STABILITY OF A MONETARY ECONOMY

WITH INFLATIONARY EXPECTATIONS

by

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This thesis explores the role of inflationary expectations in the dynamics of a general equilibrium, macroeconomic system. The analysis attempts to synthesize and extend the monetary models of Philip Cagan, Lloyd Metzler, and Don Patinkin, and determine the stability properties of such extended models.

Chapter 1 traces the history of the importance of inflationary expectations in the works of macro-economists. Although long recognized as important by the "monetarists," especially Irving Fisher, the importance of price expectations is now readily acknowledged by the "Keynesian" School of macro-economists. Chapter 2 examines the dynamics and properties of the Cagan model in detail, carefully indicating the assumptions which will later be relaxed in order to treat more general models. A critical examination of the role of adaptive expectations of the price level is presented in this chapter.

Chapter 3 rigorously details the comparative statics of a Keynesian model by analyzing equilibrium in both the "asset" and "commodity" market as a "stock" and "flow" equilibrium. The following chapter discusses the dynamic adjustment of such a Keynesian model and the role of inflationary expectations is determined to be a key aspect of the determination of price behavior in the commodity market.

Chapter 5 synthesizes the dynamic adjustment mechanism developed earlier into a full Keynesian model with both fixed and endogenous real income and a Fisherian, classical model of economic adjustment. The stability conditions of these general models are compared to those of the simple Cagan model discussed in Chapter 2.

Chapter 6 examines the properties of proportional monetary policy in the context of the models developed in the previous chapter. Computer simulations of these policies are provided. In particular, it is shown that counter-cyclical monetary policy on the money rate of interest is the most effective proportional policy for damping the economy to equilibrium. The final, seventh chapter explains why the above policy is tantamount to the stabilization of a broader monetary aggregate in an economy consisting of a competitive, unregulated banking industry. The last chapter also extend the Keynesian system to allow for both a "long" and "short" interest rate and hence allows for a "lag" in the effect of monetary policy on the real economy.

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TABLE OF CONTENTS

TABLES, FIGURES, AND CHARTS

TABLES:

FIGURES:

CHARTS:

CHAPTER 1

INTRODUCTION

In the opening address for the Conference of University Professors held in England in September, 1968, Professor John Hicks remarked:

> "The 'wealth effect' has only been thought to be a sufficient stabiliser because the 'psychological' effect of the price movement has been neglected. As soon as prices move sufficiently for people to extrapolate -- to base their expectations of future prices not upon current prices but upon the way prices has been changing - a destabilising force is set up which is bound to swamp the much weaker stabilising power of the 'wealth effect.' That is the basic cause of the instability."[1]

Although Hicks' final sentence may seem strongly worded, the existence of non-stationary price expectations has increasingly played an integral and vital role in the theory of macro-economic equilibrium. A strong impetus to this movement was the economic experience of the last half of the 1960's, a period marked by historically high interest rates and high rates of inflation. This evidence strongly suggested that the market rate of interest, without correction for anticipated rates of inflation, was not the appropriate variable to represent the opportunity cost of holding real goods (durable commodities and capital) through time.

The importance of inflationary expectations is acknowledged by both the "Keynesian" and the "monetarist" schools of thought. Milton Friedman declared in his essay, "The Optimum Quantity of Money," in

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[1] Sir John Hicks, "Automatists, Hawtreyans, and Keynesians," address reprinted in the *Journal of Money, Credit, and Banking,* vol. 1, no. 3, (August, 1969), p. 308.

1969, that

> "This necessity for overshooting in the rate of price change and in the rate of income change is in my opinion the key element in monetary theories of cyclical fluctuations ... little can be said about the details without much more precise specification of the reaction patterns of the members of the community and of the process by which they form their anticipations of price movements."[2]

It is the purpose of this thesis to explore the implications of non-stationary price expectations on the nature of macroeconomic equilibrium and particularly the dynamics by which the economy reaches such an equilibrium. It is shown that macro-models which ignore price expectations exhibit implausible steady-state and disequilibrium characteristics.

Throughout this thesis, heavy emphasis is placed on the "Keynesian" dynamic adjustment model although a classical "Fisherian" adjustment model is also presented. The role of inflationary expectations in determining the stability properties of these dynamic models is particularly examined. The stress on dynamic adjustment in this thesis arises from my belief that only when the dynamics of a model have been fully specified can one confidently predict the outcome of government counter-cyclical policy on the variables of the economy. Static models are too often not the proper framework to examine the effect of particular policies. The remainder of this chapter will give a brief historical review of economists' recognition of the significance of inflationary expectations in economic systems.

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[2] Milton Friedman, "The Optimum Quantity of Money," in *The Optimum Quantity of Money and Other Essays*, by Milton Friedman (Chicago: Aldine Publishing Co., 1969) p. 13.

Historical Review

As Irving Fisher noted in 1907[3], the concept of changes in the price level affecting the rate of interest is hardly new in economics. Fisher believed that the first reported reference to this phenomenon was written by a physician, William Douglass, in 1740, in a pamphlet published in Boston entitled "A Discourse Concerning the Currencies of the British Plantation in America." In it Douglass clearly stated that "large Emission of Paper Money does naturally rise the Interest to make good the sinking Principal."[4] Later John S. Mill, J. B. Clark, and other notable economists of the late eighteenth and nineteenth century also mentioned this fact.[5] It was not until 1896 that Irving Fisher worked out the precise mathematical relationship between inflation and interest rates.[6] A different approach is used in the mathematical appendix of this chapter to derive the fact made known by Fisher that when interest is paid continuously on a bond, the expected rate of inflation, π, must be directly added to the real cost of making the loan, r, to arrive at the money interest rate, $r_m = r + \pi$.

The strong effect of the expected rate of inflation on inflation was clearly enunciated in Knut Wicksell's *Interest and Prices*, published in 1898. Wicksell states:

> "The upward movement of prices will in some measure 'create it own draught.' When prices have been rising steadily for some time, entrepreneurs will begin to reckon

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[3]Irving Fisher, *The Rate of Interest*, (New York: Macmillan Co., 1907).

[4]*Ibid.*, p. 356.

[5]*Ibid.*, Appendix, Chap. 5, pp. 356-58.

[6]*Ibid.*, p. 361.

> on the basis not merely of the prices already attained, but of a further rise in prices. The effect on Supply and Demand is clearly the same as that of a corresponding easing of credit [lowering of the real rate of interest]. Indeed, the effect may be even greater."[7]

Although Wicksell denied that under normal circumstances such a phenomenon could create explosive inflation -

> "But so long as business continues to be conducted on normal lines, it is not to be supposed that there will be any cumulative movement of prices in the manner of an avalanche,"[8]

he later claims that,

> "The matter takes on an entirely different aspect in the case where the market is under the influence of *speculation* proper ... There is almost no limit to the rise in prices *in spite of the fact* that credit becomes more and more expensive."[9]

Irving Fisher in *The Purchasing Power of Money* published in 1922 explicitly recognized how anticipated inflation fuels inflation:

> "We all hasten to get rid of any commodity which, like ripe fruit, is spoiling in our hands. Money is no exception; when it is depreciating, holders will get rid of it as fast as possible."[10]

A. C. Pigou's 1943 article "The Classical Stationary State," began a controversy which often seemed to confuse the static consistency of an economic system with its dynamic stability. John R. Hicks particularly decried the tendency to analyze static models and hence ignore expectations. He writes in 1946 in *Value and Capital* :

> "So long as economists were content to regard the economic system in static fashion, it was reasonable to treat it as a self-righting mechanism ... As soon as we take expectations

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[7] Knut Wicksell, *Interest and Prices,* (trans. by R. F. Hahn, Reprints of Economic Classics, 1898), p. 96.

[8] *Ibid.,* p. 97.

[9] *Ibid.,* pp. 97-98.

[10] Irving Fisher, *The Purchasing Power of Money,* (New York: The Macmillan Co., 1922), p. 63.

> into account, the stability of the system is seriously weakened."[11]

In fact, Hicks flatly claims,

> "A system with an elasticity of expectations greater than unity [a price change is expected to continue in the same direction], and a constant rate of interest, is definitely unstable."[12]

Don Patinkin, although "champion" of the real balance effect for equilibrium analysis, joined with Hicks in his famous 1948 article, "Price Flexibility and Full Employment,"

> "Pigou has completed only half the task ... we can accept his proof of the *consistency* of the *static* classical system. But that still leaves completely unanswered the question of whether the classical *dynamic* system will converge to this consistent solution."[13]

And elsewhere in the article he states,

> "In dynamic analysis we must give full attention to the role played by price expectations and anticipations in general. It is quite possible that [say] the original price decline will lead to expectations of further declines. Then purchasing decisions will be postponed, aggregate demand will fall off, and the amount of unemployment increase still more."[14]

Later, in 1950, Thomas Schelling suggested the importance of inflationary expectations to cycle behavior:

> "In a personal letter Mr. Patinkin has pointed out ... that oscillations and instability may occur because of price anticipations. This possibility -- which I had denied -- I find especially interesting since it represents an 'anticipation cycle' distantly akin to the acceleration cycles discovered by Professors Samuelson and Metzler."[15]

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[11] John R. Hicks, *Value and Capital,* 2nd ed., (Oxford: The Clarendon Press, 1946), p. 256.

[12] *Ibid.,* p. 255.

[13] Don Patinkin, "Price Flexibility and Full Employment," *American Economic Review,* vol. 38, no. 4 (Sept., 1948), fn. 30, p. 561.

[14] *Ibid.,* pp. 557-8.

[15] Thomas C. Schelling, "Rejoinder," *American Economic Review,* vol. 40, no. 4 (Sept., 1950), p. 614.

The importance of price expectations has certainly not eluded Milton Friedman, leader of the "Monetarist School" at the University of Chicago, although his opinion of its importance has vacillated. Friedman mentions adverse price expectations as a disequilibrating force in 1948 in his essay, "A Monetary and Fiscal Framework for Economic Stability."[16] Later he considers these expectations to be mostly stabilizing in the early stages of mild movements in the price level and remarks in 1949, "In view of the historical instability of monetary and fiscal forces, particularly monetary, the surprising thing is that expectations in the past have not been even more destabilizing."[17] Lately, as revealed in his 1967 Presidential Address to the American Economic Association[18] and the quotation cited on p. 7, he views price expectations as an important key to the theory of the business cycle.

It was in fact Lloyd Metzler who constructed the first explicit dynamic model which simultaneously determined both the price level and the interest rate.[19] The nature of Metzler's model was very similar to that of Franco Modigliani's 1944 static model[20], but the real balance effect (Pigou effect) was added and explicit dynamics and stability

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[16] Milton Friedman, *American Economic Review*, vol. 38, no. 3, (June, 1948), pp. 245-64.

[17] Milton Friedman, "Rejoinder," to Philip Neff's comment on the above work, *American Economic Review*, vol. 39 (Sept., 1949), p. 954.

[18] Milton Friedman, "The Role of Monetary Policy," *American Economic Review*, vol. 58, no. 1, (March 1968), pp. 1-17.

[19] Lloyd Metzler, "Wealth, Saving, and the Rate of Interest," *Journal of Political Economy*, vol. 59, no. 2, (April, 1951), pp. 93-116.

[20] Franco Modigliani, "Liquidity Preference and the Theory of Interest and Money," *Econometrica*, vol. 12, no. 1, (Jan., 1944), pp. 45-88.

conditions were explored. Metzler found (as did Patinkin later[21]) that if the interest rate equilibrated the asset market and the price level equilibrated the commodity market, the system is unambiguously stable under normally accepted sign conventions for the derivatives. However, Metzler warned:

> "This does not mean, of course, that an economic system in which the saving-wealth relationship is operative will always be a stable system in reality for the equations have made no allowance for expectations, and such expectations may exert a strongly destabilizing influence on the system. If prices of commodities are rising, for example, consumers and producers may anticipate further price increases; if so, saving will probably decline and investment will increase; thereby widening the inflationary gap and accelerating the price rise ... These possibilities suggest that [this system] is stable only in a narrow sense.[22]

In the mid 1950's, two economist, Philip Cagan[23] and William Vickrey[24] independently integrated price expectations into a mathematical, dynamic specification of economic systems. From a macroeconomic standpoint, Philip Cagan's 1956 article emerged as the pathfinding work, both theoretically and empirically, in the formulation of monetary dynamics under conditions of extreme inflation. The adaptive expectation mechanism for price expectations, popularized by Cagan, replaced the Hicksian "elasticity of expectations" as the dominant expectation mechanism in subsequent growth and macro-theory articles. Because of the simplicity of the Cagan model, its full dynamics are presented first

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[21]Don Patinkin, *Money, Interest, and Prices*, 2nd, ed., (New York: Harper Row, Pub., 1965), Appendix 8.

[22]Lloyd Metzler, "Wealth, Saving, and the Rate of Interest," p. 116.

[23]Philip Cagan, "The Monetary Dynamics of Hyperinflation," in *Studies in the Quantity Theory of Money*, ed. Milton Friedman, (Chicago, Univ. of Chicago Press, 1956), pp. 25-117.

[24]William S. Vickrey, "Stability Through Inflation," in *Post-Keynesian Economics*, ed. K. Kurihara (Rutgers Univ. Press, 1954), pp. 89-122.

to highlight the principal effects of inflationary expectations on macro-equilibrium. Later chapters will develop more general models, of which the Cagan model will be a special case.

Mathematical Appendix

Chapter 1

The real presented discounted value PDV of a N-period bond paying a continuous, constant, nominal stream of interest r* from time 0 to N, and 1 at time t = N which is discounted at a constant rate δ can be represented as

$$\text{(1A.1)} \qquad PDV = \int_0^N r^* e^{-\delta t}/p(t)\, dt + e^{-\delta N}/p(N),$$

where p(t) is the price level at time t. For the bond to be issued at par, the real PDV must equal 1 and hence the required nominal rate of interest r* must be

$$\text{(1A.2)} \qquad r^* = \frac{1 - e^{-\delta N}/p(N)}{\int e^{-\delta t}/p(t)\, dt} .$$

In general, the denominator of the right hand side of 1A.2 cannot be simply integrated, but if

$$\text{(1A.3)} \qquad p(t) = e^{\pi t},$$

where π is the constant rate of expected inflation for all time durations, then

$$\text{(1A.4)} \qquad r^* = \frac{1 - e^{-(\delta+\pi)N}}{-(e^{-(\delta+\pi)N}-1)/(\delta+\pi)} = \delta + \pi ,$$

and thus the required nominal rate of interest r* is equal to the real rate of discount plus the expected rate of inflation.

CHAPTER 2

THE CAGAN MODEL

The first economist to study the phenomena of inflationary expectations in a rigorous fashion was Philip Cagan. His work, "The Monetary Dynamics of Hyperinflation," published in 1956, has served as a foundation for many of the money-growth models that appeared in the following decade.

Cagan's model consists of an economy with two asset categories - money and commodities. Commodities consist of both consumers' or consumption goods and producers' goods - the latter referred to as capital. There is no meaningful financial intermediation in this economy, no securities exist, and all capital is held directly by consumers. Hence firms play a trivial role in the Cagan economy. Later, in Chapter 5, we shall see that the role of money in the Cagan economy is essentially similar to the classical "monetarism" of Irving Fisher.

The model itself consists of two equations: an equilibrium condition in the money market and an equation representing the mechanism by which expectations of future rates of change of the price level are formed. Formally, Cagan postulated

$$M/p = L(r_o + \pi), \quad L' < 0, \tag{2.1}$$

$$\dot{\pi} = b(\dot{p}/p - \pi), \quad b > 0, \tag{2.2}$$

where π is the expected rate of price inflation; p, the price level; and r_o, the constant real rate of interest.

Eq. 2.1 represents the commonly held assumption that the demand for nominal money balances is homogeneous of degree one in the price level and negatively dependent on the opportunity cost of holding money, the market rate of interest, $r_o + \pi$. However, two more important assumptions are presented by 2.1. First, that equilibrium in the money market is maintained at each moment of time, i.e., excess demand for money (which through Walras' Law is equal to the excess supply of commodities in the Cagan economy) is always zero. Second, that the *ex ante* real rate of interest is a constant equal to the marginal product of capital. Later, when this first assumption only is relaxed, we will specify an adjustment mechanism for the price level and term the economy a Fisherian, classical economy. When the second assumption is relaxed, and the real rate of interest can deviate from the marginal physical product of capital, we shall specify a Keynesian adjustment mechanism which allows for the *ex ante* saving behavior of firms and individuals to differ.

Equation 2.2 is the adaptive expectation mechanism applied to the inflation variable. π should pertain to a future period of time relevant to the demand for money, although no distinction between long and short periods is made here. The adaptive expectation mechanism was popularized by Cagan and became widely used by other economists in both theoretical and empirical work. Alternatively, 2.2 could be written

$$2.3 \qquad \pi_t = e^{-bt} b \int_{-\infty}^{\tau} (\dot{p}/p)_\tau e^{b\tau} d\tau,$$

indicating that the expectation of the rate of inflation is the weighted sum of past rates of inflation, the weights exponentially declining into the past. Equation 2.2 can also be viewed as an error-correcting mechanism of expectations, i.e., the expected rate of inflation is revised per period of time in proportion to the difference between the actual rate of change in prices and the rate of change that was expected.

The Nature of Price Expectations

Expectations, or the manner in which expectations are formed, is a very unsettled area of economic thought. Certainly events other than just the past values of a variable determine the expected future values, viz., government actions or predicted government actions, forecasts, and values of other variables, just to mention a few. Of primary importance is the nature of the past behavior of the variable itself. If the past behavior of a variable is cyclical, an adaptive mechanism of expectation will prove to be a very poor predictor of the path of the variable through time. If the past behavior of the variable is completely random, then perhaps the simple expectation that the expected value is equal to the current value is a "rational" expectation. This would correspond to Hicks' "unit elasticity of expectations."[1] The adaptive mechanism can perhaps be best justified when there is some belief that a variable has a stable value subject to short-run random fluctuations. However, if the value of the variable persists at a new value for a long enough time, one will eventually revise his expectations completely to the new level. One can see from 2.2 and 2.3 that a step change in the rate of inflation from one level to another will bring about a smooth transition of π from the old value asymptotically to the new value.

It is a desirable characteristic of any expectation mechanism that,

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[1] Refer to Hicks, *Value and Capital*, *op. cit.*, p. 205.

if a variable x takes on a constant value x^* after some time t_o, then

$$\lim_{t\to\infty} x^e_t = x^*, \tag{2.4}$$

where x^e is the expected value of the variable. In fact, if equilibrium is defined as a state where all expectations are realized, 2.4 becomes a necessary condition for equilibrium. We shall term any expectation mechanism that obeys 2.4 to be *(steady-state) consistent*. One may suggest that the profuse use of the adaptive mechanism in rigorous economic theory reflects its qualities of providing mathematical tractability and economic realism to mathematical models.

One important characteristic of the Cagan model is that the adaptive mechanism is applied to the *rate of change* of the price level and not the price level itself. Work on the stability of general equilibrium models under adaptive expectations has been explored principally by Prof. Kenneth Arrow and various co-authors, beginning in the late 1950's.[2] The strongest result, published in 1962[3], concluded that global stability of a general equilibrium system under adaptive expectations and non-linear price adjustment was obtained if all commodities are gross substitutes and all equilibria are strictly positive. The theorem, it should be stressed, holds when adaptive expectations apply to the *level* of prices

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[2] A. C. Enthoven and Kenneth Arrow, "A Theorem on Expectations and the Stability of Equilibrium," *Econometrica*, vol. 24, no. 3, (July, 1956), pp. 288-93, and Kenneth Arrow and Marc Nerlove, "A Note on Expectations and Stability," *Econometrica*, vol. 26, no. 2 (April, 1958), pp. 297-305.

[3] Kenneth Arrow and Leonid Hurwicz, "Competitive Stability under Weak Gross Substitutability: Nonlinear Price Adjustment and Adaptive Expectations," *International Economic Review*, vol. 3, no. 2 (May, 1962), pp. 233-55.

and not the *rates of change* of prices.[4]

One could claim that there is no more justification to assume that the adaptive expectation hypothesis applies to rates of change of the price level rather than to the level of prices. However, there are persuasive arguments for its applications to rates of change. Fundamentally, since it is known that a steady-state can prevail with any given rate of monetary expansion and hence any rate of price change, an adaptive mechanism on the *level* of prices will never bring expectations in line with reality, even in the steady-state. Hence the adaptive mechanism applied to price levels is not steady-state consistent as defined in 2.4 above. Individuals are constantly fooled by the rising price level, and the expectations of future prices continaully lag behind those realized. Therefore, a "full" steady-state, where not only are all variables growing exponentially, but all expectations are justified, cannot be fulfilled under these circumstances.[5]

Is this a fundamental property of money or can it be applied to other commodities? Frank Knight noted in 1945 that

> "In wheat futures market we have the same thing -- anticipation creates changes. But there is an equilibrium there which is dependent on well-known objective facts. So have damped oscillations. But there is no definite, known equilibrium values of money. Everyone might know that Money is too high -- but the question is whether it will continue to

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[4] This point is unfortunately obscured in some literature (A. C. Enthoven, "Monetary Disequilibrium and the Dynamics of Inflation," *Economic Journal*, vol. 66, no. 262, [June, 1956], pp. 256- 70; and Patinkin, *Money, Interest, and Prices*, *op. cit.*, p. 312) leading some to suggest that Arrow's analysis, when applied to a monetary economy, is a more general case of Cagan's analysis.

[5] It is true that if $\dot{p}/p$ follows a path which in the limit does not converge to a constant, then π will never converge to $\dot{p}/p$. However, if we wish to retain the very desirable characteristic that real (per-capita) balances are constant in the steady-state, $\dot{p}/p$ must converge to a constant.

rise; one doesn't know where the breaking point is."[6]

Knight's statement implies that the answer lies in the institutional setting of monetary economics. If the money supply is rigidly based on some scarce commodity, say Gold, then there will be reason to believe that the production costs of gold will, in the long run and with given preferences, determine the price level (price of commodities in terms of gold). Hence with stationary preferences and production processes, it might be reasonable to expect the price of gold, and hence money, to be constant and not rising at some constant rate.[7]

However, once the money-commodity link is broken, as it is in most modern economies today, the concept of a stationary price of money may justifiably disappear. The complete control of the money supply by the government implies that it can establish any rate of growth of the nominal money stock it wishes, and hence any rate of growth of prices. The reasons for controlling the rate of growth of the monetary aggregate instead of level revolve around the assumption that while the nominal quantity of money may have no effect in the long run (long run neutrality), the rate of change of the nominal monetary stock may be non-neutral in its effects, and hence provide an important policy variable of the government.

The rate of growth of the money supply that the government will

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[6]Frank Knight, July 24, 1945, notes taken by Don Patinkin at the Univ. of Chicago, Economics 301, and reprinted in "The Chicago Tradition, the Quantity Theory, and Friedman," *Journal of Money, Credit, and Banking*, vol. 1, no. 1, (Feb., 1969), pp. 65-66.

[7]This line of reasoning does not suggest the superiority of a commodity standard. The production cost of commodity standards have fluctuated greatly, generating large shifts in the price of money (cf. Milton Friedman, "Commodity-Reserve Currency," *Journal of Political Economy*, vol. 59 (June, 1951), pp. 203-32.)

choose in the long run will depend on many circumstances. One might be the revenue that the government receives from the printing of money (seigniorage). It can be simply noted that, in a stationary economy, the rate of inflation is equal to the tax rate on real balances. Since $\dot{M}/M = \dot{p}/p$ in the steady state, the seigniorage, $\dot{M}/p$, of the government can be written

2.5 $$\dot{M}/p = \dot{M}/M(M/p) = \dot{p}/p(M/p).$$ [8]

This revenue might be chosen in reference to some optimal tax structure in which commodities are taxed in such a way as to minimize the dead-weight burden of taxation. Secondly, the government might believe that there exists some long run trade-off between the rate of inflation and employment, and/or the long run capital stock, and hence use its powers to create money for this goal. A third reason might be that macroeconomic instability, in the sense of the "business cycle," is different for varying rates of growth of the monetary aggregate. All these reasons will be elaborated in Chapter 6.

Analytical Properties of Cagan Model

The equilibrium properties of the Cagan equations 2.1 and 2.2 can be easily specified. For any rate of growth of the nominal money supply, $\theta = \dot{M}/M$, prices will rise at rate θ, i.e., $\dot{p}/p = \theta$. This can easily be derived by differentiating 2.1 totally with respect to time to yield

2.6 $$L/L'(\theta - \dot{p}/p) = \dot{\pi} .$$

[8] The government gains seigniorage only from the printing of non-interest bearing outside money and hence M/p refers to real outside money balances. The seigniorage resulting from the expansion of the private (inside) money stock depends on whether interest is prohibited on bank money.

If we define that in a steady-state the real money stock is constant, then $\dot{\pi}$ must equal 0 and $\dot{p}/p$ must equal θ. Hence, the equilibrium M/p is dependent on θ(if expectations are consistent) and r_o. Equation 2.1 indicates that the amount of real money in the economic system is, in the steady-state, inversely proportional to the rate at which the nominal money stock is growing. This yields the paradoxical result that to reduce once and for all the amount of real money one must be constantly increasing the quantity of nominal money and vice versa.

The local stability properties of the Cagan system are derived in the mathematical appendix. For local stability to hold around any equilibrium rate of growth of nominal money,

$$2.7 \qquad b \;<\; -L/L' \;=\; -r_m^*/E(L/r_m^*),$$

where r_m^* is the equilibrium money rate of interest, $r_o + \theta$, and $E(L/r_m^*)$ is the elasticity of the demand for money with respect to the equilibrium money rate of interest. This stability condition appears in all monetary literature based on adaptive expectations,[9] and will henceforth be referred to as the Cagan Stability Condition.

Cagan specified a particular form to the demand for money function, viz.,

$$2.8 \qquad M/p \;=\; \gamma Y_o e^{-\alpha\pi},$$

where γ and α are positive constants and Y_o is the constant real income. This specification for the demand for money, coupled with the adaptive

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[9]See Miguel Sidrauski, "Inflation and Economic Growth," *Journal of Political Economy*, vol. 75 (Dec., 1967), pp. 796-810; Jerome L. Stein, "The Optimum Quantity of Money," *Journal of Money, Credit, and Banking*, vol. 2 (Nov., 1970), pp. 397-419; and Duncan K. Foley and Miguel Sidrauski, *Monetary and Fiscal Policy in a Growing Economy*, (New York: Macmillan, 1970).

expectations on the rate of inflation, turn the local stability condition 2.7 into a global one, since L/L' becomes constant for all rates of interest. The Cagan specification 2.8 yields an elasticity of demand which is linearly proportional to the money rate of interest.

If the elasticity of the demand for money with respect to the interest rate is non-zero, it is easily seen that whenever the current rate of inflation is taken to be the expected rate, the economy is inherently unstable. This would correspond to a $b = \infty$ in 2.2. In this case 2.8 can be written

$$2.9 \qquad M/p = \gamma Y_o e^{-\alpha \dot{p}/p},$$

whose solution for any $\dot{M}/M = \theta$, is

$$2.10 \qquad \dot{p}/p = ce^{t/\alpha} + \theta,$$

where c is determined by initial conditions. Unless $c = 0$, i.e., the economy is right on the equilibrium path $\dot{p}/p = \theta$, the rate of inflation will either increase or decrease exponentially no matter what the rate of growth of the money stock.

The inherent instability when $\pi = \dot{p}/p$ can be easily shown to be a general global condition. If

$$2.11 \qquad M/p = L(r_o + \dot{p}/p), \quad L' < 0, \text{ then}$$

$$2.12 \qquad \partial(\dot{p}/p)/\partial p = -L/(L'p) > 0.$$

It was this very inherent instability of the money demand function that prompted Cagan to employ some lag to establish stability in his model. The lag of expectations of the rate of inflation is only one such lag, others will be examined later in the thesis.

Descriptive Properties of the Cagan Model

The dynamic properties of the Cagan model can be described intuitively. The following description applies to a tatonnement process, since in the Cagan model no trades are made outside of equilibrium. The fundamental cause of the instability is the fact that the excess demand for money balances may not be a decreasing function of the price of money. For instance, without inflationary expectations, i.e., $b = 0$, an upwards deviation of the price level from its equilibrium path will unambiguously cause a "deflationary" effect, an excess demand for money and hence an excess supply of commodities. People will try to sell their commodities for money and hence force the price level back down. In other words, the higher price level has caused a reduction in the real money balances of individuals, and their attempt to restore the original balance exerts a downward pressure on the price level. However, when $b > 0$, an upward deviation of the price level may cause individuals to expect prices to continue to rise, thus increasing the expected opportunity cost of holding money balances and couteracting the initial desire to restore the equilibrium level of money balances. If b is high enough, the opportunity cost of holding money rises so fast as to completely overwhelm the deflationary effect of a higher price level. In this case, individuals continually wish to get rid of their real balances, selling money for commodities and driving the price of commodities up even further, exacerbating the condition and leading to a never-ending upward spiral of prices.

The Cagan stability condition indicates that, for a given b and r_m^*, the higher the elasticity of demand for money with respect to the interest rate, the more likely the economy will be unstable. This is intuitively clear since, when the opportunity cost of holding money rises, a higher elasticity implies that the demand for real balances will decline greatly and this is the destabilizing counteracting effect of the system. In fact, if, as some classicists believed, the demand for real balances is completely insensitive to the interest rate, b can be any value and stability is assured.

The implication of the assumptions of the constant real rate of interest r_o should be stressed. In the Cagan economy, individuals must increase and decrease their money balances by decreasing or increasing the commodities that they demand. However, in the Cagan economy trades are made only at equilibrium prices, and prices are in equilibrium at all points of time. Inflation in the Cagan model does not arise from any "excess demand," but simply from the movement of prices necessary to constantly equilibrate supply and demand.

Individuals do not trade between money and financial assets in the Cagan economy, an act which would change the real rate of interest. Individuals only trade between money and commodities, the latter comprising producers' goods and consumers' goods. This assumption may be a good approximation for the study of hyperinflation where most financial markets break down. For most developed economies, however, as we shall examine later, the "trade-off" between money and financial securities becomes the key to establishing equilibrium - a process that requires some change in the real rate of interest to bring about this equilibrium.

Another method of explaining the dynamics of the Cagan model, a method that will be used extensively in later more complex models, is to describe the dynamics of the system between different equilibria caused by changes in the supply of money. Let us assume a Cagan economy in equilibrium with the money stock and hence the price level constant at M_o and p_o respectively. Let us suddenly at time t_1 increase the *level* of the money supply from M_o to M_1, holding the value at M_1 for all time after t_1. Before t_1, π is equal to zero (since all past values of inflation are zero). For the market to clear at t_1, when the money stock is increased, the price level must jump to $p_1 = p_o(M_1/M_o)$. $\dot{p}/p$ is hence undefined at $t = t_1$ and zero for all $t < t_1$ and $t > t_1$. The expected rate of inflation π, which integrates past values of $\dot{p}/p$, will ignore the discontinuity at $t = t_1$, and hence yield 0 for all t. There is no way that at time $t = t_1$, π can become positive, and hence p must jump to clear the market. This may be one of the disadvantages to using the adaptive expectation mechanism. The unsatisfactory dynamics are due to the equilibrium nature of this system and the "jump" in M. No meaningful description of "stability" can be given.

Instead of jumping the level of M, let us increase its growth rate to some positive θ from 0. The dynamics in this case are more reasonable. If b is equal to zero, p will just increase at rate θ. However, if b is positive, π must turn positive and drive p up, for a time, at a faster rate than θ. If the economy is stable and hence π becomes equal to θ, $\dot{p}/p$ will again settle down to θ. Real balances, in the new equilibrium, will be reduced. There is no possibility of cyclical behavior of any variable in the Cagan model.

The full dynamics of the Cagan model have been introduced in this chapter to expose the reader to the basic problem of monetary stability with inflationary expectations. The Cagan model is particularly simple since it is a totally equilibrium model, and behavior out of equilibrium, a very difficult economic topic, need not be described. The models discussed in the following chapters will explore behavior out of equilibrium.

Mathematical Appendix

Chapter 2

The Cagan model postulates:

2A.1 $\quad M/p = L(r_o + \pi), \quad L' < 0,$

2A.2 $\quad \dot{\pi} = b(\dot{p}/p - \pi), \quad b > 0.$

Taking the time derivative of 2A.1 and setting $\theta = \dot{M}/M$ yields

2A.3 $\quad \theta - \dot{p}/p = L'/L\ \dot{\pi}\ ,$

and substituting in 2A.2 gives

2A.4 $\quad \dot{\pi} + b(\pi - \theta)/(1 + bL'/L) = 0,$

which is a first order differential equation in π only. Linearizing 2A.4 around the equilibrium value $\pi^* = \theta$, yields,

2A.5 $\quad \dot{\pi} + b/(1 + bL'/L)\ \tilde{\pi} = 0,$

where $\tilde{\pi}$ represents the deviation of π from its equilibrium value θ. For 2A.5 and hence the Cagan system to be locally stable, the coefficient of $\tilde{\pi}$ must be positive and hence

2A.6 $\quad b < -L/L' = (r_o + \theta)/E(L/r^*_m),$

where $E(L/r^*_m)$ is the elasticity of the demand for money with respect to the money rate of interest at equilibrium. If

2A.7 $\quad M/p = \gamma Y_o e^{-\alpha\pi}$, then

2A.8 $\quad L'/L = -\alpha,$

and hence the coefficient of $(\pi - \theta)$ in 2A.4 is a constant $(= b/(1 - \alpha b))$ and therefore condition 2A.6 becomes a global condition. As $b \to \infty$, 2A.4 reduces to

2A.9 $\quad \dot{\pi} - (\pi - \theta)/\alpha = 0,$

which has a solution

2A.10 $\quad \pi = \dot{p}/p = ce^{t/\alpha} + \theta,$

which is necessarily unstable.

Chapter 3

Comparative Statics of a Keynesian Model

The Cagan economy analyzed in the preceding chapter consists of two assets, commodities and money. One of the important contributions of Keynesian theory was the stress on the substitution between money and financial assets issued by firms.[1] The model presented in this chapter integrates financial, firm-issued securities into the economic system and distinguishes between a stock equilibrium, where assets demanded must equal assets supplied, and a flow equilibrium, where the consumers' desired rate of security accumulation must equal the firms' desired rate of security flotation.

Asset Equilibrium

The assets in the economy described in this chapter consist of government non-interest bearing fiat debt called money and the accumulated capital of individuals. It is assumed that this capital is "intermediated" by firms who issue financial claims or securities against their assets. In equilibrium, the total value of these financial securities will equal the reproduction cost of the capital which backs these claims. These financial securities may be divided between bonds (certain f uture payments) and equity (residual payments). It has been shown, however, that under weak assumptions, that the total valuation of these

[1]Fluctuations in real income, probably the phenomenon most characteristic of Keynesian models, are discussed in Chapter 6.

assets is independent of the division.[2]

At any moment in time, *stock* or *asset* equilibrium is defined as the state where each participant in the economy is satisfied as to the division of his assets between money and financial securities. The real demand for money L and securities K^d will be written in the very general form

$$3.1 \qquad L = L(r+\pi, r, Y),$$

$$3.2 \qquad K^d = K^d(r+\pi, r, Y, p),$$

where $r + \pi = r_m$ is the money rate of interest, r is the real rate of interest, Y is real income, and p is the price level. The nature of these demand functions will be examined later. Equation 3.1 can be transformed into an excess demand equation by simply subtracting the supply of money M/p,

$$3.3 \qquad L^x(r+\pi, r, Y, p) = L - M/p,$$

where L^x is the excess demand for money. The real supply of securities will be indicated by $K = Q/r$, where Q is the (constant) flow of real income from these securities and $1/r$ is the capitalization rate (price) of this flow. Hence,

$$3.4 \qquad K^x(r+\pi, r, Y, p) = K^d - Q/r,$$

where K^x is the excess demand for securities. Because of the wealth constraint

$$3.5 \qquad W = M/p + K(r),$$

we know by Walras' Law that

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[2]See Franco Modigliani and Merton H. Miller, "The Cost of Capital, Corporation Finance, and the Theory of Investment," *American Economic Review*, vol. 48 (June, 1958), pp. 261-97; and Joseph E. Stiglitz, "A Re-Examination of the Modigliani-Miller Theorem," *American Economic Review*, vol. 59 (Dec., 1969), pp. 784-93.

3.6 $K^x + L^x = 0,$

and, in equilibrium,

3.7 $K^x = 0 \Longleftrightarrow L^x = 0.$

It is immediately noted from 3.6 that

3.8 $K_i^x = -L_i^x,$

where i represents the derivative of the excess demand with respect to the i^{th} common variable.

Let us now specify the signs of the derivatives of these excess demand functions. As the expected rate of inflation rises, the opportunity cost of holding money rises and hence we should expect $L_1^x < 0$ and hence $K_1^x > 0$. This means that the demand for securities rises with a rise in inflationary expectations. As the real rate r rises, the supply value of securities will decline and the opportunity cost of holding money will rise so excess demand for securities will certainly rise. Hence $K_r^x > 0$ and $L_r^x < 0$.[3] As real income rises one might expect both an increase in the demand for securities and money. However, one must remember that at any instant, only the proportion of assets can be changed, so one asset can only be increased at the expense of another asset. It is generally accepted that there is an increase in the demand for real balances when income rises and hence individuals will desire to switch from bonds to money. If the rise in income ΔY appears permanent, then this is equivalent to giving individuals securities valued at $\Delta Y/r$. Thus there will be a portfolio imbalance and an excess demand for money. Hence $L_3^x > 0$ and

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[3] A rise in r holding r_m constant will lead to a drop in the supply value of claims but the demand effect appears ambiguous. The supply effect is likely to overwhelm any oppositive demand effect and hence we may specify $K_2^x > 0$ and $L_2^x < 0$. These signs are not important to the analysis.

$K_3^x < 0$. As the price level rises, the supply of real money declines and hence the excess demand for money rises, $L_4^x > 0$, and $K_4^x < 0$. Table 1 summarizes the responses of the demand function to a change in r, π, Y, and p.

	MONEY			SECURITIES		
rise in	Demand	Supply	Ex. Demand	Demand	Supply	Ex. Demand
π	-	0	-	+	0	+
r	-	0	-	+	-	+
Y	+	0	+	-	0	-
p	0	-	+	-	0	-

Table 1
Asset Derivatives

Commodity Equilibrium

Before discussing equilibrium in the commodity market, let us state the production characteristics of this economy. We will initially assume a constant flow supply of product Y produced by a single aggregate production function whose inputs are the economy's resources: land, labor, capital, etc., and whose output is a commodity which can either be used as an input for future production (capital) or consumed (consumption goods).

In the commodity market, *flow* demands for both consumption and saving must be matched with the *flow* supplies of consumption goods and saving for equilibrium to hold. The flow demand for saving is defined as the flow demand for the acquisition of securities and money. The flow supply of saving is the sum of the flow supply of financial securities issued by firms and the increase or decrease in the value of real balances. The flow supply of financial securities must, through the corporate balance sheet, equal the flow demand by firms for real capital assets. It should be noted that some economists, notably Patinkin[4], have treated consumption goods as yet another item in the asset market. In this case consumers choose only among securities, money, and commodities. My analysis, resting on the framework of Josef May's clarification[5] of Patinkin's stock analysis, will consider the flow, not stock, demand for consumption or consumers' goods. This difference is the result of using continuous analysis and helps clarify the distinction between stocks and flows all-too-often confused in economic literature.

For ease in exposition, we shall assume that $\pi = 0$ and hece the expected capital gain or loss on money is nil. Hence, for constant M, the flow demand and supply for saving is identical to that of financial securities. The flow demand for consumption, or "consumption function," will depend on income, the real rate of interest, and wealth,

$$3.9 \qquad C^d = C^d(Y,r,K(r) + M/p).$$

Our investment function, or flow supply of saving or financial securities to the consumer can be written,

$$3.10 \qquad S^s = S^s(Y,r,K(r) + M/p).$$

Because of the budget constraint and the firms production constraint (remember real output is held constant), we can write,

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[4]Don Patinkin, *Money, Interest, and Prices, op. cit.*

[5]Josef May, "Period Analysis and Continuous Analysis in Patinkin's Macroeconomic Model," *Journal of Economic Theory*, vol. 2 (Mar., 1970), pp. 1-9.

$$3.11 \qquad Y = C^d + S^d = C^s + S^s.$$

We immediately have

$$3.12 \qquad C^s = Y - S^s = C^s(Y,r,K(r) + M/p),$$

the flow supply of consumer goods, and

$$3.13 \qquad S^d = Y - C^d = S^d(Y,r,K(r) + M/p),$$

the flow demand for financial securities.

Firms are considered to choose the division of their product between consumer goods and capital or producers' goods according to the investment function 3.10. Because of the production constraint, every decision to produce an investment good and issue a financial security to consumers is a decision not to produce or supply consumer goods and vice versa. Consumers also choose between the amount of income they wish to consume (their flow demand for consumers' goods) and the amount they wish to save (their flow demand for financial securities) according to their consumption function 3.9. Because of the consumers' budget constraint, every decision not to consume is a decision to save, i.e., income must be continuously allocated between consumer goods and financial securities. The fundamental Keynesian insight is that the decision mechanism by which consumers decide on the consumption-saving division is not identical with the producers' decision function by which entrepreneurs decide on the consumer good-investment good decision. In other words, the value of the financial securities producers wish to issue may not be equal to the value desired by consumers.

The excess flow demand for saving or financial securities may be written

$$3.14 \qquad S^x(Y,r,K(r) + M/p) = S^d - S^s.$$

S^x can also be termed the excess of the demand for acquiring financial securities by the consumer sector over the desire to provide such securities by the producing sector. Through the budget constraint and Walras' Law we know that

$$3.15 \qquad C^x(Y,\bar{\ },K(r) + M/p) = C^d - C^s = -S^x,$$

where C^x represents the excess demand for consumption goods over their supply.

Nature of Flow Demands

The signs of the derivatives of these demand functions need now be specified. A rigorous theoretical examination of the consumption function would specify the maximization of an intertemporal utility function subject to the stock and flow constraints mentioned above. In the following discussion we shall rely on the accepted results which rest upon this analysis. One can say that a temporary increase of income brings about a rise in desired consumption, but a rise less than the increment of income since the usual concavity of the utility function would dictate that consumption be "smoothed" over time. From the budget constraint 3.11 we have

$$3.16 \qquad C^d_Y + S^d_Y = 1, \text{ and } C^d_Y, S^d_Y > 0,$$

i.e., consumers wish to acquire an increased flow of both consumers' goods and financial securities. A permanent increase in income, say ΔY, is equivalent to receiving an increase in wealth $\Delta Y/r$. Viewed in this way, the consumer may just consume all the "interest" from his change in wealth and hence $S^d_Y = 0$. However, standard Keynesian analysis assumes that conditions 3.16 hold.

A rise in the real interest rate has two effects. Holding wealth constant, a change in the interest rate has the well-known ambiguous effect on the consumption function. This is the result of the conflicting signs of the substitution and income effect to which a change in the interest rate generates. However, a rise in the real interest rate will produce a fall in the value of securities. A fall in wealth will necessarily raise the desired rate of acquisition of securities and hence lower desired consumption, i.e., $C_3^d > 0$. It is assumed here that this latter effect of a rise in real r swamps the former ambiguous effect (the sign of C_2) and hence leads us to state $C_r^d < 0$. It immediately follows that $S_r^d > 0$, or that a rise in the real rate of interest will unambiguously cause saving to rise.

From our preceding discussion of the wealth effect, it is clear that a rise in the price level, by lowering wealth, will lower consumption demanded. This is the classical real balance effect that was used by Pigou in 1943 to rescue the consistency of the classical macro-system from the challenge of the Keynesian saving-investment inconsistency (non-intersecting functions at any positive rate of interest).[6] It should be noted that this formulation of the consumption function does not distinguish between an increase in wealth due to a decrease in the price level or due to a decrease in the real rate of interest. There is no special or "potent" wealth effect caused by an increase in real balances. In fact, in most developed economies, the total value of securities so overwhelms the money supply that the effect of a percentage change in the real rate of interest on wealth is much greater than a percentage change in the price level. Business cycle analysis also readily demonstrates that the change in the value of securities overwhelms changes in real

[6]A. C. Pigou, "The Classical Stationary State," *Economic Journal*, vol. 53, no. 212 (Dec., 1943), pp. 343-51.

balances. Despite its small empirical effect, we can analytically state that $C_p^d < 0$ and thus $S_p^d > 0$.

As in the case of consumption, specification of an investment function requires a rigorous theory of the behavior of the firm. For the Keynesian mechanism to be operative, such behavior must not be a mere reflection of consumer behavior and hence render the firm trivial. The following analysis is based on the most commonly accepted theory of firm behavior.

An exogenous shift in, say, technology, enabling greater output to be produced will be divided by the firm between consumption goods and investment goods, much like the consumer. Therefore, $S_Y^s > 0$ and $C_Y^s > 0$ with the constraint

3.17 $$C_Y^s + S_Y^s = 1.$$

Probably the most important variable to the firm is the real rate of interest r. Classical and Keynesian analysis stressed how a change in the real rate of interest would change the desired capital stock and hence investment demand. Tobin has stressed that the differential between the market value of financial securities and the reproduction cost of capital is the driving force behind investment and disinvestment.[7] This occurs when the real market rate of interest r deviates from the marginal product of capital r_o. A decrease in the real rate of interest below r_o, the marginal product of capital as determined by the production function, will cause an increase in the value of securities, and hence, with constant reproduction costs of capital, it will pay for a firm to float more securities to purchase physical capital. The opposite occurs when the real rate rises. However, a change in the price of factor inputs changes only

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[7] James Tobin, "A General Equilibrium Approach to Monetary Theory," *Journal of Money, Credit, and Banking*, vol. 1, no. 1, (Feb., 1969), pp. 15-29.

the desired ratio of factor employment and does not yield a *rate* of accumulation or decumulation of these inputs. To achieve a finite rate of investment, therefore, other constraints must be added. Eisner and Strotz[8] have stressed the cost of adjustment of the level of inputs as one explanation yielding a finite rate of accumulation of producers' goods. Profs. Franco Modigliani[9] and Charles Bischoff[10] have used a putty-clay production function with embodied technological progress to derive an investment function. The above explanations all demonstrate an inverse relationship between the real rate of interest and acquisition of capital.[11] We can thus conclude that the sign of S_r^s is negative and hence C_r^s is positive, i.e., producers will shift to the production of capital goods and issuance of financial securities and away from the production of consumption goods as the real interest rate falls.

The effect of a change of p on the consumption goods/investment goods decision is more difficult to determine. It might be reasonably claimed that lower real balances might discourage firms from expanding their real

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[8] Robert Eisner and Robert Strotz, "Determinants of Business Investment," in *Impacts of Monetary Policy*, The Commission on Money and Credit, (Englewood Cliffs, N.J.: Prentice Hall, Inc., 1963), pp. 59-337.

[9] Franco Modigliani, Albert Ando, Robert Raasche, Stephen Turnovsky, "On the Role of Expectations of Price Changes and Technological Change in an Investment Function," forthcoming.

[10] Charles W. Bischoff "A Study of Distributed Lags and Business Fixed Investment," unpublished Ph.D. dissertation, Massachusetts Institute of Technology, 1968.

[11] Another model yielding finite investment rates is that preferred by D. Foley and M. Sidrauski (*Monetary and Fiscal Policy in a Growing Economy*, *op. cit.*)which postulates a two-sector economy where the relative price of producers' to consumption goods, p_k, determines the supply response of the investment goods sector. In my model, consisting of only one sector, this effect is impossible. There is only one price for both consumers' and producers' goods, so relative prices cannot change. However, p_k in their model is quite similar to 1/r, the price of securities, in my model.

capital, hence $S_p^s < 0$ and $C_p^s > 0$, so that a rise in the price level will discourage the issuance of financial securities by firms as they shift toward the production of consumer goods.

It is now necessary to specify the sign of the derivatives of the excess demand functions for the acquisition of securities and consumption goods. Certainly $S_r^x = S_r^d - S_r^s > 0$ and $S_p^x > 0$. The sign of S_Y^x is ambiguous. If the marginal propensity to consume out of real income $C_Y^d = 1 - S_Y^d$ plus the marginal propensity to invest $S_Y^s = 1 - C_Y^s$ is greater than unity, then $S_Y^x = 1 - (C_Y^d + S_Y^s)$ is negative. Similarly, if the marginal propensity to spend out of real income (consume plus invest) is less than one, $S_Y^x > 0$. Table 2 summarizes the response of these functions to changes in variables.

	CONSUMPTION			SAVING		
rise in	Demand	Supply	Ex. Demand	Demand	Supply	Ex. Demand
Y	+	+	?	+	+	?
r	-	+	-	+	-	+
p	-	+	-	+	-	+

Table 2
Flow derivatives

It is sometimes useful to examine the demand and supply function for *all* commodities, both consumers' and producers' goods. Total demand for commodities, D, is thus defined as

$$3.18 \quad D(Y,r,K(r) + M/p) = C^d + S^s.$$

We can see unambiguously that $D_Y > 0$ and equals the marginal propensity to spend; $D_r < 0$, and $D_p < 0$. Since the supply of good, Y, is assumed to be exogenously determined, excess demand, denoted by F, is simply

3.19 $$F(Y,r,K(r) + M/p) = D - Y.$$

The sign of F_Y is ambiguous and depends on whether the marginal propensity to spend exceeds or falls short of unity.

It is easily shown that

3.20 $$F = C^x = -S^x, \text{ since}$$

3.21 $$F = C^d + S^s - Y = C^d - C^s + C^s + S^s - Y = C^x.$$

Therefore, the excess demand for commodities is equal to the excess demand for consumer goods or the excess supply of financial securities.

Static Equilibrium

Now that all the functions have been specified, we can establish the equilibrium in the economy. For such an equilibrium to exist, both the excess demand for each asset and for commodities must equal zero. In short, an economy is in equilibrium when

3.22 $$L^x = -K^x = 0, \text{ and}$$

3.23 $$F = C^x = -S^x = 0.$$

Totally differentiating the equilibrium conditions holding real income and the expected rate of inflation constant, we note the set of equilibrium points in the asset market satisfy the condition

3.24 $$dr/dp)_A = -L_p^x/L_r^x > 0,$$

and the flow market,

3.25 $$dr/dp)_F = -F_p/F_r < 0.$$

Figure 1 qualitatively sketches these equilibrium points for a given real income and expected rate of inflation.

Condition 3.22-23 determines an equilibrium r* and p* which simul-

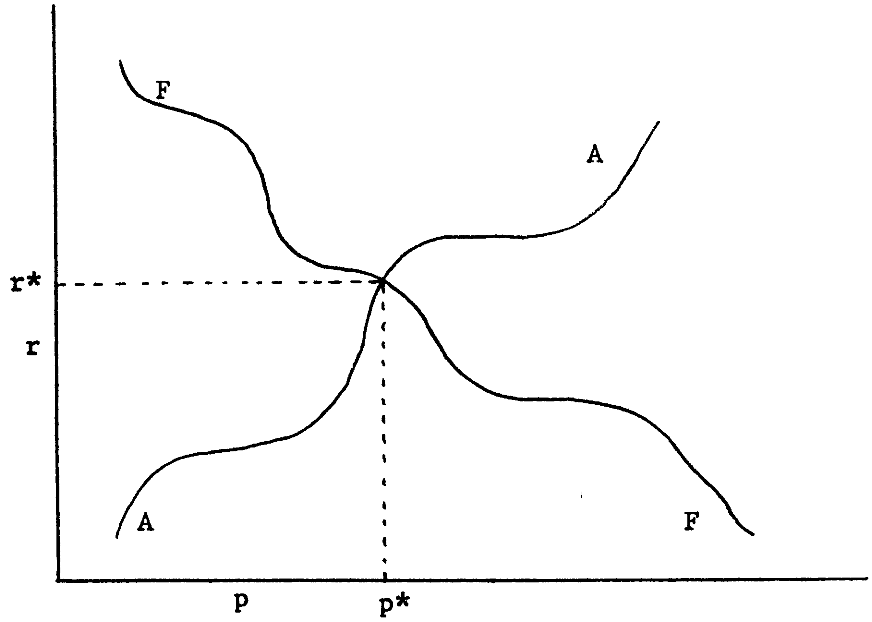

Figure 1
Static Equilibrium.

taneously clear both the asset and commodity market.[12] If the wealth effect is absent from the commodity market, i.e., $F_p = 0$, F becomes horizontal at r*. The price level is still determinate, of course, due to the upward sloping AA curve. In this case the equilibrium interest rate is determined solely in the commodity market. r* is then the result of the interaction of the productivity of capital combined with the intertemporal consumption behavior of individuals. The excess demand for money function determines the price level only. A horizontal FF curve thus means that the economy is dichotomized, i.e., neither the supply nor the demand for money affects any equilibrium values of the real variables (except real balances).

Neutrality of the money stock, i.e., the price level proportional to the stock of money, is obtained in equilibrium for a general F function.

[12]Intersection is assured, as Pigou noted, as long as the wealth effect is operative in the commodity market.

Due to the real balance effect, the nature of the demand for money or liquidity preference will play a part in determining the equilibrium real interest rate, i.e., r* is not necessarily independent of (M/p)*. However, the *supply* of nominal money is irrelevant to the determination of the equilibrium real rate. The price level is strictly proportional to the stock of money, as the rigid quantity theory claims, in long-run equilibrium.

Some economists,[13] employing intertemporal maximization techniques to the formulation

$$\text{3.26} \qquad \max \int^{\infty} U(c_t, m_t)e^{-\rho t}dt,$$

where c_t is consumption at time t, m_t is the services from a quantity of real balances (assumed proportional to the stock of real balances), and ρ is the subjective time rate of discount, have arrived at the conclusion that the long run real interest rate must equal ρ no matter what form the utility function takes. This would indicate that the economy is dichotomized in the long run and that the same quantity of capital is accumulated regardless of the quantity of real money. The FF curve is hence horizontal at ρ, the subjective rate of time preference.

It has been noted that the equilibrium curves plotted in Figure 1 hold real income and the expected rate of inflation constant. An upward shift in real income will move the AA curve upward, i.e., a higher interest rate must prevail for any given price level. This is due to our assumption that money is a superior asset with respect to an income

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[13] Miguel Sidrauski, "Rational Choice and Patterns of Growth in a Monetary Economy," *American Economic Review*, vol. 57, (May, 1967), pp. 534-44; and Paul A. Samuelson, "What Classical and Neo-classical Monetary Theory Really Was," *Canadian Journal of Economics*, vol. 1, no. 1, (Feb., 1968), pp. 1-15.

change. If the marginal propensity to spend is less than one, the FF curve will move leftward since at a given real interest rate a lower price level is required to absorb the excess supply of commodities. If the marginal propensity to spend is greater than one, the FF curve will move rightward. Therefore, an exogenous shift upwards in real income, say, has an ambiguous effect on both the price level and the real rate of interest. If the propensity to spend is unity, the equilibrium price level will fall and interest rate will rise (or at least remain constant).

An exogenous shift upwards in the expected rate of inflation will not affect the FF curves, since commodities are demanded and supplied only in real terms.[14] The asset market will certainly be affected, and the AA curve must shift downward so that at any given price level a lower real rate of interest must prevail. Hence the equilibrium price level must rise and the real rate of interest fall (or at least remain constant).

The only classes of assets considered in this model are government non-interest-bearing fiat money and claims on real physical assets, i.e., financial securities. Interest-bearing government debt has not been introduced as an additional asset. The question of whether interest-bearing government debt affects the equilibrium level of the variables depends on whether the future tax payments on the debt are completely discounted by individuals or not. If the maturity of the debt is greater than the lifetime of an individual, and we assume that individuals maximize some intertemporal utility function over their own lifetime, the interest bearing debt will, in part, appear as a net asset to these individuals,

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[14]If we consider real income as $\hat{Y} = Y_0 - \pi(M/p)$, where Y_0 is the constant real product, and $\hat{Y}$ includes the expected gain or loss on real money balances, then an upward shift of π may result in an excess supply of commodities and hence a leftward shift in the FF curve.

and will therefore affect their demands for securities, commodities, and hence all the equilibrium values of the system. In this case, creation of money by open market operations instead of deficit financing cannot be treated indifferently. These effects have been rigorously examined by Patinkin[15] and earlier by Metzler.[16] The model described in this chapter therefore only deals with the case of totally discounted interest bearing debt or no such debt at all.

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[15]Don Patinkin, *Money, Interest, and Prices*, especially pp. 288-94.

[16]Lloyd Metzler, "Wealth, Saving, and the Rate of Interest," *op. cit.*

CHAPTER 4

DYNAMIC ADJUSTMENT OF A KEYNESIAN MODEL

Some of the most interesting aspects of an economy, and most important from the standpoint of stabilization policy, are not the equilibrium values of the economic system but the *adjustment* of the system to these equilibrium values. Patinkin, as far back as 1947, claimed that the Keynesian aspects of unemployment was a disequilibrium phenomenon. He noted,

> "For the real significance of the Keynesian contribution can be realized only within the framework of *dynamic* economics. Whether or not an underemployment equilibrium exists; whether or not full employment equilibrium always will be generated in a static system -- all this is irrelevant. The fundamental issue raised by Keynesian economics is the *stability of the dynamic system*: its ability to return automatically to a full-employment equilibrium within a reasonable time (say, a year) if subjected to the customary shocks and disturbances of a peacetime economy."[1]

At the same time, Milton Friedman recognized the importance of the dynamic adjustment process for describing an economy,

> "... relative stability ... depends in addition on the number and magnitude of the disturbances to which the economy is subject, the speed with which the equilibrating forces operate, and the importance of such disequilibrating forces as adverse price expectations."[2]

Unfortunately, there is no simple, completely satisfactory way to specify the behavior of either the asset or commodity market out of equilibrium. Any exogenous shift of the functional forms, or the nominal supply of money or securities might change all the endogenous variables that affect the excess demand functions. An attempt is made in this chapter

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[1] Don Patinkin, "Price Flexibility and Full Employment," *op. cit.*, pp. 562-63.

[2] Milton Friedman, "A Monetary and Fiscal Framework for Economic Stability," *op. cit.*, pp. 261-62.

to isolate the most important variables which respond to a disequilibrium in either the asset or commodity market.

Keynes in his *General Theory* stresses the interest rate adjustment to disequilibrium in the money market and income adjustment to disequilibrium in the commodity market. The following analysis will discuss the Keynesian adjustment process assuming constant real income but including the phenomenon of inflationary expectations. In the next chapter we shall outline a classical or Fisherian model of adjustment and also endogenize real income in the spirit of Keynes.

Asset Market

The Keynesian model assumes that the real interest rate responds to excess demand in the asset market. This implies that the money interest rate adjusts to bring about equilibrium through changes in the real rate of interest. Inflationary expectations are here assumed to be determined by phenomenon exogenous to the asset market and hence are not affected by disequilibrium in this market. In general, we shall specify

$$4.1 \qquad \dot{r}_m)_\pi = \dot{r} = G(L^x), \quad G' > 0, \; G(0) = 0,$$

where G is any general monotonic function and $\dot{r}_m)_\pi$ is the rate of change in the money rate of interest holding π constant.[3] If $G' = \infty$, then r instantaneously adjust to clear the asset market.

For an example of this process let us assume that the monetary authorities inject M into an economy that was previously in asset equilibrium. This produces an excess supply of money and demand for securities. Indi-

[3]In Chapter 5, 4.1 will be generalized to include the state of excess demand in the commodity market as an argument of G.

viduals will attempt to sell their money for securities to bring about their original portfolio balance. This results in a bidding up of the price of securities and hence lowering of the interest rate. Of course, each individual's original attempt to reduce his money balances must fail on the average, but, because of the lower interest rate, individuals become satisfied with their new portfolio balance between money and securities. In other words, at the lower interest rate individuals now desire to hold just the extra amount of money the monetary authorities have created. Note that the asset equilibrium reached in the Keynesian model described above is not a full equilibrium since at the new interest rate the commodity market does not clear. However, it is precisely this short-run adjustment that Keynes had in mind when describing the liquidity preference schedule and the paths of monetary policy.

As we shall examine later in the Fisherian model, this is not the only path of adjustment available. The excess money may be used to bid up the price of commodities, and the subsequent rise in prices will lower the real quantity of money and result in the new equilibrium. This would be a full equilibrium in which both the asset and commodity market are cleared. Certainly at the formal level, excess demand in any market will affect prices of goods in other markets. However, there is overwhelming empirical evidence that suggests that in the short run, interest rate changes do equilibrate the asset market, and it may not be unreasonable to assume, due to the relatively perfect nature of financial markets, that asset prices adjust very rapidly to equilibrate supply and demand.

A more interesting, and more difficult example, involves an exogenous shift in the expected rate of inflation. As mentioned above, although π may be endogenous to the entire macro-model, it is assumed exogenous to the asset market. To analyze the effect of a shift in π, let us specify an extremely simple demand for money equation which is always in equilibrium, i.e., $G' = \infty$,

$$4.2 \qquad M/p = L(r + \pi), \quad L' < 0.$$

For a given M and p, the equilibrium *money* rate of interest is independent of the expected rate of inflation. However, the real rate of interest will decline by the exact magnitude that the expected rate of inflation rises and vice versa. A more general specification of the money demand function may be written

$$4.3 \qquad M/p = L(r + \pi, r), \quad L_1, L_2 < 0.$$

This indicates that the demand for money in inversely proportional to the real rate *given* the money rate. This would arise if there is some money held in proportion to the value of non-monetary assets, the latter being capitalized at the real rate of interest. In this case the real rate of interest will change in the opposite direction by something less than the change in the expected rate of inflation. Hence, the money rate of interest will rise as the expected rate of inflation rises and vice versa.

For a non-instantaneously clearing asset market, however, an exogenous rise in π will, in the first instant, bring about a money rate of interest higher by the expected rate of inflation. This, however, will cause an excess supply of money and drive down the real rate until the market is again cleared. In the case of fast-adjusting asset markets, however, the reader should be warned about concluding that the market rate of interest fluctuates in a one-for-one relationship with the expected rate of inflation. Such will only be true in steady state equilibrium in a dichotomized economy.

Commodity Market

In the Keynesian system with constant real income, disequilibrium in the commodity market manifests itself primarily by changes in the price level. However, the exact dynamics by which prices are formed when trading is completed out of equilibrium is not widely agreed upon.

Prof. Kenneth Arrow developed a theory in 1959[4] that one cannot view the firm as a perfect competitor when it is faced with the circumstances of trading out of equilibrium. In fact, the firm becomes both a monopolist and a monopsonist under these circumstances, setting its own price to maximize (expected) profits. Prof. Arrow states

> "Consider the case where demand exceeds supply and sellers are led to behave as monopolists. The existence of this excess both for the particular seller under consideration and for his competitors enters into the determination of the seller's *anticipated demand curve*. Given this, he sets his price so as to equate anticipated marginal revenue (possibly discounted in some form for uncertainty) to marginal cost."[5] [emphasis added]

The notion of the anticipated demands in the firm's price setting decision will be summarized in this chapter by the expected rate of inflation π, the expected rate of shifting of demand curves.

We shall now specify the general dynamics of price formation as

$$4.4 \qquad \dot{p}/p = G(F;\pi), \quad G_1 > 0, \quad G(0;0) = 0.$$

Again, we are indicating that excess demand in the asset market does not impinge upon the price level. G_1 is positive since prices should rise when there is excess demand for commodities. If $G_1 = \infty$, prices instantaneously adjust and the commodity market is in constant equilibrium.

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[4] Kenneth J. Arrow, "Towards a Theory of Price Adjustment," in *The Allocation of Economic Resources*, (Stanford: Stanford Univ. Press, 1959), pp. 41-51.

[5] *Ibid.*, p. 48.

If $G_2 \equiv 0$, and prices only respond to excess demand, we shall refer to the adjustment mechanism as *Walrasian*. Let us now define a class of excess demand functions F. Any excess demand function F will be termed *(steady-state) consistent* if, for any given constant rate of monetary growth $\dot{M}/M = \theta$,

$$4.5 \qquad \lim_{t\to\infty} F(\ ) = 0.$$

In other words, for a consistent excess demand function, the commodity market will eventually reach equilibrium (if the system is stable) for any constant rate of monetary growth. This property is similar to the previously defined consistent expectation mechanism.

If the money supply is growing at rate θ, we know that the equilibrium rate of growth of prices equals θ. If both the expectations mechanism and the excess demand function are consistent, we know that

$$4.6 \qquad \theta = G(0;\theta)$$

for all θ. This *suggests* that the G function can be written in a separable form

$$4.7 \qquad G = H(F) + \pi, \quad H(0) = 0, \; H' > 0.$$

Under this specification, the expectation mechanism is consistent if and only if the excess demand function is consistent. We shall use specification 4.7 often in our dynamic models and refer to it as the *consistent separable price formation mechanism*.

The presence or absence of steady-state consistency is an extremely important characteristic of any model. If inflationary expectations are assumed to equal zero, then the excess demand function must be inconsistent. There have been some steady-state growth models, notably by Stein[6]

[6]Jerome L. Stein, "'Neoclassical' and 'Keynes-Wicksell' Monetary Growth Models," *Journal of Money, Credit, and Banking*, vol. 1, no. 2, (May, 1969), pp. 153-71.

which have incorporated steady-state inconsistent excess demand functions. These are often termed "Keynesian" or "Keynes-Wicksell" growth models. If the money supply and hence prices, are growing at rate θ, then chronic excess demand is the cause of the inflation. There is thus a chronic excess supply of securities issued by firms over their desire to be absorbed by consumers. In Keynesian terms, *ex ante* saving continually falls short of *ex ante* investment. With constant real income, this phenomenon in a Keynesian model arises from a constant shortfall of the real rate of interest r below the marginal product of capital r_o. If real income is endogenous, and hence labor employment is allowed to fluctuate, a Keynes-Wicksell growth model will exhibit a permanent "Phillips Curve," or trade-off between employment and inflation. If the excess demand function is consistent as defined above, a vertical Phillips curve exists and there is no trade-off in the steady-state.

CHAPTER 5

FULL MODELS OF DYNAMIC ADJUSTMENT

In this chapter we shall describe two major types of dynamic models. The Keynesian model with constant real income, developed extensively in the last two chapters, specifies that disequilibrium in the asset market results in changes in the interest rate and disequilibrium in the commodity market causes changes in the price level. The Fisherian model, developed at the end of this chapter, specifies that disequilibrium in the money-commodity relationship causes changes in the price level. The Cagan model developed in Chapter 2 will be shown to be a special case of the Fisherian model.

Keynesian Model

The general Keynesian model with constant real income Y_0 that we shall consider is the combination of the adjustment mechanisms described in the previous chapter with the adaptive expectation mechanism. Specifically, we shall write system K1 as

$$\text{K1)}\quad \begin{array}{llcll} 5.1 & \dot{r} & = & \alpha(L(r+\pi) - M/p), & L' < 0 \\ 5.2 & \dot{\pi} & = & b(\dot{p}/p - \pi), & b > 0, \\ 5.3 & \dot{p}/p & = & F(r,M/p) + \pi\,, & F_1 < 0,\ F_2 > 0. \end{array}$$

The price formation mechanism is of the *separable* variety discussed in the previous chapter, where the monotonic H function is incorporated into the excess demand function F. Note that the arguments of the excess demand function have been simplified to the real rate of interest and the real balance effect. The wealth effect, induced by changes in the real rate of interest, is now included in the first argument. As long as b is strictly positive, the expectation mechanism is consistent and hence the excess demand function is consistent. When $b = 0$, i.e., inflationary expectations are totally absent, both the expectation mechanism and the excess demand function are inconsistent.

The stability of system K1 with $b = 0$ was first examined by Metzler[1] and later by Patinkin[2] in a slightly different framework. Both found the system to be locally stable around the equilibrium values $r = r^*$ and $p = p^*$ as long as the derivatives had the signs indicated. However, Metzler recognized that system K1 with $b = 0$ did not completely describe the nature of the dynamic adjustment mechanism as indicated in the introductory chapter.[3]

Let us describe the equilibrium of system K1 when b equals zero. Assume the economy is in a stationary equilibrium with $r = r^*$ and $p = p^*$ so that $F = 0$. Then allow the money supply to increase at rate θ. The excess supply of money forces the interest rate down (the real and nominal interest rates are equal at the first instant since $\pi = 0$), and produces excess demand, causing prices to rise. The new moving equilibrium is characterized by $(\dot{p}/p)^* = \theta = \dot{M}/M$, a lower interest rate and hence a higher equilibrium real balance. This result is contrary to the conclusions of all money-growth models and empirical evidence.

Instead of equation 5.3, let us introduce a simple Walrasian adjustment process, i.e.,

5.3a $\qquad \dot{p}/p = F(r, M/p), \quad F_1 < 0, \quad F_2 > 0,$

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[1] Lloyd Metzler, "Wealth, Saving, and the Rate of Interest," *op. cit.*, pp. 114-16.

[2] Don Patinkin, *Money, Interest, and Prices*, Appendix 8.

[3] It is interesting to note that this model holds M and Y constant and lets p and r "float" to equilibrate markets, while Samuelson's 1939 multiplier-accelerator model held p and r constant and let Y "float" to equilibrate markets and M "float" to keep r constant.

but retain 5.1 and 5.2 with b strictly positive. For simplicity consider the case where $F_2 = 0$, although this is not necessary to the following conclusions. In this situation a change from a stationary money supply to one rising at rate θ has an ambiguous effect on the market rate of interest. The new equilibrium will be marked by $\pi^* = \theta$, and $r^* < r_o$, where r_o is the real rate which clears the commodity market, i.e., $F(r_o) = 0$. The new equilibrium money rate of interest $r^*_m = r^* + \pi^*$ may be above or below the original equilibrium interest rate r_o. If it is above r_o, the new equilibrium real balance will be lower than the original and vice versa. In any case, chronic excess demand is exhibited in the steady-state and thus F is inconsistent.

When b is positive and the price adjustment mechanism is written in the separable form 5.3, the equilibrium response of the money interest rate to an increase in θ is unambiguous. When the real balance effect is absent from the excess demand function, the equilibrium real rate of interest is independent of θ. The equilibrium money rate equals $r_o + \theta$ and the equilibrium real balance $(M/p)^*$ is reduced. This is the result if the economy is dichotomized. If not, and the real balance effect is present in the commodity market, the new equilibrium r^* will be a declining function of θ. A decline in real balances due to an increase in θ will stimulate saving in securities, increase the capital stock, and hence move the economy along the factor price frontier. The rate of growth of the money stock may then have non-neutral effects on the economy -- changing the equilibrium real rate of interest, the capital stock, and hence income. However, in most advanced economies, these changes in the real rate are apt to be very small since the value of securities far exceeds any inflation-induced changes in real balances.

As we noted in Chapter 3, some intertemporal analysis concludes that the equilibrium capital stock and real rate of interest are independent of the rate of growth of the money stock, without denying the real balance effect. This does not indicate that economic welfare in independent of θ. The next chapter on economic stabilization will discuss this phenomenon in more detail.

Mathematical Examination of the Keynesian Model

System K1 is a third order, non-linear differential system. However, it may be linearized around the equilibrium values of $(M/p)^*$, r^*, and $\pi^* = \theta$ to be examined for local stability properties. It is useful for descriptive purposes to reduce system K1 to a second order system. This can be done by the common assumption that the speed of adjustment in the asset market is infinite, i.e., asset equilibrium is attained at all points of time. This assumption is not particularly restrictive, in view of the rather perfect nature of financial markets. The assumption $\alpha = \infty$ implies that individuals are always on their demand curve for money.[4] Prices of assets adjust, both downward and upward so quickly, especially compared to commodity prices, that it is difficult to imagine excess demands for any marketable asset as persisting more than a short period of time.

Under the assumption $\alpha = \infty$, system K1 reduces to

5.4 $$M/p = L(r+\pi), \quad L' < 0,$$

K2) 5.5 $$\dot{\pi} = b(\dot{p}/p - \pi), \quad b > 0,$$

5.6 $$\dot{p}/p = F(r, M/p) + \pi, \quad F_1 < 0,\ F_2 > 0.$$

[4]This does not necessarily imply that the demand curve is not changing over time. The analysis here, however, considers functional forms invariant over time.

The characteristic equation of the linearized version of system K2 is derived in the mathematical appendix as

5.7 $\quad \lambda^2 + \lambda[L/L'\cdot(F_1 + F_2L') + bF_1] + b(F_1 + F_2L')\cdot L/L' = 0.$

A necessary and sufficient condition for the real parts of the characteristic roots to be strictly negative is that all coefficients of λ^i be positive. The signs of the derivatives indicate that the constant term is always positive, so local stability requires that

5.8 $\quad b < -L/L' - LF_2/F_1.$[5]

Note that in the absence of the real balance effect in the commodity market, i.e., $F_2 = 0$, 5.8 reduces to the Cagan stability condition 2.7! This surprising result will be shown later to hold under still more general conditions. The second term on the right side of 5.8 can be considered the marginal effect of increased real money on the rate of inflation (LF_2) divided by the marginal effect of an increased real interest rate on the rate of inflation. If the pure interest rate effect (holding wealth constant) on excess commodity demand is zero, it is demonstrated in the appendix that 5.8 can be written

5.9 $\quad b < -r_m^*(1/E(L/r_m^*) - R),$

where $E(L/r_m^*)$ is the interest elasticity of the demand for real balances at the equilibrium market interest rate r_m^*, and R is the ratio of the value of real balances to securities, $(M/p)/K(r)$. R is very small (less than .10) for most developed economies, while the interest elasticity for the demand for money is usually found to be under one. Therefore, even when the real interest rate affects demand only through wealth, the real balance

[5]It is important to consider the units of the variables and derivatives carefully. b is measured in per unit time, so is F_2 and $1/L'$. F_1 and L are time independent.

effect F_2 has a negligible effect on the stability condition. This analytical evidence supports Hicks' presumption quoted in the first chapter that price expectations, once they become operative, can swamp the real balance effect. Hence, for mathematical tractability, we shall hereafter consider $F_2 = 0$ unless otherwise specified.

If $F_1 = -\infty$, i.e., the response in the price level to a change in the real interest rate becomes infinite, then the real rate is fixed at r^*, and the characteristic equation 5.7 reduces to

5.10 $$\lambda(L/L' + b) + bL/L' = 0.$$

This is the same differential equation derived from the linearized Cagan equations, 2A.5. The Cagan stability condition can easily be noted since the coefficient of λ must be negative.

When $F_2 = 0$, the characteristic equation of K2, 5.7, can be written as

5.11 $$\lambda^2 + \lambda F_1(L/L' + b) + F_1 bL/L' = 0.$$

Figure 2 is a phase diagram derived in the mathematical appendix of the

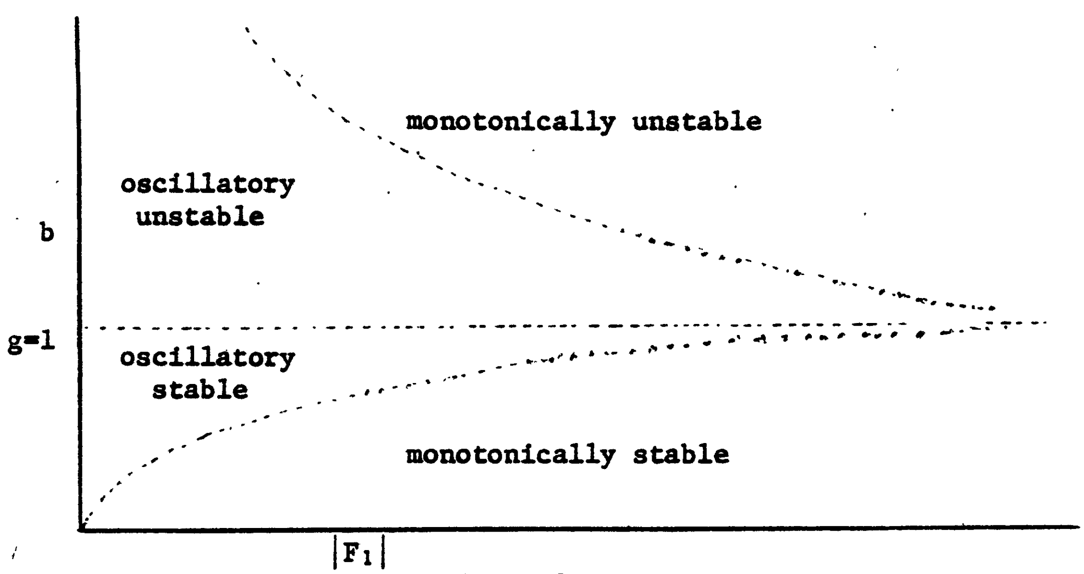

Figure 2

Phase Diagram of Model K2

economic system represented by 5.11. On the horizontal axis is $|F_1|$, the strength of the response of prices in the commodity market to changes in the real rate of interest. On the vertical axis is b, normalized as g multiples of $-L/L'$, the Cagan critical value. $g = 1$ ($b = -L/L'$) represents the dividing line between regions of stability and instability. The phase curves indicate that the slower the response of the commodity market to changes in the real rate of interest, the more likely the system is oscillatory. The faster the response, the more likely the system behaves in a monotonic manner. The limit as $F_1 \to -\infty$ is the Cagan model where the dynamics are always monotonic.

The case where the asset market does not instantaneously adjust, i.e., α is strictly less than infinity, is derived in the appendix. It can be seen that the lower the α the more likely the system is unstable for a given b. Mathematically,

5.12 $$\partial b^*/\partial \alpha > 0,$$

where b^* is the maximum value of b for which the system is still stable given any α.

Descriptive Aspects of the Keynesian Model

A verbal description of system K2 yields a more intuitive feeling for the stability properties derived above. Let the economy be in a stationary equilibrium path, $p = p^*$, $r = r^*$, $\pi^* = 0 = \dot{M}/M$. Then allow the monetary officials to inject an additional quantity of money, ΔM, into the system. The interest rate, determined only in the asset market, instantaneously responds to the excess supply of money and falls. Since, at the first instant, $\pi = 0$, all the drop in the money rate of interest is reflected by a drop in the real rate of interest r. The lower real rate r now feeds into the commodity market by increasing commodities demanded, forcing the prices of commodities upwards.

Since the asset market is always cleared, the rising price level must be associated with a rising *money* rate of interest $r + \pi$. However, since p is rising, π, the expected rate of inflation, must turn positive. Since the money interest rate, which clears the asset market, is the sum of the real rate of interest and the expected rate of inflation, it is not certain that the real rate r is in fact rising. But the real rate r must eventually rise in order to decrease excess demand and hence halt the inflation. Of course, the question of whether the real rate rises depends primarily on b and the elasticity of demand for money. If b is large, so that the expected rate of inflation quickly responds to actual inflation, and the elasticity of the money demand schedule with respect to the market interest rate is large so that decreased real balances (through rising prices) will bring only a slight increase in the money rate of interest, then prices might well rise explosively, i.e., the economy will experience a self-generating inflation with a constant money stock.

Of course, even if the real rate does rise, stability is not assured since the process could oscillate with ever greater amplitude. The major conclusion is that the rise in the money rate of interest which accompanies price rises may not reflect a rise in the real rate of interest, which affects spending. This of course has major implications for policy decisions and adds support to those who warn that interest rates appearing in the market are not the best indicator of the stance of monetary policy.

Starting again from a stationary equilibrium with a constant quantity of money, let there be an exogenous rise in π, the expected rate of inflation, which persists for some period of time before becoming adaptively formed.

A rise in the expected rate of inflation will cause inflation for two reasons. First, π is one of the components of the price formation mechanism in the model caused by the monopolistic behavior of firms out of equilibrium, and second, the real rate r must drop by the same magnitude as the rise in the expected rate of inflation for the money market to be in equilibrium. This will cause excess demand in the commodity market which is the other component of price formation. In this model, the mere expectation of inflation causes inflation and demonstrates the importance that exogenous and endogenous shifts in these expectations have on macrodynamics.

As our final example, let us examine the effects of changes in the *rate* of growth of the money stock, θ. Starting from a stationary equilibrium, the monetary authorities suddenly allow the money stock to grow constantly at rate θ. In the long run we have shown that p also grows at rate θ and the money interest rate is (approximately) equal to the original real rate plus θ. Thus real balances $(M/p)^*$ are smaller in the new equilibrium. This implies that the rate of growth of prices must have averaged more than θ during the disequilibrium period to eliminate the excess real balances. Another way of looking at this is by noting that the real rate of interest will remain depressed below the equilibrium rate even when prices have risen sufficiently to restore the original real balances. This is due to the creation of inflationary expectations. Hence prices must rise even faster than θ to restore r to its equilibrium value. A shift in the rates of growth of the nominal stock of money is much like the exogenous change in the money supply level, and may bring about a cyclical return to the new equilibrium. This is the type of adjustment regarded by

Friedman as paramount to the theory of cycles as noted in the introductory chapter.

Endogenous Real Income

Up to now we have considered real income Y to be exogenously given, specifically from the interaction of fixed inputs in our aggregate production function. Of course, a key element in any economic cycle is precisely fluctuations in real income, which is often thought to be socially more important than cycles of the price level, the real, or the money rate of interest. Keynes' *General Theory* presents the first theory of endogenous real income based on the theory of effective demand. Our theory of excess demand, resulting from disequilibrium in the commodity market, will be shown to be analogous to Keynes' theory of effective demand.

To treat real income endogenously we must refrain from considering a world of perfect certainty where perfect competition reigns perpetually in both product and factor markets. We have already mentioned that when trades are made out of equilibrium, the firm ceases to be a perfect competitor. A slight variation of the Arrow model mentioned in Chapter 4, developed primarily by Phelps[6], and Phelps and Winter,[7] treats the firms *in the short run* as both wage setters and price setters. Any price or wage offer different from the equilibrium price will not bring about an instantaneous loss or gain of all customers or employees, as postulated in

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[6]Edmund S. Phelps, "Money Wage Dynamics and Labor Market Equilibrium," in *Microeconomic Foundations of Employment and Inflation Theory*, ed. E. S. Phelps, (New York: W. W. Norton & Co., 1970), pp. 124-66.

[7]Edmund S. Phelps and Sidney G. Winter, Jr., "Optimal Price Policy under Atomistic Competition," *Ibid.*, pp. 309-37.

a world of perfect competition. Instead, firms will experience a flow of customers or employees towards or away from the firm when a non-equilibrium price is set. In the long-run firms must set the equilibrium price and wage but may differ in their "going" offers in the short run.

In this world, as Phelps and Winter go on to prove, an increase in demand will often be accompanied by a short-run rise in employment and prices and possibly even real wages. In the Arrow model, firms are also "monopolists" in the short run (in disequilibrium) and changes in output and employment are also possible. The Phelps model also supplies a rationale for the addition of π to the price formation equation. As in the Arrow model, in the short run firms are price setters and they will change their product price in response to two types of information: real demand for their product and the expected rate of increase of the general price level. As long as firms expect prices to be rising at some fixed rate, they will continually raise their product price by that rate, knowing that the real demand for their product is unchanged.

There are many other theories of why income rises in response to excess demand. Some rely on costs of information[8], others on the absence of an intertemporal auctioneer,[9] and some on the stickiness of prices and/or wages. All these theories have the property that excess demand will elicit higher employment and output. We shall therefore specify that the deviation from the equilibrium level of real income Y* responds to excess demand in the commodity market so that

5.13 $$Y = G(F(Y,r,M/p)), \quad G' > 0, \; G(0) = 0.$$

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[8]Armen A. Alchian, "Information Costs, Pricing, and Resource Unemployment," *Ibid.*, pp. 27-52.

[9]A. Leijonhufvud, "Keynes and the Keynesians: a Suggested Interpretation," *American Economic Review*, vol. 57, no. 2, (May, 1967), pp. 401-10.

We can now modify our system to endogenize real income Y. System K2 can be written

$$
\text{K3)}\quad
\begin{array}{lrcl}
5.14 & M/p & = & L(r+\pi, Y), \quad L_1 < 0,\ L_2 > 0 \\
5.15 & \dot{\pi} & = & b(\dot{p}/p - \pi), \quad b > 0 \\
5.16 & \dot{p}/p & = & F(r) + \pi, \quad F_1 < 0 \\
5.17 & Y & = & G(r), \qquad G' < 0.
\end{array}
$$

For analytical simplicity we assume $F_2 = 0$ as before. Also we assume that the marginal propensity to spend is equal to one and hence Y is absent from the excess demand function. (Otherwise, we may solve for Y in terms of r from 5.13 and substitute). Note that income now appears as a variable in the demand for money function, and $\partial L/\partial Y$ is assumed, as usual, to be positive. Since Y is only a function of the real rate of interest, L can be rewritten as a function of both the money and real rate of interest. A fall in the real rate of interest now has a more "potent" effect on the demand for money since not only does the money rate fall but income rises and both effects increase money demanded. Excess demand for commodities now directly affects the excess demand for assets.

An interesting property of system K3 is that equations 5.16 and 5.17 together represent a dynamic representation of the Phillips Curve. Real income can be written

$$5.18 \quad Y = G(F^{-1}(\dot{p}/p - \pi)) = H(\dot{p}/p - \pi), \quad H' > 0,\ H(0) = 0.$$

This formulation represents the belief of those economists who hold that income is only affected by the difference between the actual and expected rate of inflation. Abba Lerner first stated this principle in 1949 in "The Inflationary Process: Some Theoretical Aspects." Indeed, he used the term "inflation" to indicate excess demand and not rising prices,

> "In the case of foreseen price rises there is no excess of demand. Rising prices are not only insufficient to indicate inflation; they are also unnecessary. We cannot only have rising prices without inflation but we can have inflation without rising prices."[10]

In our model K3 there is a trade-off in the short run between inflation and unemployment (or real income) but in the long run such a trade-off does not exist. The result of a long-run vertical Phillips curve is a consequence of the steady-state consistency of the excess demand function F. Non-consistent functions such as the Walrasian price adjustment mechanism mentioned in the previous chapter, will lead to a long run trade-off.

Let us examine the dynamics of system K3 when there is an exogenous increase in the rate of monetary growth θ. The money and hence real interest rate falls initially to clear the asset market. This produces excess demand in the commodity market causing instantaneously both inflation and higher real income. If the economy is stable, eventually π will equal $\dot{p}/p$ and real income will return to its former equilibrium value. Therefore a rise in the rate of monetary expansion will initially cause a rise in income and possibly a cyclical return of income to its former equilibrium level as the rate of inflation converges to the value θ.

The characteristic equation of system K3 is

5.19 $$\lambda^2 + \lambda F_1(L + bL_1)/(L_1 + L_2G') + bF_1L/(L_1 + L_2G') .$$

If $G' = 0$, K3 reduces to K2 with $F_2 = 0$. Even though income is now determined endogenously, the constraint

5.20 $$b < -L/L_1$$

is again precisely the Cagan stability condition 2.7. Therefore the deter-

[10] Abba P. Lerner, "The Inflationary Process: Some Theoretical Aspects," *The Review of Economics and Statistics*, vol. 31, no. 3, (Aug., 1949), p. 194.

mination of income by excess demand does not alter the constraint on b necessary for convergence to equilibrium. This implies that the separate effect of the real rate of interest on the demand for money, the L_2G' term, does not affect the stability condition -- only the derivative of L with respect to π. The characteristic equation may be rewritten in elasticity form as

$$5.21 \quad \lambda^2 + \lambda F_1 \cdot \frac{r_m^* + bE(L/r_m^*)}{E(L/r_m^*) + E(L/Y^*)E(Y/r^*)} + \frac{br_m^* F_1}{E(L/r_m^*) + E(L/Y^*)E(Y/r^*)}$$

where $E(x/y^*)$ represents the partial elasticity of x with respect to the equilibrium value of y.[11]

Note that the term $E(L/r_m^*) + E(L/Y^*)E(Y/r^*)$ represents the total elasticity of the demand for money with respect to the real rate of interest as it affects both the money rate of interest and income. If the system is cyclical, then only the magnitude of the coefficient of λ determines the degree of damping of the system. Hence the addition of fluctuating real income "slows" the system's return to a new equilibrium position. The reason for this becomes clear once system K3 is carefully analyzed. As compared to system K2 (which is K3 with constant real income), a rise in the money supply will bring about a smaller drop in the real rate of interest. This is due to the fact that the increased real output that is generated in K3 increases the demand for money and partially offsets the fall in r. Hence a higher real rate r will clear equation 5.14 of K3 compared to 5.4 of K2. Since excess demand is dependent on the real rate, the excess demand of system K3 is smaller than system K2. Hence prices will rise more slowly to the new equilibrium.

[11]For purposes of simplification, $E(Y^*/r^*)$ is defined as $\partial Y/\partial r^* \cdot r_m^*/Y$. If stability around $\theta = 0$ is considered, $r_m^* = r^*$.

A simulation of the economic system K3 is provided in Chart 1. The following parameter values and elasticities were chosen:

$$
\begin{aligned}
E(L/r_m^*) &= -.05 \\
E(L/Y^*) &= 1.00 \\
E(Y/r^*) &= -.03 \\
\theta &= 0. \\
r^* &= .05 \\
F_1 &= -2.00 \\
b &= .95.
\end{aligned}
\tag{5.22}
$$

The expectation parameter b was chosen to be 95% of its critical value since $-r_m^*/E(L/r_m^*) = 1.00$. Hence the simulated system should be "slightly" damped to equilibrium. This value for b indicates that at the end of one period, an individual would adjust his inflationary expectation by 61% of the difference between the old and new steady-state rates. The exogenous shock which sets the system in motion consists of a sudden, 3% rise in the level of the money stock which is maintained through time. One can see that the rate of interest drops suddenly to .035 from .05 and the rate of inflation and income jumps immediately to positive values. The variables then follow their cyclical paths, slowly damping to the new equilibrium level. The sequencing of the variables in quite interesting. The peak of real income (which corresponds to the trough of the real interest rate) precedes the peak of inflation, due to the expected rate of inflation being a component of current inflation. The peak of inflation is then followed by the peak in the money interest rate, again the π term being the reason for the lag between the money rate and the real rate. The price level peaks last in the cycle. The same sequence, of course, holds for troughs. This sequence of events seems quite impervious to changes in the parameter choices listed in 5.22, although the relative time between turning points will change.

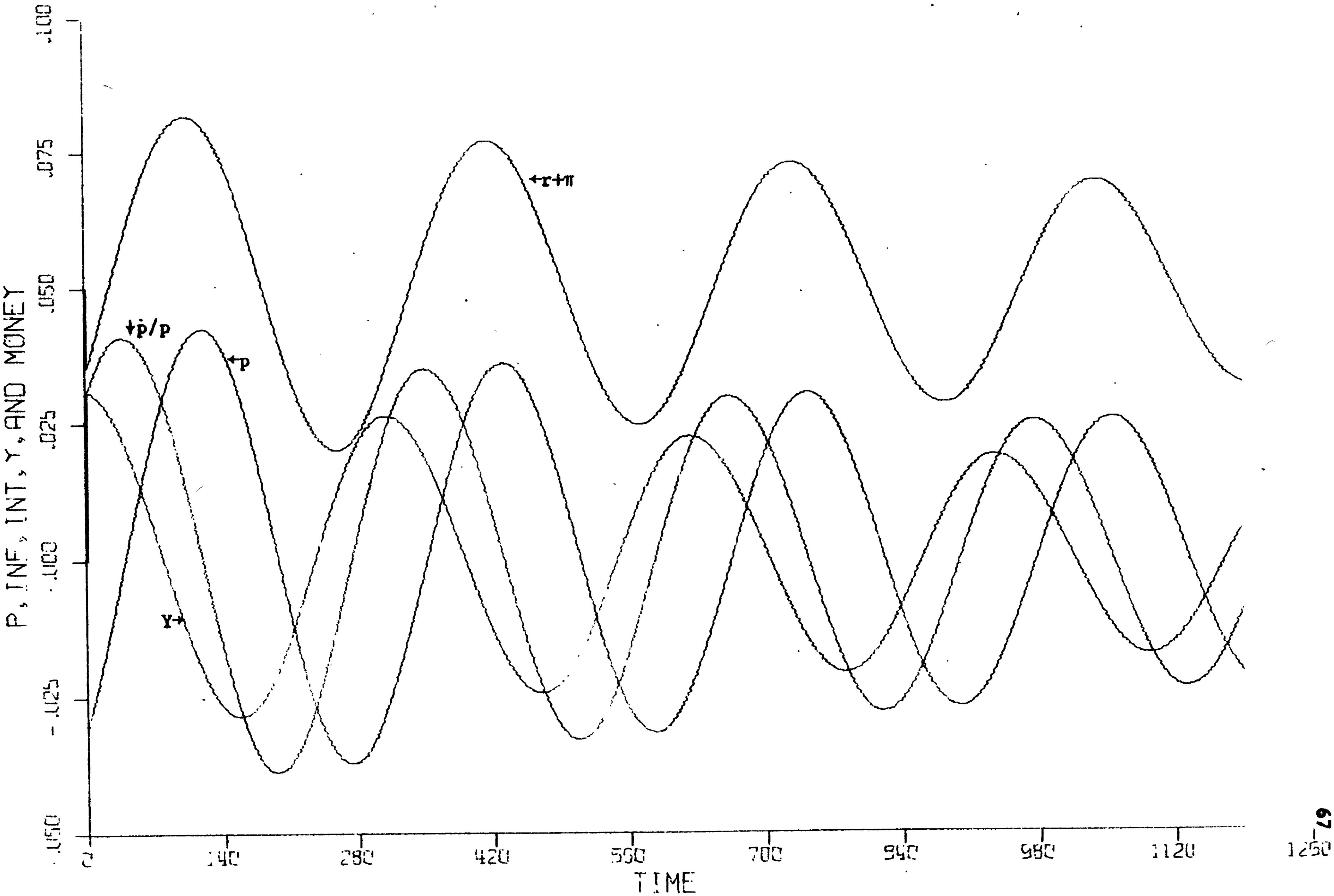

Chart 1 Simulation of K3

The Fisherian Model

We shall now specify the principal alternative form of adjustment process to the Keynesian system described above. These dynamics will be termed the "Fisherian" system since they conform closely to the principal adjustment process outlined by Irving Fisher in Chapter 4 of his 1922 classic, *The Purchasing Power of Money*.[12] The Fisherian classical system does not employ the stock-flow distinction that we have for the Keynesian model. The economy is thought to be divided into two classes of assets: money, and commodities, the latter comprising both consumers' and producers' goods. Firms, and hence financial securities, play no essential role in either the equilibrium or disequilibrium properties of the model.

For the analytic specification of the Fisherian model, we shall assume that the real rate of interest is fixed at r_0, the marginal product of capital. The money rate of interest, which includes allowance for inflationary expectations, is not fixed and becomes a key factor in Fisher's theory of interest, as mentioned in the introduction. Since there are only two assets, an excess demand for money must be matched by an excess supply of commodities and vice versa. Let us indicate the Fisherian demand for money as

5.23 $$L = L(r_0 + \pi, Y), \quad L_1 < 0, \ L_2 > 0.$$

Equilibrium would dictate that

5.24 $$M/p = L(r_0 + \pi, Y),$$

and the excess demand for money is defined as

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[12] Irving Fisher, *The Purchasing Power of Money*, (New York: The Macmillan Co., 1922).

$$5.25 \quad L^x(M/p, r_o+\pi, Y) = L - M/p, \quad L_1^x < 0,\ L_2^x < 0,\ L_3^x > 0.$$

By Walras Law, the excess demand for commodities F is equal to the negative of the excess demand for money, hence

$$5.26 \quad -L^x = F(Y, r_o+\pi, M/p), \quad F_1 < 0,\ F_2 > 0,\ F_3 > 0.$$

F_3 must be strictly positive and hence the demand for commodities is not homogeneous of degree zero in all prices. This point is strongly emphasized by Don Patinkin.[13] As long as the demand for money is positively related to income, the excess demand for commodities must be negatively related to real income and hence the marginal propensity to spend must be less than one. A higher π indicates a desired switch from money to commodities, hence $F_2 > 0$.

The dynamics of the Fisherian model specifies that an excess supply of money (excess demand for commodities) impinges on the price level in such a way as to correct the disequilibrium. With the adaptive expectation mechanism generating price expectations, and the assumption that excess demand impinges on real income as well as prices, we can represent the Fisherian system as

$$\text{F)} \quad \begin{array}{lrcll} 5.27 & \dot{p}/p & = & F(M/p - L(r_o+\pi, Y)) + \pi, & F' = F_1 > 0 \\ 5.28 & \dot{\pi} & = & b(\dot{p}/p - \pi), & b > 0 \\ 5.29 & Y & = & G(M/p - L(r_o+\pi, Y)), & G' > 0. \end{array}$$

The system specified above is steady-state consistent in both the expectations mechanism and the excess demand function F. If prices adjust instantaneously to clear the market, i.e., $F_1 = \infty$, system F reduces to the Cagan system. The same is true of the Keynesian system but the Cagan economy is a more natural descendant of that described by Fisher.

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[13] Don Patinkin, *Money, Interest, and Prices,* especially Part I.

The adjustment process described by Fisher himself is much more complicated than indicated in system F. We have only attempted to specify the principal concepts of Fisher and mold these into a full equilibrium system. Fisher described the dynamics of the adjustment process in the following manner. Let us suppose the monetary authorities increased the level of the money stock in an economy initially in equilibrium. Individuals, noting an unwanted addition to cash balances, will bid up the price of commodities (in the Keynesian system they bid up the price of securities). Although *ex ante* the real rate of interest on securities is constant, *ex post* it is not. In the beginning of the cycle, $\dot{p}/p$ exceeds π and hence firms find themselves with excess profits, since they are net debtors. This fuels their investment and pushes prices up even higher. This "accelerator" of Fisher is captured in our model as π, which allows prices to overrun their new equilibrium position. When such an overrun of prices occurs, individuals find that they are short of money balances, banks drastically hike the interest rate, and the downward part of the cyle begins. During this part of the cycle, the *ex post* real rate of interest turns out to be higher than the profits that can be earned through investment (the marginal product of capital) and hence profits fall and investment cutbacks occur. Fisher thought the system to be damped but constantly subject to exogenous shocks.

The characteristic equation of system F is derived in the mathematical appendix as

5.30 $\lambda^2 + \lambda F_1(L + bL_1)/(1 + G'L_2) + F_1bL/(1 + G'L_2)$.

The stability condition

5.31 $b < -L/L_1$,

is identical to that of the Keynesian and Cagan system. In fact, the damping characteristics of this sytem are very similar to K3 and will be described in the following chapter.

Mathematical Appendix

Chapter 5

The basic equations of the Keynesian adjustment process with instantaneous clearing of the asset market can be represented as

K2)

5A.1 $M/p = L(r + \pi), \quad L' < 0$

5A.2 $\dot{\pi} = b(\dot{p}/p - \pi), \quad b > 0$

5A.3 $\dot{p}/p = F(r,M/p) + \pi, \quad F_1 < 0, \; F_2 > 0.$

Differentiating 5A.1 with respect to time we obtain ($\theta = \dot{M}/M$)

5A.4 $\dot{r} = L(\theta - \dot{p}/p)/L' - \dot{\pi}$

or substituting 5A.2 and 5A.3 in 5A.4 we have

5A.5 $\dot{r} = -L(F(r,L) + \pi - \theta)/L' - bF(r,L),$

and combined with 5A.2

5A.6 $\dot{\pi} = bF(r,L),$

we have reduced the differential system to a function of only r and π.

Any differential system can be represented by

5A.7 $\dot{X} = A(X),$

where X is an n-vector of variables and A is a vector of functions. This system can be linearized to

5A.8 $\dot{X} = DA(X^*)\cdot\tilde{X},$

where DA(X*) is the Jacobian matrix of A(X) evaluated at the equilibrium values X* and $\tilde{X}$ is the deviations of the vector X from its equilibrium values.

System 5A.8 is globally stable and hence system 5A.7 is locally stable around X* if the characteristic roots of DA(X*), i.e., the roots λ_i of the equation

5A.9 $\det(DA(X^*) - \lambda I) = 0$

have real parts which are all negative.

Linearizing 5A.5 and 5A.6 around the equilibrium values r^*, $\pi^* = \theta$, we obtain

$$\text{5A.10} \qquad \dot{r} = [-L/L'(F_1+F_2L')-b(F_1+F_2L')]\,\tilde{r} + [-LF_2-L/L'-bF_2L']\,\tilde{\pi}$$

$$\text{5A.11} \qquad \dot{\pi} = b(F_1+F_2L')\,\tilde{r} + bF_2L'\,\tilde{\pi},$$

where $\tilde{r}$ and $\tilde{\pi}$ represent deviations from the equilibrium values. The characteristic matrix of 5A.10 and 5A.11 is

$$\text{5A.12} \qquad \begin{vmatrix} -L/L'(F_1+F_2L')-b(F_1+F_2L') - \lambda & -LF_2-L/L'-bF_2L' \\ b(F_1+F_2L') & bF_2L' - \lambda \end{vmatrix} = 0.$$

5A.12 reduces to

$$\text{5A.13} \qquad \lambda^2 + \lambda[L/L'(F_1+F_2L') + bF_1] + bL/L'(F_1+F_2L').$$

Dividing 5A.13 by F_1 and taking the limit as $F_1 \to -\infty$, yields the Cagan system. A necessary and sufficient condition for the real parts of the roots of a quadratic equation to be negative is that all the coefficients have the same sign. Since the constant term of 5A.13 is unambiguously positive, stability requires that

$$\text{5A.14} \qquad L/L'(F_1 + F_2L') + bF_1 > 0, \text{ or}$$

$$\text{5A.15} \qquad b < -L/L' - F_2L/F_1,$$

all functions evaluated at their equilibrium values r^*, $\pi^* = \theta$. Since $F(r,M/p)$ can be rewritten as $\hat{F}(r,Q/r+M/p)$, $\hat{F}_1$ represents the pure real interest rate effect of r on F and $\hat{F}_2$ would measure the wealth induced change of r on F. Hence,

$$\text{5A.16} \qquad F_1 = \hat{F}_1 - \hat{F}_2Q/r^2, \text{ and}$$

$$\text{5A.17} \qquad -F_2L/F_1 = -\hat{F}_2L/(\hat{F}_1 - \hat{F}_2Q/r^2).$$

If we assume $\hat{F}_1 = 0$, then

$$\text{5A.18} \qquad -F_2L/F_1 = -1/r \cdot L/(Q/r),$$

and hence 5A.15 can be rewritten as

5A.19 $\quad b < -r_m^*(1/E(L/r_m^*) - (M/p)^*/(Q/r)^*),$

where $E(L/r_m^*)$ is the elasticity of L with respect to r_m evaluated at equilibrium.

Let us consider the case where the asset market does not instantaneously clear but the real balance effect is nil, hence $F_2 = 0$. The Keynesian system can then be written,

5A.20 $\quad \dot{r} = \alpha(L(r+\pi) - M/p), \ \alpha > 0$

5A.21 $\quad \dot{\pi} = b(\dot{p}/p - \pi), \ \ b > 0$

5A.22 $\quad \dot{p}/p = F(r) + \pi, \quad F_1 < 0,$

where α represents the speed of adjustment in the asset market. This is an irreducible third degree differential system. Let $m = M/p$ and $\theta = \dot{M}/M$, hence

5A.23 $\quad \dot{p}/p = \theta - \dot{m}/m,$

and we can rewrite 5A.20-22 as

5A.24 $\quad \dot{r} = \alpha(L(r+\pi) - m), \ \alpha > 0$

5A.25 $\quad \dot{\pi} = bF(r)$

5A.26 $\quad \dot{m} = m(\theta - \pi - F).$

Linearizing 5A.24-26 around the equilibrium values r^*, π^*, and m^*, we have

5A.27 $\quad \dot{r} = \alpha L'\tilde{r} + \alpha L'\tilde{\pi} - \alpha\tilde{m}$

5A.28 $\quad \dot{\pi} = bF_1\tilde{r} + 0 + 0$

5A.29 $\quad \dot{m} = -mF_1\tilde{r} - m\tilde{\pi} + 0.$

The characteristic matrix is

5A.30
$$\begin{vmatrix} \alpha L' - \lambda & \alpha L' & -\alpha \\ bF_1 & -\lambda & 0 \\ -mF_1 & -m & -\lambda \end{vmatrix} = 0,$$

and the characteristic equation is

5A.31 $\lambda^3 - \alpha L'\lambda^2 + \alpha F_1(-bL' - m)\lambda - \alpha F_1 bm = 0.$

Dividing by α and taking the limit as $\alpha \rightarrow \infty$, yields 5A.13 with $F_2 = 0$. For the real roots of 5A.31 to be negative, it is a necessary, but not sufficient condition that all coefficients must be positive. If a_1, a_2, and a_3 represent the coefficients of λ^2, λ, and the constant term respectively, a necessary and sufficient condition for stability requires that

5A.32 $a_1 a_2 - a_3 > 0.$

This is often referred to as the Routhian condition. Since both a_1 and a_3 are always positive, for a_2 to be positive

5A.33 $a_2 > 0 \Longleftrightarrow b < -L/L'.$

Condition 5A.32 must be more "stringent" than the Cagan condition 5A.33. Substituting in 5A.32 yields

5A.34 $b < -L/(L' + L/(\alpha L'))$

as the local stability condition of system 5A.20-22, with all derivatives evaluated at the equilibrium point. It can be seen that

5A.35 $\partial b^*/\partial\alpha > 0,$

where b* is the maximum value of b for which 5A.34 holds.

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With $F_2 = 0$, 5A.13 reduces to

5A.36 $\lambda^2 + \lambda F_1(L/L' + b) + bF_1 L/L' = 0,$

which is of the form

5A.37 $\lambda^2 + q\beta\lambda + q\gamma = 0,$

where $q = F_1$, $\beta = (L/L' + b)$, and $\gamma = bL/L'$. The solution of 5A.37 yields

5A.38 $\lambda = .5(-q\beta \pm\sqrt{D}\,)$, where

5A.39 $D = q^2\beta^2 - 4q\gamma.$

Whenever $D < 0$, the solutions are complex and hence the system is cyclical. Define g such that

$$5A.40 \qquad b = -gL/L',$$

hence, for the system to be cyclical,

$$5A.41 \qquad F_1 > 4g/(1-g)^2 .$$

****

The Keynesian system K3 with instantaneous asset adjustment and endogenous real income yields

$$\text{K3)} \quad \begin{array}{llll} 5A.42 & M/p & = & L(r+\pi,\ Y),\quad L_1 < 0,\ L_2 > 0 \\ 5A.43 & \dot{\pi} & = & b(\dot{p}/p - \pi),\quad b > 0 \\ 5A.44 & \dot{p}/p & = & F(r) + \pi,\quad F_1 < 0 \\ 5A.45 & Y & = & G(r),\qquad G' < 0. \end{array}$$

Taking the time derivative of 5A.42 and substituting 5A.45 yields

$$5A.46 \qquad \dot{r} = L(\theta - \dot{p}/p)/(L_1+L_2G') - \dot{\pi}L_1/(L_1+L_2G'),$$

and hence system 5A.42-45 can be reduced to

$$5A.47 \qquad \dot{r} = -L(F(r)+\pi-\theta)/(L_1+L_2G') - bF(r)L_1/(L_1+L_2G')$$

$$5A.48 \qquad \dot{\pi} = bF(r).$$

Linearizing the above around the equilibrium r^*, $\pi^* = \theta$ yields

$$5A.49 \qquad \dot{r} = (-LF_1 - bL_1F_1)/(L_1+L_2G')\ \tilde{r} - L/(L_1+L_2G')\ \tilde{\pi}$$

$$5A.50 \qquad \dot{\pi} = \qquad bF_1 \qquad \tilde{r} + \quad 0.$$

The characteristic equation of 5A.49-50 is

$$5A.51 \quad \lambda^2 + \lambda F_1(L + bL_1)/(L_1+L_2G') + bF_1L/(L_1+L_2G') = 0.$$

If $G' = 0$, 5A.51 reduces to 5A.36. The stability condition for both systems are identical, requiring that

$$5A.52 \qquad b < -L/L_1,$$

where L_1 represents the derivative of L with respect to the money rate of interest holding the real rate constant ($L_1 = \partial L/\partial\pi$).

****

The modified Fisherian model may be written as

$$5A.53 \quad \dot{p}/p = F(M/p - L(r_o+\pi,Y)) + \pi, \quad F' = F_1 \gtrsim 0$$

$$\text{F)} \quad 5A.54 \quad \dot{\pi} = b(\dot{p}/p - \pi), \quad b > 0$$

$$5A.55 \quad Y = G(M/p - L(r_o+\pi,Y)), \quad G' > 0.$$

Solving for $\tilde{Y}$ implicitly from 5A.55 yields

$$5A.56 \quad \tilde{Y} = G'(\tilde{m} - L_1\tilde{\pi})/(1 + G'L_2).$$

We can also rewrite 5A.53-54 to obtain

$$5A.57 \quad \dot{m} = -mF(m - L(r_o+\pi,Y)) + (\theta - \pi)m$$

$$5A.58 \quad \dot{\pi} = bF(m - L(r_o+\pi,Y)),$$

and when 5A.56 is substituted we have a system expressed in m and π exclusively. Linearizing 5A.57-58 around m^*, $\pi^* = \theta$, we have

$$5A.59 \quad \dot{m} = -mF_1/(1 + G'L_2)\,\tilde{m} + [mF_1L_1/(1 + G'L_2) - m]\,\tilde{\pi}$$

$$5A.60 \quad \dot{\pi} = bF_1/(1 + G'L_2)\,\tilde{m} - bF_1L_1/(1 + G'L_2)\,\tilde{\pi}.$$

The characteristic equation of this system is

$$5A.61 \quad \lambda^2 + \lambda(mF_1 + bF_1L_1)/(1+G'L_2) + mbF_1/(1+G'L_2).$$

If we divide 5A.61 by F_1 and take the limit as $F_1 \to \infty$, we have the Cagan characteristic equation. The stability condition of 5A.61 is identical to the Cagan condition,

$$5A.62 \quad b < -L/L_1,$$

evaluated at the equilibrium values of π^*, m^*, and Y^*.

Chapter 6

Monetary Stabilization

The construction of a static equilibrium model of an economy yields very little information about an effective stabilization policy that the Central Bank should pursue. It is not surprising therefore, that economists who essentially agree upon the equilibrium model of an economy, say, a modified quantity theory, will disagree on the best method of stabilizing such an economy. Such was the case of the monetary economists of the University of Chicago throughout this century. Henry C. Simons believed that the primary goal of monetary policy should be the stabilization of the price level. In the depth of the Great Depression he wrote,

> "A monetary rule maintaining the constancy of some price index ... appears to afford the only promising escape from the present monetary chaos and uncertainties."[1]

Lloyd Mints later supported the Simons' policy and recommended that the aggregate money stock be pegged to the price level to accomplish Simons' aim.[2] Milton Friedman in 1948 recommended a rule which indicated that the Central Bank monetize all Federal budget deficits and demonitize all surpluses.[3] The reasoning in this work held that, if tax rates were held constant, tax revenue would fluctuate with real income

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[1]Henry C. Simons, "Rules versus Authorities in Monetary Policy," *Journal of Political Economy*, vol. 44, no. 1, (Feb., 1936), p. 30.

[2]Lloyd W. Mints, "Monetary Policy," in "A Symposium on Fiscal and Monetary Policy," *Review of Economic Statistics*, vol. 28, no. 1, (Feb., 1946), pp. 60-69.

[3]Milton Friedman, "A Monetary and Fiscal Framework for Economic Stability," *op. cit.*

and hence, with constant Federal expenditures, the Central Bank would be pursuing countercyclical monetary policy on real income.

Pursuing further empirical studies, however, Friedman recognized that the lags in the effect of monetary policy are apt to be long and uncertain. Hence his previously advocated policy of counter-cyclical policy on contemporaneous income might in fact be destabilizing. Therefore, in 1960, Friedman in *A Program for Monetary Stability* switched to recommending a constant growth rate of the monetary stock. He clearly stated,

> "There are persuasive theoretical grounds for desiring to vary the rate of growth [of money] to offset other factors. The difficulty is that, in practise, we do not know when to do so or by how much. ...It is a rule that has much to recommend it in the present state of our knowledge. But I should hope that as we operated under it we would accumulate more evidence and learn to understand more fully the working of the monetary system. As we did so, we could perhaps devise still better rules for controlling the stock of money that *could* command wider professional support and public understanding.[4]"

Friedman declared that the monetary aggregate stabilized should be at least as large as the conventionally defined money supply, currency plus demand deposits. We shall later show that under certain banking arrangements, this policy actually amounts to counter-cyclical monetary policy on the market rate of interest.

The English economist A. W. Phillips was the first to rigorously examine counter-cyclical policies in the context of a dynamic model.

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[4]Milton Friedman, *A Program for Monetary Stability*, (New York: Fordham Univ. Press, 1960), p. 98. Henry Simons stated in 1936, "A rule calling for outright fixing of the total quantity of money, however, definitely merits consideration as a perhaps preferable solution in the more distant future." ("Rules versus Authorities in Monetary Policy," *op. cit.*, p. 30). Friedman claims that if Simons knew the facts of the great monetary deflation of the early 1930's, he would have proposed this as an immediate policy goal ("The Monetary Policy of Henry Simons," *The Journal of Law and Economics*, vol. 10, Oct., 1967).

Although dealing with various forms of multiplier-accelerator models, the money stock often played a significant role in Phillips' models. In fact, he concluded that

> "Monetary policy would be a more convenient instrument [than fiscal policy] for stabilising an economy. Some may doubt whether it is a sufficiently powerful instrument; but if the right type of stabilisation policy is being applied continuously, comparatively small correcting forces are sufficient [for stability]. It is quite likely, therefore, that a monetary policy based on the principles of automatic regulating systems would be adequate to deal with all but the most severe disturbances to the economic system."[5]

The analysis in this chapter is heavily influenced by Phillips' work. In particular we shall examine two principal aspects of the stability problem. The first is the maximum value of the b parameter in the adaptive price formation mechanism which will still lead to stability of the system and second will be the degree of damping of the system to its equilibrium values. The second aspect can be made mathematically precise. Let x_t be a vector of macro-variables and x^* be their equilibrium values. For every policy P_i, there will be a time T_i associated with P_i such that, for any $\varepsilon > 0$,

6.1 $$|x_t - x^*| < \varepsilon,$$

for all $t > T_i$ if the system is stable. For a given set of initial conditions x_o and ε, a particular policy P_1 will be termed *more stable* than an alternative policy P_2 in the damping sense if

6.2 $$T_1(x_o,\varepsilon) < T_2(x_o,\varepsilon).$$

[5]A. W. Phillips, "Stabilisation Policy in a Closed Economy," *Economic Journal*, vol. 64, no. 254, (June, 1954), p. 315.

Model of Central Bank Policy

In the following analysis we shall assume that the Central Bank will completely control the stock of money. For simplicity, let us specify a real supply of money function S which is a function of the deviation of such variables as the money rate of interest, income, and real balances (the price level) from their steady-state equilibria. Analytically, we can write

6.3 $$S = S(r+\pi, Y, M/p).$$

It should be noted that S is defined to be the supply of real money although the monetary authorities can at any instant only control the nominal quantity of money. However, as long as the price level does not jump instantaneously, as is the case in all previous systems discussed except the Cagan model, a control of the nominal money supply amounts at that instant to a control of the real money supply. The function S implies that the monetary authorities react in some fashion to the levels of these variables, not their rates of change or past values. If the central bank possesses a linear supply function, they will be pursuing *proportional* policy in the terms of A. W. Phillips. However, reacting to the level of one variable might be reacting to the derivative or integral of another. In our linearized Keynesian system, for instance, $\dot{p}/p$ is a linear combination of r and π.

Some explanation of the third argument of S is warranted. If S equals M/p, then the monetary authorities keep the nominal supply of money constant and allow the real money supply to fluctuate with the price level. If S is some constant, say $(M/p)^*$, then the monetary authorities will always supply the nominal money needed to keep the real quantity constant.

Hence $S_3 = 1$ is equivalent to a fixed nominal money supply and $S_3 = 0$ to a nominal monetary policy which always exactly offsets price level changes.

Addition of this money supply function to system K3 yields

$$\text{K4)}\quad \begin{array}{lrcl} 6.4 & S(r+\pi, Y, M/p) & = & L(r+\pi, Y) \\ 6.5 & \dot{\pi} & = & b(\dot{p}/p - \pi) \\ 6.6 & \dot{p}/p & = & F(r) + \pi \\ 6.7 & Y & = & G(r). \end{array}$$

We can determine the characteristic function of K4 by linearizing around the equilibrium r^*, $\pi^* = \theta$, $(M/p)^*$, obtaining

$$6.8 \quad \lambda^2 + \lambda F_1(r_m^* S_3 + bE(L^x/r_m^*))/D + F_1 b S_3 r_m^*/D, \quad \text{where}$$

$$6.9 \quad D = E(L^x/r_m^*) + E(L^x/Y^*)\, E(Y/r^*).$$

$E(L^x/v^*)$ represents the partial elasticity of demand minus the partial elasticity of supply of money with respect to variable v evaluated at the equilibrium point v^*. When $S = M/p$, and hence $S_1 = S_2 = 0$, and $S_3 = 1$, then 6.8 reduces to 5.21, the characteristic equation of system K3. If $D < 0$, the stability condition reduces to

$$6.10 \quad b < -S_3 r_m^*/E(L^x/r_m^*).$$

If $S_3 \leq 0$, then for any positive b, the system will be unstable. In the case where $S_3 = 0$, 6.10 can be viewed as an analytical proof of the traditional claim[6] that the real supply function of the medium of exchange must be negatively pegged to a price level or the price level is indeterminate. A more generalized statement would claim that the real supply of money must be negatively pegged to some value of real balances for the system to be stable. In other words, some part of the response function must be "money illusive," i.e., respond to changes in the price

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[6]Don Patinkin, *Money, Interest, and Prices*, pp. 308-10.

level, if an equilibrium is to be stable. For example, if M/p declines below its equilibrium value for some rate of growth of the money stock, the authorities must increase M and vice versa. As mentioned earlier, the case of a constant money stock (or rate of growth) is equivalent to $S_3 = 1$. If the monetary authorities respond more strongly to changes in the price level, i.e., $S_3 > 1$, the stability constraint 6.10 will be relaxed. Any weaker response ($0 < S_3 < 1$) enhances the chances that the system exhibits instability.

The stability condition 6.10 is the intuitive extension of that derived from system K3, replacing the partial elasticity of money demand with respect to the market rate of interest with the elasticity of the excess demand function. If we assume that policy officials attempt to stabilize interest rates, resulting in $S_1 > 0$, then L_1^x (and hence $E(L^x/r_m)$) will be larger in absolute value. This indicates that, for a given b, an attempt to stabilize interest rates is more likely to result in an unstable economy. Since instability implies that all the endogenous variables tend to move away from their equilibrium values, the money rate of interest may be destabilized in the long-run by this policy. Hence, *ex ante* attempts to stabilize the money rate of interest result in *ex post* destabilization of such rates. This result has extremely important policy implications since many countries have from time to time attempted to stabilize interest rates but have had little success with this policy in the long run.

If the system K4 is cyclical, then the degree of damping to equilibrium depends only on the coefficient a_1 of λ of the characteristic equation 6.8. This is due to the fact that the "envelope" of the cycle is the function $e^{-a_1 t/2}$, which approaches zero as t approaches infinity.[7] a_1 of 6.8 is

$$6.11 \quad F_1 \frac{r^*_m S_3 + bE(L^x/r^*_m)}{E(L^x/r^*_m)+E(L^x/Y^*)E(Y/r^*)} .$$

The following conclusions may be drawn from examining the damping term 6.11.

1) The larger $|F_1|$, the faster the system damps to equilibrium. This is intuitively clear since F_1 represents the speed of response of the commodity market to disequilibrium. If $F_1 = -\infty$, then the system is always in equilibrium.

2) The larger S_3, the faster the system damps to equilibrium. This policy has been discussed with respect to the stability condition 6.10. However, it can be seen here that a policy of sharper counter response to price level deviations will result in a faster return to the equilibrium values.

3) The smaller $E(S/r_m)$, the faster the system damps to equilibrium as long as $E(L/r_m) - E(S/r_m) < 0$. $E(S/r_m)$, as S_3, affects both the stability condition and the degree of damping of the system. As mentioned earlier, if $E(S/r_m) < 0$, this amounts to a policy of *ex ante* destabilization of money interest rates but results in greater long run stability of the economy (and hence interest rates).

4) The smaller $E(S/Y)$, the *slower* the system damps to equilibrium. This yields the surprising conclusion that counter-cyclical monetary policy on real income (or pro-cyclical policy on the unemployment rate) is not only

[7]Note that the dynamics of the Cagan system imply that, around the stability point, the economy is either extremely damped (if $b < -L/L'$), or extremely explosive (if $b > -L/L'$). This may be intuitively unsatisfactory, but is characteristic of first order differential systems. Again, these singularities should be noted as $F_1 \to -\infty$.

ineffective but a hindrance to bringing the economy back to equilibrium quickly. This counter-intuitive result occurs even though money instantaneously affects real income and demonstrates the danger of extrapolating static theories to dynamic policy decisions. For values of b near the critical value, proportional policy on income has very little effect on the damping of the system. In fact, at the critical value, changes in E(S/Y) have no effect on the damping at all, as the economy perpetually oscillates with constant amplitude. The effects of E(S/Y) on damping increase as b falls.

The reason for this result rests on the same anlysis which concluded that an economy with fluctuating real income damps slower than one with a constant real income.[8] A policy of lowering the money supply when income is above equilibrium and vice versa is equivalent to a system of constant money supply with a stronger income elasticity to the demand for money, since only the *excess* demand is relevant for the analysis. Counter-cyclical policy of this type implies that the interest rate will fluctuate less than previously and therefore the movement of the price level will be slower. If the type of disturbance requires that the economy find a new equilibrium price level (due to, say, a change in the liquidity preference function or marginal product of capital), then such a policy slows the movement to a new equilibrium.[9]

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[8]See above, page 65.

[9]This does not imply that a policy of counter-cyclical policy on real income is necessarily undesirable. In our model, since real income instantaneously responds to the money supply, real income can be maintained at its equilibrium level no matter what the initial disturbance is. The policies described above are simple, proportional monetary policies on various economic variables and are in no sense the intertemporal "best" policies.

Computer Simulations

Five stabilization policies are computer simulated, all involving proportional control of the money stock dependent on some measurable economic variable. All policies have been "standardized" so that they involve approximately equal initial changes in the money stock. The policies are started at the same point in time, a time late enough to allow the exogenous shock (a 3% rise in the level of M) to cause at least one full cycle of the variables. This shock requires that the new equilibrium price level rise 3% from its previous level. The parameter values and elasticities are the same as in 5.22. The results are shown in Charts 2 through 6.

Chart 2 plots the effect of counter-cyclical monetary policy on the price level. This is equivalent to an S_3 greater than 1 in system K4. As indicated earlier, this was the policy advocated by Henry Simons. As expected from our analytical discussion of this sytem, this policy does produce a faster damping of the economy to the new equilibrium values than one of a constant money stock. After two cycles of such a policy, the deviations of the variables from their equilibrium values are only 22% of the amplitude of the deviations of a constant money stock policy. There is a slight quickening of the frequency of the cycle from this policy.

Chart 3 plots the effect of counter-cyclical monetary policy pegging the rate of inflation. This policy was not discussed analytically, but can be seen to be about equally stabilizing as the Henry Simons policy. In this case, however, a slowing of the frequency of the cycle is noticed. Since the rate of inflation is a linear combination of the expected rate

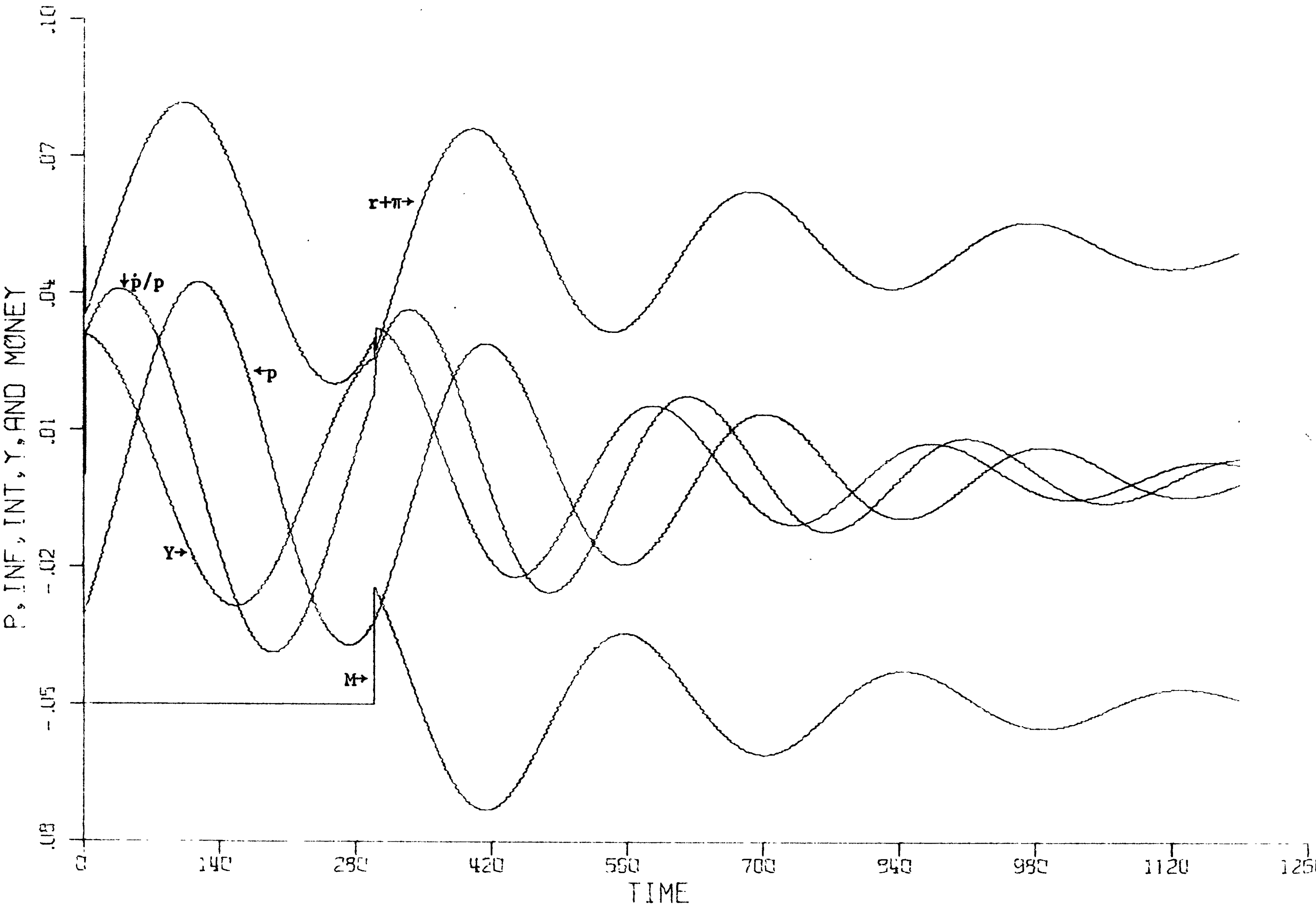

Chart 2. Countercyclical on p (Henry Simons Policy)

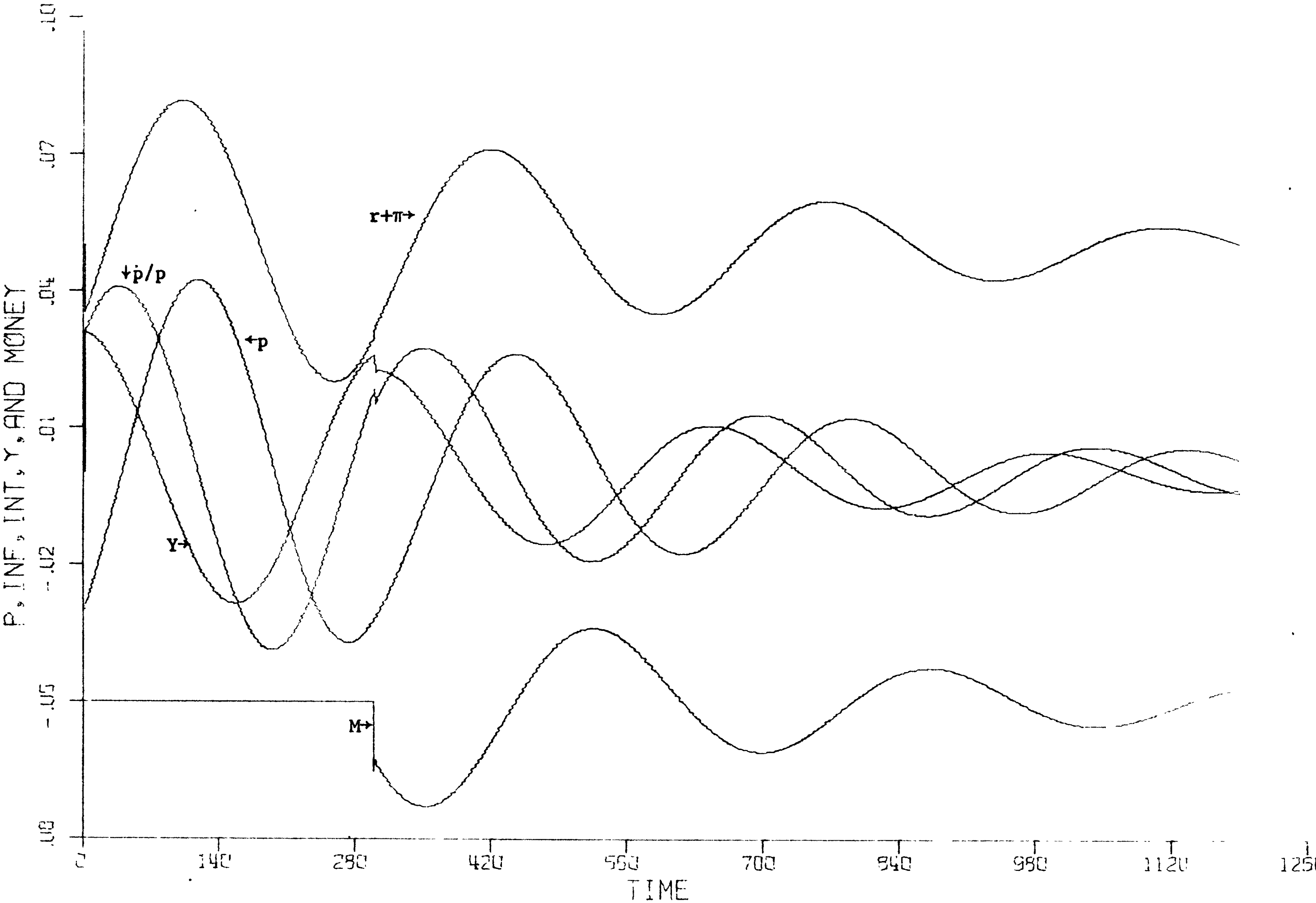

Chart 3. Countercyclical on $\dot{p}/p$

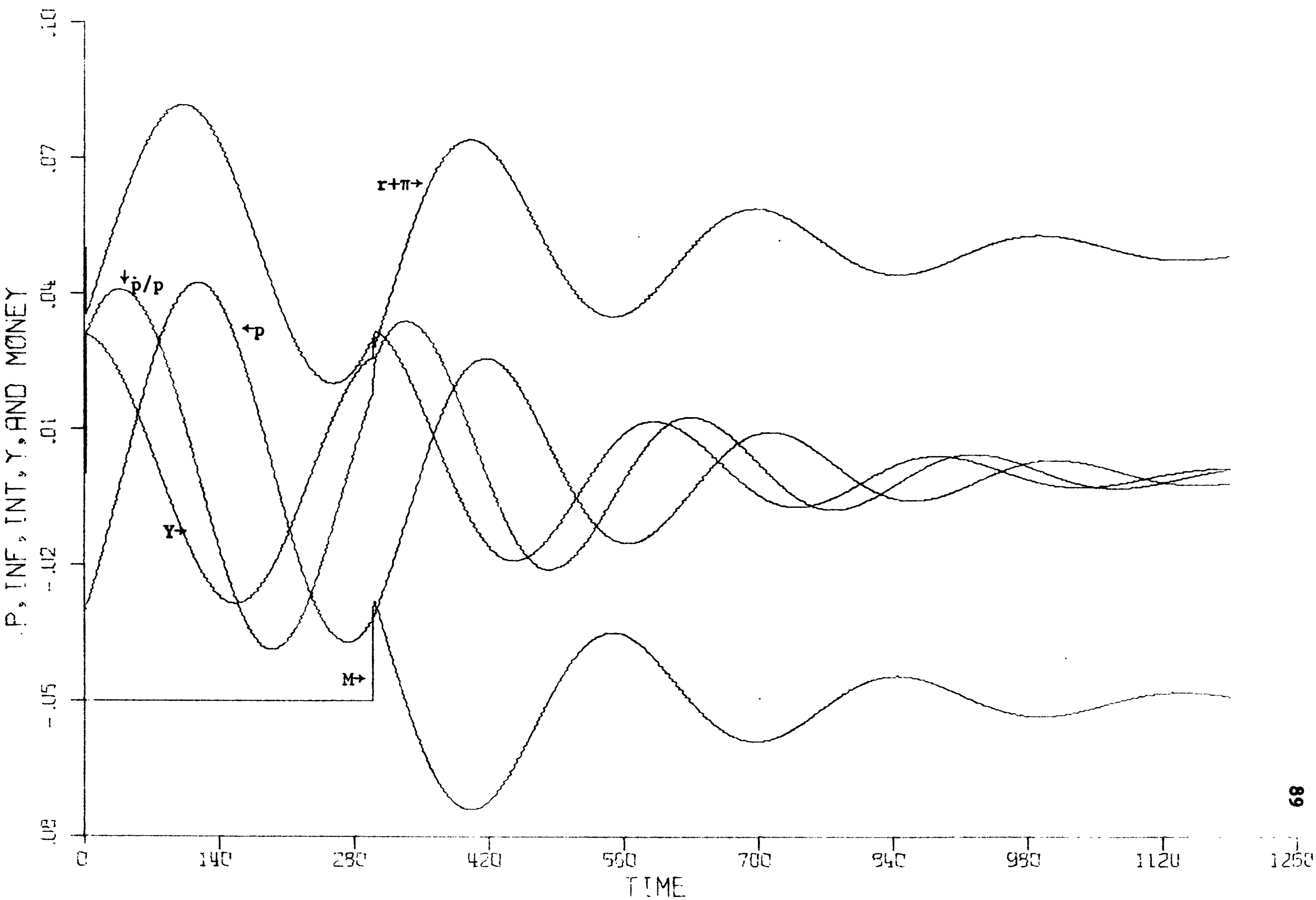

Chart 4. Countercyclical on r+π (Augmented Friedman Policy)

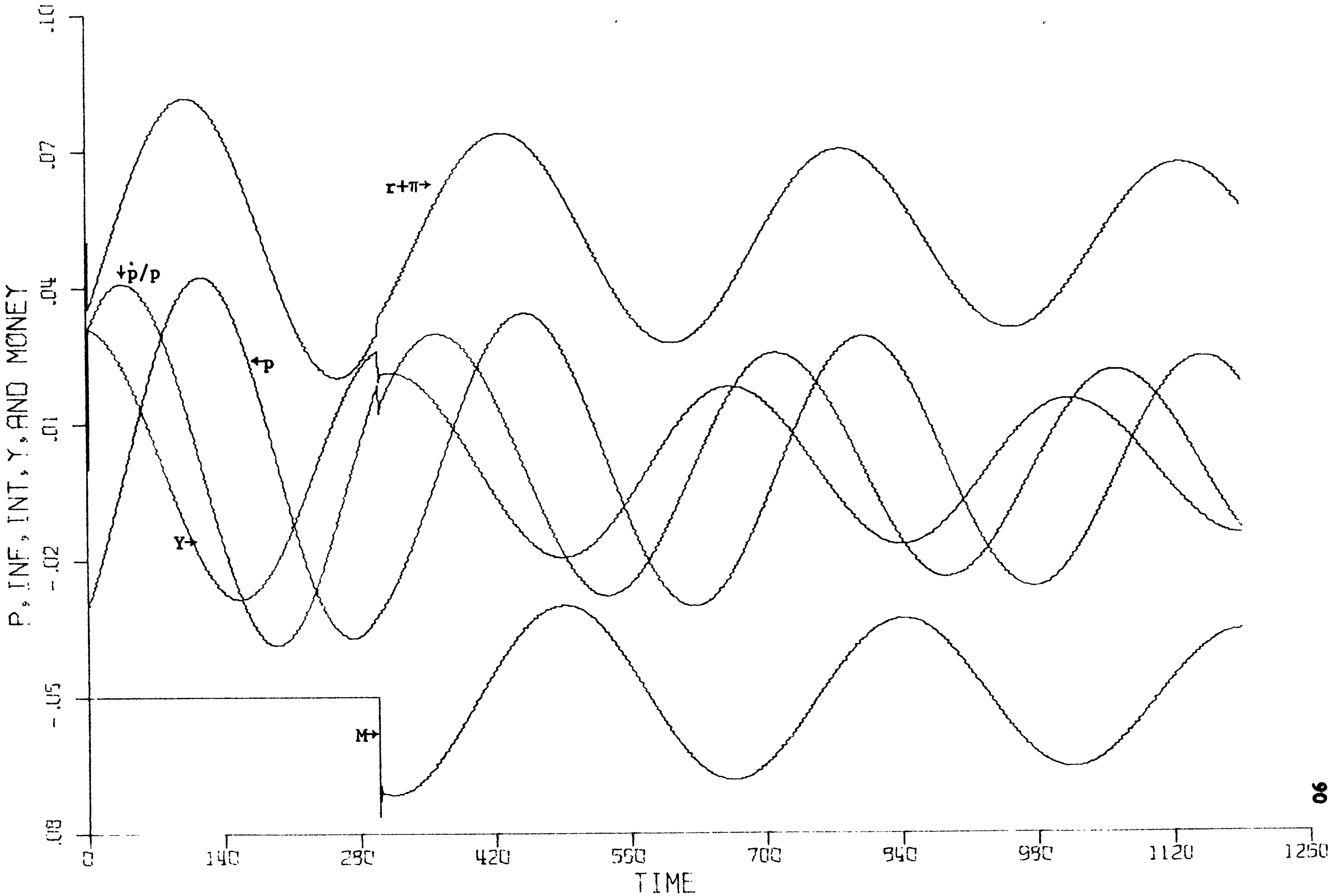

Chart 5. Countercyclical on Y (Friedman 1948 Policy)

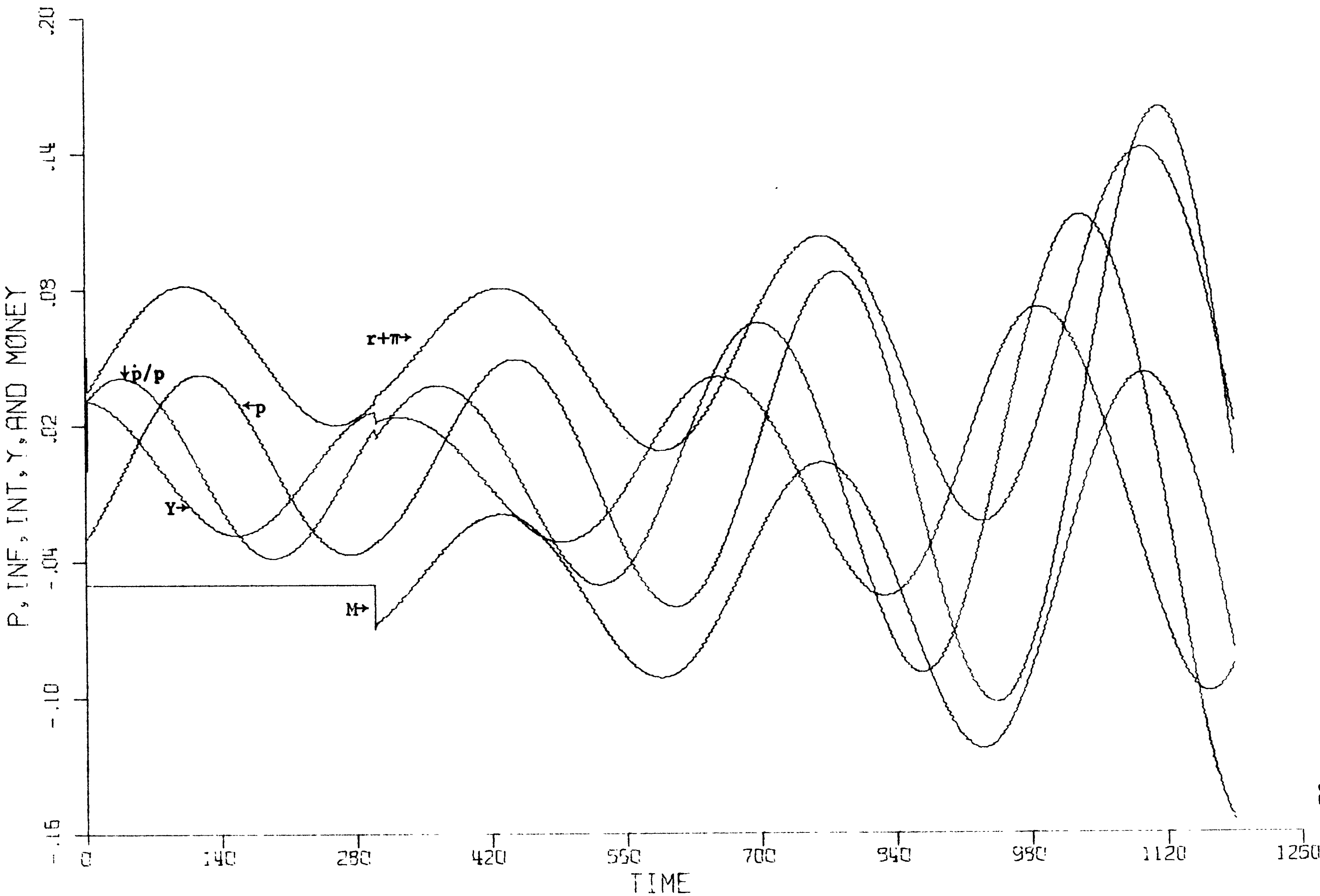

Chart 6. Procyclical on r+π (Stabilization of Interest Rate Policy)

rate of inflation and the negative of the real rate of interest (a function of real income), we certainly might expect this policy to be stabilizing since the policy implemented on either component variable alone is stabilizing.

Chart 4 plots the most stabilizing policy examined -- counter-cyclical policy on the money rate of interest. The variable $E(S/r_m)$ enter both the numerator and the denominator of the damping coefficient 6.11. The variable deviations here are damped to a level about 10% of the amplitude of the constant money stock policy. This policy represents *ex ante* destabilization of the money rate of interest since the money stock is increased when the money rate of interest is low, driving it lower, and reduced when the money rate of interest is high. This does not necessarily mean that extremes in interest rates, even in the short-run, are realized. As can be seen from our simulation, the policy,although implemented when the money interest rate was near its trough, resulted in only a small initial decrease in the money rate, a decrease which did not even bring it near its previous lows. This result holds even though we have chosen a very small elasticity of demand for money with respect to the interest rate -- .05, the low side of most empirical estimates. A higher elasticity would have caused the short-run effect on the money interest rate to be even less potent. One can see that this policy is extremely effective at *ex post* stabilization of the money rate of interest to its equilibrium value.

Chart 5 represents an attempt to stabilize the economy by implementing counter-cyclical policy on real income, a roughly pro-cyclical policy on unemployment. This policy is named the Friedman 1948 policy since he then advocated that all budget deficits and surpluses resulting from cyclical fluctuations in tax receipts be correspondingly monetized or demonitized. As discussed previously, counter-cyclical monetary policy on income is not a very effective policy when b is near its critical value as in these simulations. The frequency of the cycle is considerably slowed.

Chart 6 simulates an attempt by the Central Bank to stabilize the market rate of interest, i.e., a pro-cyclical policy on the money rate. As analyzed above, such a policy is clearly destabilizing and leads to greater fluctuations in the economy (and of course interest rates). If such a policy is pursued to extremes, it will lead to a monotonically explosive economy. As monetary economists from Wicksell[10] to Friedman[11] have indicated, a policy of interest rate stabilization (at, say, a low level) can be pursued only in the short run before other forces, specifically the rising price level, overwhelm the policy makers. *Ex ante* stabilization of interest rates means *ex post* destabilization of such rates. As is often popularly stated, interest rate fluctuations are the mechanism which "rations" credit in an economy, and any attempt to keep them too low in periods of high economic activity will just "fuel the fire."

We can now rank the damping properties of our various proportional policies according to where they occur in the cycle. The vertical axis of Figure 3 is the stabilizing or destabilizing effect of a particular policy, measured by the degree of damping to equilibrium. The horizontal axis represents the variable upon which the proportional policy is based, arranged in chronological order of their peaks and troughs. Although this

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[10] Knut Wicksell, *Interest and Prices,* Chapter 7, especially pp. 93-101.

[11] Milton Friedman, "The Role of Monetary Policy," *op. cit.*

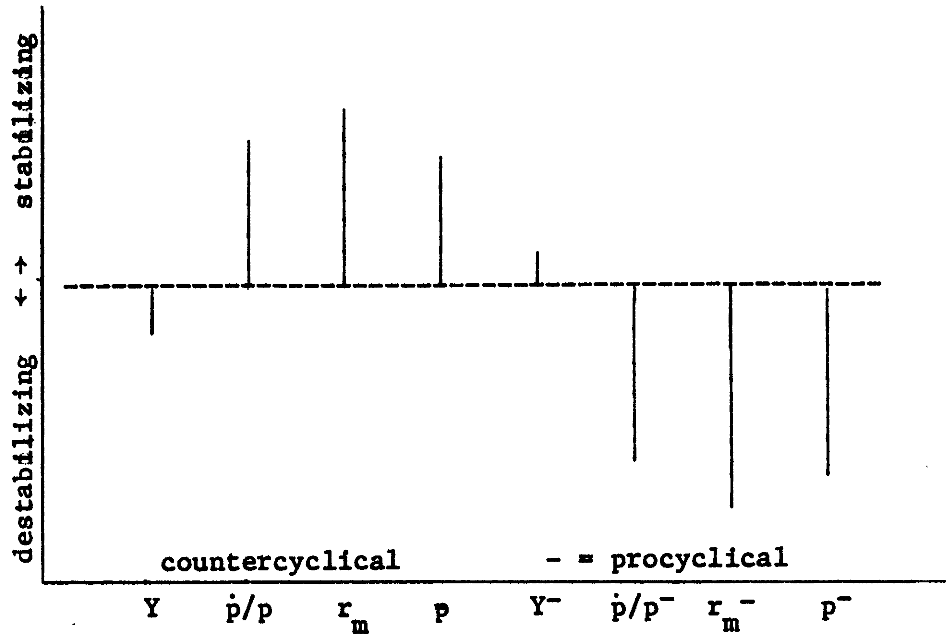

Figure 3
Stabilization Policies.

bar chart is based on a particular set of parameters, different simulations have shown it to be very robust to the choice of parameters, as long as the system remains cyclical.

One might question the importance of ranking stabilizing policies as long as they are stabilizing, since one can always increase the elasticity of supply of money with respect to a "slightly" stabilizing policy variable to make it more effective. This is certainly true if the dynamics of the economy are known with certainty, as we have assumed here. However, if there is uncertainty as to the exact nature of the economic system, extreme measures taken by policy makers should be avoided.

The dynamics of the Fisherian model derived in the preceding chapter yield the exact same qualitative conclusions with respect to counter-cyclical policy as the Keynesian model. Counter-cyclical policy on the money rate of interest both relaxes the stability condition on b and increases the degree of damping of the system. Counter-cyclical policy on real income has no effect on the stability condition, but slows the damping of the cycle. And, counter-cyclical policy on the price level (real balances) is stabilizing in both senses. These results can be easily verified by referring to 5.30 and interpreting the demand functions as excess demands.

Escalation Clauses

We have explored the dynamics of various models to discuss policies of monetary stabilization. However, it is natural to consider the subject of "escalation clauses" and their relevance to the stability analysis developed above. An escalation clause will be defined as a future contract denominated in real and not money terms, i.e., promises to pay money multiplied by some money price of a basket of representative commodities. This of course means that "escalated bonds" will be traded in terms of the real rate of interest, the payments automatically corrected *ex post* for inflation or deflation. In fact, every contract (including bank money) could be escalated except government fiat money itself. If the government paid in constant purchasing power money, then M would be worth pM at any point in time and the real supply of money would be independent of the price level. As analyzed earlier, this would mean an indeterminancy of the price level.[12] The prohibition of fluctuating, or market determined interest rates on demand deposits is *not* necessary for the determination of the price level, as will be discussed in the next chapter.

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[12] This does not mean that a fixed, non-market determined interest rate cannot be paid on fiat money (see James Tobin, "A General Equilibrium Approach to Monetary Theory," *op. cit.*).

We can conclude that a thoroughly escalated economy will still have a demand function for non-escalated money in the form $M/p = L(r+\pi)$, where π is the expected rate of depreciation of the value of money. All other equations will still hold. However, the form of the excess demand function may be altered. Since we are applying our analysis to non-tatonnement economies where trading out of equilibrium exists, we have postulated that, for one reason or another, imperfections in the commodity market prevent the price level from clearing this market at all points in time. Escalator clauses might very well speed the price level response of commodities to excess demand by eliminating contracts expressed and bargained for in money terms. This particularly applies to labor contracts which are with great difficulty adjusted downwards in money terms. We have seen from expression 6.11 that F_1, the response of excess demand to changes in the real rate of interest, is a multiplicative factor of the damping term, the coefficient of λ. As $|F_1|$ rises, the economy will move more quickly to its equilibrium. This is true even in the Fisherian model where F_1 is the speed of adjustment in the commodity/money market, and the distinction between asset and commodity equilibrium does not exist.

From the above models it can be concluded that the addition of escalator clauses, by changing F_1 in the Keynesian and Fisherian models, do not enhance the chances of instability. In fact, a faster response of prices to excess demands may well prevent income from drifting far from its equilibrium due to any exogenous shock to the demand or asset functions. The commonly held fear that escalator clauses worsen inflation can be justified if there is reason to believe that (1) aggregate demand is a function of the debtor-creditor ratio, (2) debtors have a higher marginal propensity to spend than creditors, and (3) inflation is unanticipated and thus harms creditors.[13] However, once inflation becomes anticipated, such a destabilizing process cannot be operative.

Some economists advocate the widespread adoption of escalator clauses under the hypothesis that a long-run, non-vertical Phillips curve exists. Therefore, the alleged evils of inflation will be nullified without losing the employment gains. However, in the long-run everyone is adjusted to inflation and the need for escalator clauses becomes nugatory. In the short run, the faster the economy adjusts to disequilibrium, the less time there will be such disequilibrium and hence the short-run properties of the Phillips curve may become even more transient. The preceding analysis is meant to suggest some problems that might be further explored. Unless the excess demand function for commodities is steady-state inconsistent, it is difficult to explain why the long-run Phillips curve would not be vertical.

Optimum Growth Rate of Money

Thus far we have not discussed the criteria for selecting the optimal rate of growth of money θ. Although this is appropriately discussed in the context of optimal growth models, we shall note the following considerations.

1) The values of the coefficients of the characteristic equations will usually vary with the choice of θ. For example, if b, instead of being constant, is approximately proportional to θ and the elasticity of the demand for money is constant, then the dynamics of the model are essentially independent of the rate of growth of money. If there is reason to

[13]Of course, if the above points are true, escalator clauses will lessen deflation.

believe that b rises less than θ, e.g., is constant, or that the elasticity of the demand for money with respect to the money interest rate is a declining function of such a rate,[14] then the economy will be more stable (in both the sense of the stability constraint and the damping to equilibrium) at higher equilibrium rates of inflation than at lower ones.

2) A higher rate of expansion of money will lower equilibrium real balances and hence might affect the capital stock, the labor-leisure trade-off, and hence output in an economy. As mentioned earlier,[15] for infinite horizon intertemporal maximizers, some economists have found that the capital stock is independent of θ. It is unlikely that this conclusion can be reached for individuals who are lifetime utility maximizers, but this question to my knowledge has not yet been explored.

3) The higher the rate of monetary growth, the higher the tax on cash balances. This does *not* necessarily lead to a sub-optimum position as claimed by some[16] since the tax on cash balances must be viewed in the context of an optimal tax structure which minimizes dead-weight losses. Whenever the government needs resources for the provision of public goods, and it cannot employ lump-sum taxation, it must tax commodities which results in distorting efficiency losses. Under these circumstances, it is not optimal to price each good at its marginal cost, but some amount above its marginal cost inversely proportional to its elasticity of demand. Just because fiat money is produced at zero cost by the government, therefore, does not imply that the opportunity cost of holding such fiat money should be zero at the optimum. To do this would require that the

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[14]This latter phenomenon was cited by Vickrey as a means of achieving stability through inflation(cf. Vickrey, "Stability Through Inflation," *op. cit.*).

[15]See above, p. 42.

[16]Milton Friedman, "The Optimum Quantity of Money," *op. cit.*

government receive no seigniorage from the issuance of its claims and hence no tax revenue. In fact, the lower the elasticity of demand for money, the higher would be the optimal rate of tax on real money balances.

Fuller consideration of points (2) and (3) above must essentially entail an examination of the maximization of intertemporal preference functions and are beyond the scope of this thesis. In choosing the optimal steady-state rate of growth of the money stock, consideration (1) above, which explores the stability of the system, should not be ignored.

Mathematical Appendix

Chapter 6

Specifying a supply of money function, we have model K4:

$$\text{K4)}\quad \begin{array}{llll} \text{6A.1} & S(r+\pi,Y,m) & = & L(r+\pi,Y),\quad L_1 < 0,\ L_2 > 0, \\ \text{6A.2} & \dot{\pi} & = & b(\dot{p}/p - \pi),\quad b > 0, \\ \text{6A.3} & \dot{p}/p & = & F(r) + \pi,\quad F_1 < 0, \\ \text{6A.4} & Y & = & G(r),\qquad G' < 0. \end{array}$$

Taking the time derivative of 6A.1 yields

$$\text{6A.5}\quad \dot{r} = -L_1^x bF(r)/(L_1^x+L_2^x) + S_3 m(\theta-\pi-F(r))/(L_1^x+L_2^x),$$

and we can write,

$$\text{6A.6}\quad \dot{\pi} = bF(r),$$

$$\text{6A.7}\quad \dot{m} = m(\theta - \pi - F(r)),$$

where $L^x = L - S$. Linearizing 6A.5-7 around the equilibrium r^*, $\pi^* = \theta$, m^*, we have

$$\begin{array}{llllll} \text{6A.8} & \dot{r} = & -(L_1^x bF_1 + S_3 mF_1)/(L_1^x+L_2^x) & \tilde{r} & - & mS_3/(L_1^x+L_2^x)\ \tilde{\pi} \\ \text{6A.9} & \dot{\pi} = & bF_1 & \tilde{r} & & \\ \text{6A.10} & \dot{m} = & -mF_1 & \tilde{r} & + & -m\ \tilde{\pi}\,. \end{array}$$

The characteristic equation of 6A.8-10, divided by λ, yields,

$$\text{6A.11}\quad \lambda^2 + \lambda F_1(L_1^x b + S_3 m)/(L_1^x+L_2^x) + mbF_1S_3/(L_1^x+L_2^x).$$

Hence, the stability condition is

$$\text{6A.12}\qquad b < -S_3L/L_1^x,$$

and the damping term of the system can be written

$$\text{6A.13}\qquad F_1 \cdot \frac{E(L^x/r_m^*)b + r_m^* S_3}{E(L^x/r_m^*)+E(L^x/Y^*)E(Y/r^*)}\quad.$$

Chapter 7

Extensions of the Analysis

It is natural to attempt to extend the basic models developed in Chapter 4 to conform more closely with observed phenomena and institutions without creating undue complications for the analysis. In this chapter I intend to broaden the preceding analysis along two lines. The first will be the treatment of currency and demand deposits as distinct components of money and the second will be the distinction between a "short" and a "long" interest rate.

Money Fixed Assets

In the previous analysis we have assumed that money is government fiat, non-interest bearing, legal tender. It has long been the interest of monetary economists to distinguish between "bank money" and currency, gold, or "hard" money. The expansion of bank money for a given base was often observed in the ascending stages of a business cycle, and hence had a velocity that behaved somewhat differently from the "hard" money that backed it. Recently, the empirical importance to cycle theory of bank money, or in fact, any fixed value claim, has been advanced by Gurley and Shaw.[1] Others, notably Friedman,[2] believe that the study of very broad monetary aggregates is misleading and Patinkin[3] has reemphasized

[1]John G. Gurley and Edward S. Shaw, *Money in a Theory of Finance*, (Washington: Brookings Institution, 1960).

[2]Milton Friedman, "Post-War Trends in Monetary Theory and Policy," *National Banking Review*, vol. 2, no. 1, (Sept., 1964).

[3]Don Patinkin, *Money, Interest, and Prices*, pp. 295-312.

that in equilibrium, broader monetary aggregates bear a fixed proportion to high-powered money and hence static analysis remains essentially unaltered by the addition of other money fixed claims. It is when the economy is out of equilibrium, however, that the relationships between the monetary aggregates is important and it is this phenomenon that we shall explore in depth.

In the following analysis we shall endogenize all monetary aggregates other than high-powered money. The "money" demand equations that we have previously discussed will now apply to the demand for high-powered money H (paying no or a fixed rate of interest), and hence we shall write

7.1 $$H/p = H(r+\pi, Y), \quad H_1 < 0, \; H_2 > 0,$$

where H is the direct debt (currency plus reserves) of the Central Bank. M, consisting of bank money and the currency component of H, will be assumed to be an endogenous variable depending on the rates of return on H, demand deposits, securities, and other economic variables. Hence the demand for H itself it not independent of the institutional arrangement in the economy. We shall now demonstrate in what manner total money is dependent on high-powered money H.

Let us for simplicity visualize a world in which there are only two money fixed assets: high-powered money (held by either banks or as currency by the public), and demand deposits, issued by the banking system to the public. If A represents the total money fixed assets held by the public, then we can write,

7.2 $$A = A_1 + A_2,$$

where A_1 is currency held by the public and A_2 is the demand deposits held by the public. A is hence the total traditionally defined money supply M. In addition, total government fiat money H can be written as

7.3 $\quad H = A_1 + kA_2,$

where k is the fraction of high-powered money banks wish to hold as reserves for their demand deposits. If the fraction of money the public wishes to hold as currency is h, then we can specify the system

7.4 $\quad M = A_1 + A_2$

7.5 $\quad H = A_1 + kA_2$

7.6 $\quad A_1 = hM.$

Solving we have

7.7 $\quad M = H/(h + k - hk),$

the traditional money multiplier formula.

We can now derive a more general money multiplier formula. Let A_i be the value of a particular class (e.g., demand deposits) of money fixed assets, and h_i the franction of total money fixed assets A the individual wishes to hold in form i. Hence, we may write

7.8 $\quad A_i = h_iA, \quad A = \Sigma A_i, \quad \Sigma h_i = 1.$

h_i is a fraction derived from the utility maximization of the individual and will in general be dependent on all rates of return r_i of each A_i, income, wealth, and transactions costs. Let us assume for simplicity that each banking industry issuing claim i wishes to hold k_iA_i in the form of high-powered money H. k_i is a function, as h_i, derived from banking profit maximization and dependent on all rates of return, wealth, income, and the stochastic net flows of deposits of the firms.[4] Hence, we can write

7.9 $\quad H = k_iA_i,$

where A_1 is defined to be currency and hence $k_1 = 1$. We can derive from

[4]The fact that banks may wish to hold other money fixed assets does not essentially change the analysis.

7.8 and 7.9 that

7.10 $\quad A = mH$

7.11 $\quad A_i = mh_iH$

7.12 $\quad m = 1/\sum h_ik_i,$

m being the generalized money multiplier, dependent on all rates of return, income, wealth, and other parameters. Hence, in the case of our two fixed asset world, we may set $k_1 = 1$, $k_2 = k$, $h_1 = h$, and we have from 7.12 that

7.13 $\quad m = 1/(h + (1-h)k),$

the money multiplier derived in 7.7.

h and k are in general complex function of economic variables. We must assume, as mentioned earlier, that the rate of return on H be fixed and not be a function of other market rates. Let us assume that this fixed rate is r_H, which is less than the market rate r_m for non-fixed claim assets (securities). If the banking system is competitive, the money rate of interest on demand deposits must be the market rate r_m, less an allowance for reserves; hence,

7.14 $\quad r_d = (1 - k)r_m + kr_H.$

It shall be assumed here that service charges, if any, are based only on the activity of the account and not offset against interest. The opportunity cost c for the banks and individuals of holding high-powered money is

7.15 $\quad c(H) = r_m - r_H,$

and for individuals holding demand deposits

7.16 $\quad c(D) = r_m - r_d.$

Let us assume for simplicity that the banks desired ratio of reserves to deposits, k, is a function of $r_m - r_H$, such that

7.17 $\partial k/\partial r_m < 0.$

This simply indicates that as the market rate of interest rises, the opportunity cost to the banks of holding reserves rises and hence profit maximization will require that they hold less reserves. The effect of an increase of the market rate on the currency/money ratio is more complicated. An increase in the market rate of interest will increase the opportunity cost of holding currency by the same amount. Hence,

7.18 $\partial c(H)/\partial r_m = 1.$

However, a change dr_m will result in an ambiguous change in the opportunity cost of holding demand deposits since

7.19 $\partial c(D)/\partial r_m = k + (r_m - r_H)k'.$

Since $\partial k/\partial r_m < 0$, the right side of 7.19 is not necessarily negative. This is due to the fact that a possible sharply reduced reserve ratio due to higher market rates may offset the differential $(r_m - r_H)k$ that exists between these two assets. In any case, the opportunity cost of holding currency *relative* to demand deposits must rise when the market rate of interest rises since

7.20 $k + (r_m - r_H)k' < 1.$

Even from this information one cannot positively deduce that a rise in the market rate of interest will lead to an upward shift of the currency-money ratio. If the demand for demand deposits is very elastic with respect to its opportunity cost, a small increase in such a cost may bring about a large shift out of demand deposits. Similarly, if the demand for currency is inelastic with respect to its opportunity cost, a rise in r_m will not cause a large shift from currency. We shall usually assume in our analysis that

7.21 $\partial h/\partial r_m < 0,$

i.e., a rise in the market rate of interest, by raising the opportunity cost of holding currency with respect to holding demand deposits, will induce individuals to substitute demand deposits relative to currency in their portfolio. We shall also assume that currency and demand deposits have the same elasticities with respect to income and wealth, so that changes in the latter two variables will have no effect on the currency-deposit ratio. Hence we have reduced the money multiplier m to a function of two variables, r_m and r_H, and the latter we are holding constant throughout this analysis.

Since

7.22 $\partial m/\partial r_m = m' = -(h'(1-k) + k'(1-h))/(h+k-hk)^2 > 0,$

we can easily see that a rise in the market rate of interest r_m will increase the money multiplier. This is due to two reasons: (1) the decrease in the reserves held by banks and (2) the decrease in the relative amount of currency to deposits individuals hold.[5]

It is important to consider the present institutional arrangements in the United States, specifically Regulation Q and the fixed reserve requirements. If one defines H as to equal unborrowed reserves plus currency held by the public, then it is still true that $k' < 0$, since banks may choose to hold free reserves and may borrow from the Central Bank. In general, net free or borrowed reserves will be a function of the market rate of interest as well as certain assumed fixed variables as the discount rate and "suasion" from the discount window. However, k' will certainly be much closer to zero (less negative) when required

[5]It is not necessary that h' be negative for 7.22 to hold, but if it is to hold, h' may not be so positive as to offset the reserve ratio effect.

reserve rates exist than if banks were totally free to choose their reserve levels.

Regulation Q and the Banking Act of 1935 restricts explicit payment of interest on demand deposits. However, implicit interest, in the form of credit offsets to service charges, is common and hence one can still meaningfully speak of the opportunity cost of holding demand deposits as rising less than on currency when the market rate of interest rises. As mentioned earlier, this does not guarantee that the currency-deposit ratio will fall when interest rates rise. In fact, if interest prohibition is effective, it is reasonable to assume that $h' > 0$. This is due to the fact that demand deposits are apt to be more sensitive to the interest rate than currency. In any case, the prohibition of explicit interest payment on demand deposits certainly will increase h'. This analysis suggests that the derivative of the money multiplier with respect to r_m under the present institutional arrangement is apt to be less positive than in a laissez-faire economy and might even be negative if the interest sensitivity of demand deposits relative to currency is great and interest prohibition is effective.

In the following analysis, let us assume that m' is positive. Since $M = mH$,

7.23 $$E(M/r_m) = E(m/r_m) + E(H/r_m).$$

If the monetary authorities keep the amount of high-powered money constant, so that $E(H/r_m) = 0$, then the percentage change in the money supply will be the same as the percentage change in the money multiplier m which we have assumed is dependent only on the market rate of interest r_m. The money supply and the market rate of interest would hence become coincident cyclical variables. However, there is absolutely no causal link between money and the cycle. Money is responding passively to economic forces. High powered money, the only exogenous variable of the system, is held constant.

If the monetary authorities instead keep M constant, i.e., set $E(M/r_m) = 0$, then the Central Bank must target H such that

7.24 $\quad E(H/r_m) = -E(M/r_m) < 0.$

In this case they must vary H counter-cyclically with respect to r_m in order to keep the more broadly defined M constant.

Let us now analyze the stability of an economy under a particular institutional arrangement for the supply of money. In the previous chapters we have assumed that all money (currency and demand deposits) is issued by the federal government yielding no interest. Suppose the government were to allow private, profit-maximizing banks to issue deposits but retained its right to be sole supplier of high-powered money (currency plus bank reserves). Under these circumstances, we must analyze the properties of the demand for high powered money. The demand for high-powered money is dependent on the public's demand for currency and the banks' demand for reserves. If this demand is *less* elastic than the demand for total money in the economy prior to private banking, then the economy will become more stable both with respect to a higher critical level for b, the adaptive coefficient on price expectations, and a higher degree of damping of the system to a new equilibrium.[6]

Let us now assume that with this arrangement of private banking, the government decides to regulate H such that M remains constant. As already

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[6]Empirical studies suggest that the demand for currency is quite inelastic with respect to the interest rate; the demand for reserves is also likely to be quite inelastic, especially if reserve requirements are imposed. Hence, it is not unreasonable to believe that the interest elasticity of the demand for high-powered money is less than that of total money.

explained, this requires counter-cyclical policy on H with respect to the money rate of interest. But, as analyzed in Chapter 6, this is a very stabilizing form of counter-cyclical monetary policy.

We shall now demonstrate that under a policy of constant total money supply M, the elasticity of the demand for total money with respect to the market rate of interest is equal, in equilibrium, to the elasticity of the excess demand for high-powered money. Since,

$$7.25 \qquad M^x = M^d - M^s = m(H^d - H^s),$$

taking the derivative with respect to the money rate of interest r_m yields

$$7.26 \qquad \partial M^x/\partial r_m = m(\partial H^d/\partial r_m - \partial H^s/\partial r_m) + (H^d - H^s)\partial m/\partial r_m.$$

Since we are to assume H^s is such that $\partial M^s/\partial r_m = 0$ and $H^d = H^s$ in equilibrium, 7.26 can easily be transformed to

$$7.27 \qquad E(M^d/r_m^*) = E(H^x/r_m^*),$$

where the elasticity of excess demand is defined as the elasticity of demand minus the elasticity of supply at equilibrium. Since the changes in the opportunity cost of holding demand deposits under free banking is relatively low, it can easily be seen that the total substitution against money due to a rise in the market rate will be small compared to an exclusively fiat money economy. As mentioned earlier, a low $E(H^x/r_m)$ will mean a very relaxed stability condition on b and a swift return of the economic variables to their equilibrium values.

It should be noted that a policy of holding M constant is a conscious counter-cyclical policy of high-powered money on the market rate of interest. Some writers, notably Prof. Milton Friedman, have maintained that with our present state of knowledge, our best policy is one of keeping a constant money stock.[7] In fact, he often maintained that commercial bank time deposits, as well as demand deposits, be included in a broader definition of money (M_2), and that this aggregate be maintained at a constant rate of growth.[8] Unles 100% reserves are required for these deposits, there is apt to be endogenous changes in these aggregates as market rates change, and hence the Central Bank must constantly change H to maintain a given breader aggregate. This is a specific counter-cyclical policy, i.e., *ex ante* destabilization of money rates of interest. As a result of the research outlined in this thesis, a policy of stabilizing a broader monetary aggregate is apt to be a very stabilizing form of counter-cyclical monetary policy and hence this policy has been termed the Augmented Friedman Policy. However, if interest prohibition is effective, m' is apt to be negative and stabilizing M will require pro-cyclical policy of H on the market rate of interest -- a policy more destabilizing than keeping H constant. Hence the Augmented Friedman Policy would only be effective if there were considerably less regulation of the banking system.

Short and Long Assets

Possibly one of the most observable "lags" in an economy is the lag between changes in the market rate of interest and aggregate demand. In this section I shall attempt to incorporate this lag into the model discussed in the previous chapters by introducing the concept of a "long" and "short" interest rate. The link between the two rates of interest shall

[7] See footnote 4, Chapter 6.

[8] Milton Friedman, *A Program for Monetary Stability*, p. 91.

be an adaptive mechanism, i.e., the *real* long rate shall be a distributed lag of past *real* short rates of the Koyck form (exponentially declining weights). This view, with minor variations, has been popularized by Meiselman[9] and Modigliani.[10] Analytically, if ρ is the real long rate of interest and r the real short rate, then

7.28 $\dot{\rho} = \beta(r - \rho), \quad \beta > 0.$

We can interpret the rationale for this specification in the following manner. Someone investing one dollar in a "long" bond of period N at a per period rate of interest ρ will receive

7.29 $A_L = (1 + \rho)^N$

at the end of the period. Alternatively, he will receive at the end of period N an amount A_S

7.30 $A_S = \Pi(1 + r_i), \quad 1 \leqslant i \leqslant N,$

where r_i is the uncertain i^{th} ($2 \leqslant i \leqslant N$) period short rate, by "rolling over" in short bonds. If investors are risk neutral and equate N period expected holding yields, then

7.31 $A_L = E(A_S).$[11]

If we then postulate that the expected future short rates are, through some expectation mechanism, positively related to current short rates, then a rise in the current short rate r will cause a rise in the expected future short rates and hence a rise in the long rate ρ to clear 7.31. This is of course what is approximated by relation 7.28. As the current short

[9]David Meiselman, *The Term Structure of Interest Rates,* (Englewood Cliffs, N.J., Prentice Hall, Inc., 1962).

[10]Franco Modigliani and Richard Sutch, "Innovations in Interest Rate Policy," *American Economic Review,* vol. 56, no. 2 (May, 1966), pp. 178-97.

[11]Note that this does *not* imply that they equate expected holding yields for any period other than t=N, or that $1+\rho = E\Pi(1+r_i)^{1/N}$.

rate rises, people will "revise" their expectations of future short rates to be higher and hence will require the long rate to be higher.

Note that our discussion has been only in terms of the real rate of interest, r and ρ. To speak of the money rates of interest, we must add the expected rate of inflation during the duration of the contract, which is either 1 or N periods. More realistically, then, our adaptive mechanism of inflation should perhaps be modified to the forms

7.32 $$\dot{\pi}_S = b_S(\dot{p}/p - \pi_S)$$

7.33 $$\dot{\pi}_L = b_L(\dot{p}/p - \pi_L),$$

where the subscripts S and L refer to short and long expected rates of inflation. It might be reasonable to believe that $b_S > b_L$, i.e., the adaptive mechanism weighs more recent experience more heavily for projecting inflation into the near future than into the distant future. The "money" or "market" rates of interest can now be written as $r + \pi_S$ and $\rho + \pi_L$.[12]

If the expected rate of inflation were constant for all durations, 7.28 could be rewritten in terms of money rates as

7.34 $$\dot{\rho}_m = \beta(r_m - \rho_m) + \dot{\pi} .$$

Although it is not implausible that the expected rate of inflation enter into the mechanism determining the future real short rates of interest (and for that matter the expected future real rates of interest might certainly affect the expected rate of inflation) such a complicated interaction of variables is not easily specified and beyond the scope of this thesis.

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[12] Most empirical testing on the term structure of interest rates deals with market rates because *ex ante* real rates are not accurately available. However, one must not choose a hypothesis that makes sense for real rates and then test it using uncorrected market rates.

Let us specify our extended dynamic model with the distinction between the long and short rate. The equations can be represented

$$7.35 \qquad M/p = L(r+\pi, \rho+\pi, Y), \quad L_1 < 0,\ L_2 < 0,\ L_3 > 0,$$

$$7.36 \qquad \dot{\pi} = b(\dot{p}/p - \pi), \quad b > 0,$$

$$\text{K5)} \qquad 7.37 \qquad \dot{\rho} = \beta(r - \rho), \quad \beta > 0,$$

$$7.38 \qquad \dot{p}/p = F(\rho) + \pi, \quad F_1 < 0,$$

$$7.39 \qquad Y = G(\rho), \quad G' < 0.$$

Any steady state will yields Y^*, $\pi^* = (p/p)^* = \theta$, $(M/p)^*$, and $r^* = \rho^*$. Equation 7.35 indicates that the real demand for money is dependent on the short money rate of interest, the long money rate of interest, and income. We shall assume that the variable that will change at the first instant to clear the asset market is the short rate. Equation 7.36 is the familiar adaptive mechanism on price expectations, and for analytical simplicity it is assumed that π is the expected rate of price increase for all time durations. Equation 7.37 is the mechanism by which the real long rate of interest "adapts" to changes in the real short rate, as explained above. Equation 7.38 is the dynamics of price formation in the commodity market, dependent on the real long rate of interest ρ. The rigid link between the interest rate that clears the asset market (now the short rate) and the interest rate that clears the commodity market (now the long rate) has thus been severed. Equation 7.39 indicates that excess demand for commodities affects real income as well as the rate of price increase.

The distinction between a short and long rate of interest has made model K5 a third-order differential system. The characteristic equation of our extended system, derived in the mathematical appendix is

$$7.40 \qquad \lambda^3 + \beta\lambda^2(1+(L_2+L_3G')/L_1) + \beta\lambda F_1(L/L_1+b(L_1+L_2)/L_1) + \beta F_1 bL/L_1 = 0.$$

If 7.40 is divided by β and the limit taken as β approaches infinity (hence with L_1 set equal to L_2), 7.40 becomes 5.7, the characteristic equation of system K2 with $F_2 = 0$.

A necessary condition for stability of the system is that all coefficients be positive. Since the coefficients of λ^2 and the constant term are necessarily positive, this leads us to require that

7.41 $\qquad b \; < \; -L/(L_1+L_2)$

for the coefficient of λ to be positive. This is of course just the Cagan condition, the denominator representing the derivative of the money demand function with respect to the expected rate of inflation π. However, the sufficient condition requires that the Routhian condition be met, i.e.,

7.42 $\qquad a_1a_2 - a_3 > 0,$

where a_1, a_2, and a_3 are respectively the coefficients of λ^2, λ, and λ^0 (the constant term). Hence, condition 7.41 is too weak to be sufficient and b must be restricted further for stability. As derived in the mathematical appendix, the necessary and sufficient condition for stability on parameter b in elasticity form is

7.43 $\quad b \qquad -r^*_m/(E(L/\pi^*) - R)$, where

7.44 $\quad R = (r^*_m/\beta)\cdot(E(L/r^*_m)/(E(L/\pi^*) + E(L/Y^*)E(Y/r^*)),$

and $E(L/\pi^*) = \partial L/\partial\pi \cdot r^*_m/L$. As $\beta \rightarrow \infty$, 7.43 reduces to the Cagan Stability Condition

The dynamics of this model are not too different from the dynamics of K3 discussed previously. In fact, K3 is a special case of K5 as β approaches infinity and hence the short and long interest rates become identical. An exogenous increase in the level of money will work its way through the economy in the following manner. Starting from a stationary, non-inflationary equilibrium, an increase in M brings a drop in the short real rate r. A decrease in r will then feed into equation 7.37 to produce a declining long rate ρ. One can envision this process as investors attempting to move from short term instruments, the yield on which they initially bid down when money was increased, into higher yielding long-term securities, and hence bidding up these asset prices. The falling real long rate ρ now feeds into the excess demand function for commodities, causing inflation to increase and the level of income to rise. This scenario is much more satisfactory than the previous one where an exogenous rise in M suddenly caused a high rate of inflation and a high level of income. The dynamics then follow much as described in model K3.

The explicit dynamic solution of system K5 is extremely complicated, and even if the system is cyclical and stable, there exists no simple relationship between the coefficients of the characteristic equation and the damping factor. However, computer simulations of the model yielded very similar results as reported in Chapter 6.

For the simulations it was assumed that the elasticity of the demand for money with respect to the short rate of interest equals .01 and with respect to the long rate .04, so that the sum, .05, is the same value as simulated previously. The coefficient β was chosen to be 2.00, about twice the coefficient of b, the latter chosen to be a value equal to 95% of its critical value (now slightly below 1.0). All other elasticities and parameters were the same as 5.22. The higher coefficient β reflects my belief that expected future real rates of interest respond more quickly to changes in the current real rate than expected rates of inflation respond to changes in the current rate of inflation (87% adjustment of the long rate after one period (a year) compared to 60% for the expected rate of inflation). Chart 7 graphs π, ρ, r, $\rho+\pi$, and $r+\pi$ once the system is set in motion from an exogenous 3% increase in the level of M. Note that although peaks and troughs in the real long rate lag considerably behind peaks and troughs in the real short rate, the peaks on the money long rate lag only slightly from the peaks in the money short rate. This phenomenon results from the addition of the cyclical variable π to the real rates. Since only money rates are observable, this illustrates the difficulty in separation the phenomenon of interest rate expectations from price level expectations.

Chart 8 illustrates the paths of the remaining variables, $\dot{p}/p$, p, and Y, due to the exogenous increase in money. The dynamics here have been discussed above. Note particularly that the level of inflation and income does not jump instantaneously when the level of money jumps.

Charts 9 and 10 illustrate counter-cyclical monetary policy on the level of real income and the short money rate of interest $r + \pi$. Both these policies have very similar dynamics as when applied to model K3. In particular, counter-cyclical policy is not effective in damping the cycle when applied to real income but when applied to the short money rate of interest, significant damping is attained. This latter policy would again correspond to stabilizing a broader monetary aggregate under free banking as analyzed above and is hence termed the Augmented Friedman Policy.

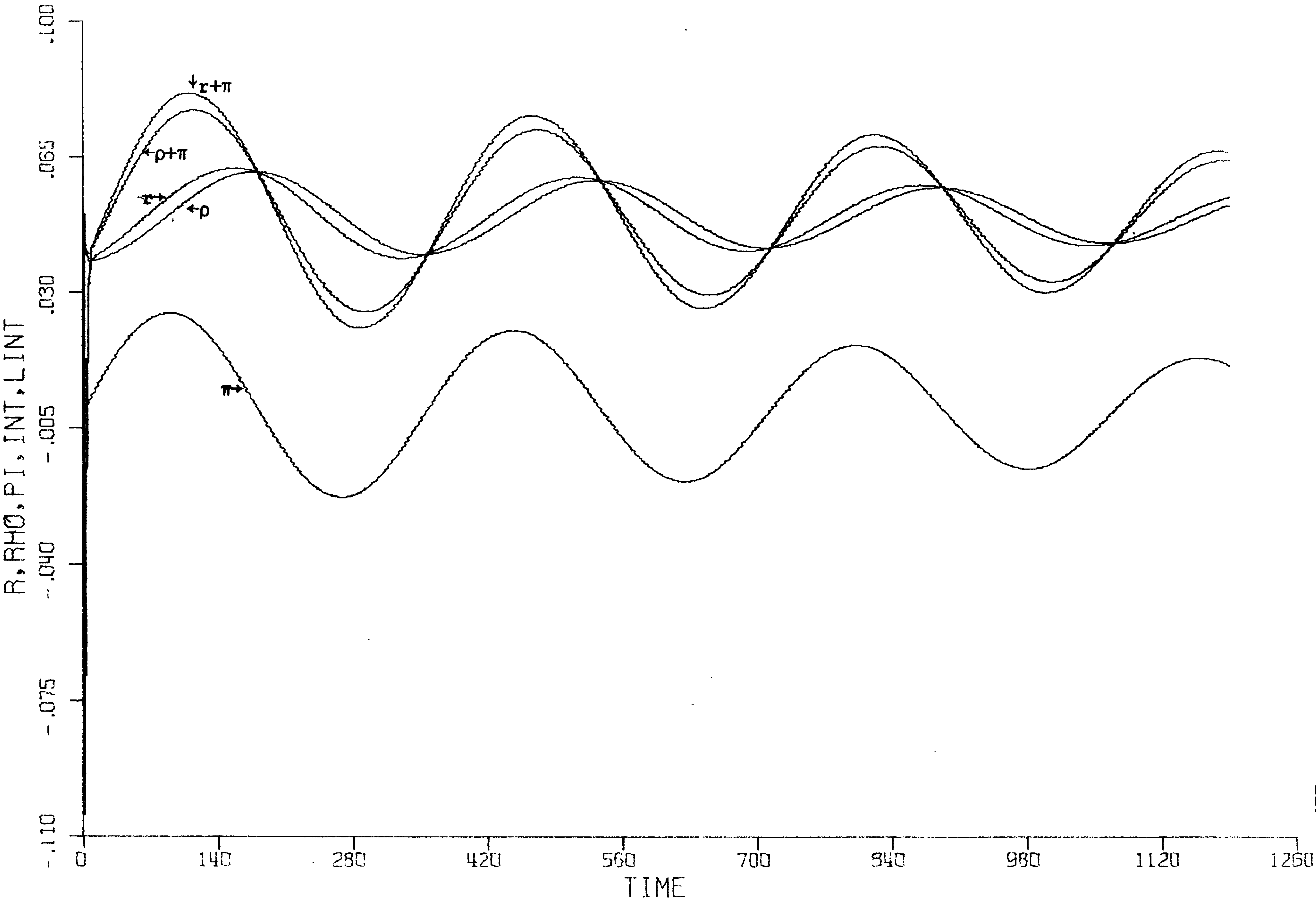

Chart 7. Simulation of K5

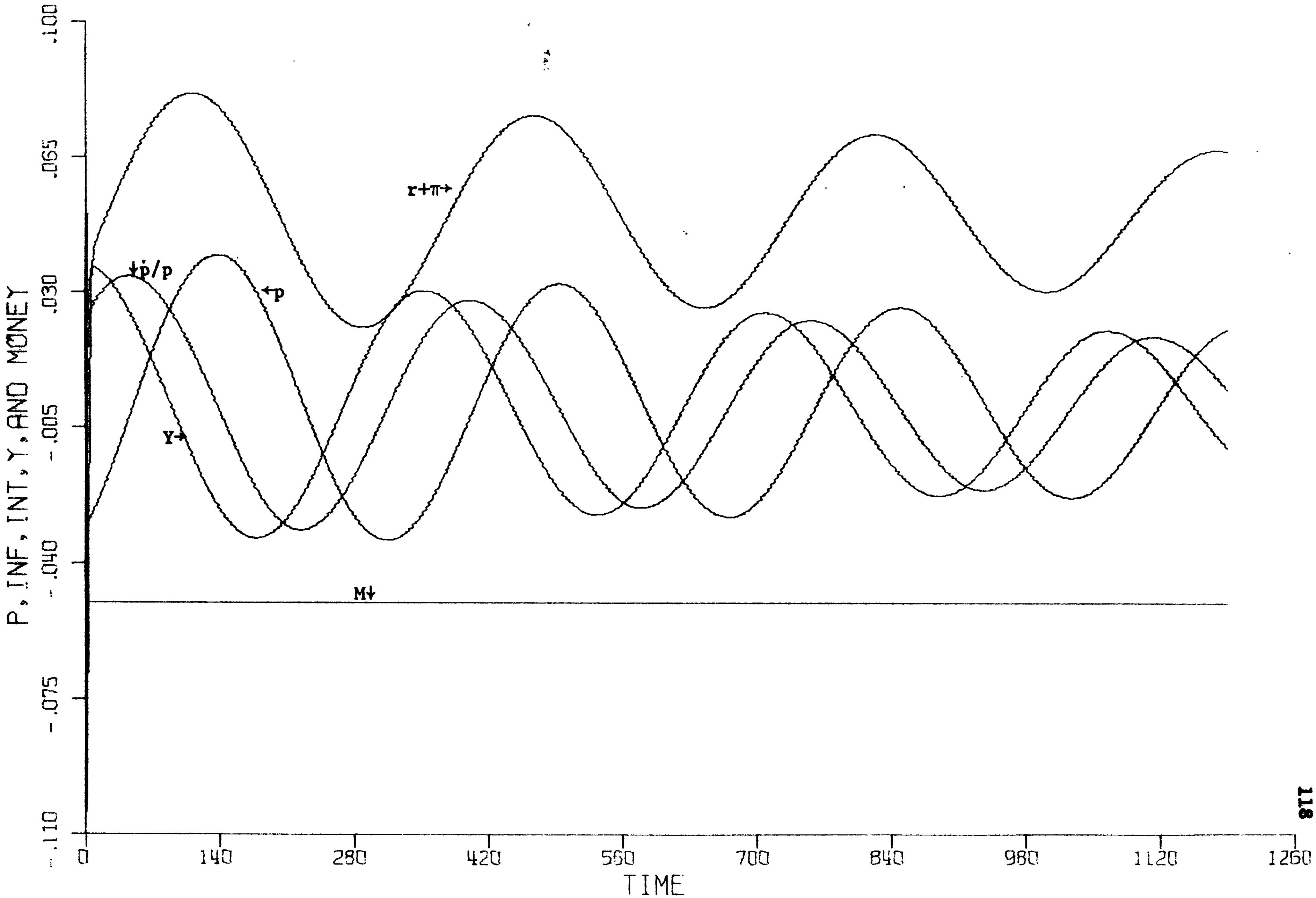

Chart 8. Simulation of K5

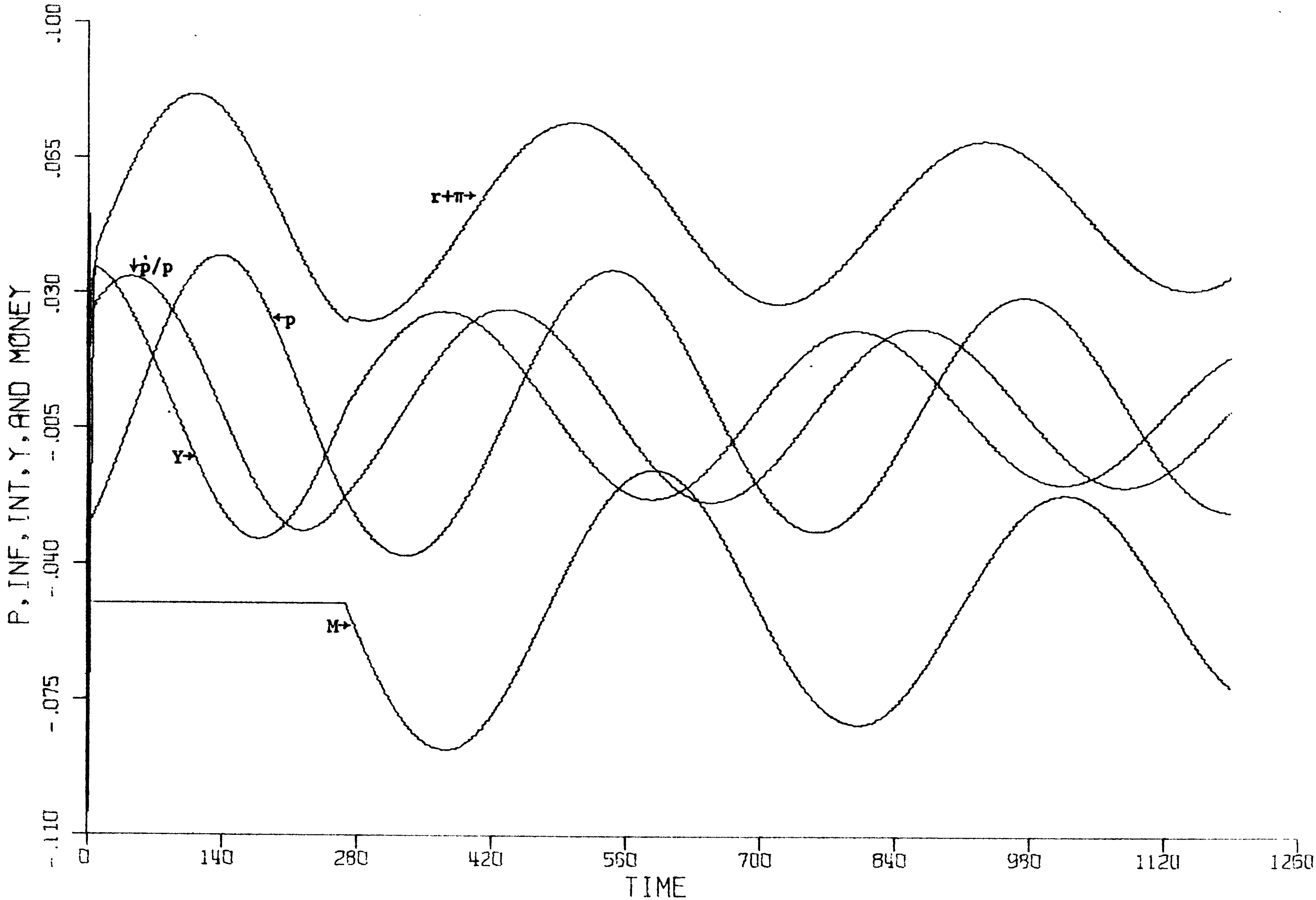

Chart 9. Countercyclical on Y (Friedman 1948 Policy)

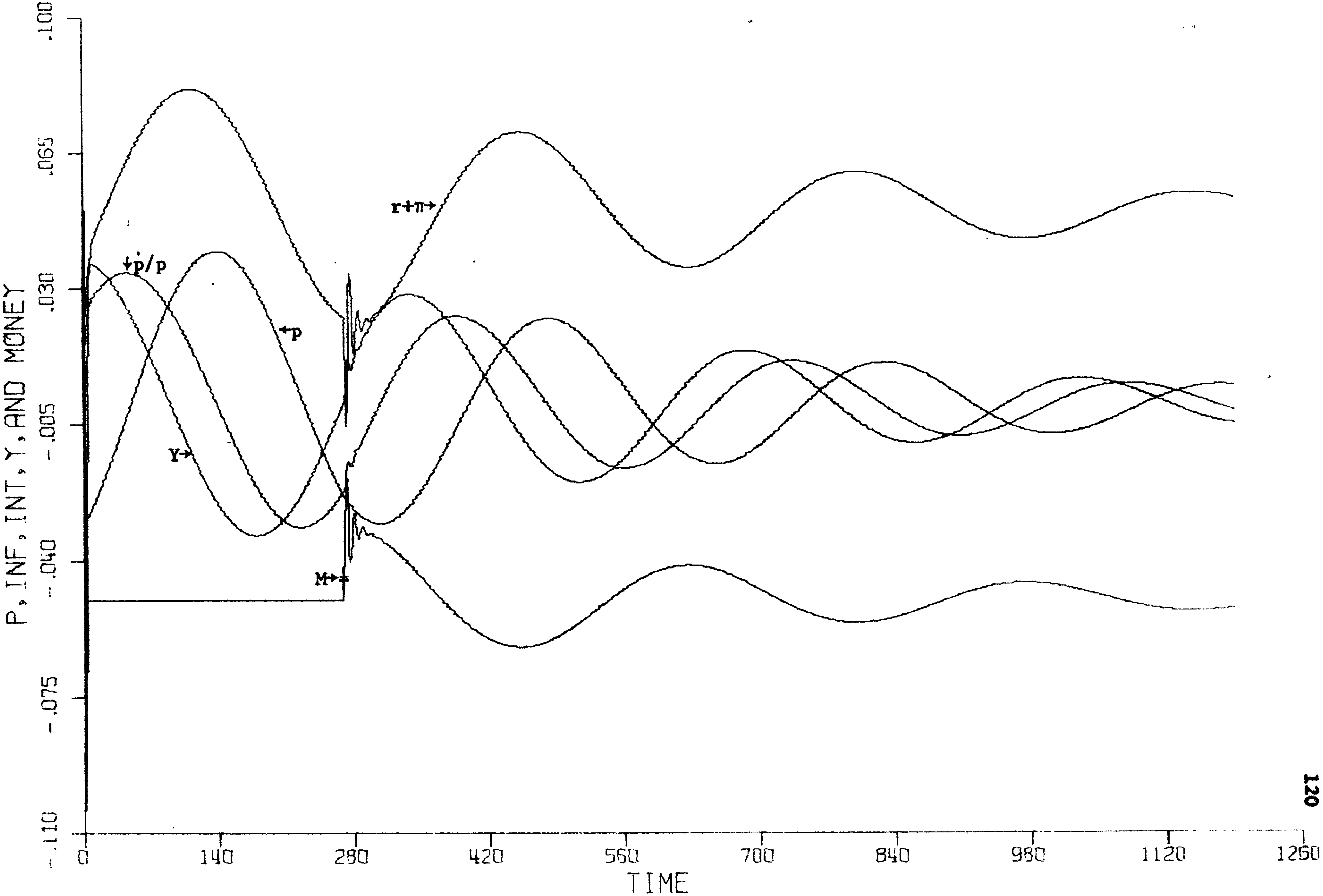

Chart 10. Countercyclical on r+π (Augmented Friedman Policy)

Mathematical Appendix

Chapter 7

Distinguishing between "long" and "short" interest rates yields the following Keynesian system:

K5)

$$7A.1 \quad M/p = L(r+\pi, \rho+\pi, Y), \quad L_1 < 0, L_2 < 0, L_3 > 0,$$

$$7A.2 \quad \dot{\pi} = b(\dot{p}/p - \pi), \quad b > 0,$$

$$7A.3 \quad \dot{\rho} = \beta(r - \rho), \quad \beta > 0,$$

$$7A.4 \quad \dot{p}/p = F(\rho) + \pi, \quad F_1 < 0,$$

$$7A.5 \quad Y = G(\rho), \quad G' < 0.$$

Taking the time derivative of 7A.1 and solving for r we have

$$7A.6 \quad r = -L(\dot{p}/p - \theta)/L_1 - (L_1+L_2)/L_1\ \dot{\pi} - (L_2+L_3G')/L_1\ \dot{\rho}.$$

Substituting for the derivatives on the right hand side we have,

$$7A.7 \quad \dot{r} = -L/L_1(F(\rho)+\pi-\theta) - (L_1+L_2)/L_1 \cdot bF(\rho) - (L_2+L_3G')/L_1 \cdot \beta(r-\rho)$$

$$7A.8 \quad \dot{\pi} = bF(\rho)$$

$$7A.9 \quad \dot{\rho} = \beta(r - \rho).$$

Linearizing 7A.7-9 around the equilibrium values $r^*=\rho^*$, $\pi^* = \theta$, we have

$$7A.10 \quad \dot{r} = -\beta(L_2+L_3G')/L_1\ \tilde{r} \ -L/L_1\ \tilde{\pi} + \beta[-LF_1/L_1-(L_1+L_2)/L_1bF_1 + (L_2+L_3G')/L_1]\ \tilde{\rho}$$

$$7A.11 \quad \dot{\pi} = bF_1\ \tilde{\rho}$$

$$7A.12 \quad \dot{\rho} = \beta\ \tilde{r} \ -\beta\ \tilde{\rho}.$$

The characteristic equation of 7A.10-12 reduces to

$$7A.13 \quad \lambda^3 + \lambda^2\beta[(L_2+L_3G')/L_1 + 1] + \lambda\beta F_1(L/L_1+b(L_1+L_2)/L_1) + \beta F_1bL/L_1.$$

For stability to prevail, all coefficients of λ^i must be positive and

$$7A.14 \quad a_1a_2 - a_3 > 0,$$

where a_i is the coefficient of λ^{3-i}. Hence

$$7A.15 \quad \beta[(L_2+L_3G')/L_1 + 1][L/L_1 + b(L_1+L_2)/L_1] - bL/L_1 < 0, \text{ and}$$

$$7A.16 \qquad b < -\frac{L}{L_1 + L_2 - \dfrac{L}{\beta(L_2+L_3G'+L_1)/L_1}} .$$

Define the elasticities such that

$$7A.17 \quad L_1 = LE(L/r_m)/r_m; \quad L_2 = LE(L/\rho_m)/\rho_m; \quad L_3G' = LE(L/Y)E(Y/r)/r_m.$$

Hence, 7A.16 can be written

$$7A.18 \quad b < -\frac{r_m^*}{E(L/r_m^*)+E(L/\rho_m^*) - \dfrac{r_m^*E(L/r_m^*)}{\beta(E(L/r_m^*)+E(L/\rho_m^*)+E(L/Y^*)E(Y/r^*))}} .$$

Since

$$7A.19 \quad E(L/\pi^*) = \partial L/\partial\pi \cdot r_m^*/L = E(L/r_m^*) + E(L/\rho_m^*),$$

where L is evaluated at $r_m^* = \rho_m^*$, 7A.18 may be rewritten

$$7A.20 \quad b < -\frac{r_m^*}{E(L/\pi^*) - \dfrac{r_m^*E(L/r_m^*)}{\beta(E(L/\pi^*) + E(L/Y^*)E(Y/r^*))}} .$$

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BIOGRAPHICAL NOTE

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