Minimal and Subminimal Logic of Negation

MSc Thesis *(Afstudeerscriptie)*

written by

Almudena Colacito
(born November 29th, 1992 in Fabriano, Italy)

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Members of the Thesis Committee:
Dr. Floris Roelofsen (Chair)
Dr. Benno van den Berg
Dr. Marta Bílková
Drs. Julia Ilin
Prof. Dr. Dick de Jongh
Prof. Dr. Yde Venema

Institute for Logic, Language and Computation

Abstract

Starting from the original formulation of minimal propositional logic proposed by Johansson, this thesis aims to investigate some of its relevant subsystems. The main focus is on negation, defined as a primitive unary operator in the language. Each of the subsystems considered is defined by means of some 'axioms of negation': different axioms enrich the negation operator with different properties. The basic logic is the one in which the negation operator has no properties at all, except the property of being functional. A Kripke semantics is developed for these subsystems, and the clause for negation is completely determined by a function between upward closed sets. Soundness and completeness with respect to this semantics are proved, both for Hilbert-style proof systems and for defined sequent calculus systems. The latter are cut-free complete proof systems and are used to prove some standard results for the logics considered (e.g., disjunction property, Craig's interpolation theorem). An algebraic semantics for the considered systems is presented, starting from the notion of Heyting algebras without a bottom element. An algebraic completeness result is proved. By defining a notion of descriptive frame and developing a duality theory, the algebraic completeness result is transferred into a frame-based completeness result which has a more generalized form than the one with respect to Kripke semantics.

Contents

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Overview

This thesis is concerned with subminimal logics, with particular focus on the negation operator. With subminimal logics we want to denote subsystems of minimal propositional logic obtained by weakening the negation operator. We give here a quick overview of the thesis, presenting its structure as well as its contents.

Chapter 1. The first chapter introduces our conceptual starting point: minimal propositional logic. After an introduction of the syntax, and a brief presentation of an associated Kripke semantics, the relation between the two versions of minimal logic is made formal. In particular, we give a proof of the fact that the two considered definitions of minimal logic are indeed equivalent. Later, the main axioms of negation are introduced, with some motivations. Their formal relation with the minimal system is studied. The chapter is concluded with some historical notes concerning minimal logic, as well as the study of minimal logic with focus on the negation operator.

Chapter 2. We start introducing the core of the thesis. We present a Kripke semantics for each of the three main subminimal systems, and we introduce some of the relevant notions (p-morphism, generated subframe, disjoint union). Completeness proofs are carried out via canonical models. The analysis goes here 'bottom-up': we start from the basic system of an arbitrary unary negation operator with no special properties, and we go on by adding axioms (and hence, properties) for the negation operator. In defining a Kripke semantics for those systems, the semantics of negation is given by a function on upward closed subsets of a partially ordered set. This reflects the fact that the negation is basically seen as a unary functional operator. The reader may notice that the axioms of negation happen to be equivalent to properties of the considered function.

Chapter 3. The third chapter deals with finite models and the disjunction property. In particular, we give two different proofs of the finite model property for the considered logics. At the beginning of the chapter we prove such a property by means of adequate sets, i.e., sets closed under subformulas. At the very end of the chapter, on the other hand, we prove the same result by means of filtrations.

The remaining part of the chapter is devoted to a semantic proof of the disjunction property. The proof goes basically as in the intuitionistic case and makes use of some preservation and invariance results. Finally, we also give a syntactic method to prove the disjunction property under negated hypothesis.

Chapter 4. After having introduced the general setting and developed a Kripke semantics for the main logical systems, the fourth chapter aims to give an introductory account of the algebraic counterpart. Starting from the notion of generalized Heyting algebra, we define the variety of *N-algebras*. We prove that every extension of the basic logic of a unary operator is complete with respect to its own algebraic counterpart. Later, we generalize the notion of Kripke frame by introducing the structures of N-*descriptive frames*. Similarly to the case of Kripke frames, the behavior of the negation on the frames is determined by a function.

In order to develop a duality result, we focus on a particular subclass of the class of N-descriptive frames: top descriptive frames, i.e., descriptive frames with a top node contained in every admissible upset. Indeed, the variety of N-algebras is the dual of the class of top descriptive frames. This allows us to obtain a frame-based completeness result which turns out to be more general than the one with respect to Kripke semantics.

Chapter 5. The last two chapters represent the proof-theoretic fragment of the thesis. We start by defining a sequent calculus system for each of the main subminimal systems we have been considering, as well as for minimal logic. The main characteristic of those systems is that they allow us to keep focusing on the negation operator. As a matter of fact, some sequent calculi for minimal logic were available already. Nonetheless, the system proposed here makes use of an alternative axiomatization of minimal logic by means of axioms of negation.

The second part of the chapter is devoted to define alternative but equivalent sequent systems, in which Weakening and Contraction are proved to be admissible rules. After proving admissibility of those rules, we show the proposed systems to be sound and complete with respect to the considered class of Kripke frames.

Chapter 6. The second proof-theoretic chapter deals with the cut elimination. We prove that the sequent calculi introduced above are cut-free complete proof systems. The proof goes in a fairly straightforward way: we prove cut admissibility first, and we obtain cut elimination as an easy consequence of that. This allows us to conclude that the proposed systems satisfy the subformula property as well as the separation property. At this point, we make use of those systems to prove some interesting results. In particular, we give a cut-free proof of the fact that, in the main logical subsystem that we study, every even number of negations is equivalent to two negations, and every odd number of negations implies one negation. We realize that the form of the negation rules we have chosen to obtain the admissibility of Contraction is indeed necessary to ensure the cut-free completeness of the calculi.

Another interesting result is a proof of the Craig Interpolation Theorem, which holds for all the systems we consider. Finally, we conclude the chapter and the whole thesis by presenting an expressiveness result. We make further use of the sequent systems to prove that there exists a sound and truthful translation of minimal propositional logic into contraposition logic. Observe that this result, together with the fact that intuitionistic logic can be translated into minimal logic, and classical logic can be translated into intuitionistic logic, gives us a chain of translation and lets us conclude that classical logic can be soundly and truthfully translated into contraposition logic.

Chapter 1

Introduction

1.1 Minimal Propositional Logic

In 1937 I. Johansson [25] developed a system, named 'minimal logic' (or 'Johansson's logic'), obtained by discarding *ex falso sequitur quodlibet* (or simply *ex falso* or *ex contradictione*) from the standard axioms for intuitionistic logic. Following [36], we call *explosive* the logical systems in which every inconsistent theory is trivial. Johansson's system can be seen as the *non-explosive* counterpart of intuitionistic logic.

In this chapter, we present minimal logic in its two equivalent formulations. Given a countable set of propositional variables, one of the formulations uses the *propositional language* of the positive fragment of intuitionistic logic, i.e., $\mathcal{L}^+ = \{\wedge, \vee, \to\}$, with an additional propositional variable f, representing *falsum*. This additional variable is often presented as the usual constant $\bot$ (see [36]). The other formulation of minimal logic makes use of the language $\mathcal{L}^+ \cup \{\neg\}$, where the unary symbol $\neg$ represents negation. Given a formula φ and a sequence $\bar{p} = (p_1, \ldots, p_n)$, the fact that all propositional variables contained in φ are in $\bar{p}$ is denoted by $\varphi(p_1, \ldots, p_n)$. We may use the term *atom* to denote propositional variables and the constant $\top$.

As the axioms corresponding to the positive fragment of intuitionistic logic we consider the following:

1. $p \to (q \to p)$
2. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
3. $(p \wedge q) \to p$
4. $(p \wedge q) \to q$
5. $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$
6. $p \to (p \vee q)$
7. $q \to (p \vee q)$
8. $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$

When defining a logic, we will list the set of Hilbert-style axioms, and we assume *modus ponens* and *uniform substitution* to be the only inference rules. We often refer to the logical system axiomatized by 1-8 in the language $\mathcal{L}^+$ as *positive logic* [36] (we may denote it as IPC^+).

Let us denote $\mathcal{L}^f$ the language corresponding to the f-version of minimal logic. We define this version as the logic axiomatized by the axioms 1-8 above. Given a formula φ, its negation $\neg\varphi$ is expressed as $\varphi \to f$. We denote this formulation of minimal logic as MPC_f. On the other hand, we present the alternative version in the language $\mathcal{L}^\neg$, as the logic axiomatized by 1-8, plus the additional axiom:

9. $((p \to q) \wedge (p \to \neg q)) \to \neg p$.

Axiom 9 expresses the fact that, for ever formula φ, its negation $\neg\varphi$ holds whenever φ leads to a contradiction. There is no indication of *what* a contradiction is. The second equivalent formulation of minimal logic will be here denoted as $\mathsf{MPC}_\neg$. Axiom 9 is referred to as *principle of contradiction*, which is indeed the original Kolmogorov's name for it [13].

We present a Kripke-style semantics for both formulations of Johansson's logic.

Definition 1 (Kripke Frame)**.** *A* propositional Kripke frame *for* MPC_f *is a triple* $\mathfrak{F} = \langle W, R, F\rangle$*, where* W *is a (non-empty) set of possible worlds,* R *is a partial order*[1] *and* $F \subseteq W$ *is an upward closed set with respect to* R*. A* propositional valuation V *is a map from the set of propositional variables to* $\mathcal{U}(W)$*, i.e., the set of upward closed subsets of* W*. A* Kripke model *for* MPC_f *is a pair* $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ *consisting of a Kripke frame and a propositional valuation.*

Given a model $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$, we call $\mathfrak{F}$ the frame underlying the model $\mathfrak{M}$. Observe that a propositional valuation V is usually defined as a function mapping every propositional variable to a subset of W. The requirement of such a subset being upward closed ensures the *persistence property* of the valuation: for every pair of possible worlds $w, v \in W$, the joint conditions $w \in V(p)$ and wRv imply $v \in V(p)$, for every propositional variable p. We often use the term *upset* to mean upward closed set.

The definition of the *forcing* (or *truth*) relation between models and formulas goes by induction on the structure of formulas.

Definition 2 (Kripke Model and Forcing Relation)**.** *Given a model* $\mathfrak{M} = \langle W, R, F, V\rangle$*, a state* $w \in W$*, and a formula* φ*, we define the* forcing *relation* $\mathfrak{M}, w \vDash \varphi$ *inductively on the structure of the formula* φ*, as follows:*

$$
\begin{array}{lcl}
\mathfrak{M}, w \vDash p & \iff & w \in V(p) \\
\mathfrak{M}, w \vDash f & \iff & w \in F \\
\mathfrak{M}, w \vDash \varphi \wedge \psi & \iff & \mathfrak{M}, w \vDash \varphi \textit{ and } \mathfrak{M}, w \vDash \psi \\
\mathfrak{M}, w \vDash \varphi \vee \psi & \iff & \mathfrak{M}, w \vDash \varphi \textit{ or } \mathfrak{M}, w \vDash \psi \\
\mathfrak{M}, w \vDash \varphi \to \psi & \iff & \forall v \in W(wRv \Rightarrow (\mathfrak{M}, v \vDash \varphi \Rightarrow \mathfrak{M}, v \vDash \psi))
\end{array}
$$

From this definition, we get a forcing condition for negated formulas:

$$\mathfrak{M}, w \vDash \neg\varphi \quad \iff \quad \forall v \in W(wRv \Rightarrow (\mathfrak{M}, v \vDash \varphi \Rightarrow v \in F)).$$

[1] Under these conditions, the pair $\langle W, R\rangle$ represents an intuitionistic Kripke frame.

A formula φ is said to be *true* on a model $\mathfrak{M}$ if the relation $\mathfrak{M}, w \vDash \varphi$ holds, for every $w \in W$. A formula φ is *true* on a frame $\mathfrak{F}$ if, for every propositional valuation V, the formula φ is true on the model $\langle \mathfrak{F}, V \rangle$. Finally, we say that a formula φ is *valid* on a class of frames $\mathcal{C}$ if it is true on every $\mathfrak{F} \in \mathcal{C}$.

We define propositional Kripke frames and models for $\mathsf{MPC}_{\neg}$ in the same way as in the case of MPC_f. The definition of the forcing relation is again the same as before, with the only difference that the clause for f is replaced by the one for negation $\neg$:

$$\mathfrak{M}, w \vDash \neg\varphi \qquad \Longleftrightarrow \qquad \forall v \in W(wRv \Rightarrow (\mathfrak{M}, v \vDash \varphi \Rightarrow v \in F)).$$

In [36], the upset F is referred to as the set of *abnormal worlds*. A Kripke frame whose set of abnormal worlds is empty, i.e., $F = \emptyset$, is called *normal*. The reader may note that a normal Kripke frame for MPC_f (or $\mathsf{MPC}_{\neg}$) simply is an intuitionistic Kripke frame. This suggests that the upset F denotes nothing more than a 'warning', a non-normal situation, with no further specifications. This is a peculiarity of Johansson's logic in its being a *paraconsistent logic*, i.e., non-explosive.

The following proposition gives some information on the relation between the two formulations of minimal logic.

Proposition 1.1.1. *For every formula φ, the following holds:*

$$\mathsf{MPC}_f \vdash f \leftrightarrow (\neg\varphi \wedge \neg\neg\varphi),$$

where $\neg\varphi$ is expressed as $\varphi \to f$.

A consequence of Proposition 1.1.1 is that the notion of '*absurdum*' expressed by the propositional variable f in MPC_f is available in the system $\mathsf{MPC}_{\neg}$ as $\neg p \wedge \neg\neg p$, where p is an arbitrary propositional variable. The result expressed by Proposition 1.1.1 will be made more precise later on, by defining effective translations between MPC_f and $\mathsf{MPC}_{\neg}$.

In 2013, Odintsov and Rybakov [37] proved MPC_f to be complete with respect to the class of Kripke models as defined in this section. The proof, via a canonical model, goes as the one for intuitionistic logic. We say that the *disjunction property* holds for a set of formulas Γ if, whenever the set contains a disjunction, then it must contain one of the disjuncts, i.e., $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$. Given a set of formulas Γ, a formula φ is a logical consequence of Γ, i.e., $\Gamma \vdash \varphi$, if there exist $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\varphi_1, \ldots, \varphi_n \vdash \varphi$. We call a *theory* a set of formulas closed under logical consequence: $\Gamma \vdash \varphi$ implies $\varphi \in \Gamma$.

Lemma 1.1.2 (Lindenbaum Lemma[2])**.** *Let Γ be a set of formulas and φ be a formula which is not a logical consequence of Γ, i.e., $\Gamma \nvdash \varphi$. There exists a theory Δ with the disjunction property, extending Γ, which does not contain φ.*

Proof. The proof goes exactly as the one for intuitionistic logic (see [4]). $\square$

[2] A more appropriate way to refer to this Lemma is as a 'Lindenbaum-type Lemma', emphasizing the fact that it is a different formulation of the same type of result stated by the original Lindenbaum Lemma.

Definition 3 (Canonical Model)**.** *The* canonical model *of* MPC_f *is a Kripke model* $\mathcal{M}_f = \langle \mathcal{W}, \mathcal{R}, \mathcal{F}, \mathcal{V} \rangle$*, where the set of worlds* $\mathcal{W}$ *is the set of* MPC_f *theories with the disjunction property, ordered by usual set-theoretic inclusion* $\mathcal{R} :=\subseteq$*. The set* $\mathcal{F} \subseteq \mathcal{W}$ *is defined as the set of theories containing* f*, i.e.,* $\{\Gamma \in \mathcal{W} \mid f \in \Gamma\}$*. The canonical valuation* $\mathcal{V}$ *is similarly defined as* $\mathcal{V}(p) := \{\Gamma \in \mathcal{W} \mid p \in \Gamma\}$*, for every* p*.*

Comparing the above notion of canonical model with the one of intuitionistic logic, the reader may note that we drop the requirement for the theories to be consistent. Having disregarded the *law of explosion*, we may allow a theory to contain f without trivializing it.

As usual when dealing with a canonical model, we want to make sure that for every propositional variable and each considered theory, the 'membership relation' coincides with the 'forcing relation'.

Lemma 1.1.3 (Truth Lemma)**.** *Given an element of the canonical model* $\Gamma \in \mathcal{W}$*, for every formula* φ*,*

$$\mathcal{M}_f, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma.$$

Proof. We should prove the statement by induction on the structure of φ. The atomic case, together with the cases for each connective $\circ \in \mathcal{L}^+$, goes as the one for intuitionistic logic (see [4]). Given that the behavior of f is exactly the same as the one of every other propositional variable, we are done. □

We say that a set of formulas Γ is *forced* at a world w in a model $\mathfrak{M}$ (and we write $\mathfrak{M}, w \vDash \Gamma$), if $\mathfrak{M}, w \vDash \psi$ for every formula $\psi \in \Gamma$. A set of formulas Γ is said to *semantically entail* a formula φ if every time the set Γ is forced at a world w in a model $\mathfrak{M}$, so is φ. We denote the relation of semantic entailment as $\Gamma \vDash \varphi$.

Theorem 1.1.4 (Soundness and Completeness)**.** *Minimal propositional logic* MPC_f *is sound and complete with respect to the class of Kripke models defined above, i.e., for every set of formula* Γ *and all formulas* φ*,*

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma \vDash \varphi.$$

Proof. Soundness of the class of Kripke models is proved by induction on the depth of the derivation $\Gamma \vdash \varphi$. The proof, as in the case of intuitionistic logic, consists in checking that each of the axioms 1-8 is valid on the class of Kripke frames and that the inference rules (i.e., *modus ponens* and uniform substitution) preserve validity. We do the proof only for one axiom, as it is very similar for the others. Consider axiom 2,

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r)).$$

The aim of the proof is to show the validity of this axiom. Thus, we consider a world w in an arbitrary Kripke frame $\mathfrak{F} = \langle W, R, F \rangle$. For every model $\mathfrak{M}$ whose underlying frame is $\mathfrak{F}$, we want

$$\mathfrak{M}, w \vDash (p \to (q \to r)) \to ((p \to q) \to (p \to r)).$$

From the definition of the forcing relation, this amounts to proving that every R-successor of w which makes the antecedent (i.e., $(p \to (q \to r))$) true, forces also the consequent (i.e., $((p \to q) \to (p \to r))$). Let v be a R-successor of w and assume that the following holds

$$\mathfrak{M}, v \vDash (p \to (q \to r)). \quad (*)$$

We again move forward and consider a world u such that vRu. Assume

$$\mathfrak{M}, u \vDash (p \to q).$$

Hence, every other R-successor of u which makes p true, makes also q true. Observe that the accessibility relation R is transitive. Thus, every R-successor of u is a R-successor of v too. This implies r to be forced in such a world, by $(*)$. Therefore,

$$\mathfrak{M}, v \vDash ((p \to q) \to (p \to r)),$$

and this proves axiom 2 to be true at world w in the model $\mathfrak{M}$. Both model and world have been chosen arbitrarily and hence, axiom 2 is indeed valid on the class of Kripke frames.

For the remaining proof of completeness, we proceed by contraposition: instead of proving directly that every formula entailed by Γ is also a logical consequence of it, we assume a formula not to be derivable from Γ and we show that it is not entailed by Γ either. Suppose φ is such that $\Gamma \nvdash \varphi$. The Lindenbaum Lemma ensures the existence of a theory Δ with the disjunction property such that $\Gamma \subseteq \Delta$ and $\varphi \notin \Delta$. By looking at the definition of canonical model, it is clear that Δ is an element of $\mathcal{W}$. Moreover, the fact that φ is not an element of Δ, together with the Truth Lemma, gives us $\mathcal{M}_f, \Delta \vDash \Gamma$ and $\mathcal{M}_f, \Delta \nvDash \varphi$. The model $\mathcal{M}_f$ being a Kripke model for MPC_f, we have shown that $\Gamma \nvDash \varphi$, as desired. $\square$

We move now to the second formulation of minimal logic. We prove that soundness and completeness hold for this version as well. The proof of soundness is, as before, a trivial matter. The only additional step we have to develop in this case is proving that also axiom 9 is valid on the considered class of Kripke frames. Completeness is proved again by means of a canonical model. Nonetheless, some preliminary results are required.

Proposition 1.1.5. *For every pair of formulas φ, ψ, the following holds:*

$$\mathsf{MPC}_\neg \vdash (\varphi \land \neg\varphi) \to \neg\psi.$$

The statement derived above will be referred to as *negative ex falso*. Despite its being a non-explosive system, Johansson's logic proves this weak form of the *ex falso quodlibet*. As it is noted in [36], this result gives us that "inconsistent theories (in minimal logic) are positive", because all negated formulas are provable in them and therefore negation makes no sense.

Proposition 1.1.6. *For every formula φ, the following holds:*

$$\mathsf{MPC}_\neg \vdash (\varphi \to \neg\varphi) \to \neg\varphi.$$

The rule whose provability is claimed in Proposition 1.1.6 is denoted here as *absorption of negation* rule.

The canonical model $\mathcal{M}_\neg$ for this version of minimal logic is built in the same way as before. The set $\mathcal{F}$ is defined in a slightly different way, according to the 'new' notion of contradiction. It still coincides with the set of worlds containing 'absurdum', which in this context means: the set of theories containing both φ and $\neg\varphi$, for some formula φ.

Lemma 1.1.7. *Given a theory $\Gamma \in \mathcal{W}$, we have that $\Gamma \in \mathcal{F}$ if and only if $\neg\psi \in \Gamma$, for all formulas ψ.*

Proof. The right-to-left direction of the statement is trivial. We focus on the other direction. Assume Γ to be in $\mathcal{F}$, and consider an arbitrary formula ψ. The definition of $\mathcal{F}$ gives us the existence of a contradiction in Γ, i.e., there is a formula φ in Γ, whose negation is also an element of Γ. The formulas φ and $\neg\varphi$ both being logical consequences of Γ, implies $\Gamma \vdash (\varphi \wedge \neg\varphi)$. Proposition 1.1.5 leads us to $\Gamma \vdash \neg\psi$, via an application of *modus ponens*. The set Γ is a theory, and hence it is closed under logical consequence. Therefore, $\neg\psi \in \Gamma$. $\square$

We recall here the claim of the Truth Lemma. The proof goes in the same way as before. This time though, the negation operator is primitive in our language, and hence the proof of the negation step is worth being unfolded. The statement of the Truth Lemma is the following: given an element of the canonical model $\Gamma \in \mathcal{W}$, for every formula φ,

$$\mathcal{M}_\neg, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma.$$

Proof. Consider the formula $\neg\varphi$. By the induction hypothesis, the statement holds for φ. The left-to-right direction is proved by contraposition. Assume $\neg\varphi \notin \Gamma$, for $\Gamma \in \mathcal{W}$. The set Γ being a theory, this gives us $\Gamma \nvdash \neg\varphi$. Proposition 1.1.6 gives us $\Gamma \vdash (\varphi \to \neg\varphi) \to \neg\varphi$. Thus, if $\Gamma \vdash (\varphi \to \neg\varphi)$ holds, by *modus ponens*, also $\Gamma \vdash \neg\varphi$, which is a contradiction. We get $\Gamma \nvdash (\varphi \to \neg\varphi)$. This is equivalent to say that the formula $\neg\varphi$ is not a logical consequence of the set $\Gamma \cup \{\varphi\}$. From the Lindenbaum Lemma, we get the existence of a theory $\Delta \in \mathcal{W}$, extending $\Gamma \cup \{\varphi\}$ and not containing $\neg\varphi$. Apply now Lemma 1.1.7, to get that Δ is not an element of $\mathcal{F}$. Moreover, $\mathcal{M}_\neg, \Delta \vDash \varphi$ by the induction hypothesis. The last two results are equivalent to $\mathcal{M}_\neg, \Delta \nvDash \neg\varphi$. The canonical model $\mathcal{M}_f$ being persistent, we conclude $\mathcal{M}_\neg, \Gamma \nvDash \neg\varphi$. For the right-to-left direction, we proceed directly. Suppose $\neg\varphi \in \Gamma$, and consider an arbitrary $\subseteq$-successor Δ of Γ. Assume $\mathcal{M}_f, \Delta \vDash \varphi$. The induction hypothesis gives us $\varphi \in \Delta$. We assumed $\neg\varphi$ to be an element of Γ, and hence, of Δ. Both φ and $\neg\varphi$ being elements of Δ, we conclude $\Delta \in \mathcal{F}$. Therefore, $\mathcal{M}_f, \Gamma \vDash \neg\varphi$ as desired. $\square$

All the tools required to prove completeness of $\mathsf{MPC}_\neg$ have been built. The remaining steps of the proof are the same as in the first presented version of

Johansson's logic. Therefore, we move forward to analyze the relation between the two formulations of minimal logic.

1.1.1 Equivalence of the two formulations of MPC

This section is dedicated to making the relation between the two given formulations of minimal logic formal. We define a way to translate MPC_f-formulas into $\mathsf{MPC}_\neg$-formulas, and vice versa. This allows us to enhance the remark made below Proposition 1.1.1. Indeed, the notion of MPC_f-*absurdum* is translated as $\neg q \wedge \neg\neg q$ in $\mathsf{MPC}_\neg$.

Definition 4. *Let φ be an arbitrary formula in MPC_f. We define a translation φ^* by recursion over the complexity of φ, as follows:*

- $p^* := p$,
- $f^* := \neg q \wedge \neg\neg q$, *where q is an arbitrary fixed propositional variable in the $\mathsf{MPC}_\neg$ language,*
- $\top^* := \top$,
- $(\varphi \circ \psi)^* := \varphi^* \circ \psi^*$, *where $\circ \in \{\wedge, \vee, \rightarrow\}$.*

We prove this translation to be sound in the following lemma.

Lemma 1.1.8. *Given a formula φ in the language of MPC_f,*

$$\mathsf{MPC}_f \vdash \varphi \Rightarrow \mathsf{MPC}_\neg \vdash \varphi^*.$$

Proof. An easy induction on the depth of the proof of φ. $\square$

At this point, we define a translation of $\mathsf{MPC}_\neg$-formulas into MPC_f-formulas. Later, after proving such a translation to be sound, we show that the two translation maps are inverses with respect to each other. This result allows us to claim the two versions of minimal logic to be equivalent.

Definition 5. *Let φ be an arbitrary formula in $\mathsf{MPC}_\neg$. We define a translation φ_* by recursion over the complexity of φ, as follows:*

- $p_* := p$,
- $\top_* := \top$,
- $(\varphi \circ \psi)_* := \varphi_* \circ \psi_*$, *where $\circ \in \{\wedge, \vee, \rightarrow\}$,*
- $(\neg\varphi)_* := \varphi_* \rightarrow f$.

Lemma 1.1.9. *Given a formula φ in the language of $\mathsf{MPC}_\neg$,*

$$\mathsf{MPC}_\neg \vdash \varphi \Rightarrow \mathsf{MPC}_f \vdash \varphi_*.$$

Proof. The structure of the proof is the same as in the previous case. We focus here on the proof of the case in which φ is the principle of contradiction. This corresponds to proving that

$$\mathsf{MPC}_f \vdash (p \to q) \wedge (p \to (q \to f)) \to (p \to f),$$

which follows trivially from the axioms of positive logic, together with *modus ponens* and the Deduction Theorem. □

The next lemma shows that the composition of the two translation results is the identity map with respect to logical consequence. In particular, given a formula φ in one of the MPC-systems, the image of φ under the composition of the two translation maps gives us a formula logically equivalent to φ in the original system. We formalize it as follows.

Lemma 1.1.10. *Let φ and ψ be, respectively, arbitrary formulas in MPC_f and $\mathsf{MPC}_\neg$. Then,*

1. $\mathsf{MPC}_f \vdash \varphi \leftrightarrow (\varphi^*)_*$
2. $\mathsf{MPC}_\neg \vdash \psi \leftrightarrow (\psi_*)^*$

Proof. The proof of this result goes by induction on the structure of the formulas φ and ψ. In both cases we focus on the negation step, i.e., we only deal with f and with $\neg\psi$, respectively. Let us start considering f. The double translation of f is of the following form:

$$(f^*)_* = (\neg q \wedge \neg\neg q)_* = ((q \to f) \wedge (q \to f) \to f).$$

Here, it is enough to employ Proposition 1.1.1 to conclude the desired result

$$\mathsf{MPC}_f \vdash f \leftrightarrow (f^*)_*.$$

Consider now $\neg\psi$, and assume by induction

$$\mathsf{MPC}_\neg \vdash \psi \leftrightarrow (\psi_*)^*.$$

Observe that

$$((\neg\psi)_*)^* = (\psi_* \to f)^* = (\psi_*)^* \to (\neg q \wedge \neg\neg q).$$

Assume $((\neg\psi)_*)^*$ within $\mathsf{MPC}_\neg$. This gives us

$$(\psi_*)^* \to (\neg q \wedge \neg\neg q),$$

which implies, by the induction hypothesis,

$$\psi \to (\neg q \wedge \neg\neg q).$$

This is equivalent to

$$(\psi \to \neg q) \wedge (\psi \to \neg\neg q),$$

which by the principle of contradiction allows us to conclude $\neg\psi$. On the other hand, assume $\neg\psi$, as well as $(\psi_*)^*$. This is equivalent to assume $\neg\psi$ and ψ, by induction hypothesis. Axiom 1 gives us that both $(q \to \psi)$ and $(q \to \neg\psi)$ are derivable in $\mathsf{MPC}_\neg$ under these assumptions. Hence, the principle of contradiction gives us

$$\neg\psi, (\psi_*)^* \vdash \neg q.$$

In a similar way, we get

$$\neg\psi, (\psi_*)^* \vdash \neg\neg q$$

as well, in $\mathsf{MPC}_\neg$. Therefore, we can get

$$\mathsf{MPC}_\neg \vdash \neg\psi \to ((\psi_*)^* \to (\neg q \wedge \neg\neg q)),$$

and we are done. □

We conclude the current section with the following fundamental result.

Theorem 1.1.11 (Equivalence of MPC_f and $\mathsf{MPC}_\neg$)**.** *Let φ and ψ be, respectively, arbitrary formulas in MPC_f and $\mathsf{MPC}_\neg$. Then,*

$$\mathsf{MPC}_f \vdash \varphi \Leftrightarrow \mathsf{MPC}_\neg \vdash \varphi^*$$

and

$$\mathsf{MPC}_\neg \vdash \psi \Leftrightarrow \mathsf{MPC}_f \vdash \psi_*.$$

Proof. An easy consequence of Lemmas 1.1.8, 1.1.9 and 1.1.10. □

Let Γ^* denote the set $\{\varphi^* \mid \varphi \in \Gamma\}$ and let Δ_* denote the set $\{\psi_* \mid \psi \in \Delta\}$, for every set Γ of MPC_f-formulas and every set Δ of $\mathsf{MPC}_\neg$-formulas. An easy generalization of the previous result allows us to claim the following.

Corollary 1.1.12. *Let Γ, φ and Δ, ψ be, respectively, arbitrary sets of formulas in MPC_f and $\mathsf{MPC}_\neg$. Then,*

$$\Gamma \vdash_{\mathsf{MPC}_f} \varphi \Leftrightarrow \Gamma^* \vdash_{\mathsf{MPC}_\neg} \varphi^*$$

and

$$\Delta \vdash_{\mathsf{MPC}_\neg} \psi \Leftrightarrow \Delta_* \vdash_{\mathsf{MPC}_f} \psi_*.$$

Proof. An easy generalization of Theorem 1.1.11. □

1.2 Weak Negation

The first two sections of this chapter aimed at introducing the general setting in which our work is developed. We are dealing with paraconsistent logical systems and the negation operator is our main focus. In 1949, D. Nelson [34] suggested a *strong* negation, whose main feature was that it distributed over conjunction. Here, we present different forms of *weak* negation instead. We present a basic system of negation, denoted as N, in which all the specific properties of the negation operator are disregarded. The negation amounts to a functional unary operator. Later, we extend this basic system by means of some 'axioms of negation': we consider some of the theorems of minimal logic (e.g., negative *ex falso*, absorption of negation) as axioms. This allows us to break down the properties of the 'usual' negation operator into a number of separate properties and to analyze their logical behavior. The main axioms considered within this work are the following:

(1) Absorption of negation: $(p \to \neg p) \to \neg p$

(2) N: $(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$

(3) Negative *ex falso*: $(p \wedge \neg p) \to \neg q$

(4) Weak contraposition: $(p \to q) \to (\neg q \to \neg p)$

All of these candidates are theorems in minimal logic. Axiom (2), denoted here as N, is an instance of a more general axiom of the form

$$(\varphi \leftrightarrow \psi) \to (k\varphi \leftrightarrow k\psi),$$

where k represents an arbitrary unary connective. This more general form of the axiom can be referred to as *uniqueness axiom* [20]. The weak contraposition axiom expresses a very basic property of negation: anti-monotonicity. The system axiomatized by means of such an axiom is of great interest: indeed, both N and negative *ex falso* are provable in this system. Moreover, the Deduction Theorem remains in force.

This section is dedicated to proving some results about these axioms. We give an alternative, equivalent, axiomatization of Johansson's system, which will be useful in Chapter 5. We show that negative *ex falso* does logically follow from weak contraposition. This will allow us to hierarchically order the logical systems defined by these axioms. From now on, we use MPC to denote the version of minimal logic axiomatized by 1-9.

Before going into proving the first relevant result of this section, we need to prove a 'substitution' result for minimal logic.

Theorem 1.2.1. *Let $\{\varphi_i\}_{i=0}^n$, $\{\psi_i\}_{i=0}^n$ be two finite families of* MPC*-formulas and let $\theta[p_0, \dots, p_n]$ be an* MPC*-formula containing variables $p_0, \dots, p_n$. Then, the following holds:*

$$\mathsf{MPC} \vdash \bigwedge_{i=1}^{n} (\varphi_i \leftrightarrow \psi_i) \to (\theta[p_0/\varphi_0, \dots, p_n/\varphi_n] \leftrightarrow \theta[p_0/\psi_0, \dots, p_n/\psi_n]).$$

Proof. The proof is a simple induction on the structure of the formula θ. In particular, it is enough to prove that the N axiom is indeed a theorem of minimal logic. Given that, the proof goes exactly the same as it does in intuitionistic logic. $\square$

The next result establishes a strong relation between minimal logic and the axioms (1) and (4).

Proposition 1.2.2. *The logical system axiomatized by 1-8, plus* absorption of negation *and* weak contraposition *is equivalent to* MPC.

Proof. The required proof amounts to showing that (1) and (4) are logical consequences of 9, and vice versa, under the intuitionistic axioms 1-8. We use Hilbert-style derivations to prove the desired results.

$\vdash (\neg q \wedge (p \rightarrow q)) \rightarrow \neg q$	Axiom 3
$\vdash \neg q \rightarrow (p \rightarrow \neg q)$	Axiom 1
$\vdash (\neg q \wedge (p \rightarrow q)) \rightarrow (p \rightarrow \neg q)$	Modus Ponens; Ded. Thm.
$\vdash (\neg q \wedge (p \rightarrow q)) \rightarrow (p \rightarrow q)$	Axiom 4
$\vdash (\neg q \wedge (p \rightarrow q)) \rightarrow ((p \rightarrow \neg q) \wedge (p \rightarrow q))$	Axiom 5; Modus Ponens
$\vdash \boldsymbol{((p \rightarrow \neg q) \wedge (p \rightarrow q)) \rightarrow \neg p}$	Axiom 9
$\vdash ((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$	Modus Ponens; Ded. Thm.
$\vdash \boldsymbol{(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)}$	Ded. Thm.

The above derivation shows that weak contraposition is a logical consequence of axiom 9. We prove the same result for absorption of negation.

$\vdash \boldsymbol{((p \rightarrow \neg q) \wedge (p \rightarrow q)) \rightarrow \neg p}$	Axiom 9
$\vdash ((p \rightarrow \neg p) \wedge (p \rightarrow p)) \rightarrow \neg p$	Substitution Instance
$\vdash \boldsymbol{(p \rightarrow \neg p) \rightarrow \neg p}$	$(p \rightarrow p)$ being a Tautology

It remains to be shown that axiom 9 is a logical consequence of (1) and (4).

$\vdash ((p \rightarrow \neg q) \wedge (p \rightarrow q)) \rightarrow (p \rightarrow q)$	Axiom 4
$\vdash \boldsymbol{(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)}$	Weak Contraposition
$\vdash ((p \rightarrow \neg q) \wedge (p \rightarrow q)) \rightarrow (\neg q \rightarrow \neg p)$	Modus Ponens; Ded. Thm.
$\vdash ((p \rightarrow \neg q) \wedge (p \rightarrow q)) \rightarrow (p \rightarrow \neg q)$	Axiom 3
$\vdash ((p \rightarrow \neg q) \wedge (p \rightarrow q)) \rightarrow (p \rightarrow \neg p)$	Modus Ponens; Ded. Thm.
$\vdash \boldsymbol{(p \rightarrow \neg p) \rightarrow \neg p}$	Absorption of Negation
$\vdash \boldsymbol{((p \rightarrow \neg q) \wedge (p \rightarrow q)) \rightarrow \neg p}$	Modus Ponens; Ded. Thm.

□

In Proposition 1.1.5 we claimed that negative *ex falso* is a theorem in Johansson's system. As we already emphasized in the previous section, this result plays an important conceptual role in the non-explosive minimal logic system: a negation contained in an inconsistent MPC theory does not contribute anything [36]. We prove now that negative *ex falso* is a logical consequence of weak contraposition. This result can be seen as showing that one of the main 'sources of explosiveness' in Johansson's system is exactly weak contraposition.

Proposition 1.2.3. *Given axioms 1-8, negative* ex falso *is a logical consequence of weak contraposition.*

Proof. The proof we unfold here is trivial, and gives the idea of how strong the correlation between weak contraposition and negative *ex falso* is. Again, we proceed by means of a Hilbert-style derivation.

$\vdash q \to (p \to q)$	Axiom 1
$\vdash \boldsymbol{(p \to q) \to (\neg q \to \neg p)}$	Weak Contraposition
$\vdash q \to (\neg q \to \neg p)$	Modus Ponens; Ded. Thm.
$\vdash \boldsymbol{(q \wedge \neg q) \to \neg p}$	Ded. Thm.

$\square$

We conclude this section giving a third alternative axiomatization of minimal propositional logic. This axiomatization is of particular interest. A study of the corresponding algebraic structures is carried out by Rasiowa in [39]. For this reason we choose to denote this axiom as R. Within the cited book, the considered algebras are referred to as *contrapositionally complemented lattices*, and the axiom is called *contraposition law.*

Proposition 1.2.4. *The logical system axiomatized by 1-8 plus*

$$(p \to \neg q) \to (q \to \neg p) \quad \text{(R)}$$

is equivalent to MPC.

Proof. We exhibit here a Hilbert-style derivation in order to obtain R. Observe that this amounts to look for a proof of $((p \to \neg q) \wedge q) \to \neg p$.

$\vdash ((p \to \neg q) \wedge q) \to (p \to q)$	Axioms 1, 4
$\vdash ((p \to \neg q) \wedge q) \to ((p \to \neg q) \wedge (p \to q))$	Axioms 3, 5
$\vdash \boldsymbol{((p \to \neg q) \wedge (p \to q)) \to \neg p}$	
$\vdash \boldsymbol{((p \to \neg q) \wedge q) \to \neg p}$	Ded. Thm.; Modus Ponens

We need now to prove that the Johansson axiom of minimal logic is a logical consequence of the considered axiom R.

$\vdash \boldsymbol{(p \to \neg q) \to (q \to \neg p)}$	R
$\vdash (\neg q \to \neg q) \to (q \to \neg\neg q)$	Substitution Instance
$\vdash (q \to \neg\neg q)$	Modus Ponens
$\vdash \neg\neg\top$	Substitution; Modus Ponens
$\vdash (p \to \neg\neg q) \to (\neg q \to \neg p)$	Substitution Instance

$\vdash (p \to q) \to (p \to \neg\neg q)$	Modus Ponens; Ded. Thm.
$\vdash (p \to q) \to (\neg q \to \neg p)$	Modus Ponens; Ded. Thm.
$\vdash (p \wedge \neg p) \to \neg q$	Proposition 1.2.3
$\vdash ((p \to q) \wedge (p \to \neg q)) \to (p \to \neg\top)$	Axioms 3, 4; NeF; Ded. Thm.
$\vdash (p \to \neg\top) \to (\neg\neg\top \to \neg p)$	CoPC Substitution Instance
$\vdash ((p \to q) \wedge (p \to \neg q)) \to (\neg\neg\top \to \neg p)$	Modus Ponens; Ded. Thm.
$\vdash \boldsymbol{((p \to q) \wedge (p \to \neg q)) \to \neg p}$	Modus Ponens; Ded. Thm.

□

In what follows, we may use the term ‘subminimal’ to refer to logical subsystems of minimal propositional logic, as well as to minimal propositional logic itself. We basically denote as subminimal every (both proper and improper) subsystem of minimal logic.

The reader may ask herself *why* we did consider exactly the axioms and principles presented in this section. Given that this work started from a study of minimal logic, the negative *ex falso* axiom is a pretty natural choice. Indeed, as already emphasized, this principle represents one of the most debatable theorems of minimal logic. It amounts to a weaker notion of ‘explosion’, and it seemed to us a natural candidate for a deeper study. On the other hand, the weak contraposition axiom can be seen as representing the ‘minimal’ requirement for a unary operator to be seen as some kind of negation.

The reader may still argue that these motivations do not exclude other principles and axioms which may have been of equal interest. For instance, one possible axiom to be considered is the one of the form:

$$\neg(p \wedge \neg p).$$

Indeed, this is a formalization of the ‘law of thought’ known as *principle of non-contradiction*, which was considered by Aristotle as ‘necessary for anyone to have who knows any of the things that are’ [23]. Other axioms that present themselves as interesting options to consider are the *distribution law*:

$$(\neg p \wedge \neg q) \to \neg(p \vee q),$$

as well as a direction of the *double negation* law:

$$p \to \neg\neg p.$$

Indeed, some basic work on these two principles has been carried out already, although it is not part of this thesis. A final possibility that I want to emphasize is the one of taking into consideration the following distribution law for double negation over conjunction:

$$\neg\neg(p \wedge q) \leftrightarrow (\neg\neg p \wedge \neg\neg q).$$

The last two principles introduced above represent two of the conditions in the definition of a *nucleus* [26].

1.2.1 Absorption of Negation: a Further Analysis

The very last part of this section wants to deal with a technical result. Although it is not really used within this thesis, the following result is worth being presented, since it could give us and the reader some useful insights concerning the relations between the different principles and axioms introduced above.

Proposition 1.2.2 gives us an alternative axiomatization of the minimal propositional system, obtained by extending the positive fragment of intuitionistic logic by means of weak contraposition and absorption of negation. Here, we give a proof of the fact that, if we substitute weak contraposition with N, we obtain again an axiomatization of minimal logic (i.e., we can prove weak contraposition as a theorem). This is a very interesting result, especially with respect to the role played by absorption of negation. As a matter of fact, the absorption of negation axiom looks innocuous and not very powerful. This result tells us the exact opposite: a functional unary operator satisfying absorption is (equivalent to) a minimal negation.

Proposition 1.2.5. *The logical system axiomatized by 1-8, plus N and* absorption of negation *proves* weak contraposition*.*

Proof. Let us consider the following syntactic derivation:

$\vdash p \to (q \leftrightarrow (p \wedge q))$	Axioms 1-8
$\vdash p \to (\neg q \leftrightarrow \neg(p \wedge q))$	N
$\vdash p \to (\neg q \to \neg(p \wedge q))$	Axiom 3
$\vdash (p \wedge q) \to (\neg q \to \neg(p \wedge q))$	
$\vdash \neg q \to ((p \wedge q) \to \neg(p \wedge q))$	
$\vdash \neg q \to \neg(p \wedge q)$	Absorption of Negation

Together with the following Hilbert-style derivation:

$\vdash (p \to q) \to ((p \wedge q) \leftrightarrow p)$	
$\vdash (p \to q) \to (\neg(p \wedge q) \leftrightarrow \neg p)$	N
$\vdash (p \to q) \to (\neg(p \wedge q) \to \neg p)$	Axiom 3

At this point, the last rows of the two derivations, together with the Deduction Theorem and *modus ponens*, give us the desired derivation of

$$(p \to q) \to (\neg q \to \neg p).$$

□

As anticipated, this result makes a further alternative axiomatization of Johansson's logic available. Although this result is not going to be used or studied within this work, it may have some interesting consequences, especially from the point of view of the expressive power of the considered systems. We will come back to this later on.

1.3 Historical Notes

> *"In both the intuitionistic and the classical logic all contradictions are equivalent. This makes it impossible to consider such entities at all in mathematics. It is not clear to me that such a radical position regarding contradiction is necessary."* (D. Nelson, [35])

Minimal logic appears for the first time under this name in an article by Ingebrigt Johansson in 1937 [25], as a weakened form of the system introduced by Arend Heyting in 1930 [24]. Indeed, the article is a result of a series of letters between Johansson and Heyting. This bunch of letters has been studied by Tim van der Molen and Dick de Jongh, and the results of this study is contained in [33]. In one of those letters, Johansson claims that the axiom $\neg p \to (p \to q)$, which we refer to as *ex falso sequitur quodlibet*, makes a dubious appearance in Heyting's system. From a constructive standpoint, the axiom looks too strong. As emphasized by van der Molen, in one of the letters Johansson writes that the axiom 'says that once $\neg p$ has been proved, q follows from p, even if this had not been the case before'. Observe that, in the new system proposed by Johansson, the positive fragment of Heyting's system remains unchanged: every positive formula can be proved in the same manner.

It is noteworthy that already Andrej Nikolaevič Kolmogorov in 1925 [29] had criticized the *ex falso quodlibet* axiom, as lacking 'any intuitive foundation'. In fact, 'it asserts something about the consequences of something impossible'. *Ex falso*, according to Kolmogorov, was not entitled to be an axiom of intuitionistic logic. Indeed, the system proposed by Johansson in [25] coincided[3] with Kolmogorov's variant of intuitionistic logic [29], which was in fact paraconsistent. As emphasized in [36], 'Kolmogorov reasonably noted that *ex falso quodlibet* has appeared only in the formal presentation of classical logic and does not occur in practical mathematical reasoning'.

The notion of minimal and subminimal negation as presented here was first studied by Dick de Jongh and Ana Lucia Vargas Sandoval in 2015; later, this study was enhanced and published as a paper with title "Subminimal Negation" [14]. There, after presenting a semantics for the two versions of minimal logic, the minimal system is analyzed as a paraconsistent logic. For this reason a study of weakened negations is launched. The only system among the subminimal ones which has been studied by de Jongh and Vargas Sandoval is the logic axiomatized by weak contraposition. In particular, a semantics for this system is developed

[3] The two systems differed from each other with respect to the language. In particular, Kolmogorov's system was formulated by using only negation (as '$\sim$') and implication as connectives. Observe that this doesn't make any essential difference in practice.

and a completeness result is proved. De Jongh and Vargas Sandoval studied the relation of contraposition logic with other principles, fixing the general setting as the one of the positive fragment of intuitionistic logic. Another result, which will be developed in a slightly different way here, has been proved by de Jongh and Vargas Sandoval: contraposition logic interprets minimal logic by means of a translation. In [14], the basic logic of a unary operator and the negative *ex falso* logic are studied. Several issues which are presented as open questions in [14] are solved in this thesis.

Subminimal systems have been studied from an algebraic perspective as well. Helena Rasiowa's work on algebraic semantics for non-classical logics [39] is the main one. The algebraic structures corresponding to minimal logic are studied there under the name of contrapositionally complemented lattices. A study of subminimal systems from an algebraic standpoint is carried out by Rodolfo C. Ertola, Adriana Galli and Marta Sagastume in [18], as well as by Ertola and Sagastume in [19]. In particular, the latter presents a subminimal system different from the ones analyzed here and studies algebraic structures denoted as weak algebras.

Chapter 2

Subminimal Systems

Since Brouwer's works [9, 8], the negation of a statement in a logical system is conceived as encoding the fact that from any proof of such a statement, a proof of a contradiction can be obtained. This notion of negation agrees with the one of paraconsistency: it does not assume that contradictions trivialize inconsistent theories. In the last century the general trend in paraconsistent logic was to consider systems with *strong* negations. Here, we want to do something different: this chapter wants to give a full introduction to the main subsystems of minimal logic, obtained by *weakening* the negation operator. We deal with a persistent unary operator and we present various axiomatizations. Each different axiomatization enriches this operator with a certain property, and this gives us the possibility of drawing a neat line and guessing *which* one is the 'minimal' property for making such an operator a negation.

The structure of this chapter is the following. We first present our basic system and define a class of Kripke-style models, with respect to which the considered system is complete. The second half of this chapter is dedicated to moving further 'up' within the hierarchy of systems that we are considering. An extension of the basic system, axiomatized by a weakened law of explosion, is presented. Finally, the most important system presented here is the logic of contraposition, i.e., the logic axiomatized by the instance of contraposition valid in intuitionistic logic.

2.1 A Basic Logic of a Unary Operator

In this section, we present the basic system in our general setting. We refer to it as 'basic logic of negation' or, even more appropriately, as 'basic logic of a unary operator'. Indeed, one can say that the unary operator $\neg$ is nothing more than a function, in this system. It has no property of negation at all. A Kripke-style semantics is introduced, in which the semantic clause for the operator $\neg$ is defined by means of an auxiliary function defined over upsets. In order to make these models satisfy the basic 'axiom of negation', a natural property for this function arises. Completeness is finally proved via a canonical model.

2.1.1 Kripke-style Semantics

In this framework, we take as a propositional language the one already considered for minimal logic with negation, i.e., $\mathcal{L}^+ \cup \{\neg\}$. The axioms for this basic system, denoted as N, are the ones of positive logic (axioms 1-8), plus the additional axiom

$$(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q). \quad \text{(N)}$$

Again, *modus ponens* and *uniform substitution* are the only available inference rules. Although the considered unary operator $\neg$ does not have the standard properties usually attributed to a negation, we shall refer to it with the term 'negation'.

At a first sight, the axiom itself suggests a possible semantics for the negation operator: the truth of a negated formula needs to be a function. In particular, in a persistent setting, the axiom even says that such a function needs to map upsets to upsets. Whenever two formulas are equivalent (i.e., two upsets coincide), so are their negations (i.e., the upsets of their negations coincide as well).

Definition 6 (Kripke Frame)**.** *A* propositional Kripke frame *for the system* N *is a triple* $\mathfrak{F} = \langle W, R, N\rangle$*, where* W *is a non-empty set of possible worlds,* R *is a partial order on* W *and* N *is a function*

$$N : \mathcal{U}(W) \to \mathcal{U}(W),$$

where $\mathcal{U}(W)$ *denotes the set of all upward closed subsets of* W*. The function* N *satisfies the following properties:*

- **P1.** *For every upset* $U \subseteq W$ *and every world* $w \in W$*,* $w \in N(U) \Leftrightarrow w \in N(U \cap R(w))$*, where* $R(w)$ *is the upset generated by* w*.*
- **P2.** *For every upset* $U \subseteq W$ *and every world* $w \in W$*, if* w *is an element of* $N(U)$*, then every* R*-successor* v *of* w *is also an element of* $N(U)$*, i.e.,* $\forall v(w \in N(U)$ *and* wRv *imply* $v \in N(U))$*.*

The two considered properties express quite natural requirements for the function N. The first one is closely related to the notion of *locality*. Indeed, the (truth) value of a formula φ in a world *only* depends on the value of such a formula on all the worlds accessible from that world. The second property ensures *persistence* for the unary operator. As the reader may observe, this requirement is already included in the definition of N being a function from upsets to upsets. Nonetheless, by stating it explicitly we make clear the fact that it is a necessary requirement and it needs to be checked when building an N-frame.

Remark. The property denoted here as **P1** is the one actually expressing the meaning of the N axiom. In particular, although the axiom basically expresses a notion of 'functionality', we need to be careful and distinguish between dealing with the axiom and dealing with the corresponding rule. Let us focus on the rule: if $\vdash \varphi \leftrightarrow \psi$ holds, then $\vdash \neg\varphi \leftrightarrow \neg\psi$ holds as well. This, in terms of Kripke frames, expresses a 'global notion'. On the other hand, the N axiom has a 'local meaning', in the following sense: we need to 'zoom in' on each node, and check that, locally, the antecedent of the axiom implies the consequent. Indeed, the latter is what is expressed by property **P1**.

Definition 7 (Kripke Model). *A* propositional Kripke model *is a quadruple* $\mathfrak{M} = \langle W, R, N, V\rangle$*, where* $\langle W, R, N\rangle$ *is a Kripke frame and the map* V *is a persistent valuation from the set of propositional variable to the upward closed subsets of* W*.*

The forcing relation is defined inductively on the structure of the formula in the same way as in the case of minimal logic. The significant difference concerns the clause for negation, in which we make use of the function N.

Definition 8 (Forcing Relation). *Given a model* $\mathfrak{M} = \langle W, R, N, V\rangle$*, a state* $w \in W$*, and a formula* φ*, we define the* forcing *relation* $\mathfrak{M}, w \vDash \varphi$ *inductively on the structure of the formula* φ*, as follows:*

$$
\begin{array}{lcl}
\mathfrak{M}, w \vDash p & \iff & w \in V(p) \\
\mathfrak{M}, w \vDash \varphi \wedge \psi & \iff & \mathfrak{M}, w \vDash \varphi \textit{ and } \mathfrak{M}, w \vDash \psi \\
\mathfrak{M}, w \vDash \varphi \vee \psi & \iff & \mathfrak{M}, w \vDash \varphi \textit{ or } \mathfrak{M}, w \vDash \psi \\
\mathfrak{M}, w \vDash \varphi \to \psi & \iff & \forall v \in W(wRv \Rightarrow (\mathfrak{M}, v \vDash \varphi \Rightarrow \mathfrak{M}, v \vDash \psi)) \\
\mathfrak{M}, w \vDash \neg\varphi & \iff & w \in N(V(\varphi)),
\end{array}
$$

where $V(\varphi)$ *denotes a 'generalized valuation', defined as the set of worlds* $w \in W$ *in which the formula* φ *is forced.*

We dedicate the next few pages to defining some notions which will turn out to be useful later on. We start defining the notion of p-morphism, which will be fundamental in Chapter 6. The basic idea is to extend the notion of intuitionistic p-morphism with a condition for N which ensures validity to be invariant under p-morphism.

Definition 9 (p-morphism). *Given a pair of* N*-frames* $\mathfrak{F} = \langle W, R, N\rangle$ *and* $\mathfrak{F}' = \langle W', R', N'\rangle$*, then* $f : W \to W'$ *is a* p-morphism *(or* bounded morphism*) from* $\mathfrak{F}$ *to* $\mathfrak{F}'$ *if*

(i.) *for each* w*,* $v \in W$*, if* wRv*, then* $f(w)R'f(v)$*;*

(ii.) *for each* $w \in W$*,* $w' \in W'$*, if* $f(w)R'w'$*, then there exists* $v \in W$ *such that* wRv *and* $f(v) = w'$*;*

(iii.) *for every upset* $V \in \mathcal{U}(W')$*, for each* $w \in W$*:* $f(w) \in N'(V)$ *if and only if* $w \in N(f^{-1}[V])$*, where* $f^{-1}[V] = \{v \in W \mid f(v) \in V\}$*.*

We can extend this notion between frames to a notion between models $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ *and* $\mathfrak{M}' = \langle \mathfrak{F}', V'\rangle$*, by adding the following requirement:*

(iv.) *for every* $p \in Prop$*, for all* $w \in W$*:* $f(w) \in V'(p) \Leftrightarrow w \in V(p)$*.*

That our notion of p-morphism is the right one will also become clear in Chapter 4 from the duality results proved there.

We prove here a result which is the first of a series of preservation and invariance results which will be proved in this section.

Lemma 2.1.1. *If f is a p-morphism from $\mathfrak{M}$ to $\mathfrak{M}'$ and $w \in W$, then for every formula φ,*

$$\mathfrak{M}', f(w) \vDash \varphi \Leftrightarrow \mathfrak{M}, w \vDash \varphi.$$

Proof. We proceed by induction on the structure of the formula φ. We focus on the negation step, which is the most interesting one from our perspective. The induction hypothesis gives us that

$$f^{-1}[V'(\varphi)] = V(\varphi),$$

which by (iii.) allows us to conclude that $w \in N(V(\varphi))$ if and only if $f(w) \in N'(V'(\varphi))$, for any $w \in W$. □

The standard semantic proof of the so-called *disjunction property* in intuitionistic logic makes use of both the notions of generated submodel and disjoint union [44]. In the next chapter, we will simulate that proof and show that our main systems enjoy the disjunction property.

Definition 10 (Generated Submodel)**.** *Given a Kripke frame $\mathfrak{F} = \langle W, R, N\rangle$ and a world $w \in W$, the* subframe generated by w, *denote as $\mathfrak{F}_w$, is the frame defined on the set of worlds $R(w)$ and such that*

$$N_w(U) = N(U) \cap R(w).$$

In a similar way, the generated submodel $\mathfrak{M}_w$ *(or* submodel generated by w*) is defined by adding a valuation to $\mathfrak{F}_w$, on the basis of the model $\mathfrak{M}$.*

When working in a setting with persistent valuations, the notion of generated submodel becomes even more essential than usual. Whenever a formula φ is satisfied in a finite model, there is indeed a 'first' (with respect to the R-accessibility relation) world in which the formula is true. In a system in which generated submodels preserve the forcing relation, the submodel generated by such a first world as a root turns out to be a model of φ (i.e., a model in which φ is true at every state). The following result shows that the notion of generated submodel just introduced preserves the truth relation.

Lemma 2.1.2. *Let $\mathfrak{M} = \langle W, R, N, V\rangle$ be a Kripke model and $w \in W$ be an arbitrary world in such a model. Let $\mathfrak{M}_w$ be the submodel generated by w. For every $v \in W$ such that $v \in R(w)$, for any formula φ,*

$$\mathfrak{M}, v \vDash \varphi \Leftrightarrow \mathfrak{M}_w, v \vDash \varphi.$$

Proof. The proof goes by induction on the structure of φ. The only interesting step is the one for negation. Assume $\mathfrak{M}, v \vDash \neg\varphi$, which, by semantics, means that $v \in N(V(\varphi))$. Since $v \in R(w)$ by assumption, we get:

$$v \in N(V(\varphi)) \cap R(w).$$

By the property **P1**, we have that this is equivalent to $v \in N(V(\varphi) \cap R(v))$. Now, v being an R-successor of w, the following holds:

$$V_w(\varphi) \cap R(v) = (V(\varphi) \cap R(w)) \cap R(v) = V(\varphi) \cap R(v).$$

Therefore, we have $v \in N_w(V(\varphi) \cap R(v))$, which gives us $v \in N_w(V_w(\varphi))$, as desired. $\square$

We define now the notion of disjoint union of models, in order to be able to 'build' a new model which carries all the information contained in the original ones. Note that we call *disjoint* two models whose domains contain no common elements.

Definition 11 (Disjoint Union)**.** *Consider a family of models $\mathfrak{M}_i = \langle W_i, R_i, N_i, V_i \rangle$ $(i \in I)$. Their* disjoint union *is the structure $\uplus_i \mathfrak{M}_i = \langle W, R, N, V \rangle$, where W is the disjoint union of the sets W_i and R is the union of the relations R_i. Moreover, the function N is defined as*

$$N : \mathcal{U}(W) \to \mathcal{U}(W),$$

$$U \mapsto \bigcup_{i \in I} N_i(U \cap W_i).$$

In addition, for each p, $V(p) = \bigcup_{i \in I} V_i(p)$.

Checking that the model just defined is indeed a model for N is a trivial matter, since the function N preserves the properties (of the N_i's) in a straightforward way. Preservation of the forcing relation is one of the most relevant characteristics of disjoint unions. We prove here that such a property holds.

Lemma 2.1.3. *Let $\mathfrak{M}_i = \langle W_i, R_i, N_i, V_i \rangle$ be a family of models of N, for some set of indexes I. Then, for each formula φ, for each $i \in I$ and for each node $w \in \mathfrak{M}_i$,*

$$\mathfrak{M}_i, w \vDash \varphi \Leftrightarrow \uplus_i \mathfrak{M}_i, w \vDash \varphi,$$

i.e., forcing relation is invariant under disjoint unions.

Proof. The proof is an induction on the structure of the formula. We again unfold only the negation step. The induction hypothesis (IH) states that, for each ψ less complex than the considered φ, $V(\psi) = \bigcup_{i \in I} V_i(\psi)$. Suppose $\mathfrak{M}_i, w \vDash \neg\varphi$, i.e., $w \in N_i(V_i(\varphi))$. The induction hypothesis ensures that this is equivalent to $w \in N_i(V(\varphi) \cap W_i)$, which by definition of N means $w \in N(V(\varphi)) \cap W_i$. Thus, $\uplus_i \mathfrak{M}_i, w \vDash \neg\varphi$. $\square$

The definitions we gave of generated submodel and disjoint union are not dependent on the properties of the function N. Indeed, they only depend on the semantic clause of negation, their main aim being to preserve the truth relation. This fact will allow us to use the same notions of generated submodel and disjoint union for all the extensions of the basic logic N which share its semantics of negation.

2.1.2 Soundness and Completeness Theorems

When defining propositional Kripke frames, we took care of making the axiom N valid on such frames. As already emphasized, the first requirement **P1** for the function N wants to express exactly what the axiom said. Moreover, observe that, given a frame $\mathfrak{F} = \langle W, R, F \rangle$, the pair $\langle W, R \rangle$ is basically an intuitionistic Kripke frame. Therefore, validity of axioms 1-8 is straightforward as well. The notion of logical consequence and semantic entailment are defined as in Chapter 1.

Theorem 2.1.4 (Soundness Theorem). *Given a set* Γ *of* N*-formulas and an arbitrary formula* φ*,*

$$\Gamma \vdash \varphi \textit{ implies } \Gamma \vDash \varphi.$$

Proof. The proof is a straightforward induction on the length of the derivation $\Gamma \vdash \varphi$. We unfold a step of the base case of the induction, the one concerning the axiom N. Assume $\Gamma \vdash (p \leftrightarrow q) \rightarrow (\neg p \leftrightarrow \neg q)$. Consider an arbitrary world w in a model $\mathfrak{M}$ and assume $\mathfrak{M}, w \vDash \Gamma$. Consider a successor of w, namely a world v, such that $\mathfrak{M}, v \vDash (p \leftrightarrow q)$. Let u be a successor of v which makes $\neg p$ true. By the semantic clause of negation, this means $u \in N(V(p))$. Property **P1** gives us $u \in N(V(p) \cap R(u))$. Moreover, persistence together with the fact that $\mathfrak{M}, v \vDash (p \leftrightarrow q)$, imply

$$V(p) \cap R(u) = V(q) \cap R(u).$$

That said, we can conclude $u \in N(V(q) \cap R(u))$, which is equivalent to $u \in N(V(q))$, i.e., $\mathfrak{M}, u \vDash \neg q$. □

The proof of completeness of the system N with respect to the considered class of Kripke frames goes via a canonical model. In order to be able to build a canonical model for this system, we will need a Lindenbaum Lemma for N. Both statement and proof are the same as the ones in Chapter 1. Hence, the reader may just refer to Lemma 1.1.2.

We can define now the canonical model.

Definition 12 (Canonical Model). *The canonical model for* N *is the quadruple* $\mathcal{M}_{\mathsf{N}} = \langle \mathcal{W}, \mathcal{R}, \mathcal{N}, \mathcal{V} \rangle$*, where* $\mathcal{W}$ *is the set of theories with the disjunction property, the accessibility relation* $\mathcal{R}$ *is the usual set-theoretic inclusion, and the valuation* $\mathcal{V}$ *is again defined as:* $\mathcal{V}(p) := \{\Gamma \in \mathcal{W} \mid p \in \Gamma\}$*. For every upset* $U \in \mathcal{U}(\mathcal{W})$*, the function* $\mathcal{N}$ *is defined as*

$$\mathcal{N}(U) := \{\Gamma \in \mathcal{W} \mid \exists \varphi \big(U \cap \mathcal{R}(\Gamma) = [\![\varphi]\!] \cap \mathcal{R}(\Gamma) \textit{ and } \neg\varphi \in \Gamma\big)\},$$

where $[\![\varphi]\!] = \{\Gamma \in \mathcal{W} \mid \varphi \in \Gamma\}$*.*

Remark. What the given definition says is that a theory Γ is in a certain $\mathcal{N}(U)$ if and only if there is a negated formula in Γ, whose valuation 'relativized' to Γ is U. This apparently convoluted definition is necessary because, up to now, the function $\mathcal{N}$ has been presented as a *total* function, over all the upsets. If one is happy with having a *partial* function in the canonical model, the definition of $\mathcal{N}$ that one obtains is extremely intuitive:

$$\mathcal{N}(\llbracket\varphi\rrbracket) := \{\Gamma \in \mathcal{W} \mid \neg\varphi \in \Gamma\}.$$

Another advantage of following this alternative approach is the fact that every extension of N has a 'good' canonical model, in the sense that its canonical model is a model for the logic. Saying it differently, if we build the canonical model exactly in the same way we do it for N, considering **L**-theories instead of N theories, we get a model validating all the theorems of **L**. This approach has been suggested by a similar approach in the context of neighborhood semantics [16, 30, 38]. Observe that this idea of 'having a good canonical model' does not imply the frame underlying the canonical model to validate the considered logic **L**. Indeed, in modal and intermediate logics, the latter are exactly the logics called *canonical*. In our setting, it seems appropriate to denote as *canonical* the logics for which, given a 'partial' canonical model, it is possible to extend the partial function $\mathcal{N}$ to a total function preserving its properties.

Before proceeding with the standard proof of completeness, we want to make sure that the canonical model is indeed a Kripke model for the system N. This amounts to proving that the function $\mathcal{N}$ satisfies the two requirements, $\mathsf{\mathbf{P1}}$ and $\mathsf{\mathbf{P2}}$.

Proposition 2.1.5. *For the function $\mathcal{N}$ as defined in Definition 12,* $\mathsf{\mathbf{P1}}$ *holds.*

Proof. The statement we have to prove is the following. Given an upward closed subset U of $\mathcal{W}$, for every theory $\Gamma \in \mathcal{W}$,

$$\Gamma \in \mathcal{N}(U) \Leftrightarrow \Gamma \in \mathcal{N}(U \cap \mathcal{R}(\Gamma)).$$

Assume $\Gamma \in \mathcal{N}(U)$. By how the canonical model was defined, this means that there exists a formula φ such that

$$U \cap \mathcal{R}(\Gamma) = \llbracket\varphi\rrbracket \cap \mathcal{R}(\Gamma) \text{ and } \neg\varphi \in \Gamma.$$

Note that $U \cap \mathcal{R}(\Gamma) = \llbracket\varphi\rrbracket \cap \mathcal{R}(\Gamma)$ is equivalent to

$$(U \cap \mathcal{R}(\Gamma)) \cap \mathcal{R}(\Gamma) = \llbracket\varphi\rrbracket \cap \mathcal{R}(\Gamma)$$

and hence, $\Gamma \in \mathcal{N}(U)$ is equivalent to $\Gamma \in \mathcal{N}(U \cap \mathcal{R}(\Gamma))$. □

Proposition 2.1.6. *For the function $\mathcal{N}$ as defined in Definition 12,* $\mathsf{\mathbf{P2}}$ *holds.*

Proof. We have to prove that

$$\Gamma \subseteq \Gamma' \text{ and } \Gamma \in \mathcal{N}(U) \Rightarrow \Gamma' \in \mathcal{N}(U).$$

For this purpose, assume that $\Gamma \subseteq \Gamma'$ are two theories in $\mathcal{W}$, and $\Gamma \in \mathcal{N}(U)$, for some $U \in \mathcal{U}(\mathcal{W})$. This means that there exists a formula φ such that

$$U \cap \mathcal{R}(\Gamma) = \llbracket\varphi\rrbracket \cap \mathcal{R}(\Gamma) \text{ and } \neg\varphi \in \Gamma \quad (*).$$

Note now that: $\Gamma \subseteq \Gamma'$ is equivalent to $\mathcal{R}(\Gamma') \subseteq \mathcal{R}(\Gamma)$. Thus, from $(*)$ we get

$$(U \cap \mathcal{R}(\Gamma)) \cap \mathcal{R}(\Gamma') = (\llbracket\varphi\rrbracket \cap \mathcal{R}(\Gamma)) \cap \mathcal{R}(\Gamma'),$$

which, by associativity of $\cap$, means $U \cap \mathcal{R}(\Gamma') = \llbracket\varphi\rrbracket \cap \mathcal{R}(\Gamma')$. Moreover, since $\Gamma \subseteq \Gamma'$ and $\neg\varphi \in \Gamma$, we also have $\neg\varphi \in \Gamma'$. Therefore, $\Gamma' \in \mathcal{N}(U)$ as desired. $\square$

The proof of completeness will now proceed in the standard way, by showing that the canonical model makes forcing relation and membership relation coincide.

Lemma 2.1.7 (Truth Lemma)**.** *Given a theory Γ in the canonical model $\mathcal{M}_{\mathsf{N}}$, for every formula φ,*

$$\mathcal{M}_{\mathsf{N}}, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma.$$

Proof. We shall argue by induction on the structure of the formula φ. The steps of the proof concerning the intuitionistic connectives and the atoms are standard. We give the proof of the negation step in detail. For the left-to-right direction, assume $\mathcal{M}_{\mathsf{N}}, \Gamma \vDash \neg\varphi$. By semantics, together with the definition of $\mathcal{N}$, this says that there exists a negated formula $\neg\psi \in \Gamma$ such that

$$\mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma) = \llbracket\psi\rrbracket \cap \mathcal{R}(\Gamma).$$

Observe that the induction hypothesis applied to φ, gives us the equality

$$\mathcal{V}(\varphi) = \llbracket\varphi\rrbracket,$$

and hence, $\llbracket\varphi\rrbracket \cap \mathcal{R}(\Gamma) = \llbracket\psi\rrbracket \cap \mathcal{R}(\Gamma)$. We want to show that $\neg\varphi \in \Gamma$. It will be sufficient to prove the following:

Claim: For every formula θ, if $\llbracket\theta\rrbracket \cap \mathcal{R}(\Gamma) = \llbracket\psi\rrbracket \cap \mathcal{R}(\Gamma)$, then $\neg\theta \in \Gamma$ holds.

In order to prove this statement, we will need to use the N axiom. First of all, observe that

$$\psi \to \theta \in \Gamma \text{ and } \theta \to \psi \in \Gamma.$$

In fact, suppose without loss of generality that $\psi \to \theta \notin \Gamma$. Then, by the Lindenbaum Lemma, there exists an extension Γ' of Γ such that $\psi \in \Gamma'$, but $\theta \notin \Gamma'$. Hence, $\Gamma' \in \llbracket\psi\rrbracket \cap \mathcal{R}(\Gamma)$, but $\Gamma' \notin \llbracket\theta\rrbracket \cap \mathcal{R}(\Gamma)$, which contradicts our assumption. Therefore, $\psi \to \theta \in \Gamma$. Now, from the fact that Γ is a theory, $\psi \leftrightarrow \theta \in \Gamma$. For the same reason, every instance of the axiom N is in Γ, which implies

$$(\psi \leftrightarrow \theta) \to (\neg\psi \leftrightarrow \neg\theta) \in \Gamma$$

and finally, $\neg\psi \leftrightarrow \neg\theta \in \Gamma$. Since, by assumption, $\neg\psi \in \Gamma$, we can conclude $\neg\theta \in \Gamma$.

It is immediate now, from the claim, to conclude $\neg\varphi \in \Gamma$. For the other direction of the statement, we have to assume $\neg\varphi \in \Gamma$. Our aim is to show that there is a formula ψ such that

$$\mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma) = [\![\psi]\!] \cap \mathcal{R}(\Gamma) \text{ and } \neg\psi \in \Gamma.$$

Again, the induction hypothesis ensures $\mathcal{V}(\varphi) = [\![\varphi]\!]$, which implies

$$\mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma) = [\![\varphi]\!] \cap \mathcal{R}(\Gamma).$$

Therefore, such a formula exists and by semantics, we can conclude $\mathcal{M}_{\mathsf{N}}, \Gamma \vDash \neg\varphi$, as desired. □

We have provided all the tools necessary to prove completeness.

Theorem 2.1.8 (Completeness Theorem)**.** *Given a set* Γ *of formulas in* N *and an arbitrary formula* φ*,*

$$\Gamma \vDash \varphi \textit{ implies } \Gamma \vdash \varphi.$$

The structure of the proof is standard, by contraposition, and it goes exactly the same as the one for minimal logic (Theorem 1.1.4).

2.2 Intermediate Systems between N and MPC

In the previous section, the basic system in our general setting has been introduced. What follows will just extend the basic notions presented there. This section is dedicated to presenting and analyzing two of the extensions of the logic N. We will move 'bottom-up', presenting first the negative *ex falso* logic. This system is obtained by adding to the N axioms (i.e., axioms 1-8 + N), the axiom NeF presented in Chapter 1. The second part of this section aims to present contraposition logic. This system, axiomatized by adding CoPC to the 'positive' axioms 1-8, turns out to be the most striking among the ones we are considering. Indeed, the unary operator $\neg$ starts here to behave as a proper negation. The semantic function N becomes *anti-monotone* in this setting.

2.2.1 Negative *ex Falso* Logic

The focus of this section will be on the weakened form of the 'law of explosion', to which we refer as negative *ex falso*. The name explains itself: the rule represents a 'negative' version of the *ex falso quodlibet*, and expresses the fact that negated formulas have no meaning at all in an inconsistent theory.

Before assuming NeF as an axiom and analyzing the resulting system, we ask ourselves whether there is a way to characterize the class of N-frames validating the formula $(p \wedge \neg p) \to \neg q$. Indeed, we want to find a property **P** such that, given an N-frame $\mathfrak{F} = \langle W, R, N\rangle$,

$$\text{NeF is valid on } \mathfrak{F} \Leftrightarrow N \text{ satisfies } \mathsf{P}.$$

The idea behind this is that, basically, we want to find a property for the function N which expresses in terms of upsets what the rule NeF says in terms of formulas.

For instance, consider a pair of propositional variables p, q and their respective valuations on an arbitrary frame, $V(p)$ and $V(q)$. Whenever a node w makes both p and $\neg p$ true (i.e., $w \in V(p) \cap N(V(p))$), we want such a node to make also $\neg q$ true (i.e., $w \in N(V(q))$). Therefore, the natural candidate as a characterizing property for N is the following.

Proposition 2.2.1. *Given an* N*-frame* $\mathfrak{F} = \langle W, R, N\rangle$*, the principle* NeF *is valid on* $\mathfrak{F}$*, if and only if* N *has the following property:*

$$\forall U, U' \in \mathcal{U}(W) : U \cap N(U) \subseteq N(U'). \quad (\text{NEF})$$

Proof. We divide the proof in its two directions. Assuming the right-hand side of the statement, we have a frame $\mathfrak{F}$ such that, for any pair of upsets U, $U' \in \mathcal{U}(W)$,

$$U \cap N(U) \subseteq N(U').$$

Add an arbitrary valuation V to $\mathfrak{F}$ and let $\mathfrak{M} = \langle W, R, V, N\rangle$ denote the resulting model. Given an arbitrary world $w \in W$, suppose one of its successors, v, satisfies the conjunction $(p \wedge \neg p)$. This means that the state v is an element of both $V(p)$ and $N(V(p))$. The set $V(p)$ is clearly an upset, V being a persistent valuation. So is $V(q)$, for any arbitrary q. Hence, the assumption ensures that $v \in V(p) \cap N(V(p))$ entails the fact that $v \in N(V(q))$, which indeed means $\mathfrak{M}, w \vDash (p \wedge \neg p) \to \neg q$. The other direction of the statement is proved by contraposition. For this, assume that we are dealing with a frame $\mathfrak{F}$ in which there exists a pair of upsets U, U' such that

$$U \cap N(U) \not\subseteq N(U').$$

Consider now a propositional valuation V on such a frame such that $V(p) := U$ and $V(q) := U'$. Note that defining such a valuation is always possible, because it indeed satisfies the persistence requirement. Now, from the fact that $U \cap N(U) \not\subseteq N(U')$, we get $V(p) \cap N(V(p)) \not\subseteq N(V(q))$. This, by semantics, implies the existence of a world w satisfying both p and $\neg p$, but not forcing $\neg q$. Therefore, $\mathfrak{M}, w \nvDash (p \wedge \neg p) \to \neg q$ and hence, NeF is not valid on the considered frame. □

In what follows, we will consider NeF as an axiom by adding it to the N-system. In fact, we conclude this section by proving soundness and completeness of such a logic with respect to the class of frames satisfying **P1**, **P2** and

NEF. For every pair of upsets $U, U' \subseteq W$: $U \cap N(U) \subseteq N(U')$.

The system axiomatized by

$$\mathsf{N} + (\boldsymbol{p} \wedge \neg \boldsymbol{p}) \to \neg \boldsymbol{q}$$

will be denoted as NeF. Soundness has essentially been proved already. The proof will be an extension of the one for the basic system, obtained by checking the base case of the induction also over the axiom NeF. In proving completeness for this system, a new definition of $\mathcal{N}$ needs to be given when building the canonical model. In particular, checking whether the canonical model of N satisfying the NeF axiom implies the function $\mathcal{N}$ to satisfy NEF is not enough[1].

Theorem 2.2.2 (Soundness Theorem). *Given a set* Γ *of* NeF*-formulas and an arbitrary formula* φ*,*

$$\Gamma \vdash \varphi \textit{ implies } \Gamma \vDash \varphi.$$

The standard Lindenbaum Lemma, as stated in Chapter 1, is still valid. Thus, we can immediately go through the construction of the canonical model. Observe that the new definition of the function $\mathcal{N}$ is an 'extension' of the one given for N: we have also to take care of the fact that a theory which contains *all* the negated formulas needs to be an element of $\mathcal{N}(U)$, for every upward closed subset U.

Definition 13 (Canonical Model). *The canonical model for* NeF *is the quadruple* $\mathcal{M}_{\mathsf{NeF}} = \langle \mathcal{W}, \mathcal{R}, \mathcal{N}, \mathcal{V} \rangle$*, where* $\mathcal{W}$ *is the set of theories with the disjunction property, the accessibility relation* $\mathcal{R}$ *is the usual set-theoretic inclusion, and the valuation* $\mathcal{V}$ *is again defined as:* $\mathcal{V}(p) := \{\Gamma \in \mathcal{W} \mid p \in \Gamma\}$*. For upset* $U \in \mathcal{U}(\mathcal{W})$*, the function* $\mathcal{N}$ *is defined as*

$$\mathcal{N}(U) := \{\Gamma \in \mathcal{W} \mid \exists\varphi\big(U \cap \mathcal{R}(\Gamma) = [\![\varphi]\!] \cap \mathcal{R}(\Gamma) \textit{ and } \neg\varphi \in \Gamma\big) \textit{ or } \forall\varphi(\neg\varphi \in \Gamma)\},$$

where $[\![\varphi]\!] = \{\Gamma \in \mathcal{W} \mid \varphi \in \Gamma\}$*.*

Is the underlying frame of the defined model indeed a NeF-frame? In order to answer this question and proceed with the proof of completeness, we check that the function $\mathcal{N}$ as defined in the previous definition satisfies **P1**, **P2** and NEF.

Proposition 2.2.3. *For the function* $\mathcal{N}$ *as defined in Definition 13,* **P1** *holds.*

Proof. We start by considering Γ to be an element of $\mathcal{N}(U)$, for some arbitrary $U \in \mathcal{U}(\mathcal{W})$. From the given definition of $\mathcal{N}$, two cases need to be considered. Note that the case in which

$$\exists\varphi\big(U \cap \mathcal{R}(\Gamma) = [\![\varphi]\!] \cap \mathcal{R}(\Gamma) \text{ and } \neg\varphi \in \Gamma\big)$$

is exactly the same as for Proposition 2.1.5. Thus, suppose that Γ is in $\mathcal{N}(U)$ because every negated formula is an element of Γ. Then, for the same reason, $\Gamma \in \mathcal{N}(U \cap \mathcal{R}(\Gamma))$, as desired. □

Proposition 2.2.4. *For the function* $\mathcal{N}$ *as defined in Definition 13,* **P2** *holds.*

Proof. Given a theory Γ, assume that Γ' is an extension of this set and that $\Gamma \in \mathcal{N}(U)$, for some upward closed subset of $\mathcal{W}$. Again, we would have to consider two cases. The situation in which

$$\exists\varphi\big(U \cap \mathcal{R}(\Gamma) = [\![\varphi]\!] \cap \mathcal{R}(\Gamma) \text{ and } \neg\varphi \in \Gamma\big)$$

[1] It is to be recalled and emphasized that, if we originally built the canonical model as suggested above Proposition 2.1.5, it would indeed be enough. All extensions of N will have proper canonical models [7].

goes exactly as in the Proposition 2.1.6. The other case follows easily from the fact that $\Gamma \subseteq \Gamma'$. Indeed, if Γ contains all the negated formulas, so does Γ'. $\square$

Proposition 2.2.5. *For the function $\mathcal{N}$ as defined in Definition 13, the property* NEF *holds.*

Proof. Consider an arbitrary $\Gamma \in \mathcal{W}$ such that $\Gamma \in U \cap \mathcal{N}(U)$, for some upset $U \subseteq \mathcal{W}$. Suppose that the reason why $\Gamma \in N(U)$ is

$$\exists\varphi\big(U \cap \mathcal{R}(\Gamma) = [\![\varphi]\!] \cap \mathcal{R}(\Gamma) \text{ and } \neg\varphi \in \Gamma\big).$$

On the other hand, we also know that $\Gamma \in U$, which implies that $\mathcal{R}(\Gamma) \subseteq U$. A consequence of this is that also $\mathcal{R}(\Gamma) \subseteq [\![\varphi]\!]$, i.e., $\varphi \in \Gamma$. Therefore, both φ and $\neg\varphi$ are elements of Γ. Since the set Γ is a theory, we have that every instance of the axiom NeF is an element of it, and hence $(\varphi \wedge \neg\varphi) \to \neg\psi \in \Gamma$, where ψ can be any formula. Again, from closure of Γ under logical consequence, $\varphi, \neg\varphi, (\varphi \wedge \neg\varphi) \to \neg\psi \in \Gamma$ gives us $\neg\psi \in \Gamma$ too, for every formula ψ. The definition of $\mathcal{N}$ ensures Γ to be in $\mathcal{N}(U')$ as well. The other case, i.e., the case in which Γ is in $\mathcal{N}(U)$ because it contains all the negated formulas, is trivial. $\square$

At this point, we are ready to state and prove the Truth Lemma for this system.

Lemma 2.2.6 (Truth Lemma)**.** *Given a theory Γ in the canonical model $\mathcal{M}_{\mathsf{NeF}}$, for every formula φ,*

$$\mathcal{M}_{\mathsf{NeF}}, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma.$$

Proof. We proceed by induction and we unfold only the negation case of the induction step. From left to right, assume that $\mathcal{M}_{\mathsf{NeF}}, \Gamma \vDash \neg\varphi$, i.e., $\Gamma \in \mathcal{N}(\mathcal{V}(\varphi))$. The definition of the function $\mathcal{N}$ leaves two options open: either

$$\exists\psi(\mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma) = [\![\psi]\!] \cap \mathcal{R}(\Gamma) \text{ and } \neg\psi \in \Gamma$$

or every negated formula $\neg\psi$ is an element of Γ. In the first case, the argument goes as the one for N (Lemma 2.1.7), the system NeF being an extension of N. On the other hand, if every negated formula is in Γ, then $\neg\varphi$ is in there too, as desired. For the right-to-left direction, the proof resembles exactly the one for the basic system N. $\square$

We conclude the section by stating the completeness result for the logic NeF. The proof goes as in the minimal logic case. In this section, we have proved that the system obtained by adding to N the negative *ex falso* axiom is sound and complete with respect to a class of frames which is a subclass of the N-frames. In particular, those frames validating NEF.

Theorem 2.2.7 (Completeness Theorem)**.** *Given a set Γ of formulas in* NeF *and an arbitrary formula φ,*

$$\Gamma \vDash \varphi \textit{ implies } \Gamma \vdash \varphi.$$

2.2.2 Contraposition Logic

The concluding part of this section and of the whole chapter wants to focus on the weak contraposition rule considered as an axiom. We again start characterizing the class of N-frames which validate CoPC. In this case, the resulting property satisfied by the function N is the so-called *anti-monotonicity*. In particular, it means that the function N reverses the set-theoretic inclusion ordering over the upsets. The section is concluded with the proofs of soundness and completeness of the system with respect to the considered class of frames, which indeed is a subclass of the class of N-frames.

Choosing a course resembling the path developed in the initial part of the section, we want to find a property which holds on a certain N-frame if and only if the contraposition rule $(p \to q) \to (\neg q \to \neg p)$ is valid on it. Considering a frame $\mathfrak{F} = \langle W, R, N \rangle$ and the upward closed subsets representing the valuations $V(p)$ and $V(q)$. The fact that the considered frame makes the implication $p \to q$ true semantically means $V(p) \subseteq V(q)$. In order to have also the consequent of the rule being true on $\mathfrak{F}$, we need the function N to reverse the given order, i.e., we need $N(V(q)) \subseteq N(V(p))$. The following proposition makes this reasoning formal.

Proposition 2.2.8. *Given an* N*-frame* $\mathfrak{F} = \langle W, R, N \rangle$*, the principle* CoPC *is valid on* $\mathfrak{F}$*, if and only if* N *has the following property:*

$$\forall U, U' \in \mathcal{U}(W) : U \subseteq U' \textit{ implies } N(U') \subseteq N(U). \quad (\textsc{anti-monotonicity})$$

Proof. For the left-to-right direction of the claim, we go by contraposition. In fact, given a certain frame $\mathfrak{F} = \langle W, R, N \rangle$, assume there exists a pair U, U' of upward closed subsets of W such that $U \subseteq U'$, but $N(U') \not\subseteq N(U)$. We equip the considered frame now with a propositional valuation V such that $V(p) = U$ and $V(q) = U'$, for some propositional variables p, q, and we denote the resulting model with $\mathfrak{M}$. By semantics, this induces the frame to make the implication $p \to q$ true. On the other hand, though, by assumption we get (at least) a world $w \in W$ such that $w \in N(U')$ while $w \notin N(U)$. This means $\mathfrak{M}, w \vDash \neg q$ although $\mathfrak{M}, w \nvDash \neg p$. Therefore, the considered frame does not make CoPC true. The other direction of the statement goes similarly. In fact, assume that ANTI-MONOTONICITY holds on some arbitrary frame $\mathfrak{F} = \langle W, R, N \rangle$. Consider a world $w \in W$ which, for some valuation V on $\mathfrak{F}$, makes $p \to q$ true. This indeed means that

$$V(p) \cap R(w) \subseteq V(q) \cap R(w). \quad (*)$$

Consider now a successor v of w, and assume $v \in N(V(q))$. The locality property of the N-frames gives us that $v \in N(V(q) \cap R(v))$. Moreover, $(*)$ implies

$$V(p) \cap R(v) \subseteq V(q) \cap R(v),$$

by associativity of $\cap$. Therefore, we can conclude by means of ANTI-MONOTONICITY that $v \in N(V(p) \cap R(v))$, which indeed gives us $v \in N(V(p))$ as required. $\square$

From now on, we use CoPC to denote the logical system obtained by adding to the axioms 1-8 (i.e., positive logic) the weak contraposition axiom,

$$(\boldsymbol{p} \rightarrow \boldsymbol{q}) \rightarrow (\neg \boldsymbol{q} \rightarrow \neg \boldsymbol{p}).$$

After having characterized the class of N-frames satisfying CoPC, we want to conclude the current chapter showing that CoPC is indeed sound and complete with respect to the class of 'anti-monotone' N-frames.

The proof of the soundness result simulates once again the intuitionistic one. Hence, we only state the result here. On the other hand, for completeness, we need to build a canonical model. The Lindenbaum Lemma as stated in Chapter 1 still allows us to build the model. Although in the case of the negative *ex falso* logic the definition of $\mathcal{N}$ in the canonical model extended the one for N, here a more radical modification of the definition is necessary.

Theorem 2.2.9 (Soundness Theorem)**.** *Given a set* Γ *of* CoPC*-formulas and an arbitrary formula* φ*,*

$$\Gamma \vdash \varphi \textit{ implies } \Gamma \vDash \varphi.$$

Definition 14 (Canonical Model)**.** *The canonical model for* CoPC *is the quadruple* $\mathcal{M}_{\mathsf{CoPC}} = \langle \mathcal{W}, \mathcal{R}, \mathcal{N}, \mathcal{V} \rangle$*, where* $\mathcal{W}$ *is the set of theories with the disjunction property, the accessibility relation* $\mathcal{R}$ *is the usual set-theoretic inclusion, and the valuation* $\mathcal{V}$ *is again defined as:* $\mathcal{V}(p) := \{\Gamma \in \mathcal{W} \mid p \in \Gamma\}$*. For every upset* $U \in \mathcal{U}(\mathcal{W})$*, the function* $\mathcal{N}$ *is defined as*

$$\mathcal{N}(U) := \{\Gamma \in \mathcal{W} \mid \forall \varphi (\llbracket \varphi \rrbracket \cap \mathcal{R}(\Gamma) \subseteq U \textit{ implies } \neg \varphi \in \Gamma)\},$$

where $\llbracket \varphi \rrbracket = \{\Gamma \in \mathcal{W} \mid \varphi \in \Gamma\}$*.*

A verification that the constructed model is indeed a CoPC-model is necessary. For this purpose, we show that the function $\mathcal{N}$ as just defined satisfies **P1**, **P2** and ANTI-MONOTONICITY. Before proceeding with the proofs, it is worth making a remark. Considering the above given definition of $\mathcal{N}$, every time we have $U \subseteq \mathcal{R}(\Gamma)$, for certain U and Γ, the definition is reduced to:

$$\Gamma \in \mathcal{N}(U) \Leftrightarrow \forall \varphi (\llbracket \varphi \rrbracket \subseteq U \text{ implies } \neg \varphi \in \Gamma).$$

This will soon turn out to be useful.

Proposition 2.2.10. *For the function* $\mathcal{N}$ *as defined in Definition 14,* **P1** *holds.*

Proof. Let Γ be an element of $\mathcal{N}(U)$, for some arbitrary $U \in \mathcal{U}(\mathcal{W})$. By definition, this is equivalent to say that Γ contains the negation of every formula φ such that

$$\llbracket \varphi \rrbracket \cap \mathcal{R}(\Gamma) \subseteq U.$$

Observe that every such a formula satisfies the equivalent property:

$$\llbracket \varphi \rrbracket \cap \mathcal{R}(\Gamma) \subseteq U \cap \mathcal{R}(\Gamma).$$

Hence, the assumption $\Gamma \in \mathcal{N}(U)$ turns out to be equivalent to $\Gamma \in \mathcal{N}(U \cap \mathcal{R}(\Gamma))$, as desired. $\square$

Here it is more convenient to show first that the function $\mathcal{N}$ is anti-monotone, and to use such a result for proving that **P2** holds.

Proposition 2.2.11. *For the function $\mathcal{N}$ as defined in Definition 14, the property of* ANTI-MONOTONICITY *holds.*

Proof. Let U and U' be two arbitrary upward closed subsets of $\mathcal{W}$ such that $U \subseteq U'$. An arbitrary Γ is in $\mathcal{N}(U')$ if and only if Γ contains the negation of every φ for which

$$[\![\varphi]\!] \cap \mathcal{R}(\Gamma) \subseteq U'$$

holds. Consider now a formula φ such that

$$[\![\varphi]\!] \cap \mathcal{R}(\Gamma) \subseteq U$$

holds instead. The assumption that $U \subseteq U'$ allows us to conclude

$$[\![\varphi]\!] \cap \mathcal{R}(\Gamma) \subseteq U \subseteq U'$$

and hence, $\neg\varphi \in \Gamma$. By definition of $\mathcal{N}$ we then conclude $\Gamma \in \mathcal{N}(U)$, which indeed is what we wanted. $\square$

Proposition 2.2.12. *For the function $\mathcal{N}$ as defined in Definition 14,* **P2** *holds.*

Proof. Given a theory Γ, assume that Γ' is an extension of Γ and that $\Gamma \in \mathcal{N}(U)$, for some upward closed $U \subseteq \mathcal{W}$. Proposition 2.2.10 tells us that $\Gamma \in \mathcal{N}(U \cap \mathcal{R}(\Gamma))$ holds as well. By simple set-theoretic properties we have that

$$U \cap R(\Gamma') \subseteq U \cap R(\Gamma)$$

and, by means of Proposition 2.2.11, we are allowed to conclude

$$\mathcal{N}(U \cap \mathcal{R}(\Gamma)) \subseteq \mathcal{N}(U \cap \mathcal{R}(\Gamma')).$$

Hence, $\Gamma \in \mathcal{N}(U \cap \mathcal{R}(\Gamma'))$ as well. Indeed,

$$U \cap \mathcal{R}(\Gamma') \subseteq \mathcal{R}(\Gamma).$$

The remark made above Proposition 2.2.10 gives us

$$\forall\varphi\big([\![\varphi]\!] \subseteq U \cap \mathcal{R}(\Gamma') \text{ implies } \neg\varphi \in \Gamma\big)$$

and hence, $\neg\varphi \in \Gamma'$. We conclude in this way that $\Gamma' \in \mathcal{N}(U \cap \mathcal{R}(\Gamma'))$ and hence, from Proposition 2.2.10, $\Gamma' \in \mathcal{N}(U)$, as desired. $\square$

At this point, we are entitled to state and prove the Truth Lemma.

Lemma 2.2.13 (Truth Lemma)**.** *Given a theory* Γ *in the canonical model* $\mathcal{M}_{\mathsf{CoPC}}$*, for every formula* φ*,*

$$\mathcal{M}_{\mathsf{CoPC}}, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma.$$

Proof. The proof will go by induction. We exhibit the negation case of the induction step. From left to right, assume that $\mathcal{M}_{\mathsf{CoPC}}, \Gamma \vDash \neg\varphi$, i.e., $\Gamma \in \mathcal{N}(\mathcal{V}(\varphi))$. Observe that, by definition of $\mathcal{N}$,

$$\forall \psi \big([\![\psi]\!] \cap \mathcal{R}(\Gamma) \subseteq \mathcal{V}(\varphi) \text{ implies } \neg\psi \in \Gamma \big).$$

The induction hypothesis gives us

$$[\![\psi]\!] \cap \mathcal{R}(\Gamma) = \mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma) \subseteq \mathcal{V}(\varphi)$$

and hence, $\neg\varphi \in \Gamma$. It remains to prove the other direction of the statement. For this purpose, assume $\neg\varphi \in \Gamma$. Recall that, by the induction hypothesis, we have the identity

$$[\![\varphi]\!] = \mathcal{V}(\varphi).$$

We prove the following claim.

Claim: For any formula ψ, if $[\![\psi]\!] \cap \mathcal{R}(\Gamma) \subseteq \mathcal{V}(\varphi)$, then $\neg\psi \in \Gamma$ holds.

Indeed, we can see that under this assumption, $\psi \to \varphi \in \Gamma$. We use the Lindenbaum Lemma to see this. Suppose $\psi \to \varphi \notin \Gamma$. Then, there exists an extension Γ' of Γ in the canonical model, such that $\psi \in \Gamma'$, while $\varphi \notin \Gamma'$ $(*)$. This gives us that $\Gamma' \in [\![\psi]\!]$, in addition to the fact that $\Gamma' \in \mathcal{R}(\Gamma)$. By assumption, this leads to $\Gamma' \in [\![\varphi]\!]$, which means $\varphi \in \Gamma'$ and contradicts $(*)$. Therefore, we conclude $\psi \to \varphi \in \Gamma$. The set Γ being a theory implies that the instance $(\psi \to \varphi) \to (\neg\varphi \to \neg\psi)$ of CoPC is in Γ as well. Thus, $\neg\varphi \to \neg\psi$ is an element of Γ and, given that $\neg\varphi \in \Gamma$, *modus ponens* gives us $\neg\psi \in \Gamma$, as desired.

The claim, together with the definition of the function $\mathcal{N}$, allows us to conclude the desired result: $\Gamma \in \mathcal{N}(\mathcal{V}(\varphi))$, i.e., $\mathcal{M}_{\mathsf{CoPC}}, \Gamma \vDash \neg\varphi$. □

We state now the completeness result for the system CoPC. The proof will go as in the minimal logic case. The logic axiomatized by the weak contraposition rule is sound and complete with respect to the class of N-frames whose function N is anti-monotone.

Theorem 2.2.14 (Completeness Theorem)**.** *Given a set* Γ *of formulas in* CoPC *and an arbitrary formula* φ*,*

$$\Gamma \vDash \varphi \textit{ implies } \Gamma \vdash \varphi.$$

The results presented in this chapter give us a Kripke-style semantics for our main subminimal systems. In the next chapter we use such semantics for proving important properties of our logics.

Observing the notions of Kripke frames and models presented in this chapter, a natural further question to be asked is the following: how does the negation operator (and hence, the N function) behave on *subframes* and *submodels*? In particular, the analysis developed here takes care only of generated subframes and submodels. Which kind of properties and conditions on negation will be preserved in the case of subframes?

Another intriguing possibility is focusing on the linear (or at least upward linear) subclass of the considered class of Kripke frames. This kind of analysis has indeed been started already [14], although no results concerning this approach have been included in this thesis. One interesting aspect of this option is related to the fact that, given a finite linear Kripke frame, each upward closed subset of it is uniquely and completely determined by its root. This suggests an alternative approach: in order to talk about a certain upset, it is sufficient to talk about the node generating it. As far as the study has been carried out, we can claim that this gives us much more information about the behavior of the function N in relation to a fixed upset of the frame (which we may call the *set of abnormal worlds*).

Chapter 3

Finite Models and the Disjunction Property

From a practical and computational point of view, a system characterized by means of finite frames and models is more suitable. Let us say that a formula is satisfiable whenever there exist a model and a world in this model which makes it true. In checking whether a formula is satisfiable or not in a system, having to deal only with finite frames makes our life easier (e.g., it usually makes logics decidable). In this chapter we give proofs of the fact that the considered subsystems of minimal logic have indeed the so-called *finite model property*: if a formula is satisfiable on an arbitrary model, then it is satisfiable on a finite model. We will present and discuss two methods for building finite models for satisfiable formulas.

Given that we are working in a setting which is essentially intuitionistic, a property which seems natural to be considered is the *disjunction property*. The disjunction property says that whenever we have a theorem of the form $A \vee B$, then we can conclude A or B to be a theorem too. In this chapter, a semantic proof of this property for all our subsystems is given. The proof resembles closely the usual intuitionistic proof of the property. Some preservation and invariance results presented in Chapter 2 are used, namely the ones concerning generated submodels and disjoint unions.

3.1 Finite Model Property

The basic system of a unary operator N not only behaves semantically well (in the sense that it is complete), but it also has a finite semantic characterization: the logic N represents the set of formulas globally forced by the class of finite frames whose function N satisfies **P1** and **P2**. Given a formula φ, let $Sub(\varphi)$ denote the set of subformulas of φ. The main notion used here is the one of *adequate set* of formulas. It consists of a 'well-behaved' set, in the following sense.

Definition 15 (Adequate Set). *A set of formulas Φ is said to be an* adequate set *if it is closed under subformulas, i.e., for every $\varphi \in \Phi$, the set $Sub(\varphi)$ is a subset of Φ.*

In what follows of this section, we use the symbol Φ to denote a *finite* adequate set. The proof of the finite model property by means of adequate sets resembles very much the proof of completeness. It requires a couple of fundamental lemmas. We first give the whole proof for the basic system N, and later we extend that proof to both contraposition and negative *ex falso* logics. For the sake of simplicity, we sometimes use FMP to denote the finite model property.

Lemma 3.1.1. *Let* Γ *be a set of formulas and* φ *be a formula such that* $\Gamma \nvdash \varphi$ *in* N *and* $\Gamma \cup \{\varphi\} \subseteq \Phi$. *There exists a set* $\Delta \subseteq \Phi$ *with the disjunction property, extending* Γ *and which does not contain* φ. *In addition,* Δ *is closed under logical* Φ*-consequence, meaning that*

$$\forall \psi \in \Phi\text{: if } \Delta \vdash \psi, \text{ then } \psi \in \Delta.$$

Proof. Consider an enumeration $\varphi_1, \varphi_2, \ldots, \varphi_m$ of the elements of Φ. We construct the required set Δ by stages. Define:

- $\Delta_0 = \Gamma$
- $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \nvdash \varphi \\ \Delta_n, & \text{otherwise} \end{cases}$

The required set of formulas will be $\Delta = \bigcup_{n \leq m} \Delta_n$. We still need to check that it satisfies the requirements. First of all, observe that $\Gamma = \Delta_0 \subseteq \Delta$. And also, $\varphi = \varphi_n$ for some $n \leq m$. Thus, since $\varphi \vdash \varphi$, the formula φ has not been added during the construction. Hence, $\varphi \notin \Delta$. Moreover, $\Delta \nvdash \varphi$. In fact, if we suppose the opposite, i.e., $\Delta \vdash \varphi$, we get that there are finitely many formulas in Δ which represent the assumptions in a proof of φ. Let $n \leq m$ be the maximum among the indices of these formulas. Thus, $\Delta_n \cup \{\varphi_n\} \vdash \varphi$, which contradicts the fact that $\varphi_n \in \Delta$. Therefore, we already get that $\Delta \nvdash \varphi$.

Consider now $\psi \in \Phi$ such that $\Delta \vdash \psi$. Since $\psi \in \Phi$, the formula ψ appears at some stage of the enumeration, i.e., $\psi = \varphi_n$ for some $n \leq m$. Suppose $\Delta_n \cup \{\varphi_n\} \vdash \varphi$. This implies $\Delta_n \vdash \varphi_n \to \varphi$ and hence, $\Delta_n \vdash \varphi$. Thus, $\Delta \vdash \varphi$, which is a contradiction. Therefore, $\psi \in \Delta$.

Finally, it remains to be proved that Δ satisfies the disjunction property. Suppose not, i.e., suppose there exists $(\psi_1 \vee \psi_2) \in \Delta$, such that $\psi_1 \notin \Delta$ and $\psi_2 \notin \Delta$. Since $\psi_1 \vee \psi_2 \in \Phi$, also $\psi_1 \in \Phi$ and $\psi_2 \in \Phi$ by closure under subformulas. This implies these two formulas to appear at some stages of the enumeration, which means that there exist $n_1, n_2 \leq m$ such that $\psi_1 = \varphi_{n_1}$ and $\psi_2 = \varphi_{n_2}$. Since neither ψ_1 nor ψ_2 is in Δ, this means that $\Delta \cup \{\psi_1\} \vdash \varphi$ and $\Delta \cup \{\psi_2\} \vdash \varphi$. Hence, $\Delta \cup \{\psi_1 \vee \psi_2\} \vdash \varphi$, which is a contradicts the fact that $\Delta \nvdash \varphi$. $\square$

As the reader may have seen, the lemma just proved is a version of the Lindenbaum Lemma. The next step consists of building a finite 'canonical model' using the sets of formulas whose existence is ensured by the previous lemma. We use the notion of set of formulas *closed under logical* Φ*-consequence* to denote a set Δ such that, for every $\varphi \in \Phi$, if $\Delta \vdash \varphi$, then $\varphi \in \Delta$.

Definition 16. *Let* $\mathcal{M}_\Phi$ *be an* N *finite model built in the following way. The set of worlds, denoted as* $\mathcal{W}_\Phi$, *is the set of subsets* Γ *of* Φ *with the disjunction property which are closed under logical* Φ*-consequence. As a relation, we again* *consider the standard set-theoretic inclusion, and we denote it as $\mathcal{R}_\Phi$. Also the propositional valuation $\mathcal{V}_\Phi$ is defined as in the canonical model, i.e.,*

$$\mathcal{V}_\Phi(p) := \{\Gamma \in \mathcal{W}_\Phi \mid p \in \Gamma\}.$$

The function $\mathcal{N}_\Phi$ is defined again as in the completeness proof, for every upward closed subset U of $\mathcal{W}_\Phi$,

$$\mathcal{N}_\Phi(U) := \{\Gamma \in \mathcal{W}_\Phi \mid \exists \neg\varphi \in \Phi\big(U \cap \mathcal{R}(\Gamma) = \llbracket\varphi\rrbracket \cap \mathcal{R}(\Gamma) \text{ and } \neg\varphi \in \Gamma\big)\},$$

where $\llbracket\varphi\rrbracket = \{\Gamma \in \mathcal{W}_\Phi \mid \varphi \in \Gamma\}$.

Observe that, in defining the function $\mathcal{N}_\Phi$, the quantifiers range over the negated formulas *in* Φ. It is immediate to check that $\mathcal{M}_\Phi$ is indeed an N-model (i.e., that $\mathcal{N}_\Phi$ satisfies **P1** and **P2**). In addition, $\mathcal{M}_\Phi$ turns out to be a finite model, being a subset of the power-set $\mathcal{P}(\Phi)$ of a finite set.

As in the completeness proof, we want the forcing relation in the model $\mathcal{M}_\Phi$ to coincide with the membership one. We state here an equivalent of the Truth Lemma.

Lemma 3.1.2. *Given a set $\Gamma \in \mathcal{M}_\Phi$ and an arbitrary formula $\varphi \in \Phi$,*

$$\mathcal{M}_\Phi, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma.$$

Proof. We use induction on the structure of the formula φ in Φ. The proof resembles exactly the one for the Truth Lemma, i.e., Lemma 2.1.7. Only, everything needs to be restricted to the set Φ. $\square$

We are ready to prove the main result. We use the term *countermodel for a formula* φ to denote a model in which there is a world which does not make φ true.

Theorem 3.1.3 (Finite Model Property)**.** *For every non-theorem of* N*, i.e., for every formula φ such that $\mathsf{N} \nvdash \varphi$, there exists a finite* N*-model which is a countermodel for φ.*

Proof. Assume $\mathsf{N} \nvdash \varphi$. Consider a finite adequate set Φ such that $\varphi \in \Phi$, and consider the model $\mathcal{M}_\Phi$. By Lemma 3.1.1, we know that there is a set $\Delta \subseteq \Phi$ with the disjunction property and closed under logical Φ-consequence, such that $\varphi \notin \Delta$. By definition of the model $\mathcal{M}_\Phi$, the set of formulas Δ is a node of $\mathcal{M}_\Phi$. This implies $\mathcal{M}_\Phi, \Delta \nvDash \varphi$ and gives us a finite countermodel for φ, namely the model $\mathcal{M}_\Phi$ $\square$

Both negative *ex falso* logic and contraposition logic have the FMP and the proofs go the same way as the one for the basic system N. The main difference is given by the definition of the model $\mathcal{M}_\Phi$. In particular, we have to take care of defining the function $\mathcal{N}_\Phi$ so to make it verify the respective required properties. We first modify the previous definition of $\mathcal{M}_\Phi$ such that the resulting model is indeed a NeF-model. Later, we follow the same procedure and we get a CoPC-model. The reader may observe that, as it happened in the case of N, the way in which the functions $\mathcal{N}_\Phi$ are defined consists of 'relativising' the canonical model $\mathcal{N}$ to the set Φ.

Let us build the finite canonical model for the negative *ex falso* logic. Let $\mathcal{M}_\Phi$ be such that the set of possible worlds $\mathcal{W}_\Phi$, the relation $\mathcal{R}_\Phi$ and the valuation function $\mathcal{V}_\Phi$ are defined exactly as in the case of N. The definition of the function $\mathcal{N}_\Phi$ needs to be refined as follows. For every upward closed subset U of $\mathcal{W}_\Phi$,

$$\mathcal{N}_\Phi(U) := \{\Gamma \in \mathcal{W}_\Phi \mid \exists \neg\varphi \in \Phi\big(U \cap \mathcal{R}(\Gamma) = [\![\varphi]\!] \cap \mathcal{R}(\Gamma) \text{ and } \neg\varphi \in \Gamma\big)$$

$$\text{or } \forall \neg\varphi \in \Phi(\neg\varphi \in \Gamma)\},$$

where $[\![\varphi]\!] = \{\Gamma \in \mathcal{W}_\Phi \mid \varphi \in \Gamma\}$. Proceeding similarly to the proofs of Propositions 2.2.3, 2.2.4 and 2.2.5, it is easy to check that the defined model turns out to be a NeF-model. At this point, an equivalent of Lemma 3.1.2 is required. We do not unfold the proof here, given that it goes in the same way as the proof of the Truth Lemma for NeF. It is enough to be careful to restrict every statement about formulas to elements of Φ.

Lemma 3.1.4. *Given a set $\Gamma \in \mathcal{M}_\Phi$ and an arbitrary formula $\varphi \in \Phi$,*

$$\mathcal{M}_\Phi, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma.$$

The finite canonical model allows us to claim and prove the finite model property for the considered negative *ex falso* logic.

Theorem 3.1.5 (Finite Model Property)**.** *For every* non-theorem *of* NeF*, i.e., for every φ such that $\mathsf{NeF} \nvdash \varphi$, there exists a finite countermodel for φ.*

Proof. The proof goes exactly the same as for N. □

We dedicate the last part of this section to contraposition logic. The proof of the finite model property follows exactly the same path as the previous ones. It is necessary to build a finite canonical model for CoPC. Since the definition of the $\mathcal{N}_\Phi$ function again resembles the one given in the completeness proof, we do not give the details of the proofs.

Consider the model $\mathcal{M}_\Phi$ defined as before, substituting the definition of $\mathcal{N}_\Phi$ with the following. Given an upset $U \subseteq \mathcal{W}_\Phi$,

$$\mathcal{N}_\Phi(U) := \{\Gamma \in \mathcal{W}_\Phi \mid \forall \neg\varphi \in \Phi\big([\![\varphi]\!] \cap \mathcal{R}(\Gamma) \subseteq U \text{ implies } \neg\varphi \in \Gamma\big)\}.$$

Using the same reasoning and proofs as in the completeness case, it is straightforward to check that $\mathcal{M}_\Phi$ is here a CoPC-model, in which the relations of membership and truth coincide. Therefore, we conclude the section stating the main result also for CoPC.

Theorem 3.1.6 (Finite Model Property)**.** *For every* non-theorem *of* CoPC*, i.e., for every φ such that $\mathsf{CoPC} \nvdash \varphi$, there exists a finite countermodel for φ.*

Proof. The proof goes exactly the same as for N. □

3.1.1 Decidability via Finite Models

We say that a subminimal logic is *decidable* if the *satisfiability* (and, equivalently, the *validity*) problem of that logic is decidable. A subminimal logic is said to be *undecidable* if it is not decidable. The finite model property turns out to be a useful step for proving decidability of a certain logic. Roughly, the idea is the following. We know that all of our subminimal systems have the finite model property with respect to some class of models. Given a certain logic, we can consider a Turing machine that *recursively enumerates* all the finite models of the considered class. On the other hand, we can build a Turing machine that recursively enumerates all the proofs, by means of the axiomatization of the logic. At this point, we are in the following situation: given a formula φ, either φ is the conclusive sequent of one of the proofs generated by the second Turing machine (and hence, it is a validity[1]) or we can find a countermodel for φ among the models generated by the first Turing machine.

We state here the result as a straightforward consequence of the finite model property proved above. The structure of the proof is standard and the reader may refer to standard manuals for the details (e.g., [7]).

Theorem 3.1.7 (Decidability)**.** *The logical systems* N*,* NeF *and* CoPC *are decidable.*

3.2 Disjunction Property

In Chapter 2, some preservation and invariance results have been proved. In the current section we make use of some of those results in proving a fundamental property in a constructive setting: the disjunction property. We recall that a logic L satisfies the *disjunction property* (DP) if and only if,

$$\text{if } \vdash_{\mathsf{L}} \varphi \vee \psi, \text{ then } \vdash_{\mathsf{L}} \varphi \text{ or } \vdash_{\mathsf{L}} \psi.$$

The standard semantic technique (employed also in intuitionistic propositional logic, see [4]) for proving the disjunction property goes by contrapositive and uses complteness. Indeed, assuming the consequent of the implication to be false implies (by completeness) the existence of two countermodels $\mathfrak{M}$ and $\mathfrak{M}'$ for, respectively, φ and ψ. Lemmas 2.1.2 and 2.1.3 allow us to build a countermodel for the disjunctive formula $\varphi \vee \psi$ which, by soundness, leads to a negation of the antecedent. The proof ends up being very similar to the proof one would do in IPC. Nonetheless, when building the countermodel for the disjunction, we need to move another step to determine how to extend the function N properly.

Let $\Sigma = \{\mathfrak{M}_i \mid i \in I\}$ be a finite set of N-models and let $\langle W_\Sigma, R_\Sigma, N_\Sigma \rangle$ denote the frame underlying the disjoint union of Σ. The following result gives us the right to extend the disjoint union of Σ by adding a new 'root', i.e., a world w_0 which is a predecessor of every other world in W_Σ. In an intuitionistic propositional setting, we need to take care of extending the partial order in the obvious

[1] Observe that one has to assume also soundness and completeness with respect to the considered semantics.

way and, when dealing with models, of enhancing the propositional valuation preserving persistence. Here, when dealing with N-models, the function N_Σ has to be extended to a new function which still satisfies **P1** and **P2**.

Theorem 3.2.1. *If $\Sigma = \{\mathfrak{M}_i \mid i \in I\}$ is a set of N-models, a new N-model can be obtained by taking the disjoint union of Σ, adding a new root w_0 below it and taking*

$$N \upharpoonright W_\Sigma = N_\Sigma,$$

$$N(W) = \emptyset,$$

where $W = W_\Sigma \cup \{w_0\}$.[2]

Proof. The only thing to be proved is that N, as defined in the statement of the theorem, satisfies **P1** and **P2**. In particular, it is not relevant how the truth relation of w_0 is defined, as long as it does not violate persistency. The proof that the properties hold can be reduced to the proof that the disjoint union is well defined (i.e., Lemma 2.1.3). In fact, the root w_0 is not an element of any $N(U)$, and moreover, the newly added $N(W)$ has no elements. Hence, saying $v \in N(U)$ for some upset U is equivalent to saying $U \subseteq W_\Sigma$ and $v \in N_\Sigma(U)$. $\square$

Corollary 3.2.2 (Disjunction Property)**.** *For every pair of formulas φ, ψ, if the disjunctive formula $\varphi \vee \psi$ is an N-theorem, then φ is a theorem of N or ψ is a theorem of N.*

Proof. As anticipated, the proof goes by contraposition. Assume that neither φ nor ψ is a theorem in the N-system, i.e., $\nvdash_{\mathsf{N}} \varphi$ and $\nvdash_{\mathsf{N}} \psi$. Thus, by completeness, we know there exist models $\mathfrak{M}$ and $\mathfrak{M}'$ and nodes $w \in \mathfrak{M}$ and $w' \in \mathfrak{M}'$, such that $\mathfrak{M}, w \nVdash \varphi$ and $\mathfrak{M}', w' \nVdash \psi$. Let $\mathfrak{N}$ be the model obtained from $\mathfrak{M}$ and $\mathfrak{M}'$ as described in Theorem 3.2.1. In order not to violate persistence, the propositional valuation needs to be extended in such a way that neither φ nor ψ can be forced in w_0. Therefore, $\mathfrak{N}$ is a countermodel for $\varphi \vee \psi$. Hence, soundness gives us $\nvdash_{\mathsf{N}} \varphi \vee \psi$. $\square$

We go on, proving the same result(s) for contraposition logic, first, and later for negative *ex falso* logic. The reason why we prove immediately such a result for CoPC is because the analogue of Theorem 3.2.1 happens to be exactly the same as in N. There, the way in which N_Σ is extended satisfies in the most natural way the property of ANTI-MONOTONICITY: the biggest upward closed subset is sent by N to the smallest one. The formalization of this is given in the proof of the following result.

Theorem 3.2.3. *If $\Sigma = \{\mathfrak{M}_i \mid i \in I\}$ is a set of CoPC-models, a new CoPC-model can be obtained by taking the disjoint union of Σ adding a new root w_0 below it and taking:*

$$N \upharpoonright W_\Sigma = N_\Sigma,$$

[2] The reader may observe that in Theorem 3.2.1 we do not specify how to extend the valuation V_Σ. Indeed, there are several ways to extend this map, as long as persistence is preserved. Note also that there is more than one way to extend N_Σ as well. Nonetheless, the one presented in Theorem 3.2.1 is eventually the easier way to do it and hence, it allows us to get a result holding *for every* Σ.

$$N(W) = \emptyset,$$

where $W = W_\Sigma \cup \{w_0\}$.

Proof. It is enough to show that extending N in this way preserves ANTI - MONOTONICITY. Assume $U \subseteq U' \in \mathcal{U}(W)$ and suppose $U' \neq W$. Thus, $U, U' \subseteq W_\Sigma$ and we already know that the thesis holds from Lemma 2.1.3. Assume now $U' = W$. Note that, in this case, for each U, the upset $U \subseteq U'$. Since $N(U') = N(W) = \emptyset$ by definition, $N(U') \subseteq N(U)$ for every X and hence we are done. □

Corollary 3.2.4 (Disjunction Property). *For every pair of formulas φ, ψ, if the disjunctive formula $\varphi \vee \psi$ is an* CoPC*-theorem, then φ is a theorem of* CoPC *or ψ is a theorem of* CoPC*.*

Proof. The proof goes exactly the same as the one for N. □

The first part of this section finds its conclusion with the proof of DP for negative *ex falso* logic. The definition of N in this case turns out to be more refined. As a matter of fact, we define the extension of N_Σ in the minimal way to make the resulting N satisfy NEF.

Theorem 3.2.5. *If $\Sigma = \{\mathfrak{M}_i \mid i \in I\}$ is a set of* NeF*-models, a new* NeF*-model can be obtained by taking the disjoint union of Σ adding a new root w_0 below it and taking:*

$$N \restriction W_\Sigma = N_\Sigma,$$

$$N(W) := \{w \in W \mid w \in (V \cap N(V)) \textit{ for some } V \in \mathcal{U}(W_\Sigma)\},$$

where $W = W_\Sigma \cup \{w_0\}$.

Proof. For proving that N satisfies locality, it is enough to make sure that

$$w \in N(W) \Leftrightarrow w \in N(W \cap R(w)).$$

The rest is ensured by Lemma 2.1.3. Note that $w \neq w_0$. Hence, we have:

$$w \in N(W) \Leftrightarrow w \in V \cap N(V) \text{ for some } V \in \mathcal{U}(W_\Sigma).$$

Thus, given that the disjoint union of Σ gives a NeF-model, for each $U \in \mathcal{U}(W_\Sigma)$ we have $w \in N(U) = N_\Sigma(U)$. Since

$$W \cap R(w) = R(w) = R_\Sigma(w),$$

we have $w \in N(W \cap R(w)) = N_\Sigma(W \cap R(w))$, as desired. Let now

$$w \in N(W \cap R(w)) = N(R(w)).$$

From the fact that $w \in N(R(w))$ and also $w \in R(w)$, we can conclude $w \in N(W)$. We show now that the defined N is also persistent. Let $w \in N(U)$ and wRv. Again, $w \neq w_0$. Assume now $U \neq W$. In this case, by Lemma 2.1.3 we are done.

Next suppose $U = W$ instead. If $w \in N(W)$, this means $w \in V \cap N(V)$ for some $V \in \mathcal{U}(W_\Sigma)$. Given that both V and $N(V)$ are upsets, $v \in V \cap N(V)$ as well. Therefore, $v \in N(W)$. Finally, in order to prove NEF, assume $w \in U \cap N(U)$ for some $U \neq W$. By Lemma 2.1.3, the only thing we need to check is that $w \in N(W)$. Indeed, this holds by definition of N. On the other hand, assume $U = W$. Hence, by definition of $N(W)$, there exists $V \in \mathcal{U}(W_\Sigma)$ such that $w \in V \cap N(V)$. Again, this allows us to conclude $w \in N(U')$ for every $U' \in \mathcal{U}(W_\Sigma)$. Moreover, since $w \in N(W)$, we are done. □

Corollary 3.2.6 (Disjunction Property)**.** *For every pair of formulas φ, ψ, if the disjunctive formula $\varphi \vee \psi$ is an* NeF*-theorem, then φ is a theorem of* NeF *or ψ is a theorem of* NeF.

Proof. The proof is the same as the one for N. □

The given proofs can be generalized to a stronger version of the disjunction property. In particular, it is not necessary to assume the set of assumptions from which $\varphi \vee \psi$ is derived to be empty. It is enough to require such a set not to contain disjunctive formulas. We are going to see a syntactic proof of this generalized result in Chapter 5.

3.2.1 Slash Relation

In 1962 [28], while working on a modified notion of realizability, Kleene became aware of having obtained an inductive definition of a property of formulas which could be used to give and generalize proofs of results such as the disjunction property. This property is expressed as a relation between a set of formulas Γ and a formula φ, and is denoted as $\Gamma|\varphi$. Following the common use and, in particular, following [42], we refer to such a relation as *slash*. Although the original Kleene's definition was not closed under deduction, this issue was solved by Aczel [1] in 1968, and since then, the slash operator has been widely used to prove mathematical results within intuitionistic systems. Our general setting being intuitionistic, we want to use the slash operator to prove some interesting results.

In this section, we assume our systems to be defined by means of *axiom schemes* without the substitution rule, given that the two formulations are equivalent.

Definition 17 (Slash)**.** *Given a set of formulas Γ, we define the* slash *relation $\Gamma|\varphi$ inductively on the structure of φ, as follows:*

- $\Gamma|p$ *iff* $\Gamma \vdash p$;
- $\Gamma|\neg\varphi$ *iff* $\Gamma \vdash \neg\varphi$;
- $\Gamma|\varphi \wedge \psi$ *iff* $\Gamma|\varphi$ *and* $\Gamma|\psi$;
- $\Gamma|\varphi \vee \psi$ *iff* $\Gamma|\varphi$ *or* $\Gamma|\psi$;
- $\Gamma|\varphi \to \psi$ *iff* $\Gamma \vdash \varphi \to \psi$ *and (not* $\Gamma|\varphi$ *or* $\Gamma|\psi$*).*

In order to prove the main result of this section, we first need to claim and prove two theorems.

Theorem 3.2.7. *If* $\Gamma|\varphi$*, then* $\Gamma \vdash_{\mathsf{N}} \varphi$*.*

Proof. The proof goes by induction on the structure of φ. □

Theorem 3.2.8. *If* $\Gamma \vdash_{\mathsf{N}} \varphi$ *and* $\Gamma|\chi$*, for each* $\chi \in \Gamma$*, then* $\Gamma|\varphi$*.*

Proof. We argue by induction on the depth of the derivation $\Gamma \vdash_{\mathsf{N}} \varphi$. We only consider the case in which φ is the N axiom $(\varphi \leftrightarrow \psi) \to (\neg\varphi \leftrightarrow \neg\psi)$. Assume $\Gamma \vdash_{\mathsf{N}} (\varphi \leftrightarrow \psi) \to (\neg\varphi \leftrightarrow \neg\psi)$, and suppose for the sake of a contradiction that $\Gamma|(\varphi \leftrightarrow \psi) \to (\neg\varphi \leftrightarrow \neg\psi)$ does not hold, which means $\Gamma|\varphi \leftrightarrow \psi$ and not $\Gamma|\neg\varphi \leftrightarrow \neg\psi$. Now, the fact that $\Gamma|\neg\varphi \leftrightarrow \neg\psi$ is not valid means, without loss of generality, that $\Gamma|\neg\varphi \to \neg\psi$ is not true. By Definition 17, two options need to be considered. First, consider the case in which $\Gamma \nvdash_{\mathsf{N}} \neg\varphi \to \neg\psi$. Observe that the facts that $\Gamma \vdash_{\mathsf{N}} (\varphi \leftrightarrow \psi) \to (\neg\varphi \leftrightarrow \neg\psi)$ by assumptions, and $\Gamma \vdash_{\mathsf{N}} \varphi \leftrightarrow \psi$, give us a contradiction. On the other hand, suppose $\Gamma|\neg\varphi$ and not $\Gamma|\neg\psi$. Now, Theorem 3.2.7 gives us that $\Gamma|\neg\varphi$ implies $\Gamma \vdash_{\mathsf{N}} \neg\varphi$. Since $\Gamma \vdash_{\mathsf{N}} \neg\varphi \leftrightarrow \neg\psi$, we can conclude $\Gamma \vdash_{\mathsf{N}} \neg\psi$ by *modus ponens*, and therefore, $\Gamma|\neg\psi$. This contradicts our assumption and allows us to conclude $\Gamma|(\varphi \leftrightarrow \psi) \to (\neg\varphi \leftrightarrow \neg\psi)$. □

This is the right place to recall the notion of *admissible rule.*

Definition 18 (Admissible Rule)**.** *An inference rule* Γ/Δ *is said to be* admissible *in a logic* L *if for every substitution* σ*:*

$$\vdash_{\mathsf{L}} \sigma\alpha \text{ for every } \alpha \in \Gamma, \text{ then } \vdash_{\mathsf{L}} \sigma\varphi \text{ for some } \varphi \in \Delta.$$

The following statement allows us conclude that the rule

$$\neg\varphi \to \psi \vee \chi/(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$$

is admissible in N.

Corollary 3.2.9. *Let* φ*,* ϑ*,* ψ *and* χ *be arbitrary formulas.*

(a) If $\vartheta|\vartheta$*, then for every* ψ*,* χ*:*

$$\text{if } \vdash_{\mathsf{N}} \vartheta \to \psi \vee \chi, \text{ then } \vdash_{\mathsf{N}} \vartheta \to \psi \text{ or } \vdash_{\mathsf{N}} \vartheta \to \chi.$$

(b) If $\vdash_{\mathsf{N}} \neg\varphi \to \psi \vee \chi$*, then* $\vdash_{\mathsf{N}} \neg\varphi \to \psi$ *or* $\vdash_{\mathsf{N}} \neg\varphi \to \chi$*.*

Proof. The proof of item (a) goes for reductio. Suppose $\vartheta|\vartheta$ and $\vdash_{\mathsf{N}} \vartheta \to \psi \vee \chi$, and suppose not $\vdash_{\mathsf{N}} \vartheta \to \psi$ and not $\vdash_{\mathsf{N}} \vartheta \to \chi$. This means, by Deduction Theorem, that $\vartheta \nvdash_{\mathsf{N}} \psi$ and $\vartheta \nvdash_{\mathsf{N}} \chi$. Now we use Theorem 3.2.7, and we get that neither $\vartheta|\psi$ nor $\vartheta|\chi$ holds. Definition 17 says that this means that $\vartheta|\psi \vee \chi$ does not hold. Now, since $\vartheta|\vartheta$ by assumptions, we conclude by Theorem 3.2.8 that $\vartheta \nvdash_{\mathsf{N}} \psi \vee \chi$ which again from Deduction Theorem, gives us $\nvdash_{\mathsf{N}} \vartheta \to \psi \vee \chi$. This is a contradiction with our assumption, and hence we get the thesis.

At this point, in order to prove item (b), we make use of item (a) and hence, we first need to check that $\neg\varphi|\neg\varphi$. Since $\neg\varphi \vdash_{\mathsf{N}} \neg\varphi$, we can immediately apply item (a) and conclude the desired thesis. $\square$

Although in this section we have been mainly talking about N, the results stated and proved are indeed valid for the extensions of N as well. The only proof we need to unfold again is the one of the analogue of Theorem 3.2.8. Everything else goes exactly the same way as in the case of N. In the following results, the symbol '$\vdash$' refers either to derivations within NeF or to derivations within CoPC.

Theorem 3.2.10. *If $\Gamma \vdash \varphi$ and $\Gamma|\chi$, for each $\chi \in \Gamma$, then $\Gamma|\varphi$.*

Proof. We start by assuming that φ is a NeF axiom. Note that we already proved the result for $(\varphi \leftrightarrow \psi) \to (\neg\varphi \leftrightarrow \neg\psi)$. Hence, let φ be the negative *ex falso* axiom $(\varphi \wedge \neg\varphi) \to \neg\psi$. Assume the axiom to be a logical consequence of Γ, and suppose for reductio that $\Gamma|(\varphi\wedge\neg\varphi) \to \neg\psi$ does not hold. Hence, $\Gamma|\varphi\wedge\neg\varphi$, while $\Gamma|\neg\psi$ does not hold. Observe that $\Gamma|\varphi\wedge\neg\varphi$ implies $\Gamma \vdash_{\mathsf{NeF}} \varphi\wedge\neg\varphi$. This gives us that $\Gamma \vdash_{\mathsf{NeF}} \neg\psi$, which indeed means $\Gamma|\neg\psi$. Therefore, we obtain a contradiction, and we can conclude $\Gamma|(\varphi \wedge \neg\varphi) \to \neg\psi$.

Consider now contraposition logic CoPC, and suppose φ to be the contraposition axiom, $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$. Assume $\Gamma \vdash_{\mathsf{CoPC}} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ and suppose for the sake of a contradiction that not $\Gamma|(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$. Thus, $\Gamma|\varphi \to \psi$ and not $\Gamma|\neg\psi \to \neg\varphi$. Since $\Gamma|\varphi \to \psi$, by Theorem 3.2.7 we get that $\Gamma \vdash_{\mathsf{CoPC}} \varphi \to \psi$ and hence, $\Gamma \vdash_{\mathsf{CoPC}} \neg\psi \to \neg\varphi$. At this point we have $\Gamma|\neg\psi$ and not $\Gamma|\neg\varphi$. Observe though that, if $\Gamma|\neg\psi$, by the same reasoning we just unfolded, we get that $\Gamma \vdash_{\mathsf{CoPC}} \neg\varphi$ as well. Hence, we are able to conclude $\Gamma|\neg\varphi$, which contradicts our assumption. $\square$

We conclude this section by stating the analogue of Corollary 3.2.9 for the considered extensions of N.

Corollary 3.2.11. *Let φ, ϑ, ψ and χ be arbitrary formulas.*

(a) If $\vartheta|\vartheta$, then for every ψ, χ:

$$\text{if } \vdash \vartheta \to \psi \vee \chi, \text{ then } \vdash \vartheta \to \psi \text{ or } \vdash \vartheta \to \chi.$$

(b) If $\vdash \neg\varphi \to \psi \vee \chi$, then $\vdash \neg\varphi \to \psi$ or $\vdash \neg\varphi \to \chi$.

It is not difficult to see that, as usual, the converse of (a) of Corollary 3.2.11 holds as well.

3.3 Filtration Method

We want now to analyze a classic method for building finite models, different (although very similar) from the one presented at the beginning of the chapter. We want to apply to the canonical model the idea of identifying as many nodes as possible, by means of an equivalence relation, in order to get a finite model from an infinite one.

Let $\mathcal{M}$ be the canonical model for the basic logic N, and let Σ be a set of formulas closed under subformulas. Consider the following set:

$$\mathcal{W}_\Sigma := \{\Gamma \cap \Sigma \mid \Gamma \in \mathcal{W}\}. \quad (*)$$

Note that, in the case in which Σ is a *finite* set of formulas, then the set $\mathcal{W}_\Sigma$ is finite, since it happens to be a subset of the set of subsets $\mathcal{P}(\Sigma)$.

Definition 19 (Filtration of the Canonical Model)**.** *Given a set of formulas Σ closed under subformulas, let $\mathcal{M}_\Sigma = \langle \mathcal{W}_\Sigma, \mathcal{R}_\Sigma, \mathcal{N}_\Sigma, \mathcal{V}_\Sigma \rangle$ denote the following: the set $\mathcal{W}_\Sigma$ is defined as in $(*)$ and is ordered by the usual set-theoretic inclusion $\mathcal{R}_\Sigma :=\subseteq$; let $\mathcal{N}_\Sigma$ be a function such that, for every $U \in \mathcal{U}(\mathcal{W}_\Sigma)$, $\Gamma \cap \Sigma \in \mathcal{N}_\Sigma(U)$ if and only if there exists a negated formula $\neg\psi \in \Gamma \cap \Sigma$ such that*

$$U \cap \mathcal{R}_\Sigma(\Gamma \cap \Sigma) = \{\Delta \cap \Sigma \in \mathcal{W}_\Sigma \mid \psi \in \Delta\} \cap \mathcal{R}_\Sigma(\Gamma \cap \Sigma).$$

Finally, let $\mathcal{V}_\Sigma$ be the valuation map such that, for every $p \in \Sigma$,

$$\mathcal{V}_\Sigma(p) := \{\Gamma \cap \Sigma \in \mathcal{W}_\Sigma \mid p \in \Gamma\}.$$

Before making use of the given definition, we have to make sure the obtained model is still a model of the considered logic N.

Proposition 3.3.1. *The function $\mathcal{N}_\Sigma$ satisfies* $\mathsf{P1}$ *and* $\mathsf{P2}$*, i.e., the model $\mathcal{M}_\Sigma$ obtained from $\mathcal{M}$ still is a N-model.*

Proof. The proof of this result goes exactly as the proofs of Proposition 2.1.5 and Proposition 2.1.6. □

We have now defined and proved all the tools necessary to show that $\mathcal{M}_\Sigma$ preserves the forcing relation for the formulas in Σ.

Theorem 3.3.2. *Let $\mathcal{M}_\Sigma = \langle \mathcal{W}_\Sigma, \mathcal{R}_\Sigma, \mathcal{N}_\Sigma, \mathcal{V}_\Sigma \rangle$ be a filtration of the canonical model for N. Then, for every formula $\varphi \in \Sigma$, and every $\Gamma \in \mathcal{W}$,*

$$\mathcal{M}, \Gamma \vDash \varphi \Leftrightarrow \mathcal{M}_\Sigma, \Gamma \cap \Sigma \vDash \varphi.$$

Proof. The proof structure is an induction on the structure of φ. The first steps of the proof are standard, and hence we check in detail only the case of negation.

$\neg\varphi$: Let us assume $\mathcal{M}, \Gamma \vDash \neg\varphi$, which by construction means $\neg\varphi \in \Gamma$. Observe that we have, for each $\Delta \in \mathcal{W}$, that $\Delta \cap \Sigma \in \mathcal{V}_\Sigma(\varphi)$ if and only if $\mathcal{M}_\Sigma, \Delta \cap \Sigma \vDash \varphi$, which is equivalent to $\mathcal{M}, \Delta \vDash \varphi$ thanks to the induction hypothesis. By construction of canonical model, this means $\varphi \in \Delta$ and in particular, by closure of Σ, we get $\varphi \in \Delta \cap \Sigma$. Hence, we have that $\Delta \cap \Sigma \in \mathcal{V}_\Sigma(\varphi)$ is equivalent to

$$\Delta \cap \Sigma \in \{\Delta' \cap \Sigma \mid \varphi \in \Delta'\}.$$

From this, we can conclude that there exists a formula $\neg\chi \in \Gamma \cap \Sigma$ such that

$$\mathcal{V}_\Sigma(\varphi) \cap \mathcal{R}_\Sigma(\Gamma \cap \Sigma) = \{\Delta \cap \Sigma \mid \chi \in \Delta\} \cap \mathcal{R}_\Sigma(\Gamma \cap \Sigma).$$

In particular, in our case χ is exactly φ. Hence, $\mathcal{M}_\Sigma, \Gamma \cap \Sigma \vDash \neg\varphi$, as desired. Now, it remains to prove the other direction of the statement for $\neg\varphi$. In order to do that, assume $\mathcal{M}_\Sigma, \Gamma \cap \Sigma \vDash \neg\varphi$, which means by how we defined a filtration of the canonical model, that there is $\neg\chi \in \Gamma \cap \Sigma$ such that $\mathcal{V}_\Sigma(\varphi) \cap \mathcal{R}_\Sigma(\Gamma \cap \Sigma) = \{\Delta \cap \Sigma \mid \chi \in \Delta\} \cap \mathcal{R}_\Sigma(\Gamma \cap \Sigma)$. We want to use this fact to conclude that in the canonical model,

$$\mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma) = [\![\chi]\!] \cap \mathcal{R}(\Gamma).$$

Consider an arbitrary $\Delta \in \mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma)$, i.e., $\mathcal{M}, \Delta \vDash \varphi$ and $\Gamma \subseteq \Delta$. From the induction hypothesis, $\mathcal{M}_\Sigma, \Delta \cap \Sigma \vDash \varphi$, and also $\Gamma \cap \Sigma \subseteq \Delta \cap \Sigma$. This means that $\Delta \cap \Sigma \in \mathcal{V}_\Sigma(\varphi) \cap \mathcal{R}_\Sigma(\Gamma \cap \Sigma)$. By hypothesis, this implies $\chi \in \Delta \cap \Sigma$, and hence $\chi \in \Delta$. Thus, we conclude $\Delta \in [\![\chi]\!] \cap \mathcal{R}(\Gamma)$, and therefore we showed that

$$\mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma) \subseteq [\![\chi]\!] \cap \mathcal{R}(\Gamma).$$

Let now Δ be arbitrary in $[\![\chi]\!] \cap \mathcal{R}(\Gamma)$. Thus, $\chi \in \Delta$ and $\Gamma \subseteq \Delta$. Now, the reader may observe that this entails that $\chi \in \Delta \cap \Sigma$ and $\Gamma \cap \Sigma \subseteq \Delta \cap \Sigma$. Hence, by assumption, $\Delta \cap \Sigma \in \mathcal{V}_\Sigma(\varphi) \cap \mathcal{R}_\Sigma(\Gamma \cap \Sigma)$, which allows us to conclude that $\mathcal{M}_\Sigma, \Delta \cap \Sigma \vDash \varphi$. By the induction hypothesis, we get that $\mathcal{M}, \Delta \vDash \varphi$ and therefore, $\Delta \in \mathcal{V}(\varphi) \cap \mathcal{R}(\Gamma)$. This last sentence gives us the right-to-left inclusion. At this point, according to how the canonical model has been defined, $\Gamma \in \mathcal{N}(\mathcal{V}(\varphi))$ and so, $\mathcal{M}, \Gamma \vDash \neg\varphi$. □

Having now proved Theorem 3.3.2, we can give an alternative proof of the FMP result for N.

Theorem 3.3.3 (Finite Model Property of N via Filtration)**.** *Consider an arbitrary formula φ such that $\nvdash_{\mathsf{N}} \varphi$. Then, there exists a finite model which does not force φ. Moreover, there is such a finite model which has at most 2^n nodes, where n is the number of subformulas of φ.*

Proof. Consider the canonical model $\mathcal{M}$ for N, and let $\mathcal{M}_{Sub(\varphi)}$ be the model obtained 'filtrating' $\mathcal{M}$ by means of the finite set of subformulas of $\Sigma = Sub(\varphi)$. Observe that we have $|\mathcal{W}_{Sub(\varphi)}| \leq 2^n$, where $n = |Sub(\varphi)|$. Moreover, by the Completeness Theorem of N and by Theorem 3.3.2, we have that $\nvdash_{\mathsf{N}} \varphi$ implies $\mathcal{M}_{Sub(\varphi)} \nvDash \varphi$, and therefore N satisfies the FMP. □

Observe that the notion of filtration introduced here strictly concerns the canonical models of our systems. The next step in this direction should be to try to give a complete account of the notion of filtration.

Chapter 4

Algebraic Semantics

The study of an algebraic semantics gives extremely interesting results both in the case of intuitionistic and intermediate logics, and in the case of modal logic. For instance, a completeness result for certain intermediate and modal logics with respect to a frame-based semantics cannot be obtained directly. Introducing an algebraic semantics allows us to talk about a broader notion of frame and to prove such completeness.

This chapter is devoted to studying and presenting an algebraic semantics for the subminimal systems we have been studying and, more generally, for all the extensions of N. In our setting, the notion of negation as a functional unary operator suggests an immediate algebraic perspective. The reader may have seen the notion of Kripke frame presented in Chapter 2 as an *a posteriori* notion. Indeed, defining the function N as a map between upsets, we obtain a natural link to the algebraic semantics.

The starting point of the chapter is going to be an algebraic completeness result for the basic logic of a unary operator, to be enhanced to all the extensions of N. In order to prove such a result, we present a generalized notion of Heyting algebra, which represents the algebraic counterpart of positive logic. Later, we will give an alternative frame-based completeness result. In particular, we want to give a correlation between the defined algebras and a new notion of frame. This, as in the case of intuitionistic and modal logic, allows us to conclude frame-based completeness by means of algebraic completeness.

4.1 Generalized Heyting Algebras

As already emphasized many times in the previous chapters, the general setting in which we have been working is the positive fragment of intuitionistic logic. The basic language that we are using is the intuitionistic one, without $\bot$. This gives us immediate candidate structures to start with: the algebraic counterpart of positive logic. The considered structures are called either *generalized Heyting algebras* (e.g., [18, 19]) or *implicative lattices* (e.g., [36]). Alternatively, we find them defined as *relatively pseudo-complemented lattices* (e.g., [39]), or as *Brouwerian algebras* (e.g., [2]). We assume the reader to be familiar with the notion of lattice [3, 12].

Definition 20 (gH-algebra)**.** *A* generalized Heyting algebra *(*gH-algebra *for short)* $\langle A, \wedge, \vee, \to, 1\rangle$ *is a lattice* $\langle A, \wedge, \vee\rangle$ *such that for every pair of elements*

$a, b \in A$, the element $a \to b$ defining the supremum of the set $\{c \in A \mid a \wedge c \leq b\}$ exists.

We call $\to$ *Heyting implication* (or *implication*). Given this notion, a *Heyting algebra* can be defined as a gH-algebra with a minimum 0. Observe that in the finite case every gH-algebra is a Heyting algebra. The equation $x \to x = y \to y$ holds in every gH-algebra, and we denote $x \to x$ as 1 (which is the top element of an algebra).

At this point, before going into the study of an algebraic semantics for N, it is worth recalling the link between gH-algebras and minimal logic. Given a gH-algebra, it is possible to define on it a negation operator which satisfies the property corresponding to the principle of contradiction,

$$((x \to y) \wedge (x \to \neg y)) \to \neg x = 1. \quad (*)$$

Resembling the Hilbert-style definition of the negation operator in the MPC_f formalization, it is enough to choose an element $f \in A$ and define, for each $a \in A$,

$$\neg a := a \to f.$$

We follow [36] and denote as *j-algebra* a gH-algebra with the negation operator defined in this way. This gives us that minimal logic MPC corresponds to the variety of j-algebras. For our interests, we can see a j-algebra also as a gH-algebra equipped with a unary operator $\neg$ satisfying the property $(*)$. This approach would immediately reflect the syntactic definition of $\mathsf{MPC}_\neg$. For a detailed account of these results, see [18, 19, 36].

We want to point out here an interesting curiosity. The algebraic counterpart of minimal logic has been extensively studied by Rasiowa as the variety of *contrapositionally complemented lattices* [39]. These structures are defined as gH-algebras equipped with a unary operator $\sim$ satisfying the equation

$$(x \to \sim y) = (y \to \sim x).$$

The reader may easily see this variety to coincide with the variety of j-algebras, as can be shown via a proof similar to the one of Proposition 1.2.4.

4.1.1 Compatible Functions

There is a natural way to modify the definition of a j-algebra and get a structure corresponding to the basic logic of unary operator N. The obvious candidate consists of a gH-algebra equipped with a unary operator whose characteristic equation is the one corresponding to the N axiom. This way of defining algebras comes from and is strictly related to the notion of *compatible function*. Such a notion was widely studied and used while trying to define new intuitionistic connectives (see, for instance, [11]). As observed in [18], in Caicedo and Cignoli's paper [11] the notion of *compatible connective* for the positive fragment of intuitionistic logic is used implicitly under the following shape: a connective ∇ is compatible if and only if $\varphi \leftrightarrow \psi \vdash \nabla\varphi \leftrightarrow \nabla\psi$, for every pair of formulas φ, ψ. In this sense, it is clear that the negation operator $\neg$ as defined in N is a compatible connective.

A further study of compatibility over positive logic, from an algebraic perspective, has been carried out in [18]. In particular, the algebraic meaning of compatibility amounts to the following: a function f is a *compatible function* of an algebra $\mathfrak{A}$ if it is compatible with all the congruence relations, i.e., if given a congruence relation Θ of $\mathfrak{A}$,

$$(x, y) \in \Theta \text{ implies } (f(x), f(y)) \in \Theta.$$

From the point of view of [11, 18], it would be enough to conceive $\neg$ as a compatible function in this sense. On the other hand, the scope of this chapter is to analyze the algebraic behavior of the operator $\neg$ seen as a weak negation operator satisfying the N axiom. Therefore, starting from Lemma 2.2 of [18], a way to define the algebraic counterpart of N discloses itself.

Definition 21 (N-algebra)**.** *An* N-algebra $\langle A, \wedge, \vee, \rightarrow, \neg, 1\rangle$ *is given by a gH-algebra* $\langle A, \wedge, \vee, \rightarrow, 1\rangle$ *equipped with a unary operator* $\neg$ *such that*

$$(x \leftrightarrow y) \rightarrow (\neg x \leftrightarrow \neg y) = 1.$$

The following result about compatible functions is stated and proved in [11, 18]. For the sake of simplicity, we state it here in the language of our interest, i.e., the compatible function we refer to is denoted by $\neg$.

Lemma 4.1.1. *Let $\mathfrak{A}$ be a gH-algebra. Then, $\mathfrak{A}$ equipped with a unary operator $\neg$ is an N-algebra if and only if for every pair of elements x, y in $\mathfrak{A}$,*

$$\neg x \wedge y = \neg(x \wedge y) \wedge y.$$

In what follows, we want to formalize the connection between the logical systems we are investigating and the structure of N-algebras.

4.2 Algebraic Completeness

In this section we make the connection between the subminimal systems and the N-algebras formal. We introduce all the tools necessary to prove an algebraic completeness result. Moreover, the proved completeness result turns out to be even more general: every extension of the basic logic N is complete with respect to the corresponding variety of N-algebras. This result will be used later on to conclude a frame-based completeness result.

As a first step, we define the basic algebraic operations. These notions are basically the same as the respective ones for Heyting algebras [3], with the additional conditions for the negation operator. The conditions for the minimum element 0 are left out as well, since even when an N-algebra contains a minimum element, we do not treat it as a 'distinguished' element.

Definition 22 (N-homomorphism)**.** *Let* $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \neg, 1\rangle$ *and* $\mathfrak{A}' = \langle A', \wedge', \vee', \rightarrow', \neg', 1'\rangle$ *be N-algebras. A map* $h : A \rightarrow A'$ *is a* N-homomorphism *(or* homomorphism*) if*

- $h(a \wedge b) = h(a) \wedge' h(b)$,
- $h(a \vee b) = h(a) \vee' h(b)$,
- $h(a \to b) = h(a) \to' h(b)$,
- $h(\neg a) = \neg' h(a)$,
- $h(1) = 1'$.

We say that an N-algebra $\mathfrak{A}'$ is a *homomorphic image* of $\mathfrak{A}$ if there exists an N-homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$.

Definition 23 (Subalgebra)**.** *Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, \neg, 1\rangle$ and $\mathfrak{A}' = \langle A', \wedge', \vee', \to', \neg', 1'\rangle$ be N-algebras. We say that $\mathfrak{A}'$ is a* subalgebra *of $\mathfrak{A}$ if $A' \subseteq A$, the operations $\wedge', \vee', \to', \neg'$ are the restrictions of $\wedge, \vee, \to, \neg$ to A' and $1 = 1'$.*

This definition gives us that A' is closed under $\wedge$, $\vee$, $\to$, $\neg$ and 1. The last operation we want to define is the one of product.

Definition 24 (Product)**.** *Let $\mathfrak{A}_1 = \langle A_1, \wedge_1, \vee_1, \to_1, \neg_1, 1_1\rangle$ and $\mathfrak{A}_2 = \langle A_2, \wedge_2, \vee_2, \to_2, \neg_2, 1_2\rangle$ be N-algebras. The* product *of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is the N-algebra $\mathfrak{A}_1 \times \mathfrak{A}_2 := \langle A_1 \times A_2, \wedge, \vee, \to, \neg, 1\rangle$, where*

- $(a_1, a_2) \wedge (b_1, b_2) := (a_1 \wedge_1 b_1, a_2 \wedge_2 b_2)$,
- $(a_1, a_2) \vee (b_1, b_2) := (a_1 \vee_1 b_1, a_2 \vee_2 b_2)$,
- $(a_1, a_2) \to (b_1, b_2) := (a_1 \to_1 b_1, a_2 \to_2 b_2)$,
- $\neg(a_1, a_2) := (\neg_1 a_1, \neg_2 a_2)$,
- $1 := (1_1, 1_2)$.

Observe that the product operation can easily be generalized to define the product of arbitrary many N-algebras $\{\mathfrak{A}_i\}_{i \in I}$.

The three notions just given are very important notions in an algebraic setting. They suffice to characterize the nature of certain classes K of algebras of the same signature. Recall that a class K of algebras is said to be a *variety* if K is closed under homomorphic images, subalgebras and products [10]. One of the fundamental theorems about varieties is the so-called *Birkhoff theorem*, which gives a simple characterization of them (see [10] for the proof).

Theorem 4.2.1 (Birkhoff)**.** *A class K of algebras forms a variety if and only if K is equationally definable.*

Let $\mathcal{NA}$ denote the class of N-algebras.

Corollary 4.2.2. *$\mathcal{NA}$ is a variety.*

This result follows from the fact that the notions of lattice, gH-algebra and N-algebra can be defined equationally. The proof goes as in the case of the variety of Heyting algebras $\mathcal{HA}$, and a more detailed account of this result for the intuitionistic case can be found in [3].

4.2.1 The Lindenbaum-Tarski Construction

The proof of algebraic completeness proceeds in a very similar way to the one in the intuitionistic case. We start by defining the notion of valuation map for a given algebra. This allows us to talk about validity of formulas in an algebraic structure.

Definition 25. *Let $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \neg, 1\rangle$ be an N-algebra. A map $v : Prop \rightarrow A$ is called a* valuation *into the N-algebra $\mathfrak{A}$. This valuation can be extended from $Prop$ to the whole set of formulas $Form$ as follows:*

- $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$,
- $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$,
- $v(\varphi \rightarrow \psi) = v(\varphi) \rightarrow v(\psi)$,
- $v(\neg\varphi) = \neg v(\varphi)$,
- $v(\top) = 1$.

Given this definition, we can say that a formula φ is *true* in $\mathfrak{A}$ under a valuation v if $v(\varphi) = 1$. We denote this relation of truth as $\langle \mathfrak{A}, v\rangle \vDash \varphi$. We say that a formula φ is *valid* on $\mathfrak{A}$ if φ is true for every possible valuation in $\mathfrak{A}$, and we denote this as $\mathfrak{A} \vDash \varphi \approx 1$. This notion can be extended to all terms, and we write $\mathfrak{A} \vDash \varphi \approx \psi$ if for every valuation $v : Form \rightarrow A$ we have $v(\varphi) = v(\psi)$.

Recall that the aim of this section is to give a proof of a completeness result with respect to the variety $\mathcal{NA}$. We want to give a proof of the fact that, for every formula φ,

$$\mathsf{N} \vdash \varphi \text{ if and only if } \varphi \text{ is valid in any N-algebra.}$$

The soundness direction of this statement is proved, as in the case of soundness in Chapter 2, by a simple inductive argument on the depth of the proof. For completeness, we want to show that every non-theorem of N can be falsified in some N-algebra. Similarly to the Kripke completeness, we have to build a 'falsifying algebra', which has the role of being a 'canonical algebra'. It is denoted as Lindenbaum-Tarski construction [12]. In order to build such an algebra, we have to do some preliminary work. In particular, the idea we are going to use is the fact that the relation of logical equivalence (i.e., $\varphi \sim \psi$ if and only if $\mathsf{N} \vdash \varphi \leftrightarrow \psi$) is a congruence on the N-algebra of formulas.

Definition 26. *Let $\mathfrak{A}$ be an N-algebra. An equivalence relation $\sim$ on A is a* congruence *if it satisfies:*

- *if $a_1 \sim b_1$ and $a_2 \sim b_2$, then $a_1 \wedge a_2 \sim b_1 \wedge b_2$,*
- *if $a_1 \sim b_1$ and $a_2 \sim b_2$, then $a_1 \vee a_2 \sim b_1 \vee b_2$,*
- *if $a_1 \sim b_1$ and $a_2 \sim b_2$, then $a_1 \rightarrow a_2 \sim b_1 \rightarrow b_2$,*
- *if $a_1 \sim b_1$, then $\neg a_1 \sim \neg b_1$.*

It is not hard to see that indeed the equivalence relation of logical equivalence preserves the operations of N-algebra, and for this, one has to make use of the axiom N. Observe that this fact is strictly related to the notion of compatible function. Indeed, the notion of compatibility as a function is equivalent, from an algebraic perspective, to the one of compatibility with congruence relations.

At this point, we want to use the resulting equivalence classes under this relation as 'building blocks' for our canonical construction.

Definition 27 (Lindenbaum-Tarski Algebra)**.** *Given a set of propositional variables Prop, let $Form/{\sim}$ be the set of equivalence classes that $\sim$ induces on the set of formulas, and for each formula φ let $[\varphi]$ denote the equivalence class containing φ. The* Lindenbaum-Tarski algebra *for this language is the structure*

$$\mathfrak{A}_L := (Form/{\sim}, \wedge_L, \vee_L, \to_L, \neg_L, 1_L),$$

where the operations are defined in the natural way as follows: $[\varphi] \wedge_L [\psi] := [\varphi \wedge \psi]$, $[\varphi] \vee_L [\psi] := [\varphi \vee \psi]$, $[\varphi] \to_L [\psi] := [\varphi \to \psi]$, $\neg_L[\varphi] := [\neg\varphi]$ and the top element is $1 := [\top]$.

This structure is a well-defined N-algebra. For the details of the intuitionistic part of the proof, see [12]. The negation operator $\neg_L$ is ensured to satisfy the desired equation by how equivalence classes have been defined.

The only thing remaining to be proved is that the Lindenbaum-Tarski algebra is indeed an 'algebraic canonical model'.

Theorem 4.2.3. *Let φ be some propositional formula, and Prop a set of proposition variables containing the propositional variables occurring in φ. Then,*

$$\mathsf{N} \vdash \varphi \textit{ if and only if } \mathfrak{A}_L \vDash \varphi \approx 1.$$

Proof. The proof goes exactly as in the intuitionistic case. The reader may find it, for instance, in [12]. □

This completeness result and the whole algebraic structure can be extended to negative *ex falso*, contraposition, and minimal propositional logic. Even more, it can be extended to all the possible extensions of N. Each of these logics $\mathbf{L}$ can be associated with the variety $\mathbf{V_L}$ of the respective N-algebras, i.e., the N-algebras in which all theorems of $\mathbf{L}$ are valid. The fact that these classes of algebras are all varieties follows from Theorem 4.2.1. The Lindenbaum-Tarski construction ensures that each of these logics is complete with respect to its algebraic semantics.

Remark. The behavior of these systems with respect to their respective algebraic semantics resembles the one of the so-called intermediate logics [3, 12], with some relevant differences. In particular, if we focus on intuitionistic propositional logic, there is only one maximal extension of it, namely CPC. In our current setting, this is not the case anymore. As a matter of fact, we can have 'incompatible' ways of extending N, meaning that we can extend N to systems which do not have any common extension. Consider, for instance, the extension of N by means of the axiom $\neg p \leftrightarrow p$. As the reader may see, there is no way to extend that system to minimal propositional logic. On the other hand, we can consider another 'trivial' extension on N by means of the axiom $\neg p$, which makes all the negative formulas true. This system indeed extends minimal propositional logic, but is incomparable with IPC and CPC.

In what follows, we are going to transfer this algebraic completeness result to a frame-based completeness result. In order to do this, we give a new notion of frame, following the one of descriptive frame in intuitionistic logic [3].

4.3 Descriptive Frames

This section wants to generalize the notion of Kripke frame as defined in Chapter 2 to the new notion of descriptive frames. In order to make things easier when defining this notion, we find this the right place to make a different formulation of the locality property explicit. The following result appears as Lemma 3.2 in [18].

Lemma 4.3.1. *Consider an arbitrary poset (P, R), and let $f : \mathcal{U}(P) \to \mathcal{U}(P)$ be a function between upsets of P. Then, the following are equivalent:*

(a) For every $U \in \mathcal{U}(P)$, $p \in P$, we have $p \in f(U)$ if and only if $p \in f(U \cap R(p))$.

(b) For every $U, V \in \mathcal{U}(P)$, we have $f(U) \cap V = f(U \cap V) \cap V$.

Proof. Let us assume (a), and consider a pair of upsets U, V of P. Consider an element $p \in f(U) \cap V$. It suffices to show that $p \in f(U \cap V)$. The considered assumption, together with (a), gives us that $p \in f(U \cap R(p))$. Observe that $p \in V$, and hence:

$$(U \cap V) \cap R(p) = U \cap R(p).$$

So, we get $p \in f(U \cap V) \cap R(p)$. Applying (a) again gives us that $p \in f(U \cap V)$. On the other hand, we assume (b) and we consider that $p \in f(U)$. This is equivalent to $p \in f(U) \cap R(p)$. By (b), we get that this is equivalent to $p \in f(U \cap R(p)) \cap R(p)$, hence to $p \in f(U \cap R(p))$ and we are done. □

An immediate and useful consequence of this is the following.

Lemma 4.3.2. *Given a partially ordered set $\langle W, R\rangle$ and a map N between upsets, the triple $\mathfrak{F} = \langle W, R, N\rangle$ is an N-frame if and only if, for every pair of upsets $U, V \subseteq W$,*

$$N(U) \cap V = N(U \cap V) \cap V.$$

Proof. An immediate consequence of Lemma 4.3.1. □

We are now ready to define the notions of general and descriptive frame. These notions happen to be very similar to their intuitionistic analogues. The idea is that the valuation need not be defined on all the upsets of the partial order. As a matter of fact, it is enough to consider a family of 'admissible' upsets for the valuation. In our setting, this has consequences on the definition of the function N as well.

Definition 28 (N-general Frame)**.** *An* N-general frame *is a quadruple* $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$, *where* $\langle W, R \rangle$ *is an intuitionistic frame,* $\mathcal{P}$ *is a set of upsets of* W*, containing* W *and which is closed under* $\cup$*, (finite)* $\cap$*,* $\to$*, where* $\to$ *is defined by*

$$U \to V := \{w \in W \mid \forall v(wRv \wedge v \in U \to v \in V)\},$$

and $N : \mathcal{P} \to \mathcal{P}$ *which satisfies locality*[1]*.*

The reader may observe that every N-frame can be seen as a general frame considering $\mathcal{P} = \mathcal{U}(W)$.

The following definition is exactly the same as in the intuitionistic case.

Definition 29. *Let* $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ *be a general frame.*

1. *We say that the frame* $\mathfrak{F}$ *is* refined *if, for every* $w, v \in W$*:* $\neg(wRv)$ *implies the existence of an upset* $U \in \mathcal{P}$ *which contains* w *and does not contain* v*, i.e.,* $w \in U$ *and* $v \notin U$*.*
2. *We say that the frame* $\mathfrak{F}$ *is* compact *if for every* $\mathcal{X} \subseteq \mathcal{P}$ *and* $\mathcal{Y} \subseteq \{W \setminus U \mid U \in \mathcal{P}\}$*, if* $\mathcal{X} \cup \mathcal{Y}$ *has the* finite intersection property*, then* $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$*.*
3. *If the frame* $\mathfrak{F}$ *is refined and compact, we call it a* descriptive frame*.*

The elements of $\mathcal{P}$ are called *admissible sets*. Observe that every *finite* N-frame, equipped with the set of its upsets, can be seen as a descriptive frame.

Remark. The reader should note that we do not explicitly require the empty-set to be an element of $\mathcal{P}$. This reflects the main difference between our setting and the intuitionistic one. In particular, our language does not contain the $\perp$ constant. From both the intuitive and the algebraic perspective, the presence of the empty-set among the admissible upsets of a frame is indeed strictly related to the presence of $\perp$ in the considered language. As a matter of fact, in intuitionistic logic, $V(\perp) = \emptyset$ needs to hold for every valuation map V and in every frame $\mathfrak{F}$; therefore, it would be unreasonable not to require $\mathcal{P}$ to contain the empty-set. On the other hand, in our setting it is more natural not to explicitly require the empty-set to be in there, since it is not a 'distinguished' element among the upsets. The sense of this will be clear as soon as we get to duality theory.

The following definition is strictly related to this discussion concerning admissibility of the empty-set.

Definition 30. *A* top descriptive frame *is a descriptive frame whose partially ordered underlying set* $\langle W, R \rangle$ *has a greatest element* t *such that, for every upset* $U \in \mathcal{P}$*, we have* $t \in U$*.*

[1]In this setting, it is easier to consider as the formulation of locality the one presented in Lemma 4.3.2.

The definition of top descriptive frames happens to be fundamental in the setting we are working in. In particular, it is useful to have the following clear: the empty-set is not an admissible set in any top descriptive frame. The name 'top descriptive frame' is chosen here to point out the connection to the notions of top frame and top model of [27].

We want to modify the definitions of p-morphism, generated subframe and disjoint union, to adapt them to top descriptive frames.

Definition 31.

- *Given two top descriptive frames* $\mathfrak{F} = \langle W, R, N, \mathcal{P}\rangle$ *and* $\mathfrak{F}' = \langle W', R', N', \mathcal{P}'\rangle$*, a map* $h : W \to W'$ *is said to be a* p-morphism *between* $\mathfrak{F}$ *and* $\mathfrak{F}'$ *if* h *is a p-morphism between* $\langle W, R\rangle$ *and* $\langle W', R'\rangle$ *[3], for every* $U' \in \mathcal{P}'$ *we have that* $h^{-1}(U') \in \mathcal{P}$ *and* $W \setminus h^{-1}(W' \setminus U') \in \mathcal{P}$*, and moreover* $w \in N(h^{-1}(U'))$ *if and only if* $h(w) \in N'(U')$*.*

- *Given a top descriptive frame* $\mathfrak{F} = \langle W, R, N, \mathcal{P}\rangle$*, a* generated subframe *of* $\mathfrak{F}$ *is a top descriptive frame* $\mathfrak{F}' = \langle W', R', N', \mathcal{P}'\rangle$*, where* $\langle W', R'\rangle$ *is a generated subframe of* $\langle W, R\rangle$*, the set* $\mathcal{P}'$ *is given by* $\{U \cap W' \mid U \in \mathcal{P}\}$*, and the function* $N'(U') = N(U') \cap W'$*, for every* $U' \in \mathcal{P}'$*.*

Observe that the given definition of p-morphism ensures the top node of $\mathfrak{F}$ to be mapped to the top node of $\mathfrak{F}'$.

In order to define the right notion of disjoint union, some more work is needed. In particular, given a (finite) family of top descriptive frames, we want to define an equivalence relation which identifies the top nodes of the different frames.

Let $\{\mathfrak{F}_i\}_{i=1}^n$ be a finite[2] set of top descriptive frames, whose domains and top nodes are respectively denoted by W_i and t_i, for every i. Consider the set of worlds $\biguplus_{i=1}^n W_i$. We define an equivalence relation over such set, as follows:

$$w \sim v \Leftrightarrow (w = v \text{ or there exist } i, j \in \{1, \dots, n\} : w = t_i \text{ and } v = t_j).$$

Let us define W as the set of equivalence classes induced by this equivalence relation, i.e.,

$$W := (\biguplus_{i=1}^n W_i)/\sim .$$

We equip the set W with an ordering relation, defined as follows:

$$[w]R[v] \Leftrightarrow \exists u \in [v] : wR_i u, \text{ for some } i \in \{1, \dots, n\}.$$

The resulting structure $\langle W, R\rangle$ is a partial ordering with a maximum node, namely the equivalence class $[t_i]$. At this point, we are ready to define the notion of disjoint union of top descriptive frames. In the following definition, given a set $X \subseteq \biguplus_{i=1}^n W_i$, we denote as $X/\sim$ the following set:

$$X/\sim := \{[w] \in W \mid w \in X\}.$$

[2]Following [3], we consider only finitely many top descriptive frames, since the disjoint union of infinitely many descriptive frames is not a descriptive frame [12].

Definition 32. *Let* $\{\mathfrak{F}_i\}_{i=1}^n$ *be a finite set of top descriptive frames. The* disjoint union *of* $\{\mathfrak{F}_i\}_{i=1}^n$ *is a top descriptive frame*

$$\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle,$$

where $\langle W, R \rangle$ *is the partial order defined as above. The set of admissible upsets* $\mathcal{P}$ *is defined as*

$$\mathcal{P} := \{ \big(\uplus_{i=1}^n U_i \big) / \sim \; | U_i \in \mathcal{P}_i \},$$

and the function $N : \mathcal{P} \to \mathcal{P}$ *maps a set* U *of the form* $\bigcup_{i=1}^n U_i / \sim$ *in the following way:*

$$N(U) := [\uplus_{i=1}^n N_i(U_i)] \, / \sim .$$

Observe that the resulting structure is a top descriptive frame with top node $[t_i]$. As already said, taking the disjoint union of a family of top descriptive frames amounts to taking the disjoint union of the original partial orders and identifying their top nodes. This makes sure the resulting frame is still a top frame and ensures the disjoint union to intuitively preserve the information contained in the original frames.

Remark. The given definition of p-morphism seems to concern only the class of top descriptive frames. Indeed, such a notion is defined for top descriptive frames exactly in the same way as it would be defined for descriptive frames. If we consider the definition of p-morphism given for top descriptive frames, the top nodes are not explicitly considered anywhere; as a matter of fact, the intuitively necessary condition

$$h(t) = t',$$

where t and t' are respectively the top nodes of $\mathfrak{F}$ and $\mathfrak{F}'$ in the definition, does not need to be stated. The reason why this is the case is that the given definition already implies the top node to be preserved. Therefore, we could have given the above definition as a notion of p-morphism for descriptive frames, and we could have defined the corresponding notion for top descriptive frames as follows: a p-morphism between two top descriptive frames is just a p-morphism between the two structures seen as descriptive frames.

At this point, we want to make the correlation between top descriptive frames and N-algebras formal. To convince the reader that the resulting duality is intuitively reasonable, we will not limit ourselves to give definitions. We will try to make reasons and motivations for our choices explicit, and this will hopefully help to give a neat 'picture' of how the duality works.

4.3.1 From Frames to Algebras

Let $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ be a top descriptive frame for N. Then, the structure

$$\langle \mathcal{P}, \cap, \cup, \to, N, W \rangle$$

is an N-algebra. We refer to it as the *dual algebra* $\mathfrak{F}^*$ of the frame $\mathfrak{F}$.

The reader may note that the step from descriptive frames to N-algebras resembles the one from intuitionistic descriptive frames to Heyting algebras [3]. The additional requirement, which is the definition of the unary negation operator, is straightforward from the fact that the function N has $\mathcal{P}$ both as domain and as range.

Remark. We want to emphasize the fact that the presence of the top node t in the considered frame is what allows us to really distinguish this duality from the Heyting duality. Let us consider the positive reduct of the top descriptive frame $\mathfrak{F}$, as well as the positive reduct[3] of the dual algebra $\mathfrak{F}^*$. The unique Heyting algebra obtained by adding the empty-set to $\mathcal{P}$ is exactly the Heying algebra dual to the descriptive frame $\langle W, R, \mathcal{P} \cup \{\emptyset\}\rangle$. This analysis is carried out, from the point of view of Brouwerian algebras and the so-called *pointed Esakia spaces*, in [2]. The Esakia space 'corresponding' to the descriptive frame $\langle W, R, \mathcal{P} \cup \{\emptyset\}\rangle$ is called *unpointed Esakia reduct* [2].

4.3.2 From Algebras to Frames

At this point, we need to see how to construct a top descriptive frame from an N-algebra.

Definition 33. *Let $\mathfrak{A} = \langle A, \wedge, \vee, \to, \neg, 1\rangle$ be an N-algebra. A non-empty subset of A is called a* filter *if*

- *$a, b \in F$ implies $a \wedge b \in F$,*
- *$a \in F$ and $a \leq b$ imply $b \in F$.*

Moreover, a filter F is called a prime *filter if*

- *$a \vee b \in F$ implies $a \in F$ or $b \in F$.*

The reader may note that the improper filter $F = A$ is included in this notion of prime filter [2]. A motivation for this choice is the following. Usually, the structure of prime filters are meant (in the intuitionistic case, for instance) to play the role of consistent theories with the disjunction property. Here, we are working in a paraconsistent setting, which allows us to drop the requirement of consistency for the considered theories with disjunction property. Therefore, we can allow the improper filter $F = A$ to appear among the prime filters, since the only reason why we usually drop it is because we do not want prime filters to contain 0, i.e., to be 'inconsistent'.

We use now this notion to build a top descriptive frame from an N-algebra. Consider, as the set of worlds,

$$W_{\mathfrak{A}} := \{F \mid F \text{ is a prime filter of } \mathfrak{A}\}.$$

[3] With 'positive reduct' we mean the structure $\langle W, R, \mathcal{P}\rangle$ in the case of the frame, and the gH-algebra $\langle \mathcal{P}, \cap, \cup, \to, W\rangle$ in the case of the dual algebra. Basically, the structures we would consider in the case of positive logic.

Given $F, F' \in W_{\mathfrak{A}}$, we consider

$$FR_{\mathfrak{A}}F' \text{ if and only if } F \subseteq F'.$$

Moreover, let $\mathcal{P}$ be defined as

$$\mathcal{P} := \{\hat{a} \mid a \in A\},$$

where $\hat{a} := \{F \in W_{\mathfrak{A}} \mid a \in F\}$. Finally, consider the function $N_{\mathfrak{A}} : \mathcal{P} \to \mathcal{P}$ defined as follows:

$$N_{\mathfrak{A}}(\hat{a}) := \widehat{(\neg a)}.$$

The structure $\mathfrak{A}_* := \langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, N_{\mathfrak{A}} \rangle$ is indeed a top descriptive frame. In fact, the presence of the improper prime filter A in the set $W_{\mathfrak{A}}$ ensures the obtained descriptive frame to be a top one. Finally, the locality property of the function $N_{\mathfrak{A}}$ easily follows from the fact that $\mathfrak{A}$ is an N-algebra and, in particular, from Lemma 4.1.1. We call the obtained structure the *dual frame* of $\mathfrak{A}$.

4.4 Duality

At this stage, we are ready to illustrate and study duality between the class of top descriptive frames and the variety of N-algebras. This duality, together with the algebraic completeness proved at the beginning of the chapter, will allow us to give a frame-based completeness result for all the extensions of N.

The first important fact to be considered is that every N-algebra can be seen as the dual of some top descriptive frame, and vice versa.

Theorem 4.4.1. *Let $\mathfrak{A}$ be an N-algebra and $\mathfrak{F}$ be a top descriptive frame. Then:*

1. $\mathfrak{A} \simeq (\mathfrak{A}_*)^*$.
2. $\mathfrak{F} \simeq (\mathfrak{F}^*)_*$.

Proof. We have to build the two natural maps and to show that they are the desired isomorphisms.

1. Consider an N-algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, \neg, 1 \rangle$ and let $W_{\mathfrak{A}}$ be as defined as in Section 4.3. We can define the isomorphism g from $\mathfrak{A}$ to $(\mathfrak{A}_*)^*$ as follows:

$$g : A \to \mathcal{P}_{\mathfrak{A}},$$

$$a \mapsto \hat{a}.$$

For the positive part of the proof the reader may see [12]. The interesting part of the proof for our scope is the one concerning the preservation of negation under g. What we want to show can be formalized as

$$g(\neg a) = N_{\mathfrak{A}}(g(a)),$$

since the negation operator on the algebra $(\mathfrak{A}_*)^*$ is exactly the function $N_{\mathfrak{A}}$. Now, by how the map g is defined, we know that

$$g(\neg a) = \widehat{(\neg a)}.$$

On the other hand, we also know that

$$N_{\mathfrak{A}}(\hat{a}) = \widehat{(\neg a)}.$$

Since $g(a)$ is exactly $\hat{a}$ by definition, we conclude that the negation operator is preserved, as desired.

2. Consider a top descriptive frame $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$. We can define an isomorphism between $\mathfrak{F}$ and $(\mathfrak{F}^*)_*$ as follows:

$$f : W \to W_{\mathfrak{F}^*},$$

$$w \mapsto \{U \in \mathcal{P} \mid w \in U\}.$$

We focus here again on the part of the proof concerning the negation, while for the positive part of the proof we refer to [12]. If we consider Definition 31, what we have to prove amounts to the following: given an upset $V \subseteq \mathcal{P}_{\mathfrak{F}^*}$ and given an element $w \in W$, then

$$f(w) \in N_{\mathfrak{F}^*}(V) \Leftrightarrow w \in N(f^{-1}(V)).$$

Let $V \subseteq \mathcal{P}_{\mathfrak{F}^*}$ be an upset. We know that V is of the form $\widehat{O}$, for a certain $O \in \mathcal{P}$. In fact, the set of admissible sets $\mathcal{P}_{\mathfrak{F}^*}$ is obtained from the algebra $\mathfrak{F}^*$ by taking the sets $\hat{a}$, for every element a of the algebra $\mathfrak{F}^*$; the considered algebra being the dual of the frame $\mathfrak{F}$, every element of the algebra is an element of $\mathcal{P}$, i.e., an admissible upset of W. Now, observe that:

$$f^{-1}(V) = \{w \in W \mid f(w) \in V\} = \{w \in W \mid \{U \in \mathcal{P} \mid w \in U\} \in V\} =$$

$$= \{w \in W \mid w \in O\} = O.$$

Consider an arbitrary $w \in W$. By how the maps f and $N_{\mathfrak{F}^*}$ are defined, we have:

$$f(w) \in N_{\mathfrak{F}^*}(V) = N_{\mathfrak{F}^*}(\widehat{O}) = \widehat{N(O)}.$$

This gives us that $f(w) \in N_{\mathfrak{F}^*}(V)$ is equivalent to $w \in N(O)$. Therefore, we can conclude that

$$f(w) \in N_{\mathfrak{F}^*}(V) \Leftrightarrow w \in N(f^{-1}(V)),$$

which is the desired result. □

As already said, this duality has been studied and used, from a categorical standpoint, in [2]. The idea behind it is the following: the necessity of a top node in the frame is a consequence of the fact that we 'drop' the bottom element on the algebraic side.

At this point, we are ready to state and prove a fundamental duality result for top descriptive frames. It is worth emphasizing that the proof of this result is very similar to the intuitionistic proof [12].

Theorem 4.4.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be N-algebras and $\mathfrak{F}$ and $\mathfrak{G}$ be top descriptive* N-*frames. Let $\{\mathfrak{A}_i\}_{i=1}^n$ and $\{\mathfrak{F}_i\}_{i=1}^n$ be sets of N-algebras and top descriptive frames, respectively. Then:*

1. (a) *$\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$ if and only if $\mathfrak{A}_*$ is isomorphic to a generated subframe of $\mathfrak{B}_*$.*
   (b) *$\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ if and only if $\mathfrak{A}_*$ is isomorphic to a p-morphic image of $\mathfrak{B}_*$.*
   (c) *$(\prod_{i=1}^n \mathfrak{A}_i)_*$ is isomorphic to the disjoint union $\biguplus_{i=1}^n (\mathfrak{A}_i)_*$.*
2. (a) *$\mathfrak{F}$ is isomorphic to a generated subframe of $\mathfrak{G}$ if and only if $\mathfrak{F}^*$ is a homomorphic image of $\mathfrak{G}^*$.*
   (b) *$\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$ if and only if $\mathfrak{F}^*$ is isomorphic to a subalgebra of $\mathfrak{G}^*$.*
   (c) *$(\biguplus_{i=1}^n \mathfrak{F}_i)^*$ is isomorphic to $\prod_{i=1}^n (\mathfrak{F}_i)^*$.*

Proof. In most of the cases within this proof, we limit ourselves to proving the steps of the proof related to the negation operator. In particular, the structure of the proof and the candidate functions are exactly the same as in the intuitionistic context, and the reader may see [12] for a detailed account.

1.(a) : Let h be a surjective homomorphism from $\mathfrak{B}$ onto $\mathfrak{A}$. The considered duality maps such a homomorphism to a p-morphism, as follows:

$$f : W_{\mathfrak{A}} \to W_{\mathfrak{B}},$$

$$F \mapsto h^{-1}(F).$$

We want to show that the function f is indeed an injective p-morphism. As anticipated, we focus only on the part of the proof concerning negation. This, in the case of descriptive frame p-morphisms, amounts to prove that the condition for the function N stated in Definition 31 is satisfied. Thus, consider a prime filter $F \in W_{\mathfrak{A}}$. We want to prove that, given an upset $V \in \mathcal{P}_{\mathfrak{B}}$,

$$F \in N_{\mathfrak{A}}(f^{-1}(V)) \Leftrightarrow f(F) \in N_{\mathfrak{B}}(V).$$

Observe that $V = \hat{a}$ and that, by definition of $N_{\mathfrak{B}}$, the image $f(F) = h^{-1}(F)$ is an element of $N_{\mathfrak{B}}(V)$ if and only if $h^{-1}(F)$ is a member of $\widehat{(\neg a)}$, i.e., $\neg a \in h^{-1}(F)$ in $\mathfrak{B}$. Therefore, we get that $h(\neg a)$ is an element of F, which from the fact that h is a homomorphism, gives us that $\neg(h(a)) \in F$. Therefore, this allows us to conclude that $F \in (\widehat{\neg(h(a))}) = N_{\mathfrak{A}}(\widehat{h(a)})$, by definition of $N_{\mathfrak{A}}$. At this point, it is enough to show that

$$f^{-1}(V) = (\widehat{h(a)}).$$

Consider a filter $G \in f^{-1}(V) = f^{-1}(\hat{a})$. Then, this is equivalent to

$$a \in f(G) = h^{-1}(G).$$

Therefore, we immediately get that $h(a) \in G$, as desired.

1.(b) : Assume here $\mathfrak{A}$ to be a subalgebra of $\mathfrak{B}$. We proceed here as in the previous case: we give a candidate map and we prove that it is indeed a surjective p-morphism between $\mathfrak{B}_*$ and $\mathfrak{A}_*$. Consider the following function:

$$f : W_{\mathfrak{B}} \to W_{\mathfrak{A}},$$

$$F \mapsto F \cap A,$$

where A represents the domain of the subalgebra $\mathfrak{A}$. Let us check that the defined function f satisfies the 'negative' condition of Definition 31. Let V be an arbitrary upset in $\mathcal{P}_{\mathfrak{A}}$. Assume $f(F) \in N_{\mathfrak{A}}(V)$. This means, assuming $V = \hat{a}$ for some $a \in A$, that $\neg a \in f(F)$. We rewrite this as $\neg a \in F \cap A$. Therefore, $\neg a$ is an element of the filter F in $\mathfrak{B}$ as well, which gives us that $F \in N_{\mathfrak{B}}(\hat{a})$. On the other hand, if $F \in N_{\mathfrak{B}}(\hat{a})$, i.e., F contains $\neg a$ with respect to $\mathfrak{B}$, we also have that $\neg a$ is in both F and A, since a is an element of A. Therefore, we have shown that:

$$F \in N_{\mathfrak{B}}(\hat{a}) \Leftrightarrow F \cap A \in N_{\mathfrak{A}}(\hat{a}),$$

for every $a \in A$. The only remaining thing to check now is that $f^{-1}(\hat{a}) = \hat{a}$, i.e.,

$$\{G \in W_{\mathfrak{B}} \mid a \in G \cap A\} = \{G \in W_{\mathfrak{B}} \mid a \in G\},$$

which is trivially true.

1.(c) : We give this proof considering only a pair of algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$. An easy induction can extend the proof to finitely many algebras. Note that Definition 32 gives us that the elements of $(\mathfrak{A}_1)_* \uplus (\mathfrak{A}_2)_*$ are equivalence classes of the form $[F]$, where F is a filter of either $\mathfrak{A}_1$ or $\mathfrak{A}_2$. We denote as W the domain

$$(W_1 \uplus W_2)/\sim$$

of $(\mathfrak{A}_1)_* \uplus (\mathfrak{A}_2)_*$ and we use $W_{\mathfrak{A}_1 \times \mathfrak{A}_2}$ to denote the domain of the frame $(\mathfrak{A}_1 \times \mathfrak{A}_2)_*$. Consider the following map:

$$f : W \to W_{\mathfrak{A}_1 \times \mathfrak{A}_2},$$

$$[F_1] \mapsto \{(a_1, a_2) \mid a_1 \in F_1 \text{ and } a_2 \in A_2\}, \text{ for every } A_1 \neq F_1 \in W_1,$$

$$[F_2] \mapsto \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in F_2\}, \text{ for every } A_2 \neq F_2 \in W_2,$$

$$[A_i] \mapsto A_1 \times A_2,$$

where A_i is the domain of $\mathfrak{A}_i$. First of all, observe that the considered function f is well-defined, as well as injective. In fact, consider without loss of generality an equivalence class $[F_1]$, with F_1 filter of $\mathfrak{A}_1$, and focus on its image by means of f. Clearly, the set

$$\{(a_1, a_2) \mid a_1 \in F_1 \text{ and } a_2 \in A_2\}$$

is upward closed, as well as closed under $\wedge$. Moreover,

$$(a_1, a_2) \vee (a_3, a_4) \in f([F_1])$$

implies $a_1 \vee a_3 \in F_1$ and $a_2 \vee a_4 \in A$. The set F_1 being a filter in A_1, we get that $a_1 \in F_1$ or $a_3 \in F_1$. Therefore, the image $f([F_1])$ is a prime filter of $\mathfrak{A}_1 \times \mathfrak{A}_2$. It is immediate to see that this holds for the top nodes as well, i.e., the image of $[A_i]$ by means of the map f is the improper prime filter of $\mathfrak{A}_1 \times \mathfrak{A}_2$. In order to get surjectivity, we can check that every prime filter F of $\mathfrak{A}_1 \times \mathfrak{A}_2$ is of the form:

$$\{(a_1, a_2) \mid a_1 \in F_1 \subseteq A_1 \text{ and } a_2 \in F_2 \subseteq A_2\},$$

where F_i is a prime filter of A_i. Without loss of generality, if we consider the set

$$\{a \in A_1 \mid \text{ there is } a' \in A_2 \text{ such that } (a, a') \in F\},$$

we need to show that such a set is a filter in A_1. This follows easily from the fact that F is a filter, and allows us to claim that the map f is bijective as well. At this point, the remaining part of the proof amounts to checking the p-morphism conditions from Definition 31. The positive part of that definition can be checked smoothly, similarly to the intuitionistic case. Hence, we decide to focus only on proving that the condition concerning N holds. We need to show that, given an element $[F]$ of W, for every upset $V \in \mathcal{P}_{\mathfrak{A}_1 \times \mathfrak{A}_2}$:

$$[F] \in N(f^{-1}(V)) \Leftrightarrow f([F]) \in N_{\mathfrak{A}_1 \times \mathfrak{A}_2}(V).$$

Consider an element $F_1 \subset A_1$ such that $f([F_1]) \in N_{\mathfrak{A}_1 \times \mathfrak{A}_2}(V)$, for some upset $V \in \mathcal{P}_{\mathfrak{A}_1 \times \mathfrak{A}_2}$. Given that, by construction, $V = \hat{a}$ for some $a = (a_1, a_2) \in \mathfrak{A}_1 \times \mathfrak{A}_2$, we get that $f([F_1]) \in N_{\mathfrak{A}_1 \times \mathfrak{A}_2}(V)$ is equivalent to

$$f([F_1]) \in N_{\mathfrak{A}_1 \times \mathfrak{A}_2}\big(\widehat{(a_1, a_2)}\big) = (\widehat{\neg_1 a_1, \neg_2} a_2).$$

This can be equivalently rewritten as $(\neg_1 a_1, \neg_2 a_2) \in f([F_1])$. At this point, by using the definition of f, we obtain that

$$(\neg_1 a_1, \neg_2 a_2) \in f([F_1]) \Leftrightarrow \neg_1 a_1 \in F_1,$$

which is in turn equivalent to

$$F_1 \in N_1(\hat{a}_1).$$

Observe now that, in terms of equivalence classes with respect to $\sim$, this means:

$$[F_1] \in N_1(\hat{a}_1)/\sim .$$

Consider now the upset $N_2(\hat{a}_2)$. Given that we are dealing with disjoint unions, we get that

$$N_1(\hat{a}_1)/\sim \cup\, N_2(\hat{a}_2)/\sim\, = \{[F] \in W \mid F \in N_1(\hat{a}_1)\} \cup \{[F] \in W \mid F \in N_2(\hat{a}_2)\} =$$

$$= \{[F] \in W \mid F \in N_1(\hat{a}_1) \cup N_2(\hat{a}_2)\} = (N_1(\hat{a}_1) \cup N_2(\hat{a}_2))/\sim =$$

$$= N\big((\hat{a}_1 \cup \hat{a}_2)/\sim\big).$$

Therefore, we have

$$[F_1] \in N\big((\hat{a}_1 \cup \hat{a}_2)/\sim\big).$$

At this point, it suffices to be proved that

$$f^{-1}(V) = f^{-1}(\hat{a}) = (\hat{a}_1 \cup \hat{a}_2)/\sim .$$

Consider an element $[G] \in f^{-1}(V)$. This means that $f([G]) \in V = \widehat{(a_1, a_2)}$, i.e., $a_1 \in G$ and $a_2 \in A_2$. Then, $G \in \hat{a}_1$, and hence it is an element of $\hat{a}_1 \cup \hat{a}_2$ as well. Therefore, we get that

$$[G] \in (\hat{a}_1 \cup \hat{a}_2)/\sim .$$

On the other hand, assume $[G] \in (\hat{a}_1 \cup \hat{a}_2)/\sim$. Then, this means that either $a_1 \in G$ or $a_2 \in G$, which gives us that the pair (a_1, a_2) surely is in $f([G])$. Hence, $[G] \in f^{-1}(V)$, and we are done. We do not unfold the symmetric case concerning an element $F_2 \in W_2$, since the proof goes exactly the same way. Finally, for the case in which $F = A_i$, the equivalence class $[A_i]$ is in every upset (since it is the top node of W), and the image $A_1 \times A_2$ as well (since it is the improper filter of $\mathfrak{A}_1 \times \mathfrak{A}_2$, and hence the top node of $W_{\mathfrak{A}_1 \times \mathfrak{A}_2}$). Therefore, this step of the proof is concluded.

2.(a) : Without loss of generality, we can assume $\mathfrak{F}$ to coincide with its isomorphic generated subframe of $\mathfrak{G}$. Let us make use of V to denote the domain of $\mathfrak{F}$. Then, consider the candidate map:

$$g : \mathcal{P}_{\mathfrak{G}} \to \mathcal{P}_{\mathfrak{F}},$$

$$U \mapsto U \cap V.$$

We want to prove that this is indeed a surjective homomorphism between the dual algebras. We keep focusing only on the 'negative' part of the proof. This amounts to proving that:

$$g(N_{\mathfrak{G}}(U)) = N_{\mathfrak{F}}(g(U)),$$

for every upset $U \in \mathcal{P}_{\mathfrak{G}}$. Consider $g(N_{\mathfrak{G}}(U))$, for some $U \in \mathcal{P}_{\mathfrak{G}}$. By how we defined g, we have that

$$g(N_{\mathfrak{G}}(U)) = N_{\mathfrak{G}}(U) \cap V.$$

Observe that, by Lemma 4.3.2,

$$N_{\mathfrak{G}}(U) \cap V = N_{\mathfrak{G}}(U \cap V) \cap V.$$

At this point, we want to make use of the actual definition of $N_{\mathfrak{F}}$, for which we refer to Definition 31. Such a definition gives us that

$$N_{\mathfrak{F}}(U \cap V) = N_{\mathfrak{G}}(U \cap V) \cap V.$$

Therefore, we are able to conclude

$$g(N_{\mathfrak{G}}(U)) = N_{\mathfrak{G}}(U) \cap V = N_{\mathfrak{F}}(U \cap V) = N_{\mathfrak{F}}(g(U)),$$

as desired.

2.(b) : Let now p be a p-morphism from $\mathfrak{G}$ onto $\mathfrak{F}$. Our aim here is to make use of such p-morphism p to build an isomorphism between $\mathfrak{F}^*$ and a subalgebra of $\mathfrak{G}^*$. Consider the following map:

$$g : \mathcal{P}_{\mathfrak{F}} \to \mathcal{P}_{\mathcal{G}},$$

$$U \mapsto p^{-1}(U).$$

Let us focus once again on the 'negative' part of the proof. Consider an arbitrary $U \in \mathcal{P}_{\mathfrak{F}}$, in order to show that

$$g(N_{\mathfrak{G}}(U)) = N_{\mathfrak{F}}(g(U)).$$

Observe that $g(N_{\mathfrak{F}}(U))$ is defined as $p^{-1}(N_{\mathfrak{F}}(U))$. This means that $w \in g(N_{\mathfrak{F}}(U))$ if and only if $p(w) \in N_{\mathfrak{F}}(U)$. Consider now the fact that p is a p-morphism, and hence, we have that

$$p(w) \in N_{\mathfrak{F}}(U) \Leftrightarrow w \in N_{\mathfrak{G}}(p^{-1}(U)),$$

by Definition 31. Indeed, this is exactly equivalent to saying

$$w \in g(N_{\mathfrak{F}}(U)) \Leftrightarrow w \in N_{\mathfrak{G}}(g(U)),$$

and hence, we are done.

2.(c) : As in the case of 1.(c), we give the proof just for two frames. It can be extended to finitely many frames by induction. Let $\mathcal{P}_{\mathfrak{F}_1 \uplus \mathfrak{F}_2}$ denote the set of admissible upsets of the disjoint union of $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Observe that every upset U in $\mathcal{P}_{\mathfrak{F}_1 \uplus \mathfrak{F}_2}$ is a set of equivalence classes $[w] \in (W_1 \uplus W_2)/\sim$. In particular, the upsets $U \in \mathcal{P}_{\mathfrak{F}_1 \uplus \mathfrak{F}_2}$ are of the form $(U_1 \uplus U_2)/\sim$, where $U_i \in \mathcal{P}_i$. Hence, consider now a function defined in the following way:

$$g : \mathcal{P}_{\mathfrak{F}_1 \uplus \mathfrak{F}_2} \to \mathcal{P}_1 \times \mathcal{P}_2,$$

$$U \mapsto (U_1, U_2),$$

for each $U = (U_1 \uplus U_2)/\sim$. The function g is clearly well-defined and bijective. Moreover, the maximum element W of the algebra $\mathcal{P}_{\mathfrak{F}_1 \uplus \mathfrak{F}_2}$ is given by the quotient $(W_1 \uplus W_2)/\sim$ and is therefore mapped to the pair (W_1, W_2). The positive part of the proof goes similarly to the intuitionistic case, and hence, we only focus on the homomorphism condition concerning the negation operator. We want to check that

$$g(N(U)) = N'(g(U)),$$

where N comes from $\mathfrak{F}_1 \uplus \mathfrak{F}_2$, and N' comes from $\mathfrak{A}_{\mathfrak{F}_1} \times \mathfrak{A}_{\mathfrak{F}_2}$ and is defined as

$$N'(V_1, V_2) = (N_1(V_1), N_2(V_2)),$$

with N_i being the function of $\mathfrak{F}_i$ and $(V_1, V_2) \in \mathcal{P}_1 \times \mathcal{P}_2$. Let U represent an upset of the form $(U_1 \uplus U_2)/\sim$, for some $U_1 \in \mathcal{P}_1$ and $U_2 \in \mathcal{P}_2$. Recall that the function N is defined in the following way:

$$N(U) = \big(N_1(U_1) \uplus N_2(U_2)\big)/\sim .$$

Hence, by definition of the map g,

$$g(N(U)) = (N_1(U_1), N_2(U_2)).$$

Thus, we have the following chain of equalities:

$$g(N(U)) = (N_1(U_1), N_2(U_2)) = N'(U_1, U_2) = N'(g(U)).$$

This gives us the desired results, and the proof of the whole theorem is concluded. $\square$

The above duality results give us a clear picture of the relation between the class of top descriptive frames and the class of N-algebras. We are now ready to employ this duality to get the desired completeness result.

4.5 Completeness

This last section wants to finalize the frame-based completeness result for all the extensions of the basic logic N. The idea is that each of these extensions is complete with respect to a certain variety of algebras, which is itself the dual of a class of top descriptive frames. This duality preserves validity of formulas and allows us to get a completeness result with respect to the considered class of top descriptive frames. Even more: every descriptive frame can have associated a unique top descriptive frame which makes exactly the same formulas valid. Therefore, the completeness result can be extended to the class of all descriptive frames as defined in Definition 29.

Let us define the notion of model corresponding to descriptive frames.

Definition 34. *A* descriptive model *for* N *is a pair* $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ *such that* $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ *is a descriptive frame and* $V : Prop \to \mathcal{P}$ *is a* descriptive valuation *map.*

Validity of formulas in a descriptive frame is defined in the same way as in the case of Kripke frames. The reader may note that, given a top descriptive frame, the top node makes every formula of the language true.

The correspondence between top descriptive frames and N-algebras can be extended to a correspondence between top descriptive models and N-algebras with valuations. We make this statement formal with the following result. Recall that we use here the notation $\langle \mathfrak{A}, v \rangle \vDash \varphi$ to say that the formula φ is true in $\mathfrak{A}$ under the valuation v. In a similar way, we use $\langle \mathfrak{A}_*, V \rangle \vDash \varphi$ to mean that the formula φ is satisfied at every node of the dual frame $\mathfrak{A}_*$ under the valuation V.

Lemma 4.5.1. *Let* $\mathfrak{A}$ *be an N-algebra. Then,*

$$\langle \mathfrak{A}, v \rangle \vDash \varphi \Leftrightarrow \langle \mathfrak{A}_*, V \rangle \vDash \varphi,$$

where $V(p) = \widehat{v(p)}$.

Proof. The first thing to be proved here is that the relation between V and v can be extended to all formulas, i.e.,

$$V(\varphi) = \widehat{v(\varphi)}.$$

The proof goes by induction on the structure of φ. We leave the positive steps to the reader, who can find them in [6]. We unfold here the step concerning negation.

$$\widehat{v(\neg\varphi)} = \widehat{\neg v(\varphi)} = N(\widehat{v(\varphi)}) = N(V(\varphi)) = V(\neg\varphi).$$

Observe that the first equality comes from Definition 25, while the second one is an application of the definition of N from an algebra to its dual frame. By induction hypothesis we can use the fact that $V(\varphi) = \widehat{v(\varphi)}$. Finally, the last equality comes from the fact that the function N exactly defines the valuation of negated formulas. At this point, we can use this result to get:

$$\langle \mathfrak{A}, v \rangle \vDash \varphi \Leftrightarrow v(\varphi) = 1 \Leftrightarrow V(\varphi) = \widehat{1} = W_{\mathfrak{A}} \Leftrightarrow \langle \mathfrak{A}_*, V \rangle \vDash \varphi.$$

Therefore, every algebra makes true exactly the same formulas as its dual top descriptive frame, as desired. □

At this point, we have got all the necessary tools to prove the frame-based completeness result. Given an arbitrary class $\mathcal{C}$ of N-algebras, let us denote as $\mathcal{C}_*$ the class of the corresponding top descriptive frames, i.e.,

$$\mathcal{C}_* := \{\mathfrak{A}_* \mid \mathfrak{A} \in \mathcal{C}\}.$$

Theorem 4.5.2. *Every extension* $\mathbf{L}$ *of the basic logic of a unary operator* N *is sound and complete with respect to the class of top descriptive frames* $(\mathbf{V_L})_*$.

Proof. Let $\mathbf{L}$ be one of the considered logical systems. Then,

$$\begin{aligned}\mathbf{L} \vdash \varphi \Leftrightarrow \mathbf{V_L} \vDash \varphi \Leftrightarrow \mathfrak{A} \vDash \varphi, &\text{ for any } \mathfrak{A} \in \mathbf{V_L} \\ \Leftrightarrow \langle \mathfrak{A}, v\rangle \vDash \varphi, &\text{ for any } \mathfrak{A} \in \mathbf{V_L} \text{ and any valuation } v \\ \Leftrightarrow \langle \mathfrak{A}_*, V\rangle \vDash \varphi, &\text{ for any } \mathfrak{A} \in (\mathbf{V_L})_* \text{ and any valuation } V \\ \Leftrightarrow \mathfrak{A}_* \vDash \varphi, &\text{ for any } \mathfrak{A} \in (\mathbf{V_L})_* \Leftrightarrow (\mathbf{V_L})_* \vDash \varphi,\end{aligned}$$

which is the desired completeness result. □

Remark. We give an argument here to prove that, given an arbitrary descriptive frame, there exists a corresponding top descriptive frame on which exactly the same formulas are valid. In order to see this, assume we have a descriptive frame $\mathfrak{F} = \langle W, R, N, \mathcal{P}\rangle$, and consider the corresponding top descriptive frame $\mathfrak{F}_T = \langle W_T, R_T, N_T, \mathcal{P}_T\rangle$ built as follows: $W_T = W \cup \{t\}$, where t is a new node; the relation R_T is obtained from R by adding the fresh node to the top, i.e.,

$$R_T = R \cup \{(w, t) \mid w \in W\};$$

the set of admissible upsets is obtained as follows

$$\mathcal{P}_T = \{U \cup \{t\} \mid U \in \mathcal{P}\};$$

and finally, the function N_T is naturally obtained from N in the following way: given an element $V = U \cup \{t\} \in \mathcal{P}_T$, define $N_T(V)$ as $N(U) \cup \{t\}$. The fact that this new frame has the same positive validities (i.e., formulas not containing $\neg$) as the original frame follows easily from §3 of [5][4]. We want here to deal with negative validities as well, and to show they are invariant under the addition of the top node. It is enough to prove that for every world $w \in W$,

$$\langle \mathfrak{F}, V\rangle, w \vDash \neg\varphi \Leftrightarrow \langle \mathfrak{F}_T, V_T\rangle, w \vDash \neg\varphi,$$

for every valuation map V, where $V_T(p) = V(p) \cup \{t\}$ for every $p \in \mathit{Prop}$. The whole proof goes by induction, hence we need to unfold an induction step. The

[4]In fact, by how $\mathcal{P}_T$ has been built, we can conclude that every propositional variable is true at the top node (indeed, the empty-set is *not* an element of $\mathcal{P}_T$). Observe that, in addition to that, we can even claim that *every* formula is true at the top node t, since t is an element of every upset.

induction hypothesis allows us to assume:

$$\langle \mathfrak{F}, V \rangle, w \vDash \varphi \Leftrightarrow \langle \mathfrak{F}_T, V_T \rangle, w \vDash \varphi.$$

The induction hypothesis can be rewritten as

$$V_T(\varphi) = V(\varphi) \cup \{t\}.$$

Recall that $\langle \mathfrak{F}, V \rangle, w \vDash \neg\varphi$ means, in terms of valuation sets, that $w \in N(V(\varphi))$. By definition of N_T, we know that

$$N_T(V_T(\varphi)) = N(V(\varphi)) \cup \{t\}.$$

This gives us immediately the desired conclusion, since we can say that for every $w \in W$,

$$w \in N(V(\varphi)) \Leftrightarrow w \in N_T(V_T(\varphi)).$$

With this result, we are able to claim that the completeness-direction of Theorem 4.5.2 amounts to a proof of completeness with respect to the class of descriptive frames. Moreover, the basic logic of a unary operator N is indeed sound and complete with respect to the class of descriptive frames as defined in Definition 29.

Let us emphasize a further interesting fact concerning finite frames. Consider the more general notion of *positive morphism* as defined in [5]. We can enhance this notion in order to get a corresponding notion of partial descriptive morphism for descriptive frames as defined in Definition 29. In particular, given two descriptive frames $\mathfrak{F} = \langle W, R, N, \mathcal{P} \rangle$ and $\mathfrak{F}' = \langle W', R', N', \mathcal{P}' \rangle$, we can consider a notion of function between $\langle W, R \rangle$ and $\langle W', R' \rangle$ as defined in Definition 20 of [5], and require the following additional properties: for every $U' \in \mathcal{P}'$, we want $h^*(U')$ to be in $\mathcal{P}$, where

$$h^*(U') := W \setminus R^{-1}(h^{-1}(W' \setminus U'));$$

moreover, we want

$$h^*(N'(U')) = N(h^*(U')).$$

Given this notion, for every finite frame $\mathfrak{F}$ we can build a partial descriptive morphism from the corresponding top frame $\mathfrak{F}_T$ which is onto over $\mathfrak{F}$. As a matter of fact, it is enough to consider the identity map for all the non-top nodes of $\mathfrak{F}$.

These last results and observations conclude the algebraic account of the considered systems. We have defined the variety of structures corresponding to the basic logic N and we have proved algebraic completeness with respect to such a variety. The second part of the chapter was devoted to build some tools, such as the notion of general and descriptive frames, in order to exhibit a duality theory for the considered algebras. The final section of the chapter makes use of the obtained tools to transfer the completeness from the algebraic side to the frame-based counterpart.

Several further issues naturally disclose themselves. A first option is to analyze the topological and categorical counterpart of the considered duality. As already said, the positive reduct of the structures we are considering is given by the structure of pointed Esakia spaces [2]. Obviously, we would need to equip such structures with a map behaving as the function N and simulating the locality property. Another immediate question to be investigated concerns the so-called universal models [3]. In particular, two different notions of universal model for positive logic are introduced in [5], and one of them corresponds to the positive part of the notion of top descriptive frame introduced here. It could be interesting to try to extend the study from [5] to the more general setting presented within this work. Formulas similar to the Jankov-de Jongh formulas seem to be of interest as well.

More generally, here we just scratched the surface of algebraic semantics: we defined and introduced the general setting and its relation both with the Kripke-style semantics and with the syntax. Many intriguing questions are still waiting for an answer.

Chapter 5

Sequent Calculi

The sequent calculus is a formalism which was introduced by Gentzen in 1935 [21, 22] and, for this reason, it is often referred to as *Gentzen system*. He considered this formalism to be more manageable and practical than natural deduction, for which the *normalization* technique was not yet available, and he proved the fundamental *cut elimination* result. Nowadays, the two formal systems of sequent calculus and natural deduction are often considered equally good alternatives in the area of structural proof theory, and the choice between them and their variants depends on particular applications of the proof theory one has in mind. The currently available technique of normalization characterizing natural deduction is comparable to the one of cut elimination developed by Gentzen. Sequent calculi with the subformula property and the separation property are especially worth being studied from a computational point of view.

Chapter 5 and Chapter 6 represent the 'proof-theoretic' side of this thesis. In the current chapter, we define sequent calculus systems for all the subminimal logics we have been considering. Indeed, the sequent calculi presented here turn out to be cut-free systems with the subformula property. We introduce here sequent systems of type **G1** [44]. Later on, we move forward to **G3**-systems and we prove completeness with respect to the Kripke semantics defined in Chapter 1 and Chapter 2. We choose to closely follow the exposition of the sequent proof theory for intuitionistic and minimal logic as it is presented in [44].

5.1 The G1-systems

We begin the section introducing some useful notation. The variable p will range over the set of propositional letters. The variables α, β and φ will range over all formulas, while Γ, Δ will represent finite[1] *multisets*[2] of formulas [32]. A multiset of formulas can be simply conceived as a set of formulas in which the number of occurrences of a formula *does* matter: we want $\{\varphi, \varphi\}$ to be different than $\{\varphi\}$. For the formal definition of *multiset*, refer to [32].

[1] We allow the sets Γ, Δ to be empty.

[2] We have made a design choice and based sequents on multisets instead of *sets* of formulas, thus we are not concealing contractions in the notation. It is convenient to deal with multisets rather than with sets of formulas when considering backward proof-search procedures and complexity-related problems, which we only briefly mention here and which are part of the future work.

Definition 35 (Sequent)**.** *A* sequent *for a language $\mathcal{L}$ is an ordered pair of finite multisets of $\mathcal{L}$-formulas. The sequent given by the ordered pair* $(\Gamma,\ \Delta)$ *is represented as:*

$$\Gamma \Rightarrow \Delta.$$

If Δ *contains* at most one *formula, the sequent is referred to as* single-conclusion.

Observe that, when representing a sequent, the notation

$$\Gamma, \Gamma'$$

denotes the multiset of formulas obtained as the union of the multisets Γ and Γ'.

As customary, we denote the left rule for a connective $\circ$ as $\circ\mathsf{L}$ and the right rule as $\circ\mathsf{R}$. We refer to the rules in which the connective is simultaneously introduced to the left and to the right by means of suitable names, denoting the intended meaning of the rule.

Definition 36 (**G1**-systems)**.** *Proofs (alternatively, deductions or derivations) are finite trees with a singe root, labelled with sequents. The axioms are at the leaves of the tree. The axioms and rules for the* **G1***-systems are the following:*

Logical Rules for the Positive Fragment

$$\mathsf{Ax}\quad p \Rightarrow p \qquad\qquad \top\mathsf{R}\quad \Rightarrow \top$$

$$\wedge\mathsf{L}\ \frac{\Gamma, \alpha, \beta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi} \qquad\qquad \wedge\mathsf{R}\ \frac{\Gamma \Rightarrow \alpha \qquad\qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

$$\vee\mathsf{L}\ \frac{\Gamma, \alpha \Rightarrow \varphi \qquad\qquad \Gamma, \beta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta \Rightarrow \varphi} \qquad\qquad \vee\mathsf{R}\ \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ , i = 1, 2$$

$$\rightarrow\mathsf{L}\ \frac{\Gamma \Rightarrow \alpha \qquad\qquad \Gamma, \beta \Rightarrow \varphi}{\Gamma, \alpha \rightarrow \beta \Rightarrow \varphi} \qquad\qquad \rightarrow\mathsf{R}\ \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}$$

Logical Rules for the $\neg$ Operator

$$\mathsf{N}\ \frac{\Gamma, \alpha \Rightarrow \beta \qquad\qquad \Gamma, \beta \Rightarrow \alpha}{\Gamma, \neg\alpha \Rightarrow \neg\beta} \qquad\qquad \mathsf{NeF}\ \frac{\Gamma, \Rightarrow \alpha}{\Gamma, \neg\alpha \Rightarrow \neg\beta}$$

$$\mathsf{CoPC}\ \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma, \neg\beta \Rightarrow \neg\alpha} \qquad\qquad \mathsf{An}\ \frac{\Gamma, \alpha \Rightarrow \neg\alpha}{\Gamma \Rightarrow \neg\alpha}$$

Structural Rules of Weakening (W) and Contraction (C)

$$\mathsf{W}\ \frac{\Gamma \Rightarrow \varphi}{\Gamma, \alpha \Rightarrow \varphi} \qquad\qquad \mathsf{C}\ \frac{\Gamma, \alpha, \alpha \Rightarrow \varphi}{\Gamma, \alpha \Rightarrow \varphi}$$

The system **G1n** *for the basic logic of unary operator* N *is obtained from the positive system by adding the Weakening and Contraction rules, together with the rule* N*. The systems for negative* ex falso *logic* **G1nef** *and contraposition logic* **G1copc** *are obtained, respectively, by adding the rule* NeF *to the sequent system for* N *and by substituting the rule* N *with the rule* CoPC *in* **G1n**. *Finally, the system for minimal propositional logic, denoted here as* $\mathbf{G1m}_{\neg}$*, is obtained by adding* An *to the system* **G1copc**.

The multiset Γ and the formula φ occurring in the rules will be referred to as *context*. Concerning the conclusion of each rule, the formula not in the context is the *principal formula*, i.e., the one whose main connective has just been introduced. Observe that Weakening and Contraction are indeed quite natural rules to consider, if one interprets a sequent of the form $\Gamma \Rightarrow \varphi$ to mean

$$\text{“} \bigwedge \Gamma \text{ implies the formula } \varphi \text{”.}$$

Given that the Weakening and Contraction rules do not introduce any connective, they are called *structural rules*.

In the two-premise rules, the contexts in both premises are the same (except in the case of $\to \mathsf{L}$, because of the restriction to have one formula in the consequent). All the rules are indeed *context-sharing*.

The systems introduced obey the so-called *subformula property*: in every derivation of a sequent $\Gamma \Rightarrow \varphi$, only subformulas of Γ, φ appear. It is easy to see, for each rule of the systems from Definition 36, that the premises only contain subformulas of the conclusion. Probably the most important consequence of this is the *separation property* for such systems, providing their completeness. The separation property denotes the fact that the proof of a sequent $\Gamma \Rightarrow \varphi$ requires logical rules only for the logical operators which actually occur in the sequent.

Let us give some brief comments on the rules of the system. Note that the conjunction, disjunction and implication rules and the two axioms simply represent the Gentzen **G1**-system for positive logic [44]. For this reason, we may refer to such system as *positive system*. The only rules which need additional explanations are the introduction rules for negation.

Remark. The aim of the negation rules is the one of simulating the action of the respective logical axioms. The rules N, NeF and CoPC introduce principal formulas both on the left and on the right-hand side of the conclusive sequent. On the other hand, the An-rule is a right rule.

There are already many different variants of Gentzen systems for minimal logic. Some of them are presented in [44] and they are generally obtained from the corresponding intuitionistic calculus by dropping the rule concerning $\bot$. The sequent system for minimal logic defined here uses the result proved in Proposition 1.2.2:

an alternative axiomatization of MPC is given by CoPC, together with An. In particular, this system allows us to keep focusing on the negation operator. Hence, we denote it as $\mathbf{G1m}_{\neg}$, following [44] and emphasizing the role of negation.

In what follows, we want to focus on showing that we can define alternative systems, which are closed under the structural rules of Weakening and Contraction.

5.2 Absorbing the Structural Rules

In this section we want to give alternative but equivalent Gentzen systems obtained by 'absorbing' Weakening and Contraction into the rules of the system. This version of the systems has computational advantages, namely it has advantages in the upside down procedure of proof-search for a given sequent.

We start by defining the alternative systems. Following [44], we refer to them as **G3**-systems.

Definition 37 (**G3**-systems)**.** *The axioms and rules for the* **G3***-systems are the following:*

$$\mathsf{Ax}\ \ \Gamma, p \Rightarrow p \qquad\qquad \top\mathsf{R}\ \ \Gamma \Rightarrow \top$$

$$\wedge\mathsf{L}\ \frac{\Gamma, \alpha, \beta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta \Rightarrow \varphi} \qquad\qquad \wedge\mathsf{R}\ \frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

$$\vee\mathsf{L}\ \frac{\Gamma, \alpha \Rightarrow \varphi \qquad \Gamma, \beta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta \Rightarrow \varphi} \qquad\qquad \vee\mathsf{R}\ \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ , i = 1, 2$$

$$\to\mathsf{L}\ \frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \qquad \Gamma, \beta \Rightarrow \varphi}{\Gamma, \alpha \to \beta \Rightarrow \varphi} \qquad\qquad \to\mathsf{R}\ \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}$$

$$\mathsf{N}\ \frac{\Gamma, \neg\alpha, \alpha \Rightarrow \beta \qquad \Gamma, \neg\alpha, \beta \Rightarrow \alpha}{\Gamma, \neg\alpha \Rightarrow \neg\beta} \qquad\qquad \mathsf{NeF}\ \frac{\Gamma, \neg\alpha \Rightarrow \alpha}{\Gamma, \neg\alpha \Rightarrow \neg\beta}$$

$$\mathsf{CoPC}\ \frac{\Gamma, \neg\beta, \alpha \Rightarrow \beta}{\Gamma, \neg\beta \Rightarrow \neg\alpha} \qquad\qquad \mathsf{An}\ \frac{\Gamma, \alpha \Rightarrow \neg\alpha}{\Gamma \Rightarrow \neg\alpha}$$

We denote as **G3n** *the system for the basic logic* N*, obtained from the positive system by adding the rule* N*. The system* **G3nef** *is defined as* **G3n** + NeF*. Substituting the rule* N *with the rule* CoPC *in* **G3n***, we obtain a sequent system for contraposition logic* **G3copc***. The* **G3***-system for minimal propositional logic is obtained by adding* An *to the system* **G3copc***.*

We emphasize here the main differences between the systems from Definition 36 and the ones in Definition 37 above. As already anticipated, the structural rules of Weakening and Contraction are absorbed into the rules of the systems. In particular, Weakening is absorbed into the axioms, as we allow non-empty context Γ. Moreover, Contraction needs to be built in the $\to \mathsf{L}$ rule, as well as in the non-right rules for $\neg$. The usefulness of these technicalities is going to be clear later, when dealing with admissibility issues.

Let us introduce some notation. We denote as *depth* of a derivation of a sequent $\Gamma \Rightarrow \varphi$ the length (i.e., the number of inference steps) of a maximal branch within the tree representing the derivation. Observe that the depth of an axiom is $n = 0$. Also observe that the depth of a derivation whose last inference step is obtained by means of a one-premise rule is $n = m+1$, where m denotes the depth of the premise. Finally, if the last inference rule is a two-premise rule, the depth n is the maximum $\max\{m_1, m_2\} + 1$, where m_1 and m_2 represent, respectively, the depth of the derivation of the left and of the right premise sequent. In this and in the next chapter, we write $\vdash_n \Gamma \Rightarrow \varphi$ as "the sequent $\Gamma \Rightarrow \varphi$ has been obtained via a deduction of depth n". Moreover, we write $\vdash_{\leq n} \Gamma \Rightarrow \varphi$ as "the sequent $\Gamma \Rightarrow \varphi$ has been obtained via a deduction of depth *at most* n".

In order to be able to use Weakening and Contraction while working within the **G3**-systems, we have to show that these structural rules are *admissible* in the considered systems. The definition given in Chapter 3 corresponds to a general notion of admissibility. In a sequent calculus setting, a rule is said to be admissible if the existence of a derivation for the premises implies the existence of a deduction for the conclusion sequent. Similarly, a rule is said to be *depth-preserving admissible* if admissibility preserves the depth, i.e., if the premises are provable via a deduction of depth n, then the conclusive sequent is derivable via a proof whose depth is *at most* n. Therefore, what we have to show amounts to proving that the systems are depth-preserving closed under the structural rules of Weakening and Contraction.

Theorem 5.2.1 (Admissibility of Weakening Rule)**.** *The Weakening rule is admissible in the considered sequent calculus systems. In particular, it is depth-preserving admissible.*

Proof. Here, we show that whenever there is a derivation $\vdash_n \Gamma \Rightarrow \varphi$ of depth n, it is possible to obtain a derivation $\vdash_{\leq n} \Gamma, \alpha \Rightarrow \varphi$ by means of the appropriate rules in each system. We prove the considered result by induction on the depth n of the derivations.

<u>Base case</u>: Consider the case in which the depth n of the derivation is 0, i.e., the sequent $\Gamma \Rightarrow \varphi$ is an axiom. Two options need to be considered. First, suppose that φ is a propositional variable p and $p \in \Gamma$. In this case, $\Gamma, \alpha \Rightarrow p$ is still an axiom and therefore it is derivable in the system. The second case goes similarly. Suppose φ is $\top$. Of course, $\Gamma, \alpha \Rightarrow \top$ is an axiom too and hence, it is derivable.

Now, we need to take care of the induction step: $n > 0$. As the induction hypothesis, let us assume that $\vdash_m \Gamma \Rightarrow \varphi$ always implies $\vdash_{\leq m} \Gamma, \alpha \Rightarrow \varphi$, for each $m < n$. In order to cover all the possible cases, it is necessary to consider all the possibilities for the rule applied as the last step of the derivation of $\Gamma \Rightarrow \varphi$.

$\wedge\mathsf{L}$: Suppose that the sequent $\Gamma \Rightarrow \varphi$ has been obtained by applying the left rule for the conjunction at the last step. Thus, Γ is of the form $\Gamma_1, \beta_1 \wedge \beta_2$, and the last inference of the derivation looks like:

$$\wedge\mathsf{L}\ \frac{\Gamma_1, \beta_1, \beta_2 \Rightarrow \varphi}{\Gamma_1, \beta_1 \wedge \beta_2 \Rightarrow \varphi}$$

This derivation has depth n. As we can see, the sequent $\Gamma_1, \beta_1, \beta_2 \Rightarrow \varphi$ has a derivation whose depth is $(n-1)$, and hence, it is possible to apply the induction hypothesis to such a sequent. From this, we get a derivation of depth *at most* $(n-1)$ of a sequent of the form $\Gamma_1, \beta_1, \beta_2, \alpha \Rightarrow \varphi$. By considering such a sequent as the premise for an inference of $\wedge\mathsf{L}$, we obtain a derivation of depth *at most* n whose last step is the following:

$$\wedge\mathsf{L}\ \frac{\Gamma_1, \beta_1, \beta_2, \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \wedge \beta_2, \alpha \Rightarrow \varphi}$$

where $\Gamma_1, \alpha, \varphi$ represent the context. Therefore, the sequent $\Gamma, \alpha \Rightarrow \varphi$ is indeed derivable in the system, and the derivation has depth at most n.

$\wedge\mathsf{R}$: Suppose that the sequent $\Gamma \Rightarrow \varphi$ has been obtained by the right rule for the conjunction as the last step. Thus, φ is of the form $\beta_1 \wedge \beta_2$, and the last inference of the derivation looks like:

$$\wedge\mathsf{R}\ \frac{\Gamma \Rightarrow \beta_1 \qquad\qquad \Gamma \Rightarrow \beta_2}{\Gamma \Rightarrow \beta_1 \wedge \beta_2}$$

The given derivation has depth n. Exactly as in the previous case, we can apply induction hypothesis on both sequents in the premise. Indeed, the depth of the derivation for each sequent is $< n$. In particular, denoting with m_1 and m_2 the depths of derivation for, respectively, the left and the right premise, we have $n = \max\{m_1, m_2\} + 1$. By the induction hypothesis, we get two derivations of the following forms:

$$\vdash_{\leq m_1} \Gamma, \alpha \Rightarrow \beta_1 \quad , \quad \vdash_{\leq m_2} \Gamma, \alpha \Rightarrow \beta_2.$$

It is clear how now we can apply the right rule for conjunction again, obtaining:

$$\wedge\mathsf{R}\ \frac{\Gamma, \alpha \Rightarrow \beta_1 \qquad\qquad \Gamma, \alpha \Rightarrow \beta_2}{\Gamma, \alpha \Rightarrow \beta_1 \wedge \beta_2}$$

where Γ, α is here the context. Therefore, the sequent $\Gamma, \alpha \Rightarrow \varphi$ is indeed derivable in the system. The depth of such a derivation turns out to be $\leq \max\{m_1, m_2\} + 1 = n$.

$\vee\mathsf{L}$: Suppose that the sequent $\Gamma \Rightarrow \varphi$ has been obtained by an application of the left rule for the disjunction as the last step. Thus, Γ is of the form $\Gamma_1, \beta_1 \vee \beta_2$, and the last inference of the derivation looks like:

$$\vee\mathsf{L}\ \frac{\Gamma_1, \beta_1 \Rightarrow \varphi \qquad\qquad \Gamma_1, \beta_2 \Rightarrow \varphi}{\Gamma_1, \beta_1 \vee \beta_2 \Rightarrow \varphi}$$

The depth of the given derivation is n. The depth of the derivation for each sequent in the premise is again $< n$, say m_1 and m_2, respectively. By applying the induction hypothesis to both such sequents, we get two derivations:

$$\vdash_{\leq m_1} \Gamma_1, \beta_1, \alpha \Rightarrow \varphi \quad , \quad \vdash_{\leq m_2} \Gamma_1, \beta_2, \alpha \Rightarrow \varphi.$$

We can apply the rule for disjunction, considering $\Gamma_1, \alpha, \varphi$ as the context. We obtain:

$$\vee\mathsf{L}\ \frac{\Gamma_1, \beta_1, \alpha \Rightarrow \varphi \qquad\qquad \Gamma_1, \beta_2, \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \vee \beta_2, \alpha \Rightarrow \varphi}$$

The depth of such a derivation turns out to be $\leq \max\{m_1, m_2\} + 1 = n$ and hence, the sequent $\Gamma, \alpha \Rightarrow \varphi$ is derivable in the system.

$\vee\mathsf{R}$: Suppose that the sequent $\Gamma \Rightarrow \varphi$ has been obtained by an application of the rule for the disjunction on the right as the last step. Thus, φ is of the form $\beta_1 \vee \beta_2$, and without loss of generality, we assume the last inference to look like:

$$\vee\mathsf{R}\ \frac{\Gamma \Rightarrow \beta_1}{\Gamma \Rightarrow \beta_1 \vee \beta_2}$$

This derivation has depth n. The depth of the derivation for the premise sequent is $(n-1)$. By applying the induction hypothesis, we get

$$\vdash_{\leq (n-1)} \Gamma, \alpha \Rightarrow \beta_1.$$

It is clear how now we can apply the right rule for disjunction again, considering Γ, α as the context. We obtain:

$$\vee\mathsf{R}\ \frac{\Gamma, \alpha \Rightarrow \beta_1}{\Gamma, \alpha \Rightarrow \beta_1 \vee \beta_2}$$

The depth of such a derivation turns out to be $\leq n$ and hence, the sequent $\Gamma, \alpha \Rightarrow \varphi$ is derivable in the system and the depth of the proof is preserved.

$\rightarrow \mathsf{L}$: Assume that the sequent $\Gamma \Rightarrow \varphi$ has been obtained by the left rule for implication as the last step. Thus, Γ is of the form $\Gamma_1, \beta_1 \rightarrow \beta_2$, and the last inference of the derivation looks like:

$$\rightarrow \mathsf{L}\ \frac{\Gamma_1, \beta_1 \rightarrow \beta_2 \Rightarrow \beta_1 \qquad\qquad \Gamma_1, \beta_2 \Rightarrow \varphi}{\Gamma_1, \beta_1 \rightarrow \beta_2 \Rightarrow \varphi}$$

The given derivation has depth n. Denote with m_1 and m_2 the depths of derivation for, respectively, the left and the right premise. By applying the induction hypothesis to both such sequents, we get two derivations

$$\vdash_{\leq m_1} \Gamma_1, \beta_1 \to \beta_2, \alpha \Rightarrow \beta_1 \quad , \quad \vdash_{\leq m_2} \Gamma_1, \beta_2, \alpha \Rightarrow \varphi.$$

We apply the rule for implication, considering $\Gamma_1, \alpha, \varphi$ as the context, and we obtain

$$\to \mathsf{L}\ \frac{\Gamma_1, \beta_1 \to \beta_2, \alpha \Rightarrow \beta_1 \qquad\qquad \Gamma_1, \beta_2, \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \to \beta_2, \alpha \Rightarrow \varphi}$$

The sequent $\Gamma, \alpha \Rightarrow \varphi$ turns out to be derivable in the system, by means of a proof with no increased depth.

$\to \mathsf{R}$: Suppose that the sequent $\Gamma \Rightarrow \varphi$ has been obtained by an application of the rule for implication on the right as the last step. Thus, φ is of the form $\beta_1 \to \beta_2$, and we assume the last inference to look like:

$$\to \mathsf{R}\ \frac{\Gamma, \beta_1 \Rightarrow \beta_2}{\Gamma \Rightarrow \beta_1 \to \beta_2}$$

Such a derivation has depth n, while the depth of the derivation for the sequent in the premise is $(n-1)$. By the induction hypothesis, we obtain

$$\vdash_{\leq (n-1)} \Gamma, \beta_1, \alpha \Rightarrow \beta_2.$$

Now, by applying the right rule for implication again, with Γ, α as the context, we get:

$$\to \mathsf{R}\ \frac{\Gamma, \beta_1, \alpha \Rightarrow \beta_2}{\Gamma, \alpha \Rightarrow \beta_1 \to \beta_2}$$

The depth of such a derivation is indeed $\leq n$ and hence, the sequent $\Gamma, \alpha \Rightarrow \varphi$ is derivable in the system and the depth of the deduction is preserved.

N : Let Γ be of the form $\Gamma_1, \neg\beta_1$, and let the last inference of the derivation look like

$$\mathsf{N}\ \frac{\Gamma_1, \neg\beta_1, \beta_1 \Rightarrow \beta_1 \qquad\qquad \Gamma_1, \neg\beta_1, \beta_2 \Rightarrow \beta_1}{\Gamma_1, \neg\beta_1 \Rightarrow \neg\beta_2}$$

The depth of this derivation is n. Denote with m_1 and m_2 the depths of derivation for, respectively, the left and the right premise, and apply the induction hypothesis to both premises, to get

$$\vdash_{\leq m_1} \Gamma_1, \neg\beta_1, \beta_1, \alpha \Rightarrow \beta_2 \quad , \quad \vdash_{\leq m_2} \Gamma_1, \neg\beta_1, \beta_2, \alpha \Rightarrow \beta_1.$$

Now, via another application of N with Γ_1, α as the context, we have

$$\mathsf{N}\ \frac{\Gamma_1, \neg\beta_1, \beta_1, \alpha \Rightarrow \beta_2 \qquad\qquad \Gamma_1, \neg\beta_1, \beta_2, \alpha \Rightarrow \beta_1}{\Gamma_1, \neg\beta_1, \alpha \Rightarrow \neg\beta_2}$$

The depth of such a derivation is $\leq \max\{m_1, m_2\} + 1 = n$. Thus, the sequent $\Gamma, \alpha \Rightarrow \varphi$ turns out to be derivable in the system via a proof of depth at most n, as desired.

NeF : Assume that the last inference rule applied to get $\Gamma \Rightarrow \varphi$ is NeF. Thus, Γ is of the form $\Gamma_1, \neg\beta_1$, while φ is $\neg\beta_2$ for some formula β_2. The inference looks as follows:

$$\mathsf{NeF}\ \frac{\Gamma_1, \neg\beta_1 \Rightarrow \beta_1}{\Gamma_1, \neg\beta_1 \Rightarrow \neg\beta_2}$$

By the induction hypothesis, there exists a derivation of depth at most $(n-1)$ of the sequent $\Gamma_1, \neg\beta_1, \alpha \Rightarrow \beta_1$. Therefore, by simply applying NeF again with context Γ_1, α, we obtain $\Gamma_1, \neg\beta_1, \alpha \Rightarrow \neg\beta_2$, which corresponds exactly to $\Gamma, \alpha \Rightarrow \varphi$. Observe that the obtained derivation has depth at most n.

CoPC : Consider the case in which the last rule applied was CoPC, as follows:

$$\mathsf{CoPC}\ \frac{\Gamma_1, \neg\beta_1, \beta_2 \Rightarrow \beta_1}{\Gamma_1, \neg\beta_1 \Rightarrow \neg\beta_2}$$

Given that the premise of this rule has a derivation of depth $(n-1)$, we can apply the induction hypothesis on such a premise sequent and get

$$\vdash_{\leq (n-1)} \Gamma_1, \neg\beta_1, \beta_2, \alpha \Rightarrow \beta_1.$$

Another application of CoPC on the obtained sequent leads us to get the desired derivation $\vdash_{\leq n} \Gamma_1, \neg\beta_1, \alpha \Rightarrow \neg\beta_2$.

An : The only step we need to take care of is the one in which the sequent $\Gamma \Rightarrow \varphi$ is the conclusion of an application of An. Consider the following step of a derivation:

$$\mathsf{An}\ \frac{\Gamma, \beta \Rightarrow \neg\beta}{\Gamma \Rightarrow \neg\beta}$$

The induction hypothesis ensures the existence of a derivation of depth $\leq (n-1)$ of the sequent $\Gamma, \beta, \alpha \Rightarrow \neg\beta$. Hence, an application of An gives us a derivation of depth $\leq n$ of the sequent $\Gamma, \alpha \Rightarrow \neg\beta$, as desired. $\square$

As the reader may have seen, the proof we gave above is not particularly involved, and it works in a straightforward way also for the negation rules. In our proof, we have been careful in ensuring admissibility to be depth-preserving. The reasons for requiring this particular property are mainly computational, since this requirement allows the usage of Weakening without increasing the ‘complexity’ of the derivation.

At this point, one of the useful notions is the one of *invertible rule*. A rule is said to be invertible if for every instance of the rule, the premise sequents are derivable if and only if the conclusion sequent is. Enhancing the proof of the Inversion Lemma that we find in [44], we would get that, in the **G3**-systems, the only non-invertible rules are $\vee$L, $\rightarrow$ L with respect to the left premise, and the newly added rules N, NeF and CoPC. Indeed, all the other rules are *depth-preserving invertible.*

We state and prove admissibility of Contraction and the proof is similar to the one given for Weakening. Nonetheless, the required induction this time is a double induction, both on the depth of derivations and on the complexity of the formula α. We recall the notion of *weight* (or *complexity*) of a formula α. This notion is defined inductively on the structure of the formula. Propositional variables and the constant $\top$ have null weight, i.e., $w(p) = w(\top) = 0$ for every $p \in Prop$. For every binary connective $\circ \in \{\wedge, \vee, \rightarrow\}$, the weight $w(\alpha_1 \circ \alpha_2) = w(\alpha_1) + w(\alpha_2) + 1$. Finally, for negation we define $w(\neg\alpha_1) = w(\alpha_1) + 1$.

Theorem 5.2.2 (Admissibility of Contraction Rule)**.** *The Contraction rule is admissible in the considered sequent calculus systems. In particular, it is depth-preserving admissible.*

Proof. We will show that, whenever we start from a derivation of the form $\vdash_n \Gamma, \alpha, \alpha \Rightarrow \varphi$ in one of the **G3**-systems, then there is also a derivation of depth at most n of the sequent $\Gamma, \alpha \Rightarrow \varphi$. The structure of the proof is an induction on the pair (n, m), where $n = w(\alpha)$ and m represents the depth of the derivation tree. The pairs are considered to be ordered by means of the lexicographic ordering[3].

$(0, 0)$: As a base case, we consider the situation in which the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is an axiom (i.e., the derivation has depth 0), and α is either a propositional variable p or $\top$. Assume first α to be p. In this case, we have three different cases: the one in which φ is p, the one in which φ is a propositional variable q such that $q \in \Gamma$, and the one in which φ is just $\top$. It is immediately clear that by considering only one instance of p in the antecedent, the resulting sequent will still be an axiom in all the three cases. Assume now that α represents $\top$. In this case, the options to be considered are just two: the one in which φ represents $\top$, and the one in which φ is a propositional variable $q \in \Gamma$. Again, in both cases, considering only one occurrence of $\top$ in the antecedent gives a sequent which is still an axiom. Therefore, whenever we have an axiom of the form $\Gamma, \alpha, \alpha \Rightarrow \varphi$, and α has null weight, the sequent of the form $\Gamma, \alpha \Rightarrow \varphi$ is still an axiom, and hence, the conclusion of C is derivable in the system in the form of an axiom, i.e., via a derivation of depth 0.

$(n, m) > (0, 0)$: Here we are considering three different cases. They indeed amount to distinguish the case in which neither of the instances of α is principal from the case in which one instance of α is indeed principal. It is immediate to see that, if α is not principal and $\Gamma, \alpha \Rightarrow \varphi$ is an axiom, the resulting sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is an axiom as well. Now, we assume the sequent not to be an axiom, and we cover all the possibilities for the inference rule applied as the last step of the derivation. Note that in the cases of left rules, the principal formula will be

[3]Given a set of ordered t-uples of natural numbers $(n_1, \ldots, n_t)$, we say that

$$(n_1, \ldots, n_t) <_L (m_1, \ldots, m_t),$$

where $<_L$ denotes the *lexicographic ordering*, if there is an index $i \in \{1, \ldots, t\}$ such that $n_i < m_i$ and, for every $j < i$, $n_j = m_j$, where $<$ is the standard ordering over $\mathbb{N}$.

either an element of Γ or an instance of α. In the case in which the principal formula is an element β of Γ, we denote Γ as Γ_1, β.

$\wedge\mathsf{L}$: Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ comes as the conclusion of an application of the conjunction rule on the left and also that $\underline{\alpha \text{ is not principal}}$ in such an inference, as follows:

$$\wedge\mathsf{L}\ \frac{\Gamma_1, \beta_1, \beta_2, \alpha, \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \wedge \beta_2, \alpha, \alpha \Rightarrow \varphi}$$

The premise sequent comes from a derivation of depth $(m-1)$. The lexicographic ordering ensures that $(n, m-1) < (n, m)$ and we can thus apply the induction hypothesis on such a sequent, obtaining a derivation $\vdash_{\leq (m-1)} \Gamma_1, \beta_1, \beta_2, \alpha \Rightarrow \varphi$. By considering this sequent as the premise for an application of $\wedge\mathsf{L}$ rule, we get a derivation of depth at most m of the sequent $\Gamma_1, \beta_1 \wedge \beta_2, \alpha \Rightarrow \varphi$, which is exactly $\Gamma, \alpha \Rightarrow \varphi$.

Assume now that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ comes again as the conclusion of an application of the conjunction rule on the left, but in this case, $\underline{\alpha \text{ is principal}}$ in such an inference. Here, α will be the a conjunction of the form $\alpha_1 \wedge \alpha_2$, and the last step will look as:

$$\wedge\mathsf{L}\ \frac{\Gamma, \alpha_1, \alpha_2, \alpha \Rightarrow \varphi}{\Gamma, \alpha, \alpha \Rightarrow \varphi}$$

The premise sequent comes from a derivation of depth $(m-1)$. Note that the left conjunction rule is indeed one of the depth-preserving invertible rules. Therefore, from $\vdash_{(m-1)} \Gamma, \alpha_1, \alpha_2, \alpha \Rightarrow \varphi$, we can get a derivation $\vdash_k \Gamma, \alpha_1, \alpha_2, \alpha_1, \alpha_2 \Rightarrow \varphi$, where $k \leq (m-1)$. Now, n_1 and n_2 being respectively the weight of α_1 and α_2, the following holds: $(k, n_1) < (m, n)$ and $(k, n_2) < (m, n)$. Thus, by applying the induction hypothesis (to both α_1 and α_2), we get $\vdash_{\leq (m-1)} \Gamma, \alpha_1, \alpha_2 \Rightarrow \varphi$. By considering this sequent as the premise for an application of $\wedge\mathsf{L}$ rule, we get a derivation of depth at most m of the sequent $\Gamma_1, \alpha_1 \wedge \alpha_2 \Rightarrow \varphi$, which is exactly $\Gamma, \alpha \Rightarrow \varphi$.

$\wedge\mathsf{R}$: Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ comes as the conclusion of an application of the right conjunction rule, as follows:

$$\wedge\mathsf{R}\ \frac{\Gamma_1, \alpha, \alpha \Rightarrow \beta_1 \qquad \Gamma_1, \alpha, \alpha \Rightarrow \beta_2}{\Gamma_1, \alpha, \alpha \Rightarrow \beta_1 \wedge \beta_2}$$

Let m_1, m_2 be the depths of the derivation whose conclusion is, respectively, the left and the right sequent of the premise. Given that $m = \max\{m_1, m_2\} + 1$, we get that $(n, m_1) < (n, m)$ and also $(n, m_2) < (n, m)$. The induction hypothesis allows us to get

$$\vdash_{\leq m_1} \Gamma_1, \alpha \Rightarrow \beta_1 \quad , \quad \vdash_{\leq m_2} \Gamma_1, \alpha \Rightarrow \beta_2.$$

By considering this sequents as the premise for an application of $\wedge\mathsf{R}$ rule with context Γ_1, α, it is possible to obtain a derivation of depth at most m of the sequent $\Gamma_1, \alpha \Rightarrow \beta_1 \wedge \beta_2$, which is exactly $\Gamma, \alpha \Rightarrow \varphi$.

$\vee\mathsf{L}$: Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is the conclusion of an application of the left disjunction rule and that α is not a principal formula, as follows,

$$\vee\mathsf{L}\ \frac{\Gamma_1, \beta_1, \alpha, \alpha \Rightarrow \varphi \qquad\qquad \Gamma_1, \beta_2, \alpha, \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \vee \beta_2, \alpha, \alpha \Rightarrow \varphi}$$

Let m_1, m_2 be the depths of the derivation whose conclusion is, respectively, the left and the right sequent of the premise. We have $(n, m_1) < (n, m)$ and also $(n, m_2) < (n, m)$, and hence, the induction hypothesis allows us to obtain the following derivations

$$\vdash_{\leq m_1} \Gamma_1, \beta_1, \alpha \Rightarrow \varphi \ \ , \ \ \vdash_{\leq m_2} \Gamma_1, \beta_2, \alpha \Rightarrow \varphi.$$

By applying $\vee\mathsf{L}$ to these sequents as premises, with context $\Gamma_1, \alpha, \varphi$, it is possible to obtain a derivation of depth at most m of the sequent $\Gamma_1, \beta_1 \vee \beta_2, \alpha \Rightarrow \varphi$, which is $\Gamma, \alpha \Rightarrow \varphi$ indeed.

Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is the conclusion of an application of the left disjunction rule and that α is a principal formula is such an inference:

$$\vee\mathsf{L}\ \frac{\Gamma, \alpha_1, \alpha \Rightarrow \varphi \qquad\qquad \Gamma, \alpha_2, \alpha \Rightarrow \varphi}{\Gamma, \alpha, \alpha \Rightarrow \varphi}$$

Let m_1, m_2 be the depths of the derivation whose conclusion is, respectively, the left and the right sequent of the premise. Note that the left disjunction rule is indeed one of the depth-preserving invertible rules. Hence, from $\vdash_{m_1} \Gamma, \alpha_1, \alpha \Rightarrow \varphi$, we can get a derivation $\vdash_{k_1} \Gamma, \alpha_1, \alpha_1 \Rightarrow \varphi$, where $k_1 \leq m_1$. Moreover, from $\vdash_{m_2}$ $\Gamma, \alpha_2, \alpha \Rightarrow \varphi$, we can get a derivation $\vdash_{k_2} \Gamma, \alpha_2, \alpha_2 \Rightarrow \varphi$, where $k_2 \leq m_2$. Now, n_1 and n_2 being respectively the weights of α_1 and α_2, we have $(k_1, n_1) < (m, n)$ and $(k_1, n_2) < (m, n)$. Thus, by applying the induction hypothesis (to both the obtained sequents), we get $\vdash_{\leq m_1} \Gamma, \alpha_1 \Rightarrow \varphi$ and also $\vdash_{\leq m_2} \Gamma, \alpha_2 \Rightarrow \varphi$. By considering these sequents as the premises for an application of $\vee\mathsf{L}$ rule, we get a derivation of depth at most m of the sequent $\Gamma_1, \alpha_1 \vee \alpha_2 \Rightarrow \varphi$, which is exactly $\Gamma, \alpha \Rightarrow \varphi$.

$\vee\mathsf{R}$: Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ comes as the conclusion of an application of the disjunction rule on the right. Without loss of generality, we consider the last step of the derivation as follows:

$$\vee\mathsf{R}\ \frac{\Gamma, \alpha, \alpha \Rightarrow \beta_1}{\Gamma, \alpha, \alpha \Rightarrow \beta_1 \vee \beta_2}$$

The premise sequent comes from a derivation of depth $(m-1)$. The lexicographic ordering ensures that $(n, m-1) < (n, m)$ and we can thus apply the induction hypothesis on such a sequent, obtaining a derivation of depth at most $(m-1)$, i.e., $\vdash_{\leq (m-1)} \Gamma, \alpha \Rightarrow \beta_1$. By considering this sequent as the premise for an application of $\vee\mathsf{R}$ rule, we get a derivation of depth at most m of the sequent $\Gamma, \alpha \Rightarrow \beta_1 \vee \beta_2$, which represents $\Gamma, \alpha \Rightarrow \varphi$, as desired.

$\rightarrow \mathsf{L}$: Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is the conclusion of an application of the left implication rule, in which $\underline{\alpha \text{ is not principal}}$:

$$\rightarrow \mathsf{L}\ \frac{\Gamma_1, \beta_1 \rightarrow \beta_2, \alpha, \alpha \Rightarrow \beta_1 \qquad\qquad \Gamma_1, \beta_2, \alpha, \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \rightarrow \beta_2, \alpha, \alpha \Rightarrow \varphi}$$

Let m_1, m_2 be the depths of the derivations of the sequents in the premise. We have $(n, m_2) < (n, m)$. The induction hypothesis allows us to get

$$\vdash_{\leq m_1} \Gamma_1, \beta_1 \rightarrow \beta_2, \alpha \Rightarrow \beta_1 \quad , \quad \vdash_{\leq m_2} \Gamma_1, \beta_2, \alpha \Rightarrow \varphi.$$

By applying $\rightarrow \mathsf{L}$ to these sequents as premises, with context $\Gamma_1, \alpha, \varphi$, it is possible to obtain a derivation of depth at most m of the sequent $\Gamma_1, \beta_1 \rightarrow \beta_2, \alpha \Rightarrow \varphi$, which turns out to be $\Gamma, \alpha \Rightarrow \varphi$ indeed.

Assume now instead, that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is the conclusion of an application of the left implication rule in which $\underline{\alpha \text{ is principal}}$:

$$\rightarrow \mathsf{L}\ \frac{\Gamma, \alpha_1 \rightarrow \alpha_2, \alpha \Rightarrow \alpha_1 \qquad\qquad \Gamma, \alpha_2, \alpha \Rightarrow \varphi}{\Gamma, \alpha, \alpha \Rightarrow \varphi}$$

Let m_1, m_2 be the depths of the derivation whose conclusion is, respectively, the left and the right sequent of the premise. Note that the left implication rule is depth-preserving invertible with respect to the right premise. Consider first the left premise. Observe that $(n, m_1) < (n, m)$ and thus, the induction hypothesis can be applied. Therefore, from $\vdash_{m_1} \Gamma, \alpha, \alpha \Rightarrow \alpha_1$, we get a derivation $\vdash_{k_1} \Gamma, \alpha \Rightarrow \alpha_1$, where $k_1 \leq m_1$. Take now into consideration the right premise of the inference. By means of depth-preserving inversion on the right, $\vdash_{m_2} \Gamma, \alpha_2, \alpha \Rightarrow \varphi$ gives us $\vdash_{k_2} \Gamma, \alpha_2, \alpha_2 \Rightarrow \varphi$, where $k_2 \leq m_2$. Now, n_2 being be the weight of α_2, the following holds: $(k_2, n_2) < (m, n)$. Thus, by applying the induction hypothesis, we get $\vdash_{\leq m_2} \Gamma, \alpha_2 \Rightarrow \varphi$. Now, we can use $\rightarrow \mathsf{L}$ again to obtain

$$\rightarrow \mathsf{L}\ \frac{\vdash_{\leq m_1} \Gamma, \alpha_1 \rightarrow \alpha_2 \Rightarrow \alpha_1 \qquad\qquad \vdash_{\leq m_2} \Gamma, \alpha_2 \Rightarrow \varphi}{\Gamma, \alpha_1 \rightarrow \alpha_2 \Rightarrow \varphi}$$

In this way, we have obtained a derivation of depth at most m of the sequent $\Gamma_1, \alpha_1 \rightarrow \alpha_2 \Rightarrow \varphi$, which is exactly $\Gamma, \alpha \Rightarrow \varphi$.

$\rightarrow \mathsf{R}$: Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is obtained by applying implication rule on the right. The last step of the derivation looks as follows:

$$\rightarrow \mathsf{R}\ \frac{\Gamma, \beta_1, \alpha, \alpha \Rightarrow \beta_2}{\Gamma, \alpha, \alpha \Rightarrow \beta_1 \rightarrow \beta_2}$$

The premise sequent comes from a derivation of depth $(m-1)$ and hence, the lexicographic ordering ensures $(n, m-1) < (n, m)$. We can apply the induction hypothesis on such a sequent, obtaining a derivation $\vdash_{\leq (m-1)} \Gamma, \beta_1, \alpha \Rightarrow \beta_1$. By considering this sequent as the premise for an application of $\to \mathsf{R}$ rule, it is possible to obtain a derivation of depth at most m of the sequent $\Gamma, \alpha \Rightarrow \beta_1 \to \beta_2$, which is $\Gamma, \alpha \Rightarrow \varphi$, as desired.

N : Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is the conclusion of an application of the N-rule, but $\underline{\alpha \text{ is not principal}}$, as follows:

$$\mathsf{N}\ \frac{\Gamma_1, \neg\beta_1, \beta_1, \alpha, \alpha \Rightarrow \beta_2 \qquad\qquad \Gamma_1, \neg\beta_1, \beta_2, \alpha, \alpha \Rightarrow \beta_1}{\Gamma_1, \neg\beta_1, \alpha, \alpha \Rightarrow \neg\beta_2}$$

Let m_1, m_2 be the depths of the derivations of the sequents in the premise. Given that $m = \max\{m_1, m_2\} + 1$, we get that $(n, m_1) < (n, m)$ and also $(n, m_2) < (n, m)$. The induction hypothesis allows us to get the following derivations:

$$\vdash_{\leq m_1} \Gamma_1, \neg\beta_1, \beta_1, \alpha \Rightarrow \beta_2 \ \ , \ \ \vdash_{\leq m_2} \Gamma_1, \neg\beta_1, \beta_2, \alpha \Rightarrow \beta_1.$$

By means of an application of the N-rule with context Γ_1, α, it is possible to obtain a derivation of depth at most m of the sequent $\Gamma_1, \neg\beta_1, \alpha \Rightarrow \neg\beta_2$, which turns out to be $\Gamma, \alpha \Rightarrow \varphi$ indeed.

Consider the case in which the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ is the conclusion of an application of the N-rule and $\underline{\alpha \text{ is a principal formula}}$. This means that α is of the form $\neg\alpha_1$, φ is of the form $\neg\beta$ and the inference goes as follows:

$$\mathsf{N}\ \frac{\Gamma, \neg\alpha_1, \alpha_1, \alpha \Rightarrow \beta \qquad\qquad \Gamma, \neg\alpha_1, \beta, \alpha \Rightarrow \alpha_1}{\Gamma, \alpha, \alpha \Rightarrow \neg\beta}$$

Let m_1, m_2 be the depths of the derivations of the sequents in the premise. Given that $m = \max\{m_1, m_2\} + 1$, we get that $(n, m_1) < (n, m)$ and also $(n, m_2) < (n, m)$. The induction hypothesis allows us to get

$$\vdash_{\leq m_1} \Gamma, \neg\alpha_1, \alpha_1 \Rightarrow \beta \ \ , \ \ \vdash_{\leq m_2} \Gamma, \neg\alpha_1, \beta \Rightarrow \alpha_1.$$

By means of an application of the N-rule with context Γ, it is possible to obtain a derivation of depth at most m of the sequent $\Gamma, \neg\alpha_1 \Rightarrow \neg\beta$, which turns out to be $\Gamma, \alpha \Rightarrow \varphi$, as desired.

NeF : Assume that the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ has been obtained by applying NeF and that $\underline{\alpha \text{ is not principal}}$, as follows:

$$\mathsf{NeF}\ \frac{\Gamma_1, \neg\beta_1, \alpha, \alpha \Rightarrow \beta_1}{\Gamma_1, \neg\beta_1, \alpha, \alpha \Rightarrow \neg\beta_2}$$

for some formula β_2. The depth of the derivation of the premise is $(m - 1)$. The pair $(n, m - 1)$ is lexicographically smaller than the pair (n, m), and this gives us the possibility of applying the induction hypothesis on such a sequent, getting $\Gamma_1, \neg\beta_1, \alpha \Rightarrow \beta_1$. Now, an application of NeF with context Γ_1, α leads us to the required sequent:

$$\mathsf{NeF}\ \frac{\Gamma_1, \neg\beta_1, \alpha \Rightarrow \beta_1}{\Gamma_1, \neg\beta_1, \alpha \Rightarrow \neg\beta_2}$$

Indeed, the conclusion of such an inference is exactly $\Gamma, \alpha \Rightarrow \varphi$, as desired.

The second case covers the possibility for one of the occurrences of α to be principal. Therefore, α is $\neg\alpha_1$ for some α_1, and the last step of the derivation looks like:

$$\mathsf{NeF}\ \frac{\Gamma, \neg\alpha_1, \alpha \Rightarrow \alpha_1}{\Gamma, \neg\alpha_1, \alpha \Rightarrow \neg\beta}$$

Given that the premise comes from a derivation of depth $(m-1)$, we can apply the induction hypothesis and obtain a sequent of the form $\Gamma, \neg\alpha_1 \Rightarrow \alpha_1$ which, via an application of NeF, gives us exactly $\Gamma, \neg\alpha_1 \Rightarrow \neg\beta$.

CoPC : Let the first case be the one in which the sequent $\Gamma, \alpha, \alpha \Rightarrow \varphi$ has been obtained by CoPC, and the formula α is not principal:

$$\mathsf{CoPC}\ \frac{\Gamma, \neg\beta_1, \beta_2, \alpha, \alpha \Rightarrow \beta_1}{\Gamma, \neg\beta_1, \alpha, \alpha \Rightarrow \neg\beta_2}$$

The depth of the derivation of the premise is $(m-1)$. Given that the pair $(n, m-1)$ is lexicographically smaller than (n, m), we can apply the induction hypothesis and get $\vdash_{\leq (m-1)} \Gamma, \neg\beta_1, \alpha \Rightarrow \beta_1$. Via an application of CoPC with context Γ, α, we obtain the desired derivation $\vdash_{\leq m} \Gamma, \neg\beta_1, \alpha \Rightarrow \neg\beta_2$.

Consider now CoPC as the last rule applied to get $\Gamma, \alpha, \alpha \Rightarrow \varphi$, and one of the instance of α is a principal formula of the form $\neg\alpha_1$. This case goes as follows:

$$\mathsf{CoPC}\ \frac{\Gamma_1, \neg\alpha_1, \alpha, \beta \Rightarrow \alpha_1}{\Gamma_1, \neg\alpha_1, \alpha \Rightarrow \neg\beta}$$

The premise sequent comes from a derivation of depth $(m-1)$, which lets us apply the induction hypothesis on such a sequent and get

$$\vdash_{\leq (m-1)} \Gamma_1, \neg\alpha_1, \beta \Rightarrow \alpha_1,$$

which is exactly the general form of the premise of CoPC; hence, by applying the latter, we obtain $\vdash_{\leq m} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\beta$ and hence, we are done.

An : Consider the case in which the sequent $\Gamma, \alpha, \alpha \Rightarrow \neg\beta$ is obtained by means of an application of An, in the following way:

$$\mathsf{An}\ \frac{\Gamma, \alpha, \alpha, \beta \Rightarrow \neg\beta}{\Gamma, \alpha, \alpha \Rightarrow \neg\beta}$$

By the induction hypothesis, there is a MPC derivation of the sequent $\Gamma, \alpha, \beta \Rightarrow \neg\beta$ whose depth is $\leq (m-1)$. Simply applying An to such a sequent, we get the desired derivation $\vdash_{\leq m} \Gamma, \alpha \Rightarrow \neg\beta$. $\square$

The closure under structural rules that we have just proved allows us to claim the following equivalence result.

Theorem 5.2.3. *Let* $* \in \{\mathbf{n}, \mathbf{nef}, \mathbf{copc}, \mathbf{m}_\neg\}$. *Then, the following holds:*

$$\mathbf{G1}* \vdash \Gamma \Rightarrow \varphi \Leftrightarrow \mathbf{G3}* \vdash \Gamma \Rightarrow \varphi.$$

Proof. The proof is an induction on the depth of the derivations, and it is straightforward from Theorem 5.2.1 and Theorem 5.2.2. □

5.3 The Cut Rule

As already anticipated, the systems we have introduced so far obey the so-called *subformula property.* The following form of 'transitivity' of the $\Rightarrow$ operator violates this property:

$$\mathsf{Cut}\ \frac{\Gamma \Rightarrow \alpha \qquad\qquad \Gamma', \alpha \Rightarrow \varphi}{\Gamma, \Gamma' \Rightarrow \varphi}$$

The above rule is called *Cut rule.* The formula α to which the rule is applied is called *cutformula.* Observe that the contexts of the two sequents in the premise are explicitly given as two different contexts Γ and Γ'. Since Weakening and Contraction are admissible rules of our systems, it would have been equally effective to consider a context-sharing Cut rule, i.e.,

$$\mathsf{Cut}_{\mathsf{cs}}\ \frac{\Gamma \Rightarrow \alpha \qquad\qquad \Gamma, \alpha \Rightarrow \varphi}{\Gamma \Rightarrow \varphi}$$

The reader may easily realize the reason why adding the Cut rule to the considered systems would violate the subformula property. Indeed, the cutformula α is not necessarily a subformula of Γ, Γ', φ. We are nevertheless going to use such a rule while building proofs, given that the following result holds for the considered systems.

Theorem 5.3.1 (Cut Elimination)**.** *Every sequent* $\Gamma \Rightarrow \varphi$ *which is provable in one of the* **G3**-*systems* + Cut, *is also provable in the same* **G3**-*system without* Cut.

We keep the proof and the discussion of this result for the next chapter.

5.4 Equivalence of G-systems and Hilbert systems

In order to start using the **G3**-systems for effective proofs, we want to make sure that they are a sound and complete systems for our logics. The proof amounts to showing that the Gentzen systems we have defined are equivalent to the logical systems obtained from the Hilbert-style axiomatizations. For each of the considered systems **G**, we want to show that having a derivation $\vdash_{\mathbf{G}} \Gamma \Rightarrow \varphi$ is equivalent to having a Hilbert-style proof of φ from assumptions in Γ. Given a Gentzen system **G**, let $\mathbf{H_G}$ denote the corresponding Hilbert system.

Theorem 5.4.1. *Given a finite multiset of formulas* Γ *and a formula* φ*, we have that* $\Gamma \Rightarrow \varphi$ *is a derivable sequent in the considered* $\mathbf{G}$*-system if and only if* $\Gamma \vdash_{\mathbf{H_G}} \varphi$ *within the respective Hilbert system* $\mathbf{H_G}$*.*

Proof. We consider the left-to-right direction. The proof goes by induction on the depth of the derivation of the sequent $\Gamma \Rightarrow \varphi$.

Consider the case in which $\Gamma \Rightarrow \varphi$ is an axiom. As we can see, there are two options for this to be the case. The first option is that φ is p, for $p \in \Gamma$. In this case, since $p \vdash p$, we get the desired Hilbert-style derivation. In the second case, φ is $\top$. This also leads us immediately to the conclusion, given that $\vdash \top$ holds.

For the cases concerning the rules from the positive system, together with Weakening and Contraction, we refer to [44]. We focus on the cases in which the last introduced formulas contain negation as the principal connective.

Assume that $\Gamma \Rightarrow \varphi$ comes as a conclusion of an inference of N. Thus, Γ consists of a set of formulas Γ_1, and a formula $\neg\alpha$. The consequent φ is the negation of a formula as well, and we shall denote it as $\neg\beta$. The last step of the derivation looks as follows:

$$\mathsf{N}\ \frac{\Gamma_1, \alpha \Rightarrow \beta \qquad\qquad \Gamma_1, \beta \Rightarrow \alpha}{\Gamma_1, \neg\alpha \Rightarrow \neg\beta}$$

The depth of such a deduction is $n > 0$. Therefore, by the induction hypothesis we can claim the existence of Hilbert-style derivations $\Gamma_1, \alpha \vdash \beta$ and $\Gamma_1, \beta \vdash \alpha$ within the N-system. At this point, by using the axiom N, *modus ponens* and the Deduction Theorem, it is immediate to obtain a proof

$$\Gamma_1, \neg\alpha \vdash \neg\beta$$

within the N-system, as desired.

Assume the sequent $\Gamma \Rightarrow \varphi$ to be obtained from an application of the rule NeF. In this case, the set Γ consists of a set of formulas Γ_1 and a formula $\neg\alpha$, while the consequent φ is the negation of a formula $\neg\beta$. The last step of the derivation looks as follows,

$$\mathsf{NeF}\ \frac{\Gamma_1 \Rightarrow \alpha}{\Gamma_1, \neg\alpha \Rightarrow \neg\beta}$$

Hence, the induction hypothesis gives us a Hilbert-style proof of $\Gamma_1 \vdash \alpha$ within NeF. That said, we obtain a derivation of the following form:

$$\Gamma_1, \neg\alpha \vdash (\alpha \wedge \neg\alpha),$$

which allows us to conclude

$$\Gamma_1, \neg\alpha \vdash \neg\beta,$$

which indeed is the desired derivation.

Assume now that the sequent $\Gamma \Rightarrow \varphi$ happens to be derived via an application of the rule CoPC. In this case, the set Γ consists of a set of formulas Γ_1 and a formula $\neg\beta$, and the consequent φ is the negation of a formula $\neg\alpha$. The last step of the derivation looks in the following way:

$$\mathsf{CoPC}\ \frac{\Gamma_1, \alpha \Rightarrow \beta}{\Gamma_1, \neg\beta \Rightarrow \neg\alpha}$$

Now, the induction hypothesis allows us to get a Hilbert-style proof of $\Gamma_1, \alpha \vdash \beta$ within the logical system CoPC. Finally, the axiom, together with *modus ponens* and the Deduction Theorem, gives us

$$\Gamma_1, \neg\beta \vdash \neg\alpha.$$

Suppose now the considered sequent $\Gamma \Rightarrow \varphi$ is the conclusion of an application of An. If this is the case, the formula φ is a negated formula $\neg\alpha$, and the last step of the proof looks as follows:

$$\mathsf{An}\ \frac{\Gamma, \alpha \Rightarrow \neg\alpha}{\Gamma \Rightarrow \neg\alpha}$$

By the induction hypothesis, we know that the derivation $\Gamma, \alpha \vdash \neg\alpha$ can be obtained within the Hilbert MPC-system. Therefore, by means of the axiom An, we can conclude $\Gamma \vdash \neg\alpha$, as desired.

For the right-to-left direction, it is enough to prove that each of the considered axioms is derivable by means of the respective sequent calculus, and that the sequent systems are closed under *modus ponens* and uniform substitution.

We start by proving that the sequent whose left-hand side is empty and whose right-hand side is given by the axiom N is derivable by means of the rule N. Consider the sequent $\Rightarrow$ N, i.e.,

$$\Rightarrow [(p \to q) \wedge (q \to p)] \to [(\neg p \to \neg q) \wedge (\neg q \to \neg p)].$$

Going backward, by means of the right implication rule $\to\mathsf{R}$, the left conjunction rule $\wedge\mathsf{L}$ and again, the right implication rule, we get the following two sequents:

$$p \to q, q \to p, \neg p \Rightarrow \neg q \text{ and } p \to q, q \to p, \neg q \Rightarrow \neg p.$$

They are completely symmetric. Therefore, we will unfold the proof for one of them, given that the other one behaves in the same way. Now, we go backward by means of N, as follows:

$$\mathsf{N}\ \frac{p \to q, q \to p, \neg p, p \Rightarrow q \qquad\qquad p \to q, q \to p, \neg p, q \Rightarrow p}{p \to q, q \to p, \neg p \Rightarrow \neg q}$$

We first consider the derivation of the left premise which, going backward, looks like

$$\to \mathsf{L}\ \frac{p \to q, q \to p, \neg p, p \Rightarrow p \qquad\qquad q \to p, \neg p, p, q \Rightarrow q}{p \to q, q \to p, \neg p, p \Rightarrow q}$$

The obtained premises are both axioms. For the derivation of the right premise, similarly we have

$$\to \mathsf{L}\ \frac{p \to q, q \to p, \neg p, q \Rightarrow q \qquad\qquad p \to q, \neg p, q, p \Rightarrow p}{p \to q, q \to p, \neg p, q \Rightarrow p}$$

which again leads backward to two axioms. Therefore, the axiom N is provable in the system **G3n**.

We repeat the same proof for the negative *ex falso* axiom, and we get the following derivation. We start from

$$(p \wedge \neg p) \Rightarrow \neg q$$

and we proceed bacward by means of the conjunction rule on the left. This gives us

$$p, \neg p \Rightarrow \neg q.$$

At this point, via a backward application of the NeF-rule, we get the axiom

$$p, \neg p \Rightarrow p,$$

and we are done. Thus, we have shown that the axiom is derivable within the system **G3nef**. In particular, this derivation, together with the previous one, proves that both the NeF axioms are provable in the **G3nef**-system.

In order to show that the weak contraposition axiom is derivable within **G3copc**, consider the following proof. From the sequent

$$p \to q \Rightarrow \neg q \to \neg p,$$

by means of the right implication rule we go backward and we get the sequent

$$p \to q, \neg q \Rightarrow \neg p.$$

The contraposition rule CoPC leads us to the sequent

$$p \to q, \neg q, p \Rightarrow q.$$

Via a final application of $\to \mathsf{L}$, we get two axioms,

$$p \to q, \neg q, p \Rightarrow p \text{ and } \neg q, p, q \Rightarrow q.$$

The given derivation allows us to conclude that the contraposition axiom is provable within **G3copc**.

The last axiom which needs to be checked is the absorption of negation axiom. Indeed, the derivation of it goes as follows: we start from the sequent

$$(p \to q) \wedge (q \to p) \Rightarrow \neg p.$$

By applied backward $\wedge\mathsf{L}$ and An subsequently, we get a sequent of the form

$$(p \to q), (p \to \neg q), p \Rightarrow \neg p.$$

Finally, an application of the implication rule on the left leads us to the axiom

$$(p \to q), (p \to \neg q), p \Rightarrow p$$

on the left, as well as to the sequent

$$(p \to q), p, \neg q \Rightarrow \neg p$$

on the right. The reader may observe that the right leaf of the proof-tree, namely $(p \to q), p, \neg q \Rightarrow \neg p$, is a weakened instance of the contraposition axiom. Therefore, by the above proof we get derivability of An within the system $\mathbf{G3m}_\neg$, as desired.

At this point, we have to show that the **G**-systems are closed under *modus ponens* and uniform substitution. This part of the proof is standard. For the details we refer to [44][4]. □

[4] Observe that the fact that *modus ponens* can be 'simulated' by the rules in the sequent systems makes use of Theorem 5.3.1. Indeed, this step of the proof needs to use the Cut rule, and hence, it needs Cut to be admissible in the considered sequent calculi.

Chapter 6

Cut Elimination and Applications

This last chapter is dedicated to admissibility and elimination of the *Cut rule*. We want to show that adding the Cut rule to the **G3**-systems does not make new sequents provable. The proof essentially amounts to showing that the systems are closed under Cut. As already anticipated, the technique of *cut elimination* was developed by Gentzen in 1935 [21, 22]. Nowadays, the technique generally used for proving the 'redundancy' of the Cut rule in a calculus is a semantic one. An advantage of this semantic approach is that it usually leads to establishing completeness with respect to the considered semantics. Nonetheless, in case of logics with a more complicated semantics, a syntactic strategy turns out to be more efficient and, sometimes, to be the only technique actually available.

We begin the chapter by proving the considered sequent calculi to be cut-free. Later, we make use of these cut-free systems and prove fundamental results, as disjunction property and Craig's Interpolation Theorem. We conclude the chapter with an interesting result: contraposition logic interprets minimal propositional logic by means of a natural translation. We will see that such a translation is somehow absorbed into the rule An, and this will make things easier through the proof.

6.1 Closure under Cut

Within this chapter, we often call an application of the Cut rule a *cut*. A deduction in the **G3**-systems which contains no cuts is called a *cut-free* deduction.

We exhibit here a syntactic argument that ensures the considered sequent systems to be closed under Cut.

Theorem 6.1.1 (Admissibility of Cut rule)**.** *The Cut rule is admissible in the considered sequent calculus systems.*

Proof. We will show that whenever we consider a Cut inference,

$$\mathsf{Cut}\ \frac{\Gamma \Rightarrow \alpha \qquad\qquad \Gamma', \alpha \Rightarrow \varphi}{\Gamma, \Gamma' \Rightarrow \varphi}$$

and we assume that its premises have cut-free proofs, the conclusion has a cut-free proof as well. The structure of the following proof resembles the one for admissibility of Weakening: it is a double induction on the pair (n, m), where n is the weight of the cutformula α; this time, m denotes the sum of the depths of the derivation of the premises, i.e., if $\vdash_{m_1} \Gamma \Rightarrow \alpha$ and $\vdash_{m_2} \Gamma', \alpha \Rightarrow \varphi$, then

$m = m_1 + m_2$. We consider a base case and an induction step, which can be seen as encoding three different cases. Given an application of Cut with associated pair (n, m), the induction hypothesis states that any other application of Cut with associated pair $(n', m') < (n, m)$ can be replaced by a cut-free derivation of the same sequent. Observe that we are going to make use of the fact that Weakening and Contraction have been proved to be admissible rules.

$(0, 0)$: In this setting, the cutformula α is either $p \in \Gamma$ or the constant $\top$, while the premises of the rule are both axioms. We shall go through all the possibilities. Consider first the case in which α is a propositional variable $p \in \Gamma$. Then, suppose φ to be p. In this case, the application of the rule gives $\Gamma, \Gamma' \Rightarrow p$ as a conclusion. Such a sequent is an axiom as well, given that $p \in \Gamma$. Therefore, it is derivable by means of a cut-free proof. Suppose now that φ is $q \in \Gamma'$. Again, given that the resulting sequent after the cut is $\Gamma, \Gamma' \Rightarrow q$, and $q \in \Gamma'$, such a sequent is still an axiom. Finally, by assuming φ to be $\top$, it is immediate to see that the resulting formula is an axiom, given that its right-hand side formula is $\top$. We should consider here also the possibility of α being $\top$. It is indeed straightforward to see that the conclusion sequent of the Cut will be an axiom.

$(n, m) > (0, 0)$: This induction step encodes three cases.

First, we consider the case in which *at least one* of the premises is an axiom. Let the right premise sequent $\Gamma', \alpha \Rightarrow \varphi$ be an axiom. This means that either $\varphi = \alpha = p$ or $\varphi = q \in \Gamma'$ or even $\varphi = \top$. The last two options immediately imply the sequent $\Gamma, \Gamma' \Rightarrow \varphi$ to be an axiom too. If the first option holds, i.e., α is the principal formula p, we can conclude $\Gamma, \Gamma' \Rightarrow \varphi$ by applying Weakening to the left premise of Cut. Now, assume the left premise sequent to be an axiom. We need to take care of all the possibilities for the right premise sequent. Nonetheless, we can assume the latter not to be an axiom, because otherwise we would get back to the previous case. Observe that the cutformula α in this setting is either p or $\top$, which implies the impossibility for α to be principal in $\Gamma', \alpha \Rightarrow \varphi$. We need to consider all the inference rules, because the principal formula of the sequent $\Gamma', \alpha \Rightarrow \varphi$ can either be an element of Γ' or φ (or both, as in the case of N, for instance).

$\wedge\mathsf{L}$: Consider the following last two steps of the derivation, assuming the right premise sequent to come from a conjunction left rule

$$\mathsf{Cut}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \wedge\mathsf{L}\ \dfrac{\vdash_{(m_2-1)} \Gamma_2, \gamma_1, \gamma_2, \alpha \Rightarrow \varphi}{\Gamma_2, \gamma_1 \wedge \gamma_2, \alpha \Rightarrow \varphi}}{\Gamma, \Gamma_2, \gamma_1 \wedge \gamma_2 \Rightarrow \varphi}$$

We can apply the induction hypothesis (IH) in the following way, given that the pair $(0, m_2 - 1)$ is lexicographically smaller than the pair $(0, m_2)$:

$$\mathsf{IH}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{(m_2-1)} \Gamma_2, \gamma_1, \gamma_2, \alpha \Rightarrow \varphi}{\Gamma, \Gamma_2, \gamma_1, \gamma_2 \Rightarrow \varphi}$$

Observe that this amounts to *permuting the cut upwards* and it is one of the main techniques used in the current proof. Indeed, instead of applying the Cut after the conjunction rule, the induction hypothesis allows us to apply the Cut first, and then conclude the desired sequent by means of the conjunction rule. In fact, via an application of $\wedge\mathsf{L}$ with context Γ, Γ_2 we can conclude

$$\wedge\mathsf{L}\ \frac{\Gamma, \Gamma_2, \gamma_1, \gamma_2 \Rightarrow \varphi}{\Gamma, \Gamma_2, \gamma_1 \wedge \gamma_2 \Rightarrow \varphi}$$

$\vee\mathsf{L}$: Suppose now that the right premise sequent is the conclusion of a disjunction left rule, as follows

$$\mathsf{Cut}\ \frac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vee\mathsf{L}\ \dfrac{\vdash_{l_1} \Gamma_2, \gamma_1, \alpha \Rightarrow \varphi \qquad \vdash_{l_2} \Gamma_2, \gamma_2, \alpha \Rightarrow \varphi}{\Gamma_2, \gamma_1 \vee \gamma_2, \alpha \Rightarrow \varphi}}{\Gamma', \Gamma_2, \gamma_1 \vee \gamma_2 \Rightarrow \varphi}$$

where l_1, l_2 are such that $m_2 = \max\{l_1, l_2\} + 1$. A double permutation of the cut upward gives us the following two sequents,

$$\mathsf{IH}\ \frac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{l_1} \Gamma_2, \gamma_1, \alpha \Rightarrow \varphi}{\Gamma, \Gamma_2, \gamma_1 \Rightarrow \varphi}$$

$$\mathsf{IH}\ \frac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{l_2} \Gamma_2, \gamma_2, \alpha \Rightarrow \varphi}{\Gamma, \Gamma_2, \gamma_2 \Rightarrow \varphi}$$

Observe that they can be used as premises for an application of $\vee\mathsf{L}$ with context Γ, Γ_2, to conclude the desired sequent $\Gamma, \Gamma_2, \gamma_1 \vee \gamma_2 \Rightarrow \varphi$.

$\to \mathsf{L}$: Consider the following last two steps of the derivation

$$\to \mathsf{L}\ \frac{\vdash_{l_1} \Gamma_2, \gamma_1 \to \gamma_2, \alpha \Rightarrow \gamma_1 \qquad \vdash_{l_2} \Gamma_2, \gamma_2, \alpha \Rightarrow \varphi}{\Gamma_2, \gamma_1 \to \gamma_2, \alpha \Rightarrow \varphi}$$

$$\mathsf{Cut}\ \frac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{m_2} \Gamma_2, \gamma_1 \to \gamma_2, \alpha \Rightarrow \varphi}{\Gamma, \Gamma_2, \gamma_1 \to \gamma_2 \Rightarrow \varphi}$$

with l_1 and l_2 representing depths which are strictly smaller than m_2. Let us apply the induction hypothesis, permuting again the cut upward as follows

$$\mathsf{IH}\ \frac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{l_1} \Gamma_2, \gamma_1 \to \gamma_2, \alpha \Rightarrow \gamma_1}{\Gamma, \Gamma_2, \gamma_1 \to \gamma_2 \Rightarrow \gamma_1}$$

$$\mathsf{IH}\ \frac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{l_2} \Gamma_2, \gamma_2, \alpha \Rightarrow \varphi}{\Gamma, \Gamma_2, \gamma_2 \Rightarrow \varphi}$$

The obtained sequents can indeed be used as premises for applying $\to \mathsf{L}$ with context Γ, Γ_2. This leads us to $\Gamma, \Gamma_2, \gamma_1 \to \gamma_2 \Rightarrow \varphi$, which is the desired sequent.

$\wedge\mathsf{R}$: Consider the following last two steps of the derivation

$$\mathsf{Cut}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \wedge\mathsf{R}\ \dfrac{\vdash_{l_1} \Gamma', \alpha \Rightarrow \gamma_1 \qquad \vdash_{l_2} \Gamma', \alpha \Rightarrow \gamma_2}{\Gamma', \alpha \Rightarrow \gamma_1 \wedge \gamma_2}}{\Gamma, \Gamma' \Rightarrow \gamma_1 \wedge \gamma_2}$$

where the depths l_1, l_2 are strictly smaller than m_2. In a similar way as before, we can apply the induction hypothesis twice and get

$$\mathsf{IH}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{l_1} \Gamma', \alpha \Rightarrow \gamma_1}{\Gamma, \Gamma' \Rightarrow \gamma_1}$$

$$\mathsf{IH}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{l_2} \Gamma', \alpha \Rightarrow \gamma_2}{\Gamma, \Gamma' \Rightarrow \gamma_2}$$

Now, via an application of $\wedge\mathsf{R}$ with the sequents obtained above as premises and context Γ, Γ' we get $\Gamma, \Gamma' \Rightarrow \gamma_1 \wedge \gamma_2$, as desired.

$\vee\mathsf{R}$: Assume that the right premise sequent has been obtained as the conclusion of a right disjunction rule, from a premise whose derivation depth is $(m_2 - 1)$,

$$\mathsf{Cut}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vee\mathsf{R}\ \dfrac{\vdash_{(m_2-1)} \Gamma', \alpha \Rightarrow \gamma_1}{\Gamma', \alpha \Rightarrow \gamma_1 \vee \gamma_2}}{\Gamma, \Gamma' \Rightarrow \gamma_1 \vee \gamma_2}$$

The induction hypothesis this time gives us a sequent which can be used as a premise for a right disjunction rule,

$$\mathsf{IH}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{(m_2-1)} \Gamma', \alpha \Rightarrow \gamma_1}{\Gamma, \Gamma' \Rightarrow \gamma_1}$$

Now, an application of $\vee\mathsf{R}$ with context Γ, Γ' gives us the desired sequent $\Gamma, \Gamma' \Rightarrow \gamma_1 \vee \gamma_2$.

$\rightarrow\mathsf{R}$: The case in which the right premise comes from an inference of $\rightarrow\mathsf{R}$ resembles exactly the previous one. Indeed, the last step of the derivation is

$$\mathsf{Cut}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \rightarrow\mathsf{R}\ \dfrac{\vdash_{(m_2-1)} \Gamma', \gamma_1, \alpha \Rightarrow \gamma_2}{\Gamma', \alpha \Rightarrow \gamma_1 \rightarrow \gamma_2}}{\Gamma, \Gamma' \Rightarrow \gamma_1 \rightarrow \gamma_2}$$

The induction hypothesis gives us the possibility of permuting the cut upward again, as follows

$$\to\mathsf{R}\ \dfrac{\mathsf{IH}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{(m_2-1)} \Gamma', \gamma_1, \alpha \Rightarrow \gamma_2}{\Gamma, \Gamma', \gamma_1 \Rightarrow \gamma_2}}{\Gamma, \Gamma' \Rightarrow \gamma_1 \to \gamma_2}$$

N : We consider here the case in which the rule introducing the right premise sequent is N, and assume l_1, l_2 to be strictly smaller than m_2:

$$\mathsf{Cut}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \mathsf{N}\ \dfrac{\vdash_{l_1} \Gamma_2, \neg\gamma_1, \gamma_1, \alpha \Rightarrow \gamma_2 \qquad \vdash_{l_2} \Gamma_2, \neg\gamma_1, \gamma_2, \alpha \Rightarrow \gamma_1}{\Gamma_2, \neg\gamma_1, \alpha \Rightarrow \neg\gamma_2}}{\Gamma, \Gamma_2, \neg\gamma_1 \Rightarrow \neg\gamma_2}$$

Let us apply the induction hypothesis in the following way

$$\mathsf{IH}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{l_1} \Gamma_2, \neg\gamma_1, \gamma_1, \alpha \Rightarrow \gamma_2}{\Gamma, \Gamma_2, \neg\gamma_1, \gamma_1 \Rightarrow \gamma_2}$$

$$\mathsf{IH}\ \dfrac{\vdash_0 \Gamma \Rightarrow \alpha \qquad \vdash_{l_2} \Gamma_2, \neg\gamma_1, \gamma_2, \alpha \Rightarrow \gamma_1}{\Gamma, \Gamma_2, \neg\gamma_1, \gamma_2 \Rightarrow \gamma_1}$$

and conclude via an application of N with premises the obtained sequents and with context Γ, Γ':

$$\mathsf{N}\ \dfrac{\Gamma, \Gamma_2, \neg\gamma_1, \gamma_1 \Rightarrow \gamma_2 \qquad \Gamma, \Gamma_2, \neg\gamma_1, \gamma_2 \Rightarrow \gamma_1}{\Gamma, \Gamma_2, \neg\gamma_1 \Rightarrow \neg\gamma_2}$$

NeF : This step of the proof goes as follows:

$$\mathsf{Cut}\ \dfrac{\vdash_{m_1} \Gamma \Rightarrow \alpha \qquad \mathsf{NeF}\ \dfrac{\vdash_{(m_2-1)} \Gamma_2, \neg\gamma_1, \alpha \Rightarrow \gamma_1}{\vdash_{m_2} \Gamma_2, \neg\gamma_1, \alpha \Rightarrow \neg\gamma_2}}{\Gamma, \Gamma_2, \neg\gamma_1 \Rightarrow \neg\gamma_2}$$

A simple permutation of the cut upward gives us the desired sequent. In fact, given that $(n, m_1+m_2-1) < (n, m_1+m_2)$, we can apply the induction hypothesis as follows

$$\mathsf{NeF}\ \dfrac{\mathsf{IH}\ \dfrac{\vdash_{m_1} \Gamma \Rightarrow \alpha \qquad \vdash_{(m_2-1)} \Gamma_2, \neg\gamma_1, \alpha \Rightarrow \gamma_1}{\Gamma, \Gamma_2, \neg\gamma_1 \Rightarrow \gamma_1}}{\Gamma, \Gamma_2, \neg\gamma_1 \Rightarrow \neg\gamma_2}$$

CoPC : Consider the following last two steps of the derivation:

$$\mathsf{Cut}\ \dfrac{\vdash_{m_1} \Gamma \Rightarrow \alpha \qquad \mathsf{CoPC}\ \dfrac{\vdash_{(m_2-1)} \Gamma_2, \neg\gamma_1, \gamma_2, \alpha \Rightarrow \gamma_1}{\vdash_{m_2} \Gamma_2, \neg\gamma_1, \alpha \Rightarrow \neg\gamma_2}}{\Gamma, \Gamma_2, \neg\gamma_1 \Rightarrow \neg\gamma_2}$$

We apply now the induction hypothesis on the pair $(0, m_1 + m_2 - 1)$, permuting the cut upward, in the following way

$$\mathsf{IH}\ \dfrac{\vdash_{m_1} \Gamma \Rightarrow \alpha \qquad \vdash_{(m_2-1)} \Gamma_2, \neg\gamma_1, \gamma_2, \alpha \Rightarrow \gamma_1}{\mathsf{CoPC}\ \dfrac{\Gamma, \Gamma_2, \neg\gamma_1, \gamma_2 \Rightarrow \gamma_1}{\Gamma, \Gamma_2, \neg\gamma_1 \Rightarrow \neg\gamma_2}}$$

which gives us the desired cut-free proof.

An : Consider the following case:

$$\mathsf{Cut}\ \dfrac{\vdash_{m_1} \Gamma \Rightarrow \alpha \qquad \mathsf{An}\ \dfrac{\vdash_{(m_2-1)} \Gamma', \beta, \alpha \Rightarrow \neg\beta}{\vdash_{m_2} \Gamma', \alpha \Rightarrow \neg\beta}}{\Gamma, \Gamma' \Rightarrow \neg\beta}$$

The induction hypothesis allows us to get the following derivation, easily permuting the cut upward using derivations such that $(m_1 + m_2 - 1) < m$.

$$\mathsf{IH}\ \dfrac{\vdash_{m_1} \Gamma \Rightarrow \alpha \qquad \vdash_{(m_2-1)} \Gamma', \beta, \alpha \Rightarrow \neg\beta}{\mathsf{An}\ \dfrac{\Gamma, \Gamma', \beta \Rightarrow \neg\beta}{\Gamma, \Gamma', \Rightarrow \neg\beta}}$$

In this way, we get the desired sequent by means of a cut-free proof.

We have covered all the possibilities assuming that (at least) one of the premises is an axiom. At this point, we consider the case in which neither of the premises is an axiom, and the cutformula α is *not* principal in *at least one* of them. Observe that in checking all the possible rules for the right premise sequent, we have already covered the case in which α is not a principal formula in such a sequent. In fact, we never used the assumption that $\Gamma \Rightarrow \alpha$ was an axiom, except for claiming that α could not be principal in the right sequent. Therefore, the remaining cases are the ones in which α is not principal in the left sequent. In this case, the left premise comes from an application of a left rule.

$\wedge\mathsf{L}$: Consider the following last two steps of the derivation:

$$\mathsf{Cut}\ \dfrac{\wedge\mathsf{L}\ \dfrac{\vdash_{(m_1-1)} \Gamma_1, \beta_1, \beta_2 \Rightarrow \alpha}{\vdash_{m_1} \Gamma_1, \beta_1 \wedge \beta_2 \Rightarrow \alpha} \qquad \vdash_{m_2} \Gamma', \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \wedge \beta_2, \Gamma' \Rightarrow \varphi}$$

Let us apply the induction hypothesis as follows:

$$\mathsf{IH}\ \dfrac{\vdash_{(m_1-1)} \Gamma_1, \beta_1, \beta_2 \Rightarrow \alpha \qquad \vdash_{m_2} \Gamma', \alpha \Rightarrow \varphi}{\wedge\mathsf{L}\ \dfrac{\Gamma_1, \beta_1, \beta_2, \Gamma' \Rightarrow \varphi}{\Gamma_1, \beta_1 \wedge \beta_2, \Gamma' \Rightarrow \varphi}}$$

We have shown that the conclusion obtained by Cut rule could be alternatively obtained in the original system.

$\vee$L : Consider the following last two steps of the derivation:

$$\dfrac{\dfrac{\vdash_{k_1} \Gamma_1, \beta_1 \Rightarrow \alpha \qquad \vdash_{k_2} \Gamma_1, \beta_2 \Rightarrow \alpha}{\Gamma_1, \beta_1 \vee \beta_2 \Rightarrow \alpha}\,\vee\mathsf{L} \qquad \vdash_{m_2} \Gamma', \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \vee \beta_2, \Gamma' \Rightarrow \varphi}\,\mathsf{Cut}$$

where the depths k_1, k_2 are strictly smaller than m_1. We can apply the induction hypothesis, to permute the cut upward with respect to the application of the disjunction left rule

$$\mathsf{IH}\ \dfrac{\vdash_{k_1} \Gamma_1, \beta_1 \Rightarrow \alpha \qquad \vdash_{m_2} \Gamma', \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1, \Gamma' \Rightarrow \varphi}$$

$$\mathsf{IH}\ \dfrac{\vdash_{k_2} \Gamma_1, \beta_2 \Rightarrow \alpha \qquad \vdash_{m_2} \Gamma', \alpha \Rightarrow \varphi}{\Gamma_1, \beta_2, \Gamma' \Rightarrow \varphi}$$

Now, via an application of $\vee$L with context $\Gamma_1, \Gamma', \varphi$ we get:

$$\vee\mathsf{L}\ \dfrac{\Gamma_1, \beta_1, \Gamma' \Rightarrow \varphi \qquad \Gamma_1, \beta_2, \Gamma' \Rightarrow \varphi}{\Gamma_1, \beta_1 \vee \beta_2, \Gamma' \Rightarrow \varphi}$$

$\rightarrow$ L : Consider the following last two steps of the derivation,

$$\rightarrow\mathsf{L}\ \dfrac{\vdash_{k_1} \Gamma_1, \beta_1 \rightarrow \beta_2 \Rightarrow \beta_1 \qquad \vdash_{k_2} \Gamma_1, \beta_2 \Rightarrow \alpha}{\Gamma_1, \beta_1 \rightarrow \beta_2 \Rightarrow \alpha}$$

$$\mathsf{Cut}\ \dfrac{\vdash_{m_1} \Gamma_1, \beta_1 \rightarrow \beta_2 \Rightarrow \alpha \qquad \vdash_{m_2} \Gamma', \alpha \Rightarrow \varphi}{\Gamma_1, \beta_1 \rightarrow \beta_2, \Gamma' \Rightarrow \varphi}$$

with k_1 and k_2 denoting depths of derivations strictly smaller than m_1. Let us apply the induction hypothesis as follows,

$$\mathsf{IH}\ \dfrac{\vdash_{k_2} \Gamma_1, \beta_2 \Rightarrow \alpha \qquad \vdash_{m_2} \Gamma', \alpha \Rightarrow \varphi}{\Gamma_1, \beta_2, \Gamma' \Rightarrow \varphi}$$

Moreover, an application of the Weakening rule leads us to the following derivation $\vdash_{k_1} \Gamma_1, \beta_1 \rightarrow \beta_2, \Gamma' \Rightarrow \beta_1$, which allows for an application of $\rightarrow$ L with context $\Gamma_1, \Gamma', \varphi$

$$\rightarrow\mathsf{L}\ \dfrac{\Gamma_1, \beta_1 \rightarrow \beta_2, \Gamma' \Rightarrow \beta_1 \qquad \Gamma_1, \beta_2, \Gamma' \Rightarrow \varphi}{\Gamma_1, \beta_1 \rightarrow \beta_2, \Gamma' \Rightarrow \varphi}$$

Assuming that α is not principal in the left premise sequent, we have covered all the possibilities. The remaining case to be checked is the one in which neither of the premises is an axiom, and the formula α is principal in *both* of them. Observe that in this case the left premise sequent is the conclusion of $\circ\mathsf{R}$ if and only if the right premise comes as the conclusion of $\circ\mathsf{L}$, where $\circ \in \{\wedge, \vee, \rightarrow\}$. In a similar way for negation, the left sequent comes as the conclusion of a negation rule if and only if also the right premise sequent does.

$\wedge$: Consider the last steps of the considered derivation, assuming that k_1 and k_2 are such that $m_1 = \max\{k_1, k_2\} + 1$, as follows

$$\mathsf{Cut}\ \dfrac{\wedge\mathsf{R}\ \dfrac{\vdash_{k_1} \Gamma \Rightarrow \alpha_1 \qquad \vdash_{k_2} \Gamma \Rightarrow \alpha_2}{\vdash_{m_1} \Gamma \Rightarrow \alpha_1 \wedge \alpha_2} \qquad \wedge\mathsf{L}\ \dfrac{\vdash_{(m_2-1)} \Gamma', \alpha_1, \alpha_2 \Rightarrow \varphi}{\vdash_{m_2} \Gamma', \alpha_1 \wedge \alpha_2 \Rightarrow \varphi}}{\Gamma, \Gamma' \Rightarrow \varphi}$$

At this point we can apply the induction hypothesis two consecutive times

$$\mathsf{IH}\ \dfrac{\Gamma \Rightarrow \alpha_2 \qquad \mathsf{IH}\ \dfrac{\Gamma \Rightarrow \alpha_1 \qquad \Gamma, \alpha_1, \alpha_2 \Rightarrow \varphi}{\Gamma, \Gamma', \alpha_2 \Rightarrow \varphi}}{\Gamma, \Gamma, \Gamma' \Rightarrow \varphi}$$

It is worth to be noted that for these applications of the induction hypothesis it has not been necessary to specify the depth of the considered derivations. Indeed, in both cases the induction hypothesis can be applied because $n = w(\alpha) = w(\alpha_1) + w(\alpha_2) + 1$, and hence the pairs $(w(\alpha_1), -)$ and $(w(\alpha_2), -)$ are both lexicographically smaller than (n, m). Now, by applying Contraction to $\Gamma, \Gamma, \Gamma' \Rightarrow \varphi$, we obtain the desired sequent.

$\vee$: Consider the last steps of the derivation in which the principal formula α is a disjunction

$$\mathsf{Cut}\ \dfrac{\vee\mathsf{R}\ \dfrac{\vdash_{(m_1-1)} \Gamma \Rightarrow \alpha_1}{\vdash_{m_1} \Gamma \Rightarrow \alpha_1 \vee \alpha_2} \qquad \vee\mathsf{L}\ \dfrac{\vdash_{l_1} \Gamma', \alpha_1 \Rightarrow \varphi \qquad \vdash_{l_2} \Gamma', \alpha_2 \Rightarrow \varphi}{\vdash_{m_2} \Gamma', \alpha_1 \vee \alpha_2 \Rightarrow \varphi}}{\Gamma, \Gamma' \Rightarrow \varphi}$$

where l_1 and l_2 are strictly smaller than m_2. It is sufficient to apply the induction hypothesis to get the desired sequent,

$$\mathsf{IH}\ \dfrac{\Gamma \Rightarrow \alpha_1 \qquad \Gamma', \alpha_1 \Rightarrow \varphi}{\Gamma, \Gamma' \Rightarrow \varphi}$$

Again, for this application of the induction hypothesis it has not been necessary to specify the depth of the considered derivations. The induction hypothesis can be applied because $n = w(\alpha) = w(\alpha_1) + w(\alpha_2) + 1$, and hence the pair $(w(\alpha_1), -)$ is lexicographically smaller than (n, m).

$\to$: Consider the last steps of the proof, assuming the cutformula to be an implication,

$$\to \mathsf{L}\ \frac{\vdash_{l_1} \Gamma', \alpha_1 \to \alpha_2 \Rightarrow \alpha_1 \qquad \vdash_{l_2} \Gamma', \alpha_2 \Rightarrow \varphi}{\vdash_{m_2} \Gamma', \alpha_1 \to \alpha_2 \Rightarrow \varphi}$$

$$\mathsf{Cut}\ \frac{\to \mathsf{R}\ \dfrac{\vdash_{(m_1-1)} \Gamma, \alpha_1 \Rightarrow \alpha_2}{\vdash_{m_1} \Gamma \Rightarrow \alpha_1 \to \alpha_2} \qquad \vdash_{m_2} \Gamma', \alpha_1 \to \alpha_2 \Rightarrow \varphi}{\Gamma, \Gamma' \Rightarrow \varphi}$$

Consider the following applications of the induction hypothesis. In the first case, we apply the induction hypothesis on a subformula of α, while in the second one, we apply it on an $m' = m_1 + l_1$, which is indeed less than $m = m_1 + m_2$:

$$\mathsf{IH}\ \frac{\Gamma, \alpha_1 \Rightarrow \alpha_2 \qquad \Gamma', \alpha_2 \Rightarrow \varphi}{\Gamma, \Gamma', \alpha_1 \Rightarrow \varphi}$$

$$\mathsf{IH}\ \frac{\vdash_{m_1} \Gamma \Rightarrow \alpha_1 \to \alpha_2 \qquad \vdash_{l_1} \Gamma', \alpha_1 \to \alpha_2 \Rightarrow \alpha_1}{\Gamma, \Gamma' \Rightarrow \alpha_1}$$

Now, we use again the induction hypothesis, and this time we are legitimated by the weight of the cutformula being smaller than n, as follows:

$$\mathsf{IH}\ \frac{\Gamma, \Gamma' \Rightarrow \alpha_1 \qquad \Gamma, \Gamma', \alpha_1 \Rightarrow \varphi}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \varphi}$$

which, by Contraction, gives us the sequent $\Gamma, \Gamma' \Rightarrow \varphi$.

$\neg$: Finally, consider the case in which the cutformula is a principal negation in both premise sequents. Here, different options need to be considered. Let us start by considering the case in which both premises come as conclusions of applications of the inference rule N, as follows

$$\mathsf{N}\ \frac{\vdash_{k_1} \Gamma_1, \neg\alpha_1, \alpha_1 \Rightarrow \alpha_2 \qquad \vdash_{k_2} \Gamma_1, \neg\alpha_1, \alpha_2 \Rightarrow \alpha_1}{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2}$$

$$\mathsf{N}\ \frac{\vdash_{l_1} \Gamma', \neg\alpha_2, \alpha_2 \Rightarrow \alpha_3 \qquad \vdash_{l_2} \Gamma', \neg\alpha_2, \alpha_3 \Rightarrow \alpha_2}{\vdash_{m_2} \Gamma', \neg\alpha_2 \Rightarrow \neg\alpha_3}$$

$$\mathsf{Cut}\ \frac{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2 \qquad \vdash_{m_2} \Gamma', \neg\alpha_2 \Rightarrow \neg\alpha_3}{\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3}$$

where $k_1, k_2 < m_1$ and $l_1, l_2 < m_2$. The induction hypothesis here needs to be applied a few times. The first two applications can be carried out because the considered derivations have shorter depth than $m_1 + m_2$:

$$\mathsf{IH}\ \frac{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2 \qquad \vdash_{l_1} \Gamma', \neg\alpha_2, \alpha_2 \Rightarrow \alpha_3}{\Gamma_1, \neg\alpha_1, \Gamma', \alpha_2 \Rightarrow \alpha_3}$$

$$\mathsf{IH}\ \frac{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2 \qquad \vdash_{l_2} \Gamma', \neg\alpha_2, \alpha_3 \Rightarrow \alpha_2}{\Gamma_1, \neg\alpha_1, \Gamma', \alpha_3 \Rightarrow \alpha_2}$$

The following two applications work because of the fact that the cutformula is a subformula of α, i.e., it is α_2, and it has smaller weight than α

$$\mathsf{IH}\ \frac{\Gamma_1, \neg\alpha_1, \alpha_1 \Rightarrow \alpha_2 \qquad \Gamma_1, \neg\alpha_1, \Gamma', \alpha_2 \Rightarrow \alpha_3}{\Gamma_1, \Gamma_1, \neg\alpha_1, \neg\alpha_1, \alpha_1, \Gamma' \Rightarrow \alpha_3}$$

$$\mathsf{IH}\ \frac{\Gamma_1, \neg\alpha_1, \Gamma', \alpha_3 \Rightarrow \alpha_2 \qquad \Gamma_1, \neg\alpha_1, \alpha_2 \Rightarrow \alpha_1}{\Gamma_1, \Gamma_1, \neg\alpha_1, \neg\alpha_1, \Gamma', \alpha_3 \Rightarrow \alpha_1}$$

After an application of Contraction on both the sequents we have obtained, we can apply N again and get:

$$\mathsf{N}\ \frac{\Gamma_1, \neg\alpha_1, \Gamma', \alpha_1 \Rightarrow \alpha_3 \qquad \Gamma_1, \neg\alpha_1, \Gamma', \alpha_3 \Rightarrow \alpha_1}{\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3}$$

which leads to the desired consequent.

We go on by assuming the left premise to be the conclusion of an application of N; the right premise sequent, on the other hand, comes from an application of NeF. Here, α is $\neg\alpha_2$.

$$\mathsf{N}\ \frac{\vdash_{k_1} \Gamma_1, \neg\alpha_1, \alpha_1 \Rightarrow \alpha_2 \qquad \vdash_{k_2} \Gamma_1, \neg\alpha_1, \alpha_2 \Rightarrow \alpha_1}{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2}$$

$$\mathsf{NeF}\ \frac{\vdash_{(m_2-1)} \Gamma', \neg\alpha_2 \Rightarrow \alpha_2}{\vdash_{m_2} \Gamma', \neg\alpha_2 \Rightarrow \neg\alpha_3}$$

$$\mathsf{Cut}\ \frac{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2 \qquad \vdash_{m_2} \Gamma', \neg\alpha_2 \Rightarrow \neg\alpha_3}{\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3}$$

where $k_1, k_2 < m_1$. We can apply the induction hypothesis with respect to shorter derivations

$$\mathsf{IH}\ \frac{\vdash_{(m_1)} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2 \qquad \vdash_{(m_2-1)} \Gamma', \neg\alpha_2 \Rightarrow \alpha_2}{\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \alpha_2}$$

At this point, we have to apply the induction hypothesis again, this time on a simpler cutformula

$$\mathsf{IH}\ \frac{\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \alpha_2 \qquad \Gamma_1, \neg\alpha_1, \alpha_2 \Rightarrow \alpha_1}{\Gamma_1, \Gamma_1, \neg\alpha_1, \neg\alpha_1, \Gamma' \Rightarrow \alpha_1}$$

By means of Contraction and NeF, we get exactly the desired sequent $\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3$.

Consider now the case in which the left premise sequent is the conclusion of a NeF inference, while the right one is the conclusion of N.

$$\mathsf{NeF}\ \frac{\vdash_{(m_1-1)} \Gamma_1, \neg\alpha_1 \Rightarrow \alpha_1}{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2}$$

$$\mathsf{N}\ \frac{\vdash_{l_1} \Gamma', \neg\alpha_2, \alpha_2 \Rightarrow \alpha_3 \qquad \vdash_{l_2} \Gamma', \neg\alpha_2, \alpha_3 \Rightarrow \alpha_2}{\vdash_{m_2} \Gamma', \neg\alpha_2 \Rightarrow \neg\alpha_3}$$

$$\mathsf{Cut}\ \frac{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2 \qquad \vdash_{m_2} \Gamma', \neg\alpha_2 \Rightarrow \neg\alpha_3}{\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3}$$

where $l_1, l_2 < m_2$. Observe that it is immediate, applying Weakening and NeF to $\Gamma_1, \neg\alpha_1 \Rightarrow \alpha_1$, to obtain the sequent $\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3$, as desired. To conclude the whole proof, consider the case in which both premise sequents come as conclusions of NeF,

$$\mathsf{Cut}\ \frac{\mathsf{NeF}\ \dfrac{\vdash_{(m_1-1)} \Gamma_1, \neg\alpha_1 \Rightarrow \alpha_1}{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2} \qquad \mathsf{NeF}\ \dfrac{\vdash_{(m_2-1)} \Gamma', \neg\alpha_2 \Rightarrow \alpha_2}{\vdash_{m_2} \Gamma', \neg\alpha_2 \Rightarrow \neg\alpha_3}}{\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3}$$

Again, here, via an application of Weakening to $\Gamma_1, \neg\alpha_1 \Rightarrow \alpha_1$, followed by an inference of NeF, we get the desired result.

We have basically covered all the cases concerning both **G3n** and **G3nef**. At this point, the two conclusive cases of the proof are about the rules of CoPC and An. Let the last two steps of the derivations look as follows:

$$\mathsf{Cut}\ \frac{\mathsf{CoPC}\ \dfrac{\vdash_{(m_1-1)} \Gamma_1, \neg\alpha_1, \alpha_2 \Rightarrow \alpha_1}{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2} \qquad \mathsf{CoPC}\ \dfrac{\vdash_{(m_2-1)} \Gamma', \neg\alpha_2, \alpha_3 \Rightarrow \alpha_2}{\vdash_{m_2} \Gamma', \neg\alpha_2 \Rightarrow \neg\alpha_3}}{\vdash_{m_2} \Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3}$$

The induction hypothesis can be applied because we are dealing with shorter derivations, as $(n, m_1 + m_2 - 1) < (n, m_1 + m_2)$:

$$\mathsf{IH}\ \frac{\vdash_{m_1} \Gamma_1, \neg\alpha_1 \Rightarrow \neg\alpha_2 \qquad \vdash_{(m_2-1)} \Gamma', \neg\alpha_2, \gamma \Rightarrow \alpha_2}{\Gamma_1, \neg\alpha_1, \Gamma', \alpha_3 \Rightarrow \alpha_2}$$

The second application of the induction hypothesis goes here on a 'smaller' cut-formula α_2:

$$\dfrac{\Gamma_1, \neg\alpha_1, \Gamma', \alpha_3 \Rightarrow \alpha_2 \qquad \Gamma_1, \neg\alpha_1, \alpha_2 \Rightarrow \alpha_1}{\dfrac{\Gamma_1, \Gamma_1, \neg\alpha_1, \neg\alpha_1, \Gamma', \alpha_3 \Rightarrow \alpha_1}{\Gamma_1, \Gamma_1, \neg\alpha_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3}\ \mathsf{CoPC}}\ \mathsf{IH}$$

which, via an application of Contraction, leads us to the desired sequent $\Gamma_1, \neg\alpha_1, \Gamma' \Rightarrow \neg\alpha_3$ via a cut-free proof.

Take now into consideration the case in which $\alpha = \neg\alpha_1$ is a principal formula in both sequents, and the left premise comes via an application of An.

$$\dfrac{\mathsf{An}\ \dfrac{\vdash_{(m_1-1)} \Gamma, \alpha_1 \Rightarrow \neg\alpha_1}{\vdash_{m_1} \Gamma \Rightarrow \neg\alpha_1} \qquad \mathsf{CoPC}\ \dfrac{\vdash_{(m_2-1)} \Gamma', \neg\alpha_1, \alpha_2 \Rightarrow \alpha_1}{\vdash_{m_2} \Gamma', \neg\alpha_1 \Rightarrow \neg\alpha_2}}{\Gamma, \Gamma' \Rightarrow \neg\alpha_2}\ \mathsf{Cut}$$

Consider the following two applications of induction hypothesis:

$$\mathsf{IH}\ \dfrac{\vdash_{(m_1-1)} \Gamma, \alpha_1 \Rightarrow \neg\alpha_1 \qquad \vdash_{m_2} \Gamma', \neg\alpha_1 \Rightarrow \neg\alpha_2}{\Gamma, \alpha_1, \Gamma' \Rightarrow \neg\alpha_2}$$

$$\mathsf{IH}\ \dfrac{\vdash_{m_1} \Gamma \Rightarrow \neg\alpha_1 \qquad \vdash_{(m_2-1)} \Gamma', \neg\alpha_1, \alpha_2 \Rightarrow \alpha_1}{\Gamma, \Gamma', \alpha_2 \Rightarrow \alpha_1}$$

Note that here we used the induction hypothesis on the base of the fact that the pairs $(n, m_1 + m_2 - 1)$ and $(n, m_1 - 1 + m_2)$ are lexicographically smaller than the pair (n, m), where $m = m_1 + m_2$. Via a last application of the induction hypothesis, plus Weakening followed by An, we get:

$$\mathsf{IH}\ \dfrac{\Gamma, \Gamma', \alpha_2 \Rightarrow \alpha_1 \qquad \Gamma, \alpha_1, \Gamma' \Rightarrow \neg\alpha_2}{\mathsf{W}\ \dfrac{\Gamma, \Gamma, \Gamma', \Gamma', \alpha_2 \Rightarrow \neg\alpha_2}{\mathsf{An}\ \dfrac{\Gamma, \Gamma', \alpha_2 \Rightarrow \neg\alpha_2}{\Gamma, \Gamma' \Rightarrow \neg\alpha_2}}}$$

and we can conclude that both **G3copc** and $\mathbf{G3m}_\neg$ are closed under Cut.

□

In the cut admissibility theorem we have proved that, given cut-free derivations of the premises of a cut, there is a cut-free derivation of the conclusion. The main induction was on the *weight of the cutformula*. In order to get an actual proof of a *cut elimination* result, one has to consider a measure of the 'cut-complexity' (often called *cutrank*, see [44]) of the whole derivation of a given sequent, and by induction on such a measure one proves that, indeed, *all* the cuts within the given derivation can be eliminated.

Definition 38 (Cutrank)**.** *The* rank *of a cut on a formula φ is defined as the successor of the weight of the formula φ, i.e., $w(\varphi)+1$. The* cutrank *of a derivation amounts to the maximum of the ranks of the cutformulas occurring in the considered derivation.*

The cut elimination theorem says that given a proof (possibly containing cuts) of a sequent, there is a cut-free proof of that same sequent. The main induction is now on the *cutrank of the derivation*, and the induction step is that, given the derivations of the premises of a cut with cutrank bounded by the weight of the cutformula, there is a proof of the conclusion of the cut with cutrank bounded by the weight of the cutformula. The proof goes *exactly* as the one we gave for cut admissibility (i.e., it uses the same local transformation steps). Moreover, it is possible to extract an effective, non-deterministic algorithm from the proof above.

We recall here the statement of the Theorem 5.3.1.

Theorem 5.3.1 Every sequent $\Gamma \Rightarrow \varphi$ which is provable in one of the **G3**-systems + Cut, is also provable in the same **G3**-system without Cut.

Proof. For a detailed analysis, see the proof of cut elimination for intuitionistic logic in [44]. □

The results proved in this section give us cut-free sequent calculi in which both Contraction and Weakening are admissible rules. We are ready to finally use the systems for effective proofs about the considered logical systems.

6.2 Some Applications

In this section, we want to convince ourselves and the reader that the systems we are working with are, in some ways, 'good' systems. The fact that we have proved them to be cut-free sequent calculi already gives us strong reasons to believe that they can be used to obtain interesting results. We want to exploit the strength of the systems by proving fundamental results for the considered logic. In particular, we make use of the **G3**-systems to give an alternative proof of an essential result already presented in Chapter 3: the disjunction property. After briefly discussing the decidability result, we conclude this section by using our calculus to give a proof Craig's Interpolation Theorem for the considered logical systems.

Whenever one wants to show that a certain proof system is indeed a useful tool for proving results in a 'constructive' setting, the disjunction property turns out to be the perfect candidate for being the required witness. In Chapter 3 we gave a semantic proof of the disjunction property, making use of some preservation and invariance results. Here, we state a slightly more general result than the one proved semantically: *disjunction property under hypothesis* [44]. The semantic proof given in Chapter 3 can indeed be generalized to prove also this version of the result. The structure of the semantic proof would remain the same, although some more preservation properties would need to be proved.

Theorem 6.2.1 (Disjunction Property under Hypothesis)**.** *Consider a finite multiset of formulas Γ which does not contain any disjunction. Then, if $\Gamma \vdash \varphi \vee \psi$, it follows that $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$. Moreover, such a property is depth-preserving.*

Proof. The proof goes exactly as the one for the intuitionistic case [44]. Indeed, the newly added rules introduce principal formulas both on the left and on the right-hand side of the sequent, and thus, they need not to be considered in the proof. □

Before moving to the interpolation results, we give here an interesting example of cut-free proofs by using the **G3copc**-system. The results we are going to prove will give us a way to work with negated formulas within the contraposition and minimal systems. As a matter of fact, the presence of contraposition ensures that every even number on negations is equivalent to two negations; moreover, every odd number of negations implies one negation.

Theorem 6.2.2. *Let $n \in \mathbb{N}$ be an arbitrary natural number such that $n \geq 1$. Then, we have that*

$$\neg^{(2n+1)}p \to \neg p \textit{ and } \neg^{(2n)}p \leftrightarrow \neg\neg p \textit{ are theorems of contraposition logic},$$

where $\neg^{(m)}$ denotes m nested applications of the negation operator for any natural number m.

Proof. The proof goes by induction on $n \in \mathbb{N}$. This means that, for the base case, we need to prove that $\neg\neg\neg p \to \neg p$ is a theorem. The proof-search procedure goes as follows: we start from the sequent we want to derive, i.e.,

$$\neg\neg\neg p \Rightarrow \neg p.$$

The only rule we can apply backward clearly is CoPC. Therefore, we get the sequent

$$\neg\neg\neg p, p \Rightarrow \neg\neg p.$$

Again, the only possible move to do backward is to apply CoPC, with principal formulas $\neg\neg\neg p$ and $\neg\neg p$. This gives us the sequent

$$\neg\neg\neg p, p, \neg p \Rightarrow \neg\neg p.$$

We can conclude now this procedure by applying the same rule backward once again, this time with principal formulas $\neg p$ and $\neg\neg p$. Indeed, we get the following sequent:

$$\neg\neg\neg p, p, \neg p, \neg p \Rightarrow p,$$

which is an axiom. Therefore, the proof-search procedure ends, and we have obtained the following derivation of the desired sequent within **G3copc**:

$$\cfrac{\cfrac{\cfrac{\neg\neg\neg p, \neg p, \neg p, p \Rightarrow p}{p, \neg\neg\neg p, \neg p \Rightarrow \neg\neg p}}{\neg\neg\neg p, p \Rightarrow \neg\neg p}}{\neg\neg\neg p \Rightarrow \neg p}$$

The reader may observe that this proof only makes use of the contraposition rule. Moreover, the proof-search procedure makes active use of the Contraction built into the rules and of the fact that we are working with multisets of formulas. At this point, let us assume by the induction hypothesis that we have both the desired results for n, i.e.,

$$\neg^{(2n+1)}p \to \neg p \text{ and } \neg^{(2n)}p \leftrightarrow \neg\neg p \text{ are theorems of contraposition logic.}$$

Thus, the sequent

$$\neg^{(2n+1)}p \Rightarrow \neg p$$

is derivable within the system. By weakening this sequent with $\neg\neg p$, we can conclude by CoPC that the sequent

$$\neg\neg p \Rightarrow \neg^{(2(n+1))}p$$

is derivable. Hence,

$$\neg\neg p \to \neg^{(2(n+1))}p$$

is a theorem of contraposition logic. On the other hand, by uniformly substituting[1] $\neg q$ in $\neg^{(2n+1)}p \to \neg p$, we get that

$$\neg^{(2(n+1))}q \to \neg\neg q$$

is a theorem as well. Finally, from the derivability of the sequent

$$\neg\neg p \Rightarrow \neg^{(2(n+1))}p,$$

we can apply Weakening and CoPC and conclude

$$\neg^{(2(n+1)+1)}p \Rightarrow \neg\neg\neg p.$$

At this point, we employ the closure of the system under Cut and we obtain

$$\neg^{(2(n+1)+1)}p \Rightarrow \neg p.$$

This gives us the theorem $\neg^{(2(n+1)+1)}p \to \neg p$, as desired. $\square$

[1]Uniform substitution is a rule from the Hilbert-style definition of CoPC. Nonetheless, we have proved in Theorem 5.4.1 that it can be simulated by means of the sequent system, and hence, for the sake of simplicity, we refer to it.

Note that the previous theorem allows us to conclude the same result for minimal propositional logic as well, **G3copc** being a subsystem of $\mathbf{G3m}_{\neg}$.

6.2.1 Decidability via Sequent Calculi

In Chapter 3, we have discussed in the standard way that the finite model property ensures the studied subminimal systems to be decidable. Here, we want to give a decision method which exploits the proof systems defined in Chapter 5. The argument is standard and simulates the proof of this result for intuitionistic logic as it is given in [44].

Theorem 6.2.3 (Decidability via Sequent Calculi)**.** *The logical systems* N*,* NeF*,* CoPC *and* MPC *are decidable.*

Proof. The idea of the proof amounts to the following: given a sequent $\Gamma \Rightarrow \varphi$, we want to decide whether there exists a proof of it within the considered **G3**-system. Hence, we have to search 'bottom-up' for a proof of the considered sequent. In doing this, we take into consideration several facts.

First of all, observe that every step of the searching procedure consists of looking simultaneously for proofs of *finitely* many sequents. Moreover, given a sequent $\Gamma \Rightarrow \varphi$, a 'predecessor' of that sequent is obtained by replacing that sequent with (at most) two sequents, which represent the premises of the eventually applied inference rule with $\Gamma \Rightarrow \varphi$ as a conclusion.

Given two sequents $\Gamma \Rightarrow \varphi$ and $\Delta \Rightarrow \psi$, we say that they are *equivalent* if Γ and Δ contain the same formulas without considering multiple occurrences (i.e., Γ and Δ conceived as *sets* of formulas are the same), and φ and ψ coincide. Given two finite sets of sequents $\{\Gamma_1 \Rightarrow \varphi_1, \ldots, \Gamma_n \Rightarrow \varphi_n\}$ and $\{\Delta_1 \Rightarrow \psi_1, \ldots, \Delta_m \Rightarrow \psi_m\}$, we consider them to be *equivalent* if, for every $\Gamma_i \Rightarrow \varphi_i$, there is an equivalent sequent $\Delta_j \Rightarrow \psi_j$ in the second set. This notion of 'being equivalent' gives us a way to stop the proof-search procedure. In particular, a branch of the search tree can end either because we meet an axiom along such a branch or because we encounter a problem (i.e., a finite set of sequents) which is equivalent to a problem we encountered lower down the branch. As already said, since the considered sequent calculi satisfy the subformula property, we get a bound on the depth of the search tree and, hence, the proof-search eventually terminates. □

As we have seen, the considered proof systems are decidable, and this can be proved fairly easily both syntactically and semantically. On the other hand, the calculi presented here are not terminating[2]. Therefore, we do not get any non-trivial decision procedure and we have no information about the computational complexity of this decidability problem. In order to get any kind of information, we should ensure our systems to be terminating, or at least we should provide a loop-checking technique. For instance, one could refer to [45] to find two different

[2]Consider a proof-search procedure for the sequent $\neg\neg\neg p \Rightarrow \neg\neg p$. The only possibility is to consider a backward application of CoPC, obtaining the sequent $\neg\neg\neg p, \neg p \Rightarrow \neg\neg p$. At this point, considering a further backward application of CoPC, with context $\neg p$, we get $\neg\neg\neg p, \neg p, \neg p \Rightarrow \neg\neg p$, and we can already see that the bottom-up proof-search procedure does not necessarily terminate.

PSPACE decision procedures for intuitionistic propositional logic and, then, try to mimic them. Nonetheless, the approach of modifying either the single-conclusion or the multi-conclusion calculus from [45] to adapt it to our systems doesn't seem to go through smoothly.

Let us focus briefly on minimal logic. As emphasized already, the formulation of minimal logic as MPC_f is very useful, since it gives us a standpoint from which the minimal system behaves basically as the intuitionistic one. To be more precise, we can easily see that minimal propositional logic is PSPACE-hard, since intuitionistic propositional logic is PSPACE-hard [41] and IPC can be translated into MPC_f by means of a *polynomial* translation[3] [14].

6.2.2 Craig's Interpolation Theorem

The current part of the section is dedicated to the Interpolation Theorem. The Interpolation Theorem for classical first-order logic was originally proved by Craig in 1957 [15]. The method he used to prove it is indeed a proof-theoretic one. In later years, refined and more generalized versions of the theorem have been proved by many different logicians [44]. In 1962, Schütte [40] proved the same result for intuitionistic logic, using Maehara's method of 'splitting' the contexts [31]. When conceiving minimal logic in the form of MPC_f, the above cited method of proving the Interpolation Theorem for intuitionistic logic gives the result for minimal logic as well.

Remark. Although we won't give a formal proof of this here, the reader may easily see that the sequent $\Gamma, \varphi \Rightarrow \varphi$ is derivable within the considered systems, for any multiset of formulas Γ, and any formula φ. The proof is a straightforward induction on the structure of the formula φ.

Given two sets of formulas Δ and Δ', we call *the common language of* Δ *and* Δ' the set of propositional variables $\{p_1, \ldots, p_n\}$ which appear in both Δ and Δ'.

The following result holds for all the subminimal systems analyzed here. The proof is an interesting application of the **G3**-systems.

Theorem 6.2.4 (Craig's Interpolation Theorem)**.** *Let Γ, Γ' denote arbitrary finite multisets of formulas and let φ be an arbitrary formula such that the common language of Γ and Γ', φ is not empty. If $\vdash \Gamma, \Gamma' \Rightarrow \varphi$, there exists a formula $\boldsymbol{\sigma}$ such that:*

1. *the language of $\boldsymbol{\sigma}$ is contained in the* common language *of Γ and Γ', φ,*
2. *$\vdash \Gamma \Rightarrow \boldsymbol{\sigma}$ and $\vdash \Gamma', \boldsymbol{\sigma} \Rightarrow \varphi$.*

Proof. The proof of this theorem goes via an induction on the depth of the derivation n. We show that the required interpolant $\boldsymbol{\sigma}$ exists for *every possible splitting* of the sequent. Given a sequent $\Gamma, \Gamma' \Rightarrow \varphi$, we use Γ_1 to denote a subset of Γ, and Γ_2 for a subset of Γ'. Let us first consider the base case with $n = 0$.

[3]More about this translation will be said at the end of the current chapter.

First consider the case in which the sequent $\Gamma, \Gamma' \Rightarrow \varphi$ is an axiom of the following form, with $\Gamma = \Gamma_1, p$,

$$\Gamma_1, p, \Gamma' \Rightarrow p.$$

Observe that the atom p is in the common language of Γ (i.e., Γ_1, p) and Γ', φ (i.e., Γ', p). Moreover, the sequents

$$\Gamma_1, p \Rightarrow p \text{ and } \Gamma', p \Rightarrow p$$

are both axioms, and hence provable in the systems. Thus, $\boldsymbol{\sigma} := p$. Consider now the second case, in which the axiom is of the form:

$$\Gamma, \Gamma_2, p \Rightarrow p,$$

where $\Gamma' = \Gamma_2, p$. Here, we need to be more careful. The most immediate choice is $\boldsymbol{\sigma} := \top$. In fact, $\top$ is trivially in the common language of Γ and Γ_2, p. Moreover, both sequents

$$\Gamma \Rightarrow \top \text{ and } \Gamma_2,\ p, \top \Rightarrow p$$

turn out to be axioms, and hence they are derivable. To conclude this case, we consider the last possibility, that is the one in which the sequent is of the form $\Gamma, \Gamma' \Rightarrow \top$. It is clear also in this case that the right candidate is $\boldsymbol{\sigma} := \top$.

We assume as the induction hypothesis that, whenever we have a sequent of the form $\Delta, \Delta' \Rightarrow \psi$, whose derivation is of depth $< n$, then the required interpolant $\boldsymbol{\sigma'}$ exists for every possible splitting of the sequent. Now, we consider all the possible rules which can represent the last step of the derivation of $\Gamma, \Gamma' \Rightarrow \varphi$. For every non-right rule, there are a few cases to be considered, depending on whether the (left) principal formula is an element of Γ or of Γ'.

$\wedge\mathsf{L}$: We shall first consider the case in which a conjunction has been introduced in Γ, and Γ is of the form $\Gamma_1, \alpha \wedge \beta$, as follows:

$$\wedge\mathsf{L}\ \frac{\Gamma_1, \alpha, \beta, \Gamma' \Rightarrow \varphi}{\Gamma_1, \alpha \wedge \beta, \Gamma' \Rightarrow \varphi}$$

By the induction hypothesis, we have the existence of $\boldsymbol{\sigma'}$ such that the sequents $\Gamma_1, \alpha, \beta \Rightarrow \boldsymbol{\sigma'}$ and $\Gamma', \boldsymbol{\sigma'} \Rightarrow \varphi$ are both derivable and $\boldsymbol{\sigma'}$ is in the common language of Γ_1, α, β and Γ', φ. By applying $\wedge\mathsf{L}$ on the sequent $\Gamma_1, \alpha, \beta \Rightarrow \boldsymbol{\sigma'}$, we obtain $\Gamma_1, \alpha \wedge \beta \Rightarrow \boldsymbol{\sigma'}$. Therefore, we get that $\boldsymbol{\sigma} := \boldsymbol{\sigma'}$ is the desired interpolant.

Assume now the principal formula to be introduced in Γ' and let the last inference look as follows:

$$\wedge\mathsf{L}\ \frac{\Gamma, \Gamma_2, \alpha, \beta, \Rightarrow \varphi}{\Gamma, \Gamma_2, \alpha \wedge \beta \Rightarrow \varphi}$$

By the induction hypothesis, we have the existence of $\boldsymbol{\sigma'}$ such that $\Gamma \Rightarrow \boldsymbol{\sigma'}$ and

$\Gamma_2, \alpha, \beta, \boldsymbol{\sigma'} \Rightarrow \varphi$ are both derivable sequents and $\boldsymbol{\sigma'}$ is in the common language of Γ and $\Gamma_2, \alpha, \beta, \varphi$. By applying $\wedge\mathsf{L}$ on the sequent $\Gamma_2, \alpha, \beta, \boldsymbol{\sigma'} \Rightarrow \varphi$, we obtain $\Gamma_2, \alpha \wedge \beta, \boldsymbol{\sigma'} \Rightarrow \varphi$. Therefore, we get that $\boldsymbol{\sigma} := \boldsymbol{\sigma'}$ is again the desired interpolant.

$\vee\mathsf{L}$: We consider the case in which a disjunction has been introduced in Γ, as follows:

$$\vee\mathsf{L}\ \frac{\Gamma_1, \alpha, \Gamma' \Rightarrow \varphi \qquad \Gamma_1, \beta, \Gamma' \Rightarrow \varphi}{\Gamma_1, \alpha \vee \beta, \Gamma' \Rightarrow \varphi}$$

By the induction hypothesis, we have ensured the existence of two different interpolants, $\boldsymbol{\sigma'}$ and $\boldsymbol{\sigma''}$, such that $\Gamma_1, \alpha \Rightarrow \boldsymbol{\sigma'}$, $\Gamma', \boldsymbol{\sigma'} \Rightarrow \varphi$, $\Gamma_1, \beta \Rightarrow \boldsymbol{\sigma''}$ and $\Gamma', \boldsymbol{\sigma''} \Rightarrow \varphi$ are derivable sequents and $\boldsymbol{\sigma'}$ is in the common language of Γ_1, α and Γ', φ, while $\boldsymbol{\sigma''}$ is in the common language of Γ_1, β and Γ', φ. By applying $\vee\mathsf{L}$ on the sequents $\Gamma', \boldsymbol{\sigma'} \Rightarrow \varphi$ and $\Gamma', \boldsymbol{\sigma''} \Rightarrow \varphi$, we get $\Gamma', \boldsymbol{\sigma'} \vee \boldsymbol{\sigma''} \Rightarrow \varphi$. Moreover, via an application of $\vee\mathsf{R}$ to the sequent $\Gamma_1, \alpha \Rightarrow \boldsymbol{\sigma'}$, we can conclude: $\Gamma_1, \alpha \Rightarrow \boldsymbol{\sigma'} \vee \boldsymbol{\sigma''}$. Finally, note that $\boldsymbol{\sigma'} \vee \boldsymbol{\sigma''}$ is in the common language of $\Gamma_1, \alpha \vee \beta$ and Γ', φ. Therefore, we get that $\boldsymbol{\sigma} := \boldsymbol{\sigma'} \vee \boldsymbol{\sigma''}$ is the desired interpolant.

Now, assume Γ' is of the form $\Gamma_2, \alpha \vee \beta$ and let the last inference look as

$$\vee\mathsf{L}\ \frac{\Gamma, \Gamma_2, \alpha \Rightarrow \varphi \qquad \Gamma, \Gamma_2, \beta \Rightarrow \varphi}{\Gamma, \Gamma_2, \alpha \vee \beta \Rightarrow \varphi}$$

By the induction hypothesis, we have ensured the existence of two different interpolants, $\boldsymbol{\sigma'}$ and $\boldsymbol{\sigma''}$, such that $\Gamma \Rightarrow \boldsymbol{\sigma'}$, $\Gamma_2, \alpha, \boldsymbol{\sigma'} \Rightarrow \varphi$, $\Gamma \Rightarrow \boldsymbol{\sigma''}$ and $\Gamma_2, \beta, \boldsymbol{\sigma''} \Rightarrow \varphi$ are derivable sequents and $\boldsymbol{\sigma'}$ is in the common language of Γ and $\Gamma_2, \alpha, \varphi$, while $\boldsymbol{\sigma''}$ is in the common language of Γ and Γ_2, β, φ. Again, the desired interpolant turns out to be $\boldsymbol{\sigma} := \boldsymbol{\sigma'} \vee \boldsymbol{\sigma''}$. Indeed, via a double application of the left disjunction rule, we get exactly $\Gamma_2, \alpha \vee \beta, \boldsymbol{\sigma'} \vee \boldsymbol{\sigma''} \Rightarrow \varphi$. In addition, the sequent $\Gamma \Rightarrow \boldsymbol{\sigma'} \vee \boldsymbol{\sigma''}$ is obtained by means of $\vee\mathsf{R}$. Finally, note that $\boldsymbol{\sigma'} \vee \boldsymbol{\sigma''}$ is in the common language of Γ and $\Gamma_2, \alpha \vee \beta, \varphi$.

$\rightarrow \mathsf{L}$: We consider the case in which an implication has been introduced in Γ, as follows:

$$\rightarrow \mathsf{L}\ \frac{\Gamma_1, \alpha \rightarrow \beta, \Gamma' \Rightarrow \alpha \qquad \Gamma_1, \beta, \Gamma' \Rightarrow \varphi}{\Gamma_1, \alpha \rightarrow \beta, \Gamma' \Rightarrow \varphi}$$

By the induction hypothesis, we have the existence of two different interpolants, $\boldsymbol{\sigma'}$ and $\boldsymbol{\sigma''}$, such that $\Gamma_1, \alpha \rightarrow \beta, \boldsymbol{\sigma'} \Rightarrow \alpha$, $\Gamma' \Rightarrow \boldsymbol{\sigma'}$, $\Gamma_1, \beta \Rightarrow \boldsymbol{\sigma''}$ and $\Gamma', \boldsymbol{\sigma''} \Rightarrow \varphi$ are derivable sequents and $\boldsymbol{\sigma'}$ is in the common language of $\Gamma_1, \alpha \rightarrow \beta$ and Γ', while $\boldsymbol{\sigma''}$ is in the common language of Γ_1, β and Γ', φ. Note that, by means of Weakening, from $\Gamma_1, \beta \Rightarrow \boldsymbol{\sigma''}$ we can get $\Gamma_1, \beta, \boldsymbol{\sigma'} \Rightarrow \boldsymbol{\sigma''}$, and from $\Gamma' \Rightarrow \boldsymbol{\sigma'}$ we can obtain $\Gamma', \boldsymbol{\sigma'} \rightarrow \boldsymbol{\sigma''} \Rightarrow \boldsymbol{\sigma'}$. Now, we have all the ingredients for applying a left implication rule and a right implication rule, as follows:

$$\rightarrow \mathsf{R}\ \frac{\rightarrow \mathsf{L}\ \dfrac{\Gamma_1, \alpha \rightarrow \beta, \boldsymbol{\sigma'} \Rightarrow \alpha \qquad \Gamma_1, \beta, \boldsymbol{\sigma'} \Rightarrow \boldsymbol{\sigma''}}{\Gamma_1, \alpha \rightarrow \beta, \boldsymbol{\sigma'} \Rightarrow \boldsymbol{\sigma''}}}{\Gamma_1, \alpha \rightarrow \beta \Rightarrow \boldsymbol{\sigma'} \rightarrow \boldsymbol{\sigma''}}$$

Via an application of the left implication rule, we also get:

$$\rightarrow \mathsf{L}\ \frac{\Gamma', \boldsymbol{\sigma'} \rightarrow \boldsymbol{\sigma''} \Rightarrow \boldsymbol{\sigma'} \qquad\qquad \Gamma', \boldsymbol{\sigma''} \Rightarrow \varphi}{\Gamma', \boldsymbol{\sigma'} \rightarrow \boldsymbol{\sigma''} \Rightarrow \varphi}$$

Therefore, the formula $\boldsymbol{\sigma'} \rightarrow \boldsymbol{\sigma''}$ satisfies 2. Moreover, it turns out to be in the common language of $\Gamma_1, \alpha \rightarrow \beta$ and Γ', φ. Hence, the formula $\boldsymbol{\sigma} := \boldsymbol{\sigma'} \rightarrow \boldsymbol{\sigma''}$ is indeed the desired interpolant.

Let us now focus on the case in which the principal formula happens to be in Γ', which is of the form $\Gamma_2, \alpha \rightarrow \beta$

$$\rightarrow \mathsf{L}\ \frac{\Gamma, \Gamma_2, \alpha \rightarrow \beta \Rightarrow \alpha \qquad\qquad \Gamma, \Gamma_2, \beta \Rightarrow \varphi}{\Gamma, \Gamma_2, \alpha \rightarrow \beta \Rightarrow \varphi}$$

By the induction hypothesis, we have the existence of two different interpolants, $\boldsymbol{\sigma'}$ and $\boldsymbol{\sigma''}$, such that $\Gamma \Rightarrow \boldsymbol{\sigma'}$, $\Gamma_2, \alpha \rightarrow \beta, \boldsymbol{\sigma'} \Rightarrow \alpha$, $\Gamma \Rightarrow \boldsymbol{\sigma''}$ and $\Gamma_2, \beta, \boldsymbol{\sigma''} \Rightarrow \varphi$ are derivable sequents and $\boldsymbol{\sigma'}$ is in the common language of Γ and $\Gamma_2, \alpha \rightarrow \beta$, while $\boldsymbol{\sigma''}$ is in the common language of Γ and Γ_2, β, φ. We first focus on the two sequents whose context is Γ. It is easy to see that from those, the sequent $\Gamma \Rightarrow \boldsymbol{\sigma'} \wedge \boldsymbol{\sigma''}$ is derivable. For the other two sequents, we first 'weaken' the contexts as follows

$$\mathsf{W}\ \frac{\Gamma_2, \alpha \rightarrow \beta, \boldsymbol{\sigma'} \Rightarrow \alpha}{\Gamma_2, \alpha \rightarrow \beta, \boldsymbol{\sigma'}, \boldsymbol{\sigma''} \Rightarrow \alpha}$$

and

$$\mathsf{W}\ \frac{\Gamma_2, \beta, \boldsymbol{\sigma''} \Rightarrow \alpha}{\Gamma_2, \beta, \boldsymbol{\sigma''}, \boldsymbol{\sigma'} \Rightarrow \varphi}$$

Now, via an application of $\rightarrow \mathsf{L}$, we get the sequent $\Gamma_2, \alpha \rightarrow \beta, \boldsymbol{\sigma'}, \boldsymbol{\sigma''} \Rightarrow \varphi$. From this, the left conjunction rule lets us derive $\Gamma_2, \alpha \rightarrow \beta, \boldsymbol{\sigma'} \wedge \boldsymbol{\sigma''} \Rightarrow \varphi$. Finally, we can see that the formula $\boldsymbol{\sigma'} \wedge \boldsymbol{\sigma''}$ is in the common language of Γ and $\Gamma_2, \alpha \rightarrow \beta, \varphi$. Therefore, $\boldsymbol{\sigma} := \boldsymbol{\sigma'} \wedge \boldsymbol{\sigma''}$ is an interpolant for the considered sequent.

N : The last cases which remain to be considered are the ones related to negation rules. Consider the trickiest case, i.e., the case in which the left formula introduced by means of N is an element of Γ and Γ is of the form $\Gamma_1, \neg\alpha$, as follows

$$\mathsf{N}\ \frac{\Gamma_1, \neg\alpha, \alpha, \Gamma' \Rightarrow \beta \qquad\qquad \Gamma_1, \neg\alpha, \Gamma', \beta \Rightarrow \alpha}{\Gamma_1, \neg\alpha, \Gamma' \Rightarrow \neg\beta}$$

The induction hypothesis ensures the existence of formulas $\boldsymbol{\sigma'}$ and $\boldsymbol{\sigma''}$ such that, the sequents $\Gamma_1, \neg\alpha, \alpha \Rightarrow \boldsymbol{\sigma'}$, $\Gamma', \boldsymbol{\sigma'} \Rightarrow \beta$, $\Gamma_1, \neg\alpha, \boldsymbol{\sigma''} \Rightarrow \alpha$ and $\Gamma', \beta \Rightarrow \boldsymbol{\sigma''}$ are derivable. Moreover, $\boldsymbol{\sigma'}$ is a formula in the common language of $\Gamma_1, \neg\alpha$ and Γ', β, and so is $\boldsymbol{\sigma''}$.

Claim: The formula

$$(\boldsymbol{\sigma}' \to \boldsymbol{\sigma}'') \to \big((\boldsymbol{\sigma}'' \to \boldsymbol{\sigma}') \wedge \neg\boldsymbol{\sigma}'\big)$$

is the desired interpolant.

First of all, note that such a formula is in the common language of $\Gamma_1, \neg\alpha$ and $\Gamma, \neg\beta$ as required. We still need to check that the sequents:

$$\Gamma_1, \neg\alpha \Rightarrow (\boldsymbol{\sigma}' \to \boldsymbol{\sigma}'') \to \big((\boldsymbol{\sigma}'' \to \boldsymbol{\sigma}') \wedge \neg\boldsymbol{\sigma}'\big)$$

and

$$\Gamma', (\boldsymbol{\sigma}' \to \boldsymbol{\sigma}'') \to \big((\boldsymbol{\sigma}'' \to \boldsymbol{\sigma}') \wedge \neg\boldsymbol{\sigma}'\big) \Rightarrow \neg\beta$$

are derivable in the system. Consider the sequents $\Gamma_1, \neg\alpha, \boldsymbol{\sigma}'' \Rightarrow \alpha$ and $\Gamma_1, \neg\alpha, \alpha \Rightarrow \boldsymbol{\sigma}'$ which we know are derivable by the induction hypothesis. Applying Cut with cutformula α, and immediately applying Contraction, we get the sequent: $\Gamma_1, \neg\alpha, \boldsymbol{\sigma}'' \Rightarrow \boldsymbol{\sigma}'$. Indeed, this result, by means of $\to \mathsf{R}$, gives us a derivation of $\Gamma_1, \neg\alpha \Rightarrow \boldsymbol{\sigma}'' \to \boldsymbol{\sigma}'$. By applying Weakening to that sequent we get $\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \boldsymbol{\sigma}'' \to \boldsymbol{\sigma}'$ $(*)$. Now, starting again from the first sequent we got by induction hypothesis, we apply Weakening to get: $\Gamma_1, \neg\alpha, \boldsymbol{\sigma}'', \boldsymbol{\sigma}' \Rightarrow \alpha$. Moreover, note that the sequent $\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'', \boldsymbol{\sigma}' \Rightarrow \boldsymbol{\sigma}'$ can clearly be checked to be derivable. We can now apply $\to \mathsf{L}$ and get the following inference:

$$\to \mathsf{L}\ \frac{\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'', \boldsymbol{\sigma}' \Rightarrow \boldsymbol{\sigma}' \qquad\qquad \Gamma_1, \neg\alpha, \boldsymbol{\sigma}'', \boldsymbol{\sigma}' \Rightarrow \alpha}{\Gamma_1, \neg\alpha, \boldsymbol{\sigma}', \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \alpha}$$

Considering now the obtained sequent, and the sequent we get by a different application of Weakening to $\Gamma_1, \neg\alpha, \alpha \Rightarrow \boldsymbol{\sigma}'$, which looks like $\Gamma_1, \neg\alpha, \alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \boldsymbol{\sigma}'$, we can apply N as follows:

$$\mathsf{N}\ \frac{\Gamma_1, \neg\alpha, \alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \boldsymbol{\sigma}' \qquad\qquad \Gamma_1, \neg\alpha, \boldsymbol{\sigma}', \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \alpha}{\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \neg\boldsymbol{\sigma}'}$$

At this point, we can use the sequent we just obtained as a premise, together with the sequent denoted by $(*)$, ad apply $\wedge\mathsf{R}$ and $\to \mathsf{R}$ as follows:

$$\to \mathsf{R}\ \frac{\wedge\mathsf{R}\ \dfrac{\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \boldsymbol{\sigma}'' \to \boldsymbol{\sigma}' \qquad\qquad \Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \neg\boldsymbol{\sigma}'}{\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \to \boldsymbol{\sigma}'' \Rightarrow \big((\boldsymbol{\sigma}'' \to \boldsymbol{\sigma}') \wedge \neg\boldsymbol{\sigma}'\big)}}{\Gamma_1, \neg\alpha \Rightarrow (\boldsymbol{\sigma}' \to \boldsymbol{\sigma}'') \to \big((\boldsymbol{\sigma}'' \to \boldsymbol{\sigma}') \wedge \neg\boldsymbol{\sigma}'\big)}$$

As we can see, we just got a derivation of one of the two desired sequents. We still need to check that indeed also the other sequent is derivable. We start in the same way as before, by applying Cut, this time with β as the cutformula, to the sequents $\Gamma', \boldsymbol{\sigma}' \Rightarrow \beta$ and $\Gamma', \beta \Rightarrow \boldsymbol{\sigma}''$. The resulting sequent has the following form: $\Gamma', \boldsymbol{\sigma}' \Rightarrow \boldsymbol{\sigma}''$, and by means of the right implication rule, we can get a derivation of $\Gamma' \Rightarrow \boldsymbol{\sigma'} \to \boldsymbol{\sigma''}$. We want to use this sequent, weakened as

$$\Gamma', (\boldsymbol{\sigma'} \to \boldsymbol{\sigma''}) \to \big((\boldsymbol{\sigma''} \to \boldsymbol{\sigma'}) \land \neg\boldsymbol{\sigma'}\big) \Rightarrow \boldsymbol{\sigma'} \to \boldsymbol{\sigma''},$$

as a premise for the last inference rule. Therefore, we need to get a derivation of the following sequent:

$$\Gamma', \big((\boldsymbol{\sigma''} \to \boldsymbol{\sigma'}) \land \neg\boldsymbol{\sigma'}\big) \Rightarrow \neg\beta,$$

to be used as the other premise. Recall that, by the induction hypothesis, we have got derivations of $\Gamma', \boldsymbol{\sigma'} \Rightarrow \beta$ and $\Gamma', \beta \Rightarrow \boldsymbol{\sigma''}$. By Weakening, we can derive: $\Gamma', \boldsymbol{\sigma''} \to \boldsymbol{\sigma'}, \neg\boldsymbol{\sigma'}, \boldsymbol{\sigma'} \Rightarrow \beta$ and $\Gamma', \boldsymbol{\sigma''} \to \boldsymbol{\sigma'}, \neg\boldsymbol{\sigma'}, \beta \Rightarrow \boldsymbol{\sigma''}$. Moreover, as already seen before, the sequent $\Gamma', \neg\boldsymbol{\sigma'}, \beta, \boldsymbol{\sigma'} \Rightarrow \boldsymbol{\sigma'}$ is derivable, because it is just a 'generalized' form of the axiom $\Delta, p \Rightarrow p$. That said, consider the following derivation:

$$\to\mathsf{L}\ \frac{\Gamma', \boldsymbol{\sigma''} \to \boldsymbol{\sigma'}, \neg\boldsymbol{\sigma'}, \beta \Rightarrow \boldsymbol{\sigma''} \qquad \Gamma', \neg\boldsymbol{\sigma'}, \beta, \boldsymbol{\sigma'} \Rightarrow \boldsymbol{\sigma'}}{\Gamma', \boldsymbol{\sigma'} \to \boldsymbol{\sigma''}, \neg\boldsymbol{\sigma'}, \beta \Rightarrow \boldsymbol{\sigma'}}$$

$$\mathsf{N}\ \frac{\Gamma', \boldsymbol{\sigma''} \to \boldsymbol{\sigma'}, \neg\boldsymbol{\sigma'}, \boldsymbol{\sigma'} \Rightarrow \beta \qquad \Gamma', \boldsymbol{\sigma'} \to \boldsymbol{\sigma''}, \neg\boldsymbol{\sigma'}, \beta \Rightarrow \boldsymbol{\sigma'}}{\land\mathsf{L}\ \dfrac{\Gamma', \boldsymbol{\sigma''} \to \boldsymbol{\sigma'}, \neg\boldsymbol{\sigma'} \Rightarrow \neg\beta}{\Gamma', \big((\boldsymbol{\sigma''} \to \boldsymbol{\sigma'}) \land \neg\boldsymbol{\sigma'}\big) \Rightarrow \neg\beta}}$$

Indeed, the conclusion of this derivation is the sequent we wanted. We can finally apply the last step of the derivation, which is a left implication rule, to get the desired final sequent. Considering as premises

$$\Gamma', (\boldsymbol{\sigma'} \to \boldsymbol{\sigma''}) \to \big((\boldsymbol{\sigma''} \to \boldsymbol{\sigma'}) \land \neg\boldsymbol{\sigma'}\big) \Rightarrow \boldsymbol{\sigma'} \to \boldsymbol{\sigma''}$$

$$\text{and } \Gamma', \big((\boldsymbol{\sigma''} \to \boldsymbol{\sigma'}) \land \neg\boldsymbol{\sigma'}\big) \Rightarrow \neg\beta,$$

which we have shown to be derivable sequents, we can conclude:

$$\Gamma', (\boldsymbol{\sigma'} \to \boldsymbol{\sigma''}) \to \big((\boldsymbol{\sigma''} \to \boldsymbol{\sigma'}) \land \neg\boldsymbol{\sigma'}\big) \Rightarrow \neg\beta.$$

The proof we have just given shows that

$$\boldsymbol{\sigma} := (\boldsymbol{\sigma'} \to \boldsymbol{\sigma''}) \to \big((\boldsymbol{\sigma''} \to \boldsymbol{\sigma'}) \land \neg\boldsymbol{\sigma'}\big)$$

works as interpolant for the considered sequent.

Let us conclude this part of the proof by considering the following case, which is the one in which the left formula introduced by N is an element of Γ':

$$\mathsf{N}\ \frac{\Gamma, \Gamma_2, \neg\alpha, \alpha \Rightarrow \beta \qquad \Gamma, \Gamma_2, \neg\alpha, \beta \Rightarrow \alpha}{\Gamma, \Gamma_2, \neg\alpha \Rightarrow \neg\beta}$$

By the induction hypothesis, we have that the sequents $\Gamma \Rightarrow \boldsymbol{\sigma'}$, $\Gamma_2, \neg\alpha, \alpha, \boldsymbol{\sigma'} \Rightarrow \beta$,

$\Gamma \Rightarrow \boldsymbol{\sigma}''$ and $\Gamma_2, \neg\alpha, \beta, \boldsymbol{\sigma}'' \Rightarrow \alpha$ are derivable, for some formula $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ such that $\boldsymbol{\sigma}'$ is in the common language of Γ and $\Gamma_2, \neg\alpha, \beta$, and so is $\boldsymbol{\sigma}''$. First of all, note that from the two sequents with context Γ it is possible to derive, by means of $\wedge\mathsf{R}$, the following sequent $\Gamma \Rightarrow \boldsymbol{\sigma}' \wedge \boldsymbol{\sigma}''$. On the other hand, similarly to what we did in the previous induction case, we apply Weakening to the other two sequents to get $\Gamma_2, \neg\alpha, \alpha, \boldsymbol{\sigma}', \boldsymbol{\sigma}'' \Rightarrow \beta$ and $\Gamma_2, \neg\alpha, \beta, \boldsymbol{\sigma}'', \boldsymbol{\sigma}' \Rightarrow \alpha$. At this point, the rule N with these sequents as premises gives us the sequent $\Gamma_2, \neg\alpha, \boldsymbol{\sigma}', \boldsymbol{\sigma}'' \Rightarrow \neg\beta$ as a conclusion. Via an application of $\wedge\mathsf{L}$, we can see the sequent $\Gamma_2, \neg\alpha, \boldsymbol{\sigma}' \wedge \boldsymbol{\sigma}'' \Rightarrow \neg\beta$ to be derivable. In addition, $\boldsymbol{\sigma}' \wedge \boldsymbol{\sigma}''$ is a formula in the common language of Γ and $\Gamma_2, \neg\alpha, \neg\beta$. Therefore, we can conclude that $\boldsymbol{\sigma} := \boldsymbol{\sigma}' \wedge \boldsymbol{\sigma}''$ is an interpolant for the considered sequent.

NeF: Given that NeF is a non-right rule, we have to deal with the two options of the left-principal formula being introduced in Γ or in Γ'. Consider the case in which the left formula introduced by means of NeF is an element of Γ, as follows:

$$\mathsf{NeF}\ \frac{\Gamma_1, \neg\alpha, \Gamma' \Rightarrow \alpha}{\Gamma_1, \neg\alpha, \Gamma' \Rightarrow \neg\beta}$$

for some arbitrary β. Observe that Γ is of the form $\Gamma_1, \neg\alpha$. The induction hypothesis gives us the existence of a formula $\boldsymbol{\sigma}'$ and derivations in NeF of $\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \Rightarrow \alpha$ and $\Gamma' \Rightarrow \boldsymbol{\sigma}'$. Moreover, such a formula $\boldsymbol{\sigma}'$ is in the common language of $\Gamma_1, \neg\alpha$ and Γ'. Consider now the following derivations: first of all, by applying Weakening to the sequent whose context is Γ', we can obtain $\Gamma', \neg\boldsymbol{\sigma}' \Rightarrow \boldsymbol{\sigma}'$ and, from there, via an application of NeF we get $\Gamma', \neg\boldsymbol{\sigma}' \Rightarrow \neg\beta$. By Weakening applied again to the same sequent, we also can get a sequent of the form: $\Gamma', \boldsymbol{\sigma}' \to \neg\boldsymbol{\sigma}' \Rightarrow \boldsymbol{\sigma}'$. The last two sequents can be used as premises for an application of $\to\mathsf{L}$:

$$\to\mathsf{L}\ \frac{\Gamma', \boldsymbol{\sigma}' \to \neg\boldsymbol{\sigma}' \Rightarrow \boldsymbol{\sigma}' \qquad\qquad \Gamma', \neg\boldsymbol{\sigma}' \Rightarrow \neg\beta}{\Gamma', \boldsymbol{\sigma}' \to \neg\boldsymbol{\sigma}' \Rightarrow \neg\beta}$$

On the other hand, by applying NeF to the sequent $\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \Rightarrow \alpha$, we can get a conclusion of the form $\Gamma_1, \neg\alpha, \boldsymbol{\sigma}' \Rightarrow \neg\boldsymbol{\sigma}'$. The right rule for implication gives us a derivation in NeF of $\Gamma_1, \neg\alpha \Rightarrow \boldsymbol{\sigma}' \to \neg\boldsymbol{\sigma}'$. Clearly, the formula $\boldsymbol{\sigma}' \to \neg\boldsymbol{\sigma}'$ is in the common language of $\Gamma_1, \neg\alpha$ and $\Gamma', \neg\beta$. Hence, the desired interpolant is $\boldsymbol{\sigma} := \boldsymbol{\sigma}' \to \neg\boldsymbol{\sigma}'$.

Now, the second and last case we have to take care of here, is the one in which the last inference step is given by the same rule, but this time the left-principal formula has been introduced in Γ', which is $\Gamma_2, \neg\alpha$ in the following way:

$$\mathsf{NeF}\ \frac{\Gamma, \Gamma_2, \neg\alpha \Rightarrow \alpha}{\Gamma, \Gamma_2, \neg\alpha \Rightarrow \neg\beta}$$

for some arbitrary β. By the induction hypothesis, we know that there is a formula $\boldsymbol{\sigma}'$ in the common language of Γ and $\Gamma_2, \neg\alpha$, such that the sequents $\Gamma \Rightarrow \boldsymbol{\sigma}'$ and $\Gamma_2, \neg\alpha, \boldsymbol{\sigma}' \Rightarrow \alpha$ are derivable. From these, the sequent $\Gamma_2, \neg\alpha, \boldsymbol{\sigma}' \Rightarrow \neg\beta$ is provable too (by NeF). Therefore, $\boldsymbol{\sigma} := \boldsymbol{\sigma}'$ works as an interpolant.

CoPC: To extend the considered proof to the contraposition system, it is enough to slightly modify the proof already done for N, substituting the step concerning the rule N, with the one related to the newly added rule CoPC. Given that such a rule is non-right, we have to deal with the two options of the principal formula introduced to the left being either an element of Γ or of Γ'. Consider the case in which the left formula has been introduced by means of CoPC and it is an element of Γ,

$$\mathsf{CoPC}\ \frac{\Gamma_1, \neg\beta, \Gamma', \alpha \Rightarrow \beta}{\Gamma_1, \neg\beta, \Gamma' \Rightarrow \neg\alpha}$$

The multiset Γ is of the form $\Gamma_1, \neg\beta$. The induction hypothesis gives us the existence of a formula $\boldsymbol{\sigma'}$ and derivations in CoPC of $\Gamma_1, \neg\beta, \boldsymbol{\sigma'} \Rightarrow \beta$ and $\Gamma', \alpha \Rightarrow \boldsymbol{\sigma'}$. Moreover, such a formula $\boldsymbol{\sigma'}$ is in the common language of $\Gamma_1, \neg\beta$ and Γ', α. First of all, we can apply CoPC on $\Gamma_1, \neg\beta, \boldsymbol{\sigma'} \Rightarrow \beta$, to get $\Gamma_1, \neg\beta \Rightarrow \neg\boldsymbol{\sigma'}$. Moreover, after weakening the sequent $\Gamma', \alpha \Rightarrow \boldsymbol{\sigma'}$ to get $\Gamma', \alpha, \neg\boldsymbol{\sigma'} \Rightarrow \boldsymbol{\sigma'}$, we can apply CoPC again and obtain the sequent $\Gamma', \neg\boldsymbol{\sigma'} \Rightarrow \neg\alpha$. The desired interpolant is indeed $\boldsymbol{\sigma} := \neg\boldsymbol{\sigma'}$. The last case we have to take care of, is the one in which the last inference step is represented by the same rule, but the principal formula on the left has been introduced in $\Gamma' = \Gamma_2, \neg\beta$, as follows

$$\mathsf{CoPC}\ \frac{\Gamma, \Gamma_2, \neg\beta, \alpha \Rightarrow \beta}{\Gamma, \Gamma_2, \neg\beta \Rightarrow \neg\alpha}$$

By the induction hypothesis, we know that there is a formula $\boldsymbol{\sigma'}$ in the common language of Γ and $\Gamma_2, \neg\beta, \alpha$, such that the sequents $\Gamma \Rightarrow \boldsymbol{\sigma'}$ and $\Gamma_2, \neg\beta, \alpha, \boldsymbol{\sigma'} \Rightarrow \beta$ are derivable in CoPC. From these, the sequent $\Gamma_2, \neg\beta, \boldsymbol{\sigma'} \Rightarrow \neg\alpha$ is provable too. Therefore, $\boldsymbol{\sigma} := \boldsymbol{\sigma'}$ works as an interpolant.

The remaining steps of the proof concern the right inference rules. Indeed, we need to prove the statement assuming that the last formula introduced in the sequent $\Gamma, \Gamma' \Rightarrow \varphi$ is exactly φ.

$\wedge\mathsf{R}$: Assume that the last inference in the derivation of the considered sequent looks as follows:

$$\wedge\mathsf{R}\ \frac{\Gamma, \Gamma' \Rightarrow \alpha \qquad \Gamma, \Gamma' \Rightarrow \beta}{\Gamma, \Gamma' \Rightarrow \alpha \wedge \beta}$$

The induction hypothesis ensures the existence of two formulas, $\boldsymbol{\sigma'}$ and $\boldsymbol{\sigma''}$ such that the sequents $\Gamma \Rightarrow \boldsymbol{\sigma'}$, $\Gamma', \boldsymbol{\sigma'} \Rightarrow \alpha$, $\Gamma \Rightarrow \boldsymbol{\sigma''}$ and $\Gamma', \boldsymbol{\sigma''} \Rightarrow \beta$ are provable. Moreover, $\boldsymbol{\sigma'}$ is in the common language of Γ and Γ', α, while $\boldsymbol{\sigma''}$ is in the common language of Γ and Γ', β. It is easy to see that the sequent $\Gamma \Rightarrow \boldsymbol{\sigma'} \wedge \boldsymbol{\sigma''}$ is derivable too. Consider now the sequents whose context contains Γ'. By applying Weakening to both of them, we get $\Gamma', \boldsymbol{\sigma'}, \boldsymbol{\sigma''} \Rightarrow \alpha$ and $\Gamma', \boldsymbol{\sigma''}, \boldsymbol{\sigma'} \Rightarrow \beta$, which are valid premises for an application of $\wedge\mathsf{R}$, whose conclusion is $\Gamma', \boldsymbol{\sigma''}, \boldsymbol{\sigma'} \Rightarrow \alpha \wedge \beta$. From this sequent, via $\wedge\mathsf{L}$, we obtain a derivation of $\Gamma', \boldsymbol{\sigma'} \wedge \boldsymbol{\sigma''} \Rightarrow \alpha \wedge \beta$. Given that the formula $\boldsymbol{\sigma'} \wedge \boldsymbol{\sigma''}$ is in the common language of Γ and $\Gamma', \alpha \wedge \beta$, we can conclude that $\boldsymbol{\sigma} := \boldsymbol{\sigma'} \wedge \boldsymbol{\sigma''}$ is the desired formula.

$\vee$R : Assume without loss of generality, that the last inference in the derivation of the considered sequent looks as follows:

$$\vee\mathsf{R}\ \frac{\Gamma, \Gamma' \Rightarrow \alpha}{\Gamma, \Gamma' \Rightarrow \alpha \vee \beta}$$

The induction hypothesis ensures the existence of a formula, $\boldsymbol{\sigma'}$ such that the sequents $\Gamma \Rightarrow \boldsymbol{\sigma'}$, $\Gamma', \boldsymbol{\sigma'} \Rightarrow \alpha$ are provable. Moreover, the variables of $\boldsymbol{\sigma'}$ are in the common language of Γ and Γ', α. From this, it follows that $\boldsymbol{\sigma'}$ is also in the common language of Γ and $\Gamma', \alpha \vee \beta$. It is immediate to see that $\Gamma', \boldsymbol{\sigma'} \Rightarrow \alpha \vee \beta$ is derivable. Therefore, the required interpolant is $\boldsymbol{\sigma} := \boldsymbol{\sigma'}$.

$\rightarrow$ R : The last rule we need to check to conclude the induction step (and the whole proof) is the implication right rule. Consider

$$\rightarrow\mathsf{R}\ \frac{\Gamma, \Gamma', \alpha \Rightarrow \beta}{\Gamma, \Gamma' \Rightarrow \alpha \rightarrow \beta}$$

The induction hypothesis gives us derivations of $\Gamma \Rightarrow \boldsymbol{\sigma'}$ and $\Gamma', \alpha, \boldsymbol{\sigma'} \Rightarrow \beta$, for some formula $\boldsymbol{\sigma'}$ in the common language of Γ and Γ', α, β. Similarly to the previous case, the sequent $\Gamma', \boldsymbol{\sigma'} \Rightarrow \alpha \rightarrow \beta$ turns out to be provable, by means of $\rightarrow$ R. Therefore, given that $\boldsymbol{\sigma'}$ is in the common language of Γ and $\Gamma', \alpha \rightarrow \beta$ by assumption, we can conclude $\boldsymbol{\sigma} := \boldsymbol{\sigma'}$.

An: We only need to check the step of induction in which the sequent $\Gamma, \Gamma' \Rightarrow \varphi$ is obtained by means of An. The rule An being a right rule, there is exactly one case to be considered, i.e.,

$$\mathsf{An}\ \frac{\Gamma, \Gamma', \alpha \Rightarrow \neg\alpha}{\Gamma, \Gamma' \Rightarrow \neg\alpha}$$

The induction hypothesis ensures the existence of a derivation of the sequents $\Gamma \Rightarrow \boldsymbol{\sigma'}$ and $\Gamma', \alpha, \boldsymbol{\sigma'} \Rightarrow \neg\alpha$ for some formula $\boldsymbol{\sigma'}$ in the common language of Γ and $\Gamma', \neg\alpha$. An application of An on the sequent $\Gamma', \alpha, \boldsymbol{\sigma'} \Rightarrow \neg\alpha$, gives us exactly $\Gamma', \boldsymbol{\sigma'} \Rightarrow \neg\alpha$ as desired. Hence, the formula $\boldsymbol{\sigma} := \boldsymbol{\sigma'}$ is an interpolant for the considered sequent. □

6.3 Translating MPC into CoPC

We conclude this chapter and the whole proof-theoretic section of this work, exploiting the sequent calculi we have presented in order to translate MPC into contraposition logic. Recall that the *negative translation* from classical logic into intuitionistic logic ensures intuitionistic logic to have at least the same *expressive power* and *consistency strength* of classical logic [43]. A similar thing happens with Gödel's translation of IPC into the modal logic S4 [12]. Here, we establish a similar translation and we use sequent calculi to give a proof of the fact that such a translation is indeed a *sound* and *truthful* one.

First, we define the translation $\varphi^\sim$ for every MPC-formula φ, inductively on the structure of the formula.

Definition 39. *Let φ be an arbitrary formula in MPC. We define a translation $\varphi^\sim$ by recursion over the complexity of φ, as follows:*

- $p^\sim := p$,
- $\top^\sim := \top$,
- $(\varphi \circ \psi)^\sim := \varphi^\sim \circ \psi^\sim$, *where* $\circ \in \{\wedge, \vee, \rightarrow\}$,
- $(\neg\varphi)^\sim := \varphi^\sim \rightarrow \neg\varphi^\sim$.

The reader should note that, in the given definition, the connectives on the left-hand side of the ':=' symbol, are to be considered in the language of MPC, while the connectives present on the right-hand side are to be read as CoPC connectives. For the sake of simplicity, we allow this abuse of notation. In what follows, we denote with $\Gamma^\sim$ the set of formulas $\{\varphi^\sim \mid \varphi \in \Gamma\}$, for every set Γ of MPC-formulas.

We need here to prove a preliminary result. In fact, given that contraposition logic is a subsystem of minimal propositional logic, a proof of the following lemma is sufficient to ensure the translation to be truthful: it states that MPC proves every formula to be equivalent to its translation. We give a proof of this fact by means of the sequent calculus system $\mathbf{G3m}_\neg$.

Lemma 6.3.1. *For every formula φ,*

$$\mathsf{MPC} \vdash \varphi \leftrightarrow \varphi^\sim.$$

Proof. The proof goes by induction on the structure of φ. We skip the base case, since every atom is easily proved to be equivalent to its translation just by means of the axioms. We unfold the proof for one direction of the implication case, and later, for the negation case. Let $\varphi \rightarrow \psi$ be the considered formula in the language of MPC. We want:

$$(\varphi \rightarrow \psi) \Rightarrow (\varphi \rightarrow \psi)^\sim.$$

The derivation goes as follows

$$\dfrac{\dfrac{\varphi \rightarrow \psi, \varphi^\sim \Rightarrow \varphi \qquad \varphi^\sim, \psi \Rightarrow \psi^\sim}{\varphi \rightarrow \psi, \varphi^\sim \Rightarrow \psi^\sim}\,{\rightarrow}\mathsf{L}}{\varphi \rightarrow \psi \Rightarrow \varphi^\sim \rightarrow \psi^\sim}\,{\rightarrow}\mathsf{R}$$

Indeed, the leaves of the given derivation are derivable in MPC by induction hypothesis and Weakening, and hence we are done. We conclude the proof by showing the result for $\neg\varphi$. Observe that, as anticipated in the introduction to this chapter, this step of the induction is made easier by the fact that An is one of the rule of our system. In fact, in order to get a proof of

$$\varphi^\sim \rightarrow \neg\varphi^\sim \Rightarrow \varphi,$$

we can consider a derivation built in the following way

$$\dfrac{\dfrac{\varphi^\sim \to \neg\varphi^\sim, \varphi \Rightarrow \varphi^\sim \qquad \varphi, \neg\varphi^\sim \Rightarrow \neg\varphi}{\varphi^\sim \to \neg\varphi^\sim, \varphi \Rightarrow \neg\varphi}\,{\to}\mathsf{L}}{\varphi^\sim \to \neg\varphi^\sim \Rightarrow \neg\varphi}\,\mathsf{An}$$

Finally, consider the sequent $\neg\varphi \Rightarrow (\neg\varphi)^\sim$, which is

$$\neg\varphi \Rightarrow \varphi^\sim \to \neg\varphi^\sim.$$

By applying backward the right implication rule and, successively CoPC, we obtain a derivation of the following form

$$\dfrac{\dfrac{\neg\varphi, \varphi^\sim, \varphi^\sim \Rightarrow \varphi}{\neg\varphi, \varphi^\sim \Rightarrow \neg\varphi^\sim}\,\mathsf{CoPC}}{\neg\varphi \Rightarrow \varphi^\sim \to \neg\varphi^\sim}\,{\to}\mathsf{R}$$

The proofs for the remaining cases (i.e., conjunction, disjunction) are very similar to the one for implication. □

Let us now go through the proof of the main theorem of the section.

Theorem 6.3.2 (Soundness and Truthfulness)**.** *The defined translation $\sim$ is sound and truthful, i.e.,*

$$\mathsf{MPC} \vdash \Gamma \Rightarrow \varphi \textit{ if and only if } \mathsf{CoPC} \vdash \Gamma^\sim \Rightarrow \varphi^\sim,$$

for every finite multiset Γ and every formula φ.

Proof. The proof goes by induction on the depth n of the derivation. Let us focus on the left-to-right direction. Here, we assume the sequent $\Gamma \Rightarrow \varphi$ to be derivable in MPC, and we aim to obtain a derivation of its 'translation' in CoPC.

Base case: Consider the case in which the sequent $\Gamma \Rightarrow \varphi$ is an axiom. There are two ways in which this can happen: $\varphi = p$ and $p \in \Gamma$, or $\varphi = \top$. In the first case, the sequent $\Gamma_1^\sim, p^\sim \Rightarrow p^\sim$ coincides with $\Delta, p \Rightarrow p$, where $\Delta = \Gamma_1^\sim$ is a set of CoPC-formulas. This is an axiom in CoPC, and hence, it is derivable. By means of the same reasoning, given that $\top^\sim = \top$, we get that $\Gamma^\sim \Rightarrow \top^\sim$ is a CoPC axiom.

Now, we need to consider the induction step, i.e., all the cases in which the sequent $\Gamma \Rightarrow \varphi$ is not an axiom and it has been obtained as a conclusion of some inference rule.

$\wedge$L : Suppose the sequent $\Gamma \Rightarrow \varphi$ to be obtained via an application of $\wedge$L, as follows:

$$\dfrac{\Gamma_1, \alpha, \beta \Rightarrow \varphi}{\Gamma_1, \alpha \wedge \beta \Rightarrow \varphi}\,\wedge\mathsf{L}$$

We know the sequent $\Gamma_1^\sim, \alpha^\sim, \beta^\sim \Rightarrow \varphi^\sim$ to be derivable in CoPC by induction hypothesis. Therefore, also $\Gamma_1^\sim, \alpha^\sim \wedge \beta^\sim \Rightarrow \varphi^\sim$ happens to be provable in CoPC.

Given that $(\alpha \wedge \beta)^\sim = \alpha^\sim \wedge \beta^\sim$ and that $\Gamma^\sim = \Gamma_1^\sim, (\alpha \wedge \beta)^\sim$, we can conclude that $\Gamma^\sim \Rightarrow \varphi^\sim$ is provable in CoPC.

$\vee\mathsf{L}$: Assume now that the considered sequent comes from an application of the disjunction left rule in MPC:

$$\vee\mathsf{L}\ \frac{\Gamma_1, \alpha \Rightarrow \varphi \qquad\qquad \Gamma_1, \beta \Rightarrow \varphi}{\Gamma_1, \alpha \vee \beta \Rightarrow \varphi}$$

The induction hypothesis ensures there are derivations in CoPC whose conclusions are, respectively, $\Gamma_1^\sim, \alpha^\sim \Rightarrow \varphi^\sim$ and $\Gamma_1^\sim, \beta^\sim \Rightarrow \varphi^\sim$. An application of $\vee\mathsf{L}$ in CoPC leads to a proof of the sequent $\Gamma_1^\sim, \alpha^\sim \vee \beta^\sim \Rightarrow \varphi^\sim$. Indeed, such a sequent is equivalent to $\Gamma_1^\sim, (\alpha \vee \beta)^\sim \Rightarrow \varphi^\sim$ and hence, we get a proof of $\Gamma^\sim \Rightarrow \varphi^\sim$ in CoPC.

$\rightarrow \mathsf{L}$: Suppose that the sequent comes from an application of the implication left rule in MPC:

$$\rightarrow \mathsf{L}\ \frac{\Gamma_1, \alpha \rightarrow \beta \Rightarrow \alpha \qquad\qquad \Gamma_1, \beta \Rightarrow \varphi}{\Gamma_1, \alpha \rightarrow \beta \Rightarrow \varphi}$$

The induction hypothesis ensures there are derivations in CoPC whose conclusions are, respectively, $\Gamma_1^\sim, (\alpha \rightarrow \beta)^\sim \Rightarrow \alpha^\sim$ and $\Gamma_1^\sim, \beta^\sim \Rightarrow \varphi^\sim$. Indeed, this is equivalent to have derivations of $\Gamma_1^\sim, \alpha^\sim \rightarrow \beta^\sim \Rightarrow \alpha^\sim$ and $\Gamma_1^\sim, \beta^\sim \Rightarrow \varphi^\sim$ and hence, by means of $\rightarrow \mathsf{L}$, we can get a proof of the sequent $\Gamma_1^\sim, \alpha^\sim \rightarrow \beta^\sim \Rightarrow \varphi^\sim$ in CoPC. This sequent is exactly $\Gamma_1^\sim, (\alpha \rightarrow \beta)^\sim \Rightarrow \varphi^\sim$ and hence, we obtained a derivation of $\Gamma^\sim \Rightarrow \varphi^\sim$ in CoPC.

$\wedge\mathsf{R}$: Assume here that $\Gamma \Rightarrow \varphi$ has been obtained from the conjuction right rule in MPC:

$$\wedge\mathsf{R}\ \frac{\Gamma \Rightarrow \alpha \qquad\qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

By the induction hypothesis, we also have derivations in CoPC whose conclusions are, respectively, $\Gamma^\sim \Rightarrow \alpha^\sim$ and $\Gamma^\sim \Rightarrow \beta^\sim$. An application of $\wedge\mathsf{R}$ in CoPC leads to a proof of the sequent $\Gamma^\sim \Rightarrow \alpha^\sim \wedge \beta^\sim$, i.e., $\Gamma^\sim \Rightarrow (\alpha \wedge \beta)^\sim$. Therefore, we have a deduction of $\Gamma^\sim \Rightarrow \varphi^\sim$ in CoPC.

$\vee\mathsf{R}$: Suppose that the sequent $\Gamma \Rightarrow \varphi$ has been obtained by means of an application of $\vee\mathsf{R}$, without loss of generality, as follows:

$$\vee\mathsf{R}\ \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}$$

We know that the sequent $\Gamma^\sim \Rightarrow \alpha^\sim$ is derivable in CoPC, by the induction hypothesis. Therefore, also $\Gamma^\sim \Rightarrow \alpha^\sim \vee \beta^\sim$ is provable in CoPC. Given that $(\alpha \vee \beta)^\sim = \alpha^\sim \vee \beta^\sim$, we can conclude that there is a derivation in CoPC whose conclusion is $\Gamma^\sim \Rightarrow \varphi^\sim$.

$\to \mathsf{R}$: Assume the sequent $\Gamma \Rightarrow \varphi$ to be obtained via an application of $\to \mathsf{R}$ in the following way:

$$\to \mathsf{R}\ \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}$$

We know by the induction hypothesis that the sequent $\Gamma^\sim, \alpha^\sim \Rightarrow \beta^\sim$ is derivable in CoPC. Therefore, also $\Gamma^\sim \Rightarrow \alpha^\sim \to \beta^\sim$ is provable in CoPC. Given that $(\alpha \to \beta)^\sim = \alpha^\sim \to \beta^\sim$, we can conclude that $\Gamma^\sim \Rightarrow \varphi^\sim$ is derivable in CoPC.

CoPC : Let us assume here that the inference rule with conclusion $\Gamma \Rightarrow \varphi$ is CoPC. Then, the last step of the derivation looks as follows:

$$\mathsf{CoPC}\ \frac{\Gamma_1, \neg\alpha, \beta \Rightarrow \alpha}{\Gamma_1, \neg\alpha \Rightarrow \neg\beta}$$

The induction hypothesis leads us to have a derivation in CoPC of the sequent $\Gamma_1^\sim, (\neg\alpha)^\sim, \beta^\sim \Rightarrow \alpha^\sim$. By how the translation was defined, this means that we have a CoPC derivation of $\Gamma_1^\sim, (\alpha^\sim \to \neg\alpha^\sim), \beta^\sim \Rightarrow \alpha^\sim$ $(*)$. First of all, we apply Weakening to get $\Gamma_1^\sim, (\alpha^\sim \to \neg\alpha^\sim), \neg\alpha^\sim, \beta^\sim \Rightarrow \alpha^\sim$. By applying an inference of CoPC to this sequent, we obtain the sequent $\Gamma_1^\sim, (\alpha^\sim \to \neg\alpha^\sim), \neg\alpha^\sim \Rightarrow \neg\beta^\sim$ $(**)$. At this point, consider the following application of $\to \mathsf{L}$ in CoPC: after having weakened both $(*)$ and $(**)$ to obtain, respectively, $\Gamma_1^\sim, (\alpha^\sim \to \neg\alpha^\sim), (\alpha^\sim \to \neg\alpha^\sim), \beta^\sim \Rightarrow \alpha^\sim$ and $\Gamma_1^\sim, (\alpha^\sim \to \neg\alpha^\sim), \neg\alpha^\sim, \beta^\sim \Rightarrow \neg\beta^\sim$, the left implication rule gives us the sequent $\Gamma_1^\sim, (\alpha^\sim \to \neg\alpha^\sim), (\alpha^\sim \to \neg\alpha^\sim), \beta^\sim \Rightarrow \neg\beta^\sim$ as the conclusion. An application of Contraction, followed by an inference of $\to \mathsf{R}$, leads us to get a derivation in CoPC of

$$\Gamma_1^\sim, (\alpha^\sim \to \neg\alpha^\sim) \Rightarrow (\beta^\sim \to \neg\beta^\sim),$$

which is exactly $\Gamma_1^\sim, (\neg\alpha)^\sim \Rightarrow (\neg\beta)^\sim$, as desired.

An : Consider now the case in which the sequent $\Gamma \Rightarrow \varphi$ has been obtained via an application of An, as follows:

$$\mathsf{An}\ \frac{\Gamma, \alpha \Rightarrow \neg\alpha}{\Gamma \Rightarrow \neg\alpha}$$

By the induction hypothesis, we have the existence of a CoPC derivation for the sequent $\Gamma^\sim, \alpha^\sim \Rightarrow (\neg\alpha)^\sim$, i.e., $\Gamma^\sim, \alpha^\sim \Rightarrow (\alpha^\sim \to \neg\alpha^\sim)$. We use here invertibility of $\to \mathsf{R}$, to get: $\Gamma^\sim, \alpha^\sim, \alpha^\sim \Rightarrow \neg\alpha^\sim$. By subsequent applications of Contraction first, and $\to \mathsf{R}$ later, we get $\Gamma^\sim \Rightarrow (\alpha^\sim \to \neg\alpha^\sim)$ to be derivable in CoPC. This gives us a derivation of $\Gamma^\sim \Rightarrow (\neg\alpha)^\sim$, which is exactly what we wanted.

To conclude the proof we need to take care of the right-to-left direction. Here, we use Lemma 6.3.1 and the fact that Cut is admissible in MPC. In fact, $\mathsf{CoPC} \vdash \Gamma^\sim \Rightarrow \varphi^\sim$ implies $\mathsf{MPC} \vdash \Gamma^\sim \Rightarrow \varphi^\sim$. Moreover, by Lemma 6.3.1, we also have

$$\mathsf{MPC} \vdash \Gamma \Rightarrow \Gamma^\sim \text{ and } \mathsf{MPC} \vdash \varphi^\sim \Rightarrow \varphi.$$

The multiset $\Gamma^{\sim}$ is finite, so we can apply Cut on every formula $\psi^{\sim}$, for $\psi \in \Gamma$, to get a derivation in MPC of the sequent $\Gamma \Rightarrow \varphi^{\sim}$. Again, using Cut with cutformula $\varphi^{\sim}$, we can conclude

$$\mathsf{MPC} \vdash \Gamma \Rightarrow \varphi$$

which indeed is the desired derivation. □

The result we have just proved turns out to be very important if one considers the chain of logics between the classical propositional system and contraposition logic, i.e.,

$$\mathsf{CPC} - \mathsf{IPC} - \mathsf{MPC} - \mathsf{CoPC}.$$

The translation from classical propositional logic to intuitionistic logic is well-known. On the other hand though, a way of translating intuitionistic logic into minimal logic was missing. Dick de Jongh and Tim van der Molen have found, in a letter from Johansson to Heyting from 1935, that indeed such a translation exists. If one considers minimal logic as MPC_f, a translation, which we refer to as hj, can be defined on implication as follows:

$$(\varphi \to \psi)^{hj} := \varphi^{hj} \to (\psi^{hj} \vee f).$$

Leaving all the other connectives unchanged, this is a translation of intuitionistic logic into minimal logic. The proof makes explicit use of the fact that in minimal logic the disjunction property holds. The argument can indeed be found originally in the letter from Johansson to Heyting.

Remark. The considered translation from minimal propositional logic into contraposition logic is *not* polynomial. Therefore, unlike the case of MPC, we cannot conclude anything about the computational complexity of the satisfiability problem for CoPC based on the proposed translation.

Before going into concluding this chapter, we want the reader to note the following. We have emphasized within the current chapter how the proposed translation has the intuitive scope of 'adding' absorption of negation to weak contraposition, in order to get Johansson's logic. This idea clearly makes use of Proposition 1.2.2. As a matter of fact, as proved in Proposition 1.2.5, the system axiomatized by weak contraposition and An is equivalent to the system axiomatized by N and An. Therefore, the option of extending the above translation to a translation of MPC into N naturally arises. Although it is not developed here, we find this analysis interesting and worth being carried on.

The last two chapters have been useful to introduce sequent calculus systems and start working with and within them. Observe that, in general, the option of considering a multi-conclusion calculus instead of a single-conclusion one may turn out to be useful to analyze the linear extension of the logics we are dealing with [17].

The fact that we are working within non-terminating sequent calculi has impact on several possible results. For instance, given that we have provided a proof of Craig's Interpolation Theorem, an immediate further problem to be analyzed is the so-called *Uniform Interpolation Theorem*. As a matter of fact, a proof-theoretic argument for such a result requires a terminating sequent calculus.

Chapter 7

Conclusions

This thesis provided a first analysis and study of some logical subsystems of minimal propositional logic from different perspectives.

In the first place, we gave an overview of our starting point, i.e., Johansson's minimal logic. We introduced two different versions of it and we recalled a Kripke semantics for each of the two versions. This analysis allowed us to start focusing on the negation operator and to introduce the axioms and the systems that turned out to be the core of the thesis. We have emphasized the relation of those axioms with the minimal logic system. Later, we started a study of the three main subminimal systems of our interest (the basic system, contraposition logic and negative *ex falso* logic), presenting a Kripke semantics for each of them and defining the basic notions of p-morphism, generated subframe and disjoint union. The third chapter was devoted to go deeper into the study of the Kripke semantics for those systems. We proved that the considered logics satisfy indeed the finite model property, as well as the disjunction property. In both cases, we gave semantic proofs of the considered results. Moreover, we introduced the notion of slash relation in order to prove that a form of disjunction property under negated hypothesis holds as well.

The second main part of the thesis, although it is covered by only one chapter, is an introduction to the algebraic semantics for the considered systems. An algebraic completeness result with respect to the variety of N-algebras is proved. This result is extended, not only for the particular systems of interests but for the other extensions of the basic system as well. By defining a notion of descriptive frame, we were able to transfer the algebraic completeness result into a frame-based completeness result which has a more generalized form than the one proved in Chapter 2. Indeed, we developed a duality theory, with particular focus on a subclass of the class of descriptive frames previously defined.

The last part of the thesis is devoted to the development of a proof-theory for the main logical systems. We presented two equivalent kinds of sequent systems. The second ones, denoted as **G3**-systems, are proved to be cut-free complete proof systems and are used here to prove interesting results. In addition to a proof of Craig's Interpolation Theorem and the disjunction property, an example of cut-free proof in our favorite system, contraposition logic, is given to obtain the following result: any even number of negations is equivalent to double negations, while any odd number of negations implies one negation. Finally, we gave a result which tells us something about the (relative) expressive power of two of the relevant systems. In particular, we used the calculi to prove that minimal propositional logic can be translated into contraposition logic in a sound and truthful way.

This work represents a starting point for a much deeper analysis of the considered systems. In particular, the systems studied here represent just different examples of logical systems within the lattice of systems between N and minimal propositional logic. Such a lattice seems to be of interest in itself. Moreover, as suggested already at the end of Chapter 1, many other principles can legitimately be assumed as axioms and further studied. This approach could lead to more generalized results. For instance, we could consider the system introduced in Proposition 1.2.5 and study it from different perspectives. In particular, a sequent calculus for such a logic would be available already from the tools presented in Chapter 5 and Chapter 6.

One interesting question with respect to the algebraic semantics can be expressed as follows: in which cases do the considered axioms lead to a *unique* possible unary operator $\neg$? As a matter of fact, in the case of an N-algebra, the unary operator $\neg$ can be defined in many different ways, as long as it satisfies the given axiom. This direction of study could for instance give us more information about the relation between the structures of N-algebra and the Heyting algebras. Another intriguing option which could lead us to give a more uniform account of the considered logical systems consists of deepening the relation between the algebraic and the proof-theoretic counterparts of the logics.

From a more conceptual point of view, one could study the notion of *incompatibility* of two formulas. This notion can be seen as being strictly related to minimal propositional logic, since it can be formally written as $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$. One could ask herself whether there exists a weaker 'logic of incompatibility', and this question can be deepened by studying various 'axioms of negation'.

More generally, this work aimed to raise interest in the study of weak negations. Clearly, many different and intriguing problems still need to be questioned and, eventually, answered.

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