Extending simple monotone drawings*

Jan Kynčl[1] and Jan Soukup[1]

1 **Department of Applied Mathematics, Charles University, Faculty of Mathematics and Physics, Malostranské nám. 25, 118 00 Praha 1, Czech Republic**
`{kyncl,soukup}@kam.mff.cuni.cz`

Abstract

We prove the following variant of Levi's Enlargement Lemma: for an arbitrary arrangement $\mathcal{A}$ of x-monotone pseudosegments in the plane and a pair of points a, b with distinct x-coordinates and not on the same pseudosegment, there exists a simple x-monotone curve with endpoints a, b that intersects every curve of $\mathcal{A}$ at most once. As a consequence, every simple monotone drawing of a graph can be extended to a simple monotone drawing of a complete graph.

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1 Introduction

Given $k \geq 1$, a finite set $\mathcal{A}$ of simple curves in the plane is called an *arrangement of k-strings* if every pair of the curves of $\mathcal{A}$ intersects at most k times, and every intersection point is a proper crossing or a common endpoint. An arrangement of 1-strings is also called an *arrangement of pseudosegments*, and each curve in the arrangement is called a *pseudosegment*. In this paper, we represent simple curves as subsets of the plane that are homeomorphic images of a closed interval.

A simple curve γ in the plane is *x-monotone*, shortly *monotone*, if γ intersects every line parallel to the y-axis at most once.

Given an arrangement $\mathcal{A}$ of monotone pseudosegments in the plane and a pair of points a, b with distinct x-coordinates and not on the same pseudosegment, we say that $\mathcal{A}$ is (a, b)-*extendable* if there exists a monotone curve with endpoints a, b that intersects every curve of $\mathcal{A}$ at most once. We say that $\mathcal{A}$ is *extendable* if it is (a, b)-extendable for all possible choices of a and b.

Our main result is the following.

▶ **Theorem 1.1.** *Every arrangement of monotone pseudosegments in the plane is extendable.*

The proof of Theorem 1.1 can be turned into an algorithm: the new pseudosegment extending an arrangement $\mathcal{A}$ and joining two given points a and b is constructed in at most $|\mathcal{A}|$ steps. Starting with an initial curve from a to b, in each step the curve is locally rerouted along one pseudosegment of $\mathcal{A}$.

A drawing of a graph in the plane is *simple* if every pair of edges has at most one common point, either a common endpoint or a proper crossing. A drawing of a graph is *monotone* if every edge is drawn as a monotone curve and no two vertices share the same x-coordinate. We have the following direct consequence of Theorem 1.1, illustrated in Figure 1.

▶ Corollary 1.2. Every simple monotone drawing of a graph in the plane can be extended to a simple monotone drawing of the complete graph with the same set of vertices.

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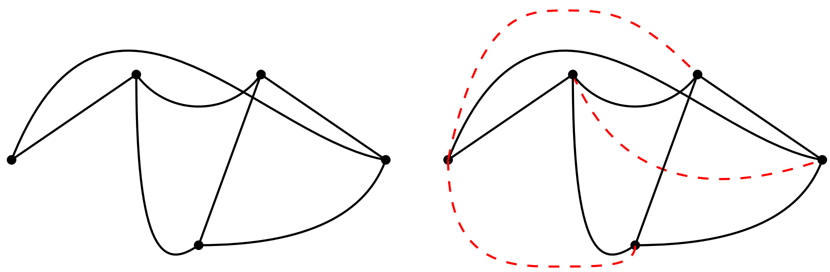

Figure 1 Left: a simple monotone drawing of a graph. Right: an extension of the drawing on the left to a simple monotone drawing of a complete graph. The added edges are dashed.

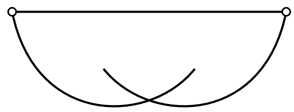

Figure 2 An example of an arrangement of three pseudosegments that cannot be extended to pseudolines forming a pseudoline arrangement.

In the full version of this article we also study the extendability problem for cylindrically monotone arrangements. We show that extending an arrangement of cylindrically monotone pseudosegments is not always possible; in fact, the corresponding decision problem is NP-hard.

We prove Theorem 1.1 in Section 2.

1.1 Related results

A *pseudoline* in the plane is an image of a Euclidean line under a homeomorphism of the plane; in other words, a pseudoline is a homeomorphic image of the set $\mathbb{R}$, unbounded in both directions. An *arrangement of pseudolines* is a finite set of pseudolines such that every pair of them has exactly one crossing, and no other common intersection point. Pseudolines are also often defined in the projective plane, as nonseparating simple closed curves.

Levi's Enlargement Lemma [9] states that for every arrangement of pseudolines and every pair of points a, b not on the same pseudoline, one can draw a new pseudoline through a and b, crossing every curve from the given arrangement exactly once. The lemma has several alternative proofs in the literature [3, 10].

By a classical result of Goodman [6], [5, Theorem 5.1.4], every arrangement of pseudolines can be transformed by a homeomorphism of the plane into an arrangement of monotone pseudolines, or a so-called *wiring diagram*. Therefore, monotone arrangements of pseudosegments can be considered as a generalization of pseudoline arrangements. On the other hand, Figure 2 shows an example that not every monotone arrangement of pseudosegments can be seen as a "restriction" of a pseudoline arrangement, and so Theorem 1.1 does not easily follow from Levi's Lemma. See Arroyo, Bensmail and Richter [1, Figure 2] for more examples. Since a pseudoline (in the projective plane) can be considered as a union of two pseudosegments, Theorem 1.1 can also be considered as a generalization of "a half" of Levi's Lemma.

A simple drawing of the disjoint union of two 2-paths that cannot be extended to a simple drawing of K_6 was constructed by Eggelton [4, Diagram 15(ii)] and later rediscovered by the first author [8, Figure 9]. Later a few more examples of non-extendable simple drawings were constructed [7, Figures 1, 10]. None of these drawings are homeomorphic to monotone drawings, which follows, for example, from Theorem 1.1.

Arroyo et al. [2] showed that it is NP-hard to decide, given an arrangement $\mathcal{A}$ of pseudosegments and a pair of points a, b, whether a and b can be joined by a simple curve crossing each pseudosegment of $\mathcal{A}$ at most once. Our NP-hardness proof in the full version is a simple adaptation of this result to cylindrically monotone arrangements.

2 Monotone arrangements in the plane

We start with a few definitions and tools for analyzing x-monotone arrangements. Given a pair of points a, b in the plane, we write $a \prec b$ if a has a smaller x-coordinate than b. Clearly, $\prec$ is a strict linear order on the points of any monotone curve.

We can naturally talk about objects lying "below" and "above" monotone curves. Let a, b be points such that $a \prec b$. For any monotone curve γ we denote by $\gamma[a, b]$ and $\gamma(a, b)$ the subset of γ formed by the points x of γ satisfying $a \preceq x \preceq b$ and $a \prec x \prec b$, respectively. Similarly, for an arrangement $\mathcal{B}$ of monotone pseudosegments we denote by $\mathcal{B}[a, b]$ the arrangement of pseudosegments where we replace each $\gamma \in \mathcal{B}$ by $\gamma[a, b]$.

By *consecutive intersections* of two monotone curves with finitely many intersections we mean consecutive intersections with respect to their x-coordinates. Let α, β be two monotone curves with finitely many intersections. Let a, b be two consecutive intersections of α, β such that $a \prec b$. Then the only intersections of $\alpha[a, b]$ with $\beta[a, b]$ are the points a and b. In this case we say that the curves α and β form a *bigon*. Furthermore, if $\alpha(a, b)$ lies above $\beta(a, b)$ we say that α and β form an *α-top*, or equivalently, a *β-bottom* bigon.

The *lower envelope* $\mathrm{low}(\mathcal{U})$ of a set $\mathcal{U}$ of curves is the set of all points p of these curves such that no other point of any curve of $\mathcal{U}$ with the same x-coordinate as p is below p. Note that if $\mathcal{U}$ is an arrangement of monotone pseudosegments, then $\mathrm{low}(\mathcal{U})$ is a finite union of connected parts of pseudosegments.

2.1 Proof of Theorem 1.1

Let $\mathcal{A}$ be an arrangement of monotone pseudosegments. Let a, b, with $a \prec b$, be points that are not on the same pseudosegment of $\mathcal{A}$. We need to find a monotone curve from a to b that intersects every curve of $\mathcal{A}$ at most once. Since every curve of $\mathcal{A}$ is monotone, we can without loss of generality assume that $\mathcal{A} = \mathcal{A}[a, b]$.

Let $\mathcal{A}'$ be an arrangement of monotone pseudosegments formed by all pseudosegments of $\mathcal{A}$ together with three new segments τ_1, τ_2, τ_3, defined as follows. The segment τ_1 is an almost vertical segment starting in a and ending in some new point to the right of a and above all pseudosegments of $\mathcal{A}$. Similarly, τ_3 is an almost vertical segment ending in b and starting in some new point to the left of b and above all pseudosegments of $\mathcal{A}$. Finally, τ_2 is a horizontal segment crossing τ_1 and τ_3, and lying entirely above all pseudosegments of $\mathcal{A}$; see Figure 3. In this way, $\mathrm{low}(\{\tau_1, \tau_2, \tau_3\})$ is a monotone curve connecting a and b "from above", so that every pseudosegment $\gamma \in \mathcal{A}$ intersects it at most twice. Furthermore no $\gamma \in \mathcal{A}$ forms a γ-top bigon with $\mathrm{low}(\{\tau_1, \tau_2, \tau_3\})$ (it can only form a γ-bottom bigon).

In order to find an extending curve we do the following. We find a nonempty subset $\mathcal{U} \subseteq \mathcal{A}'$ of pseudosegments such that the lower envelope of $\mathcal{U}$ is a monotone curve connecting a to b, intersecting every pseudosegment of $\mathcal{A}' \setminus \mathcal{U}$ at most once. Furthermore, we find $\mathcal{U}$ so that no pseudosegment α touches $\mathrm{low}(\mathcal{U})$ from below in an inner point of α. After finding such $\mathcal{U}$, a new pseudosegment connecting a and b can clearly be drawn slightly below the

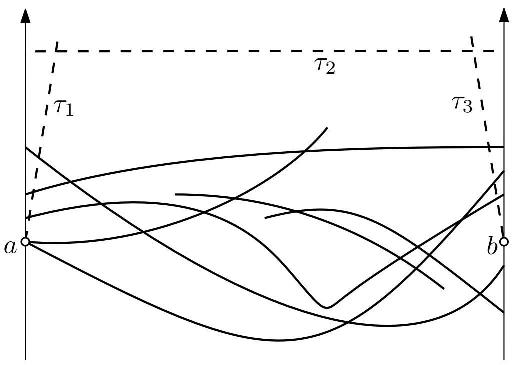

Figure 3 An arrangement of monotone pseudosegments with three added segments τ_1, τ_2, τ_3 connecting points a, b "from above".

lower envelope of $\mathcal{U}$ and will indeed intersect every pseudosegment of $\mathcal{A}'$ at most once. Thus, if such $\mathcal{U}$ exists, $\mathcal{A}'$, and consequently $\mathcal{A}$, is (a, b)-extendable.

We find $\mathcal{U}$ inductively. We start with $\mathcal{U}_0 = \{\tau_1, \tau_2, \tau_3\}$ and always look at the lower envelope of $\mathcal{U}_i$. In the ith step we select an arbitrary pseudosegment γ_i of $\mathcal{A}' \setminus \mathcal{U}_{i-1}$ intersecting $\mathrm{low}(\mathcal{U}_{i-1})$ at least twice. If there is no such γ_i then $\mathcal{U} = \mathcal{U}_{i-1}$ and we are done. Otherwise, we set $\mathcal{U}_i = \mathcal{U}_{i-1} \cup \{\gamma_i\}$. The number of pseudosegments is finite, so this process finishes with a set $\mathcal{U}$ such that the lower envelope of $\mathcal{U}$ intersects every pseudosegment of $\mathcal{A}' \setminus \mathcal{U}$ at most once.

Additionally, we prove that the induction preserves the following invariants for every $\mathcal{U}_i$.

(I1) No pseudosegment α of $\mathcal{A}' \setminus \mathcal{U}_i$ forms an α-top bigon with $\mathrm{low}(\mathcal{U}_i)$.

(I2) No pseudosegment α of $\mathcal{A}' \setminus \mathcal{U}_i$ touches $\mathrm{low}(\mathcal{U}_i)$ from below in an inner point of α.

(I3) The lower envelope of $\mathcal{U}_i$ contains no endpoints of any pseudosegment of $\mathcal{A}'$ except for the points a and b.

(I4) The lower envelope of $\mathcal{U}_i$ is connected and contains a and b. Hence, it is a monotone curve connecting a to b.

In particular, by (I4), the lower envelope of $\mathcal{U}$ is a monotone curve connecting a to b and, by (I2), no pseudosegment α of $\mathcal{A}' \setminus \mathcal{U}$ touches $\mathrm{low}(\mathcal{U})$ from below in an inner point of α. Since $\mathrm{low}(\mathcal{U})$ intersects every pseudosegment of $\mathcal{A}' \setminus \mathcal{U}$ at most once by its construction, $\mathcal{A}$ is (a, b)-extendable by the previous discussion. Thus, it suffices to prove the correctness of these invariants to finish the proof.

The invariants hold for $\mathcal{U}_0$ by the construction of τ_1, τ_2 and τ_3. Suppose all invariants hold for $\mathcal{U}_{i-1}$. In particular, $\mathrm{low}(\mathcal{U}_{i-1})$ is a monotone curve connecting a to b by invariant (I4). We show that all invariants also hold for $\mathcal{U}_i$.

The pseudosegment γ_i intersects $\mathrm{low}(\mathcal{U}_{i-1})$ at least twice. We show that γ_i intersects $\mathrm{low}(\mathcal{U}_{i-1})$ exactly twice. Suppose, for contradiction, that there are three consecutive intersections c, d and e of γ_i with $\mathrm{low}(\mathcal{U}_{i-1})$ such that $c \prec d \prec e$. Then $\gamma_i[c, d]$ with $\mathrm{low}(\mathcal{U}_{i-1})[c, d]$ forms a bigon and so does $\gamma_i[d, e]$ with $\mathrm{low}(\mathcal{U}_{i-1})[d, e]$. By invariant (I1) both of these bigons must be $\mathrm{low}(\mathcal{U}_{i-1})$-top bigons. However, in this case γ_i touches $\mathrm{low}(\mathcal{U}_{i-1})$ from below in the point d. That is not possible by invariant (I2). Thus, γ_i intersects $\mathrm{low}(\mathcal{U}_{i-1})$ exactly twice. Furthermore, by invariant (I1), γ_i and $\mathrm{low}(\mathcal{U}_{i-1})$ form a γ_i-bottom bigon.

Let x and y be the two intersection points of γ_i and $\mathrm{low}(\mathcal{U}_{i-1})$. Refer to Figure 4. Since γ_i and $\mathrm{low}(\mathcal{U}_{i-1})$ form a γ_i-bottom bigon, the only part of the curve γ_i that lies below $\mathrm{low}(\mathcal{U}_{i-1})$ is exactly $\gamma_i(x, y)$. Thus, the lower envelope of $\mathcal{U}_{i-1} \cup \{\gamma_i\}$ is a monotone curve

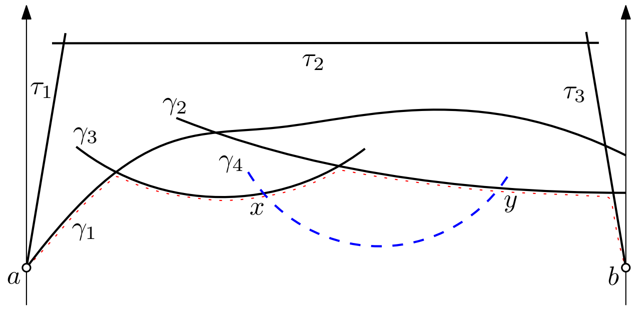

Figure 4 Induction step in the proof of Theorem 1.1. In the ith step (fourth step in the figure) we add pseudosegment γ_i (dashed) intersecting the lower envelope (dotted) of the previous segments twice. The lower envelope remains a connected curve connecting a with b and not containing any other endpoints of pseudosegments even after this addition.

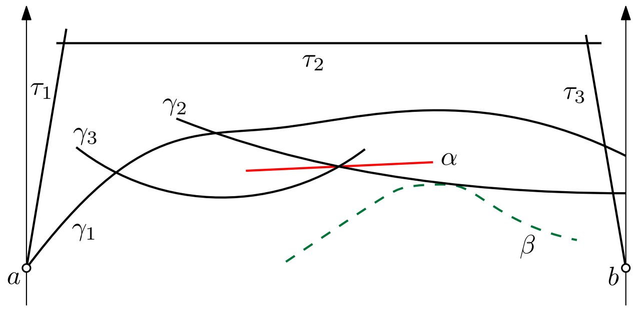

Figure 5 Induction step in the proof of Theorem 1.1. During the selection of $\mathcal{U}$ some pseudosegments may touch $\mathrm{low}(\mathcal{U})$ from above but never from below. Pseudosegment α touches $\mathrm{low}(\{\tau_1, \tau_2, \tau_3, \gamma_1, \gamma_2, \gamma_3\})$ from above. On the other hand, β cannot be in the same arrangement of pseudosegments since it touches γ_2.

connecting a and b. Therefore, invariant (I4) holds also for $\mathcal{U}_i$.

Since γ_i is a pseudosegment, its subset $\gamma_i(x, y)$ contains no endpoint of any pseudosegment of $\mathcal{A}'$. Since $\mathrm{low}(\mathcal{U}_i) \setminus \gamma_i(x, y) \subseteq \mathrm{low}(\mathcal{U}_{i-1})$ and invariant (I3) holds for $\mathcal{U}_{i-1}$, invariant (I3) also holds for $\mathcal{U}_i$.

Now, suppose that invariant (I2) does not hold, that is, there exists some pseudosegment β of $\mathcal{A}' \setminus \mathcal{U}_i$ that touches $\mathrm{low}(\mathcal{U}_i)$ from below in an inner point of β. Refer to Figure 5. By the definition of an arrangement of pseudosegments, the touching point is not an endpoint of any pseudosegment of $\mathcal{A}'$. Thus, β has to touch γ_i or $\mathrm{low}(\mathcal{U}_{i-1})$ in an inner point of β, a contradiction. Hence, invariant (I2) also holds for $\mathcal{U}_i$. Note that the analogous statement for touchings from above does not hold, that is, there may exist some pseudosegment α of $\mathcal{A}' \setminus \mathcal{U}_i$ that both touches $\mathrm{low}(\mathcal{U}_i)$ from above in an inner point of α and touches none of γ_i or $\mathrm{low}(\mathcal{U}_i)$ in an inner point of α.

Finally, suppose that invariant (I1) does not hold, that is, there exists some pseudosegment ρ of $\mathcal{A}' \setminus \mathcal{U}_i$ that together with $\mathrm{low}(\mathcal{U}_i)$ forms a ρ-top bigon. Call s and t the vertices of this bigon and assume $s \prec t$. See Figure 6.

If s and t both lie on $\gamma_i[x, y]$, then ρ and γ_i intersect twice, a contradiction. Otherwise s or t does not lie on $\gamma_i[x, y]$. Without loss of generality assume that t does not lie on $\gamma_i[x, y]$ and $y \prec t$. Then s either lies on $\mathrm{low}(\mathcal{U}_{i-1})$ or below it. In both cases $\rho[s, t]$ intersects

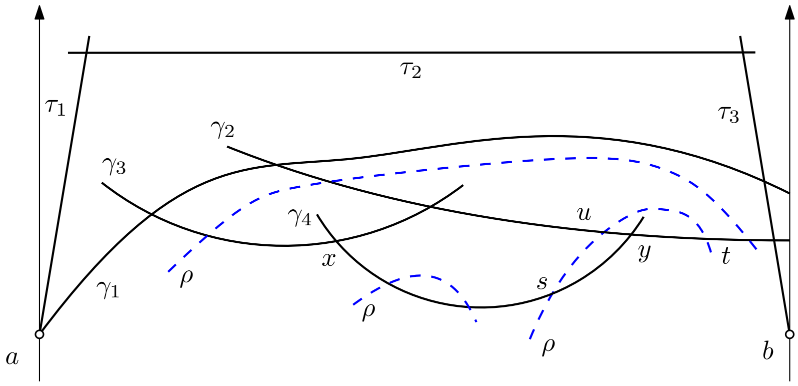

Figure 6 Induction step in the proof of Theorem 1.1. If there was some pseudosegment ρ that together with the lower envelope (low($\{\tau_1, \tau_2, \tau_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$) in the picture) formed a ρ-top bigon, it would either form a ρ-top bigon with the previous lower envelope or intersect twice the segment that was added as the last. In the picture, there are three such possible ρ's.

low($\mathcal{U}_{i-1}$) in some point other than t since $\rho[s,t]$ together with low($\mathcal{U}_i$) forms a ρ-top bigon. Denote the rightmost intersection of $\rho[s,t]$ and low($\mathcal{U}_{i-1}$) other than t by u. Then $\rho(u,t)$ lies above low($\mathcal{U}_{i-1}$) and so $\rho[u,t]$ together with low($\mathcal{U}_i$) forms a ρ-top bigon, a contradiction with invariant (I1) for $\mathcal{U}_{i-1}$. This concludes the proof of Theorem 1.1.

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