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Are Your Polyhedra the Same as My Polyhedra?

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1 Introduction

"Polyhedron" means different things to different people. There is very little in common between the meaning of the word in topology and in geometry. But even if we confine attention to geometry of the 3-dimensional Euclidean space – as we shall do from now on – "polyhedron" can mean either a solid (as in "Platonic solids", convex polyhedron, and other contexts), or a surface (such as the polyhedral models constructed from cardboard using "nets", which were introduced by Albrecht Dürer [17] in 1525, or, in a more modern version, by Aleksandrov [1]), or the 1-dimensional complex consisting of points ("vertices") and line-segments ("edges") organized in a suitable way into polygons ("faces") subject to certain restrictions ("skeletal polyhedra", diagrams of which have been presented first by Luca Pacioli [44] in 1498 and attributed to Leonardo da Vinci). The last alternative is the least usual one – but it is close to what seems to be the most useful approach to the theory of *general* polyhedra. Indeed, it does not restrict faces to be planar, and it makes possible to retrieve the other characterizations in circumstances in which they reasonably apply: If the faces of a "surface" polyhedron are simple polygons, in most cases the polyhedron is unambiguously determined by the boundary circuits of the faces. And if the polyhedron itself is without selfintersections, then the "solid" can be found from the faces. These reasons, as well as some others, seem to warrant the choice of our approach.

Before deciding on the particular choice of definition, the following facts – which I often mention at the start of courses or lectures on polyhedra – should be considered. The regular polyhedra were enumerated by the mathematicians of ancient Greece; an account of these five "Platonic solids" is the final topic of Euclid's "Elements" [18]. Although this list was considered to be complete, two millennia later Kepler [38] found two additional regular polyhedra, and in the early 1800's Poinsot [45] found these two as well as two more; Cauchy [7] soon proved that there are no others. But in the 1920's Petrie and Coxeter found (see [8]) three new regular polyhedra, and proved the completeness of that enumeration. However, in 1977 I found [21] a whole lot of new regular polyhedra, and soon thereafter Dress proved [15], [16] that one needs to add just one more polyhedron to make my list complete. Then, about ten years ago I found [22] a whole slew of new regular polyhedra, and so far nobody claimed to have found them all.

How come that results established by such accomplished mathematicians as Euclid, Cauchy, Coxeter, Dress were seemingly disproved after a while? The answer is simple – all the results mentioned are completely valid; what changed is the meaning in which the word "polyhedron" is used. As long as different people interpret the concept in different ways there is always the possibility that results true under one interpretation are false with other understandings. As a matter of fact, even slight variations in the definitions of concepts often entail significant changes in results.

In some ways the present situation concerning polyhedra is somewhat analogous to the one that developed in ancient Greece after the discovery of incommensurable quantities. Although many of the results in geometry were not affected by the existence of such quantities, it was philosophically and logically important to find a reasonable and effective approach for dealing with them. In recent years, several papers dealing with more or less general polyhedra appeared. However, the precise boundaries of the concept of polyhedra are mostly not explicitly stated, and even if explanations are given – they appear rather arbitrary and tailored to the needs of the moment [12] or else aimed at objects with great symmetry [40]. The main purpose of this paper is to present an internally consistent and quite general approach, and to illustrate its effectiveness by a number of examples.

In the detailed discussions presented in the following sections we shall introduce various restrictions as appropriate to the classes of polyhedra considered. However, I believe that in order to develop any general theory of polyhedra we should be looking for a definition that satisfies the following (admittedly somewhat fuzzy) conditions.

(i) *The generality should be restricted only for very good reasons, and not arbitrarily or because of tradition.* As an example, there is no justification for the claim that for a satisfactory theory one needs to exclude polyhedra that contain coplanar faces. (Thus, if we were to interpret the two regular star-polyhedra found by Kepler as solids – the way they are usually shown – each would be bounded by 60 congruent triangles. Since quintuplets of triangles are coplanar, these "polyhedra" would be inadmissible.) In particular, the definition should not be tailored to fit a special class of polyhedra (for example, the regular ones, or the uniform polyhedra), in such a way that it is more or less meaningless in less restricted situations (such as the absence of high symmetry).

(ii) *The combinatorial type should remain constant under continuous changes of the polyhedron.* This is in contrast to the situation concerning the usual approach to convex polyhedra, where the combinatorial type is easily seen to be discontinuous. The point is illustrated in Figure 1, where the first three diagrams show pentagonal dodecahedra that are becoming

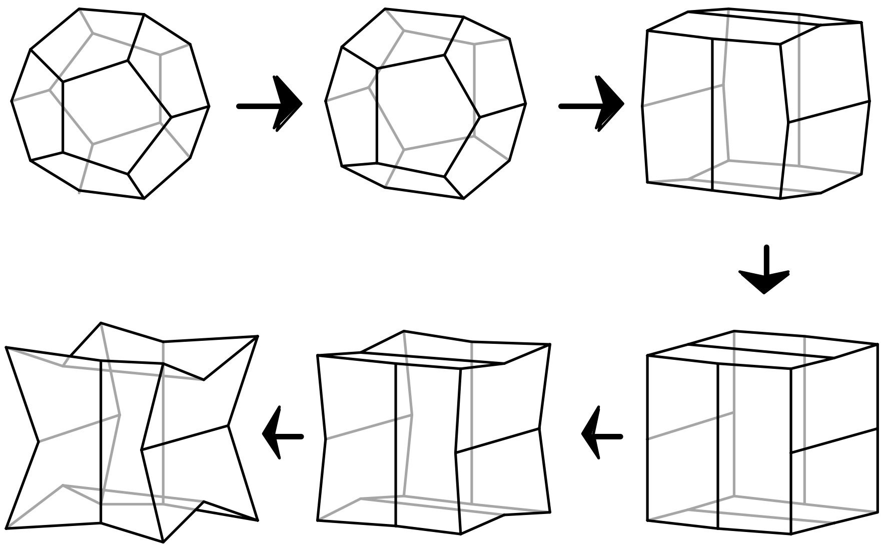

Fig. 1. The polyhedron with pairs of coplanar faces (at right in bottom row) is not a cube (even though the set of its points coincides with that of a cube) but is a pentagonal dodecahedron. It marks the transition between convex and nonconvex realizations of the same combinatorial type. Realizations of two polyhedra that have different combinatorial structure but coincide as sets of points (such as the cube and the above dodecahedron) are said to be *isomeghethic* (from Greek $\mu\varepsilon\gamma\varepsilon\theta o\sigma$ – extent, bulk).

more box-like, till pairs of faces turn coplanar for the fourth polyhedron. We wish to consider this 12-faced polyhedron as distinct from the cube, and as just a transitional step from convex to nonconvex polyhedra (as shown in the later parts of Figure 1), – all with the same combinatorial structure .

(iii) *To every combinatorial type of polyhedron there should correspond a dual type.* This is a familiar condition which is automatically satisfied by convex polyhedra, and is frequently stated as valid in all circumstances – although in fact it fails in some cases. We shall discuss this in Section 3.

Preparatory for the discussion of polyhedra, in Section 2 we consider polygons. Our working definition of "polyhedron" is presented and illustrated in Section 3. Sections 4 to 8 are devoted to the analysis of some specific classes of polyhedra that have been discussed in the literature and for which we believe the present approach provides a better and more consistent framework than previously available.

2 Polygons

Since we consider polyhedra as families of points, segments and polygons (subject to appropriate conditions), it is convenient to discuss polygons first.

Like "polyhedron", the word "polygon" has been (and still is) interpreted in various ways.

A *polygon* (specifically, an *n-gon* for some $n \geq 3$) is a cyclically ordered sequence of arbitrarily chosen points V_1, V_2, ... , V_n, (the *vertices* of the polygon), together with the segments E_i determined by pairs of vertices V_i, V_{i+1} adjacent in the cyclic order (the *edges* of the polygon). Each vertex V_i is said to be *incident* with edges E_{i-1} and E_i, and these edges only. Here and in the sequel subscripts should be understood mod n. Polygons were first considered in close to this generality by Meister [41] nearly 250 years ago. (The assertion by Günther [30, p. 25] and Steinitz [52, p. 4] that already Girard [20] had this perception of "polygon" seems unjustified.) As explicitly stressed by Meister, this definition implies that distinct vertices of a polygon may be represented by the same point, without losing their individuality, and without becoming incident with additional edges even if the point representing a vertex is situated on another edge. Hence the definition admits various unexpected possibilities: edges of length 0; collinear edges – adjacent or not; edges overlapping or coinciding in pairs or larger sets; the concurrence of three or more edges. In order to simplify the language, the locution "vertices coincide" is to be interpreted as "the *points* representing the *distinct* vertices coincide"; similarly for edges.

Most of the important writings on polygons after Meister (such as Poinsot [45], Cauchy [7], Möbius [42], Wiener [55], Steinitz [52], Coxeter [9]) formulate the definition in the same way or equivalent ones, even though in some cases certain restrictions are added; for example, Möbius insists that the polygon not be contained in a line. However, all these writers *tacitly* assume that no two vertices fall on coinciding points. It is unfortunate that Günther [30, pp. 44ff] misunderstands Meister and imputes to him the same restriction. Other authors (for example, Brückner [3, p. 1], Hess [32, p. 611]) insist explicitly in their definitions that no two vertices of a polygon are at the same point; but later Brückner [3, p. 2] gives another definition, that coincides with ours, apparently written under the impression that it has the same meaning as his earlier one. However, disallowing representation of distinct vertices by one point is a crippling restriction which, I believe, is one of the causes for the absence of an internally consistent general theory of polyhedra. (The present definition coincides with what were called *unicursal polygons* in [23], where more general objects were admitted as "polygons".)

It should be mentioned that all these authors define a polygon as being a single circuit; Möbius explicitly states that it would be contrary to the customary meaning of the word if one were to call "hexagon" the figure formed by two triangles. This seems to have had little influence on later writers. For example, without any formalities or explanations, Brückner [3, p. 6] introduces such figures as "discontinuous polygons", in contradiction to his own earlier definition of "polygon". Hess [32] has a more vague definition of polygons, and explicitly allows "discontinuous" ones.

Since the number of "essentially different" shapes possible for n-gons increases very rapidly with increasing n, it is reasonable and useful to consider various special classes which can be surveyed more readily. The historically and practically most important classes are defined by symmetries, that is, by isometric transformations of the plane of the polygon that map the polygon onto itself. In case some of the vertices coincide, symmetries should be considered as consisting of an isometry paired with a permutation of the vertices. Thus, the *quadrangle* in Figure 2 does not admit a 120° rotational symmetry, but it admits a reflection in a vertical mirror paired with the permutation (12)(34). Clearly, all symmetries of any polygon form a group, its *symmetry group*.

A polygon is called *isogonal* [*isotoxal, regular*] provided its vertices [edges, flags] form a single orbit under its symmetry group. (A flag is a pair consisting of a vertex and one of the edges incident with it.) It is easily proved that a polygon is regular if and only if it is both isogonal and isotoxal. Moreover, if $n \geq 3$ is odd, every isogonal n-gon is regular, as is every isotoxal one. The more interesting situation of even n is illustrated for $n = 6$ in Figure 3. Similar illustrations of the possibilities for other values of n appear in [23], [24], [25].

Two consequences of the above definition of polygons deserve to be specifically mentioned; both are evident in Figure 3, and become even more pronounced for larger n. First, all isogonal n-gons fit into a small number of continua, and so do all isotoxal n-gons. If polygons having some coinciding vertices were excluded, the continua would be artificially split into several components, and the continuity would largely disappear. Second, the number of regular polygons would be considerably decreased. Under our definitions, for every pair of integers n and d, with $0 \leq d \leq n/2$, there exists a regular n-gon, denoted by its *Schläfli symbol* $\{n/d\}$. The construction of polygons $\{n/d\}$ inscribed in a unit circle can be described as follows (this was first for-

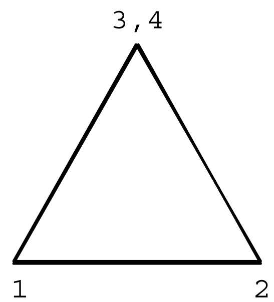

Fig. 2. This polygon looks like an equilateral triangle, but is in fact a quadrangle with two coinciding vertices. Besides the identity, the only symmetry it admits is a reflection (in a vertical mirror through the coinciding vertices 3 and 4) paired with the permutation (12)(34).

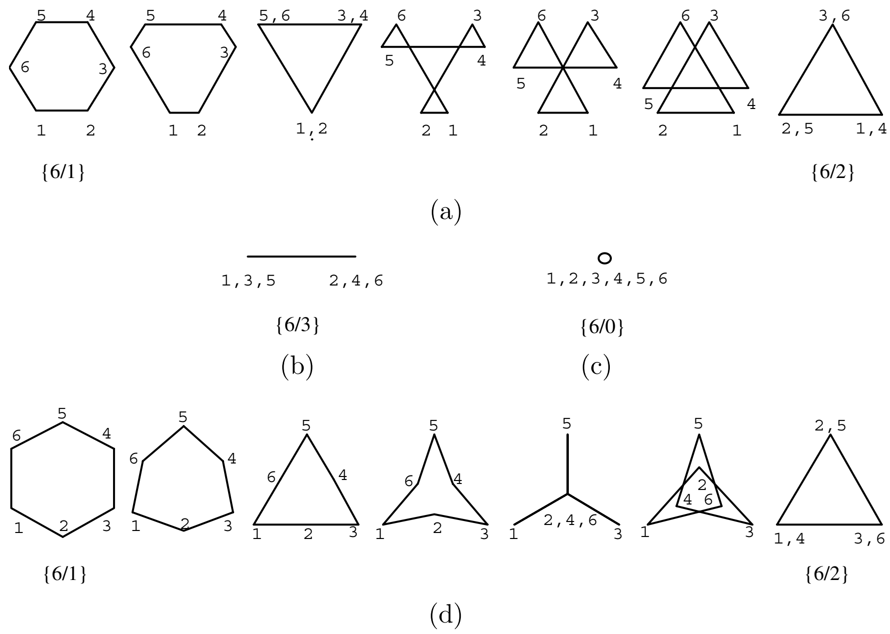

Fig. 3. Illustration of some of the polygons with high symmetry; shown is the case of hexagons. Isogonal polygons are shown in parts (a), (b), (c). Isotoxal polygons (in which all edges form one orbit under symmetries) are shown in parts (b), (c), (d). Representatives of the various shapes that isogonal or isotoxal hexagons can assume are illustrated. Regular polygons are indicated by their Schläfli symbol $\{n/d\}$.

mulated by Meister [41], and was also stated by Poinsot [45]): From a point on the circle, taken as the first vertex, advance to the next vertex by turning through an angle of $2\pi d/n$, and repeat this procedure from each resulting point till the starting point is reached at the n^{th} step. Naturally, depending on the values of n and d, some of the intermediate vertices may coincide, but their identities are determined by the number of steps that led to them. Thus, for example, $\{6/0\}$ has six coinciding vertices, $\{6/2\}$ has three pairs of coinciding vertices, and $\{6/3\}$ has two triplets of coinciding vertices; in contrast, all vertices of $\{6/1\}$ are distinct. It takes no effort to realize that all vertices of $\{n/d\}$ will be distinct if and only if n and $d > 0$ are relatively prime. This connection between geometry and number theory was stressed by Poinsot and all writers following his example, and seems to have been one of the reasons why they banished from consideration all polygons with coinciding vertices. This was done despite the fact that the various results concerning angles, areas and other properties of regular polygons remain valid regardless of the relative primeness of n and d. Moreover, allowing polygons with coinciding vertices is essential if one wishes to have continuity in the combinatorial types of polyhedra.

One other consideration requires admitting polygons – and in particular, regular polygons – with coinciding vertices. In the present paper we are concerned with unoriented polygons; however, in some situations it is convenient or necessary to assign to each polygon an orientation. This yields two oriented polygons for each unoriented $\{n/d\}$ (except if $d = 0$ or $d = n/2$). Among regular polygons it is convenient to understand that the rotations through $2\pi d/n$ yielding $\{n/d\}$ are taken in the positive orientation; then the polygon oppositely oriented to $\{n/d\}$ is $\{n/e\}$, where $e = n - d$. Thus, oriented regular polygons $\{n/d\}$ exist for all $n > d \geq 0$, and these n polygons are all distinct. The appropriateness of such a convention is made evident by its applicability in many results concerning arbitrary polygons. It would lead us too far to describe these results, which can be interpreted as consequences of the possibility of expressing every n-gon as a weighted sum (in an appropriate sense) of regular polygons. The results range from Napoleon-type theorems to the elucidations of limits of iterations of various averaging operations on polygons. Detailed information about such applications, which would not be possible under the Poinsot restriction, may be found in [2], [13], [14], [19], [39], [43], [46], [47], [48], and in their references.

3 Definition of Polyhedron

In my opinion, the most satisfying way to approach the definition of polyhedra is to distinguish between the **combinatorial structure** of a polyhedron, and the **geometric realizations** of this combinatorial structure. We start by listing the conditions under which a collection of objects called *vertices*, *edges*, and *faces* will be called an *abstract polyhedron*. The conditions involve a (primitive) relation of *incidence*, and a (derived) relation of *adjacence*. In an abstract way of thinking, an edge is a pair of vertices, and a face is a circuit of edges. More specifically, in an abstract polyhedron we have to have:

(P1) Each edge is incident with precisely two distinct vertices and two distinct faces.

Each of the two vertices is said to be *incident* (via the edge in question) with each of the two faces. Two vertices incident with an edge are said to be *adjacent*; also, two faces incident with an edge are said to be *adjacent*.

(P2) For each edge, given a vertex and a face incident with it, there is precisely one other edge incident to the same vertex and face.

This edge is said to be *adjacent* to the starting edge.

(P3f) For each face there is an integer k, such that the edges incident with the face, and the vertices incident with it via the edges, form a circuit in the sense that they can be labeled as $V_1E_1V_2E_2V_3E_3 \ldots V_{k-1}E_{k-1}V_kE_kV_1$, where each edge E_i is incident with vertices V_i and V_{i+1}, and adjacent to edges E_{i-1} and E_{i+1}. All edges and all vertices of the circuit are distinct, all subscripts are taken mod k, and $k \geq 3$.

(P3v) For each vertex there is an integer j, such that the edges incident with the vertex, and the faces incident with it via the edges, form a circuit in the sense that they can be labeled as $F_1E_1F_2E_2F_3E_3 \ldots F_{j-1}E_{j-1}F_jE_jF_1$, where each edge E_i is incident with faces F_i and F_{i+1}, and adjacent to edges E_{i-1} and E_{i+1}. All edges and all faces of the circuit are distinct, all subscripts are taken mod j, and $j \geq 3$.

Thus, each face corresponds to a simple circuit of length at least 3, and similarly for the circuits that correspond to the vertices; the latter circuits are known as *vertex stars*.

(P4) If two edges are incident with the same two vertices [faces], then the four faces [vertices] incident with the two edges are all distinct.

(P5f) Each pair F, F* of faces is connected, for some j, through a finite chain $F_1E_1F_2E_2F_3E_3 \ldots F_{j-1}E_{j-1}F_j$ of incident edges and faces, with $F_1 = F$ and $F_j = F^*$.

(P5v) Each pair V, V* of vertices is connected, for some j, through a finite chain $V_1E_1V_2E_2V_3E_3 \ldots V_{j-1}E_{j-1}V_j$ of incident edges and vertices, with $V_1 = V$ and $V_j = V^*$.

It should be noted that with this definition, the duality requirement is satisfied in an essentially trivial way: Given an abstract polyhedron, a dual abstract polyhedron is obtained by interchanging "vertices" and "faces". The formulation of the conditions (P1) to (P5) shows that they will be satisfied after such an exchange.

A *symmetry* of an abstract polyhedron is an automorphism induced by incidence-preserving permutations of the vertices, the edges, and the faces. In most cases we shall encounter, such an automorphism is already determined by a permutation of the vertices.

It is clear that the above definition could have been formulated as pertaining to a special class of cell-complexes representing 2-dimensional closed manifolds. In fact, each face may be understood as the boundary of a 2-dimensional topological disk, and the identifications determined by incidences and adjacencies determine the cell-decomposition of a manifold, which we shall call the *associated manifold* of the polyhedron. For a given abstract polyhedron we shall often refer to its associated manifold and we shall assign to the abstract polyhedron as its *genus*, or its *Euler characteristic*, the values of these functions for the associated manifold. On the other hand, cell-decompositions in general admit features that cannot occur in polyhedra; for example, our definition does not admit monogons or digons.

Equally obvious is the fact that the conditions listed above (hence the definition of an abstract polyhedron) could have been formulated in terms of lattices. Such an approach is taken by McMullen and Schulte [40], to define not only objects more general than our polyhedra, but also the analogous higher-dimensional *abstract polytopes*.

A *geometric polyhedron* or *polyhedron* for short is an image of an abstract polyhedron under a mapping in which vertices go to points, edges to segments (possibly of length 0) and faces to polygons (which are understood as circuits of incident vertices and edges). *Incidence* means that the point representing a vertex is an endpoint of a segment representing an edge, and that a segment (which represents an edge) is a member of the cycle which defines a polygon (representing a face). We say that the polyhedron is a *realization* of the *underlying* abstract polyhedron.

If all faces of a geometric polyhedron are simple polygons, we may interpret each face as a topological disk. Their totality forms a surface which may have selfintersections or overlaps. Best known examples of this kind are the two regular polyhedra first discovered by Poinsot [45] – the great icosahedron $\{3, 5/2\}$ and the great dodecahedron $\{5, 5/2\}$.

Polyhedra with the same underlying abstract polyhedron are said to be *combinatorially equivalent,* or to have the same *combinatorial type*. Realizations of two polyhedra that have different combinatorial types but coincide as sets of points are said to be *isomeghethic* (from Greek $\mu\varepsilon\gamma\varepsilon\theta o\sigma$ – extent, bulk). This term may be used in cases where we interpret the polyhedra as surfaces (such as the cube and the fourth dodecahedron in Figure 1), as well as in cases in which selfintersecting polygons necessitate interpreting faces as circuits of vertices and edges. In this sense the regular dodecahedron is isomeghetic with the uniform polyhedron in Figure 15(c).

A *symmetry* of a (geometric) polyhedron is a pairing of an isometric mapping of the polyhedron onto itself with an automorphism of the underlying abstract polyhedron. The polyhedron is *isogonal* [*isohedral*, *regular*] if its vertices [faces, flags] form one orbit under its symmetries. (A *flag* of a polyhedron is a triplet consisting of a vertex, an edge, and a face, all mutually incident.) A polyhedron is *noble* if it is both isogonal and isohedral.

Every abstract polyhedron has realizations: Nothing in the definition prevents all vertices to be represented by the same point. Clearly, such *trivial* realizations are usually of little interest, but in some contexts they need to be considered. Also, abstract polyhedra may have other *subdimensional* realizations – that is, the affine hull of a realization may well be 1- or 2-dimensional. An example of a noble polyhedron that realizes the Klein bottle in the plane is shown in Figure 4. In the remaining part of the paper we shall concentrate on full-dimensional polyhedra, that is, polyhedra with 3-dimensional affine hull.

For a given geometric polyhedron the construction of a dual polyhedron is most often carried out by applying to its faces and vertices a *polarity* (that is, a reciprocation in a sphere). From properties of this operation it follows at once that the polar of a given polyhedron is a realization of the abstract polyhedron dual to the given one. However, the possibility of carrying out the polarity depends on choosing a sphere for the inversion in such a way that its center is not contained in the plane of any face. While this is easy to accomplish in any case, the resulting shape depends strongly on the position of that center. The main problem arises in connection with polyhedra with high symmetry (for example, isogonal or uniform polyhedra) if it is desired to find a dual with the same degree of symmetry: If the only position for the center is at the centroid of the polyhedron, and the polyhedron has some faces that contain the centroid – then it is not possible to

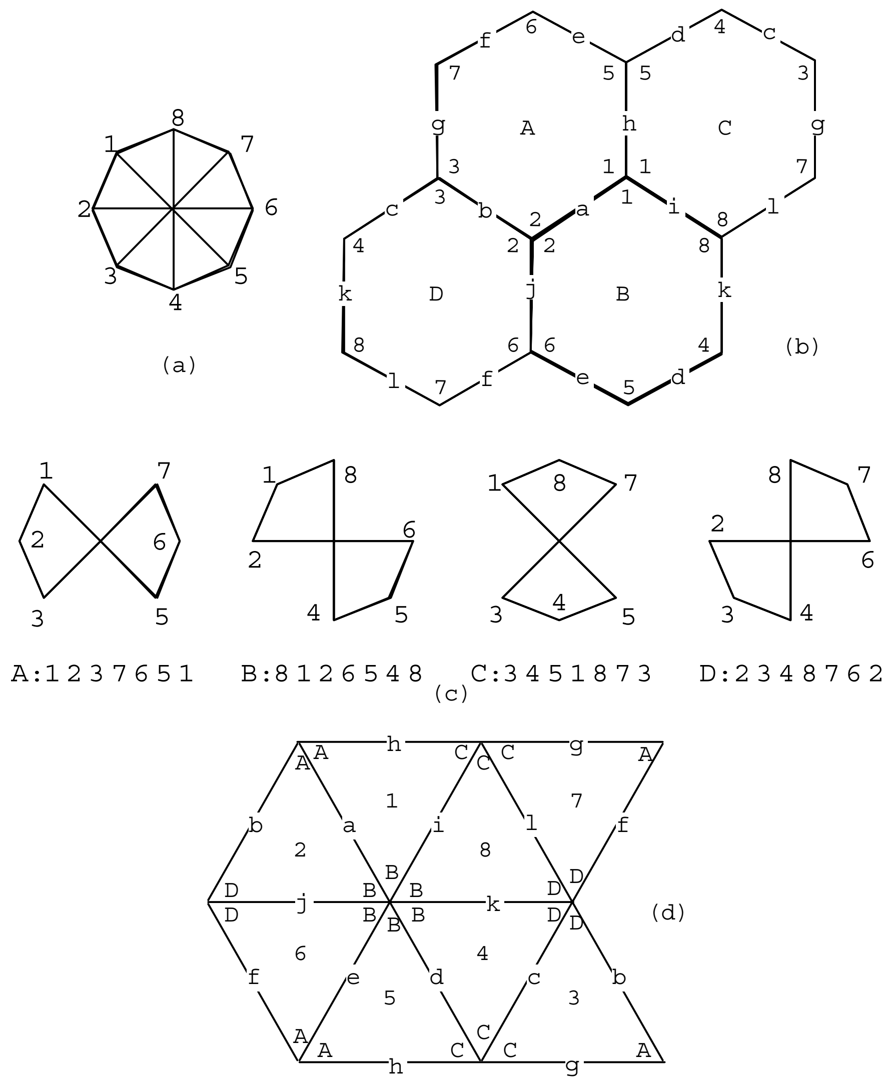

Fig. 4. A subdimensional noble polyhedron is shown in (a). The associated cell complex representing the underlying abstract polyhedron is shown in (b); the manifold is the Klein bottle. Each of the four faces of the geometric realization in (a) is shown separately in (c). Note that any two faces are both incident with two edges, but these have distinct vertices as required by condition (P4). The cell complex representing the abstract polyhedron dual to the one in (a) and (b) is shown in (d); as is easily verified, although the abstract polyhedron is noble, it admits no nontrivial realization as a noble geometric polyhedron.

find a polar polyhedron with the same symmetry. Moreover, if a polyhedron has coplanar faces [coinciding vertices] then any polar polyhedron will have coinciding vertices [coplanar faces]. All these possibilities actually occur for various interesting polyhedra. Clearly, duality-via-polarity is uninteresting for subdimensional polyhedra – it yields only trivial ones.

Our definitions are applicable to finite as well as infinite polyhedra; this enables one to include tilings and honeycombs among the objects studied. However, for the present discussion we shall restrict attention to finite polyhedra, that is, we shall assume the cardinalities of the sets of vertices, edges, and faces to be finite.

Despite the adaptability of the "skeletal" approach to such topics as polyhedra with skew polygons as faces, in the present work we shall consider only polyhedra with planar faces.

4 Regular Polyhedra

We shall now present constructions that lead to some "new" regular polyhedra. One construction which may be applied to polyhedra in general, is by the following *vertex-doubling*. Start with any abstract or geometric polyhedron. Replace each vertex by a pair of vertices, for example a green one and a red one. For each face, as you go around it, alternate between red and green vertices. If the face is an n-gon for some odd n, then there will now be a $(2n)$-gon in its place; if n is even, the vertices along the n-gon will have alternating colors – but there will be another n-gon, with vertices of the opposite colors. The collection of these new faces will be an (abstract or geometric) polyhedron if and only if there is at least one odd-sided face in the original polyhedron. If there is no such face, the resulting family of polygons does not satisfy condition (P.5f) of the definition is Section 3; instead of a polyhedron, a *compound* of two polyhedra is obtained. Dually, one can start with any polyhedron, replace each face by a pair of faces of different "colors", and take as adjacent those faces which arise from adjacent faces of the original and have different colors. This *face-doubling* gives rise to a new polyhedron if and only is there is at least one vertex of odd valence in the original polyhedron.

It should be stressed that the above comments *do not* mean that if all faces are even-sided, then there is no polyhedron in which the vertices are doubled up and all new faces have double the number of sides of the original ones. It means *only* that the above method of replacing one vertex by two vertices represented by the same point does not lead to such new polyhedra. At the end this section we shall encounter an example that illustrates this comment.

A general property of the vertex-doubling construction considered here is that it transforms regular polyhedra into regular ones, and isogonal or isohedral polyhedra into isogonal or isohedral ones, respectively. Analogously for the face-doubling construction. Probably the most interesting instances to which the vertex-doubling procedure can be applied are eight of the nine regular polyhedra (five convex and four Kepler-Poinsot) – all except the cube. The resulting "new" polyhedra are regular and can be denoted by their Schläfli symbols $\{6/2, 3\}$, $\{6/2, 4\}$, $\{6/2, 5\}$, $\{10/2, 3\}$, $\{6/2, 5/2\}$, $\{10/2, 5/2\}$, $\{10/4, 3\}$, $\{10/4, 5\}$. The face-doubling construction can be applied to all regular polyhedra except the octahedron, and yields "new" regular polyhedra $\{3, 6/2\}$, $\{4, 6/2\}$, $\{5, 6/2\}$, $\{3, 10/2\}$, $\{5/2, 6/2\}$, $\{5/2, 10/2\}$, $\{3, 10/4\}$ and $\{5, 10/4\}$. Clearly, these sixteen polyhedra form eight pairs of dual polyhedra. Moreover, the duality can be effected by a polarity (that is, by reciprocation in a suitable sphere). It should be noted that the number of combinatorial types of these regular polyhedra is smaller. Just as the combinatorial types of the icosahedron $\{3, 5\}$ and the great icosahedron $\{3, 5/2\}$ coincide, so do pairs of polyhedra $\{6/2, 5\}$ and $\{6/2, 5/2\}$, $\{10/2, 3\}$ and $\{10/4, 3\}$, $\{10/2, 5/2\}$ and $\{10/4, 5\}$, $\{5, 6/2\}$ and $\{5/2, 6/2\}$, $\{3, 10/2\}$ and $\{3, 10/4\}$, $\{5/2, 10/2\}$ and $\{5, 10/4\}$; each pair represents a single combinatorial type.

The polyhedra $\{3, 6/2\}$ and $\{6/2, 3\}$ are shown in Figure 5, where lower and upper case characters are used instead of different colors. All the other regular polyhedra listed above would appear, analogously, like their counterparts among the convex or Kepler-Poinsot polyhedra to which they are isomeghethic; however, their combinatorial structure – determined by the underlying abstract polyhedron – is different from that of the traditional ones.

A natural question that arises from these constructions is whether it is possible to perform **vertex k-tupling**, that is replace each face of a polyhedron by a polygon having k times as many sides or by a family of k polygons with the same number of sides, where $k \geq 3$. While we have seen that cases in which the doubling operation yields a polyhedron are rather easily characterized, no corresponding general result is known for ktupling. However, in case the operation is performed on a tetrahedron, there is an affirmative answer, as follows.

For a given $k \geq 2$ we may replace each vertex of the tetrahedron by k vertices; if these are denoted $\mathrm{a}_1, \ldots, \mathrm{a}_k$, $\mathrm{b}_1, \ldots, \mathrm{b}_k$, $\mathrm{c}_1, \ldots, \mathrm{c}_k$, $\mathrm{d}_1, \ldots, \mathrm{d}_k$, then the four faces are given by $[\mathrm{d}_1\mathrm{c}_1\mathrm{b}_1\ \mathrm{d}_2\mathrm{c}_2\mathrm{b}_2\ \mathrm{d}_3\mathrm{c}_3\mathrm{b}_3 \ldots \mathrm{d}_k\mathrm{c}_k\mathrm{b}_k\ \mathrm{d}_1]$, $[\mathrm{c}_1\mathrm{d}_1\mathrm{a}_1\ \mathrm{c}_k\mathrm{d}_k\mathrm{a}_k\ \mathrm{c}_{k-1}\mathrm{d}_{k-1}\mathrm{a}_{k-1} \ldots \mathrm{c}_2\mathrm{d}_2\mathrm{a}_2\ \mathrm{c}_1]$, $[\mathrm{b}_1\mathrm{a}_1\mathrm{d}_1\ \mathrm{b}_k\mathrm{a}_k\mathrm{d}_k\ \mathrm{b}_{k-1}\mathrm{a}_{k-1}\mathrm{d}_{k-1} \ldots \mathrm{b}_2\mathrm{a}_2\mathrm{d}_2\ \mathrm{b}_1]$, $[\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\ \mathrm{a}_2\mathrm{b}_2\mathrm{c}_2\ \mathrm{a}_3\mathrm{b}_3\mathrm{c}_3 \ldots \mathrm{a}_k\mathrm{b}_k\mathrm{c}_k\ \mathrm{a}_1]$; each face is of type $\{3k/k\}$. (Here, and throughout the paper, the first vertex of each face is repeated at the end to make the checking of incidences simpler; each face is described by listing its vertices in a cyclic order, and spaces are inserted to facilitate understanding the structure.) This determines an orientable polyhedron $\mathrm{P}(k)$ with $4k$ vertices, $6k$ edges, and 4 faces; hence the associated map has genus $g = k-1$. For $k = 2$ the polyhedron has as its map the regular one in Figure 5(b); no $\mathrm{P}(k)$ with $k \geq 3$ is regular, as can be checked easily. The map corresponding to $\mathrm{P}(3)$ is the only possible map of type $\{9/3, 3\}$, hence these parameters do not admit any regular map or polyhedron. On the other hand, at least

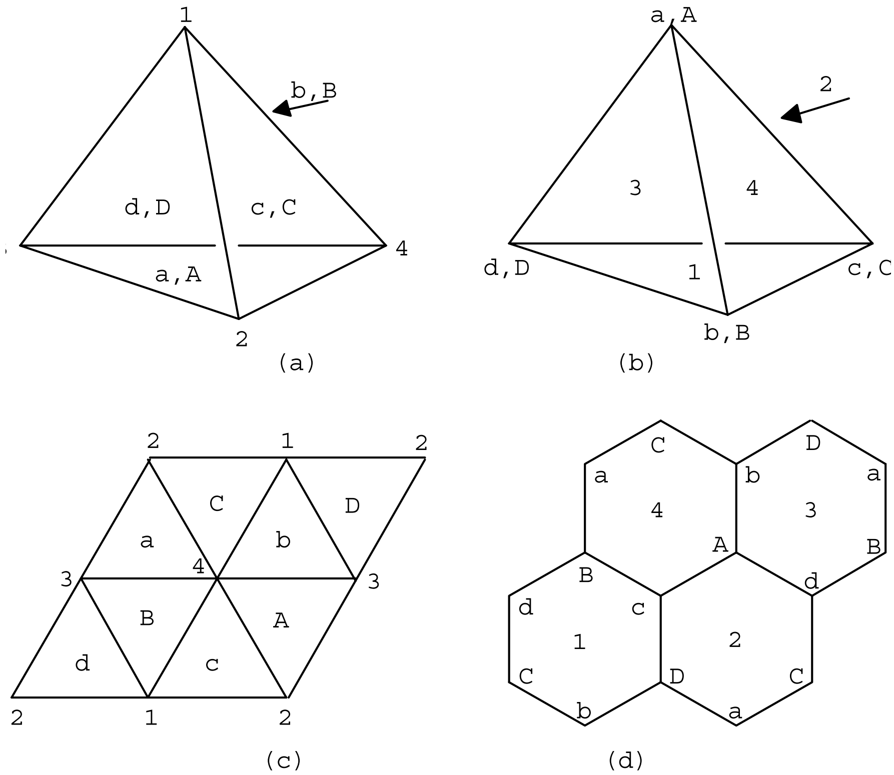

Fig. 5. The regular polyhedron in (a) was obtained by face-doubling from the regular tetrahedron, the regular polyhedron in (b), dual to it, resulted from vertex-doubling of the regular tetrahedron. The underlying abstract polyhedra are indicated by the cell complexes representing them in (c) and (d), respectively.

for $k = 4$ there is one other polyhedron $\mathrm{P}^{\#}$ with four faces of type $\{3k/k\}$, and it is regular (map W#24.22 in Wilson's catalog [56]). To distinguish between the two polyhedra we note that the faces of P(4) are $[d_1c_1b_1\ d_2c_2b_2\ d_3c_3b_3\ d_4c_4b_4\ d_1]$, $[c_1d_1a_1\ c_4d_4a_4\ c_3d_3a_3\ c_2d_2a_2\ c_1]$, $[b_1a_1d_1\ b_4a_4d_4\ b_3a_3d_3\ b_2a_2d_2\ b_1]$ and $[a_1b_1c_1\ a_2b_2c_2\ a_3b_3c_3\ a_4b_4c_4\ a_1]$, while the faces of the $\mathrm{P}^{\#}$ are $[d_1c_1b_1\ d_2c_2b_2\ d_3c_3b_3\ d_4c_4b_4\ d_1]$, $[c_1d_1a_3\ c_2d_2a_4\ c_3d_3a_1\ c_4d_4a_2\ c_1]$, $[b_1a_1d_3\ b_2a_2d_4\ b_3a_3d_1\ b_4a_4d_2\ b_1]$ and $[a_1b_1c_1\ a_2b_2c_2\ a_3b_3c_3\ a_4b_4c_4\ a_1]$. The fact that these polyhedra are not combinatorially equivalent can most easily be established from their maps, see Figure 6.

For $k \geq 5$ there probably exist polyhedra different from P(k), having faces of type $\{3k/k\}$ and $4k$ trivalent vertices. It may be conjectured that none of these polyhedra is regular. For $k \leq 16$ the validity of this can be inferred from the fact that there are no regular maps satisfying these conditions in Wilson's catalog [56], where all regular maps with at most 100 edges are listed.

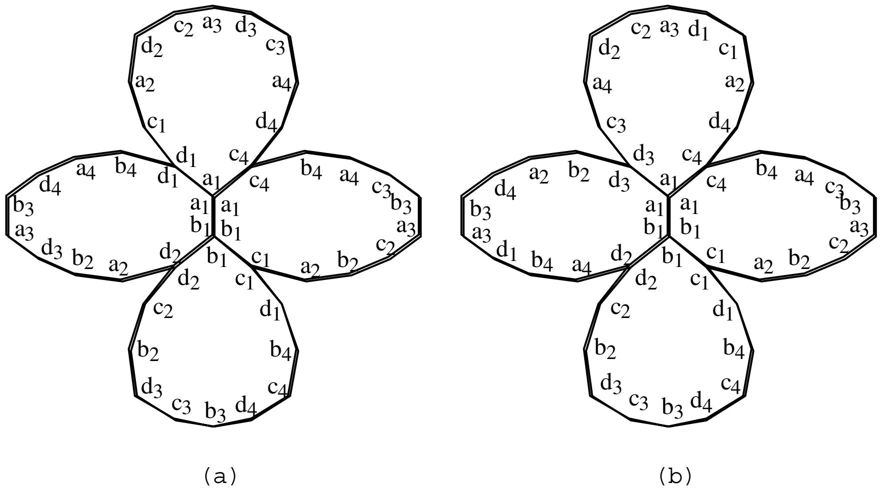

Fig. 6. The maps underlying two polyhedra, each with four faces of type $\{12/4\}$ and sixteen 3-valent vertices. The map in (b), and the corresponding polyhedron $P^{\#}$, are regular. The map of P(4) shown in (a) is not combinatorially equivalent to the map in (b). One way to see this is to observe that since (b) is regular, all its flags are equivalent. The labels of the two maps coincide for all vertices on the flags associated with vertex a_1, the edge $[a_1,b_1]$, and the face to the right of it. Hence, if the map in (a) were regular, the only possible isomorphism would preserve all labels – but this is clearly not the case with the faces on top or on the left.

Although the general vertex-doubling construction described above does not apply to the cube, there is a polyhedron obtained by doubling the vertices on the cube. This polyhedron Q and its map are shown in Figure 7. Both are regular; the polyhedron has the Schläfli symbol $\{8/2, 3\}$, and the map is W#24.21 in [56].

A construction of two infinite families of regular polyhedra should be mentioned here. The first arises from a certain vertex k-tupling of the octahedron. It is most simply described by saying that one of the triangular faces is replaced by a polygon $\{3k/k\}$, and the other triangular faces are replaced by suitable reflections of this face. The resulting polyhedron $\{3k/k, 4\}$ is easily seen to be regular; it is orientable, with 6k vertices, 12k edges and 8 faces, hence has genus $g = 3k-3$. The second family is polar to this; it arises by face k-tupling of the cube. Its Schläfli symbol is, accordingly, $\{4, 3k/k\}$. The case $k = 2$ of both families is illustrated in [27], where they are obtained by a different construction.

It may be conjectured that there is only a finite number of infinite families of full-dimensional regular polyhedra.

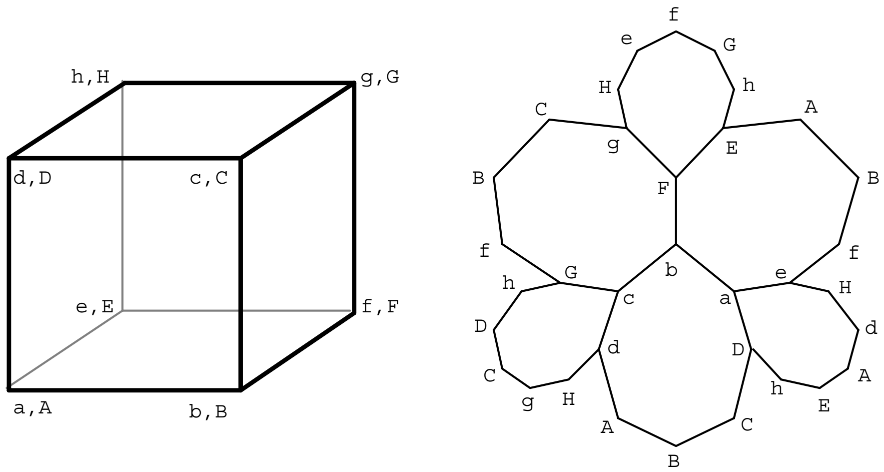

Fig. 7. Doubling-up vertices of the cube, in the way indicated in the map, yields a regular polyhedron $\{8/2, 3\}$ with 16 vertices, 24 edges and six faces. It is orientable, of genus 2. Its map is W#24.21 in the catalog [56].

5 Noble Polyhedra

Polyhedra that are noble (that is, isogonal as well as isohedral) have been studied considerably less than the slightly more symmetric regular ones. Nevertheless, they seem quite interesting. In particular, it seems that beyond finitely many infinite families of full-dimensional noble polyhedra, there exists only a finite number of individual polyhedra of this kind. This conjecture is one of the intriguing open questions concerning symmetric polyhedra.

The study of noble polyhedra was begun by Hess in the 1870's (see [33], [34], [35], [36]), and continued by Brückner [3], [4], [5], [6]. In the early papers (for example, in [33]), Hess considered noble polyhedra as generalization of regular ones. However, there seems to be no mention of these polyhedra in the literature after the works of Hess and Brückner, until [23], close to a century later. This may in part be due to a general neglect of nonconvex polyhedra during most of the 20th century, and in part to the inconsistent and clumsy exposition by Hess and Brückner of their own results.

It is obvious that all regular polyhedra are noble. It is well known that among convex polyhedra the only nonregular ones are *sphenoids*, that is, tetrahedra with congruent faces, different from equilateral triangles. From now on we shall discuss nonconvex noble polyhedra only.

We have encountered one subdimensional noble polyhedron in Figure 4, and many other examples of this kind are possible. However, the possibilities are much more restricted if we are interested in full-dimensional noble polyhedra.

Several infinite families of such polyhedra have symmetries of prisms or of anti-prisms. One family consists of the remarkable prismatic and antiprismatic *crown* polyhedra, discovered by Hess [35]; he called them *stephanoids* (from the Greek for "crown"). Their faces are selfintersecting quadrangles. Detailed descriptions and illustrations can be found in [23], where also the prismatic and antiprismatic *wreath* polyhedra (with triangular faces) and *V-faced* polyhedra are introduced and illustrated. The faces of V-faced polyhedra are full-dimensional quadrangles with vertices V_1,V_2,V_3,V_4, in which V_2 and V_4 are represented by the same point. (It should be noted that, contrary to the impression given by the illustrations in [23], the quadrangles in V-faced noble polyhedra need not be equilateral.) Hess and Brückner showed considerable ingenuity in discovering noble polyhedra, and one may wonder why they did not find the (rather simple) wreath polyhedra and V-faced polyhedra. One possible reason is that they were ignoring polyhedra with coplanar faces or coinciding vertices. However, rather inconsistently, in other instances (as mentioned below) they did allow such polyhedra, even in the same publications in which they state that vertices have to be distinct.

Probably more interesting are some of the noble polyhedra with octahedral or icosahedral symmetries. One polyhedron, a version of which was described by Hugel [37] but recognized as noble by Hess [35] and Brückner [3, p. 215], is shown in Figure 8. Its 20 vertices are the same as those of a regular dodecahedron, and its 20 faces are selfintersecting hexagons; one is indicated in Figure 8 by heavy lines. The polyhedron is *autopolar*, in the sense that its polar with respect to a suitable sphere coincides with the polyhedron itself. It is remarkable that the underlying abstract polyhedron is regular; its map is the one listed as W#60.57 in [56].

The situation is more complex concerning a noble polyhedron described by Hess [34] and Brückner [3, p. 215]. It is supposed to look like the object in Figure 9, that is, to be isomeghethic with the regular polyhedron $\{5, 5/2\}$. However, each of the twelve faces is formed by *all* segments determined by its five vertices, that is, each face is supposed to look like the union of a pentagon with a pentagram, shown by heavy lines in Figure 9. This is quite appropriate provided one allows a polygon to revisit vertices (as was the setup in [23]). In fact, in this case there are two distinct possibilities of turning the figure into an isogonal polygon (see [23]). Hess and Brückner seem to have noticed only one of these – but even this is contrary to their usual (and explicitly stated) requirement that a polygon cannot revisit any vertex. In any case, under the definitions we adopted in the present paper these are not polyhedra. However, doubling-up the vertices leads to polyhedra that are noble (as noted in [23]). One face of the first is the polygon [d F b D e B c E f C d] in the notation of Figure 9; the other 23 faces result by the application of symmetries of the icosahedron, and the interchange of lower and upper case vertices. In contrast, one face of the other polyhedron is [b C f B e F d E c D b], and the other faces are obtained analogously.

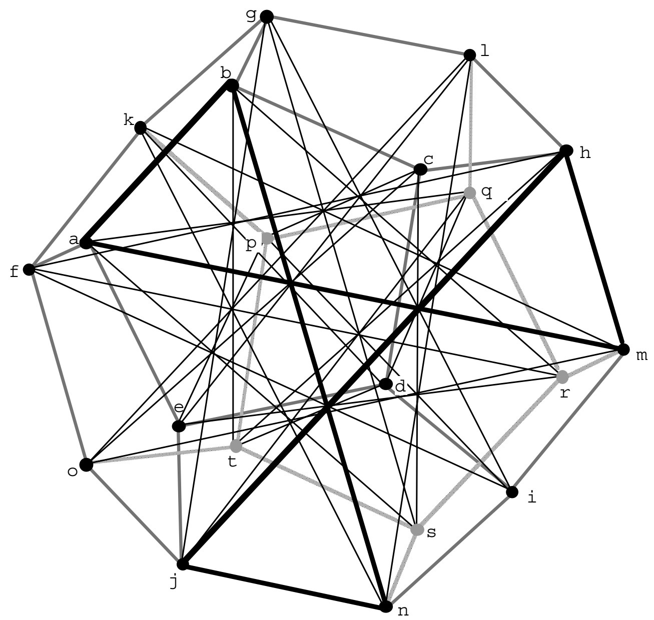

Fig. 8. An orientable selfpolar polyhedron recognized as noble by Brückner [2]. It has 20 vertices and 20 faces, one of which is emphasized; its map is not only noble, but regular, of genus 9, as observed by Prof. J. Wills. The dodecahedron serves only to guide the construction and recognition of the faces of this polyhedron.

Another remarkable invention of Brückner [3] is shown in Figure 10, which is meant to illustrate the construction of two noble polyhedra. The idea is to start with a uniform rhombicuboctahedron (shown in gray lines); the five points a, e, u, w, h are coplanar, and according to Brückner they determine two distinct polygons: [a e w a u h a] and [a e w a h u a]. The other faces are obtained by applying to each of these two the symmetries of the rhombicuboctahedron. However, each of these polygons revisits a vertex (as was allowed in [23] but not here); therefore these objects are not polyhedra in the present sense, or in the sense generally accepted by Brückner. On the other hand, as described in [23], vertex-doubling produces in each of the two cases an acceptable noble polyhedron with 48 vertices and 48 faces.

In Figure 11 are shown two additional noble polyhedra, which seem to have escaped Brückner's attention. They are generated in the same way from the uniform quasirhombi-cuboctahedron ($-3.4.4.4$) (see [W, p. 132]) as the ones described above from the rhombicuboctahedron. Although the noble polyhedra arising from Figure 11 are combinatorially equivalent to the ones in Figure 10, their metric difference can most simply be established by observing that the ratio of lengths of the longest diagonal of each face to the shortest is $2 + \sqrt{2} = 3.14142\ldots$ in Figure 11, but $1 + 1/\sqrt{2} = 1.7071\ldots$ in Figure 10.

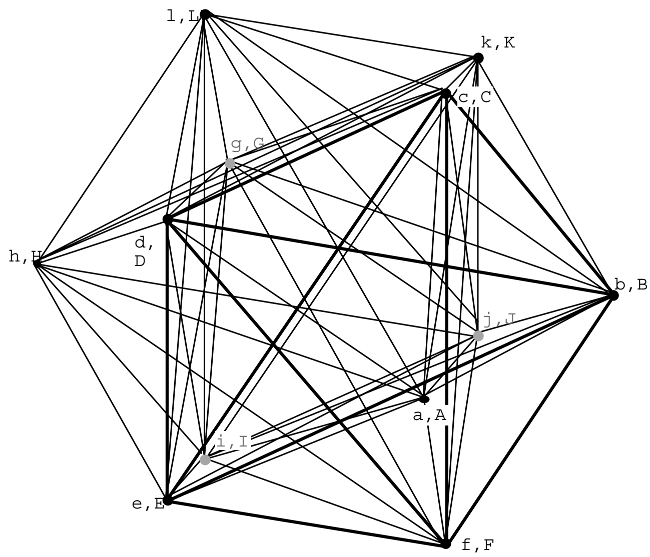

Fig. 9. The construction of noble polyhedra isomeghethic with the regular polyhedron $\{5, 5/2\}$. The emphasized decagon can be interpreted in two isogonal ways – either as [d F b D e B c E f C d] or as [b C f B e F d E c D b]; the other 23 faces result by the application of symmetries of the icosahedron, and the interchange of lower and upper case vertices. Two distinct noble polyhedra are obtained. However, if the pairs of coinciding vertices are simply identified (as done by Hess [34] and Brückner [2]), the resulting object is not a polyhedron in our sense, or in the sense ostensibly accepted by Hess and Brückner.

In view of the ingenuity with which Hess and Brückner pursued noble polyhedra, and their willingness to stretch their own rules in order to admit the ones they found, it is strange that they never mention two rather simple polyhedra (and their polars), which are shown and explained in Figure 12. The probable reason for this omission is, again, the shying away from coplanar faces or coinciding vertices.

Another way of constructing certain noble polyhedra will be mentioned in the next section.

6 Uniform Polyhedra

Uniform polyhedra are defined as isogonal polyhedra with all faces regular. They are closely related to the *Archimedean* polyhedra, studied since antiquity; the older concept requires congruence of the vertex stars instead of the more restrictive isogonality. The convex uniform polyhedra are all well known since the work of Kepler [38], but the determination of nonconvex ones was done piecemeal by many people, over close to a century, – and that only in the traditional understanding of what is a polyhedron. An illustrated list of

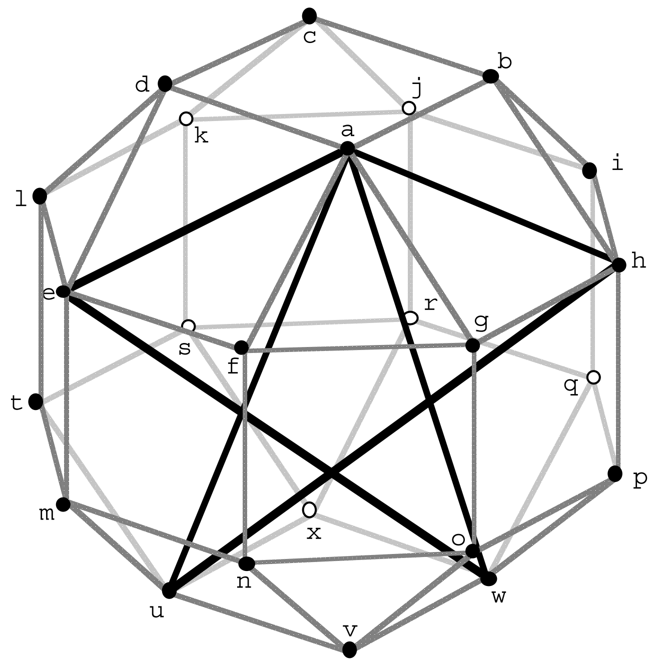

Fig. 10. The construction of two noble polyhedra, according to Brückner [2]: one of these has the face [a e w a u h a] and the other faces obtaineble by symmetries, while the other has the face [a e w a h u a] and those in its orbit. Since this involves revisiting a vertex, these are not polyhedra in our sense. However, by doubling-up vertices each leads to a noble polyhedron. The rhombicuboctahedron (3.4.4.4) serves only to guide the construction and recognition of the faces of this polyhedron.

such uniform polyhedra appears in Coxeter *et al.* [12], and the fact that this list is complete was established by Sopov [51] and Skilling [50]. Additional illustrations of all these polyhedra can be found in Wenninger [53] and Har'El [31].

It should not come as a surprise that with our definition of polyhedra there are many new possibilities for the formation of uniform ones. As with "new" regular polyhedra, the visual appearance of many "new" uniform polyhedra is somewhat disappointing – they look exactly like appropriate "old" uniform polyhedra since they are isomeghethic with them. However, their inner structure (the underlying abstract polyhedron) is different. In many cases, the abstract polyhedron admits a continuum of non-uniform isogonal realizations which do seem interesting, and in the limit become uniform; in some instances, this approach to visualization of the structure of polyhedra works for regular ones as well. Examples of both possibilities appear in [27]; one of the uniform ones is (3 . 6/2 . 3 . 6/2), another is (3 . 6/2 . 6/2), and a

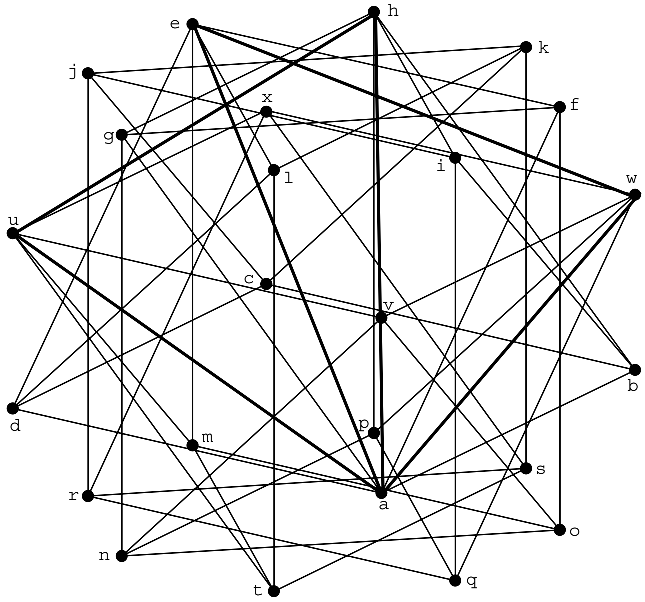

Fig. 11. Another pair of noble polyhedra, combinatorially equivalent to the ones in Figure 10, can be obtained using the vertices of the uniform quasirhombicuboctahedron (−3.4.4.4). This uniform polyhedron is combinatorially equivalent to the rhombicubocta-hedron, under the correspondence between their vertices indicated by the labels. Although the resulting noble polyhedra obtainable by doubling-up vertices are combinatorially equivalent to the the ones described in the caption of Figure 10, they are metrically distinct from them.

regular one is a 24-faced $\{4, 6/2\}$. More interesting are cases in which something genuinely new occurs. One example is (3 . 6 . 4 . 6/2 . 4 . 6), which is presented in figure 13 and its caption. Two other uniform polyhedra, with symbol (8 . 8 . 8 . 8), are shown in Figure 14. They are representative of several others that can be obtained analogously from uniform polyhedra by deleting one transitivity class of faces, and doubling-up the remaining faces. Some – though not all – of such polyhedra are noble.

Another example concerns two polyhedra the existence of which under the traditional concept of polyhedron was rejected in [12]. Discussing the possibility of existence of polyhedra with symbols t$\{5/2, 5\}$ and t$\{5/2, 3\}$ in the notation of [12], the authors say (on page 411) that "... t$\{5/2, 5\}$ consists of three coincident dodecahedra, while t$\{5/2, 3\}$ consists of two coincident great dodecahedra along with the icosahedron that has the same vertices and edges ...". (The construction in question consists of truncating the regular polyhedra to the extent of completely cutting off their "points".) While the non-acceptance of the resulting object among the uniform polyhedra in the traditional meaning is fully justified (even if not for the reason stated),

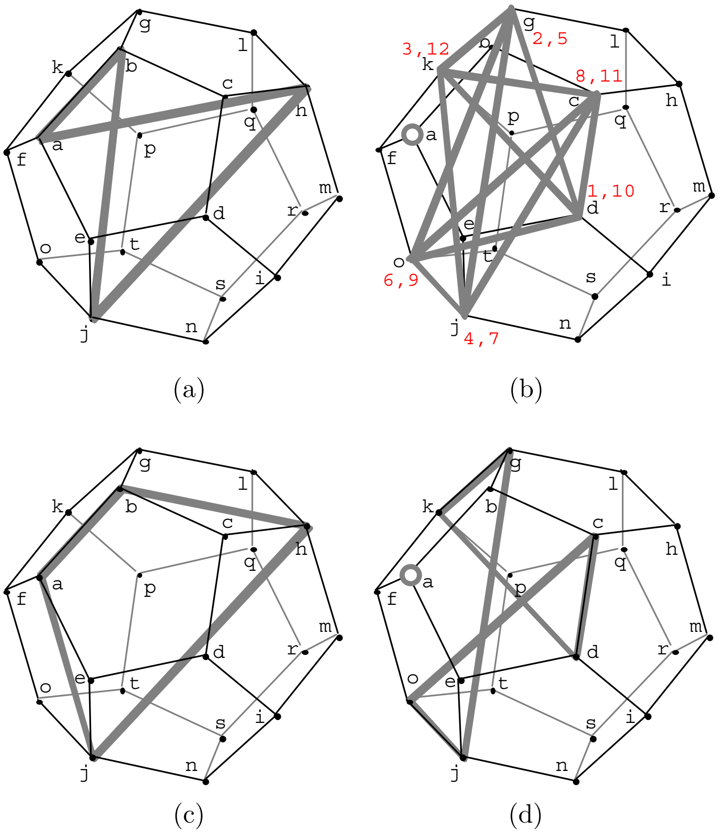

Fig. 12. Four noble polyhedra. The hexecontahedron in (a) consists of the sixty quadrangles congruent to the one emphasized, that can be inscribed in the regular dodecahedron. Its polar is an icosahedron, with twenty 12-gonal faces. The diagram in (b) shows the face which is the polar of the vertex a. Each of the 20 vertices of the icosahedron in (b) represents three coinciding vertices, while each face meets six pairs of coinciding vertices. Clearly, the coincidences here are no more against the traditional grain than the ones in the polyhedron in Figure 9. The other hexecontahedron is obtained similarly from the quadrangle in (c), while (d) shows the face of a noble icosahedron polar to the polyhedron in (c). The diagram in (d) shows the face which is the polar of the vertex a. Each of the 20 vertices of the icosahedron in (d) represents three coinciding vertices, while each face meets six pairs of coinciding vertices, each pair determining an edge of zero length. In all diagrams the dodecahedra serve only to guide the construction and recognition of the faces of the polyhedra described.

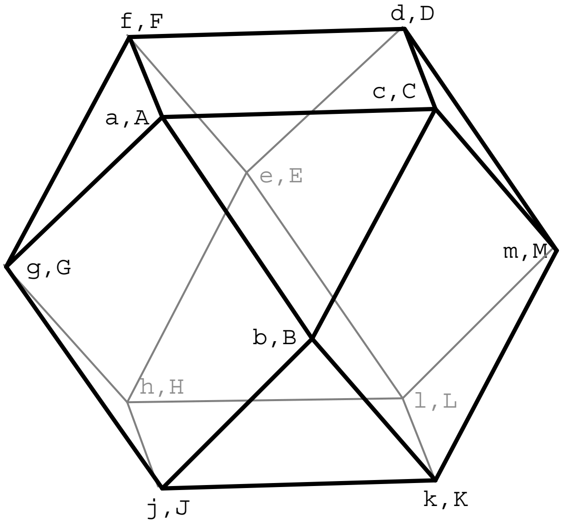

Fig. 13. A nontrivial uniform polyhedron (3 . 6 . 4 . 6/2 . 4 . 6) with 24 vertices, coinciding in pairs with the vertices of a cuboctahedron. Its faces are eight triangles: (a b c a), (A B C A), (d e f d), (D E F D), (g h j g), (G H J G), (k l m k), (K L M K); twelve squares: (a g J B a), (A G j b A), (b k M C b), (B K m c B), (c d F A c), (C D f a C), (D M l e D), (d m L E d), (E H g f E), (e h G F e), (H L k j H), (h l K J h); eight hexagons {6}: (a f e l k b a), (A F E L K B A), (b j h e d c b), (B J H E D C B), (a c m l h g a), (A C M L H G A), (d f g j k m d), (D F G J K M D); and four hexagons {6/2}: (a B c A b C a), (d E f D e F d), (g H j G h J g), (k L m K l M k). The polyhedron is orientable and of genus 9; since some of the faces pass through the center, no density at the center can be defined, and there is no polar polyhedron with the same degree of symmetry.

the uniform polyhedra t{5/2, 5} and t{5/2, 3} exist in our interpretation of "polyhedron". Indeed, as is best seen from the illustration in Figure 15, the truncation of t{5/2, 5} leads to a uniform polyhedron (5 . 10/2 . 10/2) with sixty vertices. In a similar way, the truncation of t{5/2, 3} yields a uniform polyhedron (3 . 10/2 . 10/2) with sixty vertices.

As a final example in this section, we recall that in the process of verification of the completeness of the enumeration of the uniform polyhedra in [12], Skilling [50] found one extraordinary object, the *great disnub dirhombidodecahedron*, which would have qualified as a uniform polyhedron in every respect except that it has four faces incident with some edges. However, as Skilling points out on p. 123, this object *is* a polyhedron if the exceptional edges are interpreted as two distinct edges which happen to be represented by the same segment although they are determined by different pairs of faces; in other words, it is a polyhedron in the sense adopted here. This and other "new" uniform polyhedra are discussed in greater detail in [29].

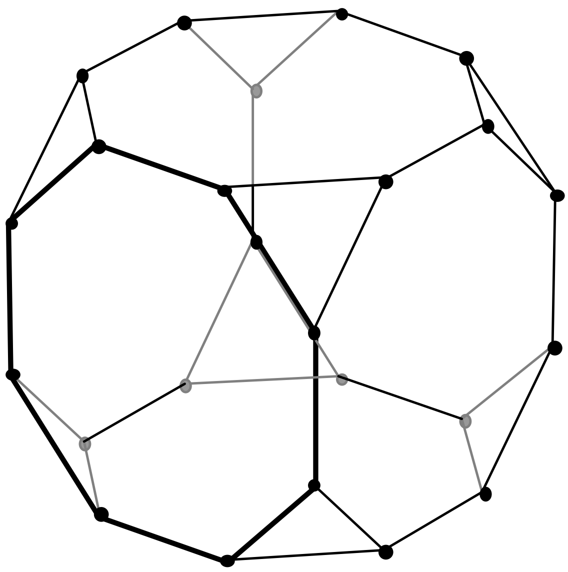

Fig. 14. Deleting all triangles from the uniform truncated cube (3 . 8 . 8), and then face-doubling the octagons, leads to two "new" polyhedra (8 . 8 . 8 . 8), which are not only uniform, but noble; moreover, they are isomeghetic. In both, each octagon has been replaced by one "red" and one "green" octagon. The two are adjacent along the four edges previously adjacent to triangles. In the first polyhedron the remaining edges are adjacent to octagons of the same color, in the second to differently colored ones. The difference between the two is that in the first, each triangular hole is surrounded by two circuits of three octagons each, while in the second one it is surrounded by one circuit of six octagons.

7 Isohedral Polyhedra with Regular Vertices

The polars of uniform polyhedra (with respect to a sphere whose center coincides with the centroid of the polyhedron) are isohedral polyhedra with regular vertex-stars. In the case of convex uniform polyhedra these isohedral polyhedra are often called *Catalan polyhedra*, although the historical justification for this seems to be ambiguous. There has been no name proposed for the general case, and, in fact, there appears to have been avoidance in considering such polyhedra. There are several reasons for this situation.

To begin with, in a number of uniform polyhedra some of the faces pass through the centroid of the polyhedron; therefore there is no polar polyhedron in either the traditional sense or in the meaning of "polyhedron" accepted here. Brückner [3, p. 191] ignores the question of polars of such polyhedra, although he claims to be systematically discussing the isogonal polyhedra *and* their polar isohedral ones. Wenninger [54] and Har'El [31] solve the problem of polars of some of the uniform polyhedra by admitting unbounded faces.

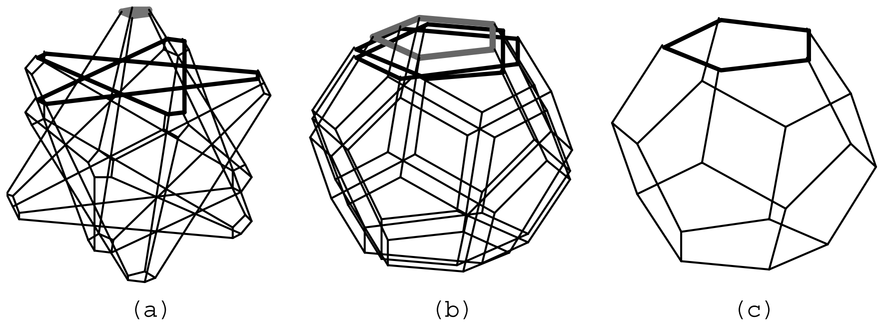

Fig. 15. The truncation of the regular polyhedron $\{5/2, 5\}$. (a) shows an early stage of the truncation; one of the pentagonal faces, and one of the decagonal faces are emphasized. (b) shows an almost complete truncation, illustrating the proximity of the emphasized pentagon and decagon. (c) is the complete truncation, in which each face of the "dodecahedron" represents one pentagon $\{5\}$ and one decagon $\{10/2\}$. Each dodecahedral vertex represents three vertices of the uniform polyhedron $(5\ .\ 10/2\ .\ 10/2)$, the truncation of $\{5/2, 5\}$. Continuation of this sequence leads to several interesting polyhedra; they will be described in detail elsewhere.

While such an approach is interesting, it certainly does not fall within the usual scope of the meaning of "isohedral polyhedron".

Another difficulty for the traditional approach is that some of the uniform polyhedra have pairs of coplanar faces; hence the polar polyhedra must have pairs of coinciding vertices – which would make them unacceptable under the traditional definition of polyhedra. However, neither in [54] nor in [31] is any mention made of this fact. The vertices which are incident with two cycles of faces are neither noticed nor explained, nor is any mention made of the fact that, for example, the uniform polyhedron (3.3.3.3.3.5/2) has 112 faces, but the purported polar shown in [54] and [31] has only 92 vertices. On the other hand, in our interpretations of polyhedra there is no problem in such cases: the two vertices of each pair are distinct, and only in the realization they happen to be represented by a single point.

8 Other Polyhedra

There are several other classes of polyhedra for which the definition of polyhedra as presented here is useful – either in clarifying and eliminating what seemed to be unexpected exceptional cases, or in enabling a complete and unambiguous determination of all members of the class.

One example of the former kind concerns the recent study by Shephard [49] of isohedral *deltahedra* (polyhedra all faces of which are equilateral triangles). After explaining one of the constructions of such polyhedra – the replacement of each face of a regular polyhedron by the mantle of a pyramid (with equilateral triangles) erected over the face as basis – the claim is made that this construction works on eight of the regular polyhedra but not on $\{5, 5/2\}$. In fact, the construction works in this case as well, and results in an isohedral hexecontahedron of type [5 . 10/2 . 10/2] that looks like the regular icosahedron to which it is isomeghethic, but has three (combinatorially distinguishable) faces over each icosahedral face. A similar construction consisting of "excavating" the pyramids is said in [49] to fail when applied to the tetrahedron "... since the construction leads to a set of twelve equilateral triangles which coincide in four sets of three." In our interpretation, the resulting polyhedron is combinatorially equivalent to the one obtained by erecting the pyramids, except that in this realization each triplet of (distinguishable) faces is represented by one triangle.

Settling on a particular definition of polyhedra makes possible the completion of enumeration of several classes of polyhedra. The determination of all face-transitive polyhedra with rectangular faces, started in [10] and [11] is being carried out in work in preparation. Also in preparation are enumerations of rhombic or parallelogram-faced isohedra (extending work in [26] and [28]) and on simplicial isohedra.

While definitions of the polyhedral concept different from the one adopted here are certainly possible, and possibly useful, at the moment there seems to be no better alternative available that is both general and internally consistent, and also satisfies the criteria set out in Section 1.

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