of course there are many (nonzero) harmonic differentials which are also in $L^2(\mathbb{D})$, but they are limits of differentials df with $f \in C^{\infty}_{\mathrm{comp}}(\mathbb{D})$ only if they vanish identically.

5. Problems

PROBLEM 6.1. *Provide the details for the L^2 extension of our proof of the Hodge decomposition in the plane.*

PROBLEM 6.2. *Discuss Hodge's theorem for the Riemann surfaces* $M = \{r_1 < |z| < r_2\}$ *where* $0 \le r_1 < r_2 \le \infty$. *Identify* $\Omega_2^1(M), \mathfrak{h}_2(M)$ *as well as* E *for each of these cases and show directly that*

$$\Omega_2^1(M) = E \oplus *E \oplus \mathfrak{h}_2(M)$$

PROBLEM 6.3. *Let* M *be a simply connected Riemann surface and let* u *be a harmonic function on* M. *Then* u *has a global harmonic conjugate on* M.

PROBLEM 6.4. *This exercise introduces and studies* Bessel functions $J_n(z)$ *with* $n \in \mathbb{Z}$, $z \in \mathbb{C}$: *they are defined as the coefficients in the Laurent expansion*

$$\exp\Big(\frac{z}{2}(\zeta - \zeta^{-1})\Big) = \sum_{n=-\infty}^{\infty} J_n(z)\zeta^n, \quad 0 < |\zeta| < \infty \tag{6.13}$$

(a) Show from the generating function (6.13) *that for each* $n \in \mathbb{Z}$ *the function* $J_n(z)$ *is entire and satisfies*

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\, d\theta \tag{6.14}$$

as an identity between entire functions. Also, prove that $J_{-n} = (-1)^n J_n$.

(b) Using (6.14) *prove that for each* $n \in \mathbb{Z}$, $w = J_n(z)$ *satisfies* Bessel's equation

$$z^2 w''(z) + z w'(z) + (z^2 - n^2)w(z) = 0 \tag{6.15}$$

This equation, which arises frequently in both mathematics and physics (as well as other applications), is the reason why Bessel functions are so important.

In what follows, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. *Also, for the remainder of (b) we allow* $n = \nu \in \mathbb{C}$ *in* (6.15). *Prove that for any* $z_0 \in \mathbb{C}^*$ *as well as* $w_0, w_1 \in \mathbb{C}$ *arbitrary there exists a unique function* $w(z)$ *defined and analytic locally around* $z = z_0$ *with the property that* $w(z_0) = w_0$, $w'(z_0) = w_1$ *and so that* (6.15) *holds on the domain of* w *(use power series around* $z = z_0$*). We refer to such a solution as a* local solution around z_0. *What happens at* $z_0 = 0$*? Show that any local solution around an arbitrary* $z_0 \in \mathbb{C}^*$ *can be analytically continued to any simply connected domain* $\Omega \subset \mathbb{C}^*$ *containing* z_0. *Moreover, show that for any simply connected domain* $\Omega \subset \mathbb{C}^*$ *there exist two linearly independent solutions* $W_0, W_1 \in \mathcal{H}(\Omega)$ *of* (6.15) *so that any local solution* w *around an arbitrary* $z_0 \in \Omega$ *is a linear combination of* W_0, W_1 *(such a pair is referred to as a* fundamental system *of solutions on* Ω*).*

Given a local solution $w(z)$ *around an arbitrary* $z_0 \in \mathbb{C}^*$, *set* $f(\zeta) = w(e^\zeta)$ *which is defined and analytic around any* ζ_0 *with* $e^{\zeta_0} = z_0$. *Derive a differential equation for* f *and use it to argue that* f *can be analytically continued to an entire function (in the language of Riemann surfaces this shows that* e^ζ uniformizes *the Riemann surface of any local solution of Bessel's equation; loosely speaking, the "worst" singularity that a solution of Bessel's equation can have at* $z = 0$ *is logarithmic).*

(c) Using either (6.13) *or* (6.14) *prove that the power series expansion of* $J_n(z)$ *around zero is*

$$J_n(z) = (z/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!(n+k)!} \tag{6.16}$$

provided $n \ge 0$. *What is the power series of* J_n *for* $n < 0$*?*

(d) Suppose the formal power series $w_n(z) = \sum_{k=0}^\infty a_{k,n} z^k$ *satisfies the ordinary differential equation* (6.15) *with some fixed integer* $n \ge 0$. *Derive a recursion relation for*

the coefficients $a_{k,n}$ *and show that up to a multiplicative constant the formal power series equals* (6.16)*, i.e.,* w_n *is a multiple of* J_n*. In particular,* J_n *is the only solution of* (6.15) *(up to multiples) which is analytic around* $z = 0$*.*

(e) Find a fundamental system of solutions of Bessel's equation (6.15) *with* $n = 0$ *on* $G = \mathbb{C} \setminus (-\infty, 0]$ *(it will help if you remember from (b) that the worst singularity at* $z = 0$ *of any solution of* (6.15) *is logarithmic). Of course you need to justify your answer. What about general* $n \geq 0$*?*

(f) We now use (6.16) *to* define J_ν *for* $\nu \in \mathbb{C}$ *by the formula*

$$J_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\, \Gamma(\nu + k + 1)} \tag{6.17}$$

where we select the principal branch of $z^\nu = e^{\nu \operatorname{Log} z}$ *for definiteness. Hence, we view* (6.17) *as an element of* $\mathcal{H}(\mathbb{C} \setminus (-\infty, 0])$*. Check that*

$$\begin{aligned} J_{\frac{1}{2}}(z) &= \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \sin z \\ J_{\frac{3}{2}}(z) &= \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \Big(\frac{\sin z}{z} - \cos z\Big) \end{aligned}$$

Show that (6.17) *agrees with the previous definition for all integer* ν*, including negative ones. Argue that* J_ν *thus defined solves* (6.15) *with* n *replaced by* $\nu \in \mathbb{C}$*, and prove that* this property fails *if we were to define* J_ν *by replacing* n *with* $\nu \in \mathbb{C}$ *in* (6.14) *(which explains why we used the power-series instead). It is worth noting that for any* $\nu \in \mathbb{C}$*, the function* $J_\nu(e^\zeta)$ *is entire in* ζ *(why?). This is in agreement with our "abstract" result from part (b).*

(g) Prove that for all $\nu \in \mathbb{C}$ *with* $\operatorname{Re} \nu > -\frac{1}{2}$ *there is the representation*

$$\begin{aligned} J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu + \frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} e^{izt} (1 - t^2)^{\nu - \frac{1}{2}}\, dt \\ &= \frac{(z/2)^\nu}{\Gamma(\nu + \frac{1}{2})\sqrt{\pi}} \int_0^{\pi} \cos(z \cos\theta) \sin^{2\nu}\theta\, d\theta \end{aligned}$$

Check directly that these integral representations satisfy the Bessel equation (6.15) *with* $n = \nu$*.*

(h) Using (6.17) *show that for any* $\nu \in \mathbb{C}$

$$\begin{aligned} J_\nu(z) &= \frac{(z/2)^\nu}{2\pi i} \int_\gamma \exp\Big(w - \frac{z^2}{4w}\Big) \frac{dw}{w^{\nu+1}} \\ &= \frac{1}{2\pi i} \int_\gamma \exp\Big(\frac{z}{2}(\zeta - \zeta^{-1})\Big) \frac{d\zeta}{\zeta^{\nu+1}} \end{aligned} \tag{6.18}$$

where γ *is a Hankel contour (see* (2.18)*). In both cases the powers involving* ν *are principal branches.* (6.18) *should of course remind you of our starting point* (6.13)*. Indeed, check that for* $\nu = n \in \mathbb{Z}$ *the representation* (6.18) *is nothing but the integral computing the* n^{th} *Laurent coefficient of* (6.13)*. Note that the power* $\zeta^{\nu+1}$ *in the denominator is single-valued if and only if* $\nu \in \mathbb{Z}$*.*

Finally, deduce from (6.18) *that*

$$J_\nu(z) = \frac{1}{2\pi i} \int_{\tilde{\gamma}} e^{z \sinh\tau - \nu\tau}\, d\tau \tag{6.19}$$

where $\widetilde{\gamma} = \operatorname{Log} \gamma$ *is the (principal) logarithm of a Hankel contour* γ*, see Figure 12.1.*

(i) Use (6.19) *to prove the* recursion relation of the Bessel functions

$$J_{\nu-1}(z) + J_{\nu+1}(z) = (2\nu/z)J_\nu(z)$$
$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z)$$

and from these that

$$J_{\nu+1}(z) = (\nu/z)J_\nu(z) - J_\nu'(z)$$
$$J_{\nu-1}(z) = (\nu/z)J_\nu(z) + J_\nu'(z)$$

In particular, $J_0'(z) = -J_1(z)$.

CHAPTER 7

Hodge's theorem and the L^2 existence theory

1. Weyl's lemma and the Hodge decomposition

In this chapter we develop some of the basic potential theory which is essential for various existence theorems on Riemann surfaces. Not only are we going to obtain the Hodge decomposition this way, but we shall also be able to prove that every Riemann surface carries a nonconstant meromorphic function. This important result is one of the ingredients in the proof of Theorem 5.19 on compact Riemann surfaces.

We shall now prove Hodge's representation (6.6). Recall that $\Omega^1_2(M;\mathbb{R})$ is the space of real-valued 1–forms ω with measurable coefficients and such that

$$\|\omega\|^2 = \int_M \omega \wedge \overline{*\omega} < \infty$$

Furthermore,

$$E := \overline{\left\{df \mid d \in \Omega^0_{\mathrm{comp}}(M;\mathbb{R})\right\}}, \quad *E := \overline{\left\{*df \mid d \in \Omega^0_{\mathrm{comp}}(M;\mathbb{R})\right\}}$$

where the closure is in the sense of $\Omega^1_2(M;\mathbb{R})$.

Theorem 7.1. *Let $\mathfrak{h}_2(M;\mathbb{R}) := \mathfrak{h}(M;\mathbb{R}) \cap \Omega^1_2(M;\mathbb{R})$. Then*

$$\Omega^1_2(M;\mathbb{R}) = E \oplus *E \oplus \mathfrak{h}_2(M;\mathbb{R})$$

We begin with the following observation.

Lemma 7.2. *Let $\alpha \in \Omega^1_2(M;\mathbb{R})$ and smooth. Then $\alpha \in E^\perp$ iff $d{*}\alpha = 0$ and $\alpha \in (*E)^\perp$ iff $d\alpha = 0$. In particular, $E \subset (*E)^\perp$ and $*E \subset E^\perp$.*

Proof. First,

$$\alpha \in E^\perp \Longleftrightarrow \alpha \in \left\{df \mid f \in C^\infty_{\mathrm{comp}}(M)\right\}^\perp$$

Moreover,

$$\begin{aligned} 0 = \langle \alpha, df\rangle = \langle *\alpha, *df\rangle &= \int_M df \wedge *\alpha \\ &= \int_M d(f{*}\alpha) - f\,d{*}\alpha = -\int_M f\,d{*}\alpha \end{aligned}$$

for all $f \in C^\infty_{\mathrm{comp}}(M)$ is the same as $d{*}\alpha = 0$. Thus, α is co-closed. The calculation for $(*E)^\perp$ is essentially the same and we skip it. $\square$

This lemma implies that

$$\Omega^1_2(M;\mathbb{R}) = E \oplus *E \oplus (E^\perp \cap (*E)^\perp)$$

and our remaining task is to identify the intersection on the right. Note that Figure 6.2 becomes clear when compared to Lemma 7.2. It is clear from Lemma 7.2 that

$$E^\perp \cap (*E)^\perp \supset \mathfrak{h}_2(M;\mathbb{R})$$

It remains to show equality here. This is very remarkable in so far as the intersection thus consists of *smooth* 1-forms. The required (elliptic) regularity ingredient in this context is the so-called Weyl lemma, see Lemma 7.4 below. Note that the following lemma concludes the proof of Theorem 7.1.

LEMMA 7.3.
$$E^{\perp} \cap (*E)^{\perp} = \mathfrak{h}_2(M;\mathbb{R})$$

PROOF. Take $\omega \in E^{\perp} \cap (*E)^{\perp}$. Then by Lemma 7.2,
$$\langle \omega, df\rangle = \langle \omega, *df\rangle = 0, \quad \forall\, f \in C^{\infty}_{\mathrm{comp}}(M)$$
and f complex-valued (say). With $\omega = u\,dz + v\,d\bar{z}$ in local coordinates $(U,z), z = x+iy$ and with f supported in U, we conclude that
$$\langle \omega, df\rangle = 2\int (u\,\overline{f_z} + v\,\overline{f_{\bar{z}}}\,)\,dxdy$$
$$\langle \omega, *df\rangle = -2i\int (u\,\overline{f_z} - v\,\overline{f_{\bar{z}}}\,)\,dxdy$$
This system is in turn equivalent to
$$\int \bar{u} f_z\,dxdy = 0, \qquad \int \bar{v} f_{\bar{z}}\,dxdy = 0$$
for all such f. Now setting $f = g_{\bar{z}}$ and $f = g_z$, respectively, where g is supported in U, yields
$$\int \bar{u}\Delta g\,dxdy = \int \bar{v}\Delta g\,dxdy = 0$$
which implies by Weyl's lemma below that u, v are harmonic and thus smooth in U. In view of Lemma 7.2, ω is both closed and co-closed and therefore harmonic. □

LEMMA 7.4. *Let $V \subset \mathbb{C}$ be open and $u \in L^1_{\mathrm{loc}}(V)$. Suppose u is* weakly harmonic*, i.e.,*
$$\int_V u\Delta\phi\,dxdy = 0 \quad \forall\, \phi \in C^{\infty}_{\mathrm{comp}}(V)$$
Then u is harmonic, i.e., $u \in C^{\infty}(V)$ and $\Delta v = 0$.

PROOF. As a first step, we prove this: suppose $\{u_n\}_{n=1}^{\infty} \subset C^{\infty}(V)$ is a sequence of harmonic functions that converges in the sense of L^1_{loc} to u_∞. Then $u_\infty \in C^{\infty}(V)$ and u_∞ is harmonic.
This follows easily from the mean-value property. Indeed, for each n, and each disk $D(z,r) \subset V$,
$$u_n(z) = \frac{1}{|D(z,r)|}\iint_{D(z,r)} u_n(\zeta)\,d\xi d\eta$$
Hence, by the assumption of L^1_{loc} convergence, $\{u_n\}_n$ is a Cauchy sequence in $C(V)$ and therefore converges uniformly on compact subsets of V to u_∞ which is thus continuous. Moreover, it inherits the mean value property
$$u_\infty(z) = \frac{1}{2\pi r}\int_{|z-\zeta|=r} u_\infty(\zeta)\,d\sigma(\zeta)$$
and is thus harmonic. Indeed, simply compare u to that harmonic function $\widetilde{u}$ which takes values u on the boundary of some disk D (we know that $\widetilde{u}$ exists from Chapter 3 since

we can solve the Dirichlet problem for the Laplacean on a disk). Now observe that $\widetilde{u}-u$ both satisfy the mean-value property and therefore also the maximum principle. Since $\widetilde{u}-u=0$ on the boundary of some disk D, necessarily also $\widetilde{u}-u=0$ on D. See also Problem 3.5, part (x).

To conclude the proof, we let $u \in L^1_{\mathrm{loc}}(V)$ be weakly harmonic and define

$$u_n(z) := (u * \phi_n)(z) \qquad \forall z \in V_{\frac{1}{n}}$$

where $\phi_n(\zeta) := n^2\phi(n\zeta)$ for all $n \geq 1$ with $\phi \geq 0$ a smooth bump function, $\mathrm{supp}(\phi) \subset \mathbb{D}$, and $\int \phi = 1$. Furthermore,

$$V_n := \{z \in V \mid \mathrm{dist}(z, \partial V) > 1/n\}$$

Since u is weakly harmonic, $\Delta u_n = 0$ on V_n and $u_n \to u$ in $L^1_{\mathrm{loc}}(V)$ by Lemma 3.3 (which applies to approximate identities in $\mathbb{R}^2$ equally well). By the previous paragraph, u is smooth and harmonic as claimed. $\square$

Let us now explain why there is always a nonzero harmonic form on a compact surface with positive genus. This goes through the standard "loop-form" construction. Thus, let $c : [0,1] \to M$ be a smooth, closed curve where M is an arbitrary Riemann surface. For

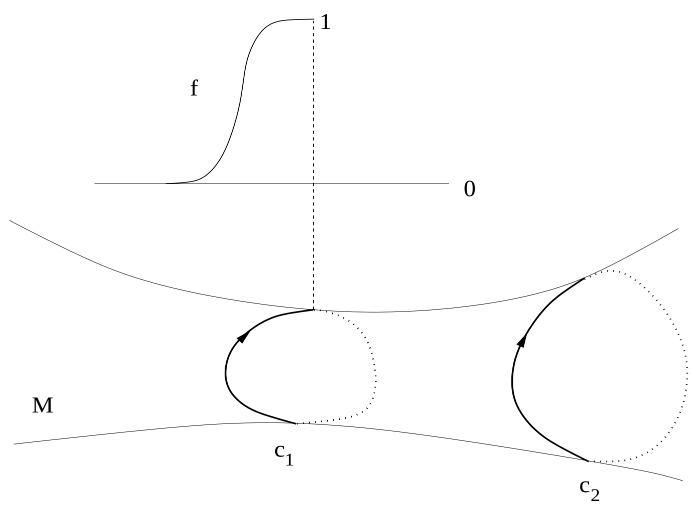

FIGURE 7.1. The function f giving rise to the loop-form η_c

simplicity, we shall also assume that $c([0,1])$ is an imbedded one–dimensional manifold and we put the natural orientation on it, i.e., $c(t)$ is oriented according to increasing t (such a closed curve will be called *loop*). Then let $\widetilde{N}_- \subset N_-$ be neighborhoods to the left of c obtained by taking the finite union of left halves (relative to c) of parametric disks centered at points of c. Furthermore, the disks used in the construction of $\widetilde{N}_-$ are assumed to be compactly contained in those for N_-. Then let f be a smooth function on N_- with $f = 1$ on $\widetilde{N}_-$, $f = 0$ on $N_- \setminus \widetilde{N}_-$ and $f = 0$ on $M \setminus N_-$, see Figure 7.1. While f is not smooth on M, the *loop-form of* c defined as

$$\eta_c := df \in \Omega^1(M;\mathbb{R})$$

is smooth and compactly supported. By construction, $d\eta_c = 0$ and the cohomology class of η_c is uniquely determined by the homology class of c, cf. Figure 7.1. First, if f_1 and f_2

are smooth functions constructed to the left of c as explained, then $f_1 - f_2 \in C^\infty_{\mathrm{comp}}(M)$ so that $df_1 - df_2$ is exact. Second, if $c_1 - c_2$ is the boundary of a compact submanifold of M, then $\eta_{c_1} - \eta_{c_2} = dh$ for some smooth, compactly supported function on M. Finally, if we had defined loop-forms $\widetilde{\eta}_c = dg$ with a function g that equals one on a neighborhhod to the right of c, then $\eta_c - \widetilde{\eta}_c$ is exact. In other words, reversing the orientation of c merely changes the sign of (the cohomology) class of η_c. The importance of loop-forms can be seen from the following simple but crucial fact:

LEMMA 7.5. *Let $\alpha \in \Omega^1$ be closed. Then*

$$\langle \alpha, *\eta_c \rangle = \int_c \alpha$$

PROOF. Compute

$$\begin{aligned}\langle \alpha, *\eta_c \rangle &= \int \alpha \wedge **\eta_c = \int_{N_-} df \wedge \alpha \\ &= \int_{N_-} d(f\alpha) = \int_{\partial N_-} f\alpha = \int_c \alpha\end{aligned}$$

as claimed. □

This property of loop-forms immediately allows us to characterize all exact forms amongst the closed ones as well as give a general topological criterion which guarantees that non-zero harmonic forms exist.

COROLLARY 7.6. *1) Let $\alpha \in \Omega^1(M;\mathbb{R})$. Then α is exact iff $\langle \alpha, \beta \rangle = 0$ for all co-closed $\beta \in \Omega^1(M;\mathbb{R})$ of compact support.*

2) Let $\alpha \in E$ be smooth. Then α is exact, i.e., $\alpha = df$ for some real-valued $f \in C^\infty(M)$.

3) Supposed the closed loop c does not separate M, i.e., $M \backslash c([0,1])$ is connected. Then there exists a closed form $\alpha \in \Omega^1(M;\mathbb{R})$ which is not exact. In particular, $\mathfrak{h}_2(M;\mathbb{R}) \neq \{0\}$.

PROOF. If $\alpha = df$ is exact, then

$$\langle \alpha, \beta \rangle = \int_M df \wedge *\beta = -\int_M f\, d{*\beta} = 0$$

for any β as in 1). Conversely, $*\eta_c$ is co-closed and compactly supported for any loop. It follows that

$$0 = \langle \alpha, *\eta_c \rangle = \int_c \alpha$$

for any loop c. Thus α is exact as claimed.

Property 2) follows from 1) via Lemma 7.2. Finally, for 3), let c^* be a closed curve in M that crosses c transversally. See Figure 7.2. This exists since $M \setminus c$ is connected. Hence,

$$\int_{c*} \eta_c = 1$$

and η_c is closed but not exact. From Theorem 7.1,

$$\eta_c = \alpha + \omega, \quad \alpha \in E, \quad \omega \in \mathfrak{h}_2(M;\mathbb{R})$$

Since ω and η_c is smooth, so is α. By 2), α is exact so $\omega \neq 0$ as desired. □

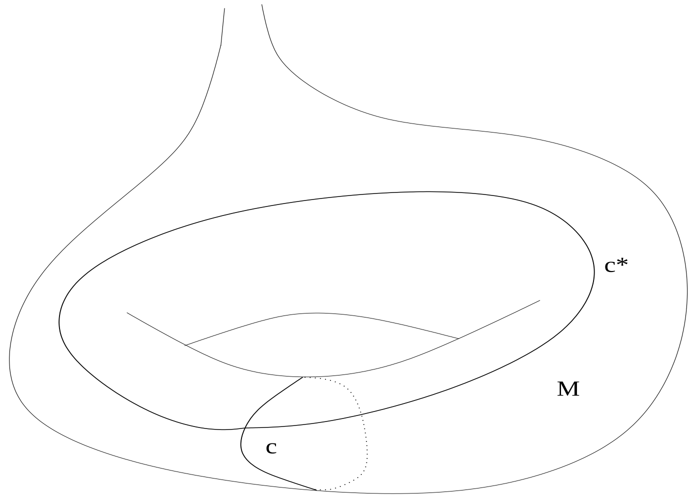

FIGURE 7.2. The curves c and c_1^*

2. Existence of nonconstant meromorphic functions

We shall now derive some very important consequences from Hodge's theorem. More precisely, we answer the fundamental question: *does a general Riemann surface carry a nonconstant meromorphic function?* We saw that in general this cannot be done with *holomorphic* functions, since compact surfaces do not allow this. However, deeper questions will still elude us here such as: *which Riemann surfaces carry a meromorphic function with exactly one simple pole?* In the following chapter we shall present the machinery (known as the Riemann-Roch theorem) needed for this purpose.

The basis for our entire existence theory will be the following result. It should be thought of as an answer to the following question: *Let $p \in M$. Can we find a function u harmonic on $M \setminus \{p\}$ so that in some parametric disk centered at p, u has a given singularity at p such as $\frac{1}{z}$ or $\log|z|$?*

Or, more generally: *let D be a parametric disk on a Riemann surface M centered at $p \in M$ and suppose h is a harmonic function on $D \setminus \{p\}$, differentiable on $\bar{D} \setminus \{p\}$. Can we find u harmonic on $M \setminus \{p\}$ with $u - h$ harmonic on* all of *D?*

These questions are part of potential theory and we shall address them by means of the Hodge theorem. As usual, M and N are arbitrary Riemann surfaces.

PROPOSITION 7.7. *Let $\bar{N} \subset M$, $\bar{N}$ compact with smooth boundary. Fix $p_0 \in N$ and h harmonic on $N \setminus \{p_0\}$ with $h \in C^1(\bar{N})$ and $\frac{\partial h}{\partial n} = 0$ on ∂N where n is some normal vector field on ∂N that never vanishes.*

Then there exists u harmonic in $M \setminus \{p_0\}$, $u - h$ harmonic on N, and $u \in \Omega_2^1(M \setminus K)$ for any compact neighborhood K of p_0. Also, u is unique up to constants.

PROOF. For the existence part, take θ a C^∞ function on N which agrees with h on $N \setminus K$ where K is an arbitrary but fixed (small) compact neighborhood of p_0. Then extend θ to M simply by setting it $= 0$ outside of N and define $d\theta = 0$ on $M \setminus N$. Note that $d\theta \in \Omega_2^1(M;\mathbb{C})$. By Hodge,

$$d\theta = \alpha + \beta, \quad \alpha \in E,\ \beta \in E^\perp$$

If $\phi \in C^\infty_{\mathrm{comp}}(M)$, then

$$\langle d\theta, d\phi\rangle = \langle \alpha, d\phi\rangle, \qquad \langle \alpha, *d\phi\rangle = 0$$

First, suppose that $\mathrm{supp}(\phi) \subset M \setminus K$. Then from $\frac{\partial h}{\partial n} = 0$ and $d * dh = 0$ on N, we

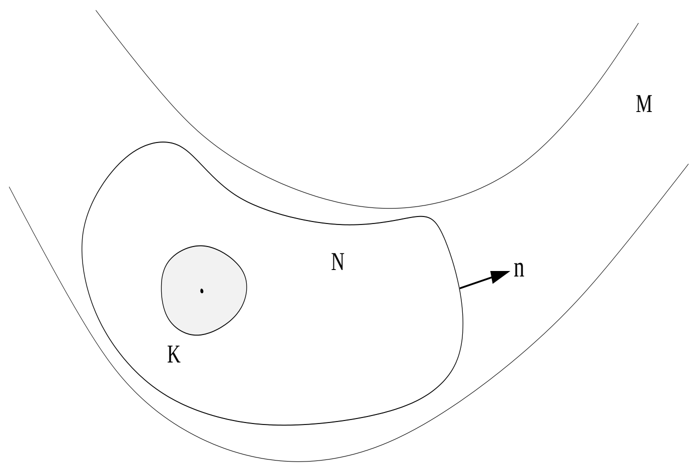

FIGURE 7.3. The sets in Proposition 7.7

obtain that

$$\langle d\phi, d\theta\rangle = \int_N d\phi \wedge *d\bar{h} = \int_{\partial N} \phi\, \overline{i_*(*dh)} = 0$$

where $i : \partial N \hookrightarrow M$ is the inclusion map. The main point to note here is that $i_*(*dh)$ is proportional to $\frac{\partial h}{\partial n} = 0$.

Hence, α is harmonic on $M \setminus K$. On the other hand, if $\mathrm{supp}(\phi) \subset N$, then

$$\langle d\theta - \alpha, d\phi\rangle = 0, \quad \langle d\theta - \alpha, *d\phi\rangle = 0$$

so that $\alpha - d\theta$ is harmonic on N. In particular, α is smooth on M and thus $\alpha = df$ with f smooth. Now set

$$u = f - \theta + h$$

By inspection, u has all the desired properties.

Finally, if v had the same properties as u, then $u - v$ would be harmonic on M and $d(u - v) \in \Omega^1_2$. In conclusion, $d(u - v) \in E \cap \mathfrak{h}_2 = \{0\}$, so $u - v = \text{const}$. □

We remark that if h were harmonic on all of N, then $h = \text{const}$ because of the Neumann condition $\frac{\partial h}{\partial n} = 0$ on ∂N. Indeed, this is merely the fact that

$$\|dh\|^2_{L^2(N)} = \int_N dh \wedge *d\bar{h} = \int_{\partial N} h\; i_*(*d\bar{h}) = 0$$

where $i : \partial N \to M$ is the inclusion and i_* the pull-back as in the proof of the previous proposition.

Furthermore, we remark that the exact same proof allows for several exceptional points $p_0, \ldots, p_k \in N$. The statement is as follows:

COROLLARY 7.8. *Let $\bar{N} \subset M$, $\bar{N}$ compact with smooth boundary. Fix finitely many points $\{p_j\}_{j=0}^k \in N$ and h harmonic on $N \setminus \{p_j\}_{j=0}^k$ with $h \in C^1(\bar{N})$ and $\frac{\partial h}{\partial n} = 0$ on ∂N where n is some normal vector field on ∂N.*

Then there exists u harmonic in $M \setminus \{p_j\}_{j=0}^k$, $u - h$ harmonic on N, and $u \in \Omega_2^1(M \setminus K)$ for any compact neighborhood K of $\{p_j\}_{j=0}^k$. Also, u is unique up to constants.

PROOF. The proof is essentially the same as that or Proposition 7.7 and we leave the details to the reader. $\square$

We can now collect a number of corollaries:

COROLLARY 7.9. *Given $n \geq 1$ and a coordinate chart (U, z) around p_0 in M with $z(p_0) = 0$ there is u harmonic on $M \setminus \{p_0\}$ with $u - z^{-n}$ harmonic on U and $du \in \Omega_2^1(M \setminus K)$ for any compact neighborhood K of p_0.*

PROOF. Simply let without loss of generality $z(U) \supset \bar{\mathbb{D}}$ and define

$$h(z) = z^{-n} + \overline{z^n} \quad \forall\, |z| \leq 1$$

The theorem applies with $\bar{N} = z^{-1}(\bar{\mathbb{D}})$ since $\frac{\partial h}{\partial n} = 0$ on $|z| = 1$. $\square$

Next, we would like to place a $\log|z|$ singularity on a Riemann surface. To apply Proposition 7.7 we need to enforce the Neumann condition $\frac{\partial h}{\partial n} = 0$. This amounts to solving the Neumann problem

$$\Delta u = 0 \;\text{ in }\; |z| < 1, \qquad \frac{\partial u}{\partial n} = -\frac{\partial}{\partial r}\log r = -1 \text{ on } |z| = 1$$

But this has no solution since the integral of -1 around $|z| = 1$ does not vanish (necessary by the divergence theorem). Now let us also note that with $M = \mathbb{C}$ the function $u(z) = \log|z|$ satisfies

$$du(z) = \frac{1}{2}\frac{dz}{\bar{z}} + \frac{1}{2}\frac{d\bar{z}}{z}$$

which is not in L^2 around $|z| = \infty$ (it barely fails). Finally, this calculation also shows that if we could place a $\log|z|$ singularity on M then this would produce a meromorphic differential $\omega = du + i * du$ with exactly one simple pole. If M is compact, then this violates the fact that the sum of the residues would have to vanish.

What all of this suggests is that we should try with *two* logarithmic singularities (in other words, instead of using a point charge, we use a dipole). This is indeed possible:

COROLLARY 7.10. *Let $p_0, p_1 \in M$ be distinct and suppose z and ζ are local coordinates around p_0 and p_1, respectively. Then there exists u harmonic on $M \setminus \{p_0, p_1\}$ with $u - \log|z|$ and $u + \log|\zeta|$ harmonic locally around p_0, p_1, respectively. Moreover, $du \in \Omega_2^1(M \setminus K)$ where K is any compact neighborhood of $\{p_0, p_1\}$.*

PROOF. For this one, assume first that p_0, p_1 are close together. Then let (U, z) be a coordinate chart with $z(p_0) = z_0 \in \mathbb{D} \setminus \{0\}$, $z(p_1) = z_1 \in \mathbb{D} \setminus \{0\}$ and $z(U) \supset \bar{\mathbb{D}}$. Define

$$h(z) = \log\left|\frac{(z - z_0)(z - z_0^*)}{(z - z_1)(z - z_1^*)}\right|$$

where z_0^*, z_1^* are the reflections of z_0, z_1 across $\partial\mathbb{D}$ (i.e., $z_j^* = \overline{z_j^{-1}}$). Then check that

$$|(z^* - z_j)(z^* - z_j^*)| = |z|^{-2}|(z - z_j)(z - z_j^*)|$$

which gives $h(z^*) = h(z)$ (why?). This in turn implies the Neumann condition $\frac{\partial h}{\partial n} = 0$. Hence, by Corollary 7.8, there exists u with all the desired properties. If p_0 and p_1 do not fall into one coordinate chart, then connect them by a chain of points that satisfy this for each adjacent pair. This yields finitely many functions u_0, u_1, u_2 etc. The desired function is the sum of all these. $\square$

To conclude our existence theory, we now state some simple but most important corollaries on meromorphic differentials.

COROLLARY 7.11. *(a) Given $n \geq 1$ and $p_0 \in M$ there exists a meromorphic differential ω with $\omega - \frac{dz}{z^{n+1}}$ holomorphic locally around p_0 (here z are local coordinates at p_0). Moreover, $\omega \in \Omega_2^1(M \setminus K)$ for every compact neighborhood K of p_0.*

(b) Let $p_0, p_1 \in M$. There exists ω meromorphic on M with $\omega - \frac{dz}{z}$ holomorphic around p_0 and $\omega + \frac{d\zeta}{\zeta}$ holomorphic around p_1, respectively (with z, ζ local coordinates). Moreover, $\omega \in \Omega_2^1(M \setminus K)$ for every compact neighborhood K of $\{p_0, p_1\}$.

PROOF. With u as in Corollary 7.9 and 7.10, respectively, we set $\alpha = du$. In the first case, $\omega = \frac{-1}{2n}(\alpha + i * \alpha)$, whereas in the second, $\omega = \alpha + i * \alpha$. $\square$

As a reality check, take $M = \mathbb{C}$ and $p_0 = 0$, say. Then for (a) we would simply obtain $\omega = \frac{dz}{z^{n+1}}$. Note that the L^2 condition holds when $n \geq 1$ but not for $n = 0$. For (b), we would take $\omega = \frac{dz}{z-p_0} - \frac{dz}{z-p_1}$. This has all the desired properties, including the L^2 condition at $z = \infty$. In the classical literature, meromorphic differentials all of whose residues vanish (as in (a)) are called *differentials of the second kind*, whereas those with simple poles are called *differentials of the third kind* (the holomorphic differentials are called *abelian* or *of the first kind*).

From the existence of differentials of the second kind we can easily derive the following specical case of the *uniformization theorem*:

COROLLARY 7.12. *Let M be compact and simply connected. Then M carries a meromorphic function of degree one. In particular, $M \simeq S^2$. In other words, up to conformal isomorphisms, there is exactly one compact simply connected Riemann surface, namely $\mathbb{C}P^1 \simeq S^2 \simeq \mathbb{C}_\infty$.*

PROOF. Let $\omega \in \mathcal{H}\Omega^1(M \setminus \{p_0\})$ where $p_0 \in M$ is arbitrary and such that $\omega = \frac{dz}{z^2}$ in local coordinates around p_0. Then set

$$f(p) := \int_{p_1}^{p} \omega$$

where the integration path connects an arbitrary but fixed $p_1 \in M \setminus \{p_0\}$ with p without passing through p_0. Since M is simply connected and since $\operatorname{res}(\omega; p_0) = 0$, it follows that f is well-defined. Clearly, f has a simply pole at p_0 and is holomorphic elsewhere. Since $\deg(f) = 1$, this map induces an isomorphism $M \to \mathbb{C}_\infty$ as desired. $\square$

This result of course raises many questions, such as: *what can one say about the non-simply connected case? More precisely, on a compact surface of genus g, what is the minimal degree that a nonconstant meromorphic function can achieve?*, *How many (in the sense of dimension) meromorphic functions are there on a compact Riemann surface which have poles at finitely many points $p_\nu \in M$ with orders at most s_ν (a positive integer)?*

The Riemann-Roch theorem in the following chapter attempts to answer these questions, at least it relates the dimension of a certain space of meromorphic functions to the dimension of a space of meromorphic differentials.

For now, let us mention that the proof of Corollary 7.12 gives the following, more precise, statement: *Define the linear space*

$$V := \{f \in \mathcal{M}(M) \;:\; f \in \mathcal{H}(M \setminus \{p_0\}), \mathrm{ord}(f, p_0) \le 1\}$$

Then $\dim(V) = 2$.

The reader will have no difficulty verifying this from the previous proof. Note that we need dimension 2 here since the constants are in V. The Riemann-Roch theorem will generalize these dimension counts to arbitrary "divisors" on compact surfaces, see the following chapter for the definition of a divisor. In Problem 7.4 we give an extension of our observation concerning $\dim(V)$ which is a special case of Riemann-Roch for genus zero.

Finally, we can now state and prove the following very satisfactory result which applies *every* Riemann surface.

THEOREM 7.13. *Let $\{p_j\}_{j=1}^J \subset M$, $J \ge 2$, and $c_j \in \mathbb{C}$ with $\sum_{j=1}^J c_j = 0$. Then there exists a meromorphic differential ω, holomorphic on $M \setminus \{p_1, p_2, \ldots, p_J\}$ so that ω has a simple pole at each p_j with residue c_j.*

PROOF. Pick any other point $p_0 \in M$ and let ω_j be meromorphic with simple poles at p_0, p_j and residues $-c_j, c_j$, respectively. The differential $\omega = \sum_{j=1}^J \omega_j$ has all the desired properties. □

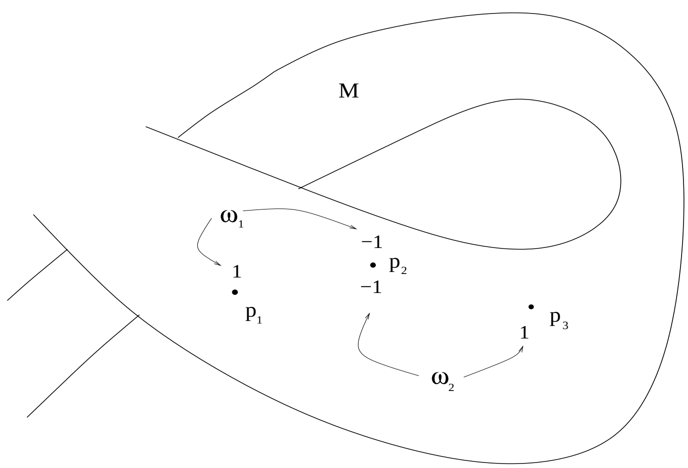

FIGURE 7.4. The construction of a meromorphic function with a pole at p_0 (and possibly other poles)

From here we immediately conclude the following remarkable result.

COROLLARY 7.14. *Every Riemann surface carries a nonconstant meromorphic function. In fact, given any distinct points $p_0, p_1, p_2 \in M$, there exists a meromorphic function f on M for which $f(p_1) = 1$, and so that f has a simple pole at p_0 and a simple* *zero at p_2. In particular, the function field $\mathcal{M}(M)$ separates points*[1] *on every Riemann surface M.*

PROOF. Take three points $p_0, p_1, p_2 \in M$ and let ω_1 be a meromorphic one-form with simple poles at p_0, p_1 and residues $1, -1$, respectively and holomorphic everywhere else. Similarly, let ω_2 be a meromorphic one-form with simple poles at p_1, p_2 and residues $-1, 1$, respectively and holomorphic everywhere else. See Figure 7.4. Now set $f = \frac{\omega_1}{\omega_2}$ where the division is well-defined in local coordinates and defines a meromorphic function. Clearly, $f(p_1) = 1$ and $f(p_2) = 0$, and the function f is not constant, meromorphic with a pole at p_0. $\square$

In the following chapter we shall study the vector space (or rather, its dimension) of meromorphic functions and differentials with zeros and poles at prescribed points.

3. Problems

PROBLEM 7.1. *Show that every compact Riemann surface admits a triangulation.*

PROBLEM 7.2. *Show that the meromorphic functions on every Riemann surface separate points in the following strong sense: given distinct points $\{p_j \ : \ 1 \le j \le n\}$ on some Riemann surface M, and n distinct values $z_j \in \mathbb{C}_\infty$, there exists a meromorphic function on M which takes the value c_j at p_j (of course $f(p_j) = \infty$ means that f has a pole at p_j).*

PROBLEM 7.3. *Let M be a Riemann surface. Here you are asked to give an alternative proof for the existence of a meromorphic differential ω which has poles of order 2 at prescribed points $p_1, \ldots, p_n \in M$ with vanishing residues and is holomorphic everywhere else by the following strategy: by linearity, $n = 1$. Pick a parametric disk (U, z) around p_1 with $z(p_1) = 0$. Then let $\omega := \chi(z)\frac{dz}{z^2}$ where χ is a smooth cut-off function which is supported in the unit disk and so that $\chi = 1$ near zero. Then ω is a smooth 1-form on $M \setminus \{p_1\}$ with the property that $\eta := \omega - i * \omega$ is smooth on M, and in fact, $\eta = 0$ near p_1. Apply the Hodge decomposition theorem to write, with a harmonic form ρ,*

$$\eta = df + *dg + \rho$$

where f, g are smooth (since η is smooth — prove it). Now use this to define a meromorphic differential with the desired properties.

PROBLEM 7.4. *Let*[2] *M be a compact Riemann surface of genus zero. Select finitely many points $\{p_\nu\}_{\nu=1}^n \subset M$ as well as a positive integer s_ν for each $1 \le \nu \le n$. Define*

$$V := \{f \in \mathcal{M}(M) \mid f \in \mathcal{H}(M \setminus \{p_\nu\}_{\nu=1}^n), \ \text{ the pole of } f \text{ at } p_\nu \text{ has order at most } s_\nu\}$$

Prove that $\dim(V) = 1 + \sum_{\nu=1}^n s_\nu$.

[1]This means that to two distinct points there exists a meromorphic function taking distinct values at these points. Note that we cannot separate points by means of holomorphic functions, in general, as they may be constant (as on a compact surface).

[2]This problem is exactly the genus zero case of the Riemann-Roch theorem for integral divisors.

CHAPTER 8

The Theorems of Riemann-Roch, Abel, and Jacobi

1. Homology bases, periods, and Riemann's bilinear relations

We will now turn to the following much deeper question: *what kind* of nonconstant meromorphic functions does a given Riemann surface admit? More precisely, if M is compact and of genus g, what can we say about the minimal degree of a meromorphic function on M? Answering this question will lead us to the Riemann-Roch theorem. To appreciate this circle of ideas, note the following: Suppose M is compact and admits a meromorphic function f of degree one. Then M defines an isomorphism between M and $\mathbb{C}P^1$. This implies that no such function exists if M has genus one or higher! On the other hand, we will show (from Riemann-Roch) that for the simply connected compact case there is such a function. In this chapter, the reader will need to know about basic topology of compact surfaces: homology, the canonical homology basis, the fundamental polygon of a compact surface of genus g. These topics are briefly discussed in Chapter 13. In addition, we will need the loop-forms from the previous chapter.

Let M be a compact Riemann surface of genus g. The intersection numbers between two closed curves γ_1, γ_2 are defined as

$$\gamma_1 \cdot \gamma_2 = \int_M \eta_{\gamma_1} \wedge \eta_{\gamma_2} = -\langle \eta_{\gamma_1}, *\eta_{\gamma_2} \rangle$$

which is always an integer. Recall that η_γ for an (oriented) loop γ is $= df$ where f is a smooth function of compact support with $f = 1$ on a small neck to the left of γ. The homology class of γ determines the cohomology class of η_γ. Hence, the intersection number is well-defined as a product between homology classes in $H_1(M;\mathbb{Z})$. Note that

$$b \cdot a = -a \cdot b, \quad (a+b) \cdot c = a \cdot c + b \cdot c$$

for any classes a, b, c.

Pick a (canonical) homology basis for the 1-cycles, and denote it by $\{A_j\}_{j=1}^{2g}$. Here $A_j = a_j$ if $1 \le j \le g$ and $A_j = b_{j-g}$ if $g+1 \le j \le 2g$ where

$$a_j \cdot b_k = \delta_{jk}, \quad a_j \cdot a_k = 0, \quad b_j \cdot b_k = 0$$

for each $1 \le j, k \le g$. Next, we define a dual basis $\{\beta_k\}_{k=1}^{2g}$ for the cohomology. It is simply

$$\beta_k = \eta_{b_k}, \quad 1 \le k \le g$$
$$\beta_k = -\eta_{a_{k-g}}, \quad g+1 \le k \le 2g$$

and satisfies the duality relation

$$\int_{A_j} \beta_k = \delta_{jk}. \tag{8.1}$$

We also record the important fact

$$\Big\{ \int_M \beta_j \wedge \beta_k \Big\}_{j,k=1}^{2g} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} =: J \tag{8.2}$$

This is due to the fact that the entries of this matrix are all possible intersection numbers of the curves a_j, b_k.

Let's collect some important properties (a form α is called real if $\alpha = \bar{\alpha}$):

LEMMA 8.1. *The real one-forms $\{\beta_j\}_{j=1}^{2g}$ are a basis of $H^1(M;\mathbb{R})$ (de-Rham space of one forms). Let α_j denote the orthogonal projection of β_j onto the harmonic forms (from the Hodge theorem). Then α_j is a real one-form, and $\{\alpha_j\}_{j=1}^{2g}$ is a basis of both $\mathfrak{h}(M;\mathbb{R})$ and $\mathfrak{h}(M;\mathbb{C})$, the real and complex-valued harmonic forms, respectively. In particular,*

$$\dim_{\mathbb{R}} \mathfrak{h}(M;\mathbb{R}) = \dim_{\mathbb{C}} \mathfrak{h}(M;\mathbb{C}) = 2g$$

The relation (8.2) *holds also for $\int_M \alpha_j \wedge \alpha_k$.*

PROOF. Since $\{A_j\}$ is a basis of $H_1(M;\mathbb{Z})$, a closed form α is exact iff

$$\int_{A_j} \alpha = 0 \quad \forall\, 1 \le j \le 2g$$

Hence the linear map $H^1(M;\mathbb{R}) \to \mathbb{R}^{2g}$

$$\alpha \mapsto \Big\{ \int_{A_j} \alpha \Big\}_{j=1}^{2g}$$

is injective. Because of (8.1) this map is also onto and is thus an isomorphism. It is called the *period map*. The exact same argument also works over $\mathbb{C}$. Since every cohomology class has a unique harmonic representative, we obtain the statements about $\mathfrak{h}$. To check that α_j is real, write

$$\beta_j = df_j + \alpha_j$$
$$\overline{\beta_j} = \beta_j = d\overline{f_j} + \overline{\alpha_j}$$

so that

$$-\alpha_j + \overline{\alpha_j} = d(f_j - \overline{f_j})$$

is both harmonic and exact, and thus zero. $\square$

Next, we find a basis for the holomorphic one-forms $\mathcal{H}\Omega^1$ (in the classical literature, these are called *differentials of the first kind*).

LEMMA 8.2. *With α_j as above, define the holomorphic differential $\omega_j = \alpha_j + i * \alpha_j$. Then $\{\omega_j\}_{j=1}^{g}$ is a basis in $\mathcal{H}\Omega^1$. In particular, $\dim_{\mathbb{C}} \mathcal{H}\Omega^1 = g$.*

PROOF. The dimension statement is immediate from

$$\mathfrak{h}(M;\mathbb{C}) = \mathcal{H}\Omega^1 \oplus \overline{\mathcal{H}\Omega^1}$$

To see the statement about the basis, we express $*$ as a matrix relative to the basis $\{\alpha_j\}_{j=1}^{2g}$. This is possible, since $*$ preserves the harmonic forms. Also, note that it preserves real forms. Hence, with $\lambda_{jk} \in \mathbb{R}$,

$$*\alpha_j = \sum_{k=1}^{2g} \lambda_{jk}\alpha_k, \quad *\mathcal{A} = \mathcal{G}\mathcal{A}, \quad \mathcal{G} = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix} \in GL(2d,\mathbb{R})$$

where $\mathcal{A}$ is the column vector with entries $\alpha_1, \ldots, \alpha_{2g}$. From $** = -\mathrm{Id}$ we deduce $\mathcal{G}^2 = -I_{2d}$. We expect Λ_2, Λ_3 to be invertible since heuristically $*$ should correspond to a switch between the a_j and the b_k curves. Indeed, we have

$$\langle \alpha_j, \alpha_\ell \rangle = \langle *\, \alpha_j, *\alpha_\ell \rangle = \sum_{k=1}^{2g} \lambda_{jk} \langle \alpha_k, *\alpha_\ell \rangle = \sum_{k=1}^{2g} \lambda_{jk} \int_M \alpha_\ell \wedge \alpha_k$$

or, in matrix notation,

$$\Gamma = \mathcal{G} J^t = \begin{bmatrix} \Lambda_2 & -\Lambda_1 \\ \Lambda_4 & -\Lambda_3 \end{bmatrix}$$

where $\Gamma = \{\langle \alpha_j, \alpha_\ell \rangle\}_{j,\ell=1}^{2g}$ is a positive definite matrix (since it is the matrix of a positive definite scalar product) and J is as above, see (8.2). Hence $\Lambda_2 > 0$ and $-\Lambda_3 > 0$; in particular, these matrices are invertible.

Suppose there were a linear relation

$$c_1\omega_1 + \ldots + c_g\omega_g = (v^t + iw^t) \cdot (\mathcal{A}_1 + i * \mathcal{A}_1) = 0$$

where $v, w \in \mathbb{R}^g$ are column vectors, and $\mathcal{A} = \binom{\mathcal{A}_1}{\mathcal{A}_2}$. By the preceding paragraph, $*\mathcal{A}_1 = \Lambda_1 \mathcal{A}_1 + \Lambda_2 \mathcal{A}_2$ so that

$$[v^t \cdot \mathcal{A}_1 - w^t \cdot (\Lambda_1 \mathcal{A}_1 + \Lambda_2 \mathcal{A}_2)] + i\big[w^t \cdot \mathcal{A}_1 + v^t \cdot (\Lambda_1 \mathcal{A}_1 + \Lambda_2 \mathcal{A}_2)] = 0$$

Since the terms in brackets are real one-forms, it follows that they both vanish. Therefore, we obtain the following relation between the linearly independent vectors $\mathcal{A}_1, \mathcal{A}_2$ of one-forms:

$$(v - \Lambda_1^t w)^t \cdot \mathcal{A}_1 = (\Lambda_2 w)^t \cdot \mathcal{A}_2, \quad (\Lambda_1^t v + w)^t \cdot \mathcal{A}_1 = -(\Lambda_2 v)^t \cdot \mathcal{A}_2$$

which finally yields $\Lambda_2 w = \Lambda_2 v = 0$, and thus $v = w = 0$ as desired. □

To proceed, we need the following remarkable identity:

LEMMA 8.3. *Let $\theta, \widetilde{\theta}$ be closed one-forms. Then*

$$\int_M \theta \wedge \widetilde{\theta} = \sum_{j=1}^{g} \Big(\int_{a_j} \theta \int_{b_j} \widetilde{\theta} - \int_{b_j} \theta \int_{a_j} \widetilde{\theta} \Big) \tag{8.3}$$

In particular, if θ is harmonic, then

$$\|\theta\|_2^2 = \sum_{j=1}^{g} \Big(\int_{a_j} \theta \int_{b_j} *\bar{\theta} - \int_{b_j} \theta \int_{a_j} *\bar{\theta} \Big)$$

PROOF. The integral on the left-hand side of (8.3) only depends on the cohomology classes of θ and $\widetilde{\theta}$, respectively. Thus, we can write

$$\theta = \sum_{j=1}^{2g} \mu_j \alpha_j + df, \quad \widetilde{\theta} = \sum_{j=1}^{2g} \widetilde{\mu}_j \alpha_j + d\widetilde{f}$$

where $\mu_j = \int_{A_j} \theta$, $\widetilde{\mu}_j = \int_{A_j} \widetilde{\theta}$. It follows that

$$\int_M \theta \wedge \widetilde{\theta} = \sum_{j,k=1}^{2g} \mu_j \widetilde{\mu}_k \int_M \alpha_j \wedge \alpha_k = \sum_{j=1}^{g} \Big(\mu_j \widetilde{\mu}_{j+g} - \mu_{j+g} \widetilde{\mu}_j \Big)$$

by (8.2). □

We now state two important corollaries:

Corollary 8.4. *Suppose $\theta \in \mathcal{H}\Omega^1$. Assume that either*

- *all a–periods vanish, i.e., $\int_{a_j} \theta = 0$ for all $1 \le j \le g$*
- *or all periods of θ are real*

Then $\theta = 0$.

Proof. From the previous lemma, since $*\theta = -i\theta$,

$$\begin{aligned}\|\theta\|^2 &= \int_M \theta \wedge \overline{*\theta} = i \int_M \theta \wedge \bar{\theta} \\ &= i \sum_{j=1}^{g} \Big(\int_{a_j} \theta \int_{b_j} \bar{\theta} - \int_{b_j} \theta \int_{a_j} \bar{\theta} \Big)\end{aligned}$$

which vanishes under either of our assumptions. ☐

Corollary 8.5. *The map*

$$\begin{cases} \omega & \mapsto \Big(\int_{a_1} \omega, \ldots, \int_{a_g} \omega \Big) \\ \mathcal{H}\Omega^1 & \to \mathbb{C}^g \end{cases}$$

is a linear isomorphism. In particular, there exists a unique basis $\{\zeta_j\}_{j=1}^g$ of $\mathcal{H}\Omega^1$ for which $\int_{a_j} \zeta_k = \delta_{jk}$.

Proof. This is an immediate consequence of the previous corollary and the fact that $\dim_{\mathbb{C}} \mathcal{H} = g$. ☐

This result raises the question what the other periods $\Pi_{jk} := \int_{b_j} \zeta_k$ look like. Before proceeding, consider the simplest example with $g = 1$, i.e., $M = \mathbb{C}/\langle 1, \tau \rangle$ where $\langle 1, \tau \rangle$ is the group generated by the translations $z \mapsto z + 1$, $z \mapsto z + \tau$ with $\operatorname{Im} \tau > 0$. It is clear that in this case the basis of the previous corollary reduces to $\zeta = dz$ with a-period 1, and b-period τ. Here we chose the a-loop to be the edge given by $z \mapsto z + 1$, and the b-loop as the edge $z \mapsto z + \tau$.

Returning to the general case, given a basis $\{\theta_j\}_{j=1}^g$ of $\mathcal{H}\Omega^1$, we call the $g \times 2g$ matrix whose j^{th} row consists of the periods of θ_j, the *period matrix* of the basis.

Lemma 8.6. *Riemann's bilinear relations: The period matrix of the basis $\{\zeta_j\}_{j=1}^g$ from above has the form*

$$(I, \Pi), \quad I = \mathrm{Id}_{g \times g}, \quad \Pi^t = \Pi, \quad \operatorname{Im} \Pi > 0$$

Proof. If $\theta, \widetilde{\theta}$ are holomorphic, then $\theta \wedge \widetilde{\theta} = 0$. Hence,

$$0 = \sum_{j=1}^{g} \Big(\int_{a_j} \theta \int_{b_j} \widetilde{\theta} - \int_{b_j} \theta \int_{a_j} \widetilde{\theta} \Big)$$

In particular, setting $\theta = \zeta_k$, $\widetilde{\theta} = \zeta_\ell$,

$$0 = \sum_{j=1}^{g} \Big(\int_{a_j} \zeta_k \int_{b_j} \zeta_\ell - \int_{a_j} \zeta_\ell \int_{b_j} \zeta_k \Big) = \Pi_{k\ell} - \Pi_{\ell k}$$

Next, we have

$$\begin{aligned}\langle \zeta_k, \zeta_j \rangle &= \int_M \zeta_k \wedge \overline{*\zeta_j} = i \int_M \zeta_k \wedge \overline{\zeta_j} \\ &= \sum_{\ell=1}^{g} \Big(\int_{a_\ell} \zeta_k \int_{b_\ell} \overline{\zeta_j} - \int_{b_\ell} \zeta_k \int_{a_\ell} \overline{\zeta_j} \Big) = 2\,\mathrm{Im}\,\Pi_{kj}\end{aligned}$$

which implies that $\mathrm{Im}\,\Pi > 0$. □

In the case of the torus from the previous example, $\Pi = \tau$ and $\mathrm{Im}\,\Pi = \mathrm{Im}\,\tau > 0$ by construction. In other words, the periods $1, \tau$ define a lattice in $\mathbb{C} = \mathbb{C}^g$ (since $g = 1$ here). This turns out to have a generalization to higher genera, leading to the *Jacobian variety*. Thus, let $\{\omega_k\}_{k=1}^g$ be *any* basis of the holomorphic differentials $\mathcal{H}\Omega^1(M)$. It turns out that the columns of the periodic matrix of this basis are linearly independent over the reals:

LEMMA 8.7. *Let $\{\omega_k\}_{k=1}^g$ be any basis of the holomorphic differentials $\mathcal{H}\Omega^1(M)$ and let $\{A_j\}_{j=1}^{2g}$ be the canonical homology basis as above. Then the $2g$ vectors in $\mathbb{C}^g$ (the columns of the period matrix)*

$$p_j := \Big(\int_{A_j} \omega_1, \dots, \int_{A_j} \omega_g \Big)^t$$

for $1 \le j \le 2g$ are linearly independent over $\mathbb{R}$. In particular,

$$L(M) := \Big\{ \sum_{j=1}^{2g} n_j p_j \mid n_j \in \mathbb{Z} \Big\}$$

is a lattice in $\mathbb{C}^g$. The quotient $J(M) := \mathbb{C}^g / L(M)$ is called the Jacobian variety *and it is a compact, commutative complex Lie group of dimension g.*

PROOF. Suppose the columns are linearly dependent over $\mathbb{R}$. Then there exist $\lambda_j \in \mathbb{R}$, $1 \le j \le 2g$ for which

$$\sum_{j=1}^{2g} \lambda_j \int_{A_j} \omega_\ell = 0 \qquad \forall\, 1 \le \ell \le g$$

In other words,

$$\langle \omega_\ell, \sum_{j=1}^{2g} \lambda_j \star \eta_{A_j} \rangle = 0 \qquad \forall\, 1 \le \ell \le g$$

Thus, for any $\omega \in \mathcal{H}\Omega^1(M)$,

$$\langle \omega, \sum_{j=1}^{2g} \lambda_j \eta_{A_j} \rangle = 0$$

and by the reality of the 1-form in the second slot here,

$$\langle \alpha, \sum_{j=1}^{2g} \lambda_j \eta_{A_j} \rangle = 0$$

for any *harmonic* differential α. But this implies that $\sum_{j=1}^{2g} \lambda_j \eta_{A_j} = 0$ and so $\lambda_j = 0$ for all $1 \leq j \leq 2g$. That $L(M)$ is a lattice simply means that $L(M)$ is a discrete subgroup of $\mathbb{C}^g$ which is clear, as are the stated properties of $J(M)$. $\square$

In the case of torus $M = \mathbb{C}/\langle 1, \tau\rangle$ we see that $J(M) \simeq M$. In fact, there is always a (canonical) holomorphic map that takes a compact Riemann surface into its Jacobian variety.

LEMMA 8.8. *The map $\phi : M \to J(M)$ defined as*

$$p \mapsto \Big(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g \Big)^t$$

where $p_0 \in M$ is an arbitrary point, is a well-defined holomorphic mapping with maximal rank one (over $\mathbb{C}$). Here the integration runs over any path that connects p_0 to p provided we chose the same path for each of the g entries of ϕ.

PROOF. Let γ and $\widetilde{\gamma}$ be two paths connecting p_0 to p. Then for all $1 \leq k \leq g$,

$$\int_\gamma \omega_k = \int_{\widetilde{\gamma}} \omega_k + \sum_{j=1}^{2g} n_j \int_{A_j} \omega_k$$

with $n_j \in \mathbb{Z}$. It follows that the two realizations of $\phi(p)$ only differ by an element of $L(M)$, as claimed. It is clear that $d\phi(p) = 0$ iff all ω_j vanish at p. But this is impossible by Corollary 8.15 below. Hence $d\phi$ has rank one as claimed. $\square$

This result of course raises a number of questions, such as: *for $g = 1$, is $M \simeq J(M)$ in the sense of conformal isomorphisms?*, or *is ϕ an imbedding for $g \geq 2$?*. We will return to these questions later in the Section on Abel's and Jacobi's theorems.

For now we return to the investigation of *bilinear relations*. The name "bilinear relation" in Riemann surface theory refers to any relation that originates by applying Lemma 8.3 to a specific choice of $\theta, \widetilde{\theta}$. Next, we wish to obtain bilinear relations for meromorphic differentials, and not just holomorphic ones. In order to do so, we first re-prove this lemma in a somewhat more intuitive fashion via Stokes theorem.

SECOND PROOF OF LEMMA 8.3. We use the fundamental polygon of the Riemann surface M, which is the polygon $\mathcal{F}$ bounded by the curves $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, \dots$ and with appropriate identifications on the boundary. Since $\mathcal{F}$ is simply connected, $\theta = df$ on $\mathcal{F}$. With some $z_0 \in \mathcal{F}$,

$$f(z) = \int_{z_0}^{z} \theta$$

Note that f does not necessarily agree at identified points. By Stokes,

$$\begin{aligned} \int_M \theta \wedge \widetilde{\theta} &= \int_{\mathcal{F}} df \wedge \widetilde{\theta} = \int_{\mathcal{F}} d(f\widetilde{\theta}) = \int_{\partial\mathcal{F}} f\widetilde{\theta} \\ &= \sum_{j=1}^{g} \Big(\int_{a_j} f\widetilde{\theta} + \int_{b_j} f\widetilde{\theta} + \int_{a_j^{-1}} f\widetilde{\theta} + \int_{b_j^{-1}} f\widetilde{\theta} \Big) \end{aligned} \tag{8.4}$$

To proceed, let $z, z' \in \partial\mathcal{F}$ be identified points on a_j, a_j^{-1}, respectively. Then

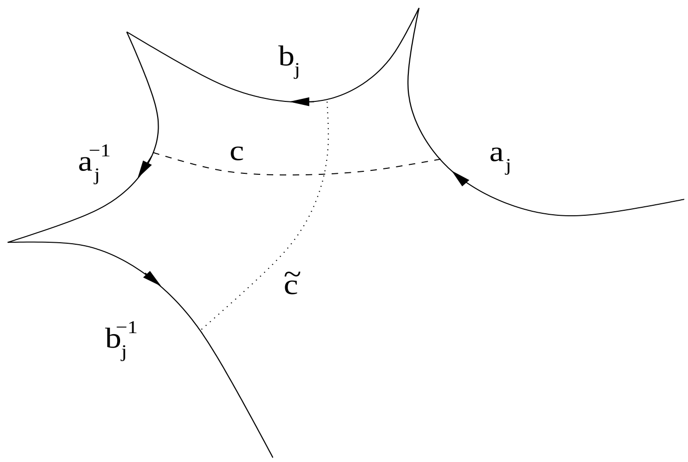

FIGURE 8.1. How the bilinear relations arise

$$\begin{aligned}\int_{a_j} f\widetilde{\theta} + \int_{a_j^{-1}} f\widetilde{\theta} &= \int_{a_j} \Big(\int_{z_0}^{z} \theta - \int_{z_0}^{z'} \theta \Big) \widetilde{\theta} \\ &= -\int_{a_j} \Big(\int_{b_j} \theta \Big) \widetilde{\theta} = -\int_{a_j} \widetilde{\theta} \int_{b_j} \theta\end{aligned}$$

In Figure 8.1 the curve c depicts the curve joining z to z'. A similar formula holds for the b_j, b_j^{-1} integrals. Plugging this into (8.4) yields (8.3). □

The importance of this method of proof lies with the fact that it applies to the case when $\widetilde{\theta}$ is a *meromorphic* differential as well. In that case we also pick up residues when applying Stokes theorem. Here's an important example, which uses our L^2 existence theory. In what follows, it will be understood automatically that the a, b–loops representing a homology basis do not pass through any pole of a meromorphic form. In particular, we regard them as *fixed* loops now rather than as homology classes.

LEMMA 8.9. *Fix some $p \in M$ and a parametric disk (U, z) at p with $z(p) = 0$, and let $n \geq 2$. Denote by $\tau_p^{(n)}$ the unique meromorphic differential, holomorphic on $M \setminus \{p\}$, with singularity*[1] $\frac{dz}{z^n}$ *locally at p, and with vanishing a-periods. Then*

$$\int_{b_\ell} \tau_p^{(n)} = \frac{2\pi i}{n-1} \, \alpha_{\ell,n-2} \quad \forall\, 1 \leq \ell \leq g \tag{8.5}$$

where $\alpha_{\ell,k}$ denotes the Taylor coefficients of ζ_ℓ locally at p, i.e.,

$$\zeta_\ell(z) = \Big(\sum_{k=0}^{\infty} \alpha_{\ell,k} \, z^k \Big) \, dz$$

in the same *local coordinates z in which $\tau_p^{(n)}$ has singularity $\frac{dz}{z^n}$.*

PROOF. To start, note that $\tau_p^{(n)} = \omega + \theta$ where ω is as in Corollary 7.11 and θ is holomorphic and chosen in such a way that the a-periods of $\tau_p^{(n)}$ vanish. To prove

[1] This means that $\tau_p^{(n)} - \frac{dz}{z^n}$ is holomorphic around p.

(8.5), pick a small positively oriented loop γ around p and let $\mathcal{F}'$ denote the fundamental polygon $\mathcal{F}$ with the disk bounded by γ deleted. Let $\zeta_\ell = df_\ell$ on $\mathcal{F}$ and thus also on $\mathcal{F}'$. Then, by the second proof of Lemma 8.3 presented above,

$$0 = \int_{\mathcal{F}'} \zeta_\ell \wedge \tau_p^{(n)} = \sum_{j=1}^{g} \Big(\int_{a_j} \zeta_\ell \int_{b_j} \tau_p^{(n)} - \int_{b_j} \zeta_\ell \int_{a_j} \tau_p^{(n)} \Big) - \int_\gamma f_\ell \, \tau_p^{(n)} \tag{8.6}$$

The vanishing of the left-hand side is the fact that $\theta \wedge \widetilde{\theta} = 0$ for holomorphic one-forms. In local coordinates,

$$\int_\gamma f_\ell \, \tau_p^{(n)} = \int_{|z|=\varepsilon} \Big(\sum_{k=0}^{\infty} \frac{\alpha_{\ell,k}}{k+1} \, z^{k+1} \Big) \frac{dz}{z^n} = \frac{2\pi i}{n-1} \, \alpha_{\ell,n-2}$$

whereas

$$\sum_{j=1}^{g} \Big(\int_{a_j} \zeta_\ell \int_{b_j} \tau_p^{(n)} - \int_{b_j} \zeta_\ell \int_{a_j} \tau_p^{(n)} \Big) = \int_{b_\ell} \tau_p^{(n)}$$

In view of (8.6) we are done. □

This lemma is needed in the proof of the Riemann-Roch theorem to which we now turn.

2. Divisors

The following terminology is standard in the field:

Definition 8.10. *A* divisor *D on M is a finite formal sum $D = \sum_\nu s_\nu p_\nu$ where $p_\nu \in M$ are distinct and $s_\nu \in \mathbb{Z}$. The* degree *of D is the integer*

$$\deg(D) = \sum_\nu s_\nu$$

If $s_\nu \geq 0$ for all ν then D is called integral. *We write $D \geq D'$ for two divisors iff $D - D' = \sum_\nu s_\nu p_\nu$ is integral. If f is a nonconstant meromorphic function on M, then we define the divisor of f as*

$$(f) = \sum_\nu \pm \mathrm{ord}(f; p_\nu) \, p_\nu$$

where the sum runs over the zeros and poles of f with the sign $\pm$ being chosen depending on whether p_ν is a zero or pole, respectively. If $f =$ const (but neither 0 nor ∞), then $(f) = 0$. In the same way, we define the divisor of a non-zero meromorphic differential:

$$(\omega) = \sum_\nu \pm \mathrm{ord}(\omega; p_\nu) \, p_\nu$$

where the sign is chosen again via the zero/pole dichotomy. Given a divisor D, we define the $\mathbb{C}$-linear space

$$L(D) = \{ f \in \mathcal{M}(M) \mid (f) \geq D \quad \text{or } f = 0 \}$$

where $\mathcal{M}(M)$ are the meromorphic functions. Analogously, we define the space

$$\Omega(D) = \{ \omega \in \mathcal{M}\Omega^1(M) \mid (\omega) \geq D \quad \text{or } \omega = 0 \}$$

where $\mathcal{M}\Omega^1(M)$ are the meromorphic differentials.

We collect some simple observations about these notions (all dimensions here are over $\mathbb{C}$):

LEMMA 8.11. *In what follows,* $\mathrm{Div}(M)$ *denotes the additive free group of divisors on* M*, a compact Riemann surface of genus* g*.*

(1) $\deg : \mathrm{Div}(M) \to \mathbb{Z}$ *is a group homomorphism.*
(2) *The map* $f \mapsto (f)$ *is a homomorphism from the multiplicative group* $\mathcal{M}(M)^*$ *of the field* $\mathcal{M}(M)$ *(which excludes* $f \equiv \infty$*) of meromorphic functions to* $\mathrm{Div}(M)$*. The image under this map is called the subgroup of* principal *divisors and the quotient* $\mathrm{Div}(M)/(\mathcal{M}(M)^*)$ *is called* divisor class group *and the conjugacy classes are called* divisor classes*. The homomorphism* deg *factors through to the divisor class group.*
(3) *The divisors of non-zero meromorphic differentials always belong to the same divisor class (called the* canonical class K*).*
(4) *If* $D \geq D'$*, then* $L(D) \subset L(D')$*.*
(5) $L(0) = \mathbb{C}$ *and* $L(D) = \{0\}$ *if* $D > 0$*.*
(6) *If* $\deg(D) > 0$*, then* $L(D) = \{0\}$*.*
(7) $\dim L(D)$ *and* $\dim \Omega(D)$ *only depend on the divisor class of* D*. Moreover,* $\dim \Omega(D) = \dim L(D - K)$ *where* K *is the canonical class.*
(8) $\Omega(0) = \mathcal{H}\Omega^1 \simeq \mathbb{C}^g$

PROOF. 1) is obvious. For 2), note that $(fg) = (f) + (g)$ and $\deg(f) = 0$ for any $f, g \in \mathcal{M}(M)^*$. For 3), observe that for any non-zero $\omega_1, \omega_2 \in \mathcal{M}\Omega^1$ the quotient $f = \frac{\omega_1}{\omega_2}$ is a non-zero meromorphic function. Since $(\omega_1) - (\omega_2) = (f)$, the statement follows. 4) is clear. For 5), note that $f \in L(D)$ with $D \geq 0$ implies that f is holomorphic and thus constant. For 6), observe that $(f) \geq D$ implies that $0 = \deg((f)) \geq \deg(D)$. For 7), suppose that $D = D' + (h)$ where h is nonconstant meromorphic. Then $f \mapsto fh$ takes $L(D')$ $\mathbb{C}$–linearly isomorphically onto $L(D)$. In particular, $\dim L(D) = \dim L(D')$. The map $\eta \mapsto \frac{\eta}{\omega}$ takes $\Omega(D)$ isomorphically onto $L(D - K)$ where $K = (\omega)$ is the canonical class, whence the dimension statement. Finally,

$$\dim \Omega(D) = \dim L(D - K) = \dim L(D' - K) = \dim \Omega(D')$$

For 8), simply note that $\Omega(0)$ consists of all holomorphic differentials. □

3. The proof of the Riemann-Roch theorem

We now state the main result of this chapter for integral divisors.

THEOREM 8.12 (Riemann-Roch). *Let* D *be an integral divisor. Then*

$$\begin{aligned} \dim L(-D) &= \deg(D) - g + 1 + \dim \Omega(D) \\ &= \deg(D) - g + 1 + \dim L(D - K) \end{aligned} \tag{8.7}$$

PROOF. By Lemma 8.11, (8.7) holds for $D = 0$. Hence, we can assume that $\deg(D) > 0$. Thus, assume that $D = \sum_{\nu=1}^{n} s_\nu p_\nu$ with $s_\nu > 0$. To expose the ideas with a minimum of technicalities, we let $s_\nu = 1$ with p_ν distinct for all $1 \leq \nu \leq n$. Let us also first assume that $g \geq 1$.

If $(f) \geq -D$, then $df \in \mathcal{M}\Omega^1$ is holomorphic on $M \setminus \bigcup_\nu \{p_\nu\}$ with $\mathrm{ord}(df, p_\nu) \geq -2$; clearly, df has zero periods and residues. Conversely, if $\eta \in \mathcal{M}\Omega^1$ has all these properties, then

$$f(q) = \int_p^q \eta$$

is well-defined where $p \in M$ is fixed and the integration is along an arbitrary curve avoiding the p_ν. It satisfies $df = \eta$ and $(f) \geq -D$. Hence,

$$\begin{aligned}\dim L(-D) &= \dim V + 1 \\ V &:= \Big\{\omega \in \mathcal{M}\Omega^1 \mid \omega \text{ has vanishing periods and residues,} \\ &\quad \omega \text{ is holomorphic on } M \setminus \bigcup_\nu \{p_\nu\}, \text{ and } \operatorname{ord}(\omega, p_\nu) \geq -2\Big\}\end{aligned}$$

To compute $\dim V$, we define for any $\underline{t} := (t_1, \ldots, t_n)$

$$\beta_{\underline{t}} := \sum_{n=1}^{n} t_\nu \tau^{(2)}_{p_\nu}$$

where $\tau^{(2)}_p$ is as in Lemma 8.9. By construction, $\beta_{\underline{t}}$ has vanishing a-periods and vanishing residues. Second, we define the map Φ as

$$\Phi \;:\; \beta_{\underline{t}} \mapsto \Big\{ \int_{b_\ell} \beta_{\underline{t}} \Big\}_{\ell=1}^{g}$$

Every $\omega \in V$ satisfies $\omega = \beta_{\underline{t}}$ for some unique $\underline{t}$ but not every $\beta_t \in V$; in fact, $V = \ker \Phi$ under this identification since the a-periods of $\tau^{(2)}_p$ vanish by construction. With $\{\zeta_\ell\}_{\ell=1}^{g}$ the basis from above,

$$\int_{b_\ell} \beta_{\underline{t}} = 2\pi i \sum_{\nu=1}^{n} t_\nu \, \alpha_{\ell,0}(p_\nu)$$

see (8.5), where

$$\zeta_\ell(z) = \Big[\sum_{j=0}^{\infty} \alpha_{\ell,j}(p_\nu) \, z^j \Big] \, dz$$

locally around p_ν. Thus, Φ is defined by the matrix

$$2\pi i \begin{bmatrix} \alpha_{1,0}(p_1) & \ldots & \alpha_{1,0}(p_n) \\ & \ldots & \\ \alpha_{g,0}(p_1) & \ldots & \alpha_{g,0}(p_n) \end{bmatrix}$$

The number of linear relations between the rows of this matrix equals

$$\dim\{\omega \in \mathcal{H}\Omega^1 \mid \omega(p_\nu) = 0 \;\; \forall \, 1 \leq \nu \leq n\}$$

which in turn equals $\dim L(D - K)$. For the latter equality, fix any non-zero $\omega \in \mathcal{M}\Omega^1$. Then $f \in L(D - K)$ iff

$$(f) \geq D - (\omega) \iff (f\omega) \geq D$$

iff $\alpha = f\omega \in \mathcal{H}\Omega^1(M)$ with $\alpha(p_\nu) = 0$ for all ν. In summary,

$$\begin{aligned}\dim L(-D) &= \dim V + 1 = \dim \ker \Phi + 1 \\ &= n - \operatorname{rank} \Phi + 1 = n - (g - \dim L(D - K)) + 1 \\ &= \deg(D) - g + 1 + \dim L(D - K)\end{aligned}$$

as claimed. Finally, if $g = 0$ then periods do not arise and for integral D with $\deg(D) > 0$ one simply has $\dim V = n = \deg(D)$ and $\dim L(D-K) = 0$ so that $\dim L(-D) = n+1 = \deg(D) + 1$ as desired.

The case of integral D which is not the sum of distinct points, the proof is only notationally more complicated. We again consider the case $g \geq 1$ first. Then, with $D = \sum_\nu s_\nu p_\nu$ and $n = \deg(D)$, consider

$$\beta_{\underline{t}} := \sum_\nu \sum_{k=2}^{s_\nu+1} t_{\nu,k}\, \tau_{p_\nu}^{(k)}, \qquad \underline{t} = \{t_{\nu,k}\}_{2\leq k\leq s_\nu+1}$$

Every ω in the linear space

$$V := \Big\{\omega \in \mathcal{M}\Omega^1 \mid \omega \text{ has vanishing periods and residues,} \\ \omega \text{ is holomorphic on } M \setminus \bigcup_\nu \{p_\nu\}, \text{ and } \operatorname{ord}(\omega, p_\nu) \geq -s_\nu - 1\Big\}$$

satisfies $\omega = \beta_{\underline{t}}$ for some $\underline{t} \in \mathbb{C}^n$. As before, we have $\dim L(-D) = \dim V + 1$. With Φ as above, $\beta_{\underline{t}} \in V$ iff $\Phi(\beta_{\underline{t}}) = 0$ so that $\dim V = \dim\ker\Phi$. From Lemma 8.9 we compute the b-periods as

$$\int_{b_\ell} \beta_{\underline{t}} = 2\pi i \sum_\nu \sum_{2\leq k\leq s_\nu+1} t_{\nu,k}\, \frac{\alpha_{\ell,k-2}(p_\nu)}{k-1}$$

For the purposes of computing dimensions, we may evidently replace $t_{\nu,k}$ with $\frac{t_{\nu,k}}{k-1}$. By the same argument as in the case $s_\nu = 1$ one now concludes that

$$\dim\ker\Phi = n - (g - \dim(D - K))$$

and (8.7) follows. Finally, the case $g = 0$ was already treated in Problem 7.4 — in fact, our argument for the $g \geq 1$ case only requires minimal changes to pass to $g = 0$ which we leave to the reader. $\square$

4. Applications and general divisors

As an immediate corollary, we obtain the following remarkable result which we already obtained before, see Corollary 7.12. Here we phrase the proof in terms of the Riemann-Roch theorem, whereas in the previous chapter we essentially gave the proof of the Riemann-Roch theorem in this special case (see also Problem 7.4).

Corollary 8.13. *Let M be compact and simply connected. Then $M \simeq \mathbb{C}P^1$ in the sense of conformal isomorphism.*

Proof. Let $D = p_0$ where $p_0 \in M$ is arbitrary. Since $g = 0$, Theorem 8.12 implies that

$$\dim L(-D) \geq 2$$

Thus, there exists a nonconstant $f \in \mathcal{M}(M)$ with a simple pole at p_0 and no other poles. Hence $\deg(f) = 1$ and f is an isomorphism between M and $\mathbb{C}P^1$. $\square$

The generalization of this proof method to higher genus yields the following statement concerning branched covers of compact Riemann surfaces.

Corollary 8.14. *Let M be compact with genus $g \geq 1$. Then M can be represented as a branched cover of $\mathbb{C}P^1$ with at most $g+1$ sheets. In other words, there exists a nonconstant meromorphic function on M of degree at most $g+1$.*

PROOF. Take $D = (g+1)p_0$. Then $\dim L(-D) \geq 2$ by Riemann-Roch. Hence, there exists $f : M \to \mathbb{C}P^1$ meromorphic, f holomorphic everywhere on $M \setminus \{p_0\}$ with $\mathrm{ord}(f, p_0) \geq -g-1$. Evidently, such an f defines the branched covering. $\square$

This shows in particular that every compact Riemann surface of genus one admits a two-sheeted branched cover. This is the missing piece in the proof of Corollary 5.20 which states that every compact Riemann surface of genus one is an elliptic curve. As the example of hyper-elliptic curves shows, there exist compact surfaces of arbitrarily large genus which can still be written as two-sheeted branched covers.

Let us give one last illustration of this proof method; to motivate the result, recall that on a simply connected surface M any holomorphic differential vanishes identically. If $g \geq 1$, we now observe completely different behavior: there is no point $p_0 \in M$ at which all holomorphic differentials vanish.

COROLLARY 8.15. *Let M be compact with $g \geq 1$. Then there is no point in M at which all $\omega \in \mathcal{H}\Omega^1$ vanish.*

PROOF. Suppose $p_0 \in M$ were such a point and set $D = p_0$. Then

$$\mathcal{H}\Omega^1(M) = \Omega(0) = \Omega(D) = L(-K) = L(D-K)$$

By Riemann-Roch therefore

$$\dim L(-D) = 1 - g + 1 + \dim L(-K) = 2 - g + g = 2$$

As in the proof of Corollary 8.13 we could now conclude that M is conformally equivalent to $\mathbb{C}P^1$, which cannot be. For example, this would contradict the Riemann-Hurwitz formula. $\square$

The significance of the previous result has to do with projective imbeddings. Indeed, for M as in the corollary, let $\{\omega_j\}_{j=1}^g$ be a basis of $\mathcal{H}\Omega^1(M)$ and define

$$\imath_M \,:\, M \to \mathbb{C}P^g,\ p \mapsto [f_1(p) : \ldots : f_g(p)]$$

where $\omega_j = f_j\,dz$ in local coordinates. This is clearly well-defined as changes of coordinates only multiply the entries by a nonzero factor; moreover, at least one f_j does not vanish. Thus, χ is well-defined and, in fact, analytic. In Problem 8.6 the reader will study the question as to when this map $\imath_M$ is in fact an imbedding. Again, the Riemann-Roch theorem is the crucial tool.

Finally, we need to prove the Riemann-Roch theorem for arbitrary divisors, and not just integral ones. The proof of this extension requires the following intermediate step which is of course an interesting result in its own right.

LEMMA 8.16. *The degree of the canonical class is given by*

$$\deg(K) = 2g - 2 = \chi(M)$$

PROOF. If $g = 0$, then take $M = \mathbb{C}_\infty$ and $\omega = dz$ in the chart $z \in \mathbb{C}$. Under the change of variables $z \mapsto \frac{1}{z}$, this transforms into $\omega = -\frac{dz}{z^2}$. Hence, $\deg(K) = -2$. If $g \geq 1$, pick any nonzero $\omega \in \mathcal{H}\Omega^1(M)$ (which can be done since this space has dimension g). Then $(\omega) = K$ is integral and by Theorem 8.12,

$$\dim L(-K) = \deg(K) - g + 1 + \dim L(0) = \deg(K) - g + 2$$

whereas $L(-K) = \mathcal{H}\Omega^1(M)$. Hence $\dim L(-K) = \dim \mathcal{H}\Omega^1(M) = g$, and $\deg(K) = 2g - 2$ as claimed.

An alternative proof based on the Riemann-Hurwitz formula is as follows: By Theorem 8.12 there exists a meromorphic function f with n simple poles for some integer $n \geq 2$ and holomorphic elsewhere. In particular, $\deg(f) = n$. Take $\omega = df$. Suppose that $p \in M$ is a branch point of f. Then p is not a pole of f and

$$\operatorname{ord}(\omega; p) = b_f(p)$$

where $b_f(p)$ is the branch number of f at p. If p is a pole of f, then $\operatorname{ord}(\omega; p) = -2$ so that

$$\deg((\omega)) = -2n + \sum_{p \in M} b_f(p)$$

By the Riemann-Hurwitz formula,

$$2(g-1) = -2n + \sum_{p \in M} b_f(p)$$

and we are done. □

Combining this with Theorem 8.12 now yields the full Riemann-Roch theorem.

Theorem 8.17. *Equation* (8.7) *holds for all divisors* D.

Proof. We already covered the case where D is equivalent to an integral divisor. Suppose D is such that $K - D$ is equivalent to an integral divisor. Then, from Theorem 8.12,

$$\dim L(D-K) = \deg(K-D) - g + 1 + \dim L(-D) = -\deg(D) + g - 1 + \dim L(-D)$$

which is the desired statement. Suppose therefore that neither D nor $K-D$ are equivalent to an integral divisor. Then

$$\dim L(-D) = \dim L(D-K) = 0$$

It remains to be shown that $\deg(D) = g - 1$. For this we write $D = D_1 - D_2$ where D_1 and D_2 are integral and have no point in common. Clearly, $\deg(D) = \deg(D_1) - \deg(D_2)$ with both degrees on the right-hand side positive. By Theorem 8.12,

$$\dim L(-D_1) \geq \deg(D_1) - g + 1 = \deg(D) + \deg(D_2) - g + 1$$

If $\deg(D) \geq g$, then $\dim L(-D_1) \geq \deg(D_2) + 1$ and there exists a function $f \in L(-D_1)$ which vanishes at all points of D_2 to the order prescribed by D_2. Indeed, this vanishing condition imposes $\deg(D_2)$ linear constraints which leaves us with one dimension in $L(-D_1)$ (for example, if $\deg(D_2) = 1$ then we use the constant function to make any nonconstant meromorphic f with $(f) \geq -D_1$ vanish at the point given by D_2). For this f,

$$(f) + D \geq -D_1 + D_2 + D = 0$$

so that $f \in L(-D)$ contrary to our assumption. This shows that $\deg(D) \leq g - 1$. Similarly,

$$\deg(K-D) = 2g - 2 - \deg(D) \leq g - 1 \Longrightarrow \deg(D) \geq g - 1$$

and we are done. □

5. The theorems of Abel and Jacobi

We shall now study the Jacobian variety $J(M) = \mathbb{C}^g/L(M)$ in more detail. Let us first recall a result for the tori $M = \mathbb{C}/\langle\omega_1, \omega_2\rangle$ which we proved in Chapter 4 in the context of elliptic functions. That is, recall Theorem 4.17 which establishes the existence of a meromorphic function on M with zeros z_j and poles ζ_k iff conditions (4.10) and (4.11) hold. Using the language of divisors, this is in agreement with the following more general result, known as Abel's theorem. As above, M is a compact Riemann surface and we let $\{\omega_j\}_{j=1}^g$ be an arbitrary basis of $\mathcal{H}\Omega^1(M)$, g the genus of M.

PROPOSITION 8.18. *Let D be a divisor on the compact Riemann surface M. There exists a meromorphic function f on M with $(f) = D$ iff the following two conditions hold:*

- $\deg(D) = 0$
- *with*[2] $D = \sum_{\nu=1}^n (p_\nu - q_\nu)$ *one has*

$$\Phi(D) := \sum_{\nu=1}^n \Big(\int_{p_\nu}^{q_\nu} \omega_1, \dots, \int_{p_\nu}^{q_\nu} \omega_g \Big) \equiv 0 \quad (\mod L(M)) \tag{8.8}$$

PROOF. If $g = 0$, then only $\deg(D) = 0$ is relevant. Thus, we are reduced to the fact that a meromorphic function exists on S^2 with poles and zeros at prescribed locations and with prescribed orders as long as the combined order of the zeros is the same as the combined order of the poles.

Let us therefore assume that $g \geq 1$. Clearly, $\deg(D) = 0$ is necessary. For (8.8), consider the map

$$\Psi \ : \ [\zeta_1 : \zeta_2] \mapsto \Phi((\zeta_1 f + \zeta_2))$$

from $\mathbb{C}P^1 \to J(M)$. Clearly, Ψ is continuous and lifts to a continuous map $\widetilde{\Psi} : \mathbb{C}P^1 \to \mathbb{C}^g$ which is, moreover, holomorphic (i.e., each component is). We conclude from the maximum principle that each component is constant whence

$$0 = \Psi([0:1]) = \Psi([1:0]) = \Phi(D)$$

as claimed. For an alternative proof of the necessity of (8.8), see Problem **??**.

For the sufficiency, we use Theorem 7.13. $\square$

6. Problems

PROBLEM 8.1. *Determine the divisor class group for S^2.*

PROBLEM 8.2. *Show that every compact surface M of genus two is a hyper-elliptic surface, i.e., it carries a meromorphic function of degree two.*

PROBLEM 8.3. *Given an example of a compact Riemann surface of genus $g \geq 3$ which is not hyper-elliptic, in other words, it cannot be written as a two-sheeted branched cover of S^2.*

[2]The p_ν do not need to be pairwise distinct, and the same holds for the q_ν. However, $p_\nu \neq q_\mu$ for any $1 \leq \nu, \mu \leq n$.

PROBLEM 8.4. *Let M be compact of genus $g \geq 1$ and let $\{\omega_j\}_{j=1}^g$ be a basis of $\mathcal{H}\Omega^1(M)$. Prove that*

$$ds^2(p) := \sum_{j=1}^{g} |\omega_j(p)|^2 \quad on \quad T_pM$$

with $p \in M$ arbitrary defines a positive definite metric on M. Show that it has nonpositive curvature. Discuss the possible vanishing of the curvature.

PROBLEM 8.5. *In this problem, you are asked to improve on Proposition 5.16 in the following way: Given z of degree n as in that proposition, prove that there exists a meromorphic function f on M which renders the polynomial in* (5.6) *irreducible. Note that this concludes the proof of Theorem 5.19.*

PROBLEM 8.6. *Here we discuss the question when the canonical map $\imath_M : M \to \mathbb{C}P^g$ defined above is in fact an imbedding.*

CHAPTER 9

The Dirichlet problem and Green functions

1. Green functions

In the notes on Hodge theory we encountered the fundamental problem of solving the so called Poisson equation

$$\Delta u = f \tag{9.1}$$

when $f \in C^2_{\mathrm{comp}}(\mathbb{R}^2)$. Such a u is not unique (add any linear function). However, we singled out the solution

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|z-\zeta|\, f(\zeta)\, d\xi d\eta \tag{9.2}$$

where $\zeta = \xi + i\eta$. The reader will easily verify that it is the *unique* solution of (9.1) with the property that $u(z) = k\log|z| + o(1)$ as $|z| \to \infty$ for some constant k. In fact, necessarily $k = \langle f\rangle$.
The function $\Gamma(z,\zeta) = \frac{1}{2\pi}\log|z-\zeta|$ is of great importance. It is called the *fundamental solution* of Δ which means that $\Delta G(\cdot,\zeta) = \delta_\zeta$ in the sense of distributions. We are now led to ask how to solve (9.1) on a bounded domain $\Omega \subset \mathbb{C}$ (for example on $\Omega = \mathbb{D}$). To obtain uniqueness – from the maximum principle – we impose a Dirichlet boundary condition $u = 0$ on $\partial\Omega$. By a *solution* of

$$\Delta u = f \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega \tag{9.3}$$

with $f \in C(\Omega)$ we mean a function $u \in C^2(\Omega)\cap C(\overline{\Omega})$ which satisfies (9.3) in the pointwise sense. To solve (9.3), we try to emulate (9.2).

Definition 9.1. *We say that $\Omega \subset \mathbb{C}$ admits a* Green function *if there exists G with the following properties:*

- $G(\cdot,\cdot) \in C(\overline{\Omega}\times\Omega\setminus\{z=\zeta\})$
- *The function $h(z,\zeta) := G(z,\zeta) - \Gamma(z,\zeta)$ is harmonic on Ω in the first variable for all $\zeta \in \Omega$, and jointly continuous on $\Omega\times\Omega$*
- $G(z,\zeta) = 0$ *for all* $(z,\zeta) \in \partial\Omega\times\Omega$

It is important to note that this definition applies to unbounded $\Omega \subset \mathbb{C}$, but in that case we view $\Omega \subset \mathbb{C}_\infty$ so that $\infty \in \partial\Omega$. In particular, we require vanishing at infinity in that case. It is clear that if G exists, then it is unique. Also, $G < 0$ in Ω by the maximum principle.

Lemma 9.2. *If Ω admits a Green function, then* (9.3) *has a unique solution for every*[1] $f \in C^2_{\mathrm{comp}}(\Omega)$ *given by*

$$u(z) = \int_\Omega G(z,\zeta)f(\zeta)\, d\zeta \qquad \forall\, z \in \Omega$$

[1]The assumptions on f in the previous lemma can be relaxed, but this does not concern us here.

PROOF. Uniqueness follows from the maximum principle. By the continuity assumptions on G, u is continuous on $\overline{\Omega}$ and satisfies $u = 0$ on $\partial\Omega$. Moreover, with $\zeta = \xi + i\eta$, we can write

$$\begin{aligned} u(z) &= \int_\Omega [G(z,\zeta) - \Gamma(z,\zeta)] f(\zeta)\, d\xi d\eta + \int_\Omega \Gamma(z,\zeta) f(\zeta)\, d\xi d\eta \\ &=: u_1(z) + u_2(z) \end{aligned}$$

The second integral on the right-hand side, which we denoted by u_2, satisfies $\Delta u_2 = f$. Indeed, setting $f = 0$ outside of Ω leads to $f \in C^2_{\mathrm{comp}}(\mathbb{R}^2)$ and we can apply the discussion centered around (9.2). As for u_1, we have

$$u_1(\cdot) = \int_\Omega h(\cdot,\zeta) f(\zeta)\, d\xi d\eta \in C(\Omega)$$

by continuity of h. Moreover, the mean value property holds:

$$\int_0^1 u_1(z_0 + re^{2\pi i\theta})\, d\theta = u_1(z_0)$$

for all $z_0 \in \Omega$ and small $r > 0$. This follows from Fubini's theorem since $h(\cdot,\zeta)$ is jointly continuous and satisfies the mean value property in the first variable. Thus, u_1 is harmonic and $\Delta u = f$ as desired. $\square$

So which $\Omega \subset \mathbb{C}$ admit a Green function? For example, take $\Omega = \mathbb{D}$. Then $G(z,0) = \frac{1}{2\pi}\log|z|$ does the trick for $\zeta = 0$. Next, we map ζ to 0 by the Möbius transformation $T(z) = \frac{z-\zeta}{1-z\bar{\zeta}}$. This yields

$$G_{\mathbb{D}}(z,\zeta) = \frac{1}{2\pi}\log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|$$

as our Green function for $\mathbb{D}$. It clearly satisfies Definition 9.1. Moreover, by inspection,

$$G_{\mathbb{D}}(z,\zeta) = G_{\mathbb{D}}(\zeta,z) \quad \forall\, z,\zeta \in \mathbb{D}$$

Now let Ω be the disk with $n \geq 1$ points removed, i.e., $\Omega = \mathbb{D}\setminus\{z_1,\ldots,z_n\}$. If G were a Green function on Ω, then for all $\zeta \in \Omega$, $z \mapsto G(z,\zeta)$ would need to be continuous in a neighborhood of z_j for all $1 \leq j \leq n$ and harmonic away from z_j. Then each z_j would constitute a removable singularity and $G(z,\zeta)$ (see Problem 3.3) therefore be harmonic in a disk around each z_j. In other words, G would be the Green function of $\mathbb{D}$ and therefore negative at each z_j violating the vanishing condition. In conclusion, Ω does not admit a Green function in the sense of Definition 9.1.

Finally, any simply connected $\Omega \subset \mathbb{C}$ for which the Riemann mapping $f : \Omega \to \mathbb{D}$ extends continuously to $\overline{\Omega}$ admits a Green function (for this it suffices to assume that $\partial\Omega$ consists of finitely many C^1 arcs). Indeed, observe that

$$G_\Omega(z,\zeta) := G_{\mathbb{D}}(f(z), f(\zeta))$$

satisfies Definition 9.1. This procedure applies to unbounded Ω, for example $\Omega = \mathbb{H}$. It automatically enforces the vanishing condition at infinity required by the fact that we view $\Omega \subset \mathbb{C}_\infty$.

2. The potential theory proof of the Riemann mapping theorem

This discussion raises the following interesting question: Is it possible to construct the Riemann mapping $\Omega \to \mathbb{D}$ from the Green function on Ω (assuming it exists)? The answer, which was found by Riemann, is "yes".

THEOREM 9.3. *Suppose $\Omega \subset \mathbb{C}_\infty$ is simply connected and admits a Green function. Then G gives rise to a bi-holomorphic mapping $f : \Omega \to \mathbb{D}$.*

PROOF. The idea is simply to write, with $\zeta \in \Omega$ fixed,

$$2\pi G(z, \zeta) = \log |z - \zeta| + \operatorname{Re} F(z)$$

where $F \in \mathcal{H}(\Omega)$ (this can be done since Ω is simply connected). Then we set

$$f_\zeta(z) := (z - \zeta) \exp(F(z)) \in \mathcal{H}(\Omega)$$

Note that f_ζ is unique up to a unimodular number. By construction,

$$|f_\zeta(z)| = \exp(\log |z - \zeta| + \operatorname{Re} F(z)) = \exp(2\pi G(z, \zeta)) < 1$$

for all $z \in \Omega$ so that $f_\zeta : \Omega \to \mathbb{D}$; furthermore, $|f_\zeta(z)| = 1$ for all $z \in \partial\Omega$ and $|f_\zeta|$ extends as a continuous mapping to all of $\overline{\Omega}$.

We claim that $f(\Omega) = \mathbb{D}$. By analyticity and since f is obviously not constant, $f(\Omega)$ is open. To show that it is closed, suppose that $f(z_n) \to w \in \mathbb{D}$. Then $z_n \to z_\infty \in \overline{\Omega}$ (if needed, pass to a subsequence of z_n, which we call z_n again; recall that we are viewing $\Omega \subset \mathbb{C}_\infty$ which is compact). If $z_\infty \in \partial\Omega$, necessarily $|w| = 1$ which is a contradiction. So $z_\infty \in \Omega$ and $f(z_n) \to f(z_\infty) = w$ which shows that $f(\Omega)$ is closed.

It remains to show that f is one-to-one. Locally around ζ this is clear (why?), but not globally on Ω. We also remark that $f_\zeta(z) = 0$ iff $z = 0$. In view of this, the logic is now as follows: suppose f_ζ is one-to-one for any $\zeta \in \Omega$. Then $T := f_\eta \circ f_\zeta^{-1} \in \operatorname{Aut}(\mathbb{D})$ is a Möbius transformation which takes $f_\zeta(\eta)$ to 0. This suggests we prove the following converse for arbitrary $\eta \in \Omega \setminus \{\zeta\}$.

Claim: Let $f_\zeta(\eta) = w$ and $T(w) = 0$, $T \in \operatorname{Aut}(\mathbb{D})$. Then $|T \circ f_\zeta| = |f_\eta|$.

If the claim holds, then we are done: assume that $f_\zeta(\eta) = f_\zeta(\widetilde{\eta}) = w \in \mathbb{D}$ and let $T(w) = 0$ were $T \in \operatorname{Aut}(\mathbb{D})$. Then

$$|T \circ f_\zeta| = |f_\eta| = |f_{\widetilde{\eta}}|$$

so that $f_{\widetilde{\eta}}(\eta) = 0$ implies $\eta = \widetilde{\eta}$ as desired. To prove the claim, note that for any $0 < \varepsilon < 1$, and some integer $k \geq 1$,

$$\begin{aligned}\log |T \circ f_\zeta(z)| &= k \log |z - \eta| + O(1) \leq \log |z - \eta| + O(1) \\ &\leq 2\pi(1 - \varepsilon) G(z, \eta)\end{aligned}$$

as $z \to \zeta$. Moreover,

$$\limsup_{z \to \zeta} \log |T \circ f_\zeta(z)| \leq 0, \quad G(z, \eta) \to 0 \ \text{ as } z \to \partial\Omega$$

Hence, on $\Omega \setminus D(\zeta, \delta)$ for all $\delta > 0$ small, we see that the harmonic function $2\pi G(\cdot, \eta)$ dominates the subharmonic function $\log |T \circ f_\zeta(\cdot)|$ on $\Omega \setminus \{\eta\}$. In conclusion,

$$\log |T \circ f_\zeta(\cdot)| \leq 2\pi G(\cdot, \eta)$$

In particular, since $T(z) = \frac{z-w}{1-z\bar{w}}$ and thus $T(0) = -w$,

$$\begin{aligned} 2\pi G(\eta,\zeta) &= \log|f_\zeta(\eta)| = \log|w| = \log|T(0)| \\ &= \log|T\circ f_\zeta(\zeta)| \le 2\pi G(\zeta,\eta) \end{aligned} \tag{9.4}$$

whence $G(\eta,\zeta) \le G(\zeta,\eta)$ which implies

$$G(\zeta,\eta) = G(\eta,\zeta) \tag{9.5}$$

This is the well-known symmetry property of the Green function. It follows that we have equality in (9.4)

$$\log|T\circ f_\zeta(\zeta)| = 2\pi G(\zeta,\eta)$$

from which we conclude via the strong maximum principle on $\Omega\setminus\{\eta\}$ that

$$\log|T\circ f_\zeta(\cdot)| = 2\pi G(\cdot,\eta) = \log|f_\eta(\cdot)|$$

as claimed. This finishes the proof. □

The importance of this argument lies with the fact that it extends from domains $\Omega\subset\mathbb{C}$ to simply connected Riemann surfaces M, at least to those that admit a *Green function* — see the following chapter for the exact definition of this concept on Riemann surfaces (we caution the reader that a Green function G on a Riemann surfaces will not necessarily conform to Definition 9.1 above in case $M\subset\mathbb{C}$).

3. Existence of Green functions via Perron's method

Let us now consider the important problem of finding the Green function on bounded domains $\Omega\subset\mathbb{C}$. Fix $\zeta\in\Omega$ and solve – if possible – the Dirichlet problem

$$\Delta u(z) = 0 \text{ in } \Omega, \quad u(z) = -\log|z-\zeta| \text{ on } \partial\Omega \tag{9.6}$$

Then $G(z,\zeta) := u(z) + \log|z-\zeta|$ is the Green function. This was Riemann's original approach, but he assumed that (9.6) always has a solution via the so-called "Dirichlet principle". In modern terms this refers to the fact that the variational problem, with $f\in C^1(\overline{\Omega})$ and $\partial\Omega$ being C^2 regular,

$$\begin{aligned} &\inf_{u\in\mathcal{A}} \int_\Omega |\nabla u|^2\,dxdy \\ &\mathcal{A} := \{u\in H^1(\Omega) \mid u-f\in H^1_0(\Omega)\} \end{aligned}$$

has a (unique) minimizer $u_0\in\mathcal{A}$ (minimizer here means that u_0 attains the infimum). Here

$$H^1(\Omega) = \{u\in L^2(\Omega) \mid \nabla u\in L^2(\Omega)\}$$

is the Sobolev space where ∇u is the distributional derivative, and $H^1_0(\Omega)\subset H^1(\Omega)$ is the subspace of vanishing trace. It a standard fact, see Evans's book, that Dirichlet's principle holds and that the minimizer u_0 is a harmonic function so that $u_0 - f\in C(\overline{\Omega})$ vanishes on $\partial\Omega$ as desired. Riemann did not have this Hilbert spaces machinery at his disposal, however.

Instead, we will use an elegant method due to Perron based on subharmonic functions. It requires less on the boundary than the variational approach. We first need to lift the concept of subharmonic functions to a general Riemann surface M.

Definition 9.4. *A function $u : M \to [-\infty, \infty)$ is subharmonic iff it is continuous and subharmonic in every chart. We denote the class of subharmonic functions on the Riemann surface M by $\mathfrak{sh}(M)$.*

Since subharmonicity is preserved under conformal transformations, this definition is meaningful. From our definition it is clear that subharmonicity is a *local* property. Hence, properties that can be checked in charts immediately lift from the planar case to Riemann surfaces. Here are two examples:

- If $u \in C^2(M)$ then u is subharmonic iff $\Delta u \geq 0$ in every chart on M.
- If $u_1, \ldots, u_k \in \mathfrak{sh}(M)$, then $\max(u_1, \ldots, u_k) \in \mathfrak{sh}(M)$ and $\sum_{j=1}^k a_j u_j \in \mathfrak{sh}(M)$ for any $a_j \geq 0$.

The following lemma collects several global properties of this class which mirror those in the planar case. We begin with the maximum principle.

Lemma 9.5. *The following properties hold for subharmonic functions:*

1. *If $u \in \mathfrak{sh}(M)$ attains its supremum on M, then $u = \mathrm{const}$.*
2. *Let h be harmonic on M and $u \in \mathfrak{sh}(M)$. If $u \leq h$ on M then either $u < h$ or $u = h$ everywhere on M.*
3. *Let $\Omega \subset M$ be a domain with compact closure in M. Suppose h is harmonic on Ω and continuous on $\overline{\Omega}$. If*

$$\limsup_{\substack{p \to q \\ p \in \Omega}} u(p) \leq h(q) \quad \forall\, q \in \partial\Omega \tag{9.7}$$

then $u \leq h$ in Ω. Equality here can only be attained in Ω if $u = h$ throughout Ω.

Proof. For 1), assume that $u \leq u(p_0)$ with $p_0 \in M$. Then

$$E = \{p \in M \mid u(p) = u(p_0)\}$$

is both open (since that is a local property and follows by considering charts) and closed. Hence $E = M$, as desired.

For 2), apply 1) to $u - h \in \mathfrak{sh}(M)$.

For 3), it suffices to consider the case $h = 0$ (otherwise consider $u - h$). If $u(p) > 0$ for any $p \in \Omega$, then u attains its supremum on Ω and is therefore constant. But this would contradict (9.7). Hence, $u \leq 0$ on Ω with equality being attained at one point forcing constancy by 2). $\square$

We remark that property 2) above characterizes $\mathfrak{sh}(M)$ and gives a nice way of defining subharmonic functions intrinsically on Riemann surfaces. For future purposes we denote property (9.7) by $u \ll h$ on $\partial\Omega$. We are now going to describe Perron's method for solving the following Dirichlet problem on a Riemann surface M:

Let $\Omega \subset M$ be a domain with $\overline{\Omega}$ compact and suppose $\phi : \partial\Omega \to \mathbb{R}$ is continuous. Find $u \in C(\overline{\Omega})$ so that u is harmonic on Ω and $u = \phi$ on $\partial\Omega$.

The first step towards the solution is furnished by showing that the upper envelope of all subharmonic functions v on Ω with $v \ll \phi$ is harmonic. The second step then addresses how to ensure that the boundary data are attained continuously. For the first step we need to following easy result.

Lemma 9.6. *Let D be a parametric disk and suppose $f \in \mathfrak{sh}(M)$ is real-valued. Let h be the harmonic function on D which has f as boundary values on ∂D. The function*

$$f_D := \begin{cases} f & \text{on } M \setminus D \\ h & \text{on } D \end{cases}$$

satisfies $f_D \in \mathfrak{sh}(M)$ and $f_D \geq f$.

Proof. It is clear that f_D is continuous. By the maximum principle, $f_D \geq f$. It is clear that f_D is subharmonic on $M \setminus \partial D$. However, if $p \in \partial D$, then we see from $f_D \geq f$ that the sub-mean value property holds at p for sufficiently small circles (relative to some chart at p). Therefore, $f_D \in \mathfrak{sh}(M)$ as claimed. □

Now for the first step in Perron's method.

Proposition 9.7. *Let ϕ be any bounded function on $\partial\Omega$. Then*

$$u = \sup\{v \mid v \in \mathfrak{sh}(\Omega),\ v \ll \phi \quad \text{on } \partial\Omega\} \tag{9.8}$$

is harmonic on Ω.

Proof. Denote the set on the right-hand side of (9.8) by $\mathcal{S}_\phi$. First note that any $v \in \mathcal{S}_\phi$ satisfies

$$\sup_\Omega v \leq \sup_{\partial\Omega} \phi < \infty$$

Moreover, replacing any $v \in \mathcal{S}_\phi$ by $\max(v, \inf_{\partial\Omega} \phi)$, we can assume that all $v \in \mathcal{S}_\phi$ are bounded below. Take any $p \in \Omega$ and a sequence of $\{v_n\}_{n=1}^\infty \subset S_\phi$ so that $v_n(p) \to u(p)$. Replacing the sequence by

$$v_1,\ \max(v_1, v_2),\ \max(v_1, v_2, v_3), \ldots$$

we may assume that $\{v_n\}_{n=1}^\infty$ is non-decreasing. In addition, by Lemma 9.6 each v_n can be assumed to be harmonic on on some parametric disk D centered at p. By Harnack's inequality on D, $v_n \to v_\infty$ uniformly on compact subsets of D with v_∞ harmonic on D.

It remains to check that $u = v_\infty$ on D. Take any $q \in D$ and let $\{w_n\}_{n=1}^\infty \subset \mathcal{S}_\phi$ with $w_n(q) \to u(q)$. As before, we can assume that w_n is harmonic and increasing on D. In fact, we can also assume that $w_n \geq v_n$ for each n. We conclude that $w_n \to w_\infty$ uniformly on compact sets with w_∞ harmonic on D and $w_\infty(p) = v_\infty(p)$. Since $w_\infty \geq v_\infty$ in D, it follows that $w_\infty = v_\infty$ on D. In particular, $u(q) = v_\infty(q)$ and we are done. □

It is worth noting that this proof has little to do with ϕ. In fact, it applies to any *Perron family* which we now define.

Definition 9.8. *A family $\mathcal{F}$ of real-valued subharmonic functions on a Riemann surface M is called* Perron family *iff*

- *for any $f, g \in \mathcal{F}$ there is $h \in \mathcal{F}$ with $h \geq \max\{f, g\}$*
- *for any parametric disk $D \subset M$ and any $v \in \mathcal{F}$ there exists $w \in \mathcal{F}$ with w harmonic on D and $w \geq v$*

Then we have the following immediate corollary of the proof of Proposition 9.7:

Lemma 9.9. *For any Perron family $\mathcal{F}$ on a Riemann surface M the function*

$$u := \sup_{v \in \mathcal{F}} v$$

is either $\equiv \infty$ or harmonic on M.

4. Behavior at the boundary

Next, we turn to the boundary behavior. For a standard example of what can go wrong on the boundary consider $\Omega = \mathbb{D} \setminus \{0\}$. Setting $\phi(0) = 0$ and $\phi(z) = 1$ for $|z| = 1$ we see that Perron's method yields $u = \text{const} = 1$; indeed, for any $\varepsilon > 0$ the function $v(z) = 1 + \varepsilon \log |z| \in \mathcal{S}_\phi$.

DEFINITION 9.10. *A* barrier *at a point $p \in \partial\Omega$ is defined to be a function β with the following properties:*

- $-\beta \in \mathfrak{sh}(\Omega)$
- $\beta \in C(\overline{\Omega})$ *and* $\beta \geq 0$ *on* $\overline{\Omega}$ *with* $\beta > 0$ *on* $\overline{\Omega} \setminus \{p\}$, *and* $\beta(p) = 0$

Any point $q \in \partial\Omega$ which admits a barrier is called regular *and $\partial\Omega$ is called regular iff all of its points are regular.*

It turns out that regularity of a boundary point is a mild condition:

LEMMA 9.11. *Suppose $p \in \partial\Omega$ satisfies an* exterior disk condition*, i.e., there exists a disk $D(z_0, \varepsilon)$ in local coordinates (U, z) around p so that*

$$z(U \cap \Omega) \cap D(z_0, \varepsilon) = \{p\}$$

Then p is regular. In particular, any C^2 boundary is regular. Moreover, suppose that $p \in \partial\Omega$ is accessible*, i.e.,*

$$z(\Omega \cap U) \subset \mathbb{D} \setminus (-1, 0]$$

in some chart (U, z) with $z(p) = 0$. Then p is regular.

PROOF. For the exterior disk condition, we define for all $q \in \Omega$,

$$\beta(q) = \begin{cases} \log(|z(q) - z_0|/\varepsilon) & \text{if } \ |z(q) - z_0| \leq \delta \\ \log(\delta/\varepsilon) & \text{if } \ |z(q) - z_0| > \delta \end{cases}$$

where $\delta > \varepsilon > 0$ are sufficiently small. If $p \in \partial\Omega$ is accessible, then we map $\mathbb{D} \setminus (-1, 0]$ conformally onto a sector of angle $\leq \pi$ which guarantees the exterior disk condition at p. □

An obvious example of a non-accessible boundary point is $p = 0$ for $\Omega = \mathbb{D} \setminus \{0\}$. The importance of barriers lies with the following fact:

PROPOSITION 9.12. *Suppose $p \in \partial\Omega$ is regular and ϕ a bounded function on $\partial\Omega$ which is continuous at p. Then the function u from Proposition 9.7 satisfies*

$$\lim_{\substack{q \to p \\ q \in \Omega}} u(q) = \phi(p)$$

In particular, if $\partial\Omega$ is regular and $\phi : \partial\Omega \to \mathbb{R}$ is continuous, then u is a solution of Dirichlet's problem on Ω with boundary data ϕ.

PROOF. Let $\mathcal{S}_\phi$ be as in Proposition 9.7. Recall that

$$\inf_{\partial\Omega} \phi \leq v \leq \sup_{\partial\Omega} \phi \tag{9.9}$$

for any $v \in \mathcal{S}_\phi$. We now claim the following: given $\varepsilon > 0$ there exists $C = C(\varepsilon)$ such that

$$v(q) - C\beta(q) \leq \phi(p) + \varepsilon \quad \forall\, q \in \Omega \tag{9.10}$$

for any $v \in \mathcal{S}_\phi$. To prove this, we let D be a small parametric disk centered at p. It can be chosen so that

$$\sup_{\partial(\Omega\cap D)} v - C\beta \leq \phi(p) + \varepsilon$$

due to the continuity of ϕ, the positivity

$$\min_{\overline{\Omega}\setminus D} \beta > 0,$$

and provided C is large enough. The maximum principle now shows that (9.10) holds on $\Omega \cap D$. On $\overline{\Omega} \setminus D$, we let C be so large that (9.10) holds due to (9.9).
In conclusion,

$$\limsup_{q\to p} u(q) \leq \phi(p) + \varepsilon$$

For the lower bound, we observe by the same arguments that

$$-C\beta + \phi(p) - \varepsilon \in \mathcal{S}_\phi$$

Hence,

$$u \geq -C\beta + \phi(p) - \varepsilon$$

so that

$$\liminf_{q\to p} u(q) \geq \phi(p) - \varepsilon$$

as desired. $\square$

We remark that the regularity of $\partial\Omega$ is also necessary for the solvability of the Dirichlet problem for general continuous boundary data; indeed, the boundary data $f(p) = |p - p_0|$ yields a barrier.

Let us make another remark concerning solving the Dirichlet problem *outside* some compact set $K \subset M$. As the example $K = \mathbb{D} \subset 2\mathbb{D}$ shows, we cannot expect unique solvability of the Dirichlet problem with data on ∂K. However, the Perron method always yields existence of bounded harmonic functions. The following result is a corollary of the proof of the preceding proposition.

COROLLARY 9.13. *Let $K \subset M$ be compact and ∂K regular. Then for any $\phi : \partial K \to \mathbb{R}$ continuous and any constant $A \geq \max_{\partial K} \phi$ there exists a harmonic function u on $\Omega := M \setminus K$ with $u \in C(\overline{\Omega})$, $u = \phi$ on $\partial\Omega$ and*

$$\min_{\partial\Omega} \phi \leq u \leq A$$

on Ω.

PROOF. Define

$$u := \sup\{v \mid v \in \mathfrak{sh}(\Omega),\ v \ll \phi \text{ on } \partial\Omega,\ v \leq A\} \tag{9.11}$$

The set on the right-hand side is a non-empty Perron family and u is harmonic on Ω and satisfies (9.11). Let β_0 be a barrier at $p \in \partial\Omega$. Let D, D' be parametric disks centered at p and D' compactly contained in D and $\bar{D}$ compact. Then for $\varepsilon > 0$ sufficiently small, the function

$$\beta := \min\{\beta_0, \varepsilon\} \text{ on } D \cap \Omega$$

is superharmonic on D with the property that $\beta = \varepsilon$ on $\Omega \cap D \setminus D'$. This shows that we can extend β to all of Ω by setting

$$\beta = \varepsilon \text{ on } \Omega \setminus D$$

The point is that we have constructed a barrier β at p which is uniformly bounded away from zero on $\Omega \setminus D$ (this is another expression of the fact that being regular is a local property around a point). Since u is bounded from above and below, the reader will have no difficulty verifying that the exact same proof as in Proposition 9.12 applies in this case. □

In the following two chapters it will become clear that the solution constructed in Corollary 9.13 is unique iff M does not admit a negative nonconstant subharmonic function (or in the terminology of the following chapter, if M is not hyperbolic). An example would be $M = \mathbb{C}$ (the reader is invited to establish uniqueness in Corollary 9.13 in that case). Note that this uniqueness is clear (as is the existence from Proposition 9.12) if M is compact. From the classification that we develop in the following two chapters it will become clear that uniqueness in Corollary 9.13 with M not compact holds iff M is conformally equivalent to $\mathbb{C}$ while it does not hold iff M is conformally to $\mathbb{D}$.

To summarize, we have solved the Dirichlet problem for all domains $\Omega \subset M$ with compact closure and regular boundary. In particular, if $M = \mathbb{C}_\infty$, any such domain admits a Green function. Moreover, if $\Omega \subset \mathbb{C}$ is simply connected, then G gives rise to a biholomorphic $f : \Omega \to \mathbb{D}$. This latter fact (the Riemann mapping theorem) we proved earlier in a completely different way which did not require any information on the boundary.

CHAPTER 10

Green functions and the classification problem

1. Green functions on Riemann surfaces

We would like to generalize the proof of the Riemann mapping theorem from the previous chapter to any Riemann surface M which admits a Green function. But what is the correct definition of a Green function G on M? Since there is no boundary, at least in the topological sense, we need to find a substitute for the the crucial vanishing condition at the boundary. One option would be to require "vanishing at infinity", i.e.,

$$\inf_{K \text{ compact}} \sup_{p \in M \setminus K} |G(p,q)| = 0$$

However, this turns out to be too restrictive. As an example, consider $M = \mathbb{D} \setminus \{0\}$. "Infinity" here is the set $\{0\} \cup \{|z| = 1\}$ but we cannot enforce vanishing at $\{0\}$. However, the Green function on $\mathbb{D}$ is — in a very precise sense — also the Green function of $\mathbb{D} \setminus \{0\}$. In fact, uniquely so, as we shall see. The issue here is that a single point is negligible (more generally, sets of *zero logarithmic capacity* are negligible). While it is of course true that this M is not simply connected, it would be unwise to introduce simple connectivity into the concept of the Green function.

As often in analysis, the correct definition of a Green function on M imposes a *minimality* condition on G. Following a time-honored tradition, we will consider *positive* Green functions rather than negative ones. Of course, this just amounts to switching the sign. In addition, we drop the factor of 2π.

Definition 10.1. *By a* Green function *with singularity at $q \in M$ we mean a real-valued function $G(p,q)$ defined on $M \setminus \{q\}$ such that*

- *$G(p,q) + \log|z|$ is harmonic locally around $p = q$ where z are local coordinates near q with $z(q) = 0$*
- *$p \mapsto G(p,q)$ is harmonic and positive on $M \setminus \{q\}$*
- *if $g(p,q)$ is any other function satisfying the previous two conditions, then $g(\cdot,q) \geq G(\cdot,q)$ on $M \setminus \{q\}$.*

It is evident that G is unique if it exists. Also, if $f : N \to M$ is a conformal isomorphism, then it is clear that $G(f(p), f(q))$ is the Green function on N with singularity at q. By the maximum principle, if G is a Green function as in the previous chapter, then $-G$ satisfies Definition 10.1. We remark that no compact surface M admits such a Green function (since $-G(\cdot,q)$ would then be a negative subharmonic function on M and therefore constant by the maximum principle). Note that $M = \mathbb{C}$ does not admit a Green function either:

Lemma 10.2. *Suppose $u < \mu$ is a subharmonic function on $\mathbb{C}$ with some constant $\mu < \infty$. Then $u = \text{const}$.*

PROOF. Let us first observe the following: suppose v is subharmonic and negative on $0 < |z| < 2$ and set $v_\varepsilon(z) := v(z) + \varepsilon \log |z|$ where $0 < \varepsilon < 1$. Then v_ε is subharmonic on $0 < |z| < 1$. Moreover, $v_\varepsilon(z) = v(z)$ for all $|z| = 1$ and $v_\varepsilon(z) \to -\infty$ as $z \to 0$. It follows from the maximum principle that $v_\varepsilon(z) \le \max_{|z|=1} v(z) < 0$ for all $0 < |z| < 1$. Now send $\varepsilon \to 0$ to conclude that $v(z) \le \max_{|z|=1} v(z) < 0$ for all $0 < |z| \le 1$.

To prove the lemma, we may assume that $u < 0$ everywhere and $\sup_{\mathbb{C}} u = 0$. Consider $u(1/z)$ on $0 < |z| < 2$. It is subharmonic and negative and therefore by the preceding paragraph

$$\sup_{|z|\ge 1} u(z) < 0.$$

It follows that $\sup_{|z|\le 1} u(z) = 0$ which is impossible. $\square$

2. Hyperbolic Riemann surfaces admit Green functions

So which M *do* admit Green functions? As we saw, $\mathbb{D}$ and thus any domain in $\mathbb{C}$ conformally equivalent to it. Note that these surfaces obviously admit negative nonconstant subharmonic functions. This suggests a classification:

DEFINITION 10.3. *A Riemann surface M is called* hyperbolic *iff it carries a nonconstant negative subharmonic function. If M is not hyperbolic and noncompact, then it is called* parabolic*.*

The logic here is as follows: using the exact same proof idea as in the Theorem 2 of the previous chapter we will show that the hyperbolic, simply connected surfaces are conformally equivalent to $\mathbb{D}$, whereas Riemann-Roch showed that the compact simply connected ones are conformally equivalent to $\mathbb{C}P^1$. This leaves the simply connected parabolic surfaces, and — you have guessed it — they are conformally equivalent to $\mathbb{C}$.

In conclusion, every simply connected Riemann surface is conformally equivalent to either $\mathbb{D}, \mathbb{C}$ or $\mathbb{C}P^1$. In the non-simply connected case, one then passes to the universal covering of M, which is again a Riemann surface $\widetilde{M}$, and proves that M is obtained from $\widetilde{M}$ by factoring by the covering group. All of these facts constitute the so-called *uniformization theorem*. It is of course a famous and central result of the field.
Now for the hard work of constructing Green functions as in Definition 10.1. We will do this via a Perron-type argument by setting

$$G(p, q) := \sup_{v \in \mathcal{G}_q} v(p), \tag{10.1}$$

the supremum being taken over the family $\mathcal{G}_q$ that we now define.

DEFINITION 10.4. *Given any $q \in M$ we define a family $\mathcal{G}_q$ of functions as follows:*

- *any v in $\mathcal{G}_q$ is subharmonic on $M \setminus \{q\}$*
- *$v + \log |z|$ is bounded above on U where (U, z) is some chart around q*
- *$v = 0$ on $M \setminus K$ for some compact $K \subset M$*

Since $0 \in \mathcal{G}_q$ we have $\mathcal{G}_q \ne \emptyset$. Note that if $G(p, q)$ is a Green function on some domain $\Omega \subset \mathbb{C}$ in the sense of the previous chapter, then

$$(-G(p, q) - \varepsilon)_+ \in \mathcal{G}_q$$

for any $\varepsilon > 0$. As another example, let $M = \mathbb{C}$ and $q = 0$. Then $-\log_-(|z|/R) \in \mathcal{G}_0$ for any $R > 0$. This shows that $G(p, 0)$ as defined in (10.1) satisfies $G(p, 0) = \infty$ for all

$p \in \mathbb{C}$. As we shall see shortly, this agrees with the fact that $\mathbb{C}$ does not admit a negative nonconstant subharmonic function. In general, one has the following result.

THEOREM 10.5. *Let $q \in M$ be fixed and let $G(p,q)$ be defined as in* (10.1). *Then either $G(p,q) = \infty$ for all $p \in M$ or $G(p,q)$ is the Green function of M with singularity at q. Moreover,*

$$\inf_{p\in M} G(p,q) = 0$$

PROOF. Observe that $\mathcal{G}_q$ is a Perron family. Hence, by the methods of the previous chapter, either $G(\cdot,q) = \infty$ identically or it is harmonic.
Next, we need to check that $p \mapsto G(p,q) + \log|z(p)|$ is harmonic locally near $p = q$. In fact, it suffices to check that

$$G(p,q) = -\log|z| + O(1) \quad \text{as } p \to q \tag{10.2}$$

where $z = z(p)$ since $G(p,q) + \log|z(p)|$ then has a removable singularity at $p = q$ as a harmonic function. If $v \in \mathcal{G}_q$, then locally around q and for any $\varepsilon > 0$,

$$v(p) + (1+\varepsilon)\log|z(p)|$$

is subharmonic and tends to $-\infty$ as $p \to q$. Therefore, by the maximum principle, for any $p \in z^{-1}(\mathbb{D} \setminus \{0\})$,

$$v(p) + (1+\varepsilon)\log|z(p)| \leq \sup_{z^{-1}(\partial\mathbb{D})} v \leq \sup_{z^{-1}(\partial\mathbb{D})} G(\cdot,q) =: k(q)$$

Hence, locally around q,

$$G(p,q) \leq -\log|z(p)| + k(q)$$

For the reverse direction, simply note that

$$v(p) = \log_+(1/|z(p)|) \in \mathcal{G}_q$$

Let $\mu = \inf_{p\in M} G(p,q) \geq 0$. If $v \in \mathcal{G}_q$, then outside some compact set K, and with $\varepsilon > 0$ arbitrary

$$v = 0 \leq G(p,q) - \mu$$

whereas

$$(1-\varepsilon)v(p) \leq G(p,q) - \mu \quad \text{as} \quad p \to q$$

By the maximum principle,

$$(1-\varepsilon)v \leq G(\cdot,q) - \mu \quad \text{on } M \setminus \{q\}$$

Letting $\varepsilon \to 0$ and by the definition of G, $G(\cdot,q) \leq G(\cdot,q) - \mu$ which implies that $\mu \leq 0$ and thus $\mu = 0$ as claimed.

Finally, suppose that $g(\cdot,q)$ satisfies the first two properties in Definition 10.1. Then for any $v \in \mathcal{G}_q$, and any $0 < \varepsilon < 1$,

$$(1-\varepsilon)v \leq g(\cdot,q)$$

by the maximum principle. It follows that $G \leq g$ as desired. □

Next, we establish the connection between M being hyperbolic and M admitting a Green function. This is subtle and introduces the important notion of *harmonic measure*.

THEOREM 10.6. *For any Riemann surface M the following are equivalent:*

- *M is hyperbolic*
- *the Green function $G(\cdot,q)$ with singularity at q exists for some $q \in M$*

- *the Green function $G(\cdot, q)$ with singularity at q exists for each $q \in M$*

Proof. We need to prove that a hyperbolic surface admits a Green function with an arbitrary singularity. The ideas are as follows: we need to show that $G(p,q) < \infty$ if $p \neq q$ which amounts to finding a "lid" for our family $\mathcal{G}_q$. In other words, we need to find a function, say $w_q(p)$, harmonic or superharmonic on $M \setminus \{q\}$ and positive there, and so that $w_q(p) = -\log|z(p)| + O(1)$ as $p \to q$. Indeed, in that case we simply observe that $v \leq w$ for every $v \in \mathcal{G}_q$. Of course, G itself is such a choice if it exists – so realistically we can only hope to make w_q superharmonic. So we need to find a subharmonic function v_1 (which would be $-w_q$) which is bounded from above and has a $\log|z(p)| + O(1)$ type singularity as $p \to q$. By assumption, there exists a negative subharmonic function v_0 on M. It, of course, does not fit the description of v_1 since it does not necessarily have the desired logarithmic singularity. So we shall need to "glue" $\log|z(p)|$ in a chart around q to a subharmonic function like v_0 which is bounded from above. However, it is hard to glue subharmonic functions. Instead, we will produce a harmonic function u that vanishes on the boundary of some parametric disk D and which is positive[1] on $M \setminus D$ (by solving the Dirichlet problem outside of D). The crucial property of u is its positivity on $M \setminus D$ and this is exactly where we invoke the nonconstancy of v_0.

The details are as follows. Pick any $q \in M$ and a chart (U, z) with $z(q) = 0$. We may assume that $D_2 := z^{-1}(2\mathbb{D})$ and its closure are contained in U. Set $D_1 := z^{-1}(\mathbb{D})$. Consider the Perron family $\mathcal{F}$ of all $v \in \mathfrak{sh}(M \setminus \bar{D}_1)$ with $v \ll 0$ on ∂D_1 and such that $0 \leq v \leq 1$ on $M \setminus D_1$. By Corollary 9.13, $u := \sup_{v \in \mathcal{F}} v$ is continuous on $M \setminus D_1$ with $u = 0$ on ∂D_1 and $0 \leq u \leq 1$. We claim that $u \not\equiv 0$. To this end, let $v_0 < 0$ be a nonconstant subharmonic function on M and set $\mu = \max_{\bar{D}_1} v_0$. Then $\mu < 0$ and

$$1 - \frac{v_0}{\mu} = 1 + \frac{v_0}{|\mu|} \in \mathcal{F}$$

Hence, $|\mu| + v_0 \leq |\mu| u$. By non-constancy of v_0,

$$\max_{\bar{D}_2} v_0 > \mu$$

so that $u > 0$ somewhere and therefore $u > 0$ everywhere on $M \setminus \bar{D}_1$. We shall now build a subharmonic function v_1, globally defined on M and bounded above, and such that v_1 behaves like $\log|z|$ around q. In fact, define

$$v_1 := \begin{cases} \log|z| & \forall\, |z| \leq 1 \\ \max\{\log|z|, ku\} & \forall\, 1 \leq |z| \leq 2 \\ ku & \forall\, z \in M \setminus D_2 \end{cases}$$

Here the constant $k > 0$ is chosen such that

$$ku > \log 2 \quad \forall\, |z| = 2$$

Due to this property, and the fact that $u = \log|z| = 0$ on $|z| = 1$, $v_1 \in C(M)$. Moreover, checking in charts reveals that v_1 is a subharmonic function off the circle $|z| = 1$. Since the sub-mean value property holds locally at every $|z| = 1$ we finally conclude that v_1 is subharmonic everywhere on M.

We are done: Indeed, any $v \in \mathcal{G}_q$ (see Definition 10.4) satisfies

$$v \leq \nu - v_1$$

[1] $1 - u$ is called the harmonic measure of ∂D relative to $M \setminus D$.

Hence, $G(p,q) \le \nu - v_1(p) < \infty$ for any $p \in M \setminus \{q\}$. □

The previous proof shows that if some compact parametric disk admits a harmonic measure, then M is hyperbolic. Let us now elucidate the important symmetry property of the Green function. We already encountered it in the previous chapter as part of the Riemann mapping theorem. However, it has nothing to do with simple connectivity as we will now see.

We begin with the following simple observation.

LEMMA 10.7. *Let M be hyperbolic and $N \subset M$ be a sub-Riemann surface with piece-wise C^2 boundary*[2] *and $\bar{N}$ compact. Then N is hyperbolic, $G_N \le G$, and $G_N(p,q) = G_N(q,p)$ for all $p,q \in N$.*

PROOF. Fix any $q \in N$ and let u_q be harmonic on N, continuous on $\bar{N}$ and with boundary data $-G(\cdot,q)$. This can be done by the results of the previous chapter. Then

$$G_N(p,q) := G(p,q) + u_q(p)$$

is the Green function on N. It follows from the maximum principle that $G_N \le G$. To prove the symmetry property, fix $p \neq q \in N$ and let $N' = N \setminus D_1 \cup D_2$ where $D_1, D_2 \subset N$ are parametric disks around p,q, respectively. Define

$$u = G_N(\cdot,p), \qquad v = G_N(\cdot,q)$$

Then, by Green's formula on N,

$$0 = \int_{\partial N'} u *dv - v *du = -\int_{\partial D_1 \cup \partial D_2} u *dv - v *du$$

Again by Green's formula, but this time on D_1 with local coordinates z, centered at p ($z(p) = 0$),

$$\begin{aligned}\int_{\partial D_1} u *dv - v *du &= \int_{\partial D_1} (u + \log|z|) *dv - v *d(u + \log|z|) \\ &\quad - \int_{\partial D_1} \log|z| *dv - v *d\log|z| \\ &= G_N(p,q)\end{aligned}$$

and similarly

$$\int_{\partial D_2} u *dv - v *du = -G_N(q,p)$$

as desired. □

To obtain the symmetry of G itself we simply take the supremum over all N as in the lemma. We will refer to those N as *admissible*.

PROPOSITION 10.8. *Let M be hyperbolic. Then the Green function is symmetric: $G(p,q) = G(q,p)$ for all $q \neq p \in M$.*

PROOF. Fix $q \in M$ and consider the family

$$\mathcal{F}_q = \{G_N(\cdot,q) \mid q \in N,\ N \text{ is admissible}\}$$

[2]This means that we can write the boundary as a finite union of C^2 curves $\gamma : [0,1] \to M$.

where we extend each G_N to be zero outside of N. This extension is subharmonic on $M \setminus \{q\}$ and $\mathcal{F}_q$ is a Perron family on $M \setminus \{q\}$:

$$\max\{G_{N_1}(\cdot,q), G_{N_2}(\cdot,q)\} \le G_{N_1 \cup N_2}(\cdot,q)$$

and $G_{N\cup D}(\cdot,q) \ge G_N(\cdot,q)$ for any parametric disk $D \subset M\setminus\{q\}$ with $G_{N\cup D}(\cdot,q)$ harmonic on D. Note that both $N_1 \cup N_2$ and $N \cup D$ are admissible. Let

$$g(\cdot,q) := \sup_{v\in\mathcal{F}_q} v \le G(\cdot,q)$$

Moreover, it is clear that

$$g(\cdot,q) \ge \sup_{v\in\mathcal{G}_q} v = G(\cdot,q)$$

Indeed, simply use that every compact $K \subset M$ is contained in an admissible N (take N to be the union of a finite open cover by parametric disks). In conclusion, $g(p,q) = G(p,q)$ which implies that $G_N(q,p) = G_N(p,q) \le G(p,q)$ for all admissible N. Hence, taking suprema, $G(q,p) \le G(p,q)$ and we are done. $\square$

3. Problems

PROBLEM 10.1. *Prove that $\mathbb{C} \setminus \{z_j\}_{j=1}^J$ is not hyperbolic in the sense of Chapter 10. Show that $\mathbb{C} \setminus (\mathbb{D} \cup \{z_j\}_{j=1}^J)$ is hyperbolic.*

CHAPTER 11

The uniformization theorem

1. The statement for simply connected surfaces

We can now state and prove the following remarkable classification result.

THEOREM 11.1. *Every simply connected surface M is conformally equivalent to either $\mathbb{C}P^1$, $\mathbb{C}$, or $\mathbb{D}$. These correspond exactly to the compact, parabolic, and hyperbolic cases, respectively.*

The compact case was already done via Riemann-Roch, whereas for the hyperbolic case we will employ the Green function technique that lead to a proof of the Riemann mapping theorem in Chapter 9.

2. Hyperbolic, simply connected, surfaces

PROOF OF THEOREM 11.1 IN THE HYPERBOLIC CASE. Let $q \in M$ and $G(p,q)$ be the Green function with singularity at q. Then there exists f_q holomorphic on M with

$$|f_q(p)| = \exp(-G(p,q)) \quad \forall\, q \in M$$

with the understanding that $f_q(p) = 0$. This follows from gluing technique of Lemma 5.5 since such a representation holds locally everywhere on M (alternatively, apply the monodromy theorem). Now $f_q : M \to \mathbb{D}$ with $f_q(p) = 0$ iff $p = q$. It remains to be shown that f_q is one-to-one since then $f_q(M)$ is a simply connected subset of $\mathbb{D}$ and therefore, by the Riemann mapping theorem, conformally equivalent to $\mathbb{D}$.

We proceed as in the planar case, see Theorem 9.3. Thus, let $p \in M$ with $q \neq p$ and T a Möbius transform with $T(f_q(p)) = 0$. We claim that $|T \circ f_q| = |f_p|$. This will then show that f_q is one-to-one (suppose $f_q(p) = f_q(p')$, then by the claim, $|f_p| = |f_{p'}|$ and thus $f_p(p') = 0$ and $p = p'$).

To prove the claim, we observe that

$$w_q := -\log|T \circ f_q| \geq G(\cdot, q) \;\text{ on } M \setminus \{q\}$$

Indeed, locally around $p = q$ we have, for some integer $k \geq 1$,

$$\log|T \circ f_q(p)| \leq k \log|z(p) - z(q)| + O(1) \;\text{ as } p \to q$$

where z are any local coordinates around q. In addition, $w_q > 0$ everywhere on M. From these properties we conclude via the maximum principle that

$$w_q \geq v \quad \forall\, v \in \mathcal{G}_q \Longrightarrow w_q \geq G(\cdot, q)$$

as desired (here $\mathcal{G}_q$ is the family from Definition 10.4). Since

$$-G(p,q) = \log|f_q(p)| = \log|T(0)| = \log|(T \circ f_q)(q)| \leq -G(q,p) = -G(p,q)$$

we obtain from the maximum principle that $w_q = G(\cdot, p)$ whence the claim. $\square$

3. Parabolic, simply connected, surfaces

It remains for us to understand the parabolic case. The logic is as follows with M simply connected: in the compact case we established the existence of a meromorphic function with a simple pole. This followed from the Riemann-Roch theorem which in turn was based (via a counting argument) on the existence of meromorphic differentials with a prescribed $\frac{dz}{z^2}$ singularity at a point.

In the hyperbolic case, we were able to place a positive harmonic function on M with a $-\log|z|$ singularity at a given point – in fact, the hyperbolic M are precisely the surfaces that allow for this. Amongst all such harmonic function we selected the minimal one (the Green function) and constructed a conformal equivalence from it.

For the parabolic case we would like to mimic the compact case by constructing a meromorphic function as before. In view of the fact that we are trying to show that M is equivalent to $\mathbb{C}$, and therefore compactifiable by the addition of one point this is reasonable. Assuming therefore that $f : M \to \mathbb{C}P^1$ is meromorphic and one-to-one, note that f cannot be onto as otherwise M would have to be compact. Without loss of generality, we can thus assume that $f : M \to \mathbb{C}$. If f were not onto $\mathbb{C}$, then by the Riemann mapping theorem we could make $f(M)$ and thus M equivalent to $\mathbb{D}$. But this would mean that M is hyperbolic. So it remains to find a suitable meromorphic function on M for which we need several more technical ingredients. The first is the maximum principle outside a compact set which establishes uniqueness in Corollary 9.13.

Proposition 11.2. *Let M be a parabolic Riemann surface and $K \subset M$ compact. Suppose u is harmonic and bounded above on $M \setminus K$, and $u \ll 0$ on ∂K. Then $u \le 0$ on $M \setminus K$.*

Proof. We have already encountered this idea in the proof of Theorem 10.6. There K was a parametric disk and we proved that a hyperbolic surface does admit what is called a harmonic measure of K — here we are trying to prove the non-existence of a harmonic measure. Since the latter was shown there to imply the existence of a Green function, we are basically done. The only issue here is that K does not need to be a parametric disk so we have to be careful when applying the Perron method because of ∂K not necessarily being regular. However, this is easily circumvented.

Suppose the proposition fails and let $u > 0$ somewhere on $M \setminus K$. Extend u to M by setting $u = 0$ on K. Then $K \subset \{u < \varepsilon\}$ is an open neighborhood of K for every $\varepsilon > 0$. This implies that for some $\varepsilon > 0$ we have $u_0 := (u-\varepsilon)_+ \in \mathfrak{sh}(M)$ and $u_0 > 0$ somewhere. Moreover, $\{u_0 = 0\} \supset \bar{D}$ for some parametric disk D. Now define

$$\mathcal{F} =: \{v \in \mathfrak{sh}(M \setminus \bar{D}) \mid v \ll 0 \text{ on } \partial D,\ 0 \le v \le 1\}$$

It is a Perron family, and from Chapter 9 we infer that

$$w := \sup_{v \in \mathcal{F}} v$$

is harmonic on $M \setminus \bar{D}$ with $w = 0$ on ∂D and $0 \le w \le 1$ on $M \setminus D$. Since $u_0 \in \mathcal{F}$ we further conclude that $w > 0$ somewhere and thus everywhere on $M \setminus D$. As in the proof of Theorem 10.6 we are now able to construct a Green function with singularity in D, contrary to our assumption of M being parabolic. $\square$

It is instructive to give an independent proof of this fact for the case of $\mathbb{C}$: we may assume that $0 \in K$. Then $v(z) := u(1/z)$ is harmonic on $\Omega := \{1/z \mid z \in \mathbb{C} \setminus K\}$ and

$v \ll 0$ on $\partial\Omega \setminus \{0\}$. Moreover, v is bounded above on Ω. Given $\delta > 0$ there exists R large so that for all $\varepsilon > 0$

$$v(z) \leq \delta - \varepsilon \log(|z|/R) \quad \forall\, z \in \Omega$$

Sending $\varepsilon \to 0$ and then $\delta \to 0$ shows that $v \leq 0$ on Ω as claimed.

Now let us make another observation concerning any u harmonic and bounded on $\mathbb{C} \setminus K$ with K compact: for any $R > 0$ with $K \subset \{|z| < R\}$ we have

$$\int\limits_{|z|=R} \frac{\partial u}{\partial n}\, d\sigma = 0 \tag{11.1}$$

Indeed, $v(z) = u(1/z)$ is bounded and harmonic on $0 < |z| < \frac{1}{R} + \varepsilon$ for some $\varepsilon > 0$. By Problem 3.3, v is necessarily harmonic on a neighborhood of zero. Hence,

$$0 = \iint\limits_{|z| \leq \frac{1}{R}} \Delta v\, dxdy = \oint\limits_{|z|=\frac{1}{R}} \frac{\partial v}{\partial n}\, d\sigma$$

and (11.1) follows. An analogous result holds on any parabolic Riemann surface, but the previous proof in $\mathbb{C}$ does not generalize to that setting. Let us give one that does generalize: Without loss of generality $K \subset \mathbb{D}$. For any $R > 1$ denote by ω_R the harmonic function so that $\omega = 1$ on $\{|z| = 1\}$ and $\omega = 0$ on $\{|z| = R\}$ (harmonic measure). This exists of course by Perron but we even have an explicit formula:

$$\omega(z) = \frac{\log(R/|z|)|}{\log R}, \quad 1 \leq |z| \leq R$$

Then, by Stokes' theorem,

$$0 = \oint\limits_{|z|=1} \frac{\partial u}{\partial r}\, d\sigma + \oint\limits_{|z|=R} u \frac{\partial \omega}{\partial r}\, d\sigma - \oint\limits_{|z|=1} u \frac{\partial \omega}{\partial r}\, d\sigma$$

Since $\frac{\partial \omega}{\partial r} = \frac{-1}{r \log R}$, it follows upon sending $R \to \infty$ that

$$0 = \oint\limits_{|z|=1} \frac{\partial u}{\partial r}\, d\sigma$$

as desired. This proof can be made to work on a general parabolic Riemann surface and we obtain the following result.

Lemma 11.3. *Let D be a parametric disk on a parabolic surface M and suppose u is harmonic and bounded on $M \setminus D$. If $u \in C^1\big(\overline{M \setminus D}\big)$, then*

$$\int_{\partial D} *du = 0$$

Proof. We say that $N \subset M$ is admissible if $\bar{N}$ is compact, $\bar{D} \subset N$, and ∂N is piecewise C^2. Then by ω_N we mean the harmonic function on $N \setminus \bar{D}$ so that $\omega = 1$ on ∂D and $\omega = 0$ on ∂N. We claim that

$$\mathcal{F} := \{\omega_N \mid N \text{ admissible}\}$$

is a Perron family on $M \setminus \bar{D}$ where we set each $\omega_N = 0$ on $M \setminus N$. In this way, each ω_N becomes subharmonic on $M \setminus \bar{D}$. To verify that $\mathcal{F}$ is a Perron family, observe that from the maximum principle,

$$\begin{aligned} \max\{\omega_{N_1}, \omega_{N_2}\} &\le \omega_{N_1 \cup N_2} \\ (\omega_N)_K &\le \omega_{N \cup K} \end{aligned}$$

where $K \subset M \setminus \bar{D}$ is any parametric disk in the second line. Since $N_1 \cup N_2$ and $N \cup K$ are again admissible, $\mathcal{F}$ is indeed such a family and

$$\omega_\infty := \sup_{v \in \mathcal{F}} v$$

is harmonic on $M \setminus D$ with $0 \le \omega_\infty \le 1$ and $\omega_\infty = 1$ on ∂D. Apply the maximum principle for parabolic surfaces as given by Proposition 11.2 to $1 - \omega_\infty$ yields $\omega_\infty = 1$ everywhere on $M \setminus D$.

Returning to any admissible N as above, we infer from Stokes theorem that

$$0 = \int_{\partial(N \setminus D)} \omega_N * du - u * d\omega_N$$

or, with suitable orientations,

$$\int_{\partial D} * du = - \int_{\partial N} u * d\omega_N + \int_{\partial D} u * d\omega_N \tag{11.2}$$

It is clear that $*d\omega_N$ is of definite sign on both ∂D and ∂N. Indeed, on these boundaries this differential form, evaluated at a tangent vector $\vec{e}$ to the boundary, is the directional derivative of ω_N along $\vec{e}^\perp$ (with a fixed sense of orientation along the boundary). Furthermore, again by Stokes,

$$\int_{\partial D} * d\omega_N = \int_{\partial N} * d\omega_N$$

In view of (11.2) and the boundedness of u it therefore suffices to show that

$$\inf_{N \text{ admissible}} \left| \int_{\partial D} * d\omega_N \right| = 0$$

However, this follows immediately from the fact that $\omega_\infty = 1$ everywhere on $M \setminus D$. $\square$

CHAPTER 12

Hints and Solutions

Solutions for Chapter 1

Problem 1.1 This is an ellipse with focal points a, b provided $k > |a-b|$, the line segment $\overline{ab}$ if $k = |a-b|$ and the empty set if $k < |a-b|$. To obtain the familiar equation of an ellipse, assume wlog that $a, b \in \mathbb{R}$ with $a+b=0$. Then square $|z-a| = k - |z+a|$, cancel the $|z|^2$, put the remaining $-2k|z+a|$ on one side and square again.

(b) For a fixed, $z \mapsto w$ is a fractional linear transformation. Hence it preserves circles. Observe that for $|z|=1$ one has $|w| = \left|\frac{z-a}{\bar{z}-\bar{a}}\right| = 1$. Since also $a \mapsto 0$, we see that $|w| < 1$ is equivalent with $|z| < 1$, $|w| = 1$ with $|z| = 1$, and $|w| > 1$ with $|z| > 1$.

Problem 1.2 With $P(z) = a_n \prod_{j=1}^n (z - z_j)$, ones has

$$P^*(z) = z^n \overline{P(\bar{z}^{-1})} = \overline{a}_n \prod_{j=1}^n (1 - \bar{z}_j z).$$

Hence, $P(z) + e^{i\theta} P^*(z) = 0$ implies that

$$\prod_{j=1}^n |z - z_j| = \prod_{j=1}^n |1 - \bar{z}_j z|$$

Hence, by 1.(b) we must have $|z| = 1$ as claimed.

Problem 1.3 Obviously, $p(1) > 0$. Suppose $|z| \le 1$, $z \ne 1$. Then multiply $P(z)$ with $1-z$:

$$(1-z)p(z) = p_0 - [(p_0 - p_1)z + (p_1 - p_2)z^2 + \ldots + (p_{n-1} - p_n)z^n + p_n z^{n+1}]$$

Since $p_j - p_{j+1} > 0$ for $0 \le j < n$ and $p_n > 0$, and since $z, z^2, \ldots, z^{n+1}$ are at most of length one but not all the same, we conclude that

$$\begin{aligned} &|(p_0 - p_1)z + (p_1 - p_2)z^2 + \ldots + (p_{n-1} - p_n)z^n + p_n z^{n+1}| \\ &< (p_0 - p_1) + (p_1 - p_2) + \ldots + (p_{n-1} - p_n) + p_n = p_0 \end{aligned}$$

which implies that $p(z) \ne 0$ in that case also.

Problem 1.4 (a) A very elegant elementary geometry proof is as follows: Suppose $\pi, \pi' \subset \mathbb{R}^3$ are planes that meet in some line ℓ_0 and let π_m be the plane through ℓ_0 which bisects the angle between π, π'. Now suppose ℓ is a line perpendicular to π_m. Then it is clear that any two other planes A, B which meet in π_m intersect at the same angle in π as they do in π'. Now apply this to the stereographic projection as follows: π is the plane $\{x_3 = 0\}$, and π' the tangent plane to S^2 at $X \in S^2$. Then consider the line ℓ through the north pole $(0,0,1)$ and the point X. Convince yourself by means of a figure that ℓ

is perpendicular to the bisector of π and π'. You can of course assume that $x_2 = 0$, say, so that everything reduces to a planar figure.

(b) Let $Z = (x_1, x_2, x_3) = \Phi^{-1}(z)$ and $W = (y_1, y_2, y_3) = \Phi^{-1}(w)$ so that

$$|Z - W|^2 = 2 - 2(x_1y_1 + x_2y_2 + x_3y_3)$$

Now express Z via z:

$$x_1 = \frac{z + \bar{z}}{1 + |z|^2},\ x_1 = \frac{-i(z - \bar{z})}{1 + |z|^2},\ x_1 = \frac{|z|^2 - 1}{1 + |z|^2}$$

and similarly for W and w to arrive at

$$d(z, w) = |Z - W| = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}$$

(c) By definition, a circle γ in S^2 is the intersection of S^2 with a plane

$$x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = k$$

Without loss of generality, $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$. In other words, the normal vector (ξ_1, ξ_2, ξ_3) corresponds to a point $w \in \mathbb{C}$ under stereographic projection (we can assume that $\xi_3 \neq 1$ for otherwise γ is the equator which goes to itself). Then by part (b) we see that $d(z, w) = \text{const}$ on γ. Using the formula for $d(z, w)$ from above we see that for fixed w, the locus of points z for which $d(z, w)$ is some given constant is a circle (or line). A line is obtained iff $(0, 0, 1) \in \gamma$.

Problem 1.5 This is done in steps: first translate, then dilate, and translate again. For the second one you also need an inversion. For the final two maps observe that the circles intersect at a right angles at both intersection points. Now choose a Möbius transformation that takes $0 \mapsto 0$ and the second intersection point to ∞ and follow it by a rotation as needed. For the final example, observe that the circles intersect at angle $60°$. So a Möbius transform will take the almond shaped intersection onto the sector $0 < \operatorname{Arg} z < \frac{\pi}{3}$. Now map this by a power $z \mapsto z^\alpha$ onto the first quadrant.

Problem 1.6 Since $0 = \sum_{j=1}^n m_j(z_j - z)$ we may assume without loss of generality that $z = 0$. A line ℓ through the origin is of the form $\operatorname{Re}(ze^{i\theta}) = 0$ for some $\theta \in [0, 2\pi)$. Since

$$0 = \sum_{j=1}^n m_j \operatorname{Re}(z_j e^{i\theta})$$

and all $m_j > 0$ we see that unless $\operatorname{Re}(z_j e^{i\theta}) = 0$ for all j (and thus all $z_j \in \ell$), necessarily $\operatorname{Re}(z_j e^{i\theta}) > 0$ for some j, as well as $\operatorname{Re}(z_j e^{i\theta}) < 0$ for others. This is equivalent with the separation property.

Problem 1.7 (a) True; let $z_j = x_j + iy_j$. Then $\sum_j x_j$ converges and since the terms are positive, also $\sum_j x_j^2$ converges, and since $\sum_j (x_j^2 - y_j^2)$ converges, therefore $\sum_j y_j^2$ converges; hence, finally, $\sum_j (x_j^2 + y_j^2)$ converges.

(b) True. Take $z_j = \frac{e^{2\pi ij\theta}}{\log(1+j)}$ where θ is irrational. The absolute divergence is clear. For the convergence, observe that for any positive integer k,

$$\sup_n \Big| \sum_{j=1}^n e^{2\pi ijk\theta} \Big| < \infty$$

Hence, "summation by parts" produces an absolutely convergent series since the logarithms yield (under the difference operation) a term of the form

$$\frac{1}{j(\log(j+1))^{k+1}}$$

which is summable for any $k \geq 1$.

Problem 1.8 We seek a function $f(x,y) = u + iv$ with the property that $v(x,y) = -v(x,-y)$. In particular, $v = 0$ on the real axis. Then

$$2u(x,y) = f(z) + f(\bar{z}) = \frac{(z+\bar{z})(1+z\bar{z})}{1+z^2+\bar{z}^2+z^2\bar{z}^2} = \frac{z}{1+z^2} + \frac{\bar{z}}{1+\bar{z}^2}$$

Hence $f(z) = \frac{z}{1+z^2}$ is what we were after.

Problem 1.9 Writing $z = re^{i\theta}$ we see that

$$w = \frac{1}{2}(r+r^{-1})\cos\theta + \frac{i}{2}(r-r^{-1})\sin\theta$$

Hence, for fixed $0 < r < 1$ we obtain an ellipse for $w = x + iy$ of the form

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \qquad a = a(r) = \frac{1}{2}(r+r^{-1}),\ b = b(r) = \frac{1}{2}(r^{-1}-r)$$

It passes through the points $(\pm a, 0)$ and $(0, \pm b)$. Now $a(r)$ and $b(r)$ are strictly decreasing as r increases from 0 to 1 and $a(1) = 1$, $b(1) = 0$. On the other hand, $a(0+) = b(0+) = \infty$. Hence, $z \mapsto w$ is a map that takes $|z| < 1$ injectively onto $\mathbb{C}_\infty \setminus [-1,1]$. The half-rays $0 < r < 1$ for fixed θ are take onto hyperbolas which are perpendicular to the family of ellipses by conformality.

By the preceding, the circle $|z| = 1$ is taken onto the segment $[-1,1]$. On the exterior $|z| > 1$ the behavior is deduced from the one on $|z| < 1$ by inversion $z \mapsto z^{-1}$ which leaves the map invariant.

Problem 1.10 (a) The condition is sufficient. Now suppose that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$. Hence $T(0) = x_1, T(1) = x_2, T(\infty) = x_3$ where $x_1, x_2, x_3 \in \mathbb{R}_\infty$. But then

$$T^{-1}(z) = [z : x_1 : x_2 : x_3]$$

has real coefficients and therefore so does T.

(b) $w = \frac{z-i}{z+i}$ takes $z \in \mathbb{R}_\infty$ onto $|w| = 1$. Conversely, $z = i\frac{1+w}{1-w}$ takes $|w| = 1$ onto $z \in \mathbb{R}_\infty$. Using the result from (a), we therefore see that the most general T with $T(\mathbb{T}) = \mathbb{T}$ is of the form

$$i\frac{1+w}{1-w} = \frac{a - ib + z(a+ib)}{c - id + z(c+id)}, \qquad a,b,c,d \in \mathbb{R}$$

By algebra, this reduces to, with $|\zeta| = 1$,

$$w = \zeta\frac{z - z_0}{1 - \bar{z}_0\, z} \tag{12.1}$$

where $z_0 = \frac{b+c+i(a-d)}{b-c-i(a+d)}$ with $a,b,c,d \in \mathbb{R}$. Note that any $z_0 \in \mathbb{C}$ can be written in this form and (12.1) is the most general representation of T. Conversely, with $z_0 \in \mathbb{C}$ arbitrary, any T as in (12.1) takes $\mathbb{T}$ onto itself.

(c) Such a T has to be amongst those from part (b) and thus of the form (12.1). The necessary and sufficient condition here is $|z_0| < 1$. See also 1, (b).

Problem 1.11 a) This is called Schwarz lemma. Apply the maximum principle to $g(z) = \frac{f(z)}{z} \in \mathcal{H}(\mathbb{D})$. Then $|g(z)| \leq 1$ on $\mathbb{D}$ and we would need to have $g = \text{const} = e^{i\theta}$ if $|g(z)| = 1$ anywhere. Since $g(0) = f'(0)$ we are done.

b) This is called Schwarz-Pick lemma. It is reduced to part (a) by two fractional linear transformations: on the domain, T sends $\mathbb{D} \to \mathbb{D}$ with $z_1 \mapsto 0$, and on the image, S sends $\mathbb{D} \to \mathbb{D}$ with $w_1 = f(z_1) \mapsto 0$. Thus,

$$T(z) = \frac{z - z_1}{1 - \bar{z}_1 z}, \qquad S(w) = \frac{w - w_1}{1 - \bar{w}_1 w}$$

and $F = S \circ f \circ T^{-1} : \mathbb{D} \to \mathbb{D}$ with $F(0) = 0$. Now apply part (a).

Problem 1.12 If $f'(z_0) = 0$, then in a small neighborhood of z_0 the function is n-to-one for some $n \geq 2$. This cannot be. The openness of $f(\Omega)$ follows from Corollary 1.22. Furthermore, f^{-1} is conformal since f is (for this we need $f' \neq 0$). Hence f^{-1} is holomorphic on $f(\Omega)$.

b) For $\mathbb{D}$, we may assume that $f(0) = 0$ after composition with a fractional linear transformation. Then $|f'(0)| \leq 1$ by the Schwarz lemma. Looking at f^{-1} we see that also $|f'(0)| \geq 1$. In conclusion, f needs to be a rotation. Thus the automorphisms are exactly the fractional linear transformation $\mathbb{D} \to \mathbb{D}$, see HW set # 1, 9(b).

Reduce $\mathbb{H}$ to $\mathbb{D}$ by a fractional linear transformation. Hence the automorphisms are all fractional linear transformations which preserve $\mathbb{H}$; by HW set #1 they are of the form $\frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$.

Finally, on all of $\mathbb{C}$: they have to be of the form $a + bz$, $a, b \in C$. You can see this easily by viewing the automorphisms as an entire function $f(z)$ with $f(0) = 0$ (wlog) and the considering the function $\frac{1}{f(\frac{1}{z})}$.

Problem 1.13 (a) First, check that the plane and disk model are isometric via the fractional linear map $w = \frac{z-i}{z+i} : \mathbb{H} \to \mathbb{D}$. Since $\operatorname{Im} z = \frac{1-|w|^2}{|1-w|^2}$ it follows that

$$\frac{1}{(\operatorname{Im} z)^2} dz d\bar{z} = \frac{4}{(1 - |w|^2)^2} dw d\bar{w}$$

as claimed. So we can work with the disk model, say. Now note that inequality (1.13) exactly means that a conformal map does not increase length in the tangent space. Therefore, it does not increase the length of curves or the distance of points.

(b) Arguing as in part (a), we see that an isometry needs to attain equality in (1.13) everywhere. In particular, by 1(b) it has to be a fractional linear transformation. Conversely, any such map attains equality in (1.13) everywhere. Hence, the isometries are precisely the automorphisms of $\mathbb{D}$ or $\mathbb{H}$ from above.

(c) Work with the $\mathbb{H}$ model. It is clear that all vertical lines are geodesics since the metric does not depend on $x = \operatorname{Re} z$. All other geodesics are obtained from this one by applying the group of isometries, i.e., the automorphisms (why?). Since those are conformal in $\mathbb{C}_\infty$, the geodesics are circles (or vertical lines) that meet $\mathbb{R}$ at a right angle.

As far as Gaussian (=sectional) curvature is concerned, it is easy to see that it most be constant; indeed, it is a metric invariant (theorema egregium by Carl G.). Since the isometries act transitively, the Gaussian curvature agrees with the value at zero (in $\mathbb{D}$) which you can compute (I think things are normalized so that it comes out as -1). For the full curvature tensor you need to compute Cristoffel symbols, which I leave to you.

Problem 1.14 The idea is the compose f with a fractional linear transformation which takes the right-hand plane onto $\mathbb{D}$ so that $a \mapsto 0$. Thus, set $Tw = \frac{w-a}{w+a}$ and consider $g = T \circ f$. Then $|g'(0)| \le 1$ with equality iff g is a rotation (# 1(a)). This is equivalent to $|f'(0)| \le 2a$ with equality iff $f(z) = a\frac{1+e^{i\theta}z}{1-e^{i\theta}z}$.

Problem 1.15 Use Morera's theorem. Thus, suppose that $\oint_{\partial T} f(z)\,dz \ne 0$ for some triangle $T \subset \Omega$. Then decompose T into 4 triangles $T_j^{(1)}$ by connecting the mid-points of each side. Repeat this to obtain a nested sequence $T_{j_n}^{(n)}$ of triangles whose diameters decrease like 2^{-n}, and thus converge to some point $z_0 \in \Omega$, and such that for some $\varepsilon > 0$

$$\Big| \oint_{T_{j_n}^{(n)}} f(z)\,dz \Big| > 4^{-n}\varepsilon \qquad \forall\, n \ge 1 \tag{12.2}$$

To obtain a contradiction, expand f around z_0:

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|)$$

The the left-hand side of (12.2) would need to be $o(4^{-n})$, which is our desired contradiction.

Solutions for Chapter 2

Problem 2.1 Follows from the maximum principle applied to the function

$$F(z) = f(z) \prod_{j=1}^{m} \frac{1 - \bar{z}_j z}{z - z_j} \in \mathcal{H}(\mathbb{D})$$

Indeed, we see that for any $|z| < 1$,

$$|F(z)| \le M \liminf_{r \to 1-} \sup_{|\zeta|=r} \prod_{j=1}^{m} \Big| \frac{1 - \bar{z}_j z}{z - z_j} \Big| = M$$

(b) Suppose $f(0) \ne 0$. Then from part (a)

$$0 < |f(0)| \le M \prod_{j=1}^{\infty} |z_j| = M \prod_{j=1}^{\infty} (1 - (1 - |z_j|))$$

so that $\sum (1 - |z_j|) < \infty$. If $f(0) = 0$, then one can move a point z_0 where $f(z_0) \ne 0$ to zero by means of an automorphism

$$w = Tz = \frac{z - z_0}{1 - \bar{z}_0 z}$$

All you need to check is that for some constant $C = C(z_0)$,

$$C^{-1} \le \frac{1 - |w|}{1 - |z|} \le C$$

uniformly in the disk. This would then show that $\sum(1 - |Tz_j|) < \infty$ is the same as $\sum(1 - |z_j|) < \infty$, as desired.

Alternatively, and somewhat more elegantly, apply part (a) to the function $g(z) = z^{-\nu} f(z)$ where ν is the order of vanishing at $z = 0$. Then $g \in \mathcal{H}(\mathbb{D})$ is bounded on $\mathbb{D}$ and vanishes at the same points as f as long as they are different from zero. Thus, we can argue as before.

Problem 2.2 If $\omega(z_j) = 0$, then

$$P(z_j) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z_j} \, d\zeta = 0$$

by Cauchy. So the minimal choice for ω is $\omega(z) = \prod_{j=1}^n (z - z_j)$. By inspection, P has degree at most $n-1$.

Problem 2.3 We will not give detailed arguments for each integral, but rather show how to divide these integrals into classes accessible by the same "trick". The reader should verify for herself or himself that the residue theorem does indeed give the stated result in each case.

- $R(x)$ is rational without poles on $\mathbb{R}$ and decaying at least like x^{-2}. Then provided $a \geq 0$,

  $$\int_{-\infty}^{\infty} e^{iax} R(x) \, dx = 2\pi i \sum_{\operatorname{Im} \zeta > 0} e^{ia\zeta} \operatorname{res}(R; \zeta)$$

  where the sum is over the residues; if $a \leq 0$ then we need to sum over all residues in the lower half plane.
- If $R(x)$ is a rational function with a simple pole at ∞ and no poles on $\mathbb{R}$ up to finitely many simple poles $\{x_j\}_{j=1}^p \subset \mathbb{R}$, then

  $$\text{P.V.} \int_{-\infty}^{\infty} e^{ix} R(x) \; dx = 2\pi i \sum_{\operatorname{Im} \zeta > 0} \operatorname{res}(R; \zeta) + \pi i \sum_{j=1}^{p} \operatorname{res}(R; x_j)$$

  An example would be P.V. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx = \pi i$ which has a principle value both around $x = 0$ and $x = \pm\infty$.
- Let $0 < \alpha < 1$ and let $R(x)$ be rational, decaying like x^{-2} at ∞, is analytic at 0 or has a simple pole there, and has no poles on $x > 0$. Then

  $$\int_0^{\infty} x^{\alpha} R(x) \; dx = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \sum_{\zeta \in \mathbb{C} \setminus [0,\infty)} \operatorname{res}(z^{\alpha} R(z); \zeta)$$

  To see this apply the residue theorem to the key-hole contour in $\mathbb{C} \setminus [0, \infty)$.
- For a rational function in cosine and sine compute

  $$\begin{aligned}\int_0^{2\pi} R(\cos\theta, \sin\theta) \, d\theta &= -i \oint_{|z|=1} R\Big(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\Big) \, \frac{dz}{z} \\ &= \oint_{|z|=1} Q(z) \, dz = 2\pi i \sum_{|\zeta|<1} \operatorname{res}(Q; \zeta)\end{aligned}$$

- $$\int_0^{\infty} R(x) \log x \; dx$$

  is computed by means of a semi circular contour with a bump around zero.

Problem 2.4 First,

$$4i\int_0^{\frac{\pi}{2}} \frac{x\,d\theta}{x^2+\sin^2\theta} = \int_{-\pi}^{\pi} \frac{ix\,d\theta}{\sin^2\theta + x^2} = \int_{-\pi}^{\pi} \frac{d\theta}{\sin\theta - ix}$$
$$= \oint_\gamma \frac{2dz}{z^2+2zx-1}$$

where γ is the unit circle and $z = e^{i\theta}$. Inside the contour there is exactly one simple pole $z_0 = -x + \sqrt{1+x^2}$ so that the integral equals

$$\frac{4\pi i}{z_0 + z_0^{-1}} = \frac{2\pi i}{\sqrt{1+x^2}}$$

b) Write

$$\int_0^{2\pi} \frac{(1+2\cos\theta)^n e^{in\theta}}{1-r-2r\cos\theta}\,d\theta = \frac{1}{i}\oint_\gamma \frac{(1+z+z^2)^n}{(1-r)z - r(1+z^2)}\,dz,$$

with γ being the unit circle. We may assume $r \neq 0$. If $-1 < r < \frac{1}{3}$, then the zeros of the quadratic polynomial in the denominator are separated by γ. Now conclude by means of the residue theorem.

Problem 2.5 Use Morera's theorem. First, F is continuous. To check that $\int_\Delta F = 0$ for any triangle $\Delta \subset G \cup G^-$ we only need to check that case where the triangle Δ intersects the real axis. Then $\Delta = \Delta^+ \cup \Delta^-$ where $\Delta^+ = \Delta \cap \mathbb{H}$ and $\Delta^- = \Delta \cap (-\mathbb{H})$. Finally,

$$\int_\Delta F = \int_{\Delta^+} F + \int_{\Delta^-} F = 0$$

by the analyticity of F in both halves (note that $z \mapsto \overline{f(\bar z)}$ is analytic as can be seen from power-series, for example).

b) Use reflection across the boundary of the circle. In analogy with part (a), this means defining

$$F(z) = \begin{cases} f(z) & z \in \mathbb{D} \\ \frac{1}{\bar f\left(\frac{1}{\bar z}\right)} & z \in \mathbb{C} \setminus \mathbb{D} \end{cases}$$

As in part (a), this can be checked to be analytic everywhere (after all, reflection is conformal up to a change of orientation - but we are changing the orientation twice – once in the domain and another time in the image – so the end result is truly conformal). Since it's also bounded, it is constant as claimed.

Problem 2.6 (a) Suppose $f(z)$ was such a bi-holomorphic map

$$f:\ \{0 < |z| < 1\} \to \Big\{\frac{1}{2} < |z| < 1\Big\}.$$

Then f has an isolated singularity at $z = 0$. Since it is bounded it thus has a removable singularity at $z = 0$ which implies that f extends to a map $F \in \mathcal{H}(\mathbb{D})$. If $F'(0) = 0$ then F, and thus also f, are n-to-one for some $n \geq 2$ in a small neighborhood $0 < |z| < \delta$. Thus $F'(0) \neq 0$, and we see that F has to be one-to-one locally around $z = 0$. Let $w_0 = F(0)$. Obviously,

$$w_0 \in \{\frac{1}{2} \leq |z| \leq 1\}$$

If $\frac{1}{2} < |w_0| < 1$, then $w_0 = f(z_0)$ for some $0 < |z_0| < 1$. But then a small disk around z_0 is mapped by f onto a small open neighborhood U of w_0; but $F^{-1}(U)$ contains a small disk around $z = 0$ as well, which would contradict that f is one-to-one. Hence $|z_0| = \frac{1}{2}$ or $|z_0| = 1$. However, this is a contradiction to $F(\mathbb{D})$ being open.

(b) Suppose $\mathbb{C}$ is conformally equivalent to a proper subdomain of $\Omega \subsetneq \mathbb{C}$. Then Ω is simply connected and therefore, by the Riemann mapping theorem, conformally equivalent to $\mathbb{D}$. But then $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$, which is impossible.

Problem 2.7 Use the argument principle on the closed curve given by $\partial(G_0 \cap \{\mathrm{Re}\, z \leq N\})$ and let $N \to \infty$. In other words, show that the index of the image of this curve under the map f is one for every point in the right half-plane when $N \to \infty$.

Problem 2.8 Use Rouche's theorem with $f(z) = \lambda - z$ and $g(z) = -e^{-z}$. Then $|g| < |f|$ on the contour $-R \leq y \leq R$ joined with $|z| = R$ as $R \to \infty$.

Problem 2.9 (a) There is a pole of order two at $z_0 = 1$ and one of order one at each of $z_1 = (-1 + i\sqrt{3})/2$, and $z_2 = (-1 - i\sqrt{3})/2$. Moreover, $r(\infty) = 0$. Hence

$$r(z) = \frac{a_0}{(z-1)^2} + \frac{a_1}{z-1} + \frac{a_2}{z-z_1} + \frac{a_3}{z-z_2} \tag{12.3}$$

The numbers a_1, a_2, a_3 are residues that you find as usual. Then a_0 is determined by evaluating at $z = 0$, for example. Obviously, there are other ways of finding these coefficients; for example, multiplying (12.3) by $(z-1)^2, z-1$ etc. and evaluating.

(b) In $|z| < 1$, write

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\Big(\frac{1}{z-2} - \frac{1}{z-1}\Big)$$

and expand the term in parentheses in a power series around $z = 0$, convergent on $|z| < 1$. Proceed analogously for the other annuli.

Problem 2.10 (a) You can differentiate under the integral sign or use Fubini and Morera. The point is of course that

$$\int_0^\infty e^{-t} t^{x-1}\, dt < \infty, \quad \forall\, x > 0$$

The functional equation as well as the integer values you get by integrating by parts.

(b) Set $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ when $\mathrm{Re}\, z > -1$ and $z \neq 0$. This gives an analytic continuation that agrees with Γ on $\mathrm{Re}\, z > 0$ by part (a). Also, it has a simple pole at $z = 0$ with residue equal to one. Now iterate this procedure to continue to $\mathrm{Re}\, z > -n$ for each positive integer n.

(c) Use the definition from (a) and write

$$\int_0^1 e^{-t} t^{z-1}\, dt = \int_0^1 \sum_{n=0}^\infty (-1)^n \frac{t^{n+z-1}}{n!}\, dt = \sum_{n=0}^\infty (-1)^n \frac{1}{n!(n+z)}$$

The convergence is not an issue since $\mathrm{Re}\, z > 0$. Obviously, the integral in (2.15) defines an entire function and the sum is a meromorphic function with simple poles at $-n$ where n is a nonnegative integer and the residue is $\frac{(-1)^n}{n!}$ as desired.

(d) Use the key-hole contour in $\mathbb{C} \setminus [0, \infty)$ which consists of a little circle of radius $\varepsilon > 0$ around zero, a large circle of radius R around zero, as well as two segments that run along the cut at $[0, \infty)$ from ε to R and back; the former as the limit of $x + i\delta$, the latter as the limit of $x - i\delta$ as $\delta \to 0$, respectively. The circles contributes $O(\varepsilon^{\mathrm{Re}\, a})$, and the large circle contributes $O(R^{\operatorname{Re} a-1})$; so they go away in the respective limits. Finally, by the residue theorem (the integrand has a simple pole at $z=-1$)

$$2\pi i e^{\pi i(a-1)} = \left(1-e^{2\pi i(a-1)}\right)\int_0^1 e^{-t}t^{a-1}\,dt$$

which gives the desired expression. It suffices to establish (2.17) for $z\in(0,1)$ by the uniqueness principle. Then

$$\begin{aligned}\Gamma(x)\Gamma(1-x) &= \int_0^\infty e^{-y}y^{x-1}\Big[y^{1-x}\int_0^\infty e^{-uy}u^{-x}\,du\Big]dy\\ &= \int_0^\infty\int_0^\infty e^{-y(1+u)}\,dy\;u^{-x}\,du = \int_0^\infty \frac{u^{-x}}{1+u}\,du\\ &= \frac{\pi}{\sin\pi(1-x)} = \frac{\pi}{\sin\pi x}\end{aligned}$$

as claimed. Since all integrands are positive and (jointly) continuous and therefore measurable, Fubini's theorem applies. Since $\Gamma(\bar z)=\overline{\Gamma(z)}$, we conclude that

$$|\Gamma(1/2+it)|^2 = \frac{2\pi}{e^{\pi t}+e^{-\pi t}}$$

(e) Write, with $\alpha,\beta>0$,

$$\begin{aligned}&\Gamma(\alpha)\Gamma(\beta)\\ &= \int_0^\infty\int_0^\infty e^{-t}t^{\alpha-1}e^{-s}s^{\beta-1}\,dsdt = \int_0^\infty\int_0^\infty e^{-t}t^{\alpha-1}e^{-ut}(ut)^{\beta-1}\,t\,dudt\\ &= \int_0^\infty\int_0^\infty e^{-t(1+u)}t^{\alpha+\beta-1}\,dtu^{\beta-1}\,du = \Gamma(\alpha+\beta)\int_0^\infty \frac{u^{\beta-1}}{(1+u)^{\alpha+\beta}}\,du\\ &= \Gamma(\alpha+\beta)\int_0^1 (1-s)^{\beta-1}s^{\alpha-1}\,ds\end{aligned}$$

where we substituted $s=\frac{1}{1+u}$ in the final step. This result extends to $\operatorname{Re}\alpha>0$ and $\operatorname{Re}\beta>0$ by the uniqueness principle.

(f) For the cosine integral, write

$$\int_0^\infty t^{z-1}\cos t\,dt = \frac12\int_0^\infty t^{z-1}e^{it}\,dt + \frac12\int_0^\infty t^{z-1}e^{-it}\,dt$$

For the first integral on the right-hand side use a contour in the first quadrant with straight segments $[\varepsilon,R]$ and $[iR,i\varepsilon]$ joined by quarter-circles of radii ε and R, respectively. For the second, use the reflection of this contour about the real axis. Putting absolute values inside the integrals shows that the circular pieces contribute nothing as $\varepsilon\to0$ and $R\to\infty$ (for this use that $0<\operatorname{Re}z<1$). In conclusion, from Cauchy's theorem,

$$\begin{aligned}&\int_0^\infty t^{z-1}e^{it}\,dt = ie^{\pi i(z-1)/2}\int_0^\infty e^{-t}t^{z-1}\,dt = e^{\pi iz/2}\Gamma(z)\\ &\int_0^\infty t^{z-1}e^{-it}\,dt = e^{-\pi iz/2}\Gamma(z)\end{aligned}$$

which gives the desired conclusion.

For the sine integral proceed in the exact same fashion in the region $0<\operatorname{Re}z<1$. To extend to the range $-1<\operatorname{Re}z<1$, note that both the left, and right-hand sides are analytic in that region (actually, it's enough to know this for one as it then follows for the other - why?). In fact, since $\sin t$ vanishes at $t = 0$ to first order, we can allow $\operatorname{Re} z > -1$ without losing convergence of the integral at $t = 0$. So the left-hand side extends analytically to the left, whereas for the right-hand side just use that the pole of Γ at $z = 0$ is canceled by the zero of $\sin(\pi z)$. For the final two identities, make the specific choices of $z = 0$ and $z = -\frac{1}{2}$, respectively, in the sine integrals and use part (d).

(g) In the integral in (2.18) the exponential decay of e^w along γ beats any polynomial growth that may come from w^{-z}. Hence, the integral converges absolutely for any Hankel contour in $\mathbb{C} \setminus (-\infty, 0]$; using Fubini and Morera thus shows that

$$F(z) = \frac{1}{2\pi i} \int_\gamma e^w w^{-z}\, dw$$

is an entire function (use that w^{-z} is entire for any fixed $w \in \mathbb{C}^*$). Moreover, $F(z)$ does not depend on the particular choice of γ by Cauchy's theorem.

Next, we argue that provided $\operatorname{Re} z < 1$ we can deform γ into a contour that runs along the cut $(-\infty, 0]$ from below and returns to $-\infty$ by running along this cut from above. Thus, this contour is $-x - i0$ followed by $-x + i0$; in the former case, x goes from ∞ to 0, and in the latter, from 0 to ∞.

The only thing that needs to be checked in this deformation is that a little circle of radius $\varepsilon > 0$ around $w = 0$ does not contribute anything in the limit $\varepsilon \to 0$. Indeed, putting absolute values into the integral yields that such a circle contributes $O(\varepsilon^{1-\operatorname{Re} z})$ and therefore vanishes in the limit since $\operatorname{Re} z < 1$. For this use that

$$|w^{-z}| = e^{-\operatorname{Re}(z \operatorname{Log} w)} = |w|^{-\operatorname{Re} z} e^{-\operatorname{Im} z \operatorname{Arg} w} \le |w|^{-\operatorname{Re} z} e^{2\pi |\operatorname{Im} z|}$$

So if $|w| = \varepsilon$, then $|w^{-z}| \le \varepsilon^{-\operatorname{Re} z} e^{2\pi|\operatorname{Im} z|}$.

The conclusion from all of this is the following: $\forall \operatorname{Re} z < 1$,

$$F(z) = \frac{-e^{-\pi i z} + e^{\pi i z}}{2\pi i} \int_0^\infty e^{-x} x^{-z}\, dx = \frac{\sin(\pi z)}{\pi} \Gamma(1 - z) = \frac{1}{\Gamma(z)}$$

by (2.17) from part (d). Since F is entire, it follows that $\frac{1}{\Gamma(z)}$ is entire, too, and the identity which we just derived holds for all $z \in \mathbb{C}$. It is important to note that in this round about way we have shown that Γ never vanishes. Can you think of any other way of proving this latter property?

Problem 2.11 The map $w \mapsto z = aw^2 + bw^3$ takes $\mathbb{D}$ onto a region $\Omega \subset \mathbb{D}$ such that $\bar{\Omega} \cap \partial\mathbb{D} = \{1\}$. Since $f(z)$ is assumed to be analytic in a neighborhood of $z = 1$, it follows that $g(w) = f(z)$ is given by a power series around $w = 0$ with radius of convergence $R' > 1$. Note that

$$g(w) = \sum_{n=0}^\infty a_n (aw^2 + bw^3)^{2^n}$$

Upon expanding and then deleting the parentheses every power of w occurs at most once in the entire series since $2.2^{n+1} > 3.2^n$. Therefore, the power series of g is exactly the one which is obtained by this process. However, since we can always introduce parentheses in a convergent series without destroying convergence, we conclude that the power series of f would need to converge for some $z > 1$ which it cannot. The generalization to other gap series is obvious.

Problem 2.12 This follows from Runge's theorem, cf. Theorem 2.29. First note that $\mathbb{C} \setminus K$ is connected. The function $\frac{1}{z}$ is holomorphic on a neighborhood of K. For any $\varepsilon > 0$ there exists a polynomial Q such that

$$\max_{z \in K} |Q(z) - 1/z| < \varepsilon$$

and thus

$$\max_{z \in K} |zQ(z) - 1| < \varepsilon$$

Hence, set $P(z) := 1 - zQ(z)$.

Problem 2.13 (a) Derive (2.19) as follows: using 25 (e) show that

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \frac{1}{h} + \int_0^1 \Big((1-t)^{z-1} - 1\Big)\frac{dt}{t} + o(1)$$

as $h \to 0$. Indeed, with $\operatorname{Re} z > 0$,

$$\begin{aligned}\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} &= \int_0^1 (1-t)^{z-h-1} t^{h-1}\, dt = \frac{1}{h} + \int_0^1 [(1-t)^{z-h-1} - 1] t^{h-1}\, dt \\ &= \frac{1}{h} + \int_0^1 [(1-t)^{z-1} - 1]\, \frac{dt}{t} + o(1)\end{aligned}$$

as claimed. To pass to the last line simply note that the second integral above is continuous at $h = 0$. Since

$$\begin{aligned}\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} &= \frac{(\Gamma(z) - h\Gamma'(z) + O(h^2))(h^{-1} + A + O(h))}{\Gamma(z)} \\ &= \frac{1}{h} + A - \frac{\Gamma'(z)}{\Gamma(z)} + O(h)\end{aligned}$$

we obtain (2.19) by equating the terms constant in h. To pass to (2.20), expand

$$\frac{dt}{t} = \frac{dt}{1-(1-t)} = \sum_{n=0}^{\infty} (1-t)^n\, dt$$

Inserting this into (2.19) and integrating term-wise yields (2.20).

For (b), note that $\frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz} \log \Gamma(z)$. This determines (2.21) up to the value of γ. To find γ, set $z = 1$ in (2.21) and notice that

$$\gamma = \lim_{N \to \infty} \sum_{n=1}^{N} \Big[\frac{1}{n} - \log\Big(1 + \frac{1}{n}\Big)\Big] = \lim_{N \to \infty} \Big[\sum_{n=1}^{N} \frac{1}{n} - \log(N+1)\Big]$$

as claimed.

For (c), write (2.21) as a limit and simplify using the limiting expression for γ.

Solutions for Chapter 3

Problem 3.1 (a) Relate $\sum_{n=0}^{\infty} s_n z^n$ to $f(z)$ via the identity

$$\frac{f(z)}{1-z} = \sum_{n=0}^{\infty} s_n z^n \quad \text{or} \quad f(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n \tag{12.4}$$

Given any $M > 0$ one has $s_n \geq M$ for all $n \geq M$. From the second identity in (12.4), it follows immediately that

$$\limsup_{z\to 1} f(z) \geq M$$

and thus $f(z) \to \infty$ as $z \to 1-$. We cannot talk about a pole as $\log(1-z) = \sum_{n=1}^\infty \frac{1}{n}\, z^n$ shows.

(b) We can assume that $s = 0$. Given $\varepsilon > 0$, show that there exists $n_0 = n_0(\varepsilon)$ such that

$$\limsup_{z\to 1} \Big| \sum_{n=n_1}^{n_2} a_n z^n \Big| < \varepsilon$$

for all $n_2 > n_1 \geq n_0$. Here the lim sup is taken over $z \in K_\alpha \cap \mathbb{D}$ with $\alpha < \pi$ fixed. This is done by summing by parts (write $a_n = s_n - s_{n-1}$ and rearrange) and noting that K_α is characterized by

$$\sup_{z\in K_\alpha\cap\mathbb{D}} \frac{|1-z|}{1-|z|} =: C_\alpha < \infty$$

(c) Consider $f(z) - \sum_{n=0}^N a_n$ where $N = [1/(1-|z|)]$. Then

$$\Big| \sum_{n=0}^{N} (1-z^n)\, a_n \Big| \leq |1-z| \sum_{n=0}^{N} n|a_n| \leq C_\alpha\, N^{-1} \sum_{n=0}^{N} n|a_n| \to 0$$

as $z \to 1$ inside K_α. Furthermore,

$$\Big| \sum_{n=N+1}^{\infty} a_n\, z^n \Big| \leq \frac{1}{1-|z|} \sup_{n\geq N} |a_n| \leq \sup_{n\geq N} n|a_n| \to 0$$

as $N \to \infty$ or $z \to 1$ in K_α.

Problem 3.2 (a) Given $\delta > 0$, there exists $K \subset \mathbb{T}$ compact with $|K| = 0$ and $|\mu|(\mathbb{T} \setminus K) < \delta$. Decompose μ as follows: for all Borel sets $E \subset \mathbb{T}$,

$$\mu(E) = \mu(E \cap K) + \mu(E \setminus K) =: \mu_1 + \mu_2$$

Then $|\mu_2| < \delta$ and evidently for all $x \in \mathbb{T} \setminus K$,

$$\lim \frac{\mu_1(I)}{|I|} \to 0 \;\text{ as }\; |I| \to 0,\; x \in I$$

Thus, it follows from the weak L^1 boundednes of the Hardy–Littlewood maximal function that

$$\Big| \Big\{ x \in \mathbb{T} \mid \limsup_{|I|\to 0,\, x\in I} \frac{|\mu|(I)}{|I|} > \gamma \Big\} \Big| \leq \frac{6}{\gamma}\, \delta$$

for each $\gamma > 0$. Letting $\delta \to 0$ and then $\gamma \to 0$, we see that the left–hand side vanishes as claimed.

(b) It suffices to show that $\Psi_n * \mu \to 0$ almost everywhere for any $d\mu \perp d\theta$, $\mu \geq 0$. As above, we split μ into two pieces: μ_1, which is supported on K, $|K| = 0$, and μ_2 which has mass at most δ. For μ_1, we note that $\lim_{n\to\infty} \Psi_n * \mu_1 = 0$ on $\mathbb{T} \setminus K$ because of (3.8). The limit in the case of μ_2 is dominated by the Hardy–Littlewood maximal function and estimated as in part (a); first send $\delta \to 0$ and then $\gamma \to 0$.

Problem 3.3 There are two crucial points here: first, although in general a harmonic function on an annulus does not have a conjugate harmonic function this failure is "one-dimensional" (compare this to $\mathcal{H}^1(\mathbb{R}^2 \setminus \{0\}) \simeq \mathbb{R}$ in the sense of de Rham cohomology

- try proving this fact with multivariable calculus). More precisely, if we subtract a multiple of $\log r = \log|z|$ then it is possible to find the conjugate harmonic function. Second, $u(z) = \log r = \log|z|$ is the only nonconstant radial harmonic function with $u(1) = 0$, $u_r(1) = 1$.

(a) Choose k such that $u - k\log r$ has a conjugate harmonic function. This happens iff the vector field

$$(-u_y, u_x) - k\big(-\frac{y}{r^2}, \frac{x}{r^2}\big)$$

is conservative, i.e., is of the form ∇v (cf. the proof of Proposition 1.28). This in turn is the same as requiring that

$$k = \frac{1}{2\pi}\oint_{|z|=r} -u_y\,dx + u_x\,dy$$

Hence, with this choice of k,

$$u(z) - k\log|z| = \operatorname{Re} f(z), \qquad f \in \mathcal{H}(\mathcal{A})$$

as claimed. Second, the mean value

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} f(re^{i\theta})\,d\theta = \operatorname{Re}\frac{1}{2\pi i}\oint_{|z|=r}\frac{f(z)}{z}\,dz$$

does not depend on $r \in (r_1, r_2)$ by Cauchy. In particular, if $r_1 = 0$ and u remains bounded as $z \to 0$, then it follows that $k = 0$.

(b) This is an immediate consequence of (a) and the simple connectivity; indeed, locally around each point $z_1 \in \Omega$ the harmonic function

$$u(z) - \log|z - z_0| = \operatorname{Re} f(\cdot, z_1)$$

where $z \mapsto f(z, z_1)$ is analytic on some small disk around z_1. Second, if the domains of $f(\cdot, z_1)$ and $f(\cdot, z_2)$ overlap, then these functions differ by an imaginary constant. Applying the monodromy theorem shows that

$$u(z) - \log|z - z_0| = \operatorname{Re} g, \qquad g \in \mathcal{H}(\Omega)$$

Define $f(z) = (z - z_0)e^{g(z)}$. It is clear that this has the desired properties ($u = \log|f|$ etc.).

Problem 3.4 (a) By assumption, we can write $u = \phi(v)$. Then $\Delta u = 0 = \phi''(v)|\nabla v|^2$ so that ϕ is linear. Hence, there is an affine relation between u and v.

(b) Let $\widetilde{u}$ be the conjugate harmonic function of u locally around z_0 normalized so that $\widetilde{u}(z_0) = 0$. Then apply (a) to v and $\widetilde{u}$ to conclude that

$$v = c_0\widetilde{u} + c_1 = c_0\widetilde{u} + v(z_0)$$

Since $|\nabla u(z_0)| = |\nabla\widetilde{u}(z_0)| = |\nabla v(z_0)|$, it follows that $c_0 = \pm 1$. Hence, $u \mp iv$ is conformal as claimed.

Problem 3.5 (i) Suppose $u(z) > M$ for some $z \in \Omega$. Then $\sup_\Omega u$ is attained in Ω. By the SMVP it follows that the set $\{z \in \Omega \mid u(z) = M\}$ is open. Since this set is also closed, it equals Ω. Hence $u = M$ contradicting the definition of M.

(ii) Apply (i) to $u - h$.

(iii) Clear from the SMVP.

(iv) Apply Jensen's inequality to the MVP and the SMVP.

(v) Since the MVP holds for h, the SMVP follows for u.

(vi) The local SMVP implies the maximum principle and that implies (ii). Now use (vi).

(vii) The monotonicity of the mean values in the radius follows from (ii). The fact that every mean value is finite can be seen like this: suppose that

$$\int_0^{2\pi} u(z_0 + r_0 e^{i\theta})\, d\theta = -\infty$$

for some z_0 and $r_0 > 0$. Then the same holds for all $0 < r < r_0$ leading to

$$\iint_{|z-z_0|<r_0} u(x,y)\, dxdy = -\infty \tag{12.5}$$

But since those $z \in \Omega$ with $u(z) > -\infty$ are dense in Ω (by the MVP since we are assuming that $u \not\equiv -\infty$), it follows from the SMVP that for z_1 arbitrarily close to z_0,

$$\int_0^{2\pi} u(z_1 + re^{i\theta})\, d\theta > -\infty$$

for all $r \in (0, r_1)$ where $r_1 := \operatorname{dist}(z_1, \partial\Omega))$. Hence,

$$\iint\limits_{|z-z_1|<r_1} u(x,y)\, dxdy > -\infty$$

contradicting (12.5) since z_1 can be chosen arbitrarily close to z_0. The other statements of (viii) are now obvious.

(viii) Consider $\iint_D \Delta u(x,y)\, dxdy$ for a disk D and relate it to the mean-value integral by the divergence theorem. In fact,

$$\frac{d}{dr}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta = \frac{1}{r}\oint\limits_{|z|=r} \frac{\partial u}{\partial n}\, d\sigma = \frac{1}{r}\iint_{|z|\le r} \Delta u\, dxdy$$

By (vii), the left-hand side is nonnegative. Thus $\Delta u \ge 0$ as claimed.

(ix) $\log|f(z)|$ is subharmonic since it is harmonic away from the discrete zeros of f. Now use (vii). For the other functions use (v).

(x) Apply (ii) to u and $-u$.

Problem 3.6 (a) This is the important Phragmen–Lindelöf principle. To prove it, let $\delta > 0$ and set

$$u_\delta(z) := u(z) - \delta|z|^{\rho_1}\cos(\rho_1\theta), \quad \rho_1 < \rho < \lambda$$

Applying the maximum principle to $\{|z| \le R\} \cap \mathcal{S}$ with large R then shows that $u_\delta \le M$ on $\mathcal{S}$. Letting $\delta \to 0$ finishes the proof. The reader should formulate for herself or himself variants of this principle, say for analytic functions on a strip $0 < \operatorname{Re} z < 1$ and bounded by M on the edges.

(b) Observe that

$$h(z) = \sum_{n=1}^{\infty} 2^{-n}\log|z - z_n| - A$$

is a negative subharmonic function on Ω provided A is large. Then for every $\delta > 0$,

$$\limsup_{z\to\zeta}(u(z) + \delta h(z)) \le M$$

and thus $u) + \delta h(z) \le M$ in Ω. Sending $\delta \to 0$ concludes the proof.

Problem 3.7 (a) If $u \in C^2(\Omega)$, then

$$\langle u, \Delta\phi\rangle = \langle \Delta u, \phi\rangle \geq 0$$

since $\Delta u \geq 0$, see Problem 3.5, (iii) and $\phi \geq 0$ by assumption. Let $\chi \geq 0$ be a radial, compactly supported, smooth bump function with $\iint \chi = 1$ and $\mathrm{supp}(\chi) \subset \mathbb{D}$. Then, with $\chi_\varepsilon(z) := \varepsilon^{-2}\chi(z/\varepsilon)$, let $u_\varepsilon := \chi_\varepsilon * u$ be defined on

$$\Omega_\varepsilon := \{z \in \Omega \mid \mathrm{dist}(z, \partial\Omega) > \varepsilon\}$$

The convolution is well-defined since $u \in L^1_{\mathrm{loc}}(\Omega)$ by Problem 3.5, (vii). Furthermore, by (vii) of the previous problem, we have $u_\varepsilon \to u$ as $\varepsilon \to 0+$ and $u_{\varepsilon_1} \geq u_{\varepsilon_2} \geq u$ if $\varepsilon_1 > \varepsilon_2$. Hence, from either the monotone or dominated convergence theorems,

$$0 \leq \langle u_\varepsilon, \Delta\phi\rangle \to \langle u, \Delta\rangle$$

as $\varepsilon \to 0+$. If $u = \log|f|$ where f is holomorphic, then

$$\mu = \sum_{f(z)=0} \nu_z\, \delta_z$$

where $\nu_z \geq 1$ is the order of vanishing of f at $z \in \Omega$.

(b) The idea is to use Green's formula, at least if $u \in C^2(\Omega)$ and then to approximate. We will employ a slightly different approach here which smooths out the logarithmic potential. Thus, let $\chi = 1$ on Ω_1 and χ smooth and compactly supported in Ω. Then, by (a), with $z \in \Omega_1$ fixed,

$$\langle u, \Delta[\chi(\cdot)\log(\varepsilon^2 + |z - \cdot|^2)]\rangle = \int \chi(\zeta)\log(\varepsilon^2 + |z-\zeta|^2)\,\mu(d\zeta) \tag{12.6}$$

Now

$$\Delta \log(\varepsilon^2 + |z|^2) = 4\frac{\partial^2}{\partial z \partial \bar{z}} \log(\varepsilon^2 + |z|^2) = \frac{4\varepsilon^2}{(\varepsilon^2 + |z|^2)^2}$$

is an approximate identity in $\mathbb{R}^2$ in the sense of Definition 3.2. Passing to the limit $\varepsilon \to 0$ in (12.6) yields, with $\zeta = \xi + i\eta$,

$$\begin{aligned} u(z) &+ \iint_{\Omega\setminus\Omega_1} u(\zeta)(\Delta\chi)(\zeta)\log|z-\zeta|\,d\xi d\eta \\ &+ \iint_{\Omega\setminus\Omega_1} u(\zeta)\nabla\chi(\zeta)\cdot\frac{z-\zeta}{|z-\zeta|^2}\,d\xi d\eta = \int \chi(\zeta)\log|z-\zeta|\,\mu(d\zeta) \end{aligned}$$

for all $z \in \Omega_1$. By putting all integrals over $\Omega \setminus \Omega_1$ into the harmonic function, we obtain (3.10) as desired. If $u = \log|f|$ with $f \in \mathcal{H}(\Omega)$, then

$$u(z) = \sum_{\zeta : f(\zeta)=0} \nu_\zeta \log|z-\zeta| + h(z)$$

where ν_ζ is the order of vanishing of f at ζ.

For an example where

$$u(z) = \int \log|z-\zeta|\;\mu(d\zeta)$$

is not continuous, take $\mu = \sum_{n=1}^\infty \frac{1}{n^2}\,\delta_{\frac{1}{n}}$. The usc property follows from Fatou's lemma.

(c) For the Jensen formula, we first observe this:

$$\int_0^1 \log|z - e^{2\pi i\theta}|\,d\theta = \begin{cases} \log|z| & \text{if } |z| > 1 \\ 0 & \text{if } |z| \le 1 \end{cases}$$

This can be done without any calculations: the integral defines a harmonic function $h(z)$ on $|z| < 1$. First, $h(0) = 0$ and h is constant on $|z| = 1$. Then, by the maximum principle, $h = \text{const} = 0$ in $|z| \le 1$. From this,

$$\int_0^1 \log|z - e^{2\pi i\theta}|\,d\theta = \log|z| + \int_0^1 \log|z^{-1} - e^{2\pi i\theta}|\,d\theta = \log|z|$$

provided $|z| > 1$ as claimed. Next, from (3.10),

$$\begin{aligned}\int_0^1 u(z + re(\theta))\,d\theta - u(z) &= \iint_{|z-\zeta|<r} \log\Big(\frac{r}{|z-\zeta|}\Big)\mu(d\zeta) \\ &= \int_0^r \frac{\mu(D(z,t))}{t}\,dt\end{aligned}$$

which is (3.11). If $u = \log|f|$ we obtain the well-known Jensen formula for analytic functions: if $f(z_0) \ne 0$, then

$$\int_0^1 \log|f(z_0 + re(\theta))|\,d\theta - \log|f(z_0)| = \sum_{\substack{|z-z_0|<r \\ f(z)=0}} \log\Big(\frac{r}{|z-z_0|}\Big)$$

It is easy to see from this that

$$\begin{aligned}\mu(\Omega_1) &\le C(\Omega_1,\Omega)(\sup_{\Omega} u - \sup_{\Omega_1} u) \\ \|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} &\le C(\Omega_2,\Omega_1,\Omega)(\sup_{\Omega} u - \sup_{\Omega_1} u)\end{aligned}$$

the inclusions $\Omega_2 \subset \Omega_1 \subset \Omega$ being compact.

Problem 3.8

(a) The bound (3.12) is obvious from the explicit form of P_r. The Harnack estimate for positive harmonic functions on $\mathbb{D}$ follows from

$$u(\rho z) = (P_r * u_\rho)(\phi)$$

where $z = re(\phi)$ and $0 < r < 1$, $0 < \rho < 1$, $u_\rho(\phi) = u(\rho e(\phi))$. Indeed, we can estimate P_r in this convolution by (3.12) and then send $\rho \to 0$. By the explicit form of the Poisson kernel, $C(r) = \frac{1+r}{1-r}$ is the best constant. For a general domain Ω, we cover any compact $K \subset \Omega$ by finitely many disks inside of Ω and then compare u at two different points $p, q \in K$ by means of a chain of these disks that passes from p to q. Since Harnack's inequality is scaling invariant, we conclude that any positive harmonic function u on $\mathbb{R}^2$ satisfies

$$\sup_{\mathbb{R}^2} u \le u(z)$$

where $z \in \mathbb{R}^2$ is arbitrary. Hence $u = \text{const}$ as claimed.

(b) Apply (a) to $\{u_m - u_n\}_{m>n\ge 1}$. If this is a Cauchy sequence at one point, then it is so uniformly on every compact subset of Ω and we are done by Problem 3.5, Part (x).

Problem 3.9 If a harmonic majorant exists, then (ii) holds by the MVP. For the converse, let h_n be harmonic on $|z| < 1 - \frac{1}{2n}$ with $h_n = u$ on $|z| = 1 - \frac{1}{2n}$. Then

$\{h_n(0)\}_{n=1}^\infty$ is increasing and bounded by (ii). By Harnack's principle, it follows that the increasing sequence h_n converges uniformly on compact sets to a function h harmonic on $\mathbb{D}$. Obviously, $h \geq u$ on $\mathbb{D}$ and h is also the least harmonic majorant. An example of a subharmonic function without a harmonic majorant would be $(P_r(\theta))^\beta$ for any $\beta > 1$ or $\exp(P_r(\theta))$.

Problem 3.10 It is obvious that (ii) implies (i). For the converse, let u be the harmonic majorant of $\log^+ |f|$ on $\mathbb{D}$. It has a harmonic conjugate $\widetilde{u}$ on $\mathbb{D}$. Define $F := u + i\widetilde{u}$, $h = e^{-F}$ and $g = fe^{-F}$. Since $u \geq 0$, it follows that $|h| \leq 1$ and $|g| \leq |f|e^{-u} \leq \exp(\log_+ |f| - u) \leq 1$.

The class of holomorphic functions that satisfy these conditions is very important; it is called the *Nevanlinna class* $N(\mathbb{D})$. It has the following property: if $f \in N(\mathbb{D})$, $f \not\equiv 0$, then

$$f(e^{i\theta}) = \lim_{z \to e^{i\theta}} f(z)$$

exists non-tangentially for (Lebesgue) almost every $\theta \in [0, 2\pi]$ and

$$\log |f(e^{i\theta})| \in L^1(d\theta).$$

In particular, $f(e^{i\theta}) \neq 0$ almost everywhere.

Problem 3.11 (a) These products are the well–known *Blaschke products.* The reader will easily check that

$$\left|1 - \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}\right| \leq C(z)(1 - |z_n|)$$

so that the product converges uniformly on compact subset of $\mathbb{D}$ to $B \in \mathcal{H}(\mathbb{D})$ with the required vanishing properties. Also, it is clear that $|B| \leq 1$.

(b) For this observe that $|B(re(\theta))/B_n(re(\theta))| \in \mathfrak{sh}(\mathbb{D})$ where B_n is the n^{th} partial product. Thus, by Problem 3.5, Part (vii),

$$\int_0^1 |B(re(\theta))/B_n(re(\theta))| \, d\theta \leq \int_0^1 |B(r'e(\theta))/B_n(r'e(\theta))| \, d\theta$$

for all $0 < r < r' < 1$. Letting $r' \to 1$ and then $n \to \infty$ implies that

$$1 \leq \int_0^1 |B(e(\theta))| \, d\theta$$

and since $|B| \leq 1$ on $\partial\mathbb{D}$ therefore $|B| = 1$ a.e.

Solutions for Chapter 4

Problem 4.1 First, note that there is $\varepsilon > 0$ so that each lattice $\omega \in \Lambda$ has the property that $D(\omega, \varepsilon) \cap \Lambda = \{\omega\}$. If $\Lambda = \{0\}$ we are done. Otherwise let ω_1 have the smallest length in $\Lambda \setminus \{0\}$. From the minimality,

$$\Lambda \cap \mathbb{R}\omega_1 = \mathbb{Z}\omega_1$$

If $\Lambda \setminus \mathbb{Z}\omega_1 \neq \emptyset$, pick $\omega_2 \Lambda \setminus \mathbb{Z}\omega_1 \neq \emptyset$ of smallest length. Using this minimality now verify that

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

Problem 4.2 For any $z \in \mathbb{C}$ let $\zeta - z \in \Lambda$ be such that ζ has the smallest size. Then $|\zeta| \leq |\zeta - \omega|$ for all $\omega \in \Lambda$ as desired. If $|z| = |z - \omega|$ for some $\omega \in \Lambda \setminus \{0\}$ then z cannot belong to the interior of the Dirichlet polygon. If two distinct interior points z_1, z_2 satisfy $z_1 - z_2 \in \Lambda$, then $|z_1| < |z_2|$ as well as $|z_2| < |z_1|$ which is a contradiction.

Problem 4.3 It is clear that $A \in GL(2,\mathbb{Z})$; otherwise it does not take lattice points to lattice points. But $|\det A| = 1$ since A^{-1} needs to have integer entries, too. Recall that we represent tori $\mathbb{C}/\Lambda$ by points $\tau \in \mathbb{H}$; simply define $\tau = \frac{\omega_2}{\omega_1}$ where we order the generating vectors ω_1, ω_2 so that $\mathrm{Im}\,(\tau) > 0$. Clearly, if $A \in PSL(2,\mathbb{Z})$, then $\tau' = A\tau$ represents the same torus as τ. In other words, τ and τ' are indistinguishable in the moduli space. On the other hand, suppose $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is a conformal equivalence. We can lift f to a map F on the universal covers, i.e., $F : \mathbb{C} \to \mathbb{C}$. Then F is an entire function. In fact, one can check that $F \in \mathrm{Aut}(\mathbb{C})$ whence $F(z) = az + b$. Furthermore, F induces an isomorphism of the groups Λ, Λ' via conjugation: $g' = F \circ g \circ F^{-1}$ with $g \in \Lambda$. From here it is easy to conclude via the first part of the problem. Indeed, one checks that $\tau' = \frac{n_1\tau + m_1}{n_2\tau + m_2}$ where $\begin{bmatrix} n_1 & m_1 \\ n_2 & m_2 \end{bmatrix} \in PSL(2,\mathbb{Z})$ as claimed.

Problem 4.4 f' has degree three and vanishes somewhere, say at w. Then $f(z + w) = f(w) + \mu z^2 + O(z^3)$ with $\mu \neq 0$ (otherwise f would have degree > 2). Since $f(z + w) - f(w) \in \mathcal{M}(M)$ is of degree 2, $z = 0$ is its unique zero. Thus,

$$g(z) := \frac{\mu}{f(z + w) - f(z)} - \wp(z) \in \mathcal{H}(M)$$

and therefore $g = \text{const}$ which implies the desired representation of f.

Problem 4.5 This follows from the Schwarz reflection principle. Indeed, if (U, ϕ) is the chart around $p \in \partial N$ as in the formulation of the problem, then we let U' be a distinct copy of U and define $\widetilde{U} := U \cup U'$ with the boundaries identified. The map ϕ is extended to U' by reflection across the real line. By the Schwarz reflection principle, the transition maps of these new charts are analytic.

Problem 4.6 Assume otherwise. Let $U \subset \mathbb{C}$ be a nonempty open set that f misses. Then it follows that f can have at most poles at the points of S which implies that f extends to an analytic function $f : M \to \mathbb{C}$ which needs to be constant.

Problem 4.7 For the fact that Φ is a homeomorphism, one uses that $\wp$ is even and $\wp'$ odd as well as the locations of the branch points as detailed in Chapter 4.

Problem 4.8 Suppose first that

Problem 4.9 (a) Since $|a| > |b|$,

$$Tz := \frac{az + \bar{b}}{bz + \bar{a}} \tag{12.7}$$

takes 0 into $\mathbb{D}$ and $\partial\mathbb{D}$ onto $\partial\mathbb{D}$. Thus, $T \in \mathrm{Aut}(\mathbb{C})$. Alternatively, compute

$$1 - |Tz|^2 = (1 - |z|^2)|T'z|$$

which implies that $|Tz| < 1$ iff $|z| < 1$. Conversely, suppose $S \in \mathrm{Aut}(\mathbb{D})$ with $|S(0)| < 1$. Choose a, b with

$$S(0) = -\frac{\bar{b}}{a} \qquad |a|^2 - |b|^2 = 1$$

and let T be as in (12.7). Then $R := T \circ S \in \mathrm{Aut}(\mathbb{D})$ with $R(0) = 0$. By Schwarz's lemma, $|R(z)| \leq |z|$ and the same holds for R^{-1}. Hence, we have $R(z) = e^{i\theta} z$ and we are done. Alternatively, suppose $Sz = \frac{az+b}{cz+d}$ with $ad - bc = 1$ preserve $\mathbb{D}$. Or, equivalently,

$$|ae^{i\theta} + b| = |ce^{i\theta} + d| \quad \forall\, \theta \in \mathbb{R}, \qquad |b| < |d|$$

which is the same as

$$|a|^2 + |b|^2 = |c|^2 + |d|^2, \quad a\bar{b} = c\bar{d}, \quad ad - bc = 1, \quad |a| > |c|,\ |b| < |d|$$

This is easily seen to amount to $d = \bar{a}, c = \bar{b}$ as claimed. That the map (4.13) defines an isomorphism of groups is a mechanical verification.

(b) One approach is to use the chordal metric $d(z, w)$ from Problem 1.4 and to verify that it is preserved iff T is represented by an element of $SU(2)$. This amounts to

$$\frac{2|z-w|}{(1+|z|^2)^{\frac{1}{2}}(1+|w|^2)^{\frac{1}{2}}} = \frac{2|Tz-Tw|}{(1+|Tz|^2)^{\frac{1}{2}}(1+|Tw|^2)^{\frac{1}{2}}}$$

or, letting $z \to w$,

$$1 + |Tz|^2 = (1 + |z|^2)|T'z|$$

An explicit calculation as in part (a) shows that this is the same as

$$a\bar{b} + c\bar{d} = 0, \quad |a|^2 + |c|^2 = |b|^2 + |d|^2 = 1, \quad ad - bc = 1$$

which in turn implies that $\bar{a} = d, c = -\bar{b}$ as desired.

The homomorphism Q from $SU(2)$ onto $SO(3)$ is

$$A \mapsto T_A \mapsto \Phi^{-1} \circ T_A \circ \Phi$$

where $A = \begin{bmatrix} a & \bar{b} \\ b & -\bar{a} \end{bmatrix}$ and Φ is the stereographic projection from Problem 1.4. With some patience the reader will verify that the unit quaternion

$$a = \cos(\omega/2) + \sin(\omega/2)(x_1 i + x_2 j + x_3 k), \qquad 0 < \omega < 2\pi,\ x_1^2 + x_2^2 + x_3^2 = 1$$

is a rotation around the axis (x_1, x_2, x_3) by the angle ω. One should think of $\mathbb{R}^3$ as the imaginary quaternions (those are the quaternions with vanishing real part). If a is the unit quaternion from above, and u an imaginary one, then $u \mapsto au\bar{a}$ is again imaginary and is precisely the aforementioned rotation in $\mathbb{R}^3$.

Solutions for Chapter 5

Problem 5.1 The fundamental groups of the surfaces in (5.12) are $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. We can view

$$\mathcal{R}S(\mathbb{C}, \mathbb{C}, \log z, 1) = \{(e^z, z) \mid z \in \mathbb{C}\},$$
$$\mathcal{R}S(\mathbb{C}, \mathbb{C}, z^{\frac{1}{n}}, 1) = \{(z^n, z) \mid z \in \mathbb{C}^*\}$$

whence the stated isomorphisms. In Chapter 5 we showed that the *ramified surface* in the second case is also $\mathbb{C}$ but for a very different reason than in the $\log z$ case.

Problem 5.2 Simply observe that

$$b_p(z) := \frac{1}{2\pi i} \oint_{|w-w_0|=r_0} w^p \frac{\partial_w f(z,w)}{f(z,w)}\, dw$$

is analytic in $D(z_0, r_1)$ for all integers $p \geq 0$. Now define

$$P(z,w) := \prod_{j=1}^{n}(w - w_j(z)), \qquad f(z, w_j(z)) = 0,\ |w_j(z) - w_0| < r_0$$

Since the coefficients of $P(z,w)$ are polynomials in the $b_p(z)$ they are also analytic. Moreover, $f(z,w)/P(z,w)$ has removable singularities at each of the $w_j(z)$ and is therefore analytic and nonvanishing.

Problem 5.3 The eigenvalues are the zeros of the characteristic polynomial. If the are all distinct, then by part (b) of the previous problem they are analytic. On the other hand, around each branch point they are analytic relative to the uniformizing variable of Lemma 5.10. Recalling how this uniformizing variable was constructed, we see that a Puiseaux series is obtained with $\ell - 1$ being the branch number at the respective branch point. In the Hermitian case, one uses that the eigenvalues are real. This is incompatible with $\ell \geq 2$; hence, the Puiseaux series are Taylor series as claimed (this fact about (real) analytic Hermitian matrices is known as Rellich's theorem).

Problem 5.4 We shall discuss the example $\sqrt[4]{\sqrt{z}-1}$ in full detail and leave it to the reader to carry out similar analyses in the other cases. We first do this the old-fashioned way without resultants etc. We expect 8 sheets since $\sqrt[8]{z}$ is what we have when z is very large. Moreover, we see that $z = \infty$ is a branch point with branching number 7. The finite branch points are $z = 0$; this is where the interior $\sqrt{z}$ branches with branching number one. However, we have four branch points in $\widetilde{\mathcal{R}S}$ each of which is rooted at $z = 0$. This comes from the fact that we will have four choices coming from the exterior $\sqrt[4]{\cdot}$. In other words, of the eight sheets, we have four pairs which form a branch rooted over $z = 0$ each of branching number 1. Finally, we need to consider the branching of that exterior $\sqrt[4]{\zeta}$. It happens at $\zeta = 0$, or in other words, at $\sqrt{z} - 1 = 0$ or $z = 1$. Notice something interesting: the three-fold branching at $z = 1$ will only happen for the positive branch of $\sqrt{z}$; indeed, the negative branch at $z = 1$ yields $\zeta = -2$ and there $\sqrt[4]{\zeta}$ doesn't branch! Write $z = 1 + \tau$ and observe that

$$\sqrt[4]{\zeta} = \sqrt[4]{\sqrt{1+\tau} - 1} = \sqrt[4]{\tau/2 + O(\tau^2)}$$

for small τ. The conclusion is that of our 8 sheets, exactly 4 will form a branch point over $z = 1$ (with branching number 3), whereas the four other sheets are unbranched at $z = 1$. Now let us compute the genus g:

$$g = 1 - 8 + \frac{1}{2}(1 + 1 + 1 + 1 + 3 + 7) = 0$$

Hence,

$$\widetilde{\mathcal{R}S}\Big(\mathbb{C}P^1, \mathbb{C}P^1, \sqrt[4]{\sqrt{z}-1}, 2\Big) \simeq S^2$$

Let us now redo the same example using the machinery developed in the previous problem. $P(w,z) = (w^4 + 1)^2 - z = 0$ is our underlying irreducible polynomial equation – so 8 sheets. Next, let's find the critical points as in part (d). These are in the z-sphere and include $z = \infty$, all zeros of the leading coefficient in w (that are none of those here since that coefficient is 1), and finally all zeros of the discriminant of $P(w,z)$ in z. Recall that these are precisely those z for which $P(\cdot, z) = 0$ and $P_w(\cdot, z)$=0 have a common solution. In our case, $P_w(w,z) = 8w^3(w^4 + 1) = 0$ iff either $w = 0$ or $w \in \{w_1, w_2, w_3, w_4\}$ where $w_j = e^{(2j+1)i\pi/4}$, $1 \leq j \leq 4$. This means that $z = 1$ or $z = 0$, respectively. You can read off a lot from this (viewing $\widetilde{\mathcal{R}S}$ somewhat imprecisely as a set of pairs (z,w)): there is a unique branch point at $(z,w) = (1,0)$ with branching number 3 (since P_w has triple zero there), and branch points at $(z,w) = (0,w_j)$ with branching number 1 for each $j = 1,2,3,4$ (in agreement with our previous analysis).

Now cut the z-sphere $\mathbb{C}P^1$ from $z = 0$ to $z = 1$ through $z = \infty$. We obtain the cut plane $A = \mathbb{C} \setminus ((-\infty, 0] \cup [1,\infty))$, which is simply connected. For each z in this set, the map $w \mapsto z = (w^4+1)^2$ is locally invertible (in fact, this is true as long as we don't hit $z = 0$ or $z = 1$). Hence, we conclude that there are 8 holomorphic functions, say $f_j(z) : A \to \mathbb{C}$, $0 \le j \le 7$ so that $z = (f_j(z)^4 + 1)^2$ identically on A. We can label these functions uniquely in terms of their behavior for large $z \in A$ where they become branches of $z^{1/8}$. Let us say that we choose

$$\lim_{y\to\infty} y^{-1/8} f_j(yi) = e^{(2j+1/2)i\pi/8}.$$

For our ramified Riemann surface $\widetilde{\mathcal{R}S}$, we note the following obvious fact: all the germs $[f_j, z]$, with $z \in A$, belong to $\widetilde{\mathcal{R}S}$ and if we lift the curve $\gamma(t) = rie^{2\pi i t}$, $0 \le t \le 1$ with $r > 1$ to $\widetilde{\mathcal{R}S}$ (i.e., perform analytic continuation along such a loop), then we see that these germs are cyclically permuted according to the cycle $(0,1,2,3,\ldots,7)$. It is a nice (and important) exercise to figure what the argument of each $f_j(x+i0)$ is for $x > 1$, $0 < x < 1$, and $x < 0$ ($|x|$ large), respectively (the argument is constant on each of the first two intervals, but not on the third). First, simply by taking $|x|$ very large,

$$\begin{aligned} f_j(x+i0) &= e^{ij\pi/4}|f_j(x+i0)| \quad \forall\, x > 1 \\ f_j(x+i0) &= e^{i\pi/8+ij\pi/4}|f_j(x+i0)| \quad \forall\, x < 0,\ |x| \to \infty \end{aligned}$$

Next, writing $z = 1+\zeta$, we obtain $f_0(z) = \sqrt[4]{\sqrt{1+\zeta}-1}$ where each of the roots is chosen to be positive (real) on the positive (real) half-axis. Then, letting $\zeta \to 0$ and $\operatorname{Im}\zeta > 0$, we observe that

$$f_0(x) = |f_0(x)|e^{\pi i/4} \quad \forall\, 0 < x < 1$$

More generally, if we pick the branch of $\sqrt{\cdot}$ with $\sqrt{x} > 0$ when $x > 0$, and the four branches of $\sqrt[4]{\cdot}$ in the natural succession (the argument increases by $\pi/4$ every time), then we obtain the branches f_0, f_2, f_4, f_6, whereas the choice of $\sqrt{x} < 0$ when $x > 0$ leads to f_1, f_3, f_5, f_7. With this in mind, you can now check that by taking ζ small as before,

$$f_{2j}(x) = |f_{2j}(x)|e^{\pi i/4 + j\pi/2} \quad \forall\, 0 < x < 1,\ j = 0,1,2,3$$

as well as

$$f_{2j+1}(x) = |f_{2j+1}(x)|e^{\pi i/4 + j\pi/2} \quad \forall\, 0 < x < 1,\ j = 0,1,2,3$$

We also see that analytic continuation along a small loop around $z = 1$ leads to the cyclic permutation $(0,2,4,6)$ (for j odd, f_j will be unchanged under analytic continuation along such a loop). Finally, from the preceding we see that analytic continuation along a small loop around $z = 0$ yields the permutation $(0,1)(2,3)(4,5)(6,7)$. All of this gives us a much more precise understanding of $\widetilde{\mathcal{R}S}$. I invite you to play with any closed curve in $\mathbb{C} \setminus \{0,1\}$ that winds around both $z = 0$ and $z = 1$ and to figure out how a germ $[f_j, 1]$ is continued analytically along that closed curve. Finally, it follows from (5.3) that $\widetilde{\mathcal{R}S}$ is simply connected. We leave it to the reader to show that $\widetilde{\mathcal{R}S}$ is simply connected without invoking relation (5.3).

Solutions for Chapter 6

Problem 6.2

Problem 6.3 Locally (in coordinates, say) every harmonic function u has a harmonic conjugate v which is unique up to a constant. Now apply Lemma 5.5 to the simply connected Riemann surface M and the family of locally defined analytic functions $u+iv$ obtained in this way. This results in a globally defined analytic function on M and thus also a global harmonic conjugate.

Problem 6.4 (a) The generating function is entire in ζ so the coefficients $J_n(z)$ are, too. Moreover, by the formula for computing Laurent coefficients,

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi i}\oint \exp\left(\frac{z}{2}(\zeta-\zeta^{-1})\right)\frac{d\zeta}{\zeta^{n+1}} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp(-in\theta + zi\sin\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - z\sin\theta)\,d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\,d\theta \end{aligned}$$

as claimed. Changing n to $-n$ and substituting $\theta+\pi$ for θ proves that $J_{-n} = (-1)^n J_n$.

(b) Simply differentiate (6.14) under the integral sign:

$$\begin{aligned} J_n'(z) &= \frac{1}{\pi}\int_0^{\pi} \sin\theta\sin(n\theta - z\sin\theta)\,d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} (n - z\cos\theta)\cos\theta\cos(n\theta - z\sin\theta)\,d\theta \end{aligned}$$

where we integrated by parts to get the second line. Also, differentiating the first line again in z yields,

$$J_n''(z) = -\frac{1}{\pi}\int_0^{\pi} \sin^2\theta\cos(n\theta - z\sin\theta)\,d\theta$$

so that

$$\begin{aligned} & z^2 J_n''(z) + zJ_n'(z) + (z^2-n^2)J_n(z) \\ &= \frac{n}{\pi}\int_0^{\pi} (z\cos\theta - n)\cos(n\theta - z\sin\theta)\,d\theta \\ &= -\frac{n}{\pi}\int_0^{\pi} \frac{d}{d\theta}\sin(n\theta - z\sin\theta)\,d\theta = 0 \end{aligned}$$

For the second part, make the power series ansatz $w(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ with $z_0 \neq = 0$. The coefficients a_0, a_1 are determined by the initial conditions $w(0)$ and $w'(0)$. Plugging this into the Bessel equation which we rewrite as

$$\begin{aligned} &[(z-z_0)^2 + 2z_0(z-z_0) + z_0^2]w''(z) + [(z-z_0)+z_0]w'(z) \\ &+ [(z-z_0)^2 + 2z_0(z-z_0) + z_0^2 - n^2]w(z) = 0 \end{aligned}$$

yields a recursion relation for the coefficients a_n. Crude estimates show that the solutions a_n of this recursion grow at most exponentially in n, so the power series will converge in some small disk around z_0, as desired.

Observe that the two solutions w_1, w_2 with $w_1(z_0) = w_2'(z_0) = 1$ and $w_1'(z_0) = w_2(z_0) = 0$ generate all solutions in a small disk around z_0: simply set

$$w(z) = w(z_0)w_1(z) + w'(z_0)w_2(z)$$

By the monodromy theorem this can be analytically continued (uniquely) to any simply connected region $G \subset \mathbb{C} \setminus \{0\}$. Hence w_1, w_2 are a fundamental system in all of G. At $z_0 = 0$ it is no longer possible to solve the Bessel equation for general initial data, and we can in general not continue analytically into the origin.

As for the final part, set $f(\zeta) = w(e^{\zeta})$. Then locally around ζ_0 the function f satisfies the ODE

$$\frac{d^2}{d\zeta^2} f + (e^{2\zeta} - n^2) f = 0$$

which admits (by power-series) analytic solutions around *every* point. Hence, again by the monodromy theorem, $f(\zeta)$ can be continued to the entire plane as an entire function.

(c) From (6.13),

$$\begin{aligned}
\sum_{\ell=0}^{\infty} \frac{1}{\ell!}\Big(\frac{z}{2}\Big)^{\ell}(\zeta - \zeta^{-1})^{\ell} &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\Big(\frac{z}{2}\Big)^{\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^{\ell-k}\zeta^{2k-\ell} \\
&= \sum_{\ell=0}^{\infty} \Big(\frac{z}{2}\Big)^{\ell} \sum_{k=0}^{\ell} \frac{(-1)^{\ell-k}}{k!(\ell-k)!}\zeta^{2k-\ell} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \Big(\frac{z}{2}\Big)^{\ell+k} \frac{(-1)^{\ell}}{k!\ell!}\zeta^{k-\ell} \\
&= \sum_{n\in\mathbb{Z}} \Big[\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \chi_{[k-\ell=n]}\Big(\frac{z}{2}\Big)^{\ell+k}\frac{(-1)^{\ell}}{k!\ell!}\Big]\zeta^n = \sum_{n=-\infty}^{\infty} J_n(z)\zeta^n
\end{aligned}$$

which gives the desired result. For negative integers use the relation $J_{-n} = (-1)^n J_n$.

(d) Simply derive a recursion relation and check that the coefficients you got in (c) are the only solution (up to a multiplicative constant). This is rather mechanical, and we skip it.

(e) The idea is to seek a solution of the form

$$\widetilde{J}_0(z) = J_0(z)\log z + \sum_{n=0}^{\infty} b_n z^n$$

Simply observe that if $w(z)$ is an analytic (around 0) solution of the Bessel equation with $n = 0$, then $\widetilde{w}(z) := w(z)\log z$ satisfies

$$z^2\widetilde{w}''(z) + z\widetilde{w}'(z) + z^2\widetilde{w}(z) = 2zw'(z)$$

The right-hand side is analytic around 0, vanishes at $z = 0$, and is even. Hence, one can uniquely solve for b_n; thus, $b_0 = 0$, and all b_n with n odd vanish. In fact, the patient reader will verify that

$$\widetilde{J}_0(z) = J_0(z)\log z - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2}\Big(\frac{z}{2}\Big)^{2n}\Big[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\Big]$$

(the usual notation for this is Y_0 which is the same up to some normalizations; after all, we can multiply by any nonzero scalar and add any multiple of J_0). A similar procedure works for all other J_n, $n \geq 1$. The reader is invited to check that J_ν and $J_{-\nu}$ with

$\nu \in \mathbb{C}\setminus\mathbb{Z}$ (as defined in part (f)) are linearly independent and thus a fundamental system for these ν. A fundamental system for $n \in \mathbb{Z}$ is then given by the limit (L'Hopital's rule)

$$J_n(z), \qquad \lim_{\nu\to n} \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$

The reader is invited to compute this for $\nu = 0$ and compare the result with $\widetilde{J}_0$ above.

(f) Verifying the formulas for $J_{\frac{1}{2}}$ and $J_{\frac{3}{2}}$ requires nothing but #4, (d) (i.e., $\Gamma(1/2) = \sqrt{\pi}$) and a comparison with the power series of cosine and sine. The fact that the definition of J_ν agrees with the previous one for nonnegative integers ν is evident. For the negative ones, use that $\frac{1}{\Gamma(z)} = 0$ at all $z \in \mathbb{Z}_0^-$. The reason that (6.14) does not yield a solution for non-integer ν is the integration by parts that was required for that purpose: we pick up non-zero boundary terms when ν is not an integer.

(g) This is proved by expanding the exponential into a power series and then by showing that you get the same series as in (6.17). Hence,

$$\begin{aligned}
&\frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_{-1}^1 e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt \\
&= \sum_{n=0}^\infty \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_{-1}^1 \frac{(izt)^n}{n!}(1-t^2)^{\nu-\frac{1}{2}}\,dt \\
&= \sum_{n=0}^\infty \frac{(iz)^{2n}}{(2n)!} \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} 2\int_0^1 t^{2n}(1-t^2)^{\nu-\frac{1}{2}}\,dt \\
&= \sum_{n=0}^\infty \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \frac{(iz)^{2n}}{(2n)!} \int_0^1 u^{n-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}\,du \\
&= \sum_{n=0}^\infty \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \frac{(-1)^n z^{2n}}{(2n)!} \frac{\Gamma(\nu+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+\nu+1)} \\
&= (z/2)^\nu \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\nu+1)}
\end{aligned}$$

where we used that

$$\Gamma(n+1/2) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$$

(use the functional equation of Γ as well as $\Gamma(1/2) = \sqrt{\pi}$). The interchange of summation and integration is justified as follows: putting absolute values inside everything (by the same argument) yields a convergent power series that converges for every $z \in \mathbb{C}$.

To verify Bessel's equation, differentiate under the integral sign (we skip this somewhat mechanical calculation).

(h) By (2.18),

$$\frac{1}{\Gamma(\nu+k+1)} = \frac{1}{2\pi i}\int_\gamma e^w w^{-(\nu+k+1)}\,dw$$

so that

$$J_\nu(z) = \frac{1}{2\pi i}\int_\gamma e^w (z/2)^\nu \sum_{k=0}^\infty \frac{(-1)^k (z/2)^{2k}}{k!} w^{-(\nu+k+1)}\, dw$$
$$= \frac{(z/2)^\nu}{2\pi i}\int_\gamma \exp\Big(w - \frac{z^2}{4w}\Big)\frac{dw}{w^{\nu+1}}$$
$$= \frac{1}{2\pi i}\int_\gamma \exp\Big(\frac{z}{2}(\zeta-\zeta^{-1})\Big)\frac{d\zeta}{\zeta^{\nu+1}}$$

where we substituted $\frac{z}{2}\zeta = w$ to pass to the last line. This proves (6.18). Finally, to pass to (6.19) substitute $\zeta = e^\tau$ in the second line of (6.18).

(i) From (6.19),

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{1}{2\pi i}\int_{\tilde\gamma} e^{z\sinh\tau - \nu\tau} 2\cosh\tau\, d\tau$$
$$= \frac{2\nu}{z}\frac{1}{2\pi i}\int_{\tilde\gamma} e^{z\sinh\tau-\nu\tau}\, d\tau = \frac{2\nu}{z}J_\nu(z)$$

as well as

$$J_{\nu-1}(z) - J_{\nu+1}(z) = \frac{1}{2\pi i}\int_{\tilde\gamma} e^{z\sinh\tau-\nu\tau} 2\sinh\tau\, d\tau = 2J_\nu'(z)$$

The second set of identities follows by adding and subtracting the two lines of the first one.

Solutions for Chapter 7

Problem 7.2 By Theorem 7.13 there exists meromorphic differentials that have simple poles with residues c_j at p_j for $1 \le j \le n$ and have a simple pole with residue $-\sum_{j=1}^n c_j$ at some other point, say p_0. By the same token there exists a meromorphic differential η with residues 1 at p_j and residue $-n$ at p_0. The ratio $\frac{\omega}{\eta}$ has the desired properties.

Problem 7.3 Follow the outline there...

Problem 7.4 Integrate up meromorphic differentials

Solutions for Chapter 8

Problem 8.1 On S^2, the principal divisors D are characterized by the condition that $\deg(D) = 0$. Indeed, on S^2 we know that a meromorphic function exists with zeros and poles at prescribed points iff the respective orders are the same (since the meromorphic functions on S^2 are precisely the rational ones). Hence two divisors are equivalent iff they have the same degree and the divisor classes are represented by $\mathbb{Z}$.

Problem **??**

Problem 8.2 Let ω be any nonzero holomorphic differential on M. Then $D = (\omega)$ satisfies $\deg(D) = 2$ by our formula for the degree of the canonical class. Hence, $\dim L(D) = 1 + \dim\Omega(D) = 2$. Here we used that $\dim\Omega(D) = 1$; clearly, $\dim\mathcal{H}\Omega^1(M) = 2$ so $\dim\Omega(D) \le 2$. In case of equality here every holomorphic differential η would need to satisfy $(\ eta) \ge D$ which contradicts Corollary 8.15. Since $\dim L(D) = 2$, we can find a nonconstant meromorphic function f with $(f) \ge D$. If $\deg(f) = 1$ this would furnish an isomorphism between M and S^2 which is impossible. Hence $\deg(f) = 2$ and we are done.

Problem 8.3

Problem 8.4 It is clear that the metric transforms correctly. It is positive definite since we know that for any point some holomorphic differential does not vanish. For the curvature, compute ...

Problem 8.5 Let z_0 be noncritical for z, i.e., $z^{-1}(z_0)$ consists of n distinct points $\{p_j\}_{j=1}^n$. We need to find a meromorphic function that "separates" these points, as we already did in the case of hyper-elliptic surfaces with $n = 2$ — there it was enough to use Corollary 7.14. Here we need the stronger Riemann-Roch theorem. In fact, use Riemann-Roch to show that for each $1 \le \ell \le n$ there exists a meromorphic function g_ℓ on M which has a pole at p_ℓ and zeros at each p_k, $1 \le k \le n$, $k \ne \ell$. Now chose n distinct complex numbers ζ_j, $1 \le j \le n$ and set

$$f := \Big(\sum_{j=1}^n \zeta_j g_j\Big)\Big(\sum_{j=1}^n g_j\Big)^{-1}$$

Check by contradiction that this f generates an irreducible polynomial via the procedure of Proposition 5.16.

Problem 8.6

Problem **??**

Problem **??**

Solutions for Chapter 10

Problem 10.1 To see that $\mathbb{C}\setminus\{z_j\}_{j=1}^J$ is not hyberbolic, use the proof idea of Lemma 10.2. To see that $\mathbb{C} \setminus (\mathbb{D} \cup \{z_j\}_{j=1}^J)$ is hyperbolic, apply $\frac{1}{z}$ to map it conformally onto $\mathbb{D}^* \setminus \{z_j^{-1}\}_{j=1}^J$ which is hyperbolic.

CHAPTER 13

Review of some facts from algebra and geometry

1. Geometry and topology

13.1. Classification of compact topological manifolds. A *topological manifold* M is a second countable, connected, Hausdorff space, which is locally homeomorphic to open subsets of the plane. A curve in M is a continuous map $c : [0,1] \to M$ and a closed curve satisfies $c(0) = c(1)$. On those curves, we define a boundary operator $\partial c := c(1) - c(0)$ which characterizes the closed curves c as all curves satisfying $\partial c = 0$. The difference of points here is a formal construction: we define a 0-cycle to be an arbitrary finite sum $\sum_j m_j p_j$ where $m_j \in \mathbb{Z}$ and $p_j \in M$. A 1-cycle is a formal finite sum of closed curves with integer coefficients $c := \sum_j n_j c_j$ where all c_j are closed curves. The boundary operator ∂ extends via linearity, i.e., $\partial c = \sum_j n_j \partial c_j$. We say that c is closed iff $\partial c = 0$ and *exact* iff $c = \partial\sigma$ where σ is a 2-cycle.

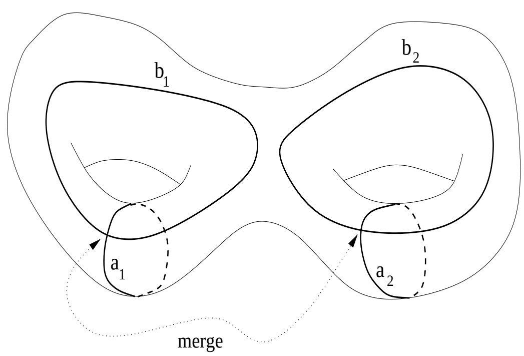

FIGURE 13.1. A homology basis

The *homology* groups of M, denoted by $H^k(M;\mathbb{Z})$ are

13.2. Differential forms. For the material here, see do Carmo [**8**], and Madsen, Tornehave [**30**]. Every differentiable manifold M carries vector fields (defined as smooth sections of the tangent bundle) and differential 1-forms (defined as smooth sections of the co-tangent bundle). In local coordinates $(x^1, x^2, \ldots, x^n)$ with $n = \dim(M)$, a 1-form is given by

$$\omega = \sum_{j=1}^{n} a_j \, dx^j =: a_j \, dx^j$$

(summation convention) with smooth a_j on some open set. If we change co-ordinates from x to y, then

$$\omega = a_j \frac{\partial x^j}{\partial y^k} \, dy^k$$

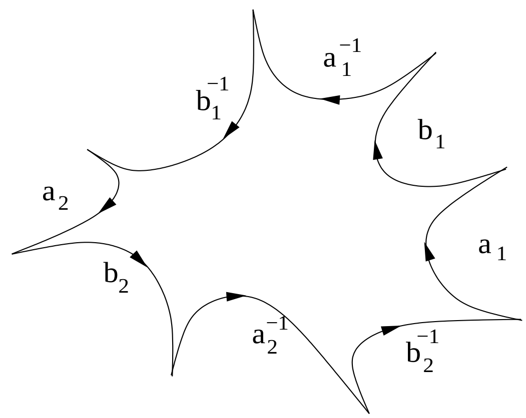

FIGURE 13.2. The fundamental polygon of a compact surface of genus g

This is referred to as a co-variant transformation rule of the coefficients. In contrast, vector fields transform contra-variantly

$$v = v^j \frac{\partial}{\partial x^j} \longrightarrow v = v^j \frac{\partial y^k}{\partial x^j} \frac{\partial}{\partial y^k}$$

More generally, the k-forms $\Omega^k(M)$ are smooth sections of the bundle $\Lambda^k(M)$ of alternating k-forms on M. In this book, $\dim(M) = 2$ and only $k = 0, 1, 2$ are relevant. However, $\Omega^0(M) = C^\infty(M)$ and $\Omega^2(M)$ are the so-called volume forms. This refers to the fact that

$$\int_M f\omega$$

is well-defined for every $f \in C^\infty(M)$ with compact support and $\omega \in \Omega^2(M)$. If M is orientable (as a Riemann surface is), then there exists a volume form ω_0 so that at every point $p \in M$ there are local co-ordinates (x, y) with $(x, y)(p) = (0, 0)$ and

$$\omega_0 = f(x, y)\, dx \wedge dy$$

with $f(0, 0) \neq 0$. This then implies that $\Omega^2(M) \simeq C^\infty(M)$ since every 2-form is a multiple of ω_0. There are two operations on forms that we shall need: the exterior product (denoted by $\wedge$) and the exterior differentiation (denoted by d). If $\omega_1 \in \Omega^{\ell_1}(M)$ and $\omega_2 \in \Omega^{\ell_2}(M)$, then

$$\omega_1 \wedge \omega_2 \in \Omega^{\ell_1 + \ell_2}(M), \qquad \omega_1 \wedge \omega_2 = (-1)^{\ell_1 \ell_2} \omega_2 \wedge \omega_1$$

as well as

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{\ell_1} \omega_1 \wedge (d\omega_2), \quad d(f\omega) = df \wedge \omega + f d\omega$$

for any $f \in \Omega^0(M)$. In local coordinates,

$$df = \frac{\partial f}{\partial x^j}\, dx^j, \qquad d(a_j\, dx^j) = \frac{\partial a_j}{\partial x^k}\, dx^k \wedge dx^j$$

From the chain rule one verifies that exterior product and differentiation commute with pullbacks. I.e., if $f : M \to N$ is a smooth map between differentiable manifolds, then the pullback f_* satisfies

$$f_*(\omega \wedge \eta) = f_*\omega \wedge f_*\eta, \qquad d(f_*\omega) = f_*(d\omega)$$

for any differential forms ω, η on N. By equality of mixed partial $d^2 f = 0$, and $d^2\omega = 0$ for any $\omega \in \Omega^1$ because $d\Omega^2 = \{0\}$ (in fact, $d^2 = 0$ always). We say that $\omega \in \Omega^1(M)$ is closed iff $d\omega = 0$ and ω is exact if $\omega = df$ for some $f \in \Omega^0(M)$. By the preceding, exact forms are closed, but the converse is not true (consider the form $d\theta$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ where θ is the polar angle). The Poincaré lemma says that *locally* all closed forms are exact. From Poincaré's lemma and Stokes' theorem one obtains that for any closed 1-form ω

$$\int_{c_1} \omega = \int_{c_2} \omega$$

if c_1, c_2 have the same end-points and are homotopic. Similarly,

$$\oint_{c_1} \omega = \oint_{c_2} \omega$$

if c_1, c_2 are closed homotopic curves.

13.3. de Rham cohomology. Of great topological importance are the de Rham spaces, which are characterized as closed forms modulo exact forms, i.e., for each $k \geq 1$,

$$H^k(M) := \{f \in \Omega^k(M) \,:\, df = 0\}/\{dg \,:\, g \in \Omega^{k-1}(M)\}$$

If M is compact, then these spaces have finite dimension. The dimensions are the Betti numbers $\beta_k(M)$ and the de Rham theorem says that they agree with the dimensions (over $\mathbb{Z}$) of the homology groups. For us the only really relevant case is $k = 1$, whereas for $k = 2$ one has $H^2(M) \simeq \mathbb{R}$ due to the orientability of M. Finally, we note that the pull back via smooth maps is well-defined on the cohomology since it commutes with the exterior differentiation. Moreover, the pull back map on the cohomology is the same for any two smooth functions which are homotopic.

13.4. The degree. Next, we recall the notion of degree from topology and check that it coincides with the degree defined in Chapter 4 for Riemann surfaces. For the sake of this paragraph alone, let M, N be n-dimensional smooth orientable, connected compact manifolds. Then integration defines a linear isomorphism

$$H^n(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$$

where $H^n(M)$ is the de Rham space of n-forms modulo exact n-forms. Let $f : M \to N$ be a smooth map and $f_* : H^n(N) \to H^n(M)$ the induced map defined via the pull-back. There exists a real number denoted by $\deg(f)$ such that

$$\int_M f_*(\omega) = \deg(f) \int_N \omega \qquad \forall \omega \in H^n(N)$$

Since the pull-back map on the cohomology spaces H^n only depends on the homotopy class, so does $\deg(f)$. It is also easy to see that it is multiplicative with regard to composition. Changing variables in charts, it is easy to verify that for any *regular* value $q \in N$ (which means that $Df(p) : T_pM \to T_qN$ is invertible for every p with $f(p) = q$)

$$\deg(f) = \sum_{p \in M:\, f(p)=q} \mathrm{Ind}(f;p)$$

where $\mathrm{Ind}(f;p) = \pm 1$ depending on whether $Df(p)$ preserves or reverses the orientation. In particular, $\deg(f) \in \mathbb{Z}$ and $\deg(f) \neq 0$ implies that f is onto. By Sard's theorem, the regular values are always dense in N (an excellent reference for all of this is [**30**]).

Returning to Riemann surfaces, we see that this notion of degree coincides exactly with the one from Chapter 4 since every analytic $f : M \to N$ necessarily preserves the orientation.

13.5. Euler characteristic. Every topological two–dimensional compact manifold M has an integer $\chi(M)$ associated with itself, called the Euler characteristic. It is defined as

$$V - E + F = \chi(M), \quad V = \text{vertices},\ E = \text{edges},\ F = \text{faces}$$

relative to an arbitrary triangulation of M (this is the *homological* characterization of

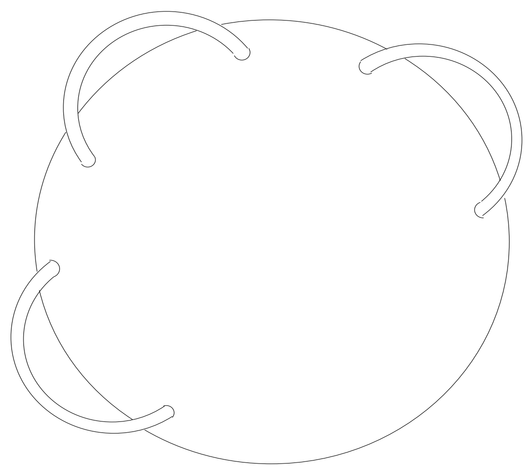

FIGURE 13.3. A sphere with k handles has genus k

$\chi(M)$); this is well-defined, in other words, $V - E + F$ does not depend on the particular choice of triangulation. Another important theorem relates the Euler characteristic with the genus g of M: if we realize M as S^2 with g handles attached, then we have the Euler–Poincaré formula

$$\chi(M) = 2 - 2g \tag{13.1}$$

Finally, let us recall the cohomological characterization of $\chi(M)$: let M be a compact, smooth two–dimensional manifold and let $H^k(M)$ denote the de Rham spaces of closed k forms modulo exact forms, $0 \le k \le 2$. Then, with $\beta_k := \dim H^k(M)$,

$$\chi(M) = \beta_0 - \beta_1 + \beta_2$$

If M is orientable (as in the case of a Riemann surface), then it is easy to see that $\beta_0 = \beta_2 = 1$. Thus, $\beta_1 = 2g$ where g is the genus.

2. Algebra

For the material here, see for example [**28**]. Given two relatively prime polynomials $P, Q \in \mathbb{C}[w, z]$, there exist $A, B \in \mathbb{C}[w, z]$ such that

$$A(w, z)P(w, z) + B(w, z)Q(w, z) = R(z) \in \mathbb{C}[z]$$

is a *nonzero* polynomial in z alone; it is called the *resultant* of P and Q. The proof of this fact is Euclid's algorithm carried out in the polynomials in $K(z)[w]$ where $K(z)$ is the quotient field of $\mathbb{C}[z]$, i.e., the field of rational functions of z. The resultant has many interesting properties, for example, if both P and Q have leading coefficient 1, then

$$R(z) = \prod_{\zeta_j, \eta_k} (\zeta_j(z) - \eta_k(z))$$

where ζ_j runs over all zeros of $P(w, z)$ and η_k runs over all zeros of $Q(w, z)$ in w, respectively. Thus, $R(z_0) = 0$ iff $P(w, z_0)$ and $Q(w, z_0)$ have a common zero in w. Moreover, with

$$P(w, z) = \sum_{j=0}^{n} a_j(z) w^j, \qquad Q(w, z) = \sum_{k=0}^{m} b_k(z) w^k$$

it follows that $R(z)$ equals the following determinant in the coefficients a_j, b_k

$$\begin{vmatrix} \overbrace{\begin{matrix} a_k & 0 & \cdots \\ a_{k-1} & a_k & \cdots \\ a_{k-2} & a_{k-1} & \cdots \\ \cdots & \cdots & \cdots \\ a_0 & a_1 & \cdots \\ 0 & a_0 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{matrix}}^{m} & \overbrace{\begin{matrix} b_m & 0 & \cdots & 0 \\ b_{m-1} & b_m & \cdots & \cdots \\ b_{m-2} & b_{m-1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}}^{n} \end{vmatrix} \tag{13.2}$$

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