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Pseudorandom functions from pseudorandom generators and CPA security

In this lecture we will see that the PRG conjecture implies the PRF conjecture. We will also see how PRFs imply an encryption scheme that is secure even when we encrypt multiple messages with the same key.

We have seen that PRF's (pseudorandom functions) are extremely useful, and we'll see some more applications of them later on. But are they perhaps too amazing to exist? Why would someone imagine that such a wonderful object is feasible? The answer is the following theorem:

Theorem 5.1 — The PRF Theorem. Suppose that the PRG Conjecture is true, then there exists a secure PRF collection $\{f_s\}_{s \in \{0,1\}^*}$ such that for every $s \in \{0,1\}^n$, f_s maps $\{0,1\}^n$ to $\{0,1\}^n$.

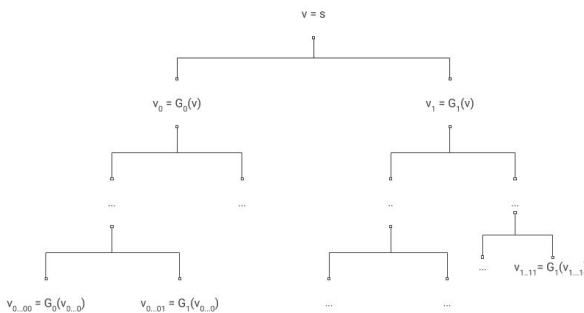


Figure 5.1: The construction of a pseudorandom function from a pseudorandom generator can be illustrated by a depth n binary tree. The root is labeled by the seed s and for every internal node v labeled by a string $x \in \{0,1\}^n$, we use that label x as a seed into the PRG G to label v 's two children. In particular, the children of v are labeled with $G_0(x)$ and $G_1(x)$ respectively. The output of the function f_s on input i is the label of the i^{th} leaf counting from left to right. Note that the numbering of leaf i is related to the bitstring representation of i and the path leaf i in the following way: we traverse to leaf i from the root by reading off the n bits of i left to right and descend into the left child of the current node for every 0 we encounter and traverse right for every 1.

Proof. We describe the proof, see also [Chapter 6 of Rosulek](#) or Section 8.5 of Katz-Lindell (section 7.5 in 2nd edition) for alternative expositions.

If the PRG Conjecture is true then in particular by the length extension theorem there exists a PRG $G : \{0, 1\}^n \rightarrow \{0, 1\}^{2n}$ that maps n bits into $2n$ bits. Let's denote $G(s) = G_0(s) \circ G_1(s)$ where \circ denotes concatenation. That is, $G_0(s)$ denotes the first n bits and $G_1(s)$ denotes the last n bits of $G(s)$.

For $i \in \{0, 1\}^n$, we define $f_s(i)$ as

$$G_{i_n}(G_{i_{n-1}}(\dots G_{i_1}(s))).$$

This corresponds to n composed applications of G_b for $b \in \{0, 1\}$. If the j^{th} bit of i 's binary string is 0 then the j^{th} application of the PRG is G_0 otherwise it is G_1 . This series of successive applications starts with the initial seed s .

This definition directly corresponds to the depiction in [Fig. 5.1](#), where the successive applications of G_b correspond to the recursive labeling procedure.

By the definition above we can see that to evaluate $f_s(i)$ we need to evaluate the pseudorandom generator n times on inputs of length n , and so if the pseudorandom generator is efficiently computable then so is the pseudorandom function. Thus, "all" that's left is to prove that the construction is secure and this is the heart of this proof.

I've mentioned before that the first step of writing a proof is convincing yourself that the statement is true, but there is actually an often more important zeroth step which is understanding what the statement actually *means*. In this case what we need to prove is the following:

We need to show that the security of the PRG G implies the security of the PRF ensemble $\{f_s\}$. Via the contrapositive, this means that we assume that there is an adversary A that can distinguish in time T a black box for $f_s(\cdot)$ from a black-box for a random function with advantage ϵ . We need to use A come up with an adversary D that can distinguish in time $\text{poly}(T)$ an input of the form $G(s)$ (where s is random in $\{0, 1\}^n$) from an input of the form y where y is random in $\{0, 1\}^{2n}$ with bias at least $\epsilon/\text{poly}(T)$.

Assume that A as above is a T -time adversary that wins in the "PRF game" with advantage ϵ . Let us consider the "lazy evaluation" implementation of the black box for A illustrated in [Fig. 5.2](#). That is, at every point in time there are nodes in the full binary tree that are labeled and nodes which we haven't yet labeled. When A makes a query i , this query corresponds to the path $i_1 \dots i_n$ in the tree. We look at the lowest (furthest away from the root) node v on this path which has been labeled by some value y , and then we continue labelling the path

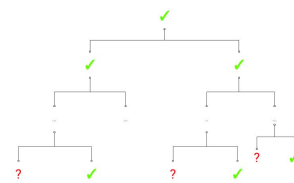


Figure 5.2: In the "lazy evaluation" implementation of the black box to the adversary, we label every node in the tree only when we need it. Subsequent traversals do not reevaluate the PRG, leading to reuse of the intermediate seeds. Thus for example, two sibling leaves will correspond to a single call to $G(x)$, where x is their parent's label, but with the left child receiving the first n bits and the right child receiving the second n bits of $G(x)$. In this figure check marks correspond to nodes that have been labeled and question marks to nodes that are still unlabeled.

from v downwards until we reach i . In other words, we label the two children of v by $G_0(y)$ and $G_1(y)$, and then if the path i involves the first child then we label its children by $G_0(G_0(y))$ and $G_1(G_0(y))$, and so on and so forth (see Fig. 5.3). Note that because $G_0(y)$ and $G_1(y)$ correspond to a single call to G , regardless of whether the traversals continues left or right (i.e. whether the current level corresponds to a value 0 or 1 in i) we label both children at the same time.

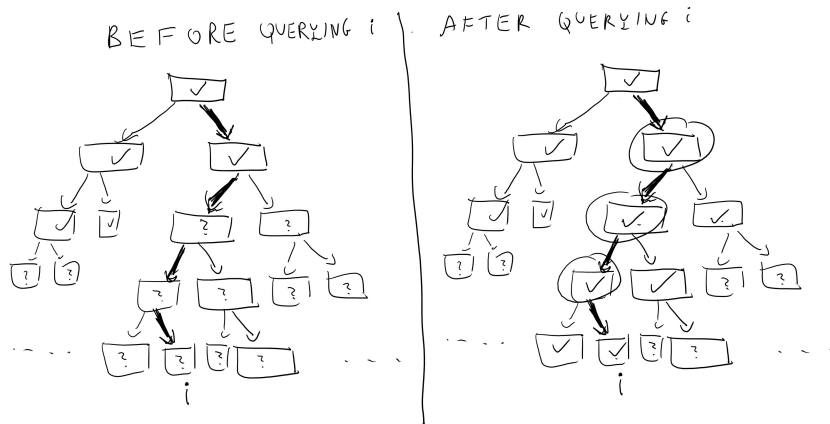


Figure 5.3: When the adversary queries i , the oracle takes the path from i to the root and computes the generator on the minimum number of internal nodes that is needed to obtain the label of the i^{th} leaf.

A moment's thought shows that this is just another (arguably cumbersome) way to describe the oracle that simply computes the map $i \mapsto f_s(i)$. And so the experiment of running A with this oracle produces precisely the same result as running A with access to $f_s(\cdot)$. Note that since A has running time at most T , the number of times our oracle will need to label an internal node is at most $T' \leq 2nT$ (since we label at most $2n$ nodes for every query i).

We now define the following T' hybrids: in the j^{th} hybrid, we run this experiment but in the first j times the oracle needs to label internal nodes then it uses independent random labels. That is, for the first j times we label a node v , instead of letting the label of v be $G_b(u)$ (where u is the parent of v , and $b \in \{0, 1\}$ corresponds to whether v is the left or right child of u), we label v by a random string in $\{0, 1\}^n$.

Note that the 0^{th} hybrid corresponds to the case where the oracle implements the function $i \mapsto f_s(i)$, while in the T'^{th} hybrid all labels are random and hence implements a random function. By the hybrid argument, if A can distinguish between the 0^{th} hybrid and the T'^{th} hybrid with bias ϵ then there must exist some j such that it distinguishes between the j^{th} hybrid (pictured in Fig. 5.4) and the $(j + 1)^{\text{st}}$ hybrid (pictured in Fig. 5.5) with bias at least ϵ/T' . We will use this j and A to break the pseudorandom generator.

We can now describe our distinguisher D (see Fig. 5.6) for the pseudorandom generator. On input a string $y \in \{0, 1\}^{2n}$ D will run

Hybrid j

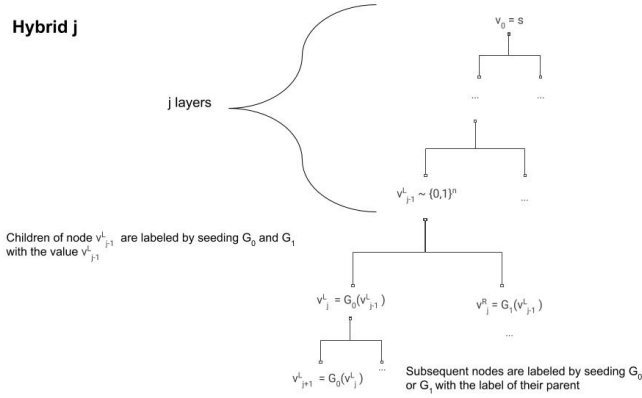


Figure 5.4: In the j^{th} hybrid the first j internal labels are drawn uniformly at random from U_n . All subsequent children's labels are produced in the usual way by seeding G with the label z of the parent and assigning the first n bits ($G_0(z)$) to the left child and the last n bits ($G_1(z)$) to the right child. For example, for some node v_{j-1}^L at the j^{th} level, we generate pseudorandom string $G(v_{j-1}^L)$ and label the left child $v_j^L = G_0(v_{j-1}^L)$ and the right child $v_j^R = G_1(v_{j-1}^L)$. Note that the labeling scheme for this diagram is different from that in the previous figures. This is simply for ease of exposition, we could still index our nodes via the path reaching them from the root.

Hybrid j+1

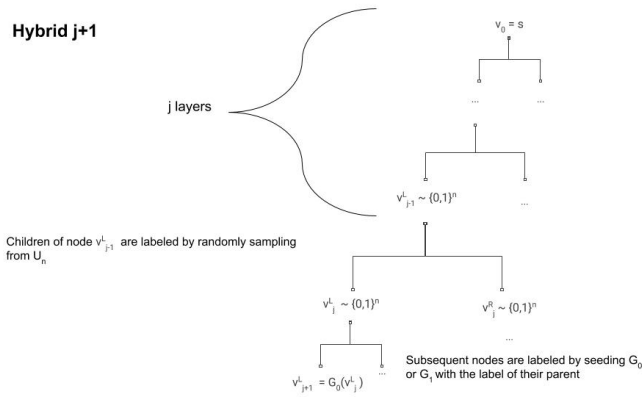


Figure 5.5: The $j + 1^{st}$ hybrid differs from the j^{th} in that the process of assigning random labels continues until the $j + 1^{st}$ step as opposed to the j^{th} . The hybrids are otherwise completely identically constructed.

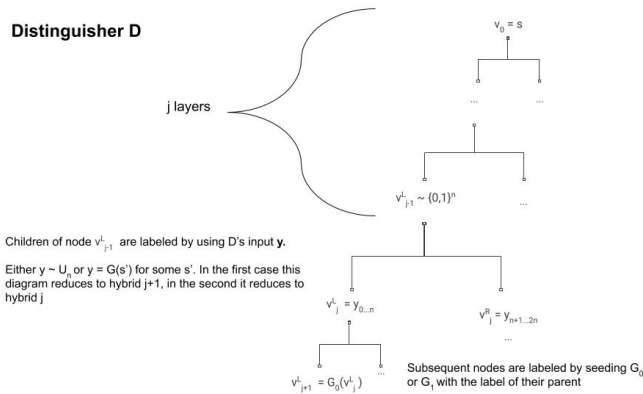


Figure 5.6: Distinguisher D is similar to hybrid j , in that the nodes in the first j layers are assigned completely random labels. When evaluating along a particular path through v_{j-1}^L , rather than labeling the two children by applying G to its label, it simply splits the input y into two strings $y_{0..n}, y_{n+1..2n}$. If y is truly random, D is identical to hybrid $j + 1$. If $y = G(s)$ for some random seed s , then D simulates hybrid j .

A and the j^{th} oracle inside its belly with one difference- when the time comes to label the j^{th} node, instead of doing this by applying the pseudorandom generator to the label of its parent v (which is what should happen in the j^{th} oracle) it uses its input y to label the two children of v .

Now, if y was completely random then we get exactly the distribution of the $j + 1^{st}$ oracle, and hence in this case D simulates internally the $j + 1^{st}$ hybrid. However, if $y = G(s)$ for some randomly sampled $s \in \{0, 1\}^n$, though it may not be obvious at first, we actually get the distribution of the j^{th} oracle.

The equivalence between hybrid j and distinguisher D under the condition that $y = G(s)$ is non obvious, because in hybrid j , the label for the children of v_{j-1}^L was supposed to be the result of applying the pseudorandom generator to the label of v_{j-1}^L and not to some other random string (see Fig. 5.6). However, because v was labeled *before* the j^{th} step then we know that it was actually labeled by a random string. Moreover, since we use lazy evaluation we know that step j is the *first* time where we actually use the value of the label of v . Hence, if at this point we *resampled* this label and used a completely independent random string s then the distribution of v_{j-1}^L and s would be *identical*.

The key observations here are:

1. The output of A does not directly depend on the internal labels, but only on the labels of the leaves (since those are the only values returned by the oracle).
2. The label for an internal vertex v is only used once, and that is for generating the labels for its children.

Hence the distribution of $y = G(s)$, for s drawn from U_n , is identical to the distribution, $G(v_{j-1}^L)$, of the j^{th} hybrid, and thus if A had advantage ϵ in breaking the PRF $\{f_s\}$ then D will have advantage ϵ/T' in breaking the PRG G thus obtaining a contradiction. ■

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Remark 5.2 — PRF's in practice. While this construction reassures us that we can rely on the existence of pseudorandom functions even on days where we remember to take our meds, this is not the construction people use when they need a PRF in practice because it is still somewhat inefficient, making n calls to the underlying pseudorandom generators. There are constructions (e.g., HMAC) based on hash functions that require stronger assumptions but can use as few as two calls to the underlying function. We will cover these constructions when we talk about hash functions and the random oracle model. One can also obtain practical constructions of PRFs from *block ciphers*, which we'll see later in this lecture.

5.1 SECURELY ENCRYPTING MANY MESSAGES - CHOSEN PLAIN-TEXT SECURITY

Let's get back to our favorite task of *encryption*. We seemed to have nailed down the definition of secure encryption, or did we?

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Try to think what kind of security guarantees are *not* provided by the notion of computational secrecy we saw in [Definition 2.6](#)

[Definition 2.6](#) talks about encrypting a *single* message, but this is not how we use encryption in the real world. Typically, Alice and Bob (or Amazon and Boaz) setup a shared key and then engage in many back and forth messages between one another. At first, we might think that this issue of a single long message vs. many short ones is merely a technicality. After all, if Alice wants to send a sequence of messages (m_1, m_2, \dots, m_t) to Bob, she can simply treat them as a single long message. Moreover, the way that *stream ciphers* work, Alice can compute the encryption for the first few bits of the message she decides what will be the next bits and so she can send the encryption of m_1 to Bob and later the encryption of m_2 . There is some truth to this sentiment, but there are issues with using stream ciphers for multiple messages. For Alice and Bob to encrypt messages in this

way, they must maintain a *synchronized shared state*. If the message m_1 was dropped by the network, then Bob would not be able to decrypt correctly the encryption of m_2 .

There is another way in which treating many messages as a single tuple is unsatisfactory. In real life, Eve might be able to have some impact on *what* messages Alice encrypts. For example, the Katz-Lindell book describes several instances in World War II where Allied forces made particular military maneuver for the sole purpose of causing the Axis forces to send encryptions of messages of the Allies' choosing. To consider a more modern example, today Google uses encryption for all of its search traffic including (for the most part) the *ads* that are displayed on the page. But this means that an attacker, by paying Google, can cause it to encrypt arbitrary text of their choosing. This kind of attack, where Eve *chooses* the message she wants to be encrypted is called a *chosen plaintext attack*. You might think that we are already covering this with our current definition that requires security for every pair of messages and so in particular this pair could be chosen by Eve. However, in the case of multiple messages, we would want to allow Eve to be able to choose m_2 after she saw the encryption of m_1 .

All that leads us to the following definition, which is a strengthening of our definition of computational security:

Definition 5.3 — Chosen Plaintext Attack (CPA) secure encryption. An encryption scheme (E, D) is *secure against chosen plaintext attack (CPA secure)* if for every polynomial time *Eve*, Eve wins with probability at most $1/2 + \text{negl}(n)$ in the game defined below:

1. The key k is chosen at random in $\{0, 1\}^n$ and fixed.
2. Eve gets the length of the key 1^n as input.¹
3. Eve interacts with E for $t = \text{poly}(n)$ rounds as follows: in the i^{th} round, Eve chooses a message m_i and obtains $c_i = E_k(m_i)$.
4. Then Eve chooses two messages m_0, m_1 , and gets $c^* = E_k(m_b)$ for $b \leftarrow_R \{0, 1\}$.
5. Eve continues to interact with E for another $\text{poly}(n)$ rounds, as in Step 3.
6. Eve *wins* if she outputs b .

Definition 5.3 is illustrated in Fig. 5.7. Our previous notion of computational secrecy (i.e., Definition 2.6) corresponds to the case that we skip Steps 3 and 5 above. Since Steps 3 and 5 only give the adversary more power (and hence is only more likely to win), CPA security (Definition 5.3) is *stronger* than computational secrecy (Definition 2.6), in the sense that every CPA secure encryption (E, D) is

¹ Giving Eve the key as a sequence of n 1's as opposed to in binary representation is a common notational convention in cryptography. It makes no difference except that it makes the input length for Eve of length n , which makes sense since we want to allow Eve to run in $\text{poly}(n)$ time.

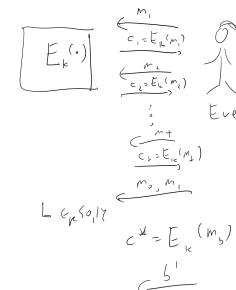


Figure 5.7: In the CPA game, Eve interacts with the encryption oracle and at the end chooses m_0, m_1 , gets an encryption $c^* = E_k(m_b)$ and outputs b' . She *wins* if $b' = b$

also computationally secure. It turns out that CPA security is *strictly stronger*, in the sense that without modification, our stream ciphers cannot be CPA secure. In fact, we have a stronger, and initially somewhat surprising theorem:

Theorem 5.4 — CPA security requires randomization. There is no CPA secure (E, D) where E is *deterministic*.

Proof. The proof is very simple: Eve will only use a single round of interacting with E where she will ask for the encryption c_1 of 0^ℓ . In the second round, Eve will choose $m_0 = 0^\ell$ and $m_1 = 1^\ell$, and get $c^* = E_k(m_b)$ she will then output 0 if and only if $c^* = c_1$. ■

This proof is so simple that you might think it shows a problem with the definition, but it is actually a real problem with security. If you encrypt many messages and some of them repeat themselves, it is possible to get significant information by seeing the repetition pattern (cue the XKCD cartoon again, see Fig. 5.8). To avoid this issue we need to use a *randomized* (or *probabilistic*) encryption, such that if we encrypt the same message twice we *won't* see two copies of the same ciphertext.² But how do we do that? Here pseudorandom functions come to the rescue:

Theorem 5.5 — CPA security from PRFs. Suppose that $\{f_s\}$ is a PRF collection where $f_s : \{0, 1\}^n \rightarrow \{0, 1\}^\ell$, then the following is a CPA secure encryption scheme: $E_s(m) = (r, f_s(r) \oplus m)$ where $r \leftarrow_R \{0, 1\}^n$, and $D_s(r, z) = f_s(r) \oplus z$.

Proof. I leave to you to verify that $D_s(E_s(m)) = m$. We need to show the CPA security property. As is usual in PRF-based constructions, we first show that this scheme will be secure if f_s was an actually random function, and then use that to derive security.

Consider the game above when played with a completely random function and let r_i be the random string chosen by E in the i^{th} round and r^* the string chosen in the last round. We start with the following simple but crucial claim:

Claim: The probability that $r^* = r_i$ for some i is at most $T/2^n$.

Proof of claim: For any particular i , since r^* is chosen independently of r_i , the probability that $r^* = r_i$ is 2^{-n} . Hence the claim follows from the union bound. QED

Given this claim we know that with probability $1 - T/2^n$ (which is $1 - \text{negl}(n)$), the string r^* is distinct from any string that was chosen before. This means that by the lazy evaluation principle, if $f_s(\cdot)$ is a completely random function then the value $f_s(r^*)$ can be thought

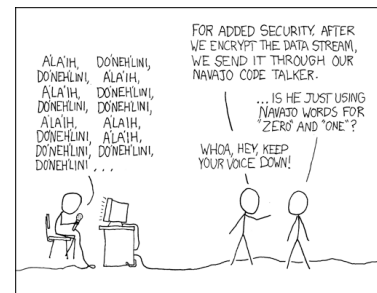


Figure 5.8: Insecurity of deterministic encryption
² If the messages are guaranteed to have *high entropy* which roughly means that the probability that a message repeats itself is negligible, then it is possible to have a secure deterministic private-key encryption, and this is sometimes used in practice. (Though often some sort of randomization or padding is added to ensure this property, hence in effect creating a randomized encryption.) Deterministic encryptions can sometimes be useful for applications such as efficient queries on encrypted databases. See [this lecture](#) in Dan Boneh’s coursera course.

of as being chosen at random in the final round independently of anything that happened before. But then $f_s(r^*) \oplus m_b$ amounts to simply using the one-time pad to encrypt m_b . That is, the distributions $f_s(r^*) \oplus m_0$ and $f_s(r^*) \oplus m_1$ (where we think of r^*, m_0, m_1 as fixed and the randomness comes from the choice of the random function $f_s(\cdot)$) are both equal to the uniform distribution U_n over $\{0, 1\}^n$ and hence Eve gets absolutely no information about b .

This shows that if $f_s(\cdot)$ was a random function then Eve would win the game with probability at most $1/2$. Now if we have some efficient Eve that wins the game with probability at least $1/2 + \epsilon$ then we can build an adversary A for the PRF that will run this entire game with black box access to $f_s(\cdot)$ and will output 1 if and only if Eve wins. By the argument above, there would be a difference of at least ϵ in the probability it outputs 1 when $f_s(\cdot)$ is random vs when it is pseudorandom, hence contradicting the security property of the PRF. ■

5.2 PSEUDORANDOM PERMUTATIONS / BLOCK CIPHERS

Now that we have pseudorandom functions, we might get greedy and want such functions with even more magical properties. This is where the notion of *pseudorandom permutations* comes in.

∴ {definition title="Pseudorandom permutations" #PRPdef} Let $\ell : \mathbb{N} \rightarrow \mathbb{N}$ be some function that is polynomially bounded (i.e., there are some $0 < c < C$ such that $n^c < \ell(n) < n^C$ for every n). A collection of functions $\{f_s\}$ where $f_s : \{0, 1\}^\ell \rightarrow \{0, 1\}^\ell$ for $\ell = \ell(|s|)$ is called a *pseudorandom permutation (PRP) collection* if:

1. It is a pseudorandom function collection (i.e., the map $s, x \mapsto f_s(x)$ is efficiently computable and there is no efficient distinguisher between $f_s(\cdot)$ with a random s and a random function).
2. Every function f_s is a permutation of $\{0, 1\}^\ell$ (i.e., a one to one and onto map).
3. There is an efficient algorithm that on input s, y returns $f_s^{-1}(y)$.

The parameter n is known as the *key length* of the pseudorandom permutation collection and the parameter $\ell = \ell(n)$ is known as the *input length* or *block length*. Often, $\ell = n$ and so in most cases you can safely ignore this distinction.

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At first look ?? might seem not to make sense, since on one hand it requires the map $x \mapsto f_s(x)$ to be a permutation, but on the other hand it can be shown that with high probability a random map $H : \{0, 1\}^\ell \rightarrow \{0, 1\}^\ell$ will *not* be a permutation. How can then such a collection be pseudorandom? The key insight is that while

a random map might not be a permutation, it is not possible to distinguish with a polynomial number of queries between a black box that computes a random function and a black box that computes a random permutation. Understanding why this is the case, and why this means that ?? is reasonable, is crucial to getting intuition to this notion, and so I suggest you pause now and make sure you understand these points.

As usual with a new concept, we want to know whether it is possible to achieve it and whether it is useful. The former is established by the following theorem:

Theorem 5.6 — PRP's from PRFs. If the PRF conjecture holds (and hence by Theorem 5.1 also if the PRG conjecture holds) then there exists a pseudorandom permutation collection.

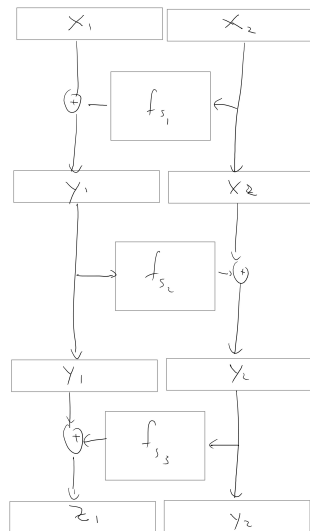


Figure 5.9: We build a PRP p on $2n$ bits from three PRFs $f_{s_1}, f_{s_2}, f_{s_3}$ on n bits by letting $p_{s_1, s_2, s_3}(x_1, x_2) = (z_1, y_2)$ where $y_1 = x_1 \oplus f_{s_1}(x_2)$, $y_2 = x_2 \oplus f_{s_2}(y_1)$ and $z_1 = f_{s_3}(y_2) \oplus y_1$.

Proof. Fig. 5.9 illustrates the construction of a pseudorandom permutation from a pseudorandom function. The construction (known as the Luby-Rackoff construction) uses several rounds of what is known as the **Feistel Transformation** that maps a function $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ into a permutation $g : \{0, 1\}^{2n} \rightarrow \{0, 1\}^{2n}$ using the map $(x, y) \mapsto (x, f(x) \oplus y)$.

Specifically, given a PRF family $\{f_s\}$ with n -bit keys, inputs, and outputs, our candidate PRP family will be called $\{p_{s_1, s_2, s_3}\}$. Here, $p_{s_1, s_2, s_3} : \{0, 1\}^{2n} \rightarrow \{0, 1\}^{2n}$ is calculated on input $(x_1, x_2) \in \{0, 1\}^{2n}$ as follows (see Fig. 5.9):

- First, map $(x_1, x_2) \mapsto (y_1, x_2)$, where $y_1 = x_1 \oplus f_{s_1}(x_2)$.

- Next, map $(y_1, x_2) \mapsto (y_1, y_2)$, where $y_2 = x_2 \oplus f_{s_2}(y_1)$.
- Next, map $(y_1, y_2) \mapsto (z_1, y_2)$, where $z_1 = y_1 \oplus f_{s_3}(y_2)$.
- Finally, output $p_{s_1, s_2, s_3}(x_1, x_2) = (z_1, y_2)$.

Each of the first three steps above corresponds to a single round of the Feistel transformation, which is easily seen to be both efficiently computable *and* efficiently invertible. In fact, we can efficiently calculate $p_{s_1, s_2, s_3}^{-1}(z_1, y_2)$ for an arbitrary string $(z_1, y_2) \in \{0, 1\}^{2n}$ by running the above three rounds of Feistel transformations in reverse order.

Thus, the real challenge in proving [Theorem 5.6](#) is not showing that $\{p_{s_1, s_2, s_3}\}$ is a valid permutation, but rather showing that it is *pseudorandom*. The details of remainder of this proof are a bit technical, and can be safely skipped on a first reading.

Intuitively, the argument goes like this. Consider an oracle \mathcal{O} for p_{s_1, s_2, s_3} that answers an adversary's query (x_1, x_2) by carrying out the three Feistel transformations outlined above and outputting (z_1, y_2) . First, we'll show that with high probability, \mathcal{O} will never encounter the same intermediate string y_1 twice, over the course of all queries (unless the adversary makes a duplicate query). Since the string y_1 , calculated in Step 1, determines the input on which f_{s_2} is evaluated in Step 2, it follows that the strings y_2 calculated in Step 2 will appear to be chosen independently and at random. In particular, *they too* will be pairwise distinct with high probability. Since the string y_2 is in turn passed as input to f_{s_3} in Step 3, it follows that the strings z_1 encountered over the course of all queries will also appear to be chosen independently and at random. Ultimately, this means that the oracle's outputs (z_1, y_2) will look like freshly independent, random strings.

To make this reasoning precise, notice first that it suffices to establish the security of a *variant* of p_{s_1, s_2, s_3} in which the pseudorandom functions f_{s_1} , f_{s_2} , and f_{s_3} used in the construction are replaced by truly random functions $h_1, h_2, h_3 : \{0, 1\}^n \rightarrow \{0, 1\}^n$. Call this variant p_{h_1, h_2, h_3} . Indeed, the assumption that $\{f_s\}$ is a PRF collection tells us that making this change has only a negligible effect on the output of an adversary with oracle access to p . With this in mind, our job is to show that for every efficient adversary A , the difference $|\Pr[A^{p_{h_1, h_2, h_3}}(1^n) = 1] - \Pr[A^{H(\cdot)}(1^n) = 1]|$ is negligible. In this expression, the first probability is taken over the choice of the random functions $h_1, h_2, h_3 : \{0, 1\}^n \rightarrow \{0, 1\}^n$ used in the Feistel transformation, and the second probability is taken over the random function $H : \{0, 1\}^{2n} \rightarrow \{0, 1\}^{2n}$. To simplify matters, suppose without loss of generality that A always makes $q(n)$ *distinct* queries to its oracle, denoted $(x_1^{(1)}, x_2^{(1)}), \dots, (x_1^{(q(n))}, x_2^{(q(n))})$ in order. Similarly, let $y_1^{(i)}, y_2^{(i)}, z_1^{(i)}$

denote the intermediate strings calculated in the three rounds of the Feistel transformation. Here, q is a polynomial in n .

Consider the case in which the adversary A is interacting with the oracle for p_{h_1, h_2, h_3} , as opposed to the random oracle. Let us say that a *collision occurs at y_1* if for some $1 \leq i < j \leq q(n)$, the string $y_1^{(i)}$ computed while answering A 's i th query coincides with the string $y_1^{(j)}$ computed while answering A 's j th query. We claim the probability that a collision occurs at y_1 is negligibly small. Indeed, if a collision occurs at y_1 , then $y_1^{(i)} = y_1^{(j)}$ for some $i \neq j$. By the construction of p_{h_1, h_2, h_3} , this means that $x_1^{(i)} \oplus h_1(x_2^{(i)}) = x_1^{(j)} \oplus h_1(x_2^{(j)})$. In particular, it cannot be the case that $x_1^{(i)} \neq x_1^{(j)}$ and $x_2^{(i)} = x_2^{(j)}$. Since we assumed that A makes distinct queries to its oracle, it follows that $x_2^{(i)} \neq x_2^{(j)}$ and hence that $h_1(x_2^{(i)})$ and $h_1(x_2^{(j)})$ are uniform and independent. In other words, $\Pr[y_1^{(i)} = y_1^{(j)}] = \Pr[x_1^{(i)} \oplus f_1(x_2^{(i)}) = x_1^{(j)} \oplus f_1(x_2^{(j)})] = 2^{-n}$. Taking a union bound over all choices of i and j , we see that the probability of a collision at y_1 is at most $q(n)^2/2^n$, which is negligible.

Next, define a *collision at y_2* , by a pair of queries $1 \leq i < j \leq q(n)$ such that $y_2^{(i)} = y_2^{(j)}$. We argue that the probability of a collision at y_2 is also negligible, provided that we condition on the overwhelmingly likely event that no collision occurs at y_1 . Indeed, if $y_1^{(i)} \neq y_1^{(j)}$ for all $i \neq j$, then $h_2(y_1^{(1)}), \dots, h_2(y_1^{(q(n))})$ are distributed independently and uniformly at random. In particular, we have $\Pr[y_2^{(i)} = y_2^{(j)} \mid \text{no collision at } y_1] = 2^{-n}$, which is negligible even after taking a union bound over all i and j . The same argument applied to the *third* round of the Feistel transformation similarly shows that, conditioned on the overwhelmingly likely event that no collision occurs at y_1 or y_2 , the strings $z_1^{(1)}, \dots, z_1^{(i)}$ for $1 \leq i \leq q(n)$ are also distributed as fresh, independent, random strings. At this point, we've shown that the adversary cannot distinguish the outputs $(z_1^{(1)}, y_2^{(1)}), \dots, (z_1^{(q(n))}, y_2^{(q(n))})$ of the oracle for p_{h_1, h_2, h_3} from the outputs of a random oracle unless an event with negligibly small probability occurs. We conclude that the collection $\{p_{h_1, h_2, h_3}\}$, and hence our original collection $\{p_{s_1, s_2, s_3}\}$, is a secure PRP collection.

For more details regarding this proof, see Section 4.5 in Boneh Shoup or Section 8.6 (7.6 in 2nd ed) in Katz-Lindell, whose proof was used as a model for ours. ■

R

Remark 5.7 — How many Feistel rounds? The construction in the proof of [Theorem 5.6](#) constructed a PRP p by performing 3 rounds of the Feistel transformation with a known PRF f . It is an interesting exercise to try to show that doing just 1 or 2 rounds of the Feis-

tel transformation *does not* suffice to achieve a PRP.
 Hint: consider an adversary that makes queries of the form
 (x_1, x_2) where x_2 is held fixed and x_1 is varied.

The more common name for a pseudorandom permutation is *block cipher* (though typically block ciphers are expected to meet additional security properties on top of being PRPs). The constructions for block ciphers used in practice don't follow the construction of [Theorem 5.6](#) (though they use some of the ideas) but have a more ad-hoc nature.

One of the first modern block ciphers was the **Data Encryption Standard (DES)** constructed by IBM in the 1970's. It is a fairly good cipher- to this day, as far as we know, it provides a pretty good number of security bits compared to its key length. The trouble is that its key is only 56 bits long, which is no longer outside the reach of modern computing power. (It turns out that subtle variants of DES are far less secure and fall prey to a technique known as **differential cryptanalysis**; the IBM designers of DES were aware of this technique but kept it secret at the behest of the NSA.)

Between 1997 and 2001, the U.S. National Institute of Standards and Technology (NIST) ran a competition to replace DES which resulted in the adoption of the block cipher Rijndael as the new **advanced encryption standard (AES)**. It has a block size (i.e., input length) of 128 bits and a key size (i.e., seed length) of 128, 196, or 256 bits.

The actual construction of AES (or DES for that matter) is not extremely illuminating, but let us say a few words about the general principle behind many block ciphers. They are typically constructed by repeating one after the other a number of very simple permutations (see [Fig. 5.10](#)). Each such iteration is called a *round*. If there are t rounds, then the key k is typically expanded into a longer string, which we think of as a t tuple of strings (k_1, \dots, k_t) via some pseudorandom generator known as the *key scheduling algorithm*. The i -th string in the tuple is known as the *round key* and is used in the i -th round. Each round is typically composed of several components: there is a "key mixing component" that performs some simple permutation based on the key (often as simply as XOR'ing the key), there is a "mixing component" that mixes the bits of the block so that bits that were initially nearby don't stay close to one another, and then there is some non-linear component (often obtained by applying some simple non-linear functions known as "S boxes" to each small block of the input) that ensures that the overall cipher will not be an affine function. Each one of these operations is an easily reversible operations, and hence decrypting the cipher simply involves running the rounds backwards.

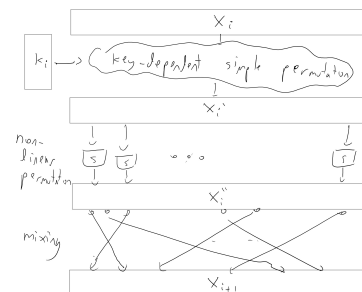


Figure 5.10: A typical round of a block cipher, k_i is the i -th round key, x_i is the block before the i -th round and x_{i+1} is the block at the end of this round.

5.3 ENCRYPTION MODES

How do we use a block cipher to actually encrypt traffic? Well we could use it as a PRF in the construction above, but in practice people use other ways.³

The most natural approach would be that to encrypt a message m , we simply use $p_s(m)$ where $\{p_s\}$ is the PRP/block cipher. This is known as the *electronic code book (ECB) mode* of a block cipher (see Fig. 5.11). Note that we can easily decrypt since we can compute $p_s^{-1}(m)$. If the PRP $\{p_s\}$ only accepts inputs of a fixed length ℓ , we can use ECB mode to encrypt a message m whose length is a multiple of ℓ by writing $m = (m_1, m_2, \dots, m_t)$, where each block m_i has length ℓ , and then encrypting each block m_i separately. The ciphertext output by this encryption scheme is $(p_s(m_1), \dots, p_s(m_t))$. A major drawback of ECB mode is that it is a *deterministic* encryption scheme and hence cannot be CPA secure. Moreover, this is actually a real problem of security on realistic inputs (see Fig. 5.12), so ECB mode should never be used.

A more secure way to use a block cipher to encrypt is the *cipher block chaining (CBC) mode*. The idea of cipher block chaining is to encrypt the blocks of a message $m = (m_1, \dots, m_t)$ sequentially. To encrypt the first block m_1 , we XOR m_1 with a random string known as the *initialization vector*, or *IV*, before applying the block cipher p_s . To encrypt one of the later blocks m_i , where $i > 1$, we XOR m_i with the *encryption* of m_{i-1} before applying the block cipher p_s . Formally, the ciphertext consists of the tuple (IV, c_1, \dots, c_t) , where IV is chosen uniformly at random and $c_i = p_s(c_{i-1} \oplus m_i)$ for $1 \leq i \leq t$ (we use the convention that $c_0 = IV$). This encryption process is depicted in Fig. 5.13. In order to decrypt (IV, c_1, \dots, c_t) , we simply calculate $m_i = p_s^{-1}(c_i) \oplus c_{i-1}$ for $1 \leq i \leq t$. Note that if we lose the block c_i to traffic in the CBC mode, then we are unable to decrypt the next block c_{i+1} , but we can recover from that point onwards.

On the one hand, CBC mode is vastly superior to a simple electronic codebook since CBC mode with a random IV is CPA secure (proving this is an excellent exercise). On the other hand, CBC mode suffers from the drawback that the encryption process cannot be parallelized: the ciphertext block c_i *must* be computed before c_{i+1} .

In the *output feedback (OFB) mode* we first encrypt the all zero string using CBC mode to get a sequence (y_1, y_2, \dots) of pseudorandom outputs that we can use as a stream cipher. To transmit a message $m \in \{0, 1\}^*$, we send the XOR of m with the bits output by this stream cipher, along with the IV used to generate the sequence. The receiver can decrypt a ciphertext (IV, c) by first using IV to recover (y_1, y_2, \dots) , and then taking the XOR of c with the appropriate number of bits

³ Partially this is because in the above construction we had to encode a plaintext of length n with a ciphertext of length $2n$ meaning an overhead of 100 percent in the communication.

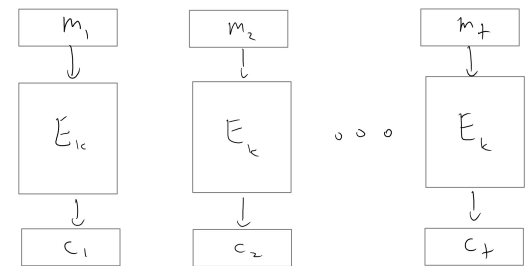


Figure 5.11: In the Electronic Codebook (ECB) mode, every message is encrypted deterministically and independently

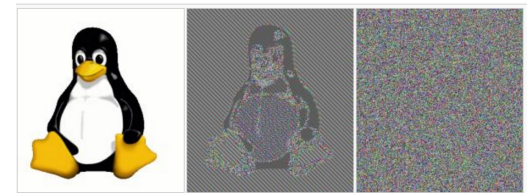


Figure 5.12: An encryption of the Linux penguin (left image) using ECB mode (middle image) vs CBC mode (right image). The ECB encryption is insecure as it reveals much structure about the original image. Image taken from Wikipedia.

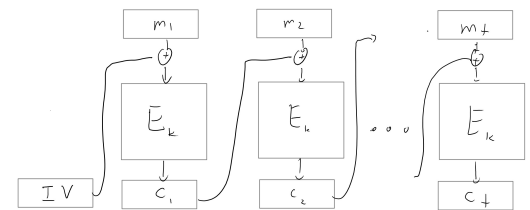


Figure 5.13: In the Cypher-Block-Chaining (CBC) the encryption of the previous message is XOR'ed into the current message prior to encrypting. The first message is XOR'ed with an *initialization vector* (IV) that if chosen randomly, ensures CPA security.

from this sequence. Like CBC mode, OFB mode is CPA secure when IV is chosen at random. Some advantages of OFB mode over CBC mode include the ability for the sender to precompute the sequence (y_1, y_2, \dots) well before the message to be encrypted is known, as well as the fact that the underlying function p_s used to generate (y_1, y_2, \dots) only needs to be a PRF (not necessarily a PRP).

Perhaps the simplest mode of operation is *counter (CTR) mode* where we convert a block cipher to a stream cipher by using the stream $p_s(IV), p_s(IV + 1), p_s(IV + 2), \dots$ where IV is a random string in $\{0, 1\}^n$ which we identify with $[2^n]$ (and perform addition modulo 2^n). That is, to encrypt a message $m = (m_1, \dots, m_t)$, we choose IV at random, and output (IV, c_1, \dots, c_t) , where $c_i = p_s(IV + i) \oplus m_i$ for $1 \leq i \leq t$. Decryption is performed similarly. For a modern block cipher, CTR mode is no less secure than CBC or OFB, and in fact offers several advantages. For example, CTR mode can easily encrypt and decrypt blocks in parallel, unlike CBC mode. In addition, CTR mode only needs to evaluate p_s once to decrypt any single block of the ciphertext, unlike OFB mode.

A fairly comprehensive study of the different modes of block ciphers is in [this document by Rogaway](#). His conclusion is that if we simply consider CPA security (as opposed to the stronger notions of *chosen ciphertext security* we'll see in the next lecture) then counter mode is the best choice, but CBC, OFB and CFB are widely implemented due to legacy reasons. ECB should not be used (except as a building block as part of a construction achieving stronger security).

5.4 OPTIONAL, ASIDE: BROADCAST ENCRYPTION

At the beginning of this chapter, we saw the proof of [Theorem 5.1](#), which states that the PRG Conjecture implies the existence of a secure PRF collection. At the heart of this proof was a rather clever construction based on a binary tree. As it turns out, similar tree constructions have been used time and again to solve many other problems in cryptography. In this section, we will discuss just one such application of these tree constructions, namely *broadcast encryption*.

Let's put ourselves in the shoes of Hollywood executives facing the following problem: we've just released a new movie for sale (in the form of a download or a Blu-ray disc), and we'd like to prevent it from being pirated. On the one hand, consumers who've purchased a copy of the movie should be able to watch it on certain approved, standalone devices such as TVs and Blu-ray players without needing an external internet connection. On the other hand, to minimize the risk of piracy, these consumers should *not* have access to the movie data itself.

One way to protect the movie data, which we model as a string x , is to provide consumers with a secure encryption $E_k(x)$ of the data. Although the secret key k used to encrypt the data is hidden from consumers, it is provided to device manufacturers so that they can embed it in their TVs and Blu-ray players in some secure, tamper-resistant manner. As long as the key k is never leaked to the public, this system ensures that only approved devices can decrypt and play a consumer's copy of the movie. For this reason, we will sometimes refer to k as the *device key*. This setup is depicted in Fig. 5.14.

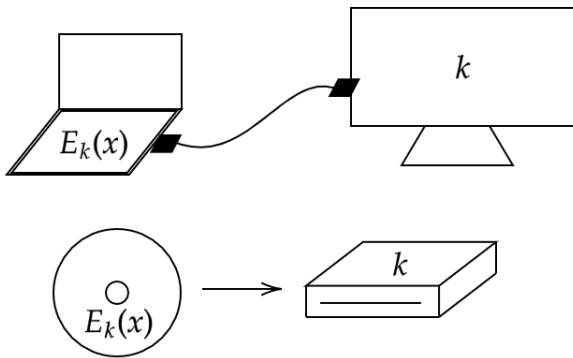


Figure 5.14: The problem setup for broadcast encryption.

Unfortunately, if we were to implement this scheme exactly as written, it would almost certainly be broken in a matter of days. After all, as soon as even a single device is hacked, the device key k would be revealed. This would allow the public to access our movie's data, as well as the data for *all future movies* we release for these devices! This latter consequence is one that we would certainly want to avoid, and doing so requires the notion of distinct, *revocable* keys:

Definition 5.8 — Broadcast Encryption Scheme. For our purposes, a *broadcast encryption scheme* consists of:

- A set of m distinct devices (or device manufacturers), each of which has access to one of the n -bit device keys k_1, \dots, k_m .
- A decryption algorithm D that receives as input a ciphertext y and a key k_i .

- An encryption algorithm E that receives as input a plaintext x , a key k_{master} , and a *revocation set* $R \subseteq [m]$ of devices (or device manufacturers) that are no longer to be trusted.

Intuitively, a broadcast encryption scheme is secure if D_{k_i} can successfully recover x from $E_{k_{master}, R}(x)$ whenever $i \notin R$, but fails to do so whenever $i \in R$. In our example of movie piracy, such an encryption scheme would allow us to *revoke* certain device keys k_i when we find out that they have been leaked. To revoke a key k_i , we would simply include $i \in R$ when encrypting all future movies. Doing so prevents k_i from being used to decrypt these movies. Crucially, revoking the key k_i of the hacked device i doesn't prevent a secure device $j \neq i$ from continuing to perform decryption on future movie releases; this is exactly what we want in our system.

For the sake of brevity, we will *not* provide a formal definition of security for broadcast encryption schemes, although this can and has been done. Instead, in the remainder of this section, we will describe a couple examples of broadcast encryption schemes, one of which makes clever use of a tree construction, as promised.

The simplest construction of a broadcast encryption scheme involves letting $k_{master} = (k_1, \dots, k_m)$ be the collection of all device keys and letting $E_{k_{master}, R}(x)$ be the concatenation over all $i \notin R$ of a secure encryption $E_{k_i}(x)$. Device i performs decryption by looking up the relevant substring $E_{k_i}(x)$ of the ciphertext and decrypting it with k_i . Intuitively, with this scheme, if x represents our movie data and there are $m \approx$ one million devices, then $E_{k_{master}, R}(x)$ is just an encryption of one million copies of the movie (one for each device key). Revoking the key k_i amounts to only encrypting 999,999 copies of all future movies, so that device i can no longer perform decryption.

Clearly, this simple solution to the broadcast encryption problem has two serious inefficiencies: the length of the master key is $O(nm)$, and the length of each encryption is $O(|x|m)$. One way to address the former problem is to use a *key derivation function*. That is, we can shorten the master key by choosing a fixed PRF collection $\{f_k\}$, and calculating each device key k_i by the rule $k_i = f_{k_{master}}(i)$. The latter problem can be addressed using a technique known as *hybrid encryption*. In hybrid encryption, we encrypt x by first choosing an ephemeral key $\hat{k} \leftarrow_R \{0, 1\}^n$, encrypting \hat{k} using each device key k_i where $i \notin R$, and then outputting the concatenation of these strings $E_{k_i}(\hat{k})$, along with a *single* encryption $E_{\hat{k}}(x)$ of the movie using the ephemeral key. Incorporating these two optimizations reduces the length of k_{master} to $O(n)$ and the length of each encryption to $O(nm + |x|)$.

keys belonging
to device 3 = 011_2

keys used after
revoking device 3

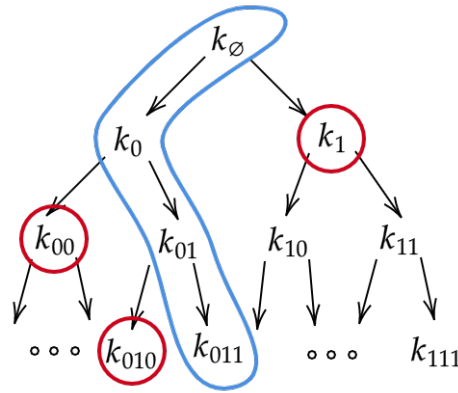


Figure 5.15: A tree based construction of broadcast encryption with revocable keys.

It turns out that we can construct a broadcast encryption scheme with even shorter ciphertexts by considering a *tree of keys* (see Fig. 5.15). The root of this tree is labeled k_\emptyset , its children are k_0 and k_1 , their children are $k_{00}, k_{01}, k_{10}, k_{11}$, and so on. The depth of the tree is $\log_2 m$, and the value of each key in the tree is decided uniformly at random, or by applying a key derivation function to a string k_{master} . Each device i receives *all* the keys on the path from the root to the i th leaf. For example, if $m = 8$, then device 011 receives the keys $k_\emptyset, k_0, k_{01}, k_{011}$.

To encrypt a message x , we carry out the following procedure: initially, when no keys have been revoked, we encrypt x using an ephemeral key \hat{k} (as described above) and encrypt \hat{k} with a single device key k_\emptyset . This is sufficient since all devices have access to k_\emptyset . In order to add a hacked device i to the revocation set, we discard all $\log_2 m$ keys belonging to device i , which comprise a root-to-leaf path in the tree. Instead of using these keys, we will make sure to encrypt all future \hat{k} 's using the *siblings* of the vertices along this path. Doing so ensures that (1) device i can no longer decrypt secure content and (2) every device $j \neq i$ can *continue* to decrypt content using at least one of the keys along the path from the root to the j th leaf. With this scheme, the total length of a ciphertext is only $O(n|R| \log_2 m + |x|)$ bits, where $|R|$ is the number of devices revoked so far. When $|R|$ is small, this bound is *much* better than what we previously achieved without a tree-based construction.

5.5 READING COMPREHENSION EXERCISES

I recommend students do the following exercises after reading the lecture. They do not cover all material, but can be a good way to check your understanding.

Exercise 5.1 Let (E, D) be the encryption scheme that we saw in Lecture 2 where $E_k(m) = G(k) \oplus m$ where $G(\cdot)$ is a pseudorandom generator. Is this scheme CPA secure?

- No it is never CPA secure.
- It is always CPA secure.
- It is sometimes CPA secure and sometimes not, depending on the properties of the PRG G



Exercise 5.2 Consider the proof constructing PRFs from PRGs. Up to an order of magnitude, how many invocations of the underlying pseudorandom generator does the pseudorandom function collection make when queried on an input $i \in \{0, 1\}^n$?

- n
- n^2
- 1
- 2^n



Exercise 5.3 In the following we identify a block cipher with a pseudorandom permutation (PRP) collection. Which of these statements is true:

- Every PRP collection is also a PRF collection
- Every PRF collection is also a PRP collection
- If $\{f_s\}$ is a PRP collection then the encryption scheme $E_s(m) = f_s(m)$ is a CPA secure encryption scheme.
- If $\{f_s\}$ is a PRF collection then the encryption scheme $E_s(m) = f_s(m)$ is a CPA secure encryption scheme.



