The quantum mind: Alternative ways of reasoning with uncertainty

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Abstract Human reasoning about and with uncertainty is often at odds with the principles of classical probability. Order effects, conjunction biases, and sure-thing inclinations suggest that an entirely different set of probability axioms could be developed and indeed may be needed to describe such habits. Recent work in diverse fields, including cognitive science, economics, and information theory, explores alternative approaches to decision theory. This work considers more expansive theories of reasoning with uncertainty while continuing to recognize the value of classical probability. In this paper, we discuss one such alternative approach, called quantum probability, and explore its applications within decision theory. Quantum probability is designed to formalize uncertainty as an ontological feature of the state of affairs, offering a mathematical model for entanglement, de/coherence, and interference, which are all concepts with unique onto-epistemological relevance for social theorists working in new and trans-materialisms. In this paper, we suggest that this work be considered part of the quantum turn in the social sciences and humanities. Our aim is to explore different models and formalizations of decision theory that attend to the situatedness of judgment. We suggest that the alternative models of reasoning explored in this article might be better suited to queries about entangled mathematical concepts and, thus, be helpful in rethinking both curriculum and learning theory.

1. Introduction

Reasoning with and about uncertainty is a complex affair. One might argue that all reasoning entails some measure of uncertainty, since choices and actions involve ambivalence, conflicted feelings, and lack of knowledge. Indeed it may be that everyday decisions, which appear fairly automatic and without deliberation, are ground in micro-perceptual calibrations of our sensory milieu, processed below the level of consciousness. If there is a kind of "probability in the wild" operating at various micro-scales, much of what we name as "reasoning" may in fact be essentially pre-individual (Hansen, 2015). The question is then whether uncertainty might be better considered in terms of a fundamental indeterminacy at work in the material world, like the random swerve proposed by Lucretius (c. 100-c. 55 BC) or quantum indeterminacy described by Niels Bohr. Philosophers of science such as Karen Barad (2007, 2012) have argued that this ontological perspective demands new ways of conceiving the mathematics of chance.

In this paper we explore quantum probability (QP) as a possible alternative formalisation of reasoning with uncertainty. QP was formulated by John Von Neumann in the 1920s as a way of describing quantum mechanics. Trueblood et al. (2011) argue that a quantum approach to human judgment is compelling because (a) judgment is a responsive action and not simply an accessing of stored information, (b) judgments disturb the learning environment and the distributed cognition across the given situation, (c) judgments impact future judgments, and thus order of judgments matters, and (d) human judgments do not follow the commutative rule of classical probability.

We explore the implications of this work for learning theory, suggesting that alternative formalizations of probability offer different insights into human reasoning. We turn to recent applications of quantum probability in psychology, exploring the ways that human judgment has been conceptualized as a kind of quantum behaviour. Researchers in diverse fields are investigating the implications of this alternative approach, including economics, information theory, cognitive science, and machine learning (see Pothos & Busemeyer 2013 for a full list of references). The idea that quantum mechanics might be a source for rethinking cognition is not new – the physicist Niels Bohr himself suggested that quantum theory might offer insight into the nature of cognition (Wang et al., 2013) as did Schrödinger in his now classic 1944 book *What is life?* We pursue this study of probability as part of our project in inclusive materialism (de Freitas & Sinclair, 2014), exploring the way that mathematical concepts are implicated in particular material and embodied practices.

Our interest in exploring the formalisms of quantum probability derives in part from our commitment to imagine the quantum at all scales. We want terms like entanglement, diffraction, and de/coherence to be more than mere metaphors, and we see a need to dig deeper into the mathematics of quantum ontology. Moreover, our work seeks to show how mathematics itself – at the level of concept, model, form – is implicated in the socio-material theories we use at other scales. For these reasons, we decided to find out what specifically was happening in mathematical quantum models. QP operates through a geometric approach, rather than a set-theoretic approach. The ‘objects’ of this geometry are vectors, which entails a vector algebra with different operators and different kinds of relationality, related to complex numbers. Most importantly for our purposes, QP offers logical and mathematical formulations of concepts such as entanglement, superposition, incompatibility and interference, which are all distinctive characteristics of quantum systems with its unique onto-epistemological associations and relevance for social theorists working in new and trans-materialisms.

2. The limits of classical probability

Hacking (1975) recounts how the mathematics of chance has always had a dual nature, being both epistemic and ontological, the first aligned with *theoretical* laws of probability and the second aligned with *empirical* experiments and the measurements of frequencies.[1] The emergence of quantum physics in the twentieth century further complicated this onto-epistemological mixture. Since quantum mechanics defies the

[1] This distinction is sometimes referred to as Bayesian versus frequentist. In this paper, we explore alternative kinds of probability (focusing on quantum models). There are many other types of probability as well and, as Gillies (2000) argues, they may each offer valid interpretations of the particular contexts from which they emerge.

causal logic of classical Newtonian science, and puts forward a new image of simultaneity and presence (of both here and not here), it breaks with classical probability models.

Classical probability (CP) is captured in the Kolmogorov axioms which stipulate that (1) probability is a non-negative real number, (2) the probability of the entire sample space is 1, and (3) the probability of the union of a countably infinite set of mutually exclusive events is equivalent to the sum of the probabilities of each of the events. Children are typically initiated into the logic of CP using examples that allow for frequency testing, so as to prove empirically that the model is accurate and true. Notably for our purposes, issues of total probability, order effects, and conditional conjunctions become pivotal learning moments in developing skills associated with reasoning about chance. These facets of CP – rules regarding the distribution of probabilities across outcomes, and the impact of order and conjunction on probability – are often a source of difficulty for children learning about probability (see, for example, Fischbein & Schwartz, 1997; Watson & Moritz (2002).

Unfortunately, or perhaps predictably, humans do not always reason according to the rules of CP (see Cosmides & Tooby (1996) on potential evolutionary reasons for this). Sutherland (2007) presents numerous examples of doctors and judges and juries violating these rules, whether it be ignoring the conditional base in a particular context, or revealing an "irrational" order bias when asked a series of questions about the likelihood of an outcome. For example, in the famous Linda problem, studied by Tversky and Kahneman (1983), participants are given this information: "Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations." Research participants are then asked "Is Linda more likely to be (a) a bank teller or (b) active in the feminist movement and a bank teller." In classical probability, the probability of the conjunction will always be less than the probability of either of the individual events: $P(A \cap B) \leq P(A)$. In other words, people should answer that it is more likely that she is a bank teller. And yet the study shows that 90% of respondents select (b). Other similar experiments related to medical judgments, predicting the results of sporting events and making risky choices show the same tendencies, revealing how these violations are widespread and persistent (Tversky & Kahneman, 1983).

It may be that we *want* people to reason according to classical probability, but a large body of evidence suggests that people do not (Kahneman et al. 1982; Tversky & Shafir 1992). Rather than simply dismiss this large body of evidence as though it pointed to human irrationality or incoherence, or offer some adhoc explanation regarding the language used in such questions, growing interest in alternative formalisations of probability has emerged. Various decision theories have attempted to address these habits of thought. Query theory, fuzzy trace theory and integration theory are all formalisations that alter the rules of CP in order to better capture the way that people reason about complex lived events. What is considered paradoxical from the perspective of CP, and what is seen as widespread violation of the laws, *may in fact be evidence of an alternative* *way of reasoning with uncertainty*: "if we are to understand the intuition behind human judgment in such situations, we have to look for an alternative probabilistic framework" (Pothos & Busemeyer, 2013, p. 257). Before unpacking some of the recent work in quantum decision theory, we first situate this work as part of a quantum turn in the social sciences.

3. The quantum turn

The philosopher of science Karen Barad (2007) argues that quantum mechanics and field theory set the stage for an entirely new kind of social science in the 21st century. She invites us to think about how quantum behaviour, which has been observed in the microscopic level of subatomic particles, can also produce effects in large-scale systems. She asks us to consider how we might live post-quantum causality? How we might mesh sub-atomic and organic temporality? In what ways can we reconceive the very notion of relationality in light of quantum science? It's important for Barad that this is not simply a matter of recognizing the way that measurements 'disturb' the behaviour of that which is studied, that is, disturb what would have happened in the absence of such a measurement (de Freitas, 2017).[2] In interpreting the surprising findings of quantum mechanics experiments, Barad contrasts two perspectives, Heisenberg and Bohr. Whereas for Heisenberg, quantum physics shows us the limits of our knowledge, and the epistemological limits of what we are able to experience or understand, for Bohr quantum science shows an inherent indeterminacy in matter. In other words, Bohr's interpretation insists that there are no separable isolated entities that can be observed from outside, and thus "entities" do not have a fixed inherent nature (wave or particle or spin). Duality of wave and particle – and *indeterminacy more generally* – is inherent to matter.

Barad (2012a) states very clearly that she is not "applying quantum physics to the social world by drawing analogies between tiny particles and people" for that would be a simplistic misuse of both theory and practice (p. 17). She emphasizes that there are not two domains (the microscopic and the macroscopic) with two different ontological principles. Thus she claims that quantum ontology is directly (and not simply analogically) relevant to every day matters—and that if this is not commonly evident, it is in part because quantum effects are very difficult to observe. Although research methods in the social sciences have begun to turn to the more-than-human in studying intra-activity, Barad (2012b) suggests we need to interrogate the very nature of causality, origin, relationality and change. Recent findings in ecology do suggest that systems operate according to a quantum model that includes complementarity and non-locality, as evidenced in causal effects that happen faster than the speed of light. For example, "birds exploit non-local connections with the earth's magnetic field to help them navigate … plants exploit quantum effects in photosynthesis … fruit flies sense of smell relies on the ability to detect quantum vibrations in smelly molecules …" (Wendt, 2015, p. 135). These examples suggest that a quantum paradigm might explain diverse kinds of behaviour and discernment (Atmanspacher, 2013; Grace & Kemp, 2013).

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[2] Indeed, Barad argues that the results of experiments conducted with the Stern-Gerlach apparatus, which measures the spin of particles along different axes, do not make sense if interpreted using Heisenberg's assumption that measurement disturbs particles (see pp. 258-263).

It was through the philosopher of mind Alexander Wendt (2015) that we were introduced to the vast research on quantum cognition, quantum brain science, and quantum probability. Wendt argues for a new quantum paradigm to better unify physical and social ontology, pursuing both a "quantum vitalism" and a "vitalist sociology" (p. 33). He argues for a "fundamental mentality" such that "mind is already inherent in every electron" and matter is intrinsically minded (p. 29-31). Such a perspective can be described as a kind of panpsychism that takes the known effect of consciousness at the macroscopic level and "scales it downward" (p. 31). He is principally focused on how quantum ontology offers an alternative theoretical framework for the study of consciousness, but not simply as a form of epistemic limitation. Like Barad, he lines up with Bohr in the debate with Heisenberg, and his project aims to show how cognition is entailed in quantum ontology.

The concept of *quantum coherence* is crucial for Wendt's approach. He suggests that life is a "macroscopic instantiation of quantum coherence". We are walking wave functions capable of carrying *coherently* incompatible potentialities; the collapse of a wave function into actuality (in the form of human action, decision and measurement) brings us into the "incoherent" world, which is separated out into individuals, actions, decisions, etc. In other words, quantum coherence designates the *virtual* side of matter where incompatible potentialities (wave *and* particular) are able to *cohere* and coexist. The state of quantum coherence is then actualized in individuated conscious states – decisions, actions, etc. According to this approach, individuation (of a conscious mind) is an after-effect of an inherent indeterminacy in matter, which is not spent or exhausted in the articulation of a decision or individual, but remains as a kind of originating vibrational energy.

We focus here on de/coherence because it is directly relevant to our later exploration of decision/actions and judgments. The coherence of organisms, suggests Wendt (2015), entails a pre-individual quantum superposition of coherent potentialities over all space-time domains (all possible actualizations), each correlated with one another and with the whole, and yet independent of the whole. In other words, the quantum coherent state, which is the pre-individual state, maximizes both global cohesion (by containing the superposition of diverse and even contradictory states) and local freedom (by fueling the active principle that individuates choice, decision and action). This richly textured virtual field of quantum coherence "underlies the sensitivity of living systems to weak signals, and their ability to intercommunicate and respond with great rapidity." (Ho in Wendt, p. 139)

Quantum coherence is a kind of *heterogeneous* holism that allows us to reconsider the nature of reason and judgment. Such an approach offers new ways of thinking about the future, remixing the temporal flow of time, and re-distributing the here-now across all possible states. This invites speculation about the means by which reasoned decisions and actions are undergone. Wendt is not arguing metaphorically, nor focusing only on the unusual behaviour of the sub-atomic, but is taking a realist position about quantum relationality. Wendt's project is somewhat different from that of new materialists such as

Barad (2007) because he is focused on the problem of consciousness and subjectivity, and because he uses quantum coherence to distinguish life from non-life. His attention to cognition, however, pushes the quantum paradigm in new directions, as we consider alternative models for reasoning with uncertainty.

What would it look like to operate according to a quantum model of reasoning? What kind of logic would accurately capture the machinations of a quantum mind? How would decisions and judgments be achieved if they followed a quantum logic rather than a classical one? The fact that cognitive models must reckon with our deeply situated knowledges makes the quantum paradigm very appealing: "In quantum physics, superposition appears puzzling: what does it mean for a particle to have a potentiality for different positions, without it actually existing at any particular position? By contrast, in psychology, superposition appears to be an intuitive way to characterize the fuzziness (the conflict, ambiguity, and ambivalence) of everyday thought" (Pothos & Busemeyer, 2013, p. 256). In the next section we turn to comparisons between classical and quantum probability models and their relevance to cognitive science. Our aim is to explore, tentatively, the possibility that quantum probability models might shed light on thinking and learning, and to gauge the extent to which QP seems to offer a better formal mathematical system for characterizing the situatedness of judgment.

We ask the reader to bear in mind, however, that any formal system will involve massive limitations and brutal simplifications. Probabilistic models of cognition are top-down – any such formalism will misrecognize much of the dynamic nature of events. But our aim is not to argue that quantum probability explains human judgment definitively – as that would go against the grain of the quantum – but rather to trouble reliance on classical probability and to invite speculation and experiment around different ways of reasoning with uncertainty.

3. Reasoning and probability

Classical (Bayesian) probability theory has strongly influenced models of cognition, inference, and learning. And yet there has always been ample evidence to suggest that human judgment rests on other kinds of inference that cannot be adequately described by the subjective 'corrections' of Bayesian probability. Wendt highlights three aspects of decision-making that arise from the work of Kahneman and colleagues in psychology. One involves the "order effect" in which the value that people ascribe to two events depends on the order in which the event is presented to them. Non-commutative reasoning is often at work when people assign probabilities to uncertain events or when people are asked to compare one thing to another, especially in cases when the events are not independent. The second is the "conjunction error" (referred to above in relation to the Linda problem) in which people think a given event A is less likely than the combination of two events A and B. Finally, the third aspect relates to the disjunction error and concerns the way people reason about unknown or hypothetical contexts. This is known as the "sure thing" principle, described by Savage (1954) as follows: "If under state of the world X, people prefer action A over action B and in state of the world ~X people prefer action A over B, then if the state of the world is unknown, a person should always prefer action A over B." Despite it being a 'sure thing', persistent evidence suggests that humans do not follow the principle in practice. Increasingly, researchers are exploring quantum probability theory to account for the preponderance of human judgments that do not follow classical probability, including the sure thing axiom of decision making (Pothos & Busemeyer, 2009), and order and conjunctive fallacies (Franco, 2009; Busemeyer et al, 2011).

Quantum probability (QP) is simply a way of assigning probabilities to events, but it does so quite differently from classical probability (CP). In other words, it offers an alternative formalization of reasoning with uncertainty. This alternative approach links uncertainty to indeterminacy in ways that open up new onto-epistemologies, better attending to the situatedness of reasoning. Knowledge states in QP are considered indeterminate in that there is a superpositioning of different possible outcomes (decisions, actions) associated or even entailed (virtually) in these states. Because such states are in fact dynamic mixtures of potentiality, they are *not* consistent with any one action or decision in the conventional sense (Pothos & Busemeyer, 2013, p. 256). Classical probability represents knowledge as a discrete set of propositions correlated to a discrete set of outcomes. Quantum probability operates in a slightly messier way, and aims to capture the con/fusion inherent to knowledge, pointing to the underlying indeterminacy of phenomena, and tapping into the specific logic of quantum entanglement.

In CP we can always form a meaningful conjunction of two propositions A and B. If we are able to determine the truth value of A and B independently, then we are able to determine the truth value of their conjunction. In quantum probability, however, there can be cases where the truth of such a conjunction is indeterminate. Our reasoning about complex events often involves incompatible insights that cannot be cobbled together to create a well-formed probability claim. In such cases, it may be that knowing the truth value of A implies not knowing the truth value of B. We might characterize this logical relation as $A \cap (B \cup \sim B)$. Another way of presenting this is the statement: "If A is true at time t1, then B is neither true nor false at time t1." This logical formulation captures Heisenberg's uncertainty principle where A and B are position and momentum – the more we measure position, the less we can say about momentum. In classical probability, incompatibility is defined in excusive terms. Two events are incompatible if they contradict each other (A and ~A). The quantum concept of incompatibility is defined somewhat differently.[3] Two events (A and B) are incompatible if there is a conditional entanglement between them that makes it impossible to pose well-formed questions about their conjunction. In order to determine $P(A \cap B)$ in such cases, quantum probability introduces sequential relations between A and B, but must also then formalise the interference of order effects. Because it attends to order effects, QP amplifies the richness of the event, and introduces sequential relations between knowledge claims, so that incompatible conjunctions can be evaluated for truth value.

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[3] Notably, Niels Bohr borrowed the notion of incompatibility from William James (Pothos & Busemeyer, 2013).

One of the major potential contributions of QP to decision theory pertains to this revised concept of incompatibility. In QP, if two parameters are incompatible it is impossible to define a single question using their conjunction. In a given case study, there can be questions about an event that are perceived as incompatible, that is, the questions concern different variables for which knowledge is not coherently integrated. Determining what kinds of questions are incompatible is extremely tricky, and is a crucial lynchpin in any theory of quantum mind. This concept of incompatibility is thus fundamental in the making of quantum probability models, and disagreements about what constitutes incompatible parameters will lead to very different models (Hampton, 2013).

Both CP and QP freeze the event in order to grapple with its possible outcomes, and in this they both fail to really capture the dynamism of the event (Behme, 2013). But, importantly, they impose very different models on that event. CP must freeze the event and treat outcomes as mutually distinct and meaningful within a single flat outcome space. All outcomes are compared within a space of complete knowledge in which being situated in one belief does not alter or curtail one's ability to determine the truth of another belief (dependent events or beliefs are extracted one from the other using the rule for conditional probability). This approach fails to reckon with the fundamental ambivalence of all thought, and the inseparability of beliefs. In other words, CP does not offer a mathematics of chance that attends holistically to the onto-indeterminacies that are immanent within all events.

In the case of the Linda example discussed in section one, CP sees only a fallacy when there is surely some 'good reason' in choosing the second option (being a bank teller and a feminist) as the higher probability. We may simultaneously hold beliefs that cannot be disentangled, and yet the question demands we do so. 1980s beliefs about bank tellers and feminists might be of this kind. CP describes Linda's condition prior to decision as a "mixed state", referencing our lack of knowledge of her actual state. In other words, CP assumes that Linda is definitely a feminist or not a feminist, and we simply do not know, and must assign probabilities to the likelihood of one over the other. QP, on the other hand, describes Linda's state prior to decision as *superposition* in which she is both, and yet neither, feminist and not-feminist: *she is in an indefinite state with regard to any such disposition, simultaneously and unconsciously entertaining a differentiated position, and uncommitted to either in some pre-individual way, perhaps in a kind of trans-feminist state*.

Before exploring the actual models, we want to emphasize the onto-epistemology entailed in the QP approach. Decision is an inventive act in QP. Responses to questions like "Is she a feminist" or "Is she a bank teller?" do not entail pre-given masked conditions yet to be observed, but are rather formulated or constructed in the act of decision. Disambiguating a superposition state is an inventive process rather than an act of revelation of previously hidden states. The virtual space of superposition does not consist of a set of choices separated-out (and then mixed up like a bag of colour marbles), but is rather a space of *holistic entanglement*. The superposition state is not the same thing as the set of all possible outcomes (each with their own probabilistic weight); superposition describes the con/fused state of potentiality where there are no determinate probabilities assigned to individual outcomes. In the description below we focus on the key issue of non-commutative actions/decisions, where order matters.

4. The geometric model

In 1900, during his famous lecture outlining twenty-three unsolved mathematical problems, David Hilbert called for a mathematical treatment of the axioms of mechanics and probability. This challenge became all the more daunting in the 1920s when quantum mechanics emerged and demanded new ways of formalizing causality and relationality. Attempts to formulate a set of mathematical axioms that might describe (and predict) quantum behaviour were proposed. Max Born, Pascual Jordan, along with Heisenberg, elaborated a matrix mechanics while Schrödinger outlined a wave mechanics. In 1926, the matrix mechanics and the wave mechanics formulations were shown to be mathematically equivalent.[4] John von Neumann, who was for a time Hilbert's assistant, developed what became the accepted mathematical formalism for quantum mechanics during the years 1926-1932. Known as "Hilbert space", this approach, which was initially developed in consort with Hilbert, focused on the amplitudes for the density of relative probabilities.[5] Hilbert spaces are vector spaces with particular properties that allow for accurate modeling of the strange behavior of sub-atomic life.

Quantum probability is based on the mathematics of Hilbert spaces, and in this section we attempt to provide an overall sense of how this model works. Using vector geometry, we begin by imagining a multi-dimensional Hilbert space of superimposed concepts, events and situations, which all co-exist despite their incompatibility. The vector space represents all possible outcomes for questions that could be asked about the system. The vector space is a richly textured space whose geometry models the nature of quantum relationality. This space is decomposable into various basis vectors that are themselves incompatible – we position these basis vectors orthonormal to each other, and together they engender this complex space of superpositioning. We then draw one particular 'knowledge state vector' that represents our current knowledge about some aspect of the situation. Each knowledge state vector is a kind of de/coherence. In the context of quantum theory, this knowledge state vector is given by an equation, a wave function ϕ, that is a superposition of two eigenstates, which are solutions of the function. The knowledge state vector can be expressed in terms of scalar multiples of the basis vectors ($\phi = a\mathrm{X} + b\mathrm{Y}$), where X and Y are the basis vectors, and a and b are called the respective amplitudes; the sum of the squares of a and b (usually complex numbers) is equal to 1. The question of where to draw the vector, with what angle, is discussed below. To determine the probability that this state vector participates in the incompatible basis vectors, we project it onto them, and then calculate its magnitude.

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[4] Though still today ambiguities remain over how to interpret the physical meaning of this mathematical equivalence (Barad, 2007).
[5] Von Neumann went on to seek alternatives to the Hilbert space model.

In the context of cognitive modelling, Pothos and Busemeyer (2013) work through the famous Linda example, which we stay with, despite its awkwardness, for the sake of continuity. Let's begin first with the question of whether Linda is a feminist. In this system there are two eigenstates (feminist/non-feminist), which correspond to two base vectors (see Figure 1a,). Linear combinations of these two base vectors produce the knowledge state vector ψ, whose length is 1, and which can form any angle with respect to the two base vectors (see Figure 1). In order to set up our geometrical model, we must first posit a wave function that contains the initial knowledge we have of the situation, which in turn enables us to determine the location of the knowledge state vector. In this context, given the description of Linda, it seems initially more likely that she is a feminist, so we decide to use the following wave function, $\psi = \sqrt{\frac{1}{4}}\psi_{\sim f} + \sqrt{\frac{3}{4}}\psi_f$, which shows the direction according to the coefficients ¼ and ¾. We've chosen the coefficients so that the original state vector is placed at a 30° angle to ψ_f , since the height of the triangle is $\sqrt{\frac{1}{4}}$ and the base of the triangle is $\sqrt{\frac{3}{4}}$. Changing the initial knowledge about the system—called the "priors"—will change the wave function, which will change the position of the knowledge state vector. If we didn't know as much about Linda, our "priors" might lead us to think that she is not more likely to be a feminist, so we might end up with a wave function such as this one: $\psi = \sqrt{\frac{1}{2}}\psi_{\sim f} + \sqrt{\frac{1}{2}}\psi_f$ which corresponds to the geometry in Figure 1b. However, it's crucial to recall that QP conceives her initial state prior to decision as a superposition in which she is both and yet neither, occupying all possible outcomes at once as a kind of trans-feminist. The initial state vector is meant to embody the entirety of these virtual potentialities, but represents a particular actualization (collapse of the wave function).

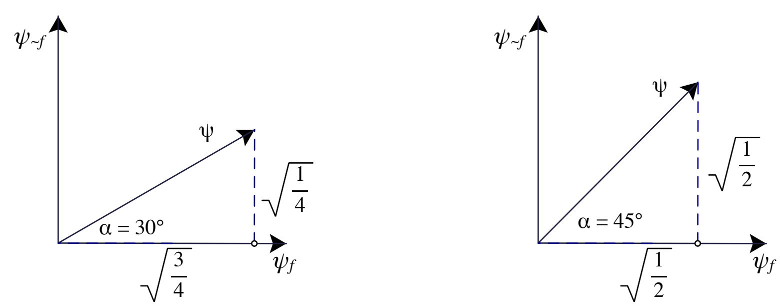

Figure 1: (a) An initial knowledge state vector for "Is Linda a feminist?" (b) a different initial knowledge state vector for "Is Linda a feminist"

When we ask the question, "Is Linda a feminist?", the probability that she is a feminist will be calculated by projecting the knowledge state vector onto the basis vector ψ_f . The probability of Linda being a feminist will be the amplitude of the knowledge state vector along ψ_f, which in the case of Figure 1a is, $\left(\sqrt{\frac{3}{4}}^2\right) = \frac{3}{4}$.

In the original Linda problem, we are actually asking a conjunction question: Is Linda a feminist and a bank teller? It's precisely in the case of conjunction that the model begins to be useful in better modeling non-commutative human reasoning. In QP, conjunction problems must be asked sequentially, so we first ask whether she's a feminist and then we ask whether she's a bank teller. Geometrically, this will involve introducing another pair of base vectors for back teller and not bank teller. The choice about how to position these relative to the feminist/~feminist basis vectors depends again on the "priors". In Figure 2a, we have shown a positioning that is based on the supposition that while being a bank teller is not the most likely of professions for a 1980s feminist, it is not that unlikely either.

In attending to order, we project sequentially, first asking whether Linda is a feminist (projecting onto ϕ_f) and then asking whether she's a bank teller (projecting onto ϕ_{bt}). We can see that the sequential projections (going from Figure 2b to Figure 2c) results in a greater amplitude compared to the direct projection onto ψ_{bt} (shown in Figure 2d).

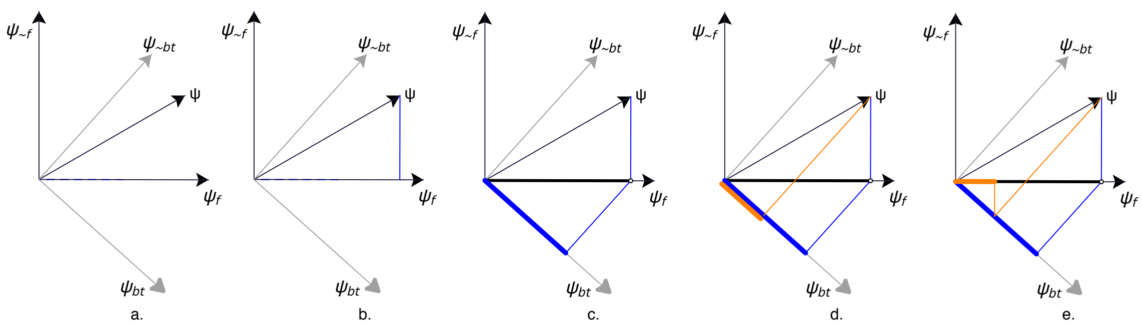

Figure 2: (a) Setting up the two subspaces for the Linda problem with four basis vectors, where v is the initial state vector (b) Projecting the knowledge state vector onto the feminist basis vector; (c) projecting the resulting vector onto the bank teller basis vector; (d) projecting the knowledge state vector onto the bank teller basis vector; (e) projecting the resulting vector onto the feminist state vector.

The difference between Figure 2c and 2d is significant. Whereas in CP, the probability of the conjunction of two propositions should be smaller or equal to the probability of each of them, the geometry in this model offers a different result, a result that aligns with most people's responses to the Linda experiment (see Kahneman et al., 1982). Further, if we had reversed the order, and started with the probability of Linda being a bank teller and then the probability of her being a feminist (projecting the state vector along ψ_{bt} onto ψ_f) we see that the result differs. In other words, the geometric model captures the order effect and kind of non-commutative logic that many people follow in reasoning about certain conjunctions, that is $P(F \cap BT) \neq P(BT \cap F)$ (compare Figure 2e with 2d). This shows how the QP model might help us with studying order effects more generally in decision theory. Although not well-captured in the geometric model, it's important to note that order effects in quantum mechanics also entail a blurring of previous decisions as new decisions are made.[6]

5. Discussion

[6] This blurring might help shed light on the important role of forgetting in learning (de Castro, 2013).

In thinking about using QP in such a context, several non-trivial structural assumptions must be made. As briefly mentioned, one must know how to place the initial knowledge state vector. When considering two parameters like feminist and bank teller, or asking two questions, one must also decide how the two subspaces are oriented to each other. These decisions will affect the actual values that are computed. Also mentioned above is the issue of how to decide whether two questions or concepts are incompatible. Pothos and Busemeyer (2013) suggest that, "a heuristic guide of whether some questions should be considered compatible is whether clarifying one is expected to interfere with the evaluation of the other" (p. 259). But the fact is that very different starting assumptions will lead to very different results (Tenton & Crupi, 2013).

The Linda problem is a classic problem in the study of human reasoning, and focuses on evaluating the likelihood of an object possessing two characteristics or qualities. We are very conscious of how the Linda problem strikes qualitative researchers as absurd, because it grossly simplifies the complex belief system that might undergird a decision about someone's disposition. Moreover, we suspect that the alternative reasoning model explored in this article might be better suited to queries about entangled mathematical concepts. Such reasoning is clearly relevant for a much broader set of situations—indeed any situation in which we ask a student to discern whether an object is a member of two classes. For example, we might consider a situation that is similar to the Linda problem, which would involve asking a student a question such as "Do you think that an arbitrary number in the 100s chart can be both even and square?" As with the QP elaboration of the Linda problem, a student might consider it just as likely for a number to be even as odd, in which case the situation will be as in Figure 1b. Then the next two basis vectors (for square and ~square) might be placed in such a way that makes the ~square basis vector quite close to the original knowledge state vector. This might represent the situation in which the student believes that there are not many numbers that are also squares. Geometrically, we would obtain a situation similar to that in Figure 2e, which is shown in Figure 3. The student's belief that a given number is even and is square is thus greater than the student's belief that the given number is square. This kind of example might even be more salient since it deals with concepts rather than dispositions (Aerts et al, 2013).

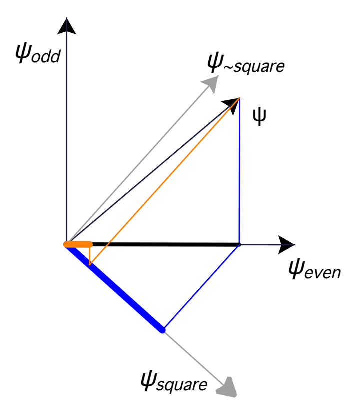

Figure 3: Geometric configuration for the question: Do you think that an arbitrary number in the 100s chart will be even and square?

In QP terms, the student's beliefs are commingled, and the act of deciding (in response to the question) is the act of measurement. The student's beliefs about the questions of evenness and squareness of a given number are incompatible if we assert that clarifying one question (yes, it is even) interferes with the evaluation of the other. Evenness and squareness are entangled concepts that are at this time—for the student—incompatible in that they cannot accurately disentangle one from the other. As they think they know or understand one (evenness) they lose sight of the other (squareness).

These kinds of formalist models are of course wholly inadequate in capturing the truly complex process of discernment (Behme, 2013). Following a more inclusive materialism, such models might be more conducive to the study of entangled concepts whose rich indeterminism might make them con/fused and related (see Aerts et al 2013 for this approach). Perhaps these kinds of models can show how children are engaged with the inherent indeterminacy of concepts, as put forward in recent work in mathematics education (de Freitas & Sinclair, 2017). Such an approach might better address the intra-active nature of the phenomenon – of the particle and the device, or the student and the question, of the concept with the other concept. Inclusive materialism (de Freitas & Sinclair, 2014) examines the way that student beliefs about number are not a property of the student, but belong to the phenomenon. The question doesn't disclose pre-existing thoughts or beliefs, but rather creates a pedagogical condition for entangled concepts to be individuated: "it is the specific material configuration that gives definition to the notion of the property in question, enacts a cut between," in our case, the two concepts (Barad, 2007, p. 264). The challenge of taking up QP in education research will be to pursue the rich implications of this onto-epistemological quantum turn.

6. Conclusion

Paradoxical findings in cognitive psychology, such as order and anchoring effects in human judgments, suggest that classical probability theory might be too limited to fully explain various aspects of human cognition (Trueblood et al., 2011, p. 1519). Increasingly, researchers are exploring quantum probability theory to account for the preponderance of human judgments that do not follow classical probability, including the sure thing axiom of decision making (Pothos & Busemeyer, 2009), and the conjunctive and disjunctive fallacies (Franco, 2009; Busemeyer et al, 2011). If ambiguity, ambivalence and fuzzy notions of truth are more accurate ways of describing our thinking, then perhaps the logic of quantum relationality offers a more suitable approach.

We suspect that quantum probability might help reevaluate developmental evidence regarding children and learning. For instance, conventional interpretations of Piagetian tasks, when children incorrectly respond to the question "which container has more water in it?", describe the children as distracted by physical and sensory parameters, and unable to abstract the formal concept of conservation (Inhelder & Piaget, 1958). We wonder if quantum probability models might shed a different light on such experiments. Perhaps these children are reasoning *through* uncertainty in ways that are at odds with classical logic and abstraction. Indeed, it may be that they are mobilizing very different onto-epistemologies in which uncertainty is directly linked to the inherent indeterminacy of concepts such as shape and number.

In practice, inferences entail complex entanglements between possible outcomes, and any discovery of new evidence interferes in the process, while embodied mental states are often a superposition of incompatible potential outcomes rather than a rational choice of one outcome over another. Classical probability fails to adequately account for this complexity, and considers all deviations from its logic as fallacious. Quantum probability, on the other hand, is designed to formalize this kind of complexity, and is derived from the behaviour of sub-atomic particles. Classical probability considers uncertainty an epistemic limitation, and accordingly, it measures degrees of certainty. Quantum probability, on the other hand, treats uncertainty as an ontological feature of the state of affairs. In this case, measures of uncertainty are *realist* measures of an environment in which outcomes are literally mixed together in varying intensity. This follows the Bohr insights in quantum physics, in which the fact that matter is both particle and wave is an inherent indeterminism, and not simply a limitation of our measurement or our human understanding.

Such foundational indeterminism means that an entangled system defies prediction when relying on the usual probability distribution of possible outcomes. The law of total probability (which is fundamental to Bayesian modelling) is violated. Order effects, conjunction biases, and sure-thing inclinations suggest that an entirely different set of probability axioms could be developed and may be needed to account for reasoning with and about uncertainty. The fuzzy logic that seems to describe our habits of reasoning is indeed physically manifest in quantum mechanics, and motivates the turn to a quantum probability to rethink cognition. This is not to cast classical probability out the window, since it's clearly a fruitful and important way of reasoning with uncertainty (Shanteau & Weiss, 2013). Quantum probability actually subsumes classical probability (as a particular case), while also addressing non-classical behaviour. Quantum decision theory builds on this alternative approach to probability, and applies it to human judgment. We hope this brief introduction to this topic will trigger more research on how the quantum mind is at work in teaching and learning.

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