TOPIC. Cumulants. Just as the generating function M of a random variable X "generates" its moments, the logarithm of M generates a sequence of numbers called the cumulants of X. Cumulants are of interest for a variety of reasons, an especially important one being the fact that the j^{th} cumulant of a sum of independent random variables is simply the sum of the j^{th} cumulants of the summands.

Definitions and examples. To begin with, suppose that X is a real random variable whose real moment generating function $M(u) = E(e^{uX})$ is finite for all u's in an open interval about 0. Since $M(0) = 1 \neq 0$ and since the composition of two functions which have power series expansions itself has a power series expansion, we may write

$$(\log M)(u) = \sum_{n=0}^{\infty} \frac{\kappa_n u^n}{n!} \tag{1}$$

for all u in some (possibly smaller) open interval about 0. The numbers $\kappa_1, \kappa_2, \ldots$ in this expansion are called the ***cumulants*** (for a reason that will be explained subsequently). Notice that

$$\kappa_n = (\log M)^{(n)}(0) \tag{2}$$

all n; in particular $\kappa_0 = \log(M(0)) = \log(1) = 0$. (1) implies (see Theorem 12.4) that for all z in some open ball about 0 in $\mathbb{C}$

$$(\log G)(z) = \sum_{n=1}^{\infty} \frac{\kappa_n z^n}{n!}; \tag{3}$$

here $G(z) = E(e^{zX})$ is the complex generating function and "log" denotes the principal branch of the complex logarithm function. In particular the characteristic function $\phi(t) = E(e^{itX})$ of X satisfies

$$K(t) := \log(\phi(t)) = \sum_{n=1}^{\infty} \frac{\kappa_n i^n t^n}{n!} \tag{4}$$

for all t in some open interval about 0 in $\mathbb{R}$, and

$$\kappa_n = K^{(n)}(0)/i^n \tag{5}$$

for all $n \in \mathbb{N}$. Since the functions $\log M$, $\log G$, and $K = \log\phi$ generate the cumulants, they are called ***cumulant generating functions*** (***CGFs***). (Some properties of cumulants and their generating functions were developed in the exercises in Section 11. None of those results are used here. Moreover there is a change in notation — in Section 11 $K(t)$ denoted $\log\bigl(M(t)\bigr)$, whereas here $K(t)$ is $\log\bigl(\phi(t)\bigr)$.)

Formula (5) is used to define the first n cumulants of X when X is only known to have an n^{th} moment, i.e., when $E(|X|^n) < \infty$. In this situation ϕ is n-times continuously differentiable and close to 1 in an open interval about $0 \in \mathbb{R}$, and so $K = \log(\phi)$ is defined and n-times continuously differentiable in that interval. Consequently the j^{th} cumulant

$$\kappa_j := K^{(j)}(0)/i^j \tag{6}$$

exists for $j = 1, \ldots, n$, and

$$K(t) = \sum_{j=1}^{n} \frac{i^j \kappa_j t^j}{j!} + o(t^n) \ \text{ as } t \to 0. \tag{7}$$

The following observation is useful for recognizing the cumulants. If X has a n^{th} moment and $\lambda_1, \ldots, \lambda_n$ are numbers such that

$$K(t) = \sum_{j=1}^{n} \frac{i^j \lambda_j t^j}{j!} + o(t^n) \ \text{ as } t \to 0, \tag{8_1}$$

then (see Exercise 1)

$$\kappa_j = \lambda_j \ \text{ for } j = 1, \ldots, n\,. \tag{8_2}$$

Example 1. (A) Suppose $X \sim N(\mu, \sigma^2)$. Then

$$\phi(t) = e^{i\mu t} e^{-\sigma^2 t^2/2} \quad \text{and} \quad K(t) = \log(\phi(t)) = i\mu t - \frac{\sigma^2 t^2}{2}$$

for t near 0. X has infinitely many cumulants since its moment generating function is finite in a neighborhood of 0 (actually, finite everywhere); formula (5) gives

$$\kappa_1 = \mu, \ \ \kappa_2 = \sigma^2, \ \ \kappa_3 = \kappa_4 = \cdots = 0. \tag{9}$$

Since the first and second cumulants of any random variables are its mean and variance (see (19) and (20)), the normal distribution has the simplest possible cumulants.

(B) Suppose $X \sim \text{Gamma}(r)$. Then

$$\phi(t) = \left(\frac{1}{1-it}\right)^r \quad \text{and}$$

$$K(t) = -r\log(1-it) = r\sum_{n=1}^{\infty}\frac{(it)^n}{n} = r\sum_{n=1}^{\infty}\frac{(n-1)!\,(it)^n}{n!}$$

for t near 0. X has cumulants of all orders. By the uniqueness of the coefficients in the power series $\sum_{n=1}^{\infty}\kappa_n i^n t^n/n!$, we get

$$\kappa_n = r(n-1)! \text{ for } n \geq 1. \tag{10}$$

(C) Suppose that $X \sim \text{Poisson}(\lambda)$. Then

$$\phi(t) = e^{\lambda(e^{it}-1)} \quad \text{and} \quad K(t) = \lambda(e^{it}-1) = \sum_{n=1}^{\infty}\frac{\lambda i^n t^n}{n!}$$

for t near 0. Evidently

$$\kappa_1 = \kappa_2 = \kappa_3 = \cdots = \lambda. \tag{11}$$

(D) Suppose that X has an unnormalized t-distribution with 3 degrees of freedom; its density has the form

$$\frac{c}{(1+x^2)^2}.$$

X has only two moments, and thus only two cumulants. It turns out (see Exercise 13.11) that $\phi(t) = e^{-|t|}(1+|t|)$ for all $t \in \mathbb{R}$. Thus

$$\begin{aligned} K(t) &= -|t| + \log\bigl(1+|t|\bigr) \\ &= -|t| + \left(|t| - \frac{|t|^2}{2} + \frac{|t|^3}{3} + \cdots\right) \quad (\text{for } |t|<1) \\ &= -\frac{t^2}{2} + o(t^2) = 0\frac{it}{1!} + 1\frac{i^2t^2}{2!} + o(t^2) = \kappa_1\frac{it}{1!} + \kappa_2\frac{i^2t^2}{2!} + o(t^2) \end{aligned}$$

as $t \to 0$. (8_2) gives

$$\kappa_1 = 0 \text{ and } \kappa_2 = 1. \tag{12}$$

κ_3, κ_4, ... are undefined. •

Properties of cumulants. This section develops some useful properties of cumulants. The n^{th} moment of cX is c^n times the n^{th} moment of X; this scaling property is shared by the cumulants.

Theorem 1 (Homogeneity). *Suppose X is a random variable with an n^{th} cumulant. Then for any $c \in \mathbb{R}$, cX has an n^{th} cumulant and*

$$\kappa_n(cX) = c^n\kappa_n(X). \tag{13}$$

Proof For t near 0 we have

$$\begin{aligned} &\phi_{cX}(t) = E(e^{itcX}) = \phi_X(ct) \\ &\implies K_{cX}(t) = K_X(ct) \\ &\implies K_{cX}^{(n)}(t) = c^n K_X^{(n)}(ct) \\ &\implies \kappa_n(cX) = K_{cX}^{(n)}(0)/i^n = c^n K_X^{(n)}(0)/i^n = c^n\kappa_n(X). \end{aligned}$$

∎

The n^{th} moment of $X+b$ is a linear combination of the first n moments of X (with what coefficients?). The situation regarding cumulants is much simpler:

Theorem 2 (Semi-invariance). *Suppose X is a random variable with an n^{th} cumulant. Then for any $b \in \mathbb{R}$, $X+b$ has an n^{th} cumulant and*

$$\kappa_n(X+b) = \begin{cases} \kappa_n(X) + b, & \text{if } n = 1, \\ \kappa_n(X), & \text{if } n > 1. \end{cases} \tag{14}$$

Proof For t near 0 we have

$$\begin{aligned} &\phi_{X+b}(t) = E(e^{it(X+b)}) = e^{itb}E(e^{itX}) = e^{itb}\phi_X(t) \\ &\implies K_{X+b}(t) = itb + K_X(t) \\ &\implies K_{X+b}^{(n)}(t) = \tfrac{d^n}{dt^n}itb + K_X^{(n)}(t) \\ &\implies (14) \text{ holds.} \end{aligned}$$

∎

Here is the reason for the name "cumulants"; note that (16) is much simpler than the corresponding relation for moments.

Theorem 3 (Cumulants accumulate). *Suppose X and Y are independent random variables, each having an n^{th} cumulant. Then $S := X + Y$ has an n^{th} cumulant, and*

$$\kappa_n(S) = \kappa_n(X) + \kappa_n(Y). \tag{15}$$

Proof $\phi_S(t) = \phi_X(t)\phi_Y(t)$, so $K_S(t) = K_X(t) + K_Y(t)$. ∎

Now we investigate the relationship between moments and cumulants. We first consider moments about 0, which we write as

$$\alpha_j := E(X^j) \tag{16}$$

for $j = 0, 1, 2, \ldots$. Note that $\alpha_0 = 1$.

Theorem 4 (The cumulant/moment connection). *Suppose X is a random variable with n moments $\alpha_1, \ldots, \alpha_n$. Then X has n cumulants $\kappa_1, \ldots, \kappa_n$, and*

$$\alpha_{r+1} = \sum_{j=0}^{r} \binom{r}{j} \alpha_j \kappa_{r+1-j} \quad \textit{for } r = 0, \ldots, n-1. \tag{17}$$

Proof For $j = 0, \ldots, n$ we have

$$\alpha_j = \phi^{(j)}(0)/i^j \quad \text{and} \quad \kappa_j = K^{(j)}(0)/i^j$$

where $\phi(t) = E(e^{itX})$ and $K(t) = \log(\phi(t))$, or, equivalently, $\phi(t) = e^{K(t)}$, for all t near 0. Differentiating this last identity gives

$$\phi'(t) = e^{K(t)}K'(t) = \phi(t)K'(t) \tag{18}$$

and evaluating this at $t = 0$ gives

$$i\alpha_1 = 1(i\kappa_1) \Longrightarrow \alpha_1 = \kappa_1 \Longrightarrow \text{(17) holds for } r = 0.$$

Differentiating (18) r times gives (see Exercise 3)

$$\phi^{(r+1)}(t) = \sum_{j=0}^{r} \binom{r}{j} \phi^{(j)}(t)\ (K')^{r-j}(t)$$

and evaluating this at $t = 0$ shows that (17) holds for $1 \le r < n$. ∎

(17): $\alpha_{r+1} = \sum_{j=0}^{r} \binom{r}{j} \alpha_j \kappa_{r+1-j}$ for $r = 0, \ldots, n-1$.

Writing out (17) for $r = 0, \ldots, 3$ produces

$$\begin{aligned} \alpha_1 &= \kappa_1, \\ \alpha_2 &= \kappa_2 + \alpha_1\kappa_1, \\ \alpha_3 &= \kappa_3 + 2\alpha_1\kappa_2 + \alpha_2\kappa_1, \\ \alpha_4 &= \kappa_4 + 3\alpha_1\kappa_3 + 3\alpha_2\kappa_2 + \alpha_3\kappa_1. \end{aligned} \tag{19}$$

These recursive formulas can be used to calculate the α's efficiently from the κ's, and vice versa. When X has mean 0, that is, when $\alpha_1 = 0 = \kappa_1$, α_j becomes

$$\mu_j := E\big((X - E(X))^j\big)$$

and formulas (19) simplify to

$$\begin{aligned} \mu_2 &= \kappa_2, & \kappa_2 &= \mu_2, \\ \mu_3 &= \kappa_3, & \kappa_3 &= \mu_3 \\ \mu_4 &= \kappa_4 + 3\kappa_2^2, & \kappa_4 &= \mu_4 - 3\mu_2^2. \end{aligned} \tag{20}$$

Since the central moments $\mu_2, \mu_3, \ldots$ and the cumulants $\kappa_2, \kappa_3, \ldots$ are unaffected by adding a constant to X, these formulas are valid even when $E(X) \neq 0$. Note that μ_2 is simply the variance of X.

Example 2. The following display exhibits the moment/cumulant connection for some important distributions:

$X \sim N(\nu, \tau^2)$		$X \sim \text{Gamma}(r)$		$X \sim \text{Poisson}(\lambda)$	
$\kappa_1 = \nu$,	$\alpha_1 = \nu$,	$\kappa_1 = r$,	$\alpha_1 = r$,	$\kappa_1 = \lambda$,	$\alpha_1 = \lambda$,
$\kappa_2 = \tau^2$,	$\mu_2 = \tau^2$,	$\kappa_2 = r$,	$\mu_2 = r$	$\kappa_2 = \lambda$,	$\mu_2 = \lambda$,
$\kappa_3 = 0$,	$\mu_3 = 0$,	$\kappa_3 = 2r$,	$\mu_3 = 2r$,	$\kappa_3 = \lambda$,	$\mu_3 = \lambda$,
$\kappa_4 = 0$,	$\mu_4 = 3\tau^4$,	$\kappa_4 = 6r$,	$\mu_4 = 6r + 3r^2$,	$\kappa_4 = \lambda$,	$\mu_4 = \lambda + 3\lambda^2$.

More on the cumulant/moment connection. Equations (19) express $\alpha_1, \ldots, \alpha_4$ recursively in terms of $\kappa_1, \ldots, \kappa_4$. By carrying out the recursions one finds that

$$\begin{aligned}
\alpha_1 &= \kappa_1, \\
\alpha_2 &= \kappa_2 + \kappa_1^2, \\
\alpha_3 &= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3, \\
\alpha_4 &= \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4.
\end{aligned}$$

Note that the formula for α_4 is a linear combination of products of κ_j's and that, including multiplicities, the subscripts j on the κ_j's in those products form the following lists:

$$[4],\ [3,1],\ [2,2],\ [2,1,1],\ \text{and } [1,1,1,1]\,.$$

These are all the possible lists of nonincreasing positive integers which add to 4; they're called the ***additive partitions*** of 4. In what follows I am going to show that for any $n \in \mathbb{N}$, α_n is a sum of the form $\sum_\pi c_\pi \kappa_\pi$ where π ranges over the additive partitions of n, c_π is a certain number depending on π, and $\kappa_\pi = \prod_{j\in\pi} \kappa_j$.

We need some notation. For a positive integer n let

$$\mathcal{P}_n \text{ be the collection of all additive partitions of } n. \tag{21}$$

By definition an element π of $\mathcal{P}_n$ has the form

$$\pi = [\underbrace{j_1, \ldots, j_1}_{m_1 \text{ times}}, \underbrace{j_2, \ldots, j_2}_{m_2 \text{ times}}, \ldots, \underbrace{j_k, \ldots, j_k}_{m_k \text{ times}}] \tag{22_1}$$

for some number k, the j_i's and m_i's being positive integers satisfying $j_1 > j_2 > \cdots > j_k$ and $n = \sum_{i=1}^k m_i j_i$. Shorthand for (22_1) is

$$\pi = [\, j_1^{m_1}, j_2^{m_2}, \ldots, j_k^{m_k}]; \tag{22_2}$$

note that in (22_2) the notation $j_i^{m_i}$ means "replicate j_i a total of m_i times", not "raise j_i to the m_i^{th} power". For example $[4^1, 2^2, 1^3]$ denotes the partition $[4,2,2,1,1,1]$ of $n = 11$; this partition has 6 el-

(22): $\pi = [\, j_1^{m_1}, j_2^{m_2}, \ldots, j_k^{m_k}] := [\underbrace{j_1, \ldots, j_1}_{m_1 \text{ times}}, \underbrace{j_2, \ldots, j_2}_{m_2 \text{ times}}, \ldots, \underbrace{j_k, \ldots, j_k}_{m_k \text{ times}}]$.

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ements, of which only 3 (namely 4, 2, and 1) are distinct. In general the partition (22) has a total of

$$\nu_\pi := m_1 + m_2 + \cdots + m_k \tag{23}$$

elements, of which $k_\pi := k$ (namely, $j_1, j_2, \ldots, j_k$) are distinct. The quantities

$$c_\pi := \frac{n!}{(j_1!)^{m_1}(j_2!)^{m_2}\cdots(j_k!)^{m_k}}\,\frac{1}{m_1!\,m_2!\,\cdots\,m_k!} \tag{24}$$

$$d_\pi := c_\pi \times (-1)^{\nu_\pi - 1}(\nu_\pi - 1)! \tag{25}$$

play an important role in what follows, as do the products

$$\alpha_\pi := \prod\nolimits_{j\in\pi} \alpha_j \quad \text{and} \quad \kappa_\pi := \prod\nolimits_{j\in\pi} \kappa_j\,. \tag{26}$$

Theorem 5. *For a random variable having moments $\alpha_1, \ldots, \alpha_n$ and cumulants $\kappa_1, \ldots, \kappa_n$,*

$$\alpha_n = \sum\nolimits_\pi c_\pi \kappa_\pi \quad \text{and} \quad \kappa_n = \sum\nolimits_\pi d_\pi \alpha_\pi\,; \tag{27}$$

the sums here are taken over the elements $\pi = [\, j_1^{m_1}, j_2^{m_2}, \ldots, j_k^{m_k}]$ of $\mathcal{P}_n$ and c_π, d_π, α_π, and κ_π are defined by (24)–(26) above.

Example 3. One of the additive partitions of 11 is $\pi = [4^1, 2^2, 1^3] = [\, j_1^{m_1}, j_2^{m_2}, j_3^{m_3}]$ for $j_1 = 4$, $j_2 = 2$, $j_3 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$. According to (27), the contribution this π makes to α_{11} is $c_\pi \kappa_\pi$ where

$$\kappa_\pi := \kappa_4 \kappa_2^2 \kappa_1^3 \quad \text{and} \quad c_\pi := \frac{11!}{4!\,(2!)^2\,(1!)^3}\,\frac{1}{1!\,2!\,3!} = 34650\,.$$

Moreover since $\nu_\pi = 1 + 2 + 3 = 6$, the contribution π makes to κ_{11} is $d_\pi \alpha_\pi$ where

$$\alpha_\pi = \alpha_4 \alpha_2^2 \alpha_1^3 \quad \text{and} \quad d_\pi = c_\pi\,(-1)^5\,5! = -4158000\,. \qquad \bullet$$

As we will see, Theorem 5 is a special case of the formula given below for the n^{th} derivative of the composition of two functions. The formula applies to real or complex valued functions of a real or complex variable. For example, one of the functions may be a characteristic function, which is (in general) a complex-valued function of a real variable, and the other may be the complex logarithm function. Recall that a function is said to ***have an n^{th} derivative at a point*** if the function is $(n-1)$-times differentiable in an open neighborhood of the point and the $(n-1)^{\text{st}}$ derivative is differentiable at the point.

Theorem 6 (Faà di Bruno's formula). *Let n be a positive integer and let f and g be two functions such that: the composite function $h := g(f)$ is defined in an open neighborhood of a point x; f has an n^{th} derivative at x; and g has an n^{th} derivative at $y := f(x)$. Then h has an n^{th} derivative at x given by the formula*

$$h^{(n)}(x) = \sum\nolimits_{\pi\in\mathcal{P}_n} c_\pi\, g^{(\nu_\pi)}(y)\Big[\prod\nolimits_{j\in\pi} f^{(j)}(x)\Big] \tag{28}$$

where c_π and ν_π are defined by (24) and (23) respectively, $f^{(j)}(x)$ is the j^{th} derivative of f at x, and $g^{(\nu)}(y)$ is the ν^{th} derivative of g at y.

Example 4. For the additive partition $[2^1, 1^1]$ of $n = 3$ one has $\nu_\pi = 1 + 1 = 2$ and $c_\pi = 3!/[(2^1 1!)(1^1 1!)] = 3$. According to Faà di Bruno's formula, the third derivative of $h = g(f)$ should contain the term $3g''\big(f(x)\big)f'(x)f''(x)$. This can be verified by direct calculation:

$$\begin{aligned} h' &= g'(f)f', \\ h'' &= g''(f)(f')^2 + g'(f)f'', \\ h''' &= \big[g'''(f)\,(f')^3 + 2g''(f)f'f''\big] + \big[g''(f)f'f'' + g'(f)f'''\big] \\ &= \quad g'''(f)(f')^3 + 3g''(f)f'f'' + g'(f)f'''. \end{aligned}$$ •

Example 5. Let X be a random variable with characteristic function ϕ, cumulant generating function $K = \log(\phi)$, moments $\alpha_1, \ldots, \alpha_n$ and cumulants $\kappa_1 \ldots \kappa_n$. $i^n\alpha_n$ is $h^{(n)}(x)$ for $x = 0$ and $h(t) = \phi(t) = e^{K(t)} = g\big(f(t)\big)$ with $f(t) = K(t)$ and $g(z) = e^z$. (28) applies because K is defined in a neighborhood of t. Since $f^{(j)}(x) = i^j\kappa_j$ and $g^{(\nu)}(z) = e^z$ equals 1 at $z = y = f(x) = 0$, Faà di Bruno's formula (28) implies

$$i^n\alpha_n = \sum\nolimits_{\pi\in\mathcal{P}_n} c_\pi\Big[\prod\nolimits_{j\in\pi} i^j\kappa_j\Big] = i^n\sum\nolimits_{\pi\in\mathcal{P}_n} c_\pi\kappa_\pi;$$

dividing through by i^n gives the LHS of (27). Similarly, the RHS of (27) follows by applying (28) for $x = 0$ to $K(t) = g\big(\phi(t)\big)$ with $g(z) = \log(z)$ and using $g^{(\nu)}(z) = (-1)^{\nu-1}(\nu-1)!/z^\nu = (-1)^{\nu-1}(\nu-1)!$ for $z = y = \phi(0) = 1$. •

Proof of Theorem 6. The method used in Example 4 shows that h has an n^{th} derivative at x, so that by Taylor's theorem

$$h(\xi) = \sum\nolimits_{j=0}^{n}\frac{h^{(j)}(x)}{j!}(\xi - x)^j + o\big(|\xi - x|^n\big) \text{ as } \xi \to x.$$

I am going to show that

$$h(\xi) = \sum\nolimits_{j=0}^{n}\frac{H_j}{j!}(\xi - x)^j + o\big(|\xi - x|^n\big) \text{ as } \xi \to x$$

for certain numbers $H_0, H_1, \ldots, H_n$, with H_n being defined by the RHS of (28). This implies $h^{(j)}(x) = H_j$ for $j = 0, \ldots, n$, and in particular that $h^{(n)}(x) = H_n$, as (28) asserts.

$h = g(f)$ (28): $h^{(n)}(x) = \sum_{\pi \in \mathcal{P}_n} c_\pi \, g^{(\nu_\pi)}\big(f(x)\big) \big[\prod_{j\in\pi} f^{(j)}(x)\big]$

Since f has an n^{th} derivative at x, Taylor's theorem implies that

$$f(\xi) = \sum\nolimits_{j=0}^{n} \frac{f_j}{j!}(\xi - x)^j + o\big(|\xi - x|^n\big) \text{ as } \xi \to x \tag{29}$$

for $f_j = f^{(j)}(x)$. Similarly, since g has an n^{th} derivative at $y = f(x)$,

$$g(\eta) = \sum\nolimits_{\nu=0}^{n} \frac{g_\nu}{\nu!}(\eta - y)^\nu + o\big(|\eta - y|^n\big) \text{ as } \eta \to y \tag{30}$$

for $g_\nu = g^{(\nu)}(y)$. Thus

$$\begin{aligned} h(\xi) = g\big(f(\xi)\big) &= \sum\nolimits_{\nu=0}^{n} \frac{g_\nu}{\nu!}\big(f(\xi) - f(x)\big)^\nu + o\big(|f(\xi) - f(x)|^n\big) \\ &= \sum\nolimits_{\nu=0}^{n} \frac{g_\nu}{\nu!}\Big(\sum\nolimits_{i=1}^{n} \frac{f_i}{i!}(\xi - x)^i + o\big(|\xi - x|^n\big)\Big)^\nu + o\big(|\xi - x|^n\big) \\ &= \sum\nolimits_{\nu=0}^{n} \frac{g_\nu}{\nu!}\Big(\sum\nolimits_{i=1}^{n} \frac{f_i}{i!}(\xi - x)^i\Big)^\nu + o\big(|\xi - x|^n\big) \\ &= \sum\nolimits_{j=0}^{n} \frac{H_j}{j!}(\xi - x)^j + o\big(|\xi - x|^n\big) \end{aligned} \tag{31}$$

as $\xi \to x$, where

$$\begin{aligned} H_n &= n! \sum\nolimits_{\nu=1}^{n} \frac{g_\nu}{\nu!}\Big[\sum\nolimits_{\substack{i_1, i_2, \ldots, i_\nu \geq 1 \\ i_1 + i_2 + \cdots + i_\nu = n}} \frac{f_{i_1} f_{i_2} \cdots f_{i_\nu}}{i_1!\, i_2! \cdots i_\nu!}\Big] \\ &= n! \sum_{\nu=1}^{n} \frac{g_\nu}{\nu!}\Big[\sum_{\substack{[j_1^{m_1}, \ldots, j_k^{m_k}] \in \mathcal{P}_n \\ m_1 + \cdots + m_k = \nu}} \frac{f_{j_1}^{m_1} \cdots f_{j_k}^{m_k}}{(j_1!)^{m_1} \cdots (j_k!)^{m_k}} \frac{\nu!}{m_1! \cdots m_k!}\Big]. \end{aligned} \tag{32}$$

The last step uses the fact that the number of ν-tuples $i_1, i_2, \ldots, i_\nu$ which contain m_1 j_1's, m_2 j_2's, ..., and m_k j_k's (for distinct $j_1, \ldots, j_k$) is given by the multinomial coefficient $\binom{\nu}{m_1 \cdots m_k} = \frac{\nu!}{m_1! \cdots m_k!}$. This completes the proof of (28). ∎

It is worth noting that (29) and (30) for arbitrary f_j's and g_ν's imply (31) with H_n given by (32). This result doesn't require f and g to be differentiable,

Exercise 1. Show that if $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are complex numbers such that

$$\sum\nolimits_{j=0}^{n} a_j t^j = \sum\nolimits_{j=0}^{n} b_j t^j + o(t^n)$$

as $t \to 0$ through $\mathbb{R}$, then $a_j = b_j$ for $j = 0, \ldots, n$. [Hint: use induction on j.] ◇

Exercise 2. (a) Suppose X and Y are independent random variables, each having an n^{th} moment. As in Theorem 3, put $S = X + Y$. Express $\alpha_n(S)$ in terms of $\alpha_j(X)$ and $\alpha_j(Y)$ for $j = 0, \ldots, n$. (b) Suppose $b \in \mathbb{R}$ and X has an n^{th} moment. Express $\alpha_n(X + b)$ in terms of $\alpha_j(X)$ for $j = 0, \ldots, n$. ◇

Exercise 3. Let (a, b) be an open subinterval of $\mathbb{R}$ and let f and g be complex-valued functions defined on (a, b). Show that if f and g are n-times differentiable, then so is $h := fg$ and

$$h^{(n)}(t) = \sum\nolimits_{j=0}^{n} \binom{n}{j} f^{(j)}(t)\, g^{(n-j)}(t) \tag{33}$$

for each $t \in (a, b)$. [Hint: use induction on n]. ◇

The next four exercises deal with the cumulants of a random variable U uniformly distributed over the interval $[-1/2, 1/2]$.

Exercise 4. Let U be as above. Show that U has CGF

$$K(t) := \log\big(E(e^{tU})\big) = \log\Big(\frac{\sinh(t/2)}{t/2}\Big) \tag{34}$$

for all real t. ($\sinh(x) := (e^x - e^{-x})/2$; take $\sinh(0)/0 := 1$.) ◇

Exercise 5. Let K be as in (34). Show that

$$K'(t) = \frac{1}{2} - \frac{1}{t} + \frac{1}{\exp(t) - 1},$$

for $t \neq 0$, while $K'(0) = 0$. ◇

Let B_0, B_1, … be the so-called ***Bernoulli numbers***, i.e., the coefficients in the power series expansion

$$\frac{t}{\exp(t)-1} = \sum_{k=0}^{\infty} \frac{B_k t^k}{k!}\,; \tag{35}$$

in particular

$$\begin{aligned} &B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \\ &B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730} \end{aligned} \tag{36}$$

while $B_3 = B_5 = \cdots = B_{11} = 0$. (See *Abramowitz and Stegun* page 804, or do "`help (bernoulli)`" in Maple.) Formula (35) holds for $|t| < 2\pi^{\dagger}$.

Exercise 6. Let U and the Bernoulli numbers be as above. By integrating $K'(s)$ from 0 to t, show that

$$K(t) = \sum_{k=2}^{\infty} \frac{B_k t^k}{k\,k!} \tag{37}$$

for $|t| < 2\pi$, and hence that the k^{th} cumulant of U is B_k/k, for $k \geq 2$; in particular all cumulants of odd order equal zero, while

$$\begin{aligned} &\kappa_2 = \frac{1}{12} \quad \kappa_4 = -\frac{1}{120}, \quad \kappa_6 = \frac{1}{252}, \\ &\kappa_8 = -\frac{1}{240}, \quad \kappa_{10} = \frac{1}{132}, \quad \kappa_{12} = -\frac{691}{32760}. \end{aligned} \tag{38}$$ ◇

† Optional: prove this. Use the fact that if f is a complex valued function which is defined and differentiable in the disk $D := \{\, z : |z - z_0| < r \,\}$, then f is infinitely differentiable in D and $f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)(z-z_0)^n/n!$ for all $z \in D$.

Exercise 7. Let U be as above. Confirm the values of κ_2, …, κ_{12} in (38) by computing the first 12 central moments of U and using the formulas for cumulants in terms of moments; use Maple or the equivalent to do the arithmetic. ◇

Exercise 8. Confirm Faà di Bruno's formula by computing the fifth derivative of $h = g(f)$ and checking the result against the RHS of ().◇

Exercise 9. For integers $1 \leq m \leq n$ let $p_{n,m}$ be the number of additive partitions of n for which the largest element is m, and let p_n be the total number of additive partitions of n. Show that

$$p_{n,m} = \begin{cases} 1, & \text{if } m = n, \\ \sum_{j=1}^{\min(m,n-m)} p_{n-m,j}, & \text{if } m < n. \end{cases}$$

and that

$$p_n = \sum_{m=1}^{n} p_{n,m}.$$

Use these relations to compute p_n for $n = 1, \ldots, 10$. ◇