A Graduate Course in Applied Cryptography

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Preface

Cryptography is an indispensable tool used to protect information in computing systems. It is used everywhere and by billions of people worldwide on a daily basis. It is used to protect data at rest and data in motion. Cryptographic systems are an integral part of standard protocols, most notably the Transport Layer Security (TLS) protocol, making it relatively easy to incorporate strong encryption into a wide range of applications.

While extremely useful, cryptography is also highly brittle. The most secure cryptographic system can be rendered completely insecure by a single specification or programming error. No amount of unit testing will uncover a security vulnerability in a cryptosystem.

Instead, to argue that a cryptosystem is secure, we rely on mathematical modeling and proofs to show that a particular system satisfies the security properties attributed to it. We often need to introduce certain plausible assumptions to push our security arguments through.

This book is about exactly that: constructing practical cryptosystems for which we can argue security under plausible assumptions. The book covers many constructions for different tasks in cryptography. For each task we define a precise security goal that we aim to achieve and then present constructions that achieve the required goal. To analyze the constructions, we develop a unified framework for doing cryptographic proofs. A reader who masters this framework will be capable of applying it to new constructions that may not be covered in the book.

Throughout the book we present many case studies to survey how deployed systems operate. We describe common mistakes to avoid as well as attacks on real-world systems that illustrate the importance of rigor in cryptography. We end every chapter with a fun application that applies the ideas in the chapter in some unexpected way.

Intended audience and how to use this book

The book is intended to be self contained. Some supplementary material covering basic facts from probability theory and algebra is provided in the appendices.

The book is divided into three parts. The first part develops *symmetric encryption* which explains how two parties, Alice and Bob, can securely exchange information when they have a shared key unknown to the attacker. The second part develops the concepts of *public-key encryption* and *digital signatures*, which allow Alice and Bob to do the same, but without having a shared, secret key. The third part is about *cryptographic protocols*, such as protocols for user identification, key exchange, and secure computation.

A beginning reader can read though the book to learn how cryptographic systems work and why they are secure. Every security theorem in the book is followed by a proof idea that explains at a high level why the scheme is secure. On a first read one can skip over the detailed proofs without losing continuity. A beginning reader may also skip over the mathematical details sections that explore nuances of certain definitions.

An advanced reader may enjoy reading the detailed proofs to learn how to do proofs in cryptography. At the end of every chapter you will find many exercises that explore additional aspects of the material covered in the chapter. Some exercises rehearse what was learned, but many exercises expand on the material and discuss topics not covered in the chapter.

Status of the book

The current draft only contains part I and the first half of part II. The remaining chapters in parts II and part III are forthcoming. We hope you enjoy this write-up. Please send us comments and let us know if you find typos or mistakes.

Citations: While the current draft is mostly complete, we still do not include citations and references to the many works on which this book is based. Those will be coming soon and will be presented in the Notes section at the end of every chapter.

Dan Boneh and Victor Shoup
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Contents

Part I

Secret key cryptography

Chapter 2

Encryption

Roughly speaking, encryption is the problem of how two parties can communicate in secret in the presence of an eavesdropper. The main goals of this chapter are to develop a meaningful and useful definition of what we are trying to achieve, and to take some first steps in actually achieving it.

2.1 Introduction

Suppose Alice and Bob share a secret key k, and Alice wants to transmit a message m to Bob over a network while maintaining the secrecy of m in the presence of an eavesdropping adversary. This chapter begins the development of basic techniques to solve this problem. Besides transmitting a message over a network, these same techniques allow Alice to store a file on a disk so that no one else with access to the disk can read the file, but Alice herself can read the file at a later time.

We should stress that while the techniques we develop to solve this fundamental problem are important and interesting, they do not by themselves solve all problems related to "secure communication."

- The techniques only provide secrecy in the situation where Alice transmits a *single* message per key. If Alice wants to secretly transmit several messages using the *same* key, then she must use methods developed in Chapter 5.

- The techniques do not provide any assurances of *message integrity*: if the attacker has the ability to modify the bits of the ciphertext while it travels from Alice to Bob, then Bob may not realize that this happened, and accept a message other than the one that Alice sent. We will discuss techniques for providing message integrity in Chapter 6.

- The techniques do not provide a mechanism that allow Alice and Bob to come to share a secret key in the first place. Maybe they are able to do this using some secure network (or a physical, face-to-face meeting) at some point in time, while the message is sent at some later time when Alice and Bob must communicate over an insecure network. However, with an appropriate infrastructure in place, there are also protocols that allow Alice and Bob to exchange a secret key even over an insecure network: such protocols are discussed in Chapters 20 and 21.

2.2 Shannon ciphers and perfect security

2.2.1 Definition of a Shannon cipher

The basic mechanism for encrypting a message using a shared secret key is called a *cipher* (or *encryption scheme*). In this section, we introduce a slightly simplified notion of a cipher, which we call a **Shannon cipher**.

A **Shannon cipher** is a pair $\mathcal{E} = (E, D)$ of functions.

- The function E (the **encryption function**) takes as input a **key** k and a **message** m (also called a **plaintext**), and produces as output a **ciphertext** c. That is,

$$c = E(k, m),$$

  and we say that c is the **encryption of** m **under** k.

- The function D (the **decryption function**) takes as input a key k and a ciphertext c, and produces a message m. That is,

$$m = D(k, c),$$

  and we say that m is the **decryption of** c **under** k.

- We require that decryption "undoes" encryption; that is, the cipher must satisfy the following **correctness property**: for all keys k and all messages m, we have

$$D(k,\ E(k,\ m)\ ) = m.$$

To be slightly more formal, let us assume that $\mathcal{K}$ is the set of all keys (the **key space**), $\mathcal{M}$ is the set of all messages (the **message space**), and that $\mathcal{C}$ is the set of all ciphertexts (the **ciphertext space**). With this notation, we can write:

$$E : \mathcal{K} \times \mathcal{M} \to \mathcal{C},$$
$$D : \mathcal{K} \times \mathcal{C} \to \mathcal{M}.$$

Also, we shall say that $\mathcal{E}$ is **defined over** $(\mathcal{K}, \mathcal{M}, \mathcal{C})$.

Suppose Alice and Bob want to use such a cipher so that Alice can send a message to Bob. The idea is that Alice and Bob must somehow agree in advance on a key $k \in \mathcal{K}$. Assuming this is done, then when Alice wants to send a message $m \in \mathcal{M}$ to Bob, she encrypts m under k, obtaining the ciphertext $c = E(k, m) \in \mathcal{C}$, and then sends c to Bob via some communication network. Upon receiving c, Bob decrypts c under k, and the correctness property ensures that $D(k, c)$ is the same as Alice's original message m. For this to work, we have to assume that c is not tampered with in transit from Alice to Bob. Of course, the goal, intuitively, is that an eavesdropper, who may obtain c while it is in transit, does not learn too much about Alice's message m — this intuitive notion is what the formal definition of security, which we explore below, will capture.

In practice, keys, messages, and ciphertexts are often sequences of bytes. Keys are usually of some fixed length; for example, 16-byte (i.e., 128-bit) keys are very common. Messages and ciphertexts may be sequences of bytes of some fixed length, or of variable length. For example, a message may be a 1GB video file, a 10MB music file, a 1KB email message, or even a single bit encoding a "yes" or "no" vote in an electronic election.

Keys, messages, and ciphertexts may also be other types of mathematical objects, such as integers, or tuples of integers (perhaps lying in some specified interval), or other, more sophisticated types of mathematical objects (polynomials, matrices, or group elements). Regardless of how fancy these mathematical objects are, in practice, they must at some point be represented as sequences of bytes for purposes of storage in, and transmission between, computers.

For simplicity, in our mathematical treatment of ciphers, we shall assume that $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{C}$ are sets of *finite* size. While this simplifies the theory, it means that if a real-world system allows messages of unbounded length, we will (somewhat artificially) impose a (large) upper bound on legal message lengths.

To exercise the above terminology, we take another look at some of the example ciphers discussed in Chapter 1.

Example 2.1. A **one-time pad** is a Shannon cipher $\mathcal{E} = (E, D)$, where the keys, messages, and ciphertexts are bit strings of the same length; that is, $\mathcal{E}$ is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where

$$\mathcal{K} := \mathcal{M} := \mathcal{C} := \{0,1\}^L,$$

for some fixed parameter L. For a key $k \in \{0,1\}^L$ and a message $m \in \{0,1\}^L$ the encryption function is defined as follows:

$$E(k, m) := k \oplus m,$$

and for a key $k \in \{0,1\}^L$ and ciphertext $c \in \{0,1\}^L$, the decryption function is defined as follows:

$$D(k, c) := k \oplus c.$$

Here, "$\oplus$" denotes bit-wise exclusive-OR, or in other words, component-wise addition modulo 2, and satisfies the following algebraic laws: for all bit vectors $x, y, z \in \{0,1\}^L$, we have

$$x \oplus y = y \oplus x, \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z, \quad x \oplus 0^L = x, \quad \text{and} \quad x \oplus x = 0^L.$$

These properties follow immediately from the corresponding properties for addition modulo 2. Using these properties, it is easy to check that the correctness property holds for $\mathcal{E}$: for all $k, m \in \{0,1\}^L$, we have

$$D(k,\ E(k,\ m)\ ) = D(k,\ k \oplus m) = k \oplus (k \oplus m) = (k \oplus k) \oplus m = 0^L \oplus m = m.$$

The encryption and decryption functions happen to be the same in this case, but of course, not all ciphers have this property. $\Box$

Example 2.2. A **variable length one-time pad** is a Shannon cipher $\mathcal{E} = (E, D)$, where the keys are bit strings of some fixed length L, while messages and ciphertexts are variable length bit strings, of length at most L. Thus, $\mathcal{E}$ is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where

$$\mathcal{K} := \{0,1\}^L \quad \text{and} \quad \mathcal{M} := \mathcal{C} := \{0,1\}^{\leq L}.$$

for some parameter L. Here, $\{0,1\}^{\leq L}$ denotes the set of all bit strings of length at most L (including the empty string). For a key $k \in \{0,1\}^L$ and a message $m \in \{0,1\}^{\leq L}$ of length ℓ, the encryption function is defined as follows:

$$E(k, m) := k[0 \mathinner{..} \ell - 1] \oplus m,$$

and for a key $k \in \{0,1\}^L$ and ciphertext $c \in \{0,1\}^{\leq L}$ of length ℓ, the decryption function is defined as follows:

$$D(k, c) := k[0 \,..\, \ell - 1] \oplus c.$$

Here, $k[0 \,..\, \ell - 1]$ denotes the truncation of k to its first ℓ bits. The reader may verify that the correctness property holds for $\mathcal{E}$. $\Box$

Example 2.3. A **substitution cipher** is a Shannon cipher $\mathcal{E} = (E, D)$ of the following form. Let Σ be a finite alphabet of symbols (e.g., the letters `A`–`Z`, plus a space symbol, ␣). The message space $\mathcal{M}$ and the ciphertext space $\mathcal{C}$ are both sequences of symbols from Σ of some fixed length L:

$$\mathcal{M} := \mathcal{C} := \Sigma^L.$$

The key space $\mathcal{K}$ consists of all permutations on Σ; that is, each $k \in \mathcal{K}$ is a one-to-one function from Σ onto itself. Note that $\mathcal{K}$ is a very large set; indeed, $|\mathcal{K}| = |\Sigma|!$ (for $|\Sigma| = 27$, $|\mathcal{K}| \approx 1.09 \cdot 10^{28}$).

Encryption of a message $m \in \Sigma^L$ under a key $k \in \mathcal{K}$ (a permutation on Σ) is defined as follows

$$E(k, m) := \big(\ k(m[0]), k(m[1]), \ldots, k(m[L-1])\ \big),$$

where $m[i]$ denotes the ith entry of m (counting from zero), and $k(m[i])$ denotes the application of the permutation k to the symbol $m[i]$. Thus, to encrypt m under k, we simply apply the permutation k component-wise to the sequence m. Decryption of a ciphertext $c \in \Sigma^L$ under a key $k \in \mathcal{K}$ is defined as follows:

$$D(k, c) := \big(\ k^{-1}(c[0]), k^{-1}(c[1]), \ldots, k^{-1}(c[L-1])\ \big).$$

Here, k^{-1} is the inverse permutation of k, and to decrypt c under k, we simply apply k^{-1} component-wise to the sequence c. The correctness property is easily verified: for a message $m \in \Sigma^L$ and key $k \in \mathcal{K}$, we have

$$\begin{aligned} D(k,\ E(k,\ m)\ ) &= D(k,\ (k(m[0]), k(m[1]), \ldots, k(m[L-1])\ ) \\ &= (k^{-1}(k(m[0])), k^{-1}(k(m[1])), \ldots, k^{-1}(k(m[L-1]))) \\ &= (m[0], m[1], \ldots, m[L-1]) = m. \quad \Box \end{aligned}$$

Example 2.4 (additive one-time pad). We may also define a "addition mod n" variation of the one-time pad. This is a cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{K} := \mathcal{M} := \mathcal{C} := \{0, \ldots, n-1\}$, where n is a positive integer. Encryption and decryption are defined as follows:

$$E(k, m) := m + k \bmod n \qquad D(k, c) := c - k \bmod n.$$

The reader may easily verify that the correctness property holds for $\mathcal{E}$. $\Box$

2.2.2 Perfect security

So far, we have just defined the basic syntax and correctness requirements of a Shannon cipher. Next, we address the question: what is a "secure" cipher? Intuitively, the answer is that a secure cipher is one for which an encrypted message remains "well hidden," even after seeing its encryption. However, turning this intuitive answer into one that is both mathematically meaningful and practically relevant is a real challenge. Indeed, although ciphers have been used for centuries, it is only in the last few decades that mathematically acceptable definitions of security have been developed.

In this section, we develop the mathematical notion of **perfect security** — this is the "gold standard" for security (at least, when we are only worried about encrypting a single message and do not care about integrity). We will also see that it is possible to achieve this level of security; indeed, we will show that the one-time pad satisfies the definition. However, the one-time pad is not very practical, in the sense that the keys must be as long as the messages: if Alice wants to send a 1GB file to Bob, they must already share a 1GB key! Unfortunately, this cannot be avoided: we will also prove that any perfectly secure cipher must have a key space at least as large as its message space. This fact provides the motivation for developing a definition of security that is weaker, but that is acceptable from a practical point of view, and which allows one to encrypt long messages using short keys.

If Alice encrypts a message m under a key k, and an eavesdropping adversary obtains the ciphertext c, Alice only has a hope of keeping m secret if the key k is hard to guess, and that means, at the very least, that the key k should be chosen at random from a large key space. To say that m is "well hidden" must at least mean that it is hard to completely determine m from c, without knowledge of k; however, this is not really enough. Even though the adversary may not know k, we assume that he does know the encryption algorithm and the distribution of k. In fact, we will assume that when a message is encrypted, the key k is always chosen at random, uniformly from among all keys in the key space. The adversary may also have some knowledge of the message encrypted — because of circumstances, he may know that the set of possible messages is quite small, and he may know something about how likely each possible message is. For example, suppose he knows the message m is either $m_0 =$ `"ATTACK␣AT␣DAWN"` or $m_1 =$ `"ATTACK␣AT␣DUSK"`, and that based on the adversary's available intelligence, Alice is equally likely to choose either one of these two messages. This, without seeing the ciphertext c, the adversary would only have a 50% chance of guessing which message Alice sent. But we are assuming the adversary does know c. Even with this knowledge, both messages may be possible; that is, there may exist keys k_0 and k_1 such that $E(k_0, m_0) = c$ and $E(k_1, m_1) = c$, so he cannot *be sure* if $m = m_0$ or $m = m_1$. However, he can still guess. Perhaps it is a property of the cipher that there are 800 keys k_0 such that $E(k_0, m_0) = c$, and 600 keys k_1 such that $E(k_1, m_1) = c$. If that is the case, the adversary's best guess would be that $m = m_0$. Indeed, the probability that this guess is correct is equal to $800/(800 + 600) \approx 57\%$, which is better than the 50% chance he would have without knowledge of the ciphertext. Our formal definition of perfect security expressly rules out the possibility that knowledge of the ciphertext increases the probability of guessing the encrypted message, or for that matter, determining *any* property of the message whatsoever.

Without further ado, we formally define perfect security. In this definition, we will consider a probabilistic experiment in which the key is drawn uniformly from the key space. We write $\mathbf{k}$ to denote the random variable representing this random key. For a message m, $E(\mathbf{k}, m)$ is another random variable, which represents the application of the encryption function to our random key and the message m. Thus, every message m gives rise to a different random variable $E(\mathbf{k}, m)$.

Definition 2.1 (perfect security). *Let $\mathcal{E} = (E, D)$ be a Shannon cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Consider a probabilistic experiment in which the random variable $\mathbf{k}$ is uniformly distributed over $\mathcal{K}$. If for all $m_0, m_1 \in \mathcal{M}$, and all $c \in \mathcal{C}$, we have*

$$\Pr[E(\mathbf{k}, m_0) = c] = \Pr[E(\mathbf{k}, m_1) = c],$$

*then we say that $\mathcal{E}$ is a **perfectly secure** Shannon cipher.*

There are a number of equivalent formulations of perfect security that we shall explore. We state a couple of these here.

Theorem 2.1. *Let $\mathcal{E} = (E, D)$ be a Shannon cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. The following are equivalent:*

(i) $\mathcal{E}$ is perfectly secure.

(ii) For every $c \in \mathcal{C}$, there exists N_c (possibly depending on c) such that for all $m \in \mathcal{M}$, we have

$$|\{k \in \mathcal{K} : E(k, m) = c\}| = N_c.$$

(iii) If the random variable $\mathbf{k}$ is uniformly distributed over $\mathcal{K}$, then each of the random variables $E(\mathbf{k}, m)$, for $m \in \mathcal{M}$, has the same distribution.

Proof. To begin with, let us restate (ii) as follows: for every $c \in \mathcal{C}$, there exists a number P_c (depending on c) such that for all $m \in \mathcal{M}$, we have $\Pr[E(\mathbf{k}, m) = c] = P_c$. Here, $\mathbf{k}$ is a random variable uniformly distributed over $\mathcal{K}$. Note that $P_c = N_c/|\mathcal{K}|$, where N_c is as in the original statement of (ii).

This version of (ii) is clearly the same as (iii).

(i) $\Longrightarrow$ (ii). We prove (ii) assuming (i). To prove (ii), let $c \in \mathcal{C}$ be some fixed ciphertext. Pick some arbitrary message $m_0 \in \mathcal{M}$, and let $P_c := \Pr[E(\mathbf{k}, m_0) = c]$. By (i), we know that for all $m \in \mathcal{M}$, we have $\Pr[E(\mathbf{k}, m) = c] = \Pr[E(\mathbf{k}, m_0) = c] = P_c$. That proves (ii).

(ii) $\Longrightarrow$ (i). We prove (i) assuming (ii). Consider any fixed $m_0, m_1 \in \mathcal{M}$ and $c \in \mathcal{C}$. (ii) says that $\Pr[E(\mathbf{k}, m_0) = c] = P_c = \Pr[E(\mathbf{k}, m_1) = c]$, which proves (i). □

As promised, we give a proof that the one-time pad (see Example 2.1) is perfectly secure.

Theorem 2.2. *The one-time pad is a perfectly secure Shannon cipher.*

Proof. Suppose that the Shannon cipher $\mathcal{E} = (E, D)$ is a one-time pad, and is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{K} := \mathcal{M} := \mathcal{C} := \{0, 1\}^L$. For any fixed message $m \in \{0, 1\}^L$ and ciphertext $c \in \{0, 1\}^L$, there is a unique key $k \in \{0, 1\}^L$ satisfying the equation

$$k \oplus m = c,$$

namely, $k := m \oplus c$. Therefore, $\mathcal{E}$ satisfies condition (ii) in Theorem 2.1 (with $N_c = 1$ for each c). □

Example 2.5. Consider again the variable length one-time pad, defined in Example 2.2. This does not satisfy our definition of perfect security, since a ciphertext has the same length as the corresponding plaintext. Indeed, let us choose an arbitrary string of length 1, call it m_0, and an arbitrary string of length 2, call it m_1. In addition, suppose that c is an arbitrary length 1 string, and that $\mathbf{k}$ is a random variable that is uniformly distributed over the key space. Then we have

$$\Pr[E(\mathbf{k}, m_0) = c] = 1/2 \quad \text{and} \quad \Pr[E(\mathbf{k}, m_1) = c] = 0,$$

which provides a direct counter-example to Definition 2.1.

Intuitively, the variable length one-time pad cannot satisfy our definition of perfect security simply because any ciphertext leaks the *length* of the corresponding plaintext. However, in some sense (which we do not make precise right now), this is the *only* information leaked. It is perhaps not clear whether this should be viewed as a problem with the cipher or with our definition of perfect security. On the one hand, one can imagine scenarios where the length of a message may vary greatly, and while we could always "pad" short messages to effectively make all messages equally long, this may be unacceptable from a practical point of view, as it is a waste of bandwidth. On the other hand, one must be aware of the fact that in certain applications, leaking just the length of a message may be dangerous: if you are encrypting a "yes" or "no" answer to a question, just the length of the obvious ASCII encoding of these strings leaks *everything*, so you better pad "no" out to three characters. □

Example 2.6. Consider again the substitution cipher defined in Example 2.3. There are a couple of different ways to see that this cipher is not perfectly secure.

For example, choose a pair of messages $m_0, m_1 \in \Sigma^L$ such that the first two components of m_0 are equal, yet the first two components of m_1 are not equal; that is,

$$m_0[0] = m_0[1] \quad \text{and} \quad m_1[0] \neq m_1[1].$$

Then for each key k, which is a permutation on Σ, if $c = E(k, m_0)$, then $c[0] = c[1]$, while if $c = E(k, m_1)$, then $c[0] \neq c[1]$. In particular, it follows that if $\mathbf{k}$ is uniformly distributed over the key space, then the distributions of $E(\mathbf{k}, m_0)$ and $E(\mathbf{k}, m_1)$ will not be the same.

Even the weakness described in the previous paragraph may seem somewhat artificial. Another, perhaps more realistic, type of attack on the substitution cipher works as follows. Suppose the substitution cipher is used to encrypt email messages. As anyone knows, an email starts with a "standard header," such as `"FROM"`. Suppose the ciphertext is $c \in \Sigma^L$ is intercepted by an adversary. The secret key is actually a permutation k on Σ. The adversary knows that

$$c[0 \dots 3] = (k(\mathtt{F}), k(\mathtt{R}), k(\mathtt{O}), k(\mathtt{M})).$$

Thus, if the original message is $m \in \Sigma^L$, the adversary can now locate all positions in m where an `F` occurs, where an `R` occurs, where an `O` occurs, and where an `M` occurs. Based just on this information, along with specific, contextual information about the message, together with general information about letter frequencies, the adversary may be able to deduce quite a bit about the original message. □

Example 2.7. Consider the additive one-time pad, defined in Example 2.4. It is easy to verity that this is perfectly secure. Indeed, it satisfies condition (ii) in Theorem 2.1 (with $N_c = 1$ for each c). □

The next two theorems develop two more alternative characterizations of perfect security. For the first, suppose an eavesdropping adversary applies some predicate ϕ to a ciphertext he has obtained. The predicate ϕ (which is a boolean-valued function on the ciphertext space) may be something very simple, like the parity function (i.e., whether the number of 1 bits in the ciphertext is even or odd), or it might be some more elaborate type of statistical test. Regardless of how clever or complicated the predicate ϕ is, perfect security guarantees that the value of this predicate on the ciphertext reveals nothing about the message.

Theorem 2.3. *Let $\mathcal{E} = (E, D)$ be a Shannon cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Consider a probabilistic experiment in which $\mathbf{k}$ is a random variable uniformly distributed over $\mathcal{K}$. Then $\mathcal{E}$ is perfectly secure if and only if for every predicate ϕ on $\mathcal{C}$, for all $m_0, m_1 \in \mathcal{M}$, we have*

$$\Pr[\phi(E(\mathbf{k}, m_0))] = \Pr[\phi(E(\mathbf{k}, m_1))].$$

Proof. This is really just a simple calculation. On the one hand, suppose $\mathcal{E}$ is perfectly secure, and let ϕ, m_0, and m_1 be given. Let $S := \{c \in \mathcal{C} : \phi(c)\}$. Then we have

$$\Pr[\phi(E(\mathbf{k}, m_0))] = \sum_{c \in S} \Pr[E(\mathbf{k}, m_0) = c] = \sum_{c \in S} \Pr[E(\mathbf{k}, m_1) = c] = \Pr[\phi(E(\mathbf{k}, m_1))].$$

Here, we use the assumption that $\mathcal{E}$ is perfectly secure in establishing the second equality. On the other hand, suppose $\mathcal{E}$ is not perfectly secure, so there exist m_0, m_1, and c such that

$$\Pr[E(\mathbf{k}, m_0) = c] \neq \Pr[E(\mathbf{k}, m_1) = c].$$

Defining ϕ to be the predicate that is true for this particular c, and false for all other ciphertexts, we see that

$$\Pr[\phi(E(\mathbf{k}, m_0))] = \Pr[E(\mathbf{k}, m_0) = c] \neq \Pr[E(\mathbf{k}, m_1) = c] = \Pr[\phi(E(\mathbf{k}, m_1))]. \quad \square$$

The next theorem states in yet another way that perfect security guarantees that the ciphertext reveals nothing about the message. Suppose that $\mathbf{m}$ is a random variable distributed over the message space $\mathcal{M}$. We do not assume that $\mathbf{m}$ is uniformly distributed over $\mathcal{M}$. Now suppose $\mathbf{k}$ is a random variable uniformly distributed over the key space $\mathcal{K}$, independently of $\mathbf{m}$, and define $\mathbf{c} := E(\mathbf{k}, \mathbf{m})$, which is a random variable distributed over the ciphertext space $\mathcal{C}$. The following theorem says that perfect security guarantees that $\mathbf{c}$ and $\mathbf{m}$ are independent random variables.

One way of characterizing this independence is to say that for each ciphertext $c \in \mathcal{C}$ that occurs with nonzero probability, and each message $m \in \mathcal{M}$, we have

$$\Pr[\mathbf{m} = m \mid \mathbf{c} = c] = \Pr[\mathbf{m} = m].$$

Intuitively, this means that after seeing a ciphertext, we have no more information about the message than we did before seeing the ciphertext.

Another way of characterizing this independence is to say that for each message $m \in \mathcal{M}$ that occurs with nonzero probability, and each ciphertext $c \in \mathcal{C}$, we have

$$\Pr[\mathbf{c} = c \mid \mathbf{m} = m] = \Pr[\mathbf{c} = c].$$

Intuitively, this means that the choice of message has no impact on the distribution of the ciphertext.

The restriction that $\mathbf{m}$ and $\mathbf{k}$ are independent random variables is sensible: in using any cipher, it is a very bad idea to choose the key in a way that depends on the message, or vice versa (see Exercise 2.16).

Theorem 2.4. *Let $\mathcal{E} = (E, D)$ be a Shannon cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Consider a random experiment in which $\mathbf{k}$ and $\mathbf{m}$ are random variables, such that*

- $\mathbf{k}$ *is uniformly distributed over $\mathcal{K}$,*

- $\mathbf{m}$ *is distributed over* $\mathcal{M}$*, and*
- $\mathbf{k}$ *and* $\mathbf{m}$ *are independent.*

Define the random variable $\mathbf{c} := E(\mathbf{k}, \mathbf{m})$*. Then we have:*

- *if* $\mathcal{E}$ *is perfectly secure, then* $\mathbf{c}$ *and* $\mathbf{m}$ *are independent;*
- *conversely, if* $\mathbf{c}$ *and* $\mathbf{m}$ *are independent, and each message in* $\mathcal{M}$ *occurs with nonzero probability, then* $\mathcal{E}$ *is perfectly secure.*

Proof. We define $\mathcal{M}^*$ to be the set of messages that occur with nonzero probability.

We begin with a simple observation. Consider any fixed $m \in \mathcal{M}^*$ and $c \in \mathcal{C}$. Then we have

$$\Pr[\mathbf{c} = c \mid \mathbf{m} = m] = \Pr[E(\mathbf{k}, m) = c \mid \mathbf{m} = m],$$

and since $\mathbf{k}$ and $\mathbf{m}$ are independent, so are $E(\mathbf{k}, m)$ and $\mathbf{m}$, and hence

$$\Pr[E(\mathbf{k}, m) = c \mid \mathbf{m} = m] = \Pr[E(\mathbf{k}, m) = c].$$

Putting this all together, we have:

$$\Pr[\mathbf{c} = c \mid \mathbf{m} = m] = \Pr[E(\mathbf{k}, m) = c]. \tag{2.1}$$

We now prove the first implication. So assume that $\mathcal{E}$ is perfectly secure. We want to show that $\mathbf{c}$ and $\mathbf{m}$ are independent. To to this, let $m \in \mathcal{M}^*$ and $c \in \mathcal{C}$ be given. It will suffice to show that

$$\Pr[\mathbf{c} = c \mid \mathbf{m} = m] = \Pr[\mathbf{c} = c].$$

We have

$$\begin{aligned}
\Pr[\mathbf{c} = c] &= \sum_{m' \in \mathcal{M}^*} \Pr[\mathbf{c} = c \mid \mathbf{m} = m'] \Pr[\mathbf{m} = m'] \qquad \text{(by total probability)} \\
&= \sum_{m' \in \mathcal{M}^*} \Pr[E(\mathbf{k}, m') = c] \Pr[\mathbf{m} = m'] \qquad \text{(by (2.1))} \\
&= \sum_{m' \in \mathcal{M}} \Pr[E(\mathbf{k}, m) = c] \Pr[\mathbf{m} = m'] \qquad \text{(by the definition of perfect security)} \\
&= \Pr[E(\mathbf{k}, m) = c] \sum_{m' \in \mathcal{M}^*} \Pr[\mathbf{m} = m'] \\
&= \Pr[E(\mathbf{k}, m) = c] \qquad \text{(probabilities sum to 1)} \\
&= \Pr[\mathbf{c} = c \mid \mathbf{m} = m] \qquad \text{(again by (2.1))}
\end{aligned}$$

This shows that $\mathbf{c}$ and $\mathbf{m}$ are independent.

That proves the first implication. For the second, we assume that $\mathbf{c}$ and $\mathbf{m}$ are independent, and moreover, that every message occurs with nonzero probability (so $\mathcal{M}^* = \mathcal{M}$). We want to show that $\mathcal{E}$ is perfectly secure, which means that for each $m_0, m_1 \in \mathcal{M}$, and each $c \in \mathcal{C}$, we have

$$\Pr[E(\mathbf{k}, m_0) = c] = \Pr[E(\mathbf{k}, m_1) = c]. \tag{2.2}$$

But we have

$$\begin{aligned}
\Pr[E(\mathbf{k}, m_0) = c] &= \Pr[\mathbf{c} = c \mid \mathbf{m} = m_0] \qquad \text{(by (2.1))} \\
&= \Pr[\mathbf{c} = c] \qquad \text{(by independence of } \mathbf{c} \text{ and } \mathbf{m}\text{)} \\
&= \Pr[\mathbf{c} = c \mid \mathbf{m} = m_1] \qquad \text{(again by independence of } \mathbf{c} \text{ and } \mathbf{m}\text{)} \\
&= \Pr[E(\mathbf{k}, m_1) = c] \qquad \text{(again by (2.1)).}
\end{aligned}$$

That shows that $\mathcal{E}$ is perfectly secure. □

2.2.3 The bad news

We have saved the bad news for last. The next theorem shows that perfect security is such a powerful notion that one can really do no better than the one-time pad: keys must be at least as long as messages. As a result, it is almost impossible to use perfectly secure ciphers in practice: if Alice wants to send Bob a 1GB video file, then Alice and Bob have to agree on a 1GB secret key in advance.

Theorem 2.5 (Shannon's theorem). *Let $\mathcal{E} = (E, D)$ be a Shannon cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. If $\mathcal{E}$ is perfectly secure, then $|\mathcal{K}| \geq |\mathcal{M}|$.*

Proof. Assume that $|\mathcal{K}| < |\mathcal{M}|$. We want to show that $\mathcal{E}$ is not perfectly secure. To this end, we show that there exist messages m_0 and m_1, and a ciphertext c, such that

$$\Pr[E(\mathbf{k}, m_0) = c] > 0, \quad \text{and} \tag{2.3}$$

$$\Pr[E(\mathbf{k}, m_1) = c] = 0. \tag{2.4}$$

Here, $\mathbf{k}$ is a random variable, uniformly distributed over $\mathcal{K}$.

To do this, choose any message $m_0 \in \mathcal{M}$, and any key $k_0 \in \mathcal{K}$. Let $c := E(k_0, m_0)$. It is clear that (2.3) holds.

Next, let

$$S := \{D(k_1, c) : k_1 \in \mathcal{K}\}.$$

Clearly,

$$|S| \leq |\mathcal{K}| < |\mathcal{M}|,$$

and so we can choose a message $m_1 \in \mathcal{M} \setminus S$.

To prove (2.4), we need to show that there is no key k_1 such that $E(k_1, m_1) = c$. Assume to the contrary that $E(k_1, m_1) = c$ for some k_1; then for this key k_1, by the correctness property for ciphers, we would have

$$D(k_1, c) = D(k_1,\ E(k_1,\ m_1)\ ) = m_1,$$

which would imply that m_1 belongs to S, which is not the case. That proves (2.4), and the theorem follows. □

2.3 Computational ciphers and semantic security

As we have seen in Shannon's theorem (Theorem 2.5), the only way to achieve perfect security is to have keys that are as long as messages. However, this is quite impractical: we would like to be able to encrypt a long message (say, a document of several megabytes) using a short key (say, a few hundred bits). The only way around Shannon's theorem is to relax our security requirements. The way we shall do this is to consider not all possible adversaries, but only *computationally feasible* adversaries, that is, "real world" adversaries that must perform their calculations on real computers using a reasonable amount of time and memory. This will lead to a weaker definition of security called **semantic security**. Furthermore, our definition of security will be flexible enough to allow ciphers with variable length message spaces to be considered secure so long as they do not leak any useful information about an encrypted message to an adversary *other than the length of message.* Also, since our focus is now on the "practical," instead of the "mathematically possible," we shall also insist that the encryption and decryption functions are themselves efficient algorithms, and not just arbitrary functions.

2.3.1 Definition of a computational cipher

A **computational cipher** $\mathcal{E} = (E, D)$ is a pair of efficient algorithms, E and D. The encryption algorithm E takes as input a key k, along with a message m, and produces as output a ciphertext c. The decryption algorithm D takes as input a key k, a ciphertext c, and outputs a message m. Keys lie in some finite key space $\mathcal{K}$, messages lie in a finite message space $\mathcal{M}$, and ciphertexts lie in some finite ciphertext space $\mathcal{C}$. Just as for a Shannon cipher, we say that $\mathcal{E}$ is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$.

Although it is not really necessary for our purposes in this chapter, we will allow the encryption function E to be a *probabilistic* algorithm (see Chapter D). This means that for fixed inputs k and m, the output of $E(k, m)$ may be one of many values. To emphasize the probabilistic nature of this computation, we write

$$c \xleftarrow{\mathrm{R}} E(k, m)$$

to denote the process of executing $E(k, m)$ and assigning the output to the program variable c. We shall use this notation throughout the text whenever we use probabilistic algorithms. Similarly, we write

$$k \xleftarrow{\mathrm{R}} \mathcal{K}$$

to denote the process of assigning to the program variable k a random, uniformly distributed element of from the key space $\mathcal{K}$. We shall use the analogous notation to sample uniformly from any finite set.

We will not see any examples of probabilistic encryption algorithms in this chapter (we will see our first examples of this in Chapter 5). Although one could allow the decryption algorithm to be probabilistic, we will have no need for this, and so will only discuss ciphers with deterministic decryption algorithms. However, it will be occasionally be convenient to allow the decryption algorithm to return a special reject value (distinct from all messages), indicating some kind of error occurred during the decryption process.

Since the encryption algorithm is probabilistic, for a given key k and message m, the encryption algorithm may output one of many possible ciphertexts; however, each of these possible ciphertexts should decrypt to m. We can state this **correctness requirement** more formally as follows: for all keys $k \in \mathcal{K}$ and messages $m \in \mathcal{M}$, if we execute

$$c \xleftarrow{\mathrm{R}} E(k, m),\ m' \leftarrow D(k, c),$$

then $m = m'$ with probability 1.

From now on, whenever we refer to a ***cipher****, we shall mean a* ***computational cipher****, as defined above. Moreover, if the encryption algorithm happens to be deterministic, then we may call the cipher a* ***deterministic cipher****.*

Observe that any deterministic cipher is a Shannon cipher; however, a computational cipher need not be a Shannon cipher (if it has a probabilistic encryption algorithm), and a Shannon cipher need not be a computational cipher (if its encryption or decryption operations have no efficient implementations).

Example 2.8. The one-time pad (see Example 2.1) and the variable length one-time pad (see Example 2.2) are both deterministic ciphers, since their encryption and decryption operations may be trivially implemented as efficient, deterministic algorithms. The same holds for the substitution cipher (see Example 2.3), provided the alphabet Σ is not too large. Indeed, in the obvious implementation, a key — which is a permutation on Σ — will be represented by an array indexed by Σ, and so we will require $O(|\Sigma|)$ space just to store a key. This will only be practical for reasonably sized Σ. The additive one-time pad discussed in Example 2.4 is also a deterministic cipher, since both encryption and decryption operations may be efficiently implemented (if n is large, special software to do arithmetic with large integers may be necessary). $\Box$

2.3.2 Definition of semantic security

To motivate the definition of semantic security, consider a deterministic cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Consider again the formulation of perfect security in Theorem 2.3. This says that for all predicates ϕ on the ciphertext space, and all messages m_0, m_1, we have

$$\Pr[\phi(E(\mathbf{k}, m_0))] = \Pr[\phi(E(\mathbf{k}, m_1))], \tag{2.5}$$

where $\mathbf{k}$ is a random variable uniformly distributed over the key space $\mathcal{K}$. Instead of insisting that these probabilities are equal, we shall only require that they are very close; that is,

$$\Big|\Pr[\phi(E(\mathbf{k}, m_0))] - \Pr[\phi(E(\mathbf{k}, m_1))]\Big| \leq \epsilon, \tag{2.6}$$

for some very small, or *negligible*, value of ϵ. By itself, this relaxation does not help very much (see Exercise 2.5). However, instead of requiring that (2.6) holds for every possible ϕ, m_0, and m_1, we only require that (2.6) holds for all messages m_0 and m_1 that can be generated by some efficient algorithm, and all predicates ϕ that can be computed by some efficient algorithm (these algorithms could be probabilistic). For example, suppose it were the case that using the best possible algorithms for generating m_0 and m_1, and for testing some predicate ϕ, and using (say) 10,000 computers in parallel for 10 years to perform these calculations, (2.6) holds for $\epsilon = 2^{-100}$. While not perfectly secure, we might be willing to say that the cipher is *secure for all practical purposes.*

Also, in defining semantic security, we address an issue raised in Example 2.5. In that example, we saw that the variable length one-time pad did not satisfy the definition of perfect security. However, we want our definition to be flexible enough so that ciphers like the variable length one-time pad, which effectively leak no information about an encrypted message other than its length, may be considered secure as well.

Now the details. To precisely formulate the definition of semantic security, we shall describe an **attack game** played between two parties: the **challenger** and an **adversary**. As we will see, the

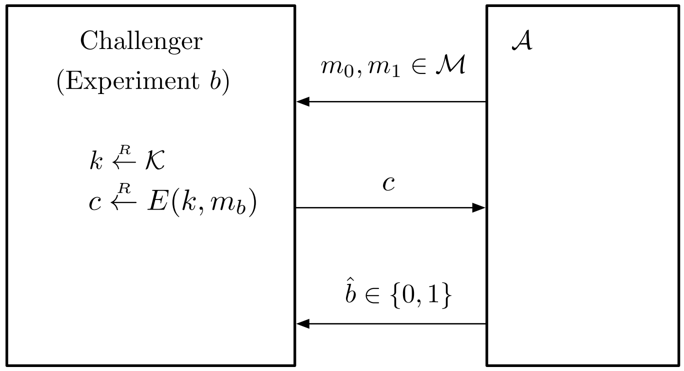


Figure 2.1: Experiment b of Attack Game 2.1

challenger follows a very simple, fixed protocol. However, an adversary $\mathcal{A}$ may follow an arbitrary (but still efficient) protocol. The challenger and the adversary $\mathcal{A}$ send messages back and forth to each other, as specified by their protocols, and at the end of the game, $\mathcal{A}$ outputs some value. Actually, our attack game for defining semantic security comprises two alternative "sub-games," or "experiments" — in both experiments, the adversary follows the same protocol; however, the challenger's behavior is slightly different in the two experiments. The attack game also defines a probability space, and this in turn defines the adversary's *advantage*, which measures the difference between the probabilities of two events in this probability space.

Attack Game 2.1 (semantic security). For a given cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define

Experiment b:

- The adversary computes $m_0, m_1 \in \mathcal{M}$, of the same length, and sends them to the challenger.
- The challenger computes $k \xleftarrow{R} \mathcal{K}$, $c \xleftarrow{R} E(k, m_b)$, and sends c to the adversary.
- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **semantic security advantage** with respect to $\mathcal{E}$ as

$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \Box$$

Note that in the above game, the events W_0 and W_1 are defined with respect to the probability space determined by the random choice of k, the random choices made (if any) by the encryption algorithm, and the random choices made (if any) by the adversary. The value $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}]$ is a number between 0 and 1.

See Fig. 2.1 for a schematic diagram of Attack Game 2.1. As indicated in the diagram, $\mathcal{A}$'s "output" is really just a final message to the challenger.

Definition 2.2 (semantic security). *A cipher $\mathcal{E}$ is* ***semantically secure*** *if for all efficient adversaries $\mathcal{A}$, the value* $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

As a formal definition, this is not quite complete, as we have yet to define what we mean by "messages of the same length", "efficient adversaries", and "negligible". We will come back to this shortly.

Let us relate this formal definition to the discussion preceding it. Suppose that the adversary $\mathcal{A}$ in Attack Game 2.1 is deterministic. First, the adversary computes in a deterministic fashion messages m_0, m_1, and then evaluates a predicate ϕ on the ciphertext c, outputting 1 if true and 0 if false. Semantic security says that the value ϵ in (2.6) is negligible. In the case where $\mathcal{A}$ is probabilistic, we can view $\mathcal{A}$ as being structured as follows: it generates a random value r from some appropriate set, and deterministically computes messages $m_0^{(r)}$, $m_1^{(r)}$, which depend on r, and evaluates a predicate $\phi^{(r)}$ on c, which also depends on r. Here, semantic security says that the value ϵ in (2.6), with m_0, m_1, ϕ replaced by $m_0^{(r)}, m_1^{(r)}, \phi^{(r)}$, is negligible — but where now the probability is with respect to a randomly chosen key and a randomly chosen value of r.

Remark 2.1. Let us now say a few words about the requirement that the messages m_0 and m_1 computed by the adversary Attack Game 2.1 be of the same length.

- First, the notion of the "length" of a message is specific to the particular message space $\mathcal{M}$; in other words, in specifying a message space, one must specify a rule that associates a length (which is a non-negative integer) with any given message. For most concrete message spaces, this will be clear: for example, for the message space $\{0,1\}^{\leq L}$ (as in Example 2.2), the length of a message $m \in \{0,1\}^{\leq L}$ is simply its length, $|m|$, as a bit string. However, to make our definition somewhat general, we leave the notion of length as an abstraction. Indeed, some message spaces may have no particular notion of length, in which case all messages may be viewed as having length 0.

- Second, the requirement that m_0 and m_1 be of the same length means that the adversary is not deemed to have broken the system just because he can effectively distinguish an encryption of a message of one length from an encryption of a message of a different length. This is how our formal definition captures the notion that an encryption of a message is allowed to leak the length of the message (but nothing else).

  We already discussed in Example 2.5 how in certain applications, leaking the just length of the message can be catastrophic. However, since there is no general solution to this problem, most real-world encryption schemes (for example, TLS) do not make any attempt at all to hide the length of the message. This can lead to real attacks. For example, Chen et al. [25] show that the lengths of encrypted messages can reveal considerable information about private data that a user supplies to a cloud application. They use an online tax filing system as their example, but other works show attacks of this type on many other systems. □

Example 2.9. Let $\mathcal{E}$ be a deterministic cipher that is perfectly secure. Then it is easy to see that for every adversary $\mathcal{A}$ (efficient or not), we have $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] = 0$. This follows almost immediately from Theorem 2.3 (the only slight complication is that our adversary $\mathcal{A}$ in Attack Game 2.1 may be probabilistic, but this is easily dealt with). In particular, $\mathcal{E}$ is semantically secure. Thus, if $\mathcal{E}$ is the one-time pad (see Example 2.1), we have $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] = 0$ for all adversaries $\mathcal{A}$; in particular, the one-time pad is semantically secure. Because the definition of semantic security is a bit more forgiving with regard to variable length message spaces, it is also easy to see that if $\mathcal{E}$ is the variable length one-time pad (see Example 2.2), then $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] = 0$ for all adversaries $\mathcal{A}$; in particular, the variable length one-time pad is also semantically secure. □

We need to say a few words about the terms "efficient" and "negligible". Below in Section 2.4 we will fill in the remaining details (they are somewhat tedious, and not really very enlightening). Intuitively, *negligible* means so small as to be "zero for all practical purposes": think of a number like 2^{-100} — if the probability that you spontaneously combust in the next year is 2^{-100}, then you would not worry about such an event occurring any more than you would an event that occurred with probability 0. Also, an *efficient adversary* is one that runs ins a"reasonable" amount time.

We introduce two other terms:

- A value N is called *super-poly* is $1/N$ is negligible.
- A *poly-bounded* value which intuitively a reasonably sized number — in particular, we can say that the running time of any efficient adversary is a poly-bounded value.

Fact 2.6. *If ϵ and ϵ' are negligible values, and Q and Q' are poly-bounded values, then:*

(i) $\epsilon + \epsilon'$ *is a negligible value,*

(ii) $Q + Q'$ *and* $Q \cdot Q'$ *are poly-bounded values, and*

(iii) $Q \cdot \epsilon$ *is a negligible value.*

For now, the reader can just take these facts as axioms. Instead of dwelling on these technical issues, we discuss an example that illustrates how one typically *uses* this definition in analyzing the security of a larger system that uses a semantically secure cipher.

2.3.3 Connections to weaker notions of security

Message recovery attacks

Intuitively, in a message recovery attack, an adversary is given an encryption of a random message, and is able to recover the message from the ciphertext with probability significantly better than random guessing, that is, probability $1/|\mathcal{M}|$. Of course, any reasonable notion of security should rule out such an attack, and indeed, semantic security does.

While this may seem intuitively obvious, we give a formal proof of this. One of our motivations for doing this is to illustrate in detail the notion of a *security reduction*, which is the main technique used to reason about the security of systems. Basically, the proof will argue that any efficient adversary $\mathcal{A}$ that can effectively mount a message recovery attack on $\mathcal{E}$ can be used to build an efficient adversary $\mathcal{B}$ that breaks the semantic security of $\mathcal{E}$; since semantic security implies that no such $\mathcal{B}$ exists, we may conclude that no such $\mathcal{A}$ exists.

To formulate this proof in more detail, we need a formal definition of a message recovery attack. As before, this is done by giving attack game, which is a protocol between a challenger and an adversary.

Attack Game 2.2 (message recovery). For a given cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, the attack game proceeds as follows:

- The challenger computes $m \xleftarrow{\text{R}} \mathcal{M}$, $k \xleftarrow{\text{R}} \mathcal{K}$, $c \xleftarrow{\text{R}} E(k, m)$, and sends c to the adversary.

- The adversary outputs a message $\hat{m} \in \mathcal{M}$.

Let W be the event that $\hat{m} = m$. We say that $\mathcal{A}$ wins the game in this case, and we define $\mathcal{A}$'s **message recovery advantage** with respect to $\mathcal{E}$ as

$$\mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := \big|\Pr[W] - 1/|\mathcal{M}|\big|. \quad \square$$

Definition 2.3 (security against message recovery). *A cipher $\mathcal{E}$ is* ***secure against message recovery*** *if for all efficient adversaries $\mathcal{A}$, the value $\mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.*

Theorem 2.7. *Let $\mathcal{E} = (E, D)$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. If $\mathcal{E}$ is semantically secure then $\mathcal{E}$ is secure against message recovery.*

Proof. Assume that $\mathcal{E}$ is semantically secure. Our goal is to show that $\mathcal{E}$ is secure against message recovery.

To prove that $\mathcal{E}$ is secure against message recovery, we have to show that every efficient adversary $\mathcal{A}$ has negligible advantage in Attack Game 2.2. To show this, we let an arbitrary but efficient adversary $\mathcal{A}$ be given, and our goal now is to show that $\mathcal{A}$'s message recovery advantage, $\mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$, is negligible. Let p denote the probability that $\mathcal{A}$ wins the message recovery game, so that

$$\mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = \big|p - 1/|\mathcal{M}|\big|.$$

We shall show how to construct an efficient adversary $\mathcal{B}$ whose semantic security advantage in Attack Game 2.1 is related to $\mathcal{A}$'s message recovery advantage as follows:

$$\mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}]. \tag{2.7}$$

Since $\mathcal{B}$ is efficient, and since we are assume $\mathcal{E}$ is semantically secure, the right-hand side of (2.7) is negligible, and so we conclude that $\mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.

So all that remains to complete the proof is to show how to construct an efficient $\mathcal{B}$ that satisfies (2.7). The idea is to use $\mathcal{A}$ as a "black box" — we do not have to understand the inner workings of $\mathcal{A}$ as at all.

Here is how $\mathcal{B}$ works. Adversary $\mathcal{B}$ generates two random messages, m_0 and m_1, and sends these to its own SS challenger. This challenger sends $\mathcal{B}$ a ciphertext c, which $\mathcal{B}$ forwards to $\mathcal{A}$, *as if it were coming from $\mathcal{A}$'s MR challenger*. When $\mathcal{A}$ outputs a message $\hat{m}$, our adversary $\mathcal{B}$ compares m_0 to $\hat{m}$, and outputs $\hat{b} = 1$ if $m_0 = \hat{m}$, and $\hat{b} = 1$ otherwise.

That completes the description of $\mathcal{B}$, which is illustrated in Fig. **??**.

Note that the running time of $\mathcal{B}$ is essentially the same as that of $\mathcal{A}$. We now analyze the $\mathcal{B}$'s SS advantage, and relate this to $\mathcal{A}$'s MR advantage.

For $b = 0, 1$, let p_b be the probability that $\mathcal{B}$ outputs 1 if $\mathcal{B}$'s SS challenger encrypts m_b. So by definition

$$\mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}] = |p_1 - p_0|.$$

On the one hand, when c is an encryption of m_0, the probability p_0 is precisely equal to $\mathcal{A}$'s probability of winning the message recovery game, so $p_0 = p$. On the other hand, when c is an encryption of m_1, the adversary $\mathcal{A}$'s output is independent of m_0, and so $p_1 = 1/|\mathcal{M}|$. It follows that

$$\mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}] = |p_1 - p_0| = \big|1/|\mathcal{M}| - p\big| = \mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}].$$

This proves (2.7). In fact, equality holds in (2.7), but that is not essential to the proof. $\square$

The reader should make sure that he or she understands the logic of this proof, as this type of proof will be used over and over again throughout the book. We shall review the important parts of the proof here, and give another way of thinking about it.

The core of the proof was establishing the following fact: for every efficient MR adversary $\mathcal{A}$ that attacks $\mathcal{E}$ as in Attack Game 2.2, there exists an efficient SS adversary $\mathcal{B}$ that attacks $\mathcal{E}$ as in Attack Game 2.1 such that

$$\mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}]. \tag{2.8}$$

We are trying to prove that if $\mathcal{E}$ is semantically secure, then $\mathcal{E}$ is secure against message recovery. In the above proof, we argued that if $\mathcal{E}$ is semantically secure, then the right-hand side of (2.8) must be negligible, and hence so must the left-hand side; since this holds for all efficient $\mathcal{A}$, we conclude that $\mathcal{E}$ is secure against message recovery.

Another way to approach the proof of the theorem is to prove the contrapositive: if $\mathcal{E}$ is *not* secure against message recovery, then $\mathcal{E}$ is *not* semantically secure. So, let us assume that $\mathcal{E}$ is not secure against message recovery. This means there exists an efficient adversary $\mathcal{A}$ whose message recovery advantage is non-negligible. Using $\mathcal{A}$ we build an efficient adversary $\mathcal{B}$ that satisfies (2.8). By assumption, $\mathrm{MR}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is non-negligible, and (2.8) implies that $\mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}]$ is non-negligible. From this, we conclude that $\mathcal{E}$ is not semantically secure.

Said even more briefly: to prove that semantic security implies security against message recovery, we show how to turn an efficient adversary that breaks message recovery into an efficient adversary that breaks semantic security.

We also stress that the adversary $\mathcal{B}$ constructed in the proof just uses $\mathcal{A}$ as a "black box." In fact, almost all of the constructions we shall see are of this type: $\mathcal{B}$ is essentially just a wrapper around $\mathcal{A}$, consisting of some simple and efficient "interface layer" between $\mathcal{B}$'s challenger and a single running instance of $\mathcal{A}$. Ideally, we want the computational complexity of the interface layer to not depend on the computational complexity of $\mathcal{A}$; however, some dependence is unavoidable: if an attack game allows $\mathcal{A}$ to make multiple queries to its challenger, the more queries $\mathcal{A}$ makes, the more work must be performed by the interface layer, but this work should just depend on the number of such queries and not on the running time of $\mathcal{A}$.

Thus, we will say adversary $\mathcal{B}$ is an **elementary wrapper** around adversary $\mathcal{A}$ when it can be structured as above, as an efficient interface interacting with $\mathcal{A}$. The salient properties are:

- If $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, and $\mathcal{A}$ is efficient, then $\mathcal{B}$ is efficient.

- If $\mathcal{C}$ is an elementary wrapper around $\mathcal{B}$ and $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, then $\mathcal{C}$ is an elementary wrapper around $\mathcal{A}$.

These notions are formalized in Section 2.4 (but again, they are extremely tedious).

Computing individual bits of a message

If an encryption scheme is secure, not only should it be hard to recover the whole message, but it should be hard to compute any partial information about the message.

We will not prove a completely general theorem here, but rather, consider a specific example.

Suppose $\mathcal{E} = (E, D)$ is a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{M} = \{0,1\}^L$. For $m \in \mathcal{M}$, we define $\mathrm{parity}(m)$ to be 1 if the number of 1's in m is odd, and 0 otherwise. Equivalently, $\mathrm{parity}(m)$ is the exclusive-OR of all the individual bits of m.

We will show that if $\mathcal{E}$ is semantically secure, then given an encryption c of a random message m, it is hard to predict parity(m). Now, since parity(m) is a single bit, any adversary can predict this value correctly with probability 1/2 just by random guessing. But what we want to show is that no efficient adversary can do significantly better than random guessing.

As a warm up, suppose there were an efficient adversary $\mathcal{A}$ that could predict parity(m) with probability 1. This means that for every message m, every key k, and every encryption c of m, when we give $\mathcal{A}$ the ciphertext c, it outputs the parity of m. So we could use $\mathcal{A}$ to build an SS adversary $\mathcal{B}$ that works as follows. Our adversary chooses two messages, m_0 and m_1, arbitrarily, but with parity$(m_0) = 0$ and parity$(m_1) = 1$. Then it hands these two messages to its own SS challenger, obtaining a ciphertext c, which it then forwards to it $\mathcal{A}$. After receiving c, adversary $\mathcal{A}$ outputs a bit $\hat{b}$, and $\mathcal{B}$ outputs this same bit $\hat{b}$ as its own output. It is easy to see that $\mathcal{B}$'s SS advantage is precisely 1: when its SS challenger encrypts m_0, it always outputs 0, and when its SS challenger encrypts m_1, it always outputs 1.

This shows that if $\mathcal{E}$ is semantically secure, there is no efficient adversary that can predict parity with probability 1. However, we can say even more: if $\mathcal{E}$ is semantically secure, there is no efficient adversary that can predict parity with probability significantly better than 1/2. To make this precise, we give an attack game:

Attack Game 2.3 (parity prediction). For a given cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, the attack game proceeds as follows:

- The challenger computes $m \xleftarrow{\text{R}} \mathcal{M}$, $k \xleftarrow{\text{R}} \mathcal{K}$, $c \xleftarrow{\text{R}} E(k, m)$, and sends c to the adversary.
- The adversary outputs $\hat{b} \in \{0, 1\}$.

Let W be the event that $\hat{b} = \text{parity}(m)$. We define $\mathcal{A}$'s **message recovery advantage** with respect to $\mathcal{E}$ as

$$\text{Parity}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := \Big|\Pr[W] - 1/2\Big|. \quad \Box$$

Definition 2.4 (parity prediction). *A cipher $\mathcal{E}$ is **secure against parity prediction** if for all efficient adversaries $\mathcal{A}$, the value* Parity$\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

Theorem 2.8. *Let $\mathcal{E} = (E, D)$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and $\mathcal{M} = \{0, 1\}^L$. If $\mathcal{E}$ is semantically secure, then $\mathcal{E}$ is secure against parity prediction.*

Proof. As in the proof of Theorem 2.7, we give a proof by reduction. In particular, we will show that for every parity prediction adversary $\mathcal{A}$ that attacks $\mathcal{E}$ as in Attack Game 2.3, there exists an SS adversary $\mathcal{B}$ that attacks $\mathcal{E}$ as in Attack Game 2.1, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{Parity}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = \frac{1}{2} \cdot \text{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}].$$

Let $\mathcal{A}$ be a parity prediction adversary that predicts parity with probability $1/2 + \epsilon$, so Parity$\mathsf{adv}[\mathcal{A}, \mathcal{E}] = |\epsilon|$.

Here is how we construct our SS adversary $\mathcal{B}$.

Our adversary $\mathcal{B}$ generates a random message m_0, and sets $m_1 \leftarrow m_0 \oplus (0^{L-1} \parallel 1)$; that is, m_1 is that same as m_0, except that the last bit is flipped. In particularly, m_0 and m_1 have opposite parity.

Our adversary $\mathcal{B}$ sends the pair m_0, m_1 to its own SS challenger, receives a ciphertext c from that challenger, and forwards c to $\mathcal{A}$. When $\mathcal{A}$ outputs a bit $\hat{b}$, our adversary $\mathcal{B}$ outputs 1 if $\hat{b} = \text{parity}(m_0)$, and outputs 0, otherwise.

For $b = 0, 1$, let p_b be the probability that $\mathcal{B}$ outputs 1 if $\mathcal{B}$'s SS challenger encrypts m_b. So by definition

$$\text{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}] = |p_1 - p_0|.$$

We claim that $p_0 = 1/2 + \epsilon$ and $p_1 = 1/2 - \epsilon$. This because regardless of whether m_0 or m_1 is encrypted, the distribution of m_b is uniform over $\mathcal{M}$, and so in case $b = 0$, our parity predictor $\mathcal{A}$ will output parity(m_0) with probability $1/2 + \epsilon$, and when $b = 1$, our parity predictor $\mathcal{A}$ with output parity(m_1) with probability $1/2 + \epsilon$, and so outputs parity(m_0) with probability $1 - (1/2 + \epsilon) = 1/2 - \epsilon$.

Therefore,

$$\text{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}] = |p_1 - p_0| = 2|\epsilon| = 2 \cdot \text{Parity}\mathsf{adv}[\mathcal{A}, \mathcal{E}],$$

which proves the theorem. $\square$

We have shown that if an adversary can effectively predict the parity of a message, then it can be used to break semantic security. Conversely, it turns out that if an adversary can break semantic security, he can effectively predict *some* predicate of the message (see Exercise 3.15).

2.3.4 Consequences of semantic security

In this section, we examine the consequences of semantic security in the context of a specific example, namely, electronic gambling. The specific details of the example are not so important, but the example illustrates how one typically uses the assumption of semantic security in applications.

Consider the following extremely simplified version of roulette, which is a game between the *house* and a *player*. The player gives the house 1 dollar. He may place one of two kinds of bets:

- "high or low," or
- "even or odd."

After placing his bet, the house chooses a random number $r \in \{0, 1, \ldots, 36\}$. The player wins if $r \neq 0$, and if

- he bet "high" and $r > 18$,
- he bet "low" and $r \leq 18$,
- he bet "even" and r is even,
- he bet "odd" and r is odd.

If the player wins, the house pays him 2 dollars (for a net win of 1 dollar), and if the player looses, the house pays nothing (for a net loss of 1 dollar). Clearly, the house has a small, but not insignificant advantage in this game: the probability that the player wins is $18/37 \approx 48.65\%$.

Now suppose that this game is played over the Internet. Also, suppose that for various technical reasons, the house publishes an encryption of r *before* the player places his bet (perhaps to be decrypted by some regulatory agency that shares a key with the house). The player is free to analyze this encryption before placing his bet, and of course, by doing so, the player could conceivably

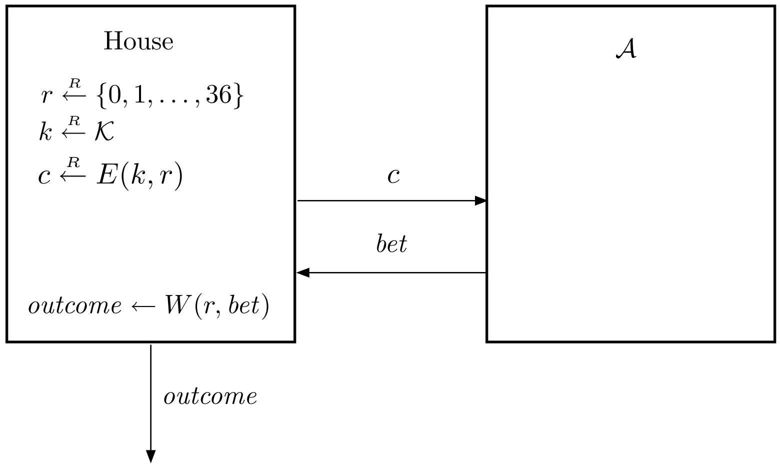


Figure 2.2: Internet roulette

increase his chances of winning. However, if the cipher is any good, the player's chances should not increase by much. Let us prove this, assuming r is encrypted using a semantically secure cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{M} = \{0, 1, \dots, 36\}$ (we shall view all messages in $\mathcal{M}$ as having the same length in this example). Also, from now in, let us call the player $\mathcal{A}$, to stress the adversarial nature of the player, and assume that $\mathcal{A}$'s strategy can be modeled as an efficient algorithm. The game is illustrated in Fig. 2.2. Here, *bet* denotes one of "high," "low," "even," "odd." Player $\mathcal{A}$ sends *bet* to the house, who evaluates the function $W(r, bet)$, which is 1 if *bet* is a winning bet with respect to r, and 0 otherwise. Let us define

$$\mathrm{IR}\mathsf{adv}[\mathcal{A}] := \big|\Pr[W(r, bet) = 1] - 18/37\big|.$$

Our goal is to prove the following theorem.

Theorem 2.9. *If $\mathcal{E}$ is semantically secure, then for every efficient player $\mathcal{A}$, the quantity* $\mathrm{IR}\mathsf{adv}[\mathcal{A}]$ *is negligible.*

As we did in Section 2.3.3, we prove this by reduction. More concretely, we shall show that for every player $\mathcal{A}$, there exists an SS adversary $\mathcal{B}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{IR}\mathsf{adv}[\mathcal{A}] = \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}]. \tag{2.9}$$

Thus, if there were an efficient player $\mathcal{A}$ with a non-negligible advantage, we would obtain an efficient SS adversary $\mathcal{B}$ that breaks the semantic security of $\mathcal{E}$, which we are assuming is impossible. Therefore, there is no such $\mathcal{A}$.

To motivate and analyze our new adversary $\mathcal{B}$, consider an "idealized" version of Internet roulette, in which instead of publishing an encryption of the actual value r, the house instead publishes an encryption of a "dummy"value, say 0. The logic of the ideal Internet roulette game is illustrated in Fig. 2.3. Note, however, that in the ideal Internet roulette game, the house still uses the *actual* value of r to determine the outcome of the game. Let p_0 be the probability that $\mathcal{A}$ wins at Internet roulette, and let p_1 be the probability that $\mathcal{A}$ wins at ideal Internet roulette.

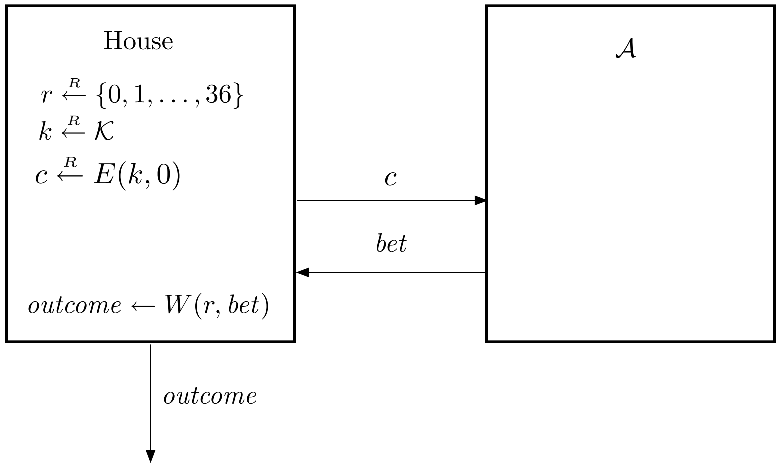


Figure 2.3: ideal Internet roulette

Our adversary $\mathcal{B}$ is designed to play in Attack Game 2.1 so that if $\hat{b}$ denotes $\mathcal{B}$'s output in that game, then we have:

- if $\mathcal{B}$ is placed in Experiment 0, then $\Pr[\hat{b} = 1] = p_0$;
- if $\mathcal{B}$ is placed in Experiment 1, then $\Pr[\hat{b} = 1] = p_1$.

The logic of adversary $\mathcal{B}$ is illustrated in Fig. 2.4. It is clear by construction that $\mathcal{B}$ satisfies the properties claimed above, and so in particular,

$$\mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}] = |p_1 - p_0|. \tag{2.10}$$

Now, consider the probability p_1 that $\mathcal{A}$ wins at ideal Internet roulette. No matter how clever $\mathcal{A}$'s strategy is, he wins with probability 18/37, since in this ideal Internet roulette game, the value of *bet* is computed from c, which is statistically independent of the value of r. That is, ideal Internet roulette is equivalent to physical roulette. Therefore,

$$\mathrm{IR}\mathsf{adv}[\mathcal{A}] = |p_1 - p_0|. \tag{2.11}$$

Combining (2.10) and (2.11), we obtain (2.9).

The approach we have used to analyze Internet roulette is one that we will see again and again. The basic idea is to replace a system component by an idealized version of that component, and then analyze the behavior of this new, idealized version of the system.

Another lesson to take away from the above example is that in reasoning about the security of a system, what we view as "the adversary" depends on what we are trying to do. In the above analysis, we cobbled together a new adversary $\mathcal{B}$ out of several components: one component was the original adversary $\mathcal{A}$, while other components were scavenged from other parts of the system (the algorithm of "the house," in this example). This will be very typical in our security analyses throughout this text. Intuitively, if we imagine a diagram of the system, at different points in the security analysis, we will draw a circle around different components of the system to identify what we consider to be "the adversary" at that point in the analysis.

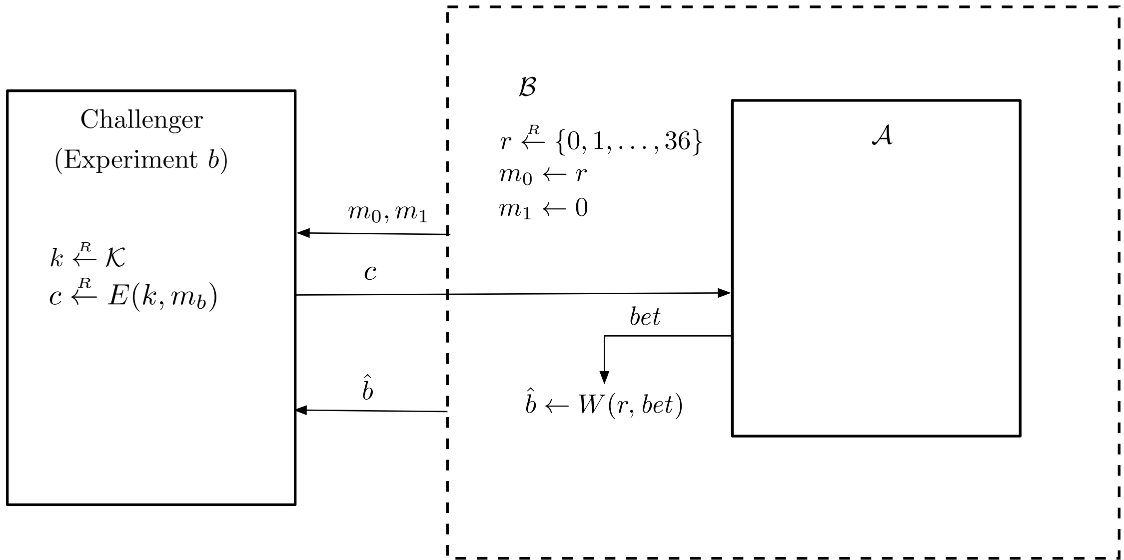


Figure 2.4: The SS adversary $\mathcal{B}$ in Attack Game 2.1

2.3.5 Bit guessing: an alternative characterization of semantic security

The example in Section 2.3.4 was a typical example of how one could use the definition of semantic security to analyze the security properties of a larger system that makes use of a semantically secure cipher. However, there is another characterization of semantic security that is typically more convenient to work with when one is trying to prove that a given cipher satisfies the definition. In this alternative characterization, we define a new attack game. The role played by the adversary is exactly the same as before. However, instead of having two different experiments, there is just a single experiment. In this **bit-guessing version** of the attack game, the challenger chooses $b \in \{0, 1\}$ at random and runs Experiment b of Attack Game 2.1; it is the adversary's goal to guess the bit b with probability significantly better than $1/2$. Here are the details:

Attack Game 2.4 (semantic security: bit-guessing version). For a given cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, the attack game runs as follows:

- The adversary computes $m_0, m_1 \in \mathcal{M}$, of the same length, and sends them to the challenger.
- The challenger computes $b \xleftarrow{R} \{0, 1\}$, $k \xleftarrow{R} \mathcal{K}$, $c \xleftarrow{R} E(k, m_b)$, and sends c to the adversary.
- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

We say that $\mathcal{A}$ **wins** the game if $\hat{b} = b$. □

Fig. 2.5 illustrates Attack Game 2.4. Note that in this game, the event that the $\mathcal{A}$ wins the game is defined with respect to the probability space determined by the random choice of b and k, the random choices made (if any) of the encryption algorithm, and the random choices made (if any) by the adversary.

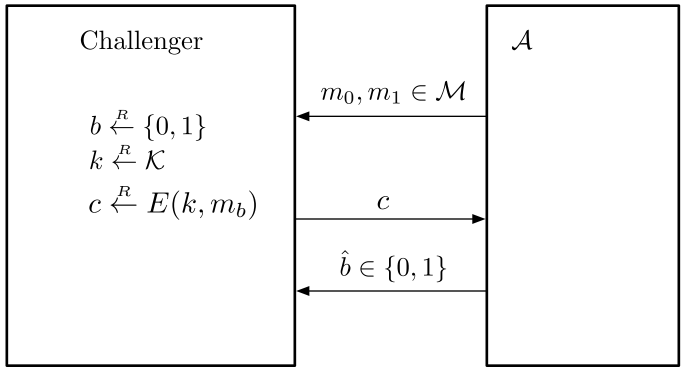


Figure 2.5: Attack Game 2.4

Of course, any adversary can win the game with probability 1/2, simply by ignoring c completely and choosing $\hat{b}$ at random (or alternatively, always choosing $\hat{b}$ to be 0, or always choosing it to be 1). What we are interested in is how much *better* than random guessing an adversary can do. If W denotes the event that the adversary wins the bit-guessing version of the attack game, then we are interested in the quantity $|\Pr[W] - 1/2|$, which we denote by $\mathsf{SSadv}^*[\mathcal{A}, \mathcal{E}]$. Then we have:

Theorem 2.10. *For every cipher $\mathcal{E}$ and every adversary $\mathcal{A}$, we have*

$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \mathsf{SSadv}^*[\mathcal{A}, \mathcal{E}]. \tag{2.12}$$

Proof. This is just a simple calculation. Let p_0 be the probability that the adversary outputs 1 in Experiment 0 of Attack Game 2.1, and let p_1 be the probability that the adversary outputs 1 in Experiment 1 of Attack Game 2.1.

Now consider Attack Game 2.4. From now on, all events and probabilities are with respect to this game. If we condition on the event that $b = 0$, then in this conditional probability space, all of the other random choices made by the challenger and the adversary are distributed in exactly the same way as the corresponding values in Experiment 0 of Attack Game 2.1. Therefore, if $\hat{b}$ is the output of the adversary in Attack Game 2.4, we have

$$\Pr[\hat{b} = 1 \mid b = 0] = p_0.$$

By a similar argument, we see that

$$\Pr[\hat{b} = 1 \mid b = 1] = p_1.$$

So we have

$$\begin{aligned}
\Pr[\hat{b} = b] &= \Pr[\hat{b} = b \mid b = 0] \Pr[b = 0] + \Pr[\hat{b} = b \mid b = 1] \Pr[b = 1] \\
&= \Pr[\hat{b} = 0 \mid b = 0] \cdot \tfrac{1}{2} + \Pr[\hat{b} = 1 \mid b = 1] \cdot \tfrac{1}{2} \\
&= \tfrac{1}{2}\Big(1 - \Pr[\hat{b} = 1 \mid b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\Big) \\
&= \tfrac{1}{2}(1 - p_0 + p_1).
\end{aligned}$$

Therefore,

$$\mathsf{SSadv}^*[\mathcal{A},\mathcal{E}] = \left|\Pr[\hat{b}=b] - \tfrac{1}{2}\right| = \tfrac{1}{2}|p_1 - p_0| = \tfrac{1}{2}\cdot\mathsf{SSadv}[\mathcal{A},\mathcal{E}].$$

That proves the theorem. $\Box$

Just as it is convenient to refer $\mathsf{SSadv}[\mathcal{A},\mathcal{E}]$ as $\mathcal{A}$'s "SS advantage," we shall refer to $\mathsf{SSadv}^*[\mathcal{A},\mathcal{E}]$ as $\mathcal{A}$'s "bit-guessing SS advantage."

A generalization

As it turns out, the above situation is quite generic. Although we do not need it in this chapter, for future reference we indicate here how the above situation generalizes. There will be a number of situations we shall encounter where some particular security property, call it "X," for some of cryptographic system, call it "$\mathcal{S}$," can be defined in terms of an attack game involving two experiments, Experiment 0 and Experiment 1, where the adversary $\mathcal{A}$'s protocol is the same in both experiments, while that of the challenger is different. For $b = 0, 1$, we define W_b to be the event that $\mathcal{A}$ outputs 1 in Experiment b, and we define

$$\mathsf{Xadv}[\mathcal{A},\mathcal{S}] := \left|\Pr[W_0] - \Pr[W_1]\right|$$

to be $\mathcal{A}$'s "X advantage." Just as above, we can always define a "bit-guessing" version of the attack game, in which the challenger chooses $b \in \{0,1\}$ at random, and then runs Experiment b as its protocol. If W is the event that the adversary's output is equal to b, then we define

$$\mathsf{Xadv}^*[\mathcal{A},\mathcal{S}] := \left|\Pr[W] - 1/2\right|$$

to be $\mathcal{A}$'s "bit-guessing X advantage."

Using exactly the same calculation as in the proof of Theorem 2.10, we have

$$\mathsf{Xadv}[\mathcal{A},\mathcal{S}] = 2\cdot\mathsf{Xadv}^*[\mathcal{A},\mathcal{S}]. \tag{2.13}$$

2.4 Mathematical details

Up until now, we have used the terms *efficient* and *negligible* rather loosely, without a formal mathematical definition:

- we required that a computational cipher have *efficient* encryption and decryption algorithms;
- for a semantically secure cipher, we required that any *efficient* adversary have a *negligible* advantage in Attack Game 2.1.

The goal of this section is to provide precise mathematical definitions for these terms. While these definitions lead to a satisfying theoretical framework for the study of cryptography as a mathematical discipline, we should warn the reader:

- the definitions are rather complicated, requiring an unfortunate amount of notation; and
- the definitions model our intuitive understanding of these terms only very crudely.

We stress that the reader may safely skip this section without suffering a significant loss in understanding. Before marching headlong into the formal definitions, let us remind the reader of what we are trying to capture in these definitions.

- First, when we speak of an efficient encryption or decryption algorithm, we usually mean one that runs very quickly, encrypting data at a rate of, say, 10–100 computer cycles per byte of data.

- Second, when we speak of an efficient adversary, we usually mean an algorithm that runs in some large, but still feasible amount of time (and other resources). Typically, one assumes that an adversary that is trying to break a cryptosystem is willing to expend many more resources than a user of the cryptosystem. Thus, 10,000 computers running in parallel for 10 years may be viewed as an upper limit on what is feasibly computable by a determined, patient, and financially well-off adversary. However, in some settings, like the Internet roulette example in Section 2.3.4, the adversary may have a much more limited amount of time to perform its computations before they become irrelevant.

- Third, when we speak of an adversary's advantage as being negligible, we mean that it is so small that it may as well be regarded as being equal to zero for all practical purposes. As we saw in the Internet roulette example, if no efficient adversary has an advantage better than 2^{-100} in Attack Game 2.1, then no player can in practice improve his odds at winning Internet roulette by more than 2^{-100} relative to physical roulette.

Even though our intuitive understanding of the term *efficient* depends on the context, our formal definition will not make any such distinction. Indeed, we shall adopt the computational complexity theorist's habit of equating the notion of an *efficient* algorithm with that of a *(probabilistic) polynomial-time* algorithm. For better and for worse, this gives us a formal framework that is independent of the specific details of any particular model of computation.

2.4.1 Negligible, super-poly, and poly-bounded functions

We begin by defining the notions of *negligible*, *super-poly*, and *poly-bounded* functions.

Intuitively, a negligible function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is one that not only tends to zero as $n \to \infty$, but does so faster than the inverse of any polynomial.

Definition 2.5. *A function $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ is called* ***negligible*** *if for all $c \in \mathbb{R}_{>0}$ there exists $n_0 \in \mathbb{Z}_{\geq 1}$ such that for all integers $n \geq n_0$, we have $|f(n)| < 1/n^c$.*

An alternative characterization of a negligible function, which is perhaps easier to work with, is the following:

Theorem 2.11. *A function $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ is negligible if and only if for all $c > 0$, we have*

$$\lim_{n\to\infty} f(n)n^c = 0.$$

Proof. Exercise. □

Example 2.10. Some examples of negligible functions:

$$2^{-n}, \quad 2^{-\sqrt{n}}, \quad n^{-\log n}.$$

Some examples of non-negligible functions:

$$\frac{1}{1000n^4 + n^2 \log n}, \quad \frac{1}{n^{100}}. \quad \Box$$

Once we have the term "negligible" formally defined, defining "super-poly" is easy:

Definition 2.6. *A function $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ is called* ***super-poly*** *if $1/f$ is negligible.*

Essentially, a poly-bounded function $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ is one that is bounded (in absolute value) by some polynomial. Formally:

Definition 2.7. *A function $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ is called* ***poly-bounded****, if there exists $c, d \in \mathbb{R}_{>0}$ such that for all integers $n \geq 0$, we have $|f(n)| \leq n^c + d$.*

Note that if f is a poly-bounded function, then $1/f$ is definitely *not* a negligible function. However, as the following example illustrates, one must take care not to draw erroneous inferences.

Example 2.11. Define $f : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ so that $f(n) = 1/n$ for all even integers n and $f(n) = 2^{-n}$ for all odd integers n. Then f is not negligible, and $1/f$ is neither poly-bounded nor super-poly. $\Box$

2.4.2 Computational ciphers: the formalities

Now the formalities. We begin by admitting a lie: when we said a computational cipher $\mathcal{E} = (E, D)$ is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{K}$ is the key space, $\mathcal{M}$ is the message space, and $\mathcal{C}$ is the ciphertext space, and with each of these spaces being finite sets, we were not telling the whole truth. In the mathematical model (though not always in real-world systems), we associate with $\mathcal{E}$ *families* of key, message, and ciphertext spaces, indexed by

- a **security parameter**, which is a positive integer, and is denoted by λ, and
- a **system parameter**, which is a bit string, and is denoted by Λ.

Thus, instead of just finite sets $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{C}$, we have families of finite sets

$$\{\mathcal{K}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \{\mathcal{M}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \text{and} \quad \{\mathcal{C}_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$

which for the purposes of this definition, we view as sets of bit strings (which may represent mathematical objects by way of some canonical encoding functions).

The idea is that when the cipher $\mathcal{E}$ is deployed, the security parameter λ is fixed to some value. Generally speaking, larger values of λ imply higher levels of security (i.e., resistance against adversaries with more computational resources), but also larger key sizes, as well as slower encryption and decryption speeds. Thus, the security parameter is like a "dial" we can turn, setting a trade-off between security and efficiency.

Once λ is chosen, a system parameter Λ is generated using an algorithm specific to the cipher. The idea is that the system parameter Λ (together with λ) gives a detailed description of a fixed instance of the cipher, with

$$(\mathcal{K}, \mathcal{M}, \mathcal{C}) = (\mathcal{K}_{\lambda,\Lambda},\ \mathcal{M}_{\lambda,\Lambda},\ \mathcal{C}_{\lambda,\Lambda}).$$

This one, fixed instance may be deployed in a larger system and used by many parties — the values of λ and Λ are public and known to everyone (including the adversary).

Example 2.12. Consider the additive one-time pad discussed in Example 2.4. This cipher was described in terms of a modulus n. To deploy such a cipher, a suitable modulus n is generated, and is made public (possibly just "hardwired" into the software that implements the cipher). The modulus n is the system parameter for this cipher. Each specific value of the security parameter determines the length, in bits, of n. The value n itself is generated by some algorithm that may be probabilistic and whose output distribution may depend on the intended application. For example, we may want to insist that n is a prime in some applications. □

Before going further, we define the notion of an efficient algorithm. For the purposes of this definition, we shall only consider algorithms A that take as input a security parameter λ, as well as other parameters whose total length is bounded by some fixed polynomial in λ. Basically, we want to say that the running time of A is bounded by a polynomial in λ, but things are complicated if A is probabilistic:

Definition 2.8 (efficient algorithm). *Let A be a an algorithm (possibly probabilistic) that takes as input a security parameter $\lambda \in \mathbb{Z}_{\geq 1}$, as well as other parameters encoded as a bit string $x \in \{0,1\}^{\leq p(\lambda)}$ for some fixed polynomial p. We call A an **efficient algorithm** if there exist a poly-bounded function t and a negligible function ϵ such that for all $\lambda \in \mathbb{Z}_{\geq 1}$, and all $x \in \{0,1\}^{\leq p(\lambda)}$, the probability that the running time of A on input (λ, x) exceeds $t(\lambda)$ is at most $\epsilon(\lambda)$.*

We stress that the probability in the above definition is with respect to the coin tosses of A: this bound on the probability must hold for all possible inputs x.[1]

Here is a formal definition that captures the basic requirements of systems that are parameterized by a security and system parameter, and introduces some more terminology. In the following definition we use the notation $\mathrm{Supp}(P(\lambda))$ to refer to the **support** of the distribution $P(\lambda)$, which is the set of all possible outputs of algorithm P on input λ.

Definition 2.9. *A **system parameterization** is an efficient probabilistic algorithm P that given a security parameter $\lambda \in \mathbb{Z}_{\geq 1}$ as input, outputs a bit string Λ, called a **system parameter**, whose length is always bounded by a polynomial in λ. We also define the following terminology:*

- *A collection $\mathbf{S} = \{\mathcal{S}_{\lambda,\Lambda}\}_{\lambda,\Lambda}$ of finite sets of bits strings, where λ runs over $\mathbb{Z}_{\geq 1}$ and Λ runs over $\mathrm{Supp}(P(\lambda))$, is called a **family of spaces with system parameterization** P, provided the lengths of all the strings in each of the sets $\mathcal{S}_{\lambda,\Lambda}$ are bounded by some polynomial p in λ.*
- *We say that $\mathbf{S}$ is **efficiently recognizable** if there is an efficient* deterministic *algorithm that on input $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, and $s \in \{0,1\}^{\leq p(\lambda)}$, determines if $s \in \mathcal{S}_{\lambda,\Lambda}$.*
- *We say that $\mathbf{S}$ is **efficiently sampleable** if there is an efficient probabilistic algorithm that on input $\lambda \in \mathbb{Z}_{\geq 1}$ and $\Lambda \in \mathrm{Supp}(P(\lambda))$, outputs an element uniformly distributed over $\mathcal{S}_{\lambda,\Lambda}$.*

[1]By not insisting that a probabilistic algorithm halts in a specified time bound with probability 1, we give ourselves a little "wiggle room," which allows us to easily do certain types of random sampling procedure that have no *a priori* running time bound, but are very unlikely to run for too long (e.g., think of flipping a coin until it comes up "heads"). An alternative approach would be to bound the *expected* running time, but this turns out to be somewhat problematic for technical reasons.

Note that this definition of an efficient algorithm does not require that the algorithm halt with probability 1 on all inputs. An algorithm that with probability $2^{-\lambda}$ entered an infinite loop would satisfy the definition, even though it does not halt with probability 1. These issues are rather orthogonal. In general, we shall only consider algorithms that halt with probability 1 on all inputs: this can more naturally be seen as a requirement on the output distribution of the algorithm, rather than on its running time.

- *We say that* $\mathbf{S}$ ***has an effective length function*** *if there is an efficient* deterministic *algorithm that on input* $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, *and* $s \in \mathcal{S}_{\lambda,\Lambda}$, *outputs a non-negative integer, called the* ***length*** *of* s.

We can now state the complete, formal definition of a computational cipher:

Definition 2.10 (computational cipher). *A* ***computational cipher*** *consists of a pair of algorithms E and D, along with three families of spaces with system parameterization P:*

$$\mathbf{K} = \{\mathcal{K}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \mathbf{M} = \{\mathcal{M}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \text{and} \quad \mathbf{C} = \{\mathcal{C}_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$

such that

1. $\mathbf{K}$, $\mathbf{M}$, *and* $\mathbf{C}$ *are efficiently recognizable.*

2. $\mathbf{K}$ *is efficiently sampleable.*

3. $\mathbf{M}$ *has an effective length function.*

4. *Algorithm E is an efficient probabilistic algorithm that on input λ, Λ, k, m, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $k \in \mathcal{K}_{\lambda,\Lambda}$, and $m \in \mathcal{M}_{\lambda,\Lambda}$, always outputs an element of $\mathcal{C}_{\lambda,\Lambda}$.*

5. *Algorithm D is an efficient* deterministic *algorithm that on input λ, Λ, k, c, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $k \in \mathcal{K}_{\lambda,\Lambda}$, and $c \in \mathcal{C}_{\lambda,\Lambda}$, outputs either an element of $\mathcal{M}_{\lambda,\Lambda}$, or a special symbol* reject $\notin \mathcal{M}_{\lambda,\Lambda}$.

6. *For all $\lambda, \Lambda, k, m, c$, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $k \in \mathcal{K}_{\lambda,\Lambda}$, $m \in \mathcal{M}_{\lambda,\Lambda}$, and $c \in \mathrm{Supp}(E(\lambda, \Lambda; k, m))$, we have $D(\lambda, \Lambda; k, c) = m$.*

Note that in the above definition, the encryption and decryption algorithms take λ and Λ as auxiliary inputs. So as to be somewhat consistent with the notation already introduced in Section 2.3.1, we write this as $E(\lambda, \Lambda; \cdots)$ and $D(\lambda, \Lambda; \cdots)$.

Example 2.13. Consider the additive one-time pad (see Example 2.12). In our formal framework, the security parameter λ determines the bit length $L(\lambda)$ of the modulus n, which is the system parameter. The system parameter generation algorithm takes as input λ and generates a modulus n of length $L(\lambda)$. The function $L(\cdot)$ should be polynomially bounded. With this assumption, it is clear that the system parameter generation algorithm satisfies its requirements. The requirements on the key, message, and ciphertext spaces are also satisfied:

1. Elements of these spaces have polynomially bounded lengths: this again follows from our assumption that $L(\cdot)$ is polynomially bounded.

2. The key space is efficiently sampleable: just choose $k \xleftarrow{\text{R}} \{0, \ldots, n-1\}$.

3. The key, message, and ciphertext spaces are efficiently recognizable: just test if a bit string s is the binary encoding of an integer between 0 and $n-1$.

4. The message space also has an effective length function: just output (say) 0. $\Box$

We note that some ciphers (for example the one-time pad) may not need a system parameter. In this case, we can just pretend that the system parameter is, say, the empty string. We also note that some ciphers do not really have a security parameter either; indeed, many industry-standard ciphers simply come ready-made with a fixed key size, with no security parameter that can be tuned. This is simply mismatch between theory and practice — that is just the way it is.

That completes our formal mathematical description of a computational cipher, in all its glorious detail.[2] The reader should hopefully appreciate that while these formalities may allow us to make mathematically precise and meaningful statements, they are not very enlightening, and mostly serve to obscure what is really going on. Therefore, in the main body of the text, we will continue to discuss ciphers using the simplified terminology and notation of Section 2.3.1, with the understanding that all statements made have a proper and natural interpretation in the formal framework discussed in this section. This will be a pattern that is repeated in the sequel: we shall mainly discuss various types of cryptographic schemes using a simplified terminology, without mention of security parameters and system parameters — these mathematical details will be discussed in a separate section, but will generally follow the same general pattern established here.

2.4.3 Efficient adversaries and attack games

In defining the notion of semantic security, we have to define what we mean by an *efficient adversary*. Since this concept will be used extensively throughout the text, we present a more general framework here.

For any type of cryptographic scheme, security will be defined using an attack game, played between an adversary $\mathcal{A}$ and a challenger: $\mathcal{A}$ follows an arbitrary protocol, while the challenger follows some simple, fixed protocol determined by the cryptographic scheme and the notion of security under discussion. Furthermore, both adversary and challenger take as input a common security parameter λ, and the challenger starts the game by computing a corresponding system parameter Λ, and sending this to the adversary.

To model these types of interactions, we introduce the notion of an **interactive machine**. Before such a machine M starts, it always gets the security parameter λ written in a special buffer, and the rest of its internal state is initialized to some default value. Machine M has two other special buffers: an *incoming message buffer* and an *outgoing message buffer*. Machine M may be invoked many times: each invocation starts when M's external environment writes a string to M's incoming message buffer; M reads the message, performs some computation, updates its internal state, and writes a string on its outgoing message buffer, ending the invocation, and the outgoing message is passed to the environment. Thus, M interacts with its environment via a simple message passing system. We assume that M may indicate that it has halted by including some signal in its last outgoing message, and M will essentially ignore any further attempts to invoke it.

We shall assume messages to and from the machine M are restricted to be of *constant length*. This is not a real restriction: we can always simulate the transmission of one long message by sending many shorter ones. However, making a restriction of this type simplifies some of the technicalities. We assume this restriction from now on, for adversaries as well as for any other type of interactive machine.

For any given environment, we can measure the **total running time** of M by counting the number of steps it performs across all invocations until it signals that it has halted. This running time depends not only on M and its random choices, but also on the environment in which M runs.[3]

[2] Note that the definition of a Shannon cipher in Section 2.2.1 remains unchanged. The claim made at the end of Section 2.3.1 that any deterministic computational cipher is also a Shannon cipher needs to be properly interpreted: for each λ and Λ, we get a Shannon cipher defined over $(\mathcal{K}_{\lambda,\Lambda}, \mathcal{M}_{\lambda,\Lambda}, \mathcal{C}_{\lambda,\Lambda})$.

[3] Analogous to the discussion in footnote 1 on page 30, our definition of an efficient interactive machine will not require that it halts with probability 1 for all environments. This is an orthogonal issue, but it will be an implicit

Definition 2.11 (efficient interactive machine). *We say that M is an* ***efficient interactive machine*** *if there exist a poly-bounded function t and a negligible function ϵ, such that for all environments (not even computationally bounded ones), the probability that the total running time of M exceeds $t(\lambda)$ is at most $\epsilon(\lambda)$.*

We naturally model an adversary as an interactive machine. An **efficient adversary** is simply an efficient interactive machine.

We can connect two interactive machines together, say M' and M, to create a new interactive machine $M'' = \langle M', M \rangle$. Messages from the environment to M'' always get routed to M'. The machine M' may send a message to the environment, or to M; in the latter case, the output message sent by M gets sent to M'. We assume that if M halts, then M' does not send it any more messages. See Fig. **??**.

Thus, when M'' is invoked, its incoming message is routed to M', and then M' and M may interact some number of times, and then the invocation of M'' ends when M' sends a message to the environment. We call M' the "open" machine (which interacts with the outside world), and M the "closed" machine (which interacts only with M').

Naturally, we can model the interaction of a challenger and an adversary by connecting two such machines together as above: the challenger becomes the open machine, and the adversary becomes the closed machine.

In our security reductions, we typically show how to use an adversary $\mathcal{A}$ that breaks some system to build an adversary $\mathcal{B}$ that breaks some other system. The essential property that we want is that if $\mathcal{A}$ is efficient, then so is $\mathcal{B}$. However, our reductions are almost always of a very special form, where $\mathcal{B}$ is a wrapper around $\mathcal{A}$, consisting of some simple and efficient "interface layer" between $\mathcal{B}$'s challenger and a single running instance of $\mathcal{A}$.

Ideally, we want the computational complexity of the interface layer to not depend on the computational complexity of $\mathcal{A}$; however, some dependence is unavoidable: the more queries $\mathcal{A}$ makes to its challenger, the more work must be performed by the interface layer, but this work should just depend on the number of such queries and not on the running time of $\mathcal{A}$.

To formalize this, we build $\mathcal{B}$ as a composed machine $\langle M', M \rangle$, where M' represents the interface layer (the "open" machine), and M represents the instance of $\mathcal{A}$ (the "closed" machine). This leads us to the following definition.

Definition 2.12 (elementary wrapper). *An interactive machine M' is called an* ***efficient interface*** *if there exists a poly-bounded function t and a negligible function ϵ, such that for all M (not necessarily computationally bounded), when we execute the composed machine $\langle M', M \rangle$ in an arbitrary environment (again, not necessarily computationally bounded), the following property holds:*

> *at every point in the execution of $\langle M', M \rangle$, if I is the number of interactions between M' and M up to at that point, and T is the total running time of M' up to that point, then the probability that $T > t(\lambda + I)$ is at most $\epsilon(\lambda)$.*

If M' is an efficient interface, and M is any machine, then we say $\langle M', M \rangle$ is an ***elementary wrapper around*** *M.*

---

requirement of any machines we consider.

Thus, we will say adversary $\mathcal{B}$ is an elementary wrapper around adversary $\mathcal{A}$ when it can be structured as above, as an efficient interface interacting with $\mathcal{A}$. Our definitions were designed to work well together. The salient properties are:

- If $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, and $\mathcal{A}$ is efficient, then $\mathcal{B}$ is efficient.
- If $\mathcal{C}$ is an elementary wrapper around $\mathcal{B}$ and $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, then $\mathcal{C}$ is an elementary wrapper around $\mathcal{A}$.

Also note that in our attack games, the challenger is typically satisfies our definition of an efficient interface. For such a challenger and any efficient adversary $\mathcal{A}$, we can view their entire interaction as a that of a single, efficient machine.

Query bounded adversaries. In the attack games we have seen so far, the adversary makes just a fixed number of queries. Later in the text, we will see attack games in which the adversary $\mathcal{A}$ is allowed to make many queries — even though there is no a priori bound on the number of queries it is allowed to make, if $\mathcal{A}$ is efficient, the number of queries will be bounded by some poly-bounded value Q (at least with all but negligible probability). In proving security for such attack games, in designing an elementary wrapper $\mathcal{B}$ from $\mathcal{A}$, it will usually be convenient to tell $\mathcal{B}$ *in advance* an upper bound Q on how many queries $\mathcal{A}$ will ultimately make. To fit this into our formal framework, we can set things up so that $\mathcal{A}$ starts out by sending a sequence of Q special messages to "signal" this query bound to $\mathcal{B}$. If we do this, then not only can $\mathcal{B}$ use the value Q in its logic, it is also allowed to run in time that depends on Q, without violating the time constraints in Definition 2.12. This is convenient, as then $\mathcal{B}$ is allowed to initialize data structures whose size may depend on Q. Of course, all of this is just a legalistic "hack" to work around technical constraints that would otherwise be too restrictive, and should not be taken too seriously. We will never make this "signaling" explicit in any of our presentations.

2.4.4 Semantic security: the formalities

In defining any type of security, we will define the adversary's advantage in the attack game as a function $\mathrm{Adv}(\lambda)$. This will be defined in terms of probabilities of certain events in the attack game: for each value of λ we get a different probability space, determined by the random choices of the challenger, and the random choices made the adversary. Security will mean that for every efficient adversary, the function $\mathrm{Adv}(\cdot)$ is negligible.

Turning now to the specific situation of semantic security of a cipher, in Attack Game 2.1, we defined the value $\mathrm{SS}\mathsf{adv}[\mathcal{A},\mathcal{E}]$. This value is actually a function of the security parameter λ. The proper interpretation of Definition 2.3 is that $\mathcal{E}$ is secure if for all efficient adversaries $\mathcal{A}$ (modeled as an interactive machine, as described above), the function $\mathrm{SS}\mathsf{adv}[\mathcal{A},\mathcal{E}](\lambda)$ in the security parameter λ is negligible (as defined in Definition 2.5). Recall that both challenger and adversary receive λ as a common input. Control begins with the challenger, who sends the system parameter to the adversary. The adversary then sends its query to the challenger, which consists of two plaintexts, who responds with a ciphertext. Finally, the adversary outputs a bit (technically, in our formal machine model, this "output" is a message sent to the challenger, and then the challenger halts). The value of $\mathrm{SS}\mathsf{adv}[\mathcal{A},\mathcal{E}](\lambda)$ is determined by the random choices of the challenger (including the choice of system parameter) and the random choices of the adversary. See Fig. 2.6 for a complete picture of Attack Game 2.1.

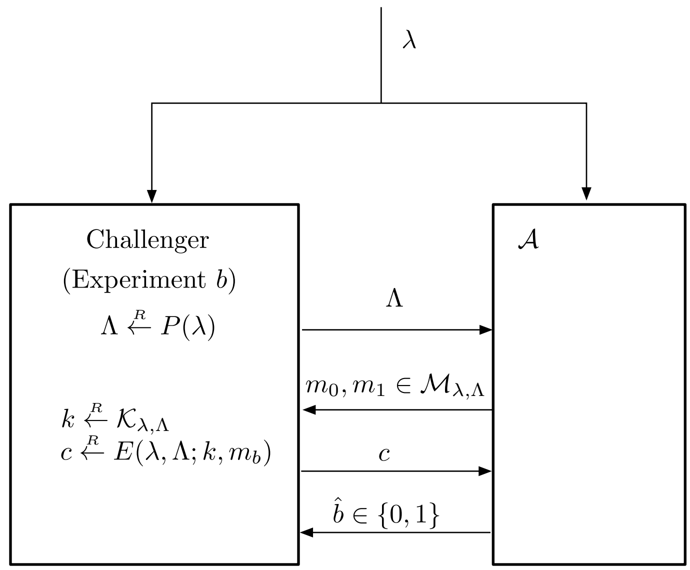


Figure 2.6: The fully detailed version of Attack Game 2.1

Also, in Attack Game 2.1, the requirement that the two messages presented by the adversary have the same length means that the length function provided in part 3 of Definition 2.10 evaluates to the same value on the two messages.

It is perhaps useful to see what it means for a cipher $\mathcal{E}$ to be insecure according to this formal definition. This means that there exists an adversary $\mathcal{A}$ such that $\mathrm{SS}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is a non-negligible function in the security parameter. This means that $\mathrm{SS}\mathsf{adv}[\mathcal{A}, \mathcal{E}](\lambda) \geq 1/\lambda^c$ for some $c > 0$ and for infinitely many values of the security parameter λ. So this does not mean that $\mathcal{A}$ can "break" $\mathcal{E}$ for all values of the security parameter, but only infinitely many values of the security parameter.

In the main body of the text, we shall mainly ignore security parameters, system parameters, and the like, but it will always be understood that all of our "shorthand" has a precise mathematical interpretation. In particular, we will often refer to certain values v as be negligible (resp., poly-bounded), which really means that v is a negligible (resp., poly-bounded) function of the security parameter.

2.5 A fun application: anonymous routing

Our friend Alice wants to send a message m to Bob, but she does not want Bob or anyone else to know that the message m is from Alice. For example, Bob might be running a public discussion forum and Alice wants to post a comment anonymously on the forum. Posting anonymously lets Alice discuss health issues or other matters without identifying herself. In this section we will assume Alice only wants to post a *single* message to the forum.

One option is for Alice to choose a proxy, Carol, send m to Carol, and ask Carol to forward the message to Bob. This clearly does not provide anonymity for Alice since anyone watching the network will see that m was sent from Alice to Carol and then from Carol to Bob. By tracing the path of m through the network anyone can see that the post came from Alice.

A better approach is for Alice to establish a shared key k with Carol and send $c := E(k, m)$ to Carol, where $\mathcal{E} = (E, D)$ is a semantically secure cipher. Carol decrypts c and forwards m to Bob. Now, someone watching the network will see one message sent from Alice to Carol and a different message sent from Carol to Bob. Nevertheless, this method still does not ensure anonymity for Alice: if on a particular day the only message that Carol receives is the one from Alice and the only message she sends goes to Bob, then an observer can link the two and still learn that the posted message came from Alice.

We solve this problem by having Carol provide a *mixing service*, that is, a service that mixes incoming messages from many different parties $A_1, \ldots, A_n$. For $i = 1, \ldots, n$, Carol establishes a secret key k_i with party A_i and each party A_i sends to Carol an encrypted message $c_i := E\big(k_i,\ \langle \mathit{destination}_i, m_i \rangle\big)$. Carol collects all n incoming ciphertexts, decrypts each of them with the correct key, and forwards the resulting plaintexts in some random order to their destinations. Now an observer examining Carol's traffic sees n messages going in and n messages going out, but cannot tell which message was sent where. Alice's message is one of the n messages sent out by Carol, but the observer cannot tell which one. We say that Alice's *anonymity set* is of size n.

The remaining problem is that Carol can still tell that Alice is the one who posted a specific message on the discussion forum. To eliminate this final risk Alice uses multiple mixing services, say, Carol and David. She establishes a secret key k_{c} with Carol and a secret key k_{d} with David. To send her message to Bob she constructs the following nested ciphertext c_2:

$$c_2 := E\big(k_{\mathrm{c}},\ E(k_{\mathrm{d}}, m)\big)\ . \tag{2.14}$$

For completeness Alice may want to embed routing information inside the ciphertext so that c_2 is actually constructed as:

$$c_2 := E\big(k_{\mathrm{c}},\ \langle \mathsf{David}, c_1 \rangle\big) \qquad \text{where} \qquad c_1 := E\big(k_{\mathrm{d}},\ \langle \mathsf{Bob}, m \rangle\big)\ .$$

Next, Alice sends c_2 to Carol. Carol decrypts c_2 and obtains the plaintext $\langle \mathsf{David}, c_1 \rangle$ which tells her to send c_1 to David. David decrypts c_1 and obtains the plaintext $\langle \mathsf{Bob}, m \rangle$ which tells him to send m to Bob. This process of decrypting a nested ciphertext, illustrated in Fig. 2.7, is similar to peeling an onion one layer at a time. For this reason this routing procedure is often called *onion routing*.

Now even if Carol observes all network traffic she cannot tell with certainty who posted a particular message on Bob's forum. The same holds for David. However, if Carol and David collude they can figure it out. For this reason Alice may want to route her message through more than two mixes. As long as one of the mixes does not collude with the others, Alice's anonymity will be preserved.

One small complication is that when Alice establishes her shared secret key k_{d} with David, she must do so without revealing her identity to David. Otherwise, David will know that c_1 came from Alice, which we do not want. This is not difficult to do, and we will see how later in the book (Section 20.14).

Security of nested encryption. To preserve Alice's anonymity it is necessary that Carol, who knows k_{c}, learn no information about m from the nested ciphertext c_2 in (2.14). Otherwise, Carol could potentially use the information she learns about m from c_2 to link Alice to her post on Bob's discussion forum. For example, suppose Carol could learn the first few characters of m from c_2 and

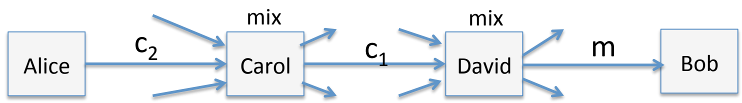


Figure 2.7: An example onion routing using two mixes

later find that there is only one post on Bob's forum starting with those characters. Carol could then link the entire post to Alice because she knows that c_2 came from Alice.

The same holds for David: it had better be the case that David, who knows $k_{\rm d}$, can learn no information about m from the nested ciphertext c_2 in (2.14).

Let us argue that if $\mathcal{E}$ is semantically secure then no efficient adversary can learn any information about m given c_2 and one of $k_{\rm c}$ or $k_{\rm d}$.

More generally, for a cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ let us define the n-way nested cipher $\mathcal{E}_n = (E_n, D_n)$ as

$$E_n\big((k_0, \dots, k_{n-1}),\ m\big) = E\big(k_{n-1},\ E(k_{n-2},\ \cdots E(k_0, m)\cdots)\big)\ .$$

Decryption applies the keys in the reverse order:

$$D_n\big((k_0, \dots, k_{n-1}),\ c\big) = D\big(k_0,\ D(k_1,\ \cdots D(k_{n-1}, c)\cdots)\big)\ .$$

Our goal is to show that if $\mathcal{E}$ is semantically secure then $\mathcal{E}_n$ is semantically secure even if the adversary is given all but one of the keys $k_0, \dots, k_{n-1}$. To make this precise, we define two experiments, Experiment 0 and Experiment 1, where for $b = 0, 1$, Experiment b is:

- The adversary gives the challenger (m_0, m_1, d) where $m_0, m_1 \in \mathcal{M}$ are equal length messages and $0 \le d < n$.
- The challenger chooses n keys $k_0, \dots, k_{n-1} \xleftarrow{R} \mathcal{K}$ and computes $c \xleftarrow{R} E_n\big((k_0, \dots, k_{n-1}),\ m_b\big)$. It sends c to the adversary along with all keys $k_0, \dots, k_{n-1}$, but excluding the key k_d.
- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

This game captures the fact that the adversary sees all keys $k_0, \dots, k_{n-1}$ except for k_d and tries to break semantic security.

We define the adversary's advantage, $\mathrm{NE}^{(\mathrm{n})}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$, as in the definition of semantic security:

$$\mathrm{NE}^{(\mathrm{n})}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = \big|\Pr[W_0] - \Pr[W_1]\big|$$

where W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b, for $b = 0, 1$. We say that $\mathcal{E}$ is semantically secure for n-way nesting if $\mathrm{NE}^{(\mathrm{n})}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.

Theorem 2.12. *For every constant $n > 0$, if $\mathcal{E} = (E, D)$ is semantically secure then $\mathcal{E}$ is semantically secure for n-way nesting.*

> *In particular, for every n-way nested adversary $\mathcal{A}$ attacking $\mathcal{E}_n$, there exists a semantic security adversary $\mathcal{B}$ attacking $\mathcal{E}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that*
>
> $$\mathrm{NE}^{(\mathrm{n})}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}]\ .$$

The proof of this theorem is a good exercise in security reductions. We leave it for Exercise 2.15.

2.6 Notes

The one time pad is due to Gilbert Vernam in 1917, although there is evidence that it was discovered earlier [10].

Citations to the literature to be added.

2.7 Exercises

2.1 (multiplicative one-time pad). We may also define a "multiplication mod p" variation of the one-time pad. This is a cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{K} := \mathcal{M} := \mathcal{C} := \{1, \ldots, p-1\}$, where p is a prime. Encryption and decryption are defined as follows:

$$E(k, m) := k \cdot m \bmod p \qquad D(k, c) := k^{-1} \cdot c \bmod p.$$

Here, k^{-1} denotes the multiplicative inverse of k modulo p. Verify the correctness property for this cipher and prove that it is perfectly secure.

2.2 (A good substitution cipher). Consider a variant of the substitution cipher $\mathcal{E} = (E, D)$ defined in Example 2.3 where every symbol of the message is encrypted using an *independent* permutation. That is, let $\mathcal{M} = \mathcal{C} = \Sigma^L$ for some a finite alphabet of symbols Σ and some L. Let the key space be $\mathcal{K} = S^L$ where S is the set of all permutations on Σ. The encryption algorithm $E(k, m)$ is defined as

$$E(k, m) := \big(\ k[0](m[0]),\ k[1](m[1]),\ \ldots,\ k[L-1](m[L-1])\ \big)$$

Show that $\mathcal{E}$ is perfectly secure.

2.3 (Chain encryption). Let $\mathcal{E} = (E, D)$ be a perfectly secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ where $\mathcal{K} = \mathcal{M}$. Let $\mathcal{E}' = (E', D')$ be a cipher where encryption is defined as $E'((k_1, k_2), m) := \big(E(k_1, k_2), E(k_2, m)\big)$. Show that $\mathcal{E}'$ is perfectly secure.

2.4 (A broken one-time pad). Consider a variant of the one time pad with message space $\{0,1\}^L$ where the key space $\mathcal{K}$ is restricted to all L-bit strings with an even number of 1's. Give an efficient adversary whose semantic security advantage is 1.

2.5 (A stronger impossibility result). This exercise generalizes Shannon's theorem (Theorem 2.5). Let $\mathcal{E}$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Suppose that $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] \le \epsilon$ for *all* adversaries $\mathcal{A}$, even including *computationally unbounded* ones. Show that $|\mathcal{K}| \ge (1-\epsilon)|\mathcal{M}|$.

2.6 (A matching bound). This exercise develops a converse of sorts for the previous exercise. For $j = 0, \ldots, L-1$, let $\epsilon = 1/2^j$. Consider the L-bit one-time pad variant $\mathcal{E}$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ where $\mathcal{M} = \mathcal{C} = \{0,1\}^L$. The key space $\mathcal{K}$ is restricted to all L-bit strings whose first $L-j$ bits are not all zero, so that $|\mathcal{K}| = (1-\epsilon)|\mathcal{M}|$. Show that:

(a) there is an efficient adversary $\mathcal{A}$ such that $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] = \epsilon/(1-\epsilon)$;

(b) for all adversaries $\mathcal{A}$, even including *computationally unbounded* ones, $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] \le \epsilon/(1-\epsilon)$.

Note: Since the advantage of $\mathcal{A}$ in part (a) is non-zero, the cipher $\mathcal{E}$ cannot be perfectly secure.

2.7 (Deterministic ciphers). In this exercise, you are asked to prove in detail the claims made in Example 2.9. Namely, show that if $\mathcal{E}$ is a deterministic cipher that is perfectly secure, then $\mathrm{SS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 0$ for every adversary $\mathcal{A}$ (bearing in mind that $\mathcal{A}$ may be probabilistic); also show that if $\mathcal{E}$ is the variable length one-time pad, then $\mathrm{SS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 0$ for all adversaries $\mathcal{A}$.

2.8 (Roulette). In Section 2.3.4, we argued that if value r is encrypted using a semantically secure cipher, then a player's odds of winning at Internet roulette are very close to those of real roulette. However, our "roulette" game was quite simple. Suppose that we have a more involved game, where different outcomes may result in different winnings. The rules are not so important, but assume that the rules are easy to evaluate (given a bet and the number r) and that every bet results in a payout of $0, 1, \ldots, n$ dollars, where n is poly-bounded. Let μ be the expected winnings in an optimal strategy for a real version of this game (with no encryption). Let μ' be the expected winnings of some (efficient) player in an Internet version of this game (with encryption). Show that $\mu \leq \mu' + \epsilon$, where ϵ is negligible, assuming the cipher is semantically secure.

Hint: You may want to use the fact that if $\mathbf{X}$ is a random variable taking values in the set $\{0, 1, \ldots, n\}$, the expected value of $\mathbf{X}$ is equal to $\sum_{i=1}^{n} \Pr[\mathbf{X} \geq i]$.

2.9. Prove Fact 2.6, using the formal definitions in Section 2.4.

2.10 (Exercising the definition of semantic security). Let $\mathcal{E} = (E, D)$ be a semantically secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{M} = \mathcal{C} = \{0, 1\}^L$. Which of the following encryption algorithms yields a semantically secure scheme? Either give an attack or provide a security proof via an explicit reduction.

(a) $E'(k, m) = 0 \parallel E(k, m)$

(b) $E'(k, m) = E(k, m) \parallel \mathrm{parity}(m)$

(c) $E'(k, m) = \mathrm{reverse}(E(k, m))$

(d) $E'(k, m) = E(k, \mathrm{reverse}(m))$

Here, for a bit string s, $\mathrm{parity}(s)$ is 1 if the number of 1's in s is odd, and 0 otherwise; also, $\mathrm{reverse}(s)$ is the string obtained by reversing the order of the bits in s, e.g., $\mathrm{reverse}(1011) = 1101$.

2.11 (Key recovery attacks). Let $\mathcal{E} = (E, D)$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. A key recovery attack is modeled by the following game between a challenger and an adversary $\mathcal{A}$: the challenger chooses a random key k in $\mathcal{K}$, a random message m in $\mathcal{M}$, computes $c \xleftarrow{R} E(k, m)$, and sends (m, c) to $\mathcal{A}$. In response $\mathcal{A}$ outputs a guess $\hat{k}$ in $\mathcal{K}$. We say that $\mathcal{A}$ wins the game if $D(\hat{k}, c) = m$ and define $\mathrm{KR}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ to be the probability that $\mathcal{A}$ wins the game. As usual, we say that $\mathcal{E}$ is secure against key recovery attacks if for all efficient adversaries $\mathcal{A}$ the advantage $\mathrm{KR}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.

(a) Show that the one-time pad is not secure against key recovery attacks.

(b) Show that if $\mathcal{E}$ is semantically secure and $\epsilon = |\mathcal{K}|/|\mathcal{M}|$ is negligible, then $\mathcal{E}$ is secure against key recovery attacks. In particular, show that for every efficient key-recovery adversary $\mathcal{A}$ there is an efficient semantic security adversary $\mathcal{B}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{KR}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}] + \epsilon$$

Hint: Your semantic security adversary $\mathcal{B}$ will output 1 with probability KR$\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ in the semantic security Experiment 0 and output 1 with probability at most ϵ in Experiment 1. Deduce from this a lower bound on SS$\mathsf{adv}[\mathcal{B}, \mathcal{E}]$ in terms of ϵ and KR$\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ from which the result follows.

(c) Deduce from part (b) that if $\mathcal{E}$ is semantically secure and $|\mathcal{M}|$ is super-poly then $|\mathcal{K}|$ cannot be poly-bounded.

Note: $|\mathcal{K}|$ can be poly-bounded when $|\mathcal{M}|$ is poly-bounded, as in the one-time pad.

2.12 (Security against message recovery). In Section 2.3.3 we developed the notion of security against message recovery. Construct a cipher that is secure against message recovery, but is not semantically secure.

2.13 (Advantage calculations in simple settings). Consider the following two experiments Experiment 0 and Experiment 1:

- In Experiment 0 the challenger flips a fair coin (probability 1/2 for HEADS and 1/2 for TAILS) and sends the result to the adversary $\mathcal{A}$.
- In Experiment 1 the challenger always sends TAILS to the adversary.

The adversary's goal is to distinguish these two experiments: at the end of each experiment the adversary outputs a bit 0 or 1 for its guess for which experiment it is in. For $b = 0, 1$ let W_b be the event that in experiment b the adversary output 1. The adversary tries to maximize its distinguishing advantage, namely the quantity

$$\left|\Pr[W_0] - \Pr[W_1]\right| \quad \in [0, 1] \ .$$

If the advantage is negligible for all efficient adversaries then we say that the two experiments are indistinguishable.

(a) Calculate the advantage of each of the following adversaries:

(i) $\mathcal{A}_1$: Always output 1.

(ii) $\mathcal{A}_2$: Ignore the result reported by the challenger, and randomly output 0 or 1 with even probability.

(iii) $\mathcal{A}_3$: Output 1 if HEADS was received from the challenger, else output 0.

(iv) $\mathcal{A}_4$: Output 0 if HEADS was received from the challenger, else output 1.

(v) $\mathcal{A}_5$: If HEADS was received, output 1. If TAILS was received, randomly output 0 or 1 with even probability.

(b) What is the maximum advantage possible in distinguishing these two experiments? Explain why.

2.14 (Permutation cipher). Consider the following cipher (E, D) defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ where $\mathcal{C} = \mathcal{M} = \{0, 1\}^\ell$ and $\mathcal{K}$ is the set of all $\ell!$ permutations of the set $\{0, \dots, \ell - 1\}$. For a key $k \in \mathcal{K}$ and message $m \in \mathcal{M}$ define $E(k, m)$ to be result of permuting the bits of m using the permutation k, namely $E(k, m) = m[k(0)]...m[k(\ell - 1)]$. Show that this cipher is not semantically secure by showing an adversary that achieves advantage 1.

2.15 (Nested encryption). For a cipher $\mathcal{E} = (E, D)$ define the nested cipher $\mathcal{E}' = (E', D')$ as

$$E'\big((k_0, k_1), m\big) = E\big(k_1, E(k_0, m)\big) \quad \text{and} \quad D'\big((k_0, k_1), c\big) = D(k_0, D(k_1, c)) \ .$$

Our goal is to show that if $\mathcal{E}$ is semantically secure then $\mathcal{E}'$ is semantically secure even if the adversary is given one of the keys k_0 or k_1.

(a) Consider the following semantic security experiments, Experiments 0 and 1: in Experiment b, for $b = 0, 1$, the adversary generates two messages m_0 and m_1 and gets back k_1 and $E'\big((k_0, k_1),\ m_b\big)$. The adversary outputs $\hat{b}$ in $\{0, 1\}$ and we define its advantage, $\mathrm{NE}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ as in the usual the definition of semantic security. Show that for every nested encryption adversary $\mathcal{A}$ attacking $\mathcal{E}'$, there exists a semantic security adversary $\mathcal{B}$ attacking $\mathcal{E}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{NE}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}] \ .$$

Draw a diagram with $\mathcal{A}$ on the right, $\mathcal{B}$ in the middle, and $\mathcal{B}$'s challenger on the left. Show the message flow between these three parties that takes place in your proof of security.

(b) Repeat part (a), but now when the adversary gets back k_0 (instead of k_1) and $E'\big((k_0, k_1), m_b\big)$ in Experiments 0 and 1. Draw a diagram describing the message flow in your proof of security as you did in part (a).

This problem comes up in the context of anonymous routing on the Internet as discussed in Section 2.5.

2.16 (Self referential encryption). Let us show that encrypting a key under itself can be dangerous. Let $\mathcal{E}$ be a semantically secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{K} \subseteq \mathcal{M}$, and let $k \xleftarrow{\text{R}} \mathcal{K}$. A ciphertext $c_* := E(k, k)$, namely encrypting k using k, is called a **self referential encryption**.

(a) Construct a cipher $\tilde{\mathcal{E}} = (\tilde{E}, \tilde{D})$ derived from $\mathcal{E}$ such that $\tilde{\mathcal{E}}$ is semantically secure, but becomes insecure if the adversary is given $\tilde{E}(k, k)$. You have just shown that semantic security does not imply security when one encrypts one's key.

(b) Construct a cipher $\hat{\mathcal{E}} = (\hat{E}, \hat{D})$ derived from $\mathcal{E}$ such that $\hat{\mathcal{E}}$ is semantically and remains semantically secure (provably) even if the adversary is given $\hat{E}(k, k)$. To prove that $\hat{\mathcal{E}}$ is semantically secure, you should show the following: for every adversary $\mathcal{A}$ that attacks $\hat{\mathcal{E}}$, there exists and adversary $\mathcal{B}$ that attacks $\mathcal{E}$ such that (i) the running time $\mathcal{B}$ is about the same as that of $\mathcal{A}$, and (ii) $\mathrm{SS}\mathsf{adv}[\mathcal{A}, \hat{\mathcal{E}}] \le \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}]$.

2.17 (Compression and encryption). Two standards committees propose to save bandwidth by combining compression (such as the Lempel-Ziv algorithm used in the zip and gzip programs) with encryption. Both committees plan on using the variable length one time pad for encryption.

- One committee proposes to compress messages before encrypting them. Explain why this is a bad idea.

  Hint: Recall that compression can significantly shrink the size of some messages while having little impact on the length of other messages.

- The other committee proposes to compress ciphertexts after encryption. Explain why this is a bad idea.

Over the years many problems have surfaced when combining encryption and compression. The CRIME [92] and BREACH [88] attacks are good representative examples.

2.18 (Voting protocols). This exercise develops a simple voting protocol based on the additive one-time pad (Example 2.4). Suppose we have t voters and a counting center. Each voter is going to vote 0 or 1, and the counting center is going to tally the votes and broadcast the total sum S. However, they will use a protocol that guarantees that no party (voter or counting center) learns anything other than S (but we shall assume that each party faithfully follows the protocol).

The protocol works as follows. Let $n > t$ be an integer. The counting center generates an encryption of 0: $c_0 \xleftarrow{R} \{0, \ldots, n-1\}$, and passes c_0 to voter 1. Voter 1 adds his vote v_1 to c_0, computing $c_1 \leftarrow c_0 + v_1 \bmod n$, and passes c_1 to voter 2. This continues, with each voter i adding v_i to c_{i-1}, computing $c_i \leftarrow c_{i-1} + v_i \bmod n$, and passing c_i to voter $i+1$, except that voter t passes c_t to the counting center. The counting center computes the total sum as $S \leftarrow c_t - c_0 \bmod n$, and broadcasts S to all the voters.

(a) Show that the protocol correctly computes the total sum.

(b) Show that the protocol is perfectly secure in the following sense. For voter $i = 1, \ldots, t$, define $View_i := (S, c_{i-1})$, which represents the "view" of voter i. We also define $View_0 := (c_0, c_t)$, which represents the "view" of the counting center. Show that for each $i = 0, \ldots, t$ and $S = 0, \ldots, t$, the following holds:

> as the choice of votes $v_1, \ldots, v_t$ varies, subject to the restrictions that each $v_j \in \{0, 1\}$ and $\sum_{j=1}^{t} v_j = S$, the distribution of $View_i$ remains the same.

(c) Show that if two voters i, j collude, they can determine the vote of a third voter k. You are free to choose the indices i, j, k.

2.19 (Two-way split keys). Let $\mathcal{E} = (E, D)$ be a semantically secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ where $\mathcal{K} = \{0,1\}^d$. Suppose we wish to split the ability to decrypt ciphertexts across two parties, Alice and Bob, so that both parties are needed to decrypt ciphertexts. For a random key k in $\mathcal{K}$ choose a random r in $\mathcal{K}$ and define $k_a := r$ and $k_b := k \oplus r$. Now if Alice and Bob get together they can decrypt a ciphertext c by first reconstructing the key k as $k = k_a \oplus k_b$ and then computing $D(k, c)$. Our goal is to show that neither Alice nor Bob can decrypt ciphertexts on their own.

(a) Formulate a security notion that captures the advantage that an adversary has in breaking semantic security given Bob's key k_b. Denote this 2-way key splitting advantage by $2\mathrm{KS}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$.

(b) Show that for every 2-way key splitting adversary $\mathcal{A}$ there is a semantic security adversary $\mathcal{B}$ such that $2\mathrm{KS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}]$.

2.20 (Simple secret sharing). Let $\mathcal{E} = (E, D)$ be a semantically secure cipher with key space $\mathcal{K} = \{0,1\}^L$. A bank wishes to split a decryption key $k \in \{0,1\}^L$ into three shares p_0, p_1, and p_2 so that two of the three shares are needed for decryption. Each share can be given to a different bank executive, and two of the three must contribute their shares for decryption to proceed. This way, decryption can proceed even if one of the executives is out sick, but at least two executives are needed for decryption.

(a) To do so the bank generates two random pairs (k_0, k_0') and (k_1, k_1') so that $k_0 \oplus k_0' = k_1 \oplus k_1' = k$. How should the bank assign shares so that any two shares enable decryption using k, but no single share can decrypt?

Hint: The first executive will be given the share $p_0 := (k_0, k_1)$.

(b) Generalize the scheme from part (a) so that 3-out-of-5 shares are needed for decryption. Reconstituting the key only uses XOR of key shares. Two shares should reveal nothing about the key k.

(c) More generally, we can design a t-out-of-w system this way for any $t < w$. How does the size of each share scale with t? We will see a much better way to do this in Section 11.6.

2.21 (Simple threshold decryption). Let $\mathcal{E} = (E, D)$ be a semantically secure cipher with key space $\mathcal{K}$. In this exercise we design a system that lets a bank split a key k into three shares p_0, p_1, and p_2 so that two of the three shares are needed for decryption, as in Exercise 2.20. However, decryption is done without ever reconstituting the complete key at a single location.

We use nested encryption from Exercise 2.15. Choose a random key $k := (k_0, k_1, k_2, k_3)$ in $\mathcal{K}^4$ and encrypt a message m as:

$$c \xleftarrow{R} \Big(E\big(k_1, E(k_0, m)\big), \quad E\big(k_4, E(k_3, m)\big) \Big).$$

(a) Construct the shares p_0, p_1, p_2 so that any two shares enable decryption, but no single share can decrypt. Hint: the first share is $p_0 := (k_0, k_3)$.

Discussion: Suppose the entities holding shares p_0 and p_2 are available to decrypt. To decrypt a ciphertext c, first send c to the entity holding p_2 to partially decrypt c. Then forward the result to the entity holding p_0 to complete the decryption. This way, decryption is done without reconstituting the complete key k at a single location.

(b) Generalize the scheme from part (a) so that 3-out-of-5 shares are needed for decryption. Explain how decryption can be done without reconstituting the key in a single location.

An encryption scheme where the key can be split into shares so that t-out-of-w shares are needed for decryption, and decryption does not reconstitute the key at a single location, is said to provide **threshold decryption**. We will see a much better way to do this in Section 11.6.

2.22 (Bias correction). Consider again the bit-guessing version of the semantic security attack game (i.e., Attack Game 2.4). Suppose an efficient adversary $\mathcal{A}$ wins the game (i.e., guesses the hidden bit b) with probability $1/2 + \epsilon$, where ϵ is non-negligible. Note that ϵ could be positive or negative (the definition of negligible works on absolute values). Our goal is to show that there is another efficient adversary $\mathcal{B}$ that wins the game with probability $1/2 + \epsilon'$, where ϵ' is non-negligible and positive.

(a) Consider the following adversary $\mathcal{B}$ that uses $\mathcal{A}$ as a subroutine in Attack Game 2.4 in the following two-stage attack. In the first stage, $\mathcal{B}$ plays challenger to $\mathcal{A}$, but $\mathcal{B}$ generates its own hidden bit b_0, its own key k_0, and eventually $\mathcal{A}$ outputs its guess-bit $\hat{b}_0$. Note that in this stage, $\mathcal{B}$'s challenger in Attack Game 2.4 is not involved at all. In the second stage, $\mathcal{B}$ restarts $\mathcal{A}$, and lets $\mathcal{A}$ interact with the "real" challenger in Attack Game 2.4, and eventually

$\mathcal{A}$ outputs a guess-bit $\hat{b}$. When this happens, $\mathcal{B}$ outputs $\hat{b} \oplus \hat{b}_0 \oplus b_0$. Note that this run of $\mathcal{A}$ is completely independent of the first — the coins of $\mathcal{A}$ and also the system parameters are generated independently in these two runs.

Show that $\mathcal{B}$ wins Attack Game 2.4 with probability $1/2 + 2\epsilon^2$.

(b) One might be tempted to argue as follows. Just construct an adversary $\mathcal{B}$ that runs $\mathcal{A}$, and when $\mathcal{A}$ outputs $\hat{b}$, adversary $\mathcal{B}$ outputs $\hat{b} \oplus 1$. Now, we do not know if ϵ is positive or negative. If it is positive, then $\mathcal{A}$ satisfies are requirements. If it is negative, then $\mathcal{B}$ satisfies our requirements. Although we do not know which one of these two adversaries satisfies our requirements, we know that one of them definitely does, and so existence is proved.

What is wrong with this argument? The explanation requires an understanding of the mathematical details regarding security parameters (see Section 2.4).

(c) Can you come up with another efficient adversary $\mathcal{B}'$ that wins the bit-guessing game with probability at least $1 + |\epsilon|/2$? Your adversary $\mathcal{B}'$ will be less efficient than $\mathcal{B}$.

Chapter 3

Stream ciphers

In the previous chapter, we introduced the notions of perfectly secure encryption and semantically secure encryption. The problem with perfect security is that to achieve it, one must use very long keys. Semantic security was introduced as a weaker notion of security that would perhaps allow us to build secure ciphers that use reasonably short keys; however, we have not yet produced any such ciphers. This chapter studies one type of cipher that does this: the stream cipher.

3.1 Pseudo-random generators

Recall the one-time pad. Here, keys, messages, and ciphertexts are all L-bit strings. However, we would like to use a key that is much shorter. So the idea is to instead use a short, ℓ-bit "seed" s as the encryption key, where ℓ is much smaller than L, and to "stretch" this seed into a longer, L-bit string that is used to mask the message (and unmask the ciphertext). The string s is stretched using some efficient, deterministic algorithm G that maps ℓ-bit strings to L-bit strings. Thus, the key space for this modified one-time pad is $\{0,1\}^\ell$, while the message and ciphertext spaces are $\{0,1\}^L$. For $s \in \{0,1\}^\ell$ and $m, c \in \{0,1\}^L$, encryption and decryption are defined as follows:

$$E(s,m) := G(s) \oplus m \quad \text{and} \quad D(s,c) := G(s) \oplus c.$$

This modified one-time pad is called a **stream cipher**, and the function G is called a **pseudo-random generator**.

If $\ell < L$, then by Shannon's Theorem, this stream cipher cannot achieve perfect security; however, if G satisfies an appropriate security property, then this cipher is semantically secure. Suppose s is a random ℓ-bit string and r is a random L-bit string. Intuitively, if an adversary cannot effectively tell the difference between $G(s)$ and r, then he should not be able to tell the difference between this stream cipher and a one-time pad; moreover, since the latter cipher is semantically secure, so should be the former. To make this reasoning rigorous, we need to formalize the notion that an adversary cannot "effectively tell the difference between $G(s)$ and r."

An algorithm that is used to distinguish a pseudo-random string $G(s)$ from a truly random string r is called a **statistical test**. It takes a string as input, and outputs 0 or 1. Such a test is called **effective** if the probability that it outputs 1 on a pseudo-random input is significantly different than the probability that it outputs 1 on a truly random input. Even a relatively small difference in probabilities, say 1%, is considered significant; indeed, even with a 1% difference, if we can obtain a few hundred independent samples, which are either all pseudo-random or all truly random, then we will be able to infer with high confidence whether we are looking at pseudo-random strings or at truly random strings. However, a non-zero but negligible difference in probabilities, say 2^{-100}, is not helpful.

How might one go about designing an effective statistical test? One basic approach is the following: given an L-bit string, calculate some statistic, and then see if this statistic differs greatly from what one would expect if the string were truly random.

For example, a very simple statistic that is easy to compute is the number k of 1's appearing in the string. For a truly random string, we would expect $k \approx L/2$. If the PRG G had some bias towards either 0-bits or 1-bits, we could effectively detect this with a statistical test that, say, outputs 1 if $|k - 0.5L| < 0.01L$, and otherwise outputs 0. This statistical test would be quite effective if the PRG G did indeed have some significant bias towards either 0 or 1.

The test in the previous example can be strengthened by considering not just individual bits, but pairs of bits. One could break the L-bit string up into $\approx L/2$ bit pairs, and count the number k_{00} of pairs 00, the number k_{01} of pairs 01, the number k_{10} of pairs 10, and the number k_{11} of pairs 11. For a truly random string, one would expect each of these numbers to be $\approx L/2 \cdot 1/4 = L/8$. Thus, a natural statistical test would be one that tests if the distance from $L/8$ of each of these numbers is less than some specified bound. Alternatively, one could sum up the squares of these distances, and test whether this sum is less than some specified bound — this is the classical χ-squared test from statistics. Obviously, this idea generalizes from pairs of bits to tuples of any length.

There are many other simple statistics one might check. However, simple tests such as these do not tend to exploit deeper mathematical properties of the algorithm G that a malicious adversary may be able to exploit in designing a statistical test specifically geared towards G. For example, there are PRG's for which the simple tests in the previous two paragraphs are completely ineffective, but yet are completely predictable, given sufficiently many output bits; that is, given a prefix of $G(s)$ of sufficient length, the adversary can compute all the remaining bits of $G(s)$, or perhaps even compute the seed s itself.

Our definition of security for a PRG formalizes the notion there should be no effective (and efficiently computable) statistical test.

3.1.1 Definition of a pseudo-random generator

A **pseudo-random generator**, or **PRG** for short, is an efficient, deterministic algorithm G that, given as input a **seed** s, computes an output r. The seed s comes from a finite **seed space** $\mathcal{S}$ and the output r belongs to a finite **output space** $\mathcal{R}$. Typically, $\mathcal{S}$ and $\mathcal{R}$ are sets of bit strings of some prescribed length (for example, in the discussion above, we had $\mathcal{S} = \{0,1\}^\ell$ and $\mathcal{R} = \{0,1\}^L$). We say that G is a PRG defined over $(\mathcal{S}, \mathcal{R})$.

Our definition of security for a PRG captures the intuitive notion that if s is chosen at random from $\mathcal{S}$ and r is chosen at random from $\mathcal{R}$, then no efficient adversary can effectively tell the difference between $G(s)$ and r: the two are **computationally indistinguishable**. The definition is formulated as an attack game.

Attack Game 3.1 (PRG). For a given PRG G, defined over $(\mathcal{S}, \mathcal{R})$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define:

Experiment b:

- The challenger computes $r \in \mathcal{R}$ as follows:

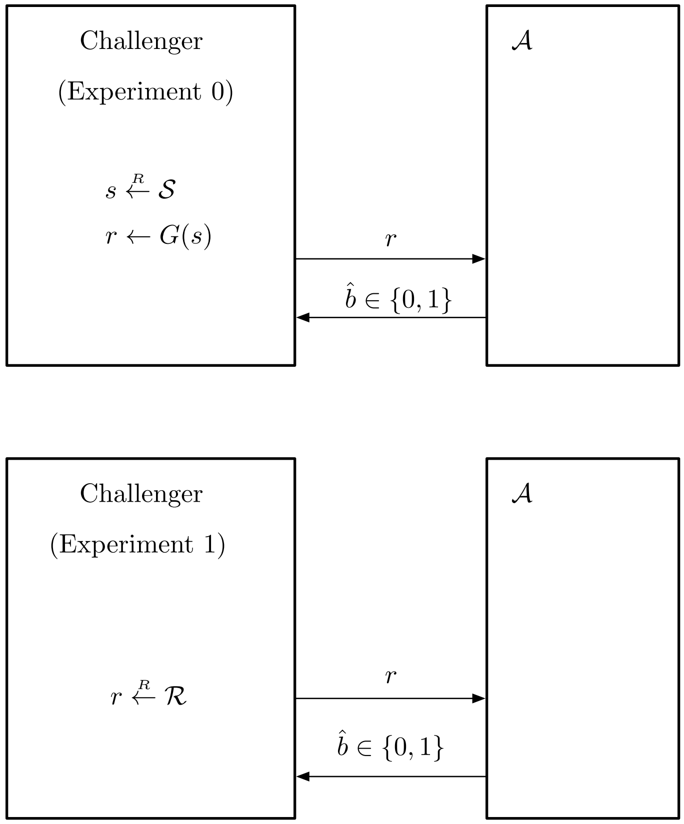


Figure 3.1: Experiments 0 and 1 of Attack Game 3.1

- if $b = 0$: $s \xleftarrow{R} \mathcal{S}$, $r \leftarrow G(s)$;
- if $b = 1$: $r \xleftarrow{R} \mathcal{R}$.

and sends r to the adversary.

- Given r, the adversary computes and outputs a bit $\hat{b} \in \{0,1\}$.

For $b = 0,1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to G as

$$\mathrm{PRG}\mathsf{adv}[\mathcal{A}, G] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \square$$

The attack game is illustrated in Fig. 3.1.

Definition 3.1 (secure PRG). *A PRG G is* ***secure*** *if the value $\mathrm{PRG}\mathsf{adv}[\mathcal{A}, G]$ is negligible for all efficient adversaries $\mathcal{A}$.*

As discussed in Section 2.3.5, Attack Game 3.1 can be recast as a "bit guessing" game, where instead of having two separate experiments, the challenger chooses $b \in \{0,1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$. In this game, we measure $\mathcal{A}$'s *bit-guessing advantage*

$\mathrm{PRG}\mathsf{adv}^*[\mathcal{A}, G]$ as $|\Pr[\hat{b} = b] - 1/2|$. The general result of Section 2.3.5 (namely, (2.13)) applies here as well:

$$\mathrm{PRG}\mathsf{adv}[\mathcal{A}, G] = 2 \cdot \mathrm{PRG}\mathsf{adv}^*[\mathcal{A}, G]. \tag{3.1}$$

We also note that a PRG can only be secure if the cardinality of the seed space is super-poly (see Exercise 3.5).

3.1.2 Mathematical details

Just as in Section 2.4, we give here more of the mathematical details pertaining to PRGs. Just like Section 2.4, this section may be safely skipped on first reading with very little loss in understanding.

First, we state the precise definition of a PRG, using the terminology introduced in Definition 2.9.

Definition 3.2 (pseudo-random generator). *A* ***pseudo-random generator*** *consists of an algorithm G, along with two families of spaces with system parameterization P:*

$$\mathbf{S} = \{\mathcal{S}_{\lambda,\Lambda}\}_{\lambda,\Lambda} \quad \textit{and} \quad \mathbf{R} = \{\mathcal{R}_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$

such that

1. $\mathbf{S}$ *and* $\mathbf{R}$ *are efficiently recognizable and sampleable.*

2. *Algorithm G is an efficient deterministic algorithm that on input λ, Λ, s, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, and $s \in \mathcal{S}_{\lambda,\Lambda}$, outputs an element of $\mathcal{R}_{\lambda,\Lambda}$.*

Next, Definition 3.1 needs to be properly interpreted. First, in Attack Game 3.1, it is to be understood that for each value of the security parameter λ, we get a different probability space, determined by the random choices of the challenger and the random choices of the adversary. Second, the challenger generates a system parameter Λ, and sends this to the adversary at the very start of the game. Third, the advantage $\mathrm{PRG}\mathsf{adv}[\mathcal{A}, G]$ is a function of the security parameter λ, and security means that this function is a negligible function.

3.2 Stream ciphers: encryption with a PRG

Let G be a PRG defined over $(\{0,1\}^\ell, \{0,1\}^L)$; that is, G stretches an ℓ-bit seed to an L-bit output. The **stream cipher $\mathcal{E} = (E, D)$ constructed from** G is defined over $(\{0,1\}^\ell, \{0,1\}^{\leq L}, \{0,1\}^{\leq L})$; for $s \in \{0,1\}^\ell$ and $m, c \in \{0,1\}^{\leq L}$, encryption and decryption are defined as follows: if $|m| = v$, then

$$E(s, m) := G(s)[0 \mathinner{..} v-1] \ \oplus \ m,$$

and if $|c| = v$, then

$$D(s, c) := G(s)[0 \mathinner{..} v-1] \ \oplus \ c.$$

As the reader may easily verify, this satisfies our definition of a cipher (in particular, the correctness property is satisfied).

Note that for the purposes of analyzing the semantic security of $\mathcal{E}$, the length associated with a message m in Attack Game 2.1 is the natural length $|m|$ of m in bits. Also, note that if v is much smaller than L, then for many practical PRGs, it is possible to compute the first v bits of $G(s)$ much faster than actually computing all the bits of $G(s)$ and then truncating.

The main result of this section is the following:

Theorem 3.1. *If G is a secure PRG, then the stream cipher $\mathcal{E}$ constructed from G is a semantically secure cipher.*

In particular, for every SS adversary $\mathcal{A}$ that attacks $\mathcal{E}$ as in Attack Game 2.1, there exists a PRG adversary $\mathcal{B}$ that attacks G as in Attack Game 3.1, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \mathrm{PRG}\mathsf{adv}[\mathcal{B}, G]. \tag{3.2}$$

Proof idea. The basic idea is to argue that we can replace the output of the PRG by a truly random string, without affecting the adversary's advantage by more than a negligible amount. However, after making this replacement, the adversary's advantage is zero. □

Proof. Let $\mathcal{A}$ be an efficient adversary attack $\mathcal{E}$ as in Attack Game 2.1. We want to show that $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}]$ is negligible, assuming that G is a secure PRG. It is more convenient to work with the bit-guessing version of the SS attack game. We prove:

$$\mathsf{SSadv}^*[\mathcal{A}, \mathcal{E}] = \mathrm{PRG}\mathsf{adv}[\mathcal{B}, G] \tag{3.3}$$

for some efficient adversary $\mathcal{B}$. Then (3.2) follows from Theorem 2.10. Moreover, by the assumption the G is a secure PRG, the quantity $\mathrm{PRG}\mathsf{adv}[\mathcal{B}, G]$ must negligible, and so the quantity $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}]$ is negligible as well.

So consider the adversary $\mathcal{A}$'s attack of $\mathcal{E}$ in the bit-guessing version of Attack Game 2.1. In this game, $\mathcal{A}$ presents the challenger with two messages m_0, m_1 of the same length; the challenger then chooses a random key s and a random bit b, and encrypts m_b under s, giving the resulting ciphertext c to $\mathcal{A}$; finally, $\mathcal{A}$ outputs a bit $\hat{b}$. The adversary $\mathcal{A}$ wins the game if $\hat{b} = b$.

Let us call this **Game 0.** The logic of the challenger in this game may be written as follows:

> Upon receiving $m_0, m_1 \in \{0,1\}^v$ from $\mathcal{A}$, for some $v \leq L$, do:
> $\quad b \xleftarrow{\mathrm{R}} \{0,1\}$
> $\quad s \xleftarrow{\mathrm{R}} \{0,1\}^\ell$, $r \leftarrow G(s)$
> $\quad c \leftarrow r[0\,..\,v-1] \oplus m_b$
> $\quad$ send c to $\mathcal{A}$.

Game 0 is illustrated in Fig. 3.2.

Let W_0 be the event that $\hat{b} = b$ in Game 0. By definition, we have

$$\mathsf{SSadv}^*[\mathcal{A}, \mathcal{E}] = |\Pr[W_0] - 1/2|. \tag{3.4}$$

Next, we modify the challenger of Game 0, obtaining new game, called **Game 1**, which is exactly the same as Game 0, except that the challenger uses a truly random string in place of a pseudo-random string. The logic of the challenger in Game 1 is as follows:

> Upon receiving $m_0, m_1 \in \{0,1\}^v$ from $\mathcal{A}$, for some $v \leq L$, do:
> $\quad b \xleftarrow{\mathrm{R}} \{0,1\}$
> $\quad r \xleftarrow{\mathrm{R}} \{0,1\}^L$
> $\quad c \leftarrow r[0\,..\,v-1] \oplus m_b$
> $\quad$ send c to $\mathcal{A}$.

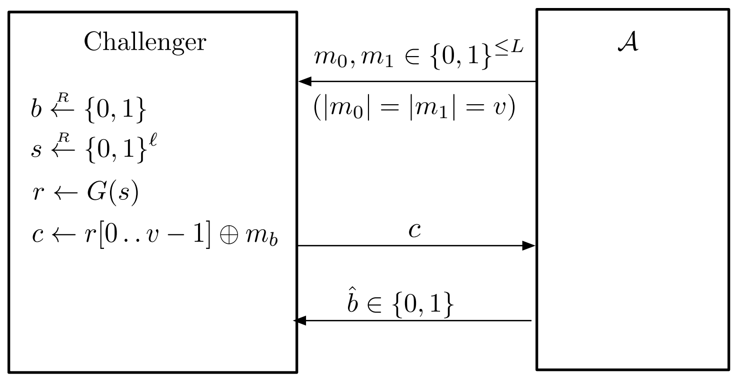


Figure 3.2: Game 0 in the proof of Theorem 3.1

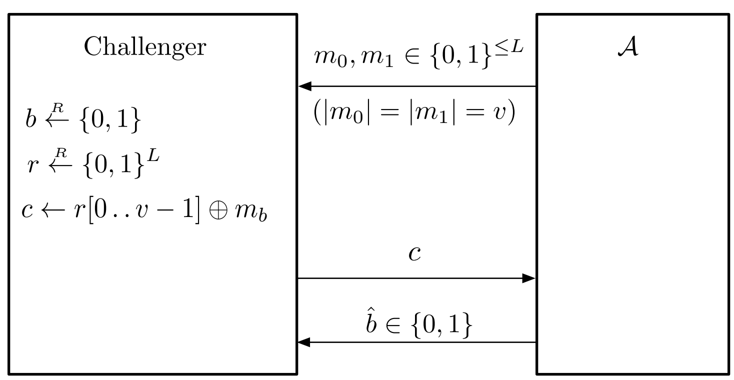


Figure 3.3: Game 1 in the proof of Theorem 3.1

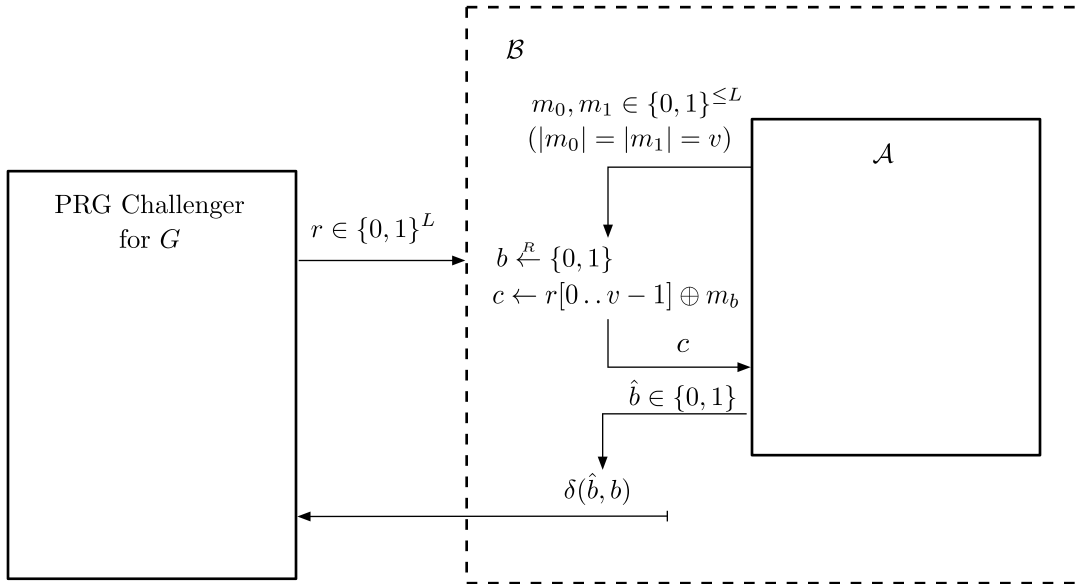


Figure 3.4: The PRG adversary $\mathcal{B}$ in the proof of Theorem 3.1

As usual, $\mathcal{A}$ outputs a bit $\hat{b}$ at the end of this game. We have highlighted the changes from Game 0 in gray. Game 1 is illustrated in Fig. 3.3.

Let W_1 be the event that $\hat{b} = b$ in Game 1. We claim that

$$\Pr[W_1] = 1/2. \tag{3.5}$$

This is because in Game 1, the adversary is attacking the variable length one-time pad. In particular, it is easy to see that the adversary's output $\hat{b}$ and the challenger's hidden bit b are independent.

Finally, we show how to construct an efficient PRG adversary $\mathcal{B}$ that uses $\mathcal{A}$ as a subroutine, such that

$$|\Pr[W_0] - \Pr[W_1]| = \mathrm{PRG}\mathsf{adv}[\mathcal{B}, G]. \tag{3.6}$$

This is actually quite straightforward. The logic of our new adversary $\mathcal{B}$ is illustrated in Fig. 3.4. Here, δ is defined as follows:

$$\delta(x,y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases} \tag{3.7}$$

Also, the box labeled "PRG Challenger" is playing the role of the challenger in Attack Game 3.1 with respect to G.

In words, adversary $\mathcal{B}$, which is a PRG adversary designed to attack G (as in Attack Game 3.1), receives $r \in \{0,1\}^L$ from its PRG challenger, and then plays the role of challenger to $\mathcal{A}$, as follows:

Upon receiving $m_0, m_1 \in \{0,1\}^v$ from $\mathcal{A}$, for some $v \leq L$, do:
  $b \xleftarrow{R} \{0,1\}$
  $c \leftarrow r[0\,..\,v-1] \oplus m_b$
  send c to $\mathcal{A}$.

Finally, when $\mathcal{A}$ outputs a bit $\hat{b}$, $\mathcal{B}$ outputs the bit $\delta(\hat{b}, b)$.

Let p_0 be the probability that $\mathcal{B}$ outputs 1 when the PRG challenger is running Experiment 0 of Attack Game 3.1, and let p_1 be the probability that $\mathcal{B}$ outputs 1 when the PRG challenger is running Experiment 1 of Attack Game 3.1. By definition, $\text{PRG}\mathsf{adv}[\mathcal{B}, G] = |p_1 - p_0|$. Moreover, if the PRG challenger is running Experiment 0, then adversary $\mathcal{A}$ is essentially playing our Game 0, and so $p_0 = \Pr[W_0]$, and if the PRG challenger is running Experiment 1, then $\mathcal{A}$ is essentially playing our Game 1, and so $p_1 = \Pr[W_1]$. Equation (3.6) now follows immediately.

Combining (3.4), (3.5), and (3.6), yields (3.3). $\Box$

In the above theorem, we reduced the security of $\mathcal{E}$ to that of G by showing that if $\mathcal{A}$ is an efficient SS adversary that attacks $\mathcal{E}$, then there exists an efficient PRG adversary $\mathcal{B}$ that attacks G, such that

$$\text{SS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq 2 \cdot \text{PRG}\mathsf{adv}[\mathcal{B}, G].$$

(Actually, we showed that equality holds, but that is not so important.) In the proof, we argued that if G is secure, then $\text{PRG}\mathsf{adv}[\mathcal{B}, G]$ is negligible, hence by the above inequality, we conclude that $\text{SS}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is also negligible. Since this holds for all efficient adversaries $\mathcal{A}$, we conclude that $\mathcal{E}$ is semantically secure.

Analogous to the discussion after the proof of Theorem 2.7, another way to structure the proof is by proving the contrapositive: indeed, if we assume that $\mathcal{E}$ is insecure, then there must be an efficient adversary $\mathcal{A}$ such that $\text{SS}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is non-negligible, and the reduction (and the above inequality) gives us an efficient adversary $\mathcal{B}$ such that $\text{PRG}\mathsf{adv}[\mathcal{B}, G]$ is also non-negligible. That is, if we can break $\mathcal{E}$, we can also break G. While logically equivalent, such a proof has a different "feeling": one starts with an adversary $\mathcal{A}$ that breaks $\mathcal{E}$, and shows how to use $\mathcal{A}$ to construct a new adversary $\mathcal{B}$ that breaks G.

The reader should notice that the proof of the above theorem follows the same basic pattern as our analysis of Internet roulette in Section 2.3.4. In both cases, we started with an attack game (Fig. 2.2 or Fig. 3.2) which we modified to obtain a new attack game (Fig. 2.3 or Fig. 3.3); in this new attack game, it was quite easy to compute the adversary's advantage. Also, we used an appropriate security assumption to show that the difference between the adversary's advantages in the original and the modified games was negligible. This was done by exhibiting a new adversary (Fig. 2.4 or Fig. 3.4) that attacked the underlying cryptographic primitive (cipher or PRG) with an advantage equal to this difference. Assuming the underlying primitive was secure, this difference must be negligible; alternatively, one could argue the contrapositive: if this difference were not negligible, the new adversary would "break" the underlying cryptographic primitive.

This is a pattern that will be repeated and elaborated upon throughout this text. The reader is urged to study both of these analyses to make sure he or she completely understands what is going on.

3.3 Stream cipher limitations: attacks on the one time pad

Although stream ciphers are semantically secure they are highly brittle and become totally insecure if used incorrectly.

3.3.1 The two-time pad is insecure

A stream cipher is well equipped to encrypt a *single* message from Alice to Bob. Alice, however, may wish to send several messages to Bob. For simplicity suppose Alice wishes to encrypt two messages m_1 and m_2. The naive solution is to encrypt both messages using the same stream cipher key s:

$$c_1 \leftarrow m_1 \oplus G(s) \qquad \text{and} \qquad c_2 \leftarrow m_2 \oplus G(s) \tag{3.8}$$

A moments reflection shows that this construction is insecure in a very strong sense. An adversary who intercepts c_1 and c_2 can compute

$$\Delta := c_1 \oplus c_2 = \big(m_1 \oplus G(s)\big) \oplus \big(m_2 \oplus G(s)\big) = m_1 \oplus m_2$$

and obtain the xor of m_1 and m_2. Not surprisingly, English text contains enough redundancy that given $\Delta = m_1 \oplus m_2$ the adversary can recover both m_1 and m_2 in the clear. Hence, the construction in (3.8) leaks the plaintexts after seeing only two sufficiently long ciphertexts.

The construction in (3.8) is jokingly called the **two-time pad**. We just argued that the two-time pad is totally insecure. In particular, **a stream cipher key should never be used to encrypt more than one message**. Throughout the book we will see many examples where a one-time cipher is sufficient. For example, when choosing a new random key for every message as in Section 5.4.1. However, in settings where a single key is used multiple times, one should never use a stream cipher directly. We build multi-use ciphers in Chapter 5.

Incorrectly reusing a stream cipher key is a common error in deployed systems. For example, a protocol called PPTP enables two parties A and B to send encrypted messages to one another. Microsoft's implementation of PPTP in Windows NT uses a stream cipher called RC4. The original implementation encrypts messages from A to B using the same RC4 key as messages from B to A [95]. Consequently, by eavesdropping on two encrypted messages headed in opposite directions an attacker could recover the plaintext of both messages.

Another amusing story about the two-time pad is relayed by Klehr [52] who describes in great detail how Russian spies in the US during World War II were sending messages back to Moscow, encrypted with the one-time pad. The system had a critical flaw, as explained by Klehr:

> During WWII the Soviet Union could not produce enough one-time pads ... to keep up with the enormous demand .... So, they used a number of one-time pads twice, thinking it would not compromise their system. American counter-intelligence during WWII collected all incoming and outgoing international cables. Beginning in 1946, it began an intensive effort to break into the Soviet messages with the cooperation of the British and by ... the Soviet error of using some one-time pads as two-time pads, was able, over the next 25 years, to break some 2900 messages, containing 5000 pages of the hundreds of thousands of messages that been sent between 1941 and 1946 (when the Soviets switched to a different system).

The decryption effort was codenamed project Venona. The Venona files are most famous for exposing Julius and Ethel Rosenberg and help give indisputable evidence of their involvement with the Soviet spy ring. Starting in 1995 all 3000 Venona decrypted messages were made public.

3.3.2 The one-time pad is malleable

Although semantic security ensures that an adversary cannot read the plaintext, it provides no guarantees for integrity. When using a stream cipher, an adversary can change a ciphertext and the modification will never be detected by the decryptor. Even worse, let us show that by changing the ciphertext, the attacker can control how the decrypted plaintext will change.

Suppose an attacker intercepts a ciphertext $c := E(s, m) = m \oplus G(s)$. The attacker changes c to $c' := c \oplus \Delta$ for some Δ of the attacker's choice. Consequently, the decryptor receives the modified message

$$D(s, c') = c' \oplus G(s) = (c \oplus \Delta) \oplus G(s) = m \oplus \Delta.$$

Hence, without knowledge of either m or s, the attacker was able to cause the decrypted message to become $m \oplus \Delta$ for Δ of the attacker's choosing. We say that stream-ciphers are **malleable** since an attacker can cause predictable changes to the plaintext. We will construct ciphers that provide both privacy and integrity in Chapter 9.

A simple example where malleability could help an attacker is an encrypted file system. To make things concrete, suppose Bob is a professor and that Alice and Molly are students. Bob's students submit their homework by email, and then Bob stores these emails on a disk encrypted using a stream cipher. An email always starts with a standard header. Simplifying things a bit, we can assume that an email from, say, Alice, always starts with the characters `From:Alice`.

Now suppose Molly is able to gain access to Bob's disk and locate the encryption of the email from Alice containing her homework. Molly can effectively steal Alice's homework, as follows. She simply XORs the appropriate five-character string into the ciphertext in positions 6 to 10, so as to change the header `From:Alice` to the header `From:Molly`. Molly makes this change by only operating on ciphertexts and without knowledge of Bob's secret key. Bob will never know that the header was changed, and he will grade Alice's homework, thinking it is Molly's, and Molly will get the credit instead of Alice.

Of course, for this attack to be effective, Molly must somehow be able to find the email from Alice on Bob's encrypted disk. However, in some implementations of encrypted file systems, file metadata (such as file names, modification times, etc) are not encrypted. Armed with this metadata, it may be straightforward for Molly to locate the encrypted email from Alice and carry out this attack.

3.4 Composing PRGs

In this section, we discuss two constructions that allow one to build new PRGs out of old PRGs. These constructions allow one to increase the size of the output space of the original PRG while at the same time preserving its security. Perhaps more important than the constructions themselves is the proof technique, which is called a **hybrid argument**. This proof technique is used pervasively throughout modern cryptography.

3.4.1 A parallel construction

Let G be a PRG defined over $(\mathcal{S}, \mathcal{R})$. Suppose that in some application, we want to use G many times. We want all the outputs of G to be computationally indistinguishable from random elements of $\mathcal{R}$. If G is a secure PRG, and if the seeds are independently generated, then this will indeed be the case.

We can model the use of many applications of G as a new PRG G'. That is, we construct a new PRG G' that applies G to n seeds, and concatenates the outputs. Thus, G' is defined over $(\mathcal{S}^n, \mathcal{R}^n)$, and for $s_1, \ldots, s_n \in \mathcal{R}$,

$$G'(s_1, \ldots, s_n) := (G(s_1), \ldots, G(s_n)).$$

We call G' the **n-wise parallel composition of** G. The value n is called a **repetition parameter**, and we require that it is a poly-bounded value.

Theorem 3.2. *If G is a secure PRG, then the n-wise parallel composition G' of G is also a secure PRG.*

In particular, for every PRG adversary $\mathcal{A}$ that attacks G' as in Attack Game 3.1, there exists a PRG adversary $\mathcal{B}$ that attacks G as in Attack Game 3.1, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{PRG}\mathsf{adv}[\mathcal{A}, G'] = n \cdot \mathrm{PRG}\mathsf{adv}[\mathcal{B}, G].$$

As a warm up, we first prove this theorem in the special case $n = 2$. Let $\mathcal{A}$ be an efficient PRG adversary that has advantage ϵ in attacking G' in Attack Game 3.1. We want to show that ϵ is negligible, under the assumption that G is a secure PRG. To do this, let us define **Game 0** to be Experiment 0 of Attack Game 3.1 with $\mathcal{A}$ and G'. The challenger in this game works as follows:

$s_1 \xleftarrow{\text{R}} \mathcal{S}$, $r_1 \leftarrow G(s_1)$
$s_2 \xleftarrow{\text{R}} \mathcal{S}$, $r_2 \leftarrow G(s_2)$
send (r_1, r_2) to $\mathcal{A}$.

Let p_0 denote the probability with which $\mathcal{A}$ outputs 1 in this game.

Next, we define **Game 1**, which is played between $\mathcal{A}$ and a challenger that works as follows:

$r_1 \xleftarrow{\text{R}} \mathcal{R}$
$s_2 \xleftarrow{\text{R}} \mathcal{S}$, $r_2 \leftarrow G(s_2)$
send (r_1, r_2) to $\mathcal{A}$.

Note that Game 1 corresponds to neither Experiment 0 nor Experiment 1 of Attack Game 3.1; rather, it is a "hybrid" experiment corresponding to something in between Experiments 0 and 1. All we have done is replaced the pseudo-random value r_1 in Game 0 by a truly random value (as highlighted). Intuitively, under the assumption that G is a secure PRG, the adversary $\mathcal{A}$ should not notice the difference. To make this argument precise, let p_1 be the probability that $\mathcal{A}$ outputs 1 in Game 1.

Let $\delta_1 := |p_1 - p_0|$. We claim that δ_1 is negligible, assuming that G is a secure PRG. Indeed, we can easily construct an efficient PRG adversary $\mathcal{B}_1$ whose advantage in attacking G in Attack Game 3.1 is precisely equal to δ_1. The adversary $\mathcal{B}_1$ works as follows:

Upon receiving $r \in \mathcal{R}$ from its challenger, $\mathcal{B}_1$ plays the role of challenger to $\mathcal{A}$, as follows:

$r_1 \leftarrow r$
$s_1 \xleftarrow{\text{R}} \mathcal{S}$, $r_2 \leftarrow G(s_2)$
send (r_1, r_2) to $\mathcal{A}$.

Finally, $\mathcal{B}_1$ outputs whatever $\mathcal{A}$ outputs.

Observe that when $\mathcal{B}_1$ is in Experiment 0 of its attack game, it perfectly mimics the behavior of the challenger in Game 0, while in Experiment 1, it perfectly mimics the behavior of the challenger in Game 1. Thus, p_0 is equal to the probability that $\mathcal{B}_1$ outputs 1 in Experiment 0 of Attack Game 3.1, while p_1 is equal to the probability that $\mathcal{B}_1$ outputs 1 in Experiment 1 of Attack Game 3.1. Thus, $\mathcal{B}_1$'s advantage in attacking G is precisely $|p_1 - p_0|$, as claimed.

Next, we define **Game 2**, which is played between $\mathcal{A}$ and a challenger that works as follows:

$r_1 \xleftarrow{R} \mathcal{R}$
$r_2 \xleftarrow{R} \mathcal{R}$
send (r_1, r_2) to $\mathcal{A}$.

All we have done is replaced the pseudo-random value r_2 in Game 1 by a truly random value (as highlighted). Let p_2 be the probability that $\mathcal{A}$ outputs 1 in Game 2. Note that Game 2 corresponds to Experiment 1 of Attack Game 3.1 with $\mathcal{A}$ and G', and so p_2 is equal to the probability that $\mathcal{A}$ outputs 1 in Experiment 1 of Attack Game 3.1 with respect to G'.

Let $\delta_2 := |p_2 - p_1|$. By an argument similar to that above, it is easy to see that δ_2 is negligible, assuming that G is a secure PRG. Indeed, we can easily construct an efficient PRG adversary $\mathcal{B}_2$ whose advantage in Attack Game 3.1 with respect to G is precisely equal to δ_2. The adversary $\mathcal{B}_2$ works as follows:

Upon receiving $r \in \mathcal{R}$ from its challenger, $\mathcal{B}_2$ plays the role of challenger to $\mathcal{A}$, as follows:

$r_1 \xleftarrow{R} \mathcal{R}$
$r_2 \leftarrow r$
send (r_1, r_2) to $\mathcal{A}$.

Finally, $\mathcal{B}_2$ outputs whatever $\mathcal{A}$ outputs.

It should be clear that p_1 is equal to the probability that $\mathcal{B}_2$ outputs 1 in Experiment 0 of Attack Game 3.1, while p_2 is equal to the probability that $\mathcal{B}_2$ outputs 1 in Experiment 1 of Attack Game 3.1.

Recalling that $\epsilon = \text{PRG}\mathsf{adv}[\mathcal{A}, G']$, then from the above discussion, we have

$$\epsilon = |p_2 - p_0| = |p_2 - p_1 + p_1 - p_0| \leq |p_1 - p_0| + |p_2 - p_1| = \delta_1 + \delta_2.$$

Since both δ_1 and δ_2 are negligible, then so is ϵ (see Fact 2.6).

That completes the proof that G' is secure in the case $n = 2$. Before giving the proof in the general case, we give another proof in the case $n = 2$. While our first proof involved the construction of two adversaries $\mathcal{B}_1$ and $\mathcal{B}_2$, our second proof combines these two adversaries into a single PRG adversary $\mathcal{B}$ that plays Attack Game 3.1 with respect to G, and which runs as follows:

upon receiving $r \in \mathcal{R}$ from its challenger, adversary $\mathcal{B}$ chooses $\omega \in \{1, 2\}$ at random, and gives r to $\mathcal{B}_\omega$; finally, $\mathcal{B}$ outputs whatever $\mathcal{B}_\omega$ outputs.

Let W_0 be the event that $\mathcal{B}$ outputs 1 in Experiment 0 of Attack Game 3.1, and W_1 be the event that $\mathcal{B}$ outputs 1 in Experiment 1 of Attack Game 3.1. Conditioning on the events $\omega = 1$ and $\omega = 2$, we have

$$\begin{aligned}
\Pr[W_0] &= \Pr[W_0 \mid \omega = 1] \Pr[\omega = 1] + \Pr[W_0 \mid \omega = 2] \Pr[\omega = 2] \\
&= \tfrac{1}{2}\Big(\Pr[W_0 \mid \omega = 1] + \Pr[W_0 \mid \omega = 2]\Big) \\
&= \tfrac{1}{2}(p_0 + p_1).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Pr[W_1] &= \Pr[W_1 \mid \omega = 1] \Pr[\omega = 1] + \Pr[W_1 \mid \omega = 2] \Pr[\omega = 2] \\
&= \tfrac{1}{2}\Big(\Pr[W_1 \mid \omega = 1] + \Pr[W_1 \mid \omega = 2]\Big) \\
&= \tfrac{1}{2}(p_1 + p_2).
\end{aligned}$$

Therefore, if δ is the advantage of $\mathcal{B}$ in Attack Game 3.1 with respect to G, we have

$$\delta = \left|\Pr[W_1] - \Pr[W_0]\right| = \left|\tfrac{1}{2}(p_1 + p_2) - \tfrac{1}{2}(p_0 + p_1)\right| = \tfrac{1}{2}|p_2 - p_0| = \epsilon/2.$$

Thus, $\epsilon = 2\delta$, and since δ is negligible, so is ϵ (see Fact 2.6).

Now, finally, we present the proof of Theorem 3.2 for general, poly-bounded n.

Proof idea. We could try to extend the first strategy outlined above from $n = 2$ to arbitrary n. That is, we could construct a sequence of $n + 1$ games, starting with a challenger that produces a sequence $(G(s_1), \ldots, G(s_n))$, of pseudo-random elements replacing elements one at a time with truly random elements of $\mathcal{R}$, ending up with a sequence $(r_1, \ldots, r_n)$ of truly random elements of $\mathcal{R}$. Intuitively, the adversary should not notice any of these replacements, since G is a secure PRG; however, proving this formally would require the construction of n different adversaries, each of which attacks G in a slightly different way. As it turns out, this leads to some annoying technical difficulties when n is not an absolute constant, but is simply poly-bounded; it is much more convenient to extend the second strategy outlined above, constructing a single adversary that attacks G "in one blow." □

Proof. Let $\mathcal{A}$ be an efficient PRG adversary that plays Attack Game 3.1 with respect to G'. We first introduce a sequence of $n + 1$ **hybrid games**, called Hybrid 0, Hybrid 1, …, Hybrid n. For $j = 0, 1, \ldots, n$, Hybrid j is a game played between $\mathcal{A}$ and a challenger that prepares a tuple of n values, the first j of which are truly random, and the remaining $n - j$ of which are pseudo-random outputs of G; that is, the challenger works as follows:

$$\begin{aligned}
&r_1 \xleftarrow{\text{R}} \mathcal{R} \\
&\qquad\vdots \\
&r_j \xleftarrow{\text{R}} \mathcal{R} \\
&s_{j+1} \xleftarrow{\text{R}} \mathcal{S},\ r_{j+1} \leftarrow G(s_{j+1}) \\
&\qquad\vdots \\
&s_n \xleftarrow{\text{R}} \mathcal{S},\ r_n \leftarrow G(s_n) \\
&\text{send } (r_1, \ldots, r_n) \text{ to } \mathcal{A}.
\end{aligned}$$

As usual, $\mathcal{A}$ outputs 0 or 1 at the end of the game. Fig. 3.5 illustrates the values prepared by the challenger in each of these $n+1$ games. Let p_j denote the probability that $\mathcal{A}$ outputs 1 in Hybrid j. Note that p_0 is also equal to the probability that $\mathcal{A}$ outputs 1 in Experiment 0 of Attack Game 3.1, while p_n is equal to the probability that $\mathcal{A}$ outputs 1 in Experiment 1. Thus, we have

$$\text{PRG}\mathsf{adv}[\mathcal{A}, G'] = |p_n - p_0|. \tag{3.9}$$

We next define a PRG adversary $\mathcal{B}$ that plays Attack Game 3.1 with respect to G, and which works as follows:

Upon receiving $r \in \mathcal{R}$ from its challenger, $\mathcal{B}$ plays the role of challenger to $\mathcal{A}$, as follows:

Hybrid 0:	$G(s_1)$	$G(s_2)$	$G(s_3)$	$\cdots$	$G(s_n)$
Hybrid 1:	r_1	$G(s_2)$	$G(s_3)$	$\cdots$	$G(s_n)$
Hybrid 2:	r_1	r_2	$G(s_3)$	$\cdots$	$G(s_n)$
$\vdots$					
Hybrid $n-1$:	r_1	r_2	r_3	$\cdots$	$G(s_n)$
Hybrid n:	r_1	r_2	r_3	$\cdots$	r_n

Figure 3.5: Values prepared by challenger in Hybrids $0, 1, \ldots, n$. Each r_i is a random element of $\mathcal{R}$, and each s_i is a random element of $\mathcal{S}$.

$\omega \xleftarrow{\text{R}} \{1, \ldots, n\}$
$r_1 \xleftarrow{\text{R}} \mathcal{R}$
$\quad\vdots$
$r_{\omega-1} \xleftarrow{\text{R}} \mathcal{R}$
$r_\omega \leftarrow r$
$s_{\omega+1} \xleftarrow{\text{R}} \mathcal{S}$, $r_{\omega+1} \leftarrow G(s_{\omega+1})$
$\quad\vdots$
$s_n \xleftarrow{\text{R}} \mathcal{S}$, $r_n \leftarrow G(s_n)$
send $(r_1, \ldots, r_n)$ to $\mathcal{A}$.

Finally, $\mathcal{B}$ outputs whatever $\mathcal{A}$ outputs.

Let W_0 be the event that $\mathcal{B}$ outputs 1 in Experiment 0 of Attack Game 3.1, and W_1 be the event that $\mathcal{B}$ outputs 1 in Experiment 1 of Attack Game 3.1. The key observation is this:

conditioned on $\omega = j$ for every fixed $j = 1, \ldots, n$, Experiment 0 of $\mathcal{B}$'s attack game is equivalent to Hybrid $j-1$, while Experiment 1 of $\mathcal{B}$'s attack game is equivalent to Hybrid j.

Therefore,

$$\Pr[W_0 \mid \omega = j] = p_{j-1} \quad \text{and} \quad \Pr[W_1 \mid \omega = j] = p_j.$$

So we have

$$\Pr[W_0] = \sum_{j=1}^{n} \Pr[W_0 \mid \omega = j] \Pr[\omega = j] = \frac{1}{n} \sum_{j=1}^{n} \Pr[W_0 \mid \omega = j] = \frac{1}{n} \sum_{j=1}^{n} p_{j-1},$$

and similarly,

$$\Pr[W_1] = \sum_{j=1}^{n} \Pr[W_1 \mid \omega = j] \Pr[\omega = j] = \frac{1}{n} \sum_{j=1}^{n} \Pr[W_1 \mid \omega = j] = \frac{1}{n} \sum_{j=1}^{n} p_j.$$

Finally, we have

$$\begin{aligned}
\text{PRG}\mathsf{adv}[\mathcal{B}, G] &= |\Pr[W_1] - \Pr[W_0]| \\
&= \left| \frac{1}{n} \sum_{j=1}^{n} p_j - \frac{1}{n} \sum_{j=1}^{n} p_{j-1} \right| \\
&= \frac{1}{n} |p_n - p_0|,
\end{aligned}$$

and combining this with (3.9), we have

$$\text{PRG}\mathsf{adv}[\mathcal{A}, G'] = n \cdot \text{PRG}\mathsf{adv}[\mathcal{B}, G].$$

Since we are assuming G is a secure PRG, it follows that $\text{PRG}\mathsf{adv}[\mathcal{B}, G]$ is negligible, and since n is poly-bounded, it follows that $\text{PRG}\mathsf{adv}[\mathcal{A}, G']$ is negligible (see Fact 2.6). That proves the theorem. □

Theorem 3.2 says that the security of a PRG degrades at most linearly in the number of times that we use it. One might ask if this bound is tight; that is, might security indeed degrade linearly in the number of uses? The answer is in fact "yes" (see Exercise 3.14).

3.4.2 A sequential construction: the Blum-Micali method

We now present a sequential construction, invented by Blum and Micali, which uses a PRG that stretches just a little, and builds a PRG that stretches an arbitrary amount.

Let G be a PRG defined over $(\mathcal{S}, \mathcal{R} \times \mathcal{S})$, for some finite sets $\mathcal{S}$ and $\mathcal{R}$. For every poly-bounded value $n \geq 1$, we can construct a new PRG G', defined over $(\mathcal{S}, \mathcal{R}^n \times \mathcal{S})$. For $s \in \mathcal{S}$, we let

$G'(s) :=$
  $s_0 \leftarrow s$
  for $i \leftarrow 1$ to n do
    $(r_i, s_i) \leftarrow G(s_{i-1})$
  output $(r_1, \ldots, r_n, s_n)$.

We call G' the **n-wise sequential composition of** G. See Fig. 3.6 for a schematic description of G' for $n = 3$.

We shall prove below in Theorem 3.3 that if G is a secure PRG, then so is G'. As a special case of this construction, suppose G is a PRG defined over $(\{0,1\}^\ell, \{0,1\}^{t+\ell})$, for some positive integers ℓ and t; that is, G stretches ℓ-bit strings to $(t+\ell)$-bit strings. We can naturally view the output space of G as $\{0,1\}^t \times \{0,1\}^\ell$, and applying the above construction, and interpreting outputs as bit strings, we get a PRG G' that stretches ℓ-bit strings to $(nt+\ell)$-bit strings.

Theorem 3.3. *If G is a secure PRG, then the n-wise sequential composition G' of G is also a secure PRG.*

> *In particular, for every PRG adversary $\mathcal{A}$ that plays Attack Game 3.1 with respect to G', there exists a PRG adversary $\mathcal{B}$ that plays Attack Game 3.1 with respect to G, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that*
>
> $$\text{PRG}\mathsf{adv}[\mathcal{A}, G'] = n \cdot \text{PRG}\mathsf{adv}[\mathcal{B}, G].$$

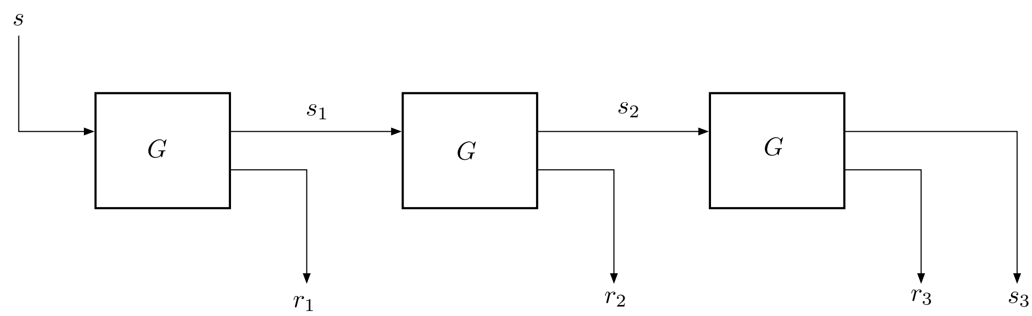


Figure 3.6: The sequential construction for $n = 3$

Proof idea. The proof of this is a hybrid argument that is very similar in spirit to the proof of Theorem 3.2. The intuition behind the proof is as follows: Consider a PRG adversary $\mathcal{A}$ who receives the $(r_1, \ldots, r_n, s_n)$ Experiment 0 of Attack Game 3.1. Since $s = s_0$ is random and G is a secure PRG, we may replace (r_1, s_1) by a completely random element of $\mathcal{R} \times \mathcal{S}$, and the probability that $\mathcal{A}$ outputs 1 in this new, hybrid game should change by only a negligible amount. Now, since s_1 is random (and again, since G is a secure PRG), we may replace (r_2, s_2) by a completely random element of $\mathcal{R} \times \mathcal{S}$, and the probability that $\mathcal{A}$ outputs 1 in this second hybrid game should again change by only a negligible amount. Continuing in this way, we may incrementally replace (r_3, s_3) through (r_n, s_n) by random elements of $\mathcal{R} \times \mathcal{S}$, and the probability that $\mathcal{A}$ outputs 1 should change by only a negligible amount after making all these changes (assuming n is poly-bounded). However, at this point, $\mathcal{A}$ outputs 1 with the same probability with which he would output 1 in Experiment 1 in Attack Game 3.1, and therefore, this probability is negligibly close to the probability that $\mathcal{A}$ outputs 1 in Experiment 0 of Attack Game 3.1.

That is the idea; however, just as in the proof of Theorem 3.2, for technical reasons, we design a single PRG adversary that attacks G. $\Box$

Proof. Let $\mathcal{A}$ be a PRG adversary that plays Attack Game 3.1 with respect to G'. We first introduce a sequence of $n + 1$ hybrid games, called Hybrid 0, Hybrid 1, ..., Hybrid n. For $j = 0, 1, \ldots, n$, we define Hybrid j to be the game played between $\mathcal{A}$ and the following challenger:

$$
\begin{aligned}
& r_1 \xleftarrow{\text{R}} \mathcal{R} \\
& \quad \vdots \\
& r_j \xleftarrow{\text{R}} \mathcal{R} \\
& s_j \xleftarrow{\text{R}} \mathcal{S} \\
& (r_{j+1}, s_{j+1}) \leftarrow G(s_j) \\
& \quad \vdots \\
& (r_n, s_n) \leftarrow G(s_{n-1}) \\
& \text{send } (r_1, \ldots, r_n, s_n) \text{ to } \mathcal{A}.
\end{aligned}
$$

As usual, $\mathcal{A}$ outputs 0 or 1 at the end of the game. See Fig. 3.7 for a schematic description of how these challengers work in the case $n = 3$. Let p_j denote the probability that $\mathcal{A}$ outputs 1 in Hybrid j. Note that p_0 is also equal to the probability that $\mathcal{A}$ outputs 1 in Experiment 0 of

Attack Game 3.1, while p_n is equal to the probability that $\mathcal{A}$ outputs 1 in Experiment 1 of Attack Game 3.1. Thus, we have

$$\text{PRG}\mathsf{adv}[\mathcal{A}, G'] = |p_n - p_0|. \tag{3.10}$$

We next define a PRG adversary $\mathcal{B}$ that plays Attack Game 3.1 with respect to G, and which works as follows:

> Upon receiving $(r, s) \in \mathcal{R} \times \mathcal{S}$ from its challenger, $\mathcal{B}$ plays the role of challenger to $\mathcal{A}$, as follows:
>
> $\omega \xleftarrow{\text{R}} \{1, \ldots, n\}$
> $r_1 \xleftarrow{\text{R}} \mathcal{R}$
> $\quad\vdots$
> $r_{\omega-1} \xleftarrow{\text{R}} \mathcal{R}$
> $(r_\omega, s_\omega) \leftarrow (r, s)$
> $(r_{\omega+1}, s_{\omega+1}) \leftarrow G(s_\omega)$
> $\quad\vdots$
> $(r_n, s_n) \leftarrow G(s_{n-1})$
> send $(r_1, \ldots, r_n, s_n)$ to $\mathcal{A}$.
>
> Finally, $\mathcal{B}$ outputs whatever $\mathcal{A}$ outputs.

Let W_0 be the event that $\mathcal{B}$ outputs 1 in Experiment 0 of Attack Game 3.1, and W_1 be the event that $\mathcal{B}$ outputs 1 in Experiment 1 of Attack Game 3.1. The key observation is this:

> *conditioned on $\omega = j$ for every fixed $j = 1, \ldots, n$, Experiment 0 of $\mathcal{B}$'s attack game is equivalent to Hybrid $j - 1$, while Experiment 1 of $\mathcal{B}$'s attack game is equivalent to Hybrid j.*

Therefore,

$$\Pr[W_0 \mid \omega = j] = p_{j-1} \quad \text{and} \quad \Pr[W_1 \mid \omega = j] = p_j.$$

The remainder of the proof is a simple calculation that is *identical* to that in the last paragraph of the proof of Theorem 3.2. □

One criteria for evaluating a PRG is its **expansion rate**: a PRG that stretches an n-bit seed to an m-bit output has expansion rate of m/n; more generally, if the seed space is $\mathcal{S}$ and the output space is $\mathcal{R}$, we would define the expansion rate as $\log|\mathcal{R}|/\log|\mathcal{S}|$. The sequential composition achieves a better expansion rate than the parallel composition. However, it suffers from the drawback that it cannot be parallelized. In fact, we can obtain the best of both worlds: a large expansion rate with a highly parallelizable construction (see Section 4.4.4).

3.4.3 Mathematical details

There are some subtle points in the proofs of Theorems 3.2 and 3.3 that merit discussion.

First, in both constructions, the underlying PRG G may have system parameters. That is, there may be a probabilistic algorithm that takes as input the security parameter λ, and outputs a system parameter Λ. Recall that a system parameter is public data that fully instantiates the

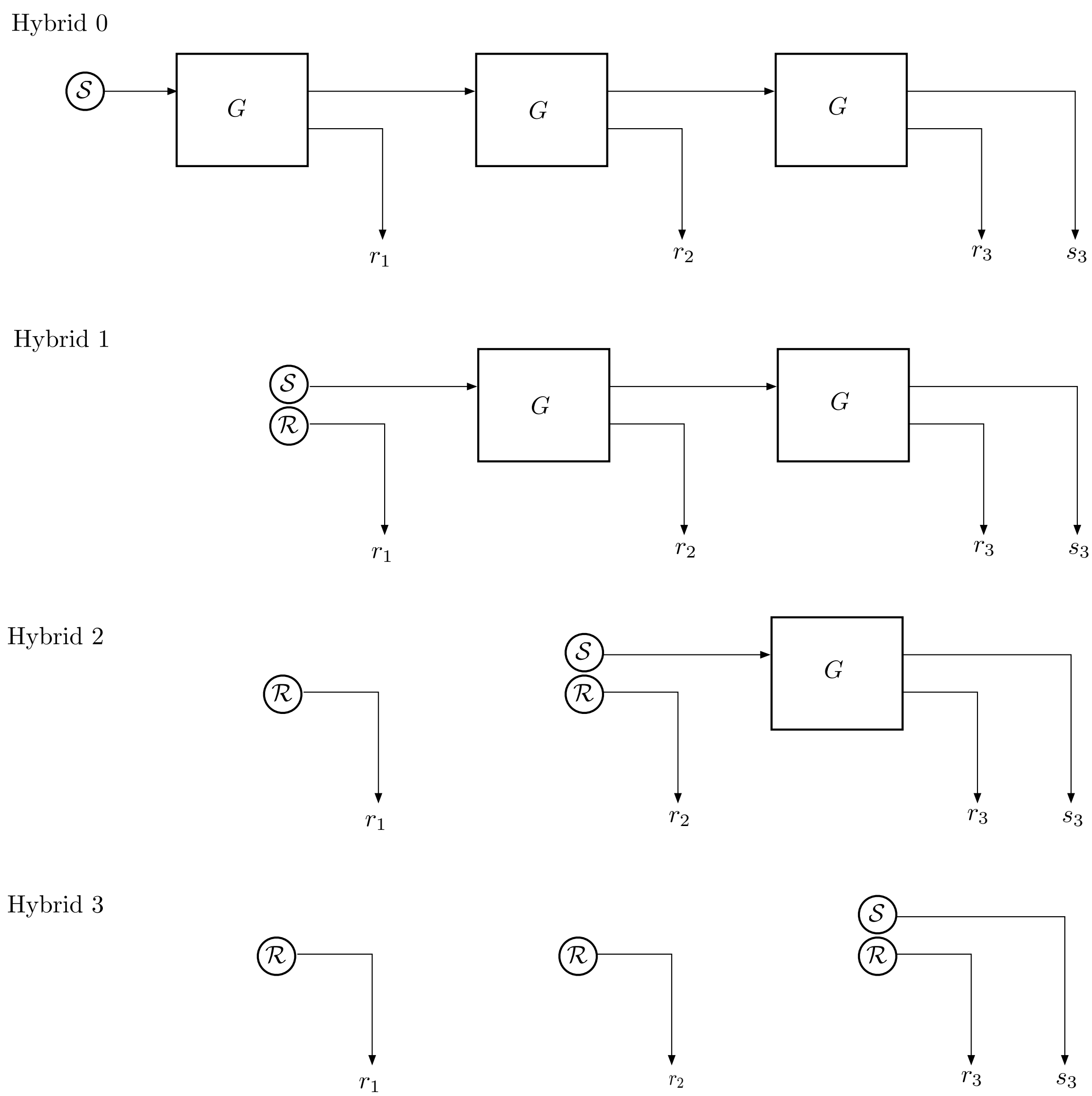


Figure 3.7: The challenger's computation in the hybrid games for $n = 3$. The circles indicate randomly generated elements of $\mathcal{S}$ or $\mathcal{R}$, as indicated by the label.

scheme (in this case, it might define the seed and output spaces). For both the parallel and sequential constructions, one could use the same system parameter for all n instances of G; in fact, for the sequential construction, this is necessary to ensure that outputs from one round may be used as inputs in the next round. The proofs of these security theorems are perfectly valid if the same system parameter is used for all instances of G, or if different system parameters are used.

Second, we briefly discuss a rather esoteric point regarding hybrid arguments. To make things concrete, we focus attention on the proof of Theorem 3.2 (although analogous remarks apply to the proof of Theorem 3.3, or any other hybrid argument). In proving this theorem, we ultimately want to show that if there is an efficient adversary $\mathcal{A}$ that breaks G', then there is an efficient adversary that breaks G. Suppose that $\mathcal{A}$ is an efficient adversary that breaks G', so that its advantage $\epsilon(\lambda)$ (which we write here explicitly as a function of the security parameter λ) with respect to G' is not negligible. This means that there exists a constant c such that $\epsilon(\lambda) \geq 1/\lambda^c$ for infinitely many λ.

Now, in the discussion preceding the proof of Theorem 3.2, we considered the special case $n = 2$, and showed that there exist efficient adversaries $\mathcal{B}_1$ and $\mathcal{B}_2$, such that $\epsilon(\lambda) \leq \delta_1(\lambda) + \delta_2(\lambda)$ for all λ, where $\delta_j(\lambda)$ is the advantage of $\mathcal{B}_j$ with respect to G. It follows that either $\delta_1(\lambda) \geq 1/2\lambda^c$ infinitely often, or $\delta_2(\lambda) \geq 1/2\lambda^c$ infinitely often. So we may conclude that either $\mathcal{B}_1$ breaks G or $\mathcal{B}_2$ breaks G (or possibly both). Thus, *there exists* an efficient adversary that breaks G: it is either $\mathcal{B}_1$ or $\mathcal{B}_2$, which one we do not say (and we do not have to). However, whichever one it is, it is a fixed adversary that is defined uniformly for all λ; that is, it is a fixed machine that takes λ as input.

This argument is perfectly valid, and extends to every *constant* n: we would construct n adversaries $\mathcal{B}_1, \ldots, \mathcal{B}_n$, and argue that for some $j = 1, \ldots, n$, adversary $\mathcal{B}_j$ must have advantage $1/n\lambda^c$ infinitely often, and thus break G. However, this argument does not extend to the case where n is a function of λ, which we now write explicitly as $n(\lambda)$. The problem is not that $1/(n(\lambda)\lambda^c)$ is perhaps too small (it is not). The problem is quite subtle, so before we discuss it, let us first review the (valid) proof that we did give. For each λ, we defined a sequence of $n(\lambda) + 1$ hybrid games, so that for each λ, we actually get a different sequence of games. Indeed, we cannot speak of a single, finite sequence of games that works for all λ, since $n(\lambda) \to \infty$. Nevertheless, we explicitly constructed a fixed adversary $\mathcal{B}$ that is defined uniformly for all λ; that is, $\mathcal{B}$ is a fixed machine that takes λ as input. The sequence of hybrid games that we define for each λ is a mathematical object for which we make no claims as to its computability — it is simply a convenient device used in the analysis of $\mathcal{B}$.

Hopefully by now the reader has at least a hint of the problem that arises if we attempt to generalize the argument for constant n to a function $n(\lambda)$. First of all, it is not even clear what it means to talk about $n(\lambda)$ adversaries $\mathcal{B}_1, \ldots, \mathcal{B}_{n(\lambda)}$: our adversaries our supposed to be fixed machines that take λ as input, and the machines themselves should not depend on λ. Such linguistic confusion aside, our proof for the constant case only shows that there exists an "adversary" that for infinitely many values of λ somehow knows the "right" value of $j = j(\lambda)$ to use in the $(n(\lambda) + 1)$-game hybrid argument — no single, constant value of j necessarily works for infinitely many λ. One can actually make sense of this type of argument if one uses a non-uniform model of computation, but we shall not take this approach in this text.

All of these problems simply go away when we use a hybrid argument that constructs a single adversary $\mathcal{B}$, as we did in the proofs of Theorems 3.2 and 3.3. However, we reiterate that the original analysis we did in the where $n = 2$, or its natural extension to every constant n, is perfectly valid. In that case, we construct a single, fixed sequence of $n + 1$ games, with each individual game uniformly defined for all λ (just as our attack games are in our security definitions), as well as a finite collection of adversaries, each of which is a fixed machine. We reiterate this because in the sequel we shall often be constructing proofs that involve finite sequences of games like this (indeed, the proof of Theorem 3.1 was of this type). In such cases, each game will be uniformly defined for all λ, and will be denoted Game 0, Game 1, etc. In contrast, when we make a hybrid argument that uses non-uniform sequences of games, we shall denote these games Hybrid 0, Hybrid 1, etc., so as to avoid any possible confusion.

3.5 The next bit test

Let G be a PRG defined over $(\{0,1\}^\ell, \{0,1\}^L)$, so that it stretches ℓ-bit strings to L-bit strings. There are a number of ways an adversary might be able to distinguish a pseudo-random output of G from a truly random bit string. Indeed, suppose that an efficient adversary were able to compute, say, the last bit of G's output, given the first $L-1$ bits of G's output. Intuitively, the existence of such an adversary would imply that G is insecure, since given the first $L-1$ bits of a truly random L-bit string, one has at best a 50-50 chance of guessing the last bit. It turns out that an interesting converse, of sorts, is also true.

We shall formally define the notion of **unpredictability** for a PRG, which essentially says that given the first i bits of G's output, it is hard to predict the next bit (i.e., the $(i+1)$-st bit) with probability significantly better that 1/2 (here, i is an adversarially chosen index). We shall then prove that unpredictability and security are equivalent. The fact that security implies unpredictability is fairly obvious: the ability to effectively predict the next bit in the pseudo-random output string immediately gives an effective statistical test. However, the fact that unpredictability implies security is quite interesting (and requires more effort to prove): it says that if there is any effective statistical test at all, then there is in fact an effective method for predicting the next bit in a pseudo-random output string.

Attack Game 3.2 (Unpredictable PRG). For a given PRG G, defined over $(\mathcal{S}, \{0,1\}^L)$, and a given adversary $\mathcal{A}$, the attack game proceeds as follows:

- The adversary sends an index i, with $0 \le i \le L-1$, to the challenger.
- The challenger computes
$$s \xleftarrow{\text{R}} \mathcal{S}, \;\; r \leftarrow G(s)$$
and sends $r[0\,..\,i-1]$ to the adversary.
- The adversary outputs $g \in \{0,1\}$.

We say that $\mathcal{A}$ **wins** if $r[i] = g$, and we define $\mathcal{A}$'s **advantage** $\text{Pred}\mathsf{adv}[\mathcal{A}, G]$ to be $|\Pr[\mathcal{A} \text{ wins}] - 1/2|$. $\square$

Definition 3.3 (Unpredictable PRG). *A PRG G is **unpredictable** if the value* $\text{Pred}\mathsf{adv}[\mathcal{A}, G]$ *is negligible for all efficient adversaries $\mathcal{A}$.*

We begin by showing the security implies unpredictability.

Theorem 3.4. *Let G be a PRG, defined over $(\mathcal{S}, \{0,1\}^L)$. If G is secure, then G is unpredictable.*

In particular, for every adversary $\mathcal{A}$ breaking the unpredictability of G, as in Attack Game 3.2, there exists an adversary $\mathcal{B}$ breaking the security G as in Attack Game 3.1, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{Pred}\mathsf{adv}[\mathcal{A}, G] = \text{PRG}\mathsf{adv}[\mathcal{B}, G].$$

Proof. Let $\mathcal{A}$ be an adversary breaking the predictability of G, and let i denote the index chosen by $\mathcal{A}$. Also, suppose $\mathcal{A}$ wins Attack Game 3.2 with probability $1/2 + \epsilon$, so that $\text{Pred}\mathsf{adv}[\mathcal{A}, G] = |\epsilon|$.

We build an adversary $\mathcal{B}$ breaking the security of G, using $\mathcal{A}$ as a subroutine, as follows:

Upon receiving $r \in \{0, 1\}^L$ from its challenger, $\mathcal{B}$ does the following:

- $\mathcal{B}$ gives $r[0 \mathinner{..} i-1]$ to $\mathcal{A}$, obtaining $\mathcal{A}$'s output $g \in \{0, 1\}$;
- if $r[i] = g$, then output 1, and otherwise, output 0.

For $b = 0, 1$, let W_b be the event that $\mathcal{B}$ outputs 1 in Experiment b of Attack Game 3.1. In Experiment 0, r is a pseudo-random output of G, and W_0 occurs if and only if $r[i] = g$, and so by definition

$$\Pr[W_0] = 1/2 + \epsilon.$$

In Experiment 1, r is a truly random bit string, but again, W_1 occurs if and only if $r[i] = g$; in this case, however, as random variables, the values of $r[i]$ and g are independent, and so

$$\Pr[W_1] = 1/2.$$

It follows that

$$\text{PRG}\mathsf{adv}[\mathcal{B}, G] = |\Pr[W_1] - \Pr[W_0]| = |\epsilon| = \text{Pred}\mathsf{adv}[\mathcal{A}, G]. \quad \square$$

The more interesting, and more challenging, task is to show that unpredictability implies security. Before getting into all the details of the proof, we sketch the high level ideas.

First, we shall employ a hybrid argument, which will essentially allow us to argue that if $\mathcal{A}$ is an efficient adversary that can effectively distinguish a pseudo-random L-bit string from a random L-bit string, then we can construct an efficient adversary $\mathcal{B}$ that can effectively distinguish

$$x_1 \cdots x_j \, x_{j+1}$$

from

$$x_1 \cdots x_j \, r,$$

where j is a randomly chosen index, $x_1, \ldots, x_L$ is the pseudo-random output, and r is a random bit. Thus, adversary $\mathcal{B}$ can distinguish the pseudo-random bit x_{j+1} from the random bit r_{j+1}, given the "side information" $x_1, \ldots, x_j$.

We want to turn $\mathcal{B}$'s distinguishing advantage into a predicting advantage. The rough idea is this: given $x_1, \ldots, x_j$, we feed $\mathcal{B}$ the string $x_1, \ldots, x_j \, r$ for a randomly chosen bit r; if $\mathcal{B}$ outputs 1, our prediction for x_{j+1} is r; otherwise, or prediction for x_{j+1} is $\bar{r}$ (the complement of r).

That this prediction strategy works is justified by the following general result, which we call the *distinguisher/predictor lemma*. The general setup is as follows. We have:

- a random variable $\mathbf{X}$, which corresponds to the "side information" $x_1, \ldots, x_j$ above, as well as any random coins used by the adversary $\mathcal{B}$;

- a 0/1-valued random variable $\mathbf{B}$, which corresponds to x_{j+1} above, and which may be correlated with $\mathbf{X}$;
- a 0/1-valued random variable $\mathbf{R}$, which corresponds to r above, and which is independent of $(\mathbf{X}, \mathbf{B})$;
- a function d, which corresponds to $\mathcal{B}$'s strategy, so that $\mathcal{B}$'s distinguishing advantage is equal to $|\epsilon|$, where $\epsilon = \Pr[d(\mathbf{X}, \mathbf{B}) = 1] - \Pr[d(\mathbf{X}, \mathbf{R}) = 1]$.

The lemma says that if we define $\mathbf{B}'$ using the predicting strategy outlined above, namely $\mathbf{B}' = \mathbf{R}$ if $d(\mathbf{X}, \mathbf{R}) = 1$, and $\mathbf{B}' = \overline{\mathbf{R}}$ otherwise, then the probability that the prediction $\mathbf{B}'$ is equal to the actual value $\mathbf{B}$ is precisely $1/2 + \epsilon$. Here is the precise statement of the lemma:

Lemma 3.5 (Distinguisher/predictor lemma). *Let $\mathbf{X}$ be a random variable taking values in some set S, and let $\mathbf{B}$ and $\mathbf{R}$ be a 0/1-valued random variables, where $\mathbf{R}$ is uniformly distributed over $\{0,1\}$ and is independent of $(\mathbf{X}, \mathbf{B})$. Let $d : S \times \{0,1\} \to \{0,1\}$ be an arbitrary function, and let*

$$\epsilon := \Pr[d(\mathbf{X}, \mathbf{B}) = 1] - \Pr[d(\mathbf{X}, \mathbf{R}) = 1].$$

Define the random variable $\mathbf{B}'$ as follows:

$$\mathbf{B}' := \begin{cases} \mathbf{R} & \text{if } d(\mathbf{X}, \mathbf{R}) = 1; \\ \overline{\mathbf{R}} & \text{otherwise.} \end{cases}$$

Then

$$\Pr[\mathbf{B}' = \mathbf{B}] = 1/2 + \epsilon.$$

Proof. We calculate $\Pr[\mathbf{B}' = \mathbf{B}]$, conditioning on the events $\mathbf{B} = \mathbf{R}$ and $\mathbf{B} = \overline{\mathbf{R}}$:

$$\begin{aligned}
\Pr[\mathbf{B}' = \mathbf{B}] &= \Pr[\mathbf{B}' = \mathbf{B} \mid \mathbf{B} = \mathbf{R}] \Pr[\mathbf{B} = \mathbf{R}] + \Pr[\mathbf{B}' = \mathbf{B} \mid \mathbf{B} = \overline{\mathbf{R}}] \Pr[\mathbf{B} = \overline{\mathbf{R}}] \\
&= \Pr[d(\mathbf{X}, \mathbf{R}) = 1 \mid \mathbf{B} = \mathbf{R}] \frac{1}{2} + \Pr[d(\mathbf{X}, \mathbf{R}) = 0 \mid \mathbf{B} = \overline{\mathbf{R}}] \frac{1}{2} \\
&= \frac{1}{2} \Big(\Pr[d(\mathbf{X}, \mathbf{R}) = 1 \mid \mathbf{B} = \mathbf{R}] + (1 - \Pr[d(\mathbf{X}, \mathbf{R}) = 1 \mid \mathbf{B} = \overline{\mathbf{R}})] \Big) \\
&= \frac{1}{2} + \frac{1}{2}(\alpha - \beta),
\end{aligned}$$

where

$$\alpha := \Pr[d(\mathbf{X}, \mathbf{R}) = 1 \mid \mathbf{B} = \mathbf{R}] \text{ and } \beta := \Pr[d(\mathbf{X}, \mathbf{R}) = 1 \mid \mathbf{B} = \overline{\mathbf{R}}].$$

By independence, we have

$$\alpha = \Pr[d(\mathbf{X}, \mathbf{R}) = 1 \mid \mathbf{B} = \mathbf{R}] = \Pr[d(\mathbf{X}, \mathbf{B}) = 1 \mid \mathbf{B} = \mathbf{R}] = \Pr[d(\mathbf{X}, \mathbf{B}) = 1].$$

To see the last equality, the result of Exercise 3.25 may be helpful.

We thus calculate that

$$\begin{aligned}
\epsilon &= \Pr[d(\mathbf{X}, \mathbf{B}) = 1] - \Pr[d(\mathbf{X}, \mathbf{R}) = 1] \\
&= \alpha - \Big(\Pr[d(\mathbf{X}, \mathbf{R}) = 1 \mid \mathbf{B} = \mathbf{R}] \Pr[\mathbf{B} = \mathbf{R}] + \Pr[d(\mathbf{X}, \mathbf{R}) = 1 \mid \mathbf{B} = \overline{\mathbf{R}}] \Pr[\mathbf{B} = \overline{\mathbf{R}}] \Big) \\
&= \alpha - \frac{1}{2}(\alpha + \beta) \\
&= \frac{1}{2}(\alpha - \beta),
\end{aligned}$$

which proves the lemma. □

Theorem 3.6. *Let G be a PRG, defined over $(\mathcal{S}, \{0,1\}^L)$. If G is unpredictable, then G is secure.*

> *In particular, for every adversary $\mathcal{A}$ breaking the security of G as in Attack Game 3.1, there exists an adversary $\mathcal{B}$, breaking the unpredictability of G as in Attack Game 3.2, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that*
>
> $$\mathrm{PRG}\mathsf{adv}[\mathcal{A}, G] = L \cdot \mathrm{Pred}\mathsf{adv}[\mathcal{B}, G].$$

Proof. Let $\mathcal{A}$ attack G as in Attack Game 3.1. Using $\mathcal{A}$, we build a predictor $\mathcal{B}$, which attacks G as in Attack Game 3.2, and works as follows:

- Choose $\omega \in \{1, \ldots, L\}$ at random.
- Send $L - \omega$ to the challenger, obtaining a string $x \in \{0,1\}^{L-\omega}$.
- Generate ω random bits $r_1, \ldots, r_\omega$, and give the L-bit string $x \parallel r_1 \cdots r_\omega$ to $\mathcal{A}$.
- If $\mathcal{A}$ outputs 1, then output r_1; otherwise, output $\overline{r}_1$.

To analyze $\mathcal{B}$, we consider $L + 1$ hybrid games, called Hybrid 0, Hybrid 1, ..., Hybrid L. For $j = 0, \ldots, L$, we define Hybrid j to be the game played between $\mathcal{A}$ and a challenger that generates a bit string r consisting of $L - j$ pseudo-random bits, followed by j truly random bits; that is, the challenger chooses $s \in \mathcal{S}$ and $t \in \{0,1\}^j$ at random, and sends $\mathcal{A}$ the bit string

$$r := G(s)[0\,..\,L - j - 1] \parallel t.$$

As usual, $\mathcal{A}$ outputs 0 or 1 at the end of the game, and we define p_j to be the probability that $\mathcal{A}$ outputs 1 in Hybrid j. Note that p_0 is the probability that $\mathcal{A}$ outputs 1 in Experiment 0 of Attack Game 3.1, while p_L is the probability that $\mathcal{A}$ outputs 1 in Experiment 1 of Attack Game 3.1.

Let W be the event that $\mathcal{B}$ wins in Attack Game 3.2 (that is, correctly predicts the next bit). Then we have

$$\begin{aligned}
\Pr[W] &= \sum_{j=1}^{L} \Pr[W \mid \omega = j] \Pr[\omega = j] \\
&= \frac{1}{L} \sum_{j=1}^{L} \Pr[W \mid \omega = j] \\
&= \frac{1}{L} \sum_{j=1}^{L} \Big(\frac{1}{2} + p_{j-1} - p_j \Big) \quad \text{(by Lemma 3.5)} \\
&= \frac{1}{2} + \frac{1}{L}(p_0 - p_L),
\end{aligned}$$

and the theorem follows. □

3.6 Case study: the Salsa and ChaCha PRGs

There are many ways to build PRGs and stream ciphers in practice. One approach builds PRGs using the Blum-Micali paradigm discussed in Section 3.4.2. Another approach, discussed more generally in the Chapter 5, builds them from a more versatile primitive called a *pseudorandom function* in counter mode. We start with a construction that uses this latter approach.

Salsa20/12 and Salsa20/20 are fast stream ciphers designed by Dan Burnstein in 2005. Salsa20/12 is one of four Profile 1 stream ciphers selected for the eStream portfolio of stream ciphers. eStream is a project that identifies fast and secure stream ciphers that are appropriate for practical use. Variants of Salsa20/12 and Salsa20/20, called ChaCha12 and ChaCha20 respectively, were proposed by Bernstein in 2008. These stream ciphers have been incorporated into several widely deployed protocols such as TLS and SSH.

Let us briefly describe the PRGs underlying the Salsa and ChaCha stream cipher families. These PRGs take as input a 256-bit seed and a 64-bit nonce. For now we ignore the nonce and simply set it to 0. We discuss the purpose of the nonce at the end of this section. The Salsa and ChaCha PRGs follow the same high level structure shown in Fig. 3.8. They make use of two components:

- A padding function denoted $\text{pad}(s, j, 0)$ that combines a 256-bit seed s with a 64-bit counter j to form a 512-bit block. The third input, a 64-bit nonce, is always set to 0 for now.
- A fixed public permutation $\pi : \{0,1\}^{512} \to \{0,1\}^{512}$.

These components are used to output $L < 2^{64}$ pseudorandom blocks, each 512 bits long, using the following algorithm (Fig. 3.8):

input: seed $s \in \{0,1\}^{256}$
1. for $j \leftarrow 0$ to $L - 1$
2. $\qquad h_j \leftarrow \text{pad}(s, j, 0) \in \{0,1\}^{512}$
3. $\qquad r_j \leftarrow \pi(h_j) \oplus h_j$
4. output $(r_0, \ldots, r_{L-1})$.

The final PRG output is $512 \cdot L$ bits long. We note that in Salsa and ChaCha the XOR on line 3 is a slightly more complicated operation: the 512-bit operands h_j and $\pi(h_j)$ are split into 16 words each 32-bits long and then added word-wise mod 2^{32}.

The design of Salsa and ChaCha is highly parallelizable and can take advantage of multiple processor cores to speed-up encryption. Moreover, it enables random access to output blocks: output block number j can be computed without having to first compute all previous blocks. Generators based on the Blum-Micali paradigm do not have these properties.

We analyze the security of the Salsa and ChaCha design in Exercise 4.23 in the next chapter, after we develop a few more tools.

The details. We briefly describe the padding function $\text{pad}(s, j, n)$ and the permutation π used in ChaCha20. The padding function takes as input a 256-bit seed $s_0, \ldots, s_7 \in \{0,1\}^{32}$, a 64-bit counter $j_0, j_1 \in \{0,1\}^{32}$, and 64-bit nonce $n_0, n_1 \in \{0,1\}^{32}$. It outputs a 512-bit block denoted

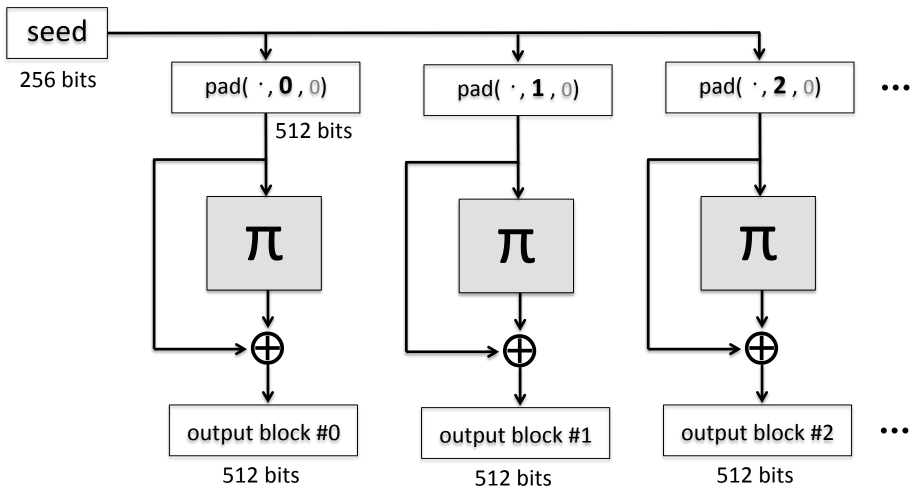


Figure 3.8: A schematic of the Salsa and ChaCha PRGs

$x_0, \ldots, x_{15} \in \{0,1\}^{32}$. The output is arranged in a 4×4 matrix of 32-bit words as follows:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \\ x_8 & x_9 & x_{10} & x_{11} \\ x_{12} & x_{13} & x_{14} & x_{15} \end{pmatrix} \longleftarrow \begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ s_0 & s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 & s_7 \\ j_0 & j_1 & n_0 & n_1 \end{pmatrix} \tag{3.11}$$

where c_0, c_1, c_2, c_3 are fixed 32-bit constants.

The permutation $\pi : \{0,1\}^{512} \to \{0,1\}^{512}$ is constructed by iterating a simpler permutation a fixed number of times. The 512-bit input to π is treated as a 4×4 array of 32-bit words denoted by $x_0, \ldots, x_{15}$. In ChaCha20 the function π is implemented by repeating the following sequence of steps ten times:

$$\begin{array}{lll} \text{QuarterRound}(x_0, x_4, x_8, x_{12}), & \text{QuarterRound}(x_1, x_5, x_9, x_{13}), & \text{QuarterRound}(x_2, x_6, x_{10}, x_{14}), \\ \text{QuarterRound}(x_3, x_7, x_{11}, x_{15}), & \text{QuarterRound}(x_0, x_5, x_{10}, x_{15}), & \text{QuarterRound}(x_1, x_6, x_{11}, x_{12}), \\ \text{QuarterRound}(x_2, x_7, x_8, x_{13}), & \text{QuarterRound}(x_3, x_4, x_9, x_{14}) & \end{array}$$

where QuarterRound($\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}$) is defined as the following sequence of steps written as C code:

```
a += b;  d ^= a;  d <<<= 16;
c += d;  b ^= c;  b <<<= 12;
a += b;  d ^= a;  d <<<= 8;
c += d;  b ^= c;  b <<<= 7;
```

The first four invocations of QuarterRound are applied to each of the first four columns. The next four invocations are applied to each of the four diagonals. This completes our description of ChaCha20, except that we still need to discuss the use of nonces.

Using nonces. While the PRGs we discussed so far only take the seed as input, many PRGs used in practice take an additional input called a *nonce*. That is, the PRG is a function $G : \mathcal{S} \times \mathcal{N} \to \mathcal{R}$

where $\mathcal{S}$ and $\mathcal{R}$ are as before and $\mathcal{N}$ is called a *nonce space*. The nonce lets us generate multiple pseudorandom outputs from a single seed s. That is, $G(s, n_0)$ is one pseudorandom output and $G(s, n_1)$ for $n_1 \neq n_0$ is another. The nonce turns the PRG into a more powerful primitive called a *pseudorandom function* discussed in the next chapter. As we will see, secure pseudorandom functions make it possible to use the same seed to encrypt multiple messages securely.

3.7 Case study: linear generators

In this section we look at two example PRGs built from linear functions. Both generators follow the Blum-Micali paradigm presented in Section 3.4.2. Our first example, called a *linear congruential generator*, is completely insecure and we present it to give an example of some beautiful mathematics that comes up when attacking PRGs. Our second example, called a *subset sum generator*, is a provably secure PRG assuming a certain version of the classic subset-sum problem is hard.

3.7.1 An example cryptanalysis: linear congruential generators

Linear congruential generators (LCG) are used in statistical simulations to generate pseudorandom values. They are fast, easy to implement, and widely deployed. Variants of LCG are used to generate randomness in early versions of `glibc`, Microsoft Visual Basic, and the Java runtime. While these generators may be sufficient for simulations they should *never* be used for cryptographic applications because they are insecure as PRGs. In particular, they are predictable: given a few consecutive outputs of an LCG generator it is easy to compute all subsequent outputs. In this section we describe an attack on LCG generators by showing a prediction algorithm.

The basic linear congruential generator is specified by four public system parameters: an integer q, two constants $a, b \in \{0, \dots, q-1\}$, and a positive integer $w \leq q$. The constant a is taken to be relatively prime to q. We use $\mathcal{S}_q$ and $\mathcal{R}$ to denote the sets:

$$\mathcal{S}_q := \{0, \dots, q-1\}; \quad \mathcal{R} := \big\{0, \dots, \lfloor (q-1)/w \rfloor\ \big\}.$$

Here $\lfloor \cdot \rfloor$ is the floor function: for a real number x, $\lfloor x \rfloor$ is the biggest integer less than or equal to x. Now, the generator $G_{\text{lcg}} : \mathcal{S}_q \to \mathcal{R} \times \mathcal{S}_q$ with seed $s \in \mathcal{S}_q$ is defined as follows:

$$G_{\text{lcg}}(s) := \big(\ \lfloor s/w \rfloor, \quad as + b \bmod q\ \big).$$

When w is a power of 2, say $w = 2^t$, then the operation $\lfloor s/w \rfloor$ simply erases the t least significant bits of s. Hence, the left part of $G_{\text{lcg}}(s)$ is the result of dropping the t least significant bits of s.

The generator G_{lcg} is clearly insecure since given $s' := as + b \bmod q$ it is straight-forward to recover s and then distinguish $\lfloor s/w \rfloor$ from random. Nevertheless, consider a variant of the Blum-Micali construction in which the final $\mathcal{S}_q$-value is not output:

$$
\begin{array}{ll}
G^{(n)}_{\text{lcg}}(s) := \qquad & s_0 \leftarrow s \\
& \text{for } i \leftarrow 1 \text{ to } n \text{ do} \\
& \qquad r_i \leftarrow \lfloor s_{i-1}/w \rfloor, \quad s_i \leftarrow a s_{i-1} + b \bmod q \\
& \text{output } (r_1, \dots, r_n).
\end{array}
$$

We refer to each iteration of the loop as a single iteration of the LCG generator and call each one of $r_1, \dots, r_n$ the output of a single iteration.

Different implementations use different system parameters q, a, b, w. For example, the `Math.random` function in the Java 8 Development Kit (JDKv8) uses $q = 2^{48}$, $w = 2^{22}$, and the hexadecimal constants $a = \text{0x5DEECE66D}, b = \text{0x0B}$. Thus, every iteration of the LCG generator outputs the top $48 - 22 = 26$ bits of the 48-bit state s_i.

The parameters used by this Java 8 generator are clearly too small for security applications since the output of the first iteration of the generator reveals all but 22 bits of the seed s. An attacker can easily recover these unknown 22 bits by exhaustive search: for every possible value of the 22 bits the attacker forms a candidate seed $\hat{s}$. It tests if $\hat{s}$ is the correct seed by comparing subsequent outputs computed from seed $\hat{s}$ to a few subsequent outputs observed from the actual generator. By trying all 2^{22} candidates (about four million) the attacker eventually finds the correct seed s and can then predict *all* subsequent outputs of the generator. This attack runs in under a second on a modern processor.

Even when the LCG parameters are sufficiently large to prevent exhaustive search, say $q = 2^{512}$, the generator $G_{\text{lcg}}^{(n)}$ is insecure and should never be used for security applications despite its wide availability in software libraries. Known attacks [40] on the LCG show that even if the generator outputs only a few bits per iteration, it is still possible to predict the entire sequence from just a few consecutive outputs. Let us see an elegant version of this attack.

Cryptanalysis. Suppose that q is large (e.g. $q = 2^{512}$) and the LCG generator $G_{\text{lcg}}^{(n)}$ outputs about half the bits of the state s per iteration, as in the Java 8 `Math.random` generator. An exhaustive search on the seed s is not possible given its size. Nevertheless, we show how to quickly predict the generator from the output of only two consecutive iterations.

More precisely, suppose that $w < \sqrt{q}/c$ for some fixed $c > 0$, say $c = 32$. This means that at every iteration the generator outputs slightly more than half the bits of the current internal state.

Suppose the attacker is given two consecutive outputs of the generator $r_i, r_{i+1} \in \mathcal{R}$. We show how it can predict the remaining sequence. The attacker knows that

$$r_i = \lfloor s_i/w \rfloor \qquad \text{and} \qquad r_{i+1} = \lfloor s_{i+1}/w \rfloor = \lfloor (as_i + b \bmod q)/w \rfloor \ .$$

for some unknown $s_i \in \mathcal{S}_q$. We have

$$r_i \cdot w + e_0 = s_i \qquad \text{and} \qquad r_{i+1} \cdot w + e_1 = (as_i + b \bmod q),$$

where e_0 and e_1 are the remainders after dividing s_i and s_{i+1} by w; in particular, $0 \le e_0, e_1 < w < \sqrt{q}/c$. The fact that e_0, e_1 are smaller than $\sqrt{q}$ is an essential ingredient of the attack. Next, let us write s in place of s_i, and eliminate the mod q by introducing an integer variable x to obtain

$$r_i \cdot w + e_0 = s \qquad \text{and} \qquad r_{i+1} \cdot w + e_1 = as + b + qx \ .$$

The values x, s, e_0, e_1 are unknown to the attacker, but it knows r_i, r_{i+1}, w, a, b. Finally, re-arranging terms to put the terms involving x and s on the left gives

$$s = r_i \cdot w + e_0 \qquad \text{and} \qquad as + qx = r_{i+1}w - b + e_1 \ . \tag{3.12}$$

We can re-write (3.12) in vector form as

$$s \cdot \begin{pmatrix} 1 \\ a \end{pmatrix} + x \cdot \begin{pmatrix} 0 \\ q \end{pmatrix} = \mathbf{g} + \mathbf{e} \qquad \text{where} \qquad \mathbf{g} := \begin{pmatrix} r_i w \\ r_{i+1}w - b \end{pmatrix} \quad \text{and} \quad \mathbf{e} := \begin{pmatrix} e_0 \\ e_1 \end{pmatrix} \ . \tag{3.13}$$

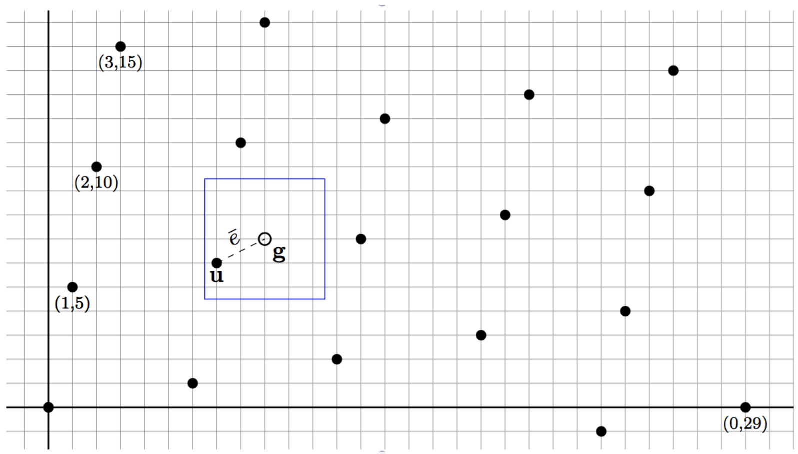


Figure 3.9: The two-dimensional lattice associated with attacking the LCG. Here the lattice is generated by the vectors $(1,5)^\intercal$ and $(0,29)^\intercal$. The attacker has a vector $\mathbf{g} = (9,7)^\intercal$ and wishes to find the closest lattice vector $\mathbf{u}$. In this picture there is indeed only one "close" lattice vector to $\mathbf{g}$.

Let $\mathbf{u} \in \mathbb{Z}^2$ denote the unknown vector $\mathbf{u} := \mathbf{g} + \mathbf{e} = s \cdot (1,a)^\intercal + x \cdot (0,q)^\intercal$. If the attacker could find $\mathbf{u}$ then he could easily recover s and x from $\mathbf{u}$ by linear algebra. Using s he could predict the rest of the PRG output. Thus, to break the generator it suffices to find the vector $\mathbf{u}$. The attacker knows the vector $\mathbf{g} \in \mathbb{Z}^2$, and moreover, he knows that $\mathbf{e}$ is short, namely $\|\mathbf{e}\|_\infty$ is at most $\sqrt{q}/c$. Therefore, he knows that $\mathbf{u}$ is "close" to $\mathbf{g}$.

We show how to find $\mathbf{u}$ from $\mathbf{g}$. Consider the set of all integer linear combinations of the vectors $(1,a)^\intercal$ and $(0,q)^\intercal$. This set, denoted by $\mathcal{L}_a$, is a subset of $\mathbb{Z}^2$ and contains vectors like $(1,a)^\intercal$, $(2,2a)^\intercal$, $(3,3a-2q)^\intercal$, and so on. The set $\mathcal{L}_a$ is illustrated in Fig. 3.9 where the solid dots in the figure are the integer linear combinations of the vectors $(1,a)^\intercal$ and $(0,q)^\intercal$. The set $\mathcal{L}_a$ is called the two-dimensional **lattice** generated by the vectors $(1,a)^\intercal$ and $(0,q)^\intercal$.

Now, the attacker has a vector $\mathbf{g} \in \mathbb{Z}^2$ and knows that his target vector $\mathbf{u} \in \mathcal{L}_a$ is close to $\mathbf{g}$. If he could find the *closest* vector in $\mathcal{L}_a$ to $\mathbf{g}$ then there is a good chance that this vector is the desired vector $\mathbf{u}$. The following lemma shows that indeed this is the case for most $a \in \mathcal{S}_q$.

Lemma 3.7. *For at least $(1 - 16/c^2) \cdot q$ of the a in $\mathcal{S}_q$, the lattice $\mathcal{L}_a \subseteq \mathbb{Z}^2$ has the following property: for every $\mathbf{g} \in \mathbb{Z}^2$ there is at most one vector $\mathbf{u} \in \mathcal{L}_a$ such that $\|\mathbf{g} - \mathbf{u}\|_\infty < \sqrt{q}/c$.*

Taking $c = 32$ in Lemma 3.7 (so that $w = \sqrt{q}/30$) shows that for 98% of the $a \in \mathcal{S}_q$ the closest vector to $\mathbf{g}$ in $\mathcal{L}_a$ is precisely the desired vector $\mathbf{u}$. Before proving the lemma, let us first complete the description of the attack.

It remains to efficiently find the closest vector to $\mathbf{g}$ in $\mathcal{L}_a$. This problem is a special case of a general problem called the **closest vector problem**: given a lattice $\mathcal{L}$ and a vector $\mathbf{g}$, find the vector in $\mathcal{L}$ that is closest to $\mathbf{g}$. When the lattice $\mathcal{L}$ is two dimensional there is an efficient polynomial time algorithm for this problem [102]. Armed with this algorithm the attacker can recover the internal state s_i of the LCG generator from just two outputs r_i, r_{i+1} of the generator and predict the remaining sequence. This attack works for 98% of the $a \in \mathcal{S}_q$.

For completeness we note that some example $a \in \mathcal{S}_q$ in the 2% where the attack fails are $a = 1$ and $a = 2$. For these a there may be many lattice vector in $\mathcal{L}_a$ close to a given $\mathbf{g}$. We leave it as a fun exercise to devise an attack that works for the a in $\mathcal{S}_q$ to which Lemma 3.7 does not apply. We conclude this section with a proof of Lemma 3.7.

Proof of Lemma 3.7. Let $\mathbf{g} \in \mathbb{Z}^2$ and suppose there are two vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ in $\mathcal{L}_a$ that are close to $\mathbf{g}$, that is, $\|\mathbf{u}_i - \mathbf{g}\|_\infty < \sqrt{q}/c$ for $i = 0, 1$. Then $\mathbf{u}_0$ and $\mathbf{u}_1$ must be close to each other. Indeed, by the triangle inequality, we have

$$\|\mathbf{u}_0 - \mathbf{u}_1\|_\infty \leq \|\mathbf{u}_0 - \mathbf{g}\|_\infty + \|\mathbf{g} - \mathbf{u}_1\|_\infty \leq 2\sqrt{q}/c \ .$$

Since any lattice is closed under addition, we see that $\mathbf{u} := \mathbf{u}_0 - \mathbf{u}_1$ is a vector in the lattice $\mathcal{L}_a$, and we conclude that $\mathcal{L}_a$ must contain a "short" vector, namely, a non-zero vector of norm at most $B := 2\sqrt{q}/c$. So let us bound the number of "bad" a's for which $\mathcal{L}_a$ contains such a short vector.

Let us first consider the case when q is prime. We show that every short vector is contained in at most one lattice $\mathcal{L}_a$ and therefore the number of bad a's is at most the number of short vectors. Let $\mathbf{t} = (s, y)^\intercal \in \mathbb{Z}^2$ be some non-zero vector such that $\|\mathbf{t}\|_\infty \leq B$. Suppose that $\mathbf{t} \in \mathcal{L}_a$ for some $a \in \mathcal{S}_q$. Then there exist integers s_a and x_a such that $s_a \cdot (1, a)^\intercal + x_a \cdot (0, q)^\intercal = \mathbf{t} = (s, y)^\intercal$. From this we obtain that $s = s_a$ and $y = as \bmod q$. Moreover, $s \neq 0$ since otherwise $\mathbf{t} = \mathbf{0}$. Since $y = as \bmod q$ and $s \neq 0$, the value of a is uniquely determined, namely, $a = ys^{-1} \bmod q$. Hence, when q is prime, every non-zero short vector $\mathbf{t}$ is contained in at most one lattice $\mathcal{L}_a$ for some $a \in \mathcal{S}_q$. It follows that the number of bad a is at most the number of short vectors, which is $(2B)^2 = 16q/c^2$.

The same bound on the number of bad a's holds when q is not prime. To see why consider a specific non-zero $s \in \mathcal{S}_q$ and let $d = \gcd(s, q)$. As above, a vector $\mathbf{t} = (s, y)^\intercal$ is contained in some lattice $\mathcal{L}_a$ only if there is an $a \in \mathcal{S}_q$ satisfying $as \equiv y \pmod{q}$. This implies that y must be a multiple of d so that we need only consider $2B/d$ possible values of y. For each such y the vector $\mathbf{t} = (s, y)^\intercal$ is in at most d lattices $\mathcal{L}_a$. Since there are $2B$ possible values for s, this shows that the number of bad a's is bounded by $d \cdot 2B/d \cdot 2B = (2B)^2$ as in the case when q is prime.

To conclude, there are at most $16q/c^2$ bad values of a in $\mathcal{S}_q$. Therefore, for $(1 - 16/c^2) \cdot q$ of the a values in $\mathcal{S}_q$, the lattice $\mathcal{L}_a$ contains no non-zero short vectors and the lemma follows. $\square$

3.7.2 The subset sum generator

We next show how to construct a pseudorandom generator from simple linear operations. The generator is secure assuming that a certain randomized version of the classic *subset sum problem* is hard.

The modular subset problem. Let q be a positive integer and set $\mathcal{S}_q := \{0, \ldots, q-1\}$. Choose n integers $\mathbf{a} := (a_0, \ldots, a_{n-1})$ in $\mathcal{S}_q$ and define the subset sum function $f_\mathbf{a} : \{0,1\}^n \to \mathcal{S}_q$ as

$$f_\mathbf{a}(\mathbf{s}) := \sum_{i : s_i = 1} a_i \bmod q \ .$$

Now, for a target integer $t \in \mathcal{S}_q$ the modular subset problem is defined as follows:

given $(q, \mathbf{a}, t)$ as input, output a vector $\mathbf{s} \in \{0,1\}^n$ such that $f_\mathbf{a}(\mathbf{s}) = t$, if one exists.

In other words, the problem is to invert the function $f_\mathbf{a}(\cdot)$ by finding a pre-image of t, if one exists. The modular subset problem is known to be $\mathcal{NP}$ hard.

The subset sum PRG. The subset problem naturally suggests the following PRG: at setup time fix an integer q and choose random integers $\vec{a} := (a_0, \ldots, a_{n-1})$ in $\mathcal{S}_q$. The PRG $G_{q,\vec{a}}$ takes a seed $\mathbf{s} \in \{0,1\}^n$ and outputs a pseudorandom value in $\mathcal{S}_q$. It is defined as

$$G_{q,\vec{a}}(\mathbf{s}) := \sum_{i=1}^{n} a_i \cdot s_i \bmod q \ .$$

The PRG expands an n bit seed to a $\log_2 q$ bits of output. Choosing an n and q so that $2n = \log_2 q$ gives a PRG whose output is twice the size of the input. We can plug this into the Blum-Micali construction to expand the output further.

While the PRG is far slower than custom constructions like ChaCha20 from Section 3.6, the work per bit of output is a single modular addition in $\mathcal{S}_q$, which may be appropriate for some applications that are not time sensitive.

Impagliazzo and Naor [56] show that attacking $G_{q,\vec{a}}$ as a PRG is as hard as solving a certain randomized variant of the modular subset sum problem. after we develop a few more tools. While there is considerable work on solving the modular subset problem, the problem appears to be hard when $2n = \log_2 q$ for large n, say $n > 1000$, which implies the security of $G_{q,\vec{a}}$ as a PRG.

Variants. Fischer and Stern [37] and others propose the following variation of the subset sum generator:

$$G_{q,A}(\mathbf{s}) := A \cdot \mathbf{s} \bmod q$$

where q is a small prime, A is a random matrix in $\mathcal{S}_q^{n \times m}$ for $n < m$, and the seed $\mathbf{s}$ is uniform in $\{0,1\}^m$. The generator maps an m-bit seed to $n \log_2 q$ bits of output. We discuss this generator further in Chapter 17.

3.8 Case study: cryptanalysis of the DVD encryption system

The Content Scrambling System (CSS) is a system used for protecting movies on DVD disks. It uses a stream cipher, called the CSS stream cipher, to encrypt movie contents. CSS was designed in the 1980's when exportable encryption was restricted to 40-bit keys. As a result, CSS encrypts movies using a 40-bit secret key. While ciphers using 40-bit keys are woefully insecure, we show that the CSS stream cipher is particularly weak and can be broken in far less time than an exhaustive search over all 2^{40} keys. It provides a fun opportunity for cryptanalysis.

Linear feedback shift registers (LFSR). The CSS stream cipher is built from two LFSRs. An n-bit LFSR is defined by a set of integers $V := \{v_1, \ldots, v_d\}$ where each v_i is in the range $\{0, \ldots, n-1\}$. The elements of V are called **tap positions**. An LFSR gives a PRG as follows (Fig. 3.10):

Input: $s = (b_{n-1}, \ldots, b_0) \in \{0,1\}^n$ and $s \neq 0^n$
Output: $y \in \{0,1\}^\ell$ where $\ell > n$

for $i \leftarrow 1 \ldots \ell$ do
  output b_0 // *output one bit*
  $b \leftarrow b_{v_1} \oplus \cdots \oplus b_{v_d}$ // *compute feedback bit*
  $s \leftarrow (b,\ b_{n-1}, \ldots,\ b_1)$ // *shift register bits to the right*

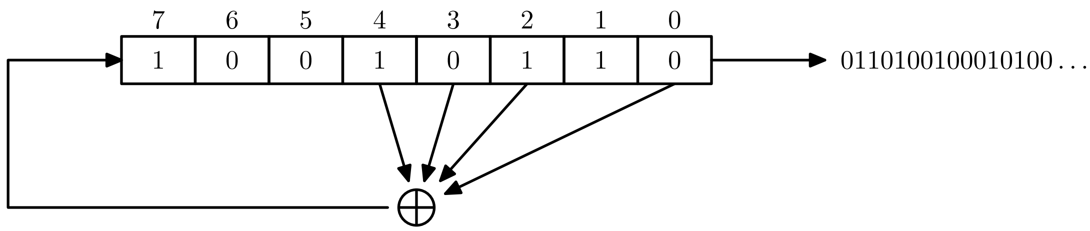


Figure 3.10: The 8 bit linear feedback shift register $\{4, 3, 2, 0\}$

The LFSR outputs one bit per clock cycle. Note that if an LFSR is started in state $s = 0^n$ then its output is degenerate, namely all 0. For this reason one of the seed bits is always set to 1.

LFSR can be implemented in hardware with few transistors. As a result, stream ciphers built from LFSR are attractive for low-cost consumer electronics such as DVD players, cell phones, and Bluetooth devices.

Stream ciphers from LSFRs. A single LFSR is completely insecure as a PRG since given n consecutive bits of its output it is trivial to compute all subsequent bits. Nevertheless, by combining several LFSRs using a non-linear component it is possible to get some (weak) security as a PRG. Trivium, one of the eStream portfolio stream ciphers, is built this way.

One approach to building stream ciphers from LFSRs is to run several LFSRs in parallel and combine their output using a non-linear operation. The CSS stream cipher, described next, combines two LFSRs using addition over the integers. The A5/1 stream cipher used to encrypt GSM cell phone traffic combines the outputs of three LFSRs. The Bluetooth E0 stream cipher combines four LFSRs using a 2-bit finite state machine. All these algorithms have been shown to be insecure and should not be used: recovering the plaintext takes far less time than an exhaustive search on the key space.

Another approach is to run a single LFSR and generate the output from a non-linear operation on its internal state. The `snow` 3G cipher used to encrypt 3GPP cell phone traffic operates this way.

The CSS stream cipher. The CSS stream cipher is built from the PRG shown in Fig. 3.11. The PRG works as follows:

Input: seed $s \in \{0, 1\}^{40}$

write $s = s_1 \| s_2$ where $s_1 \in \{0, 1\}^{16}$ and $s_2 \in \{0, 1\}^{24}$
load $1 \| s_1$ into a 17-bit LFSR
load $1 \| s_2$ into a 25-bit LFSR
$c \leftarrow 0$ // *carry bit*

repeat
  run both LFSRs for eight cycles to obtain $x, y \in \{0, 1\}^8$
  treat x, y as integers in $0 \dots 255$
  output $x + y + c \bmod 256$
  if $x + y > 255$ then $c \leftarrow 1$ else $c \leftarrow 0$ // *carry bit*
forever

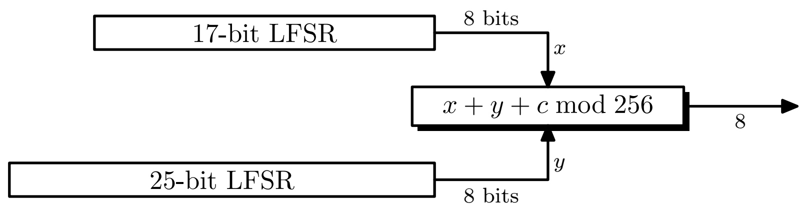


Figure 3.11: The CSS stream cipher

The PRG outputs one byte per iteration. Prepending 1 to both s_1 and s_2 ensures that the LFSRs are never initialized to the all 0 state. The taps for both LFSRs are fixed. The 17-bit LFSR uses taps $\{14, 0\}$. The 25-bit LFSR uses taps $\{12, 4, 3, 0\}$.

The CSS PRG we presented is a minor variation of CSS that is a little easier to describe, but has the same security. In the real CSS, instead of prepending a 1 to the initial seeds, one inserts the 1 in bit position 9 for the 17-bit LFSR and in bit position 22 for the 25-bit LFSR. In addition, the real CSS discards the first byte output by the 17-bit LFSR and the first two bytes output by the 25-bit LFSR. Neither issue affects the analysis presented next.

Insecurity of CSS. Given the PRG output, one can clearly recover the secret seed in time 2^{40} by exhaustive search over the seed space. We show a much faster attack that takes only 2^{16} guesses. Suppose we are given the first 100 bytes $\bar{z} := (z_1, z_2, \ldots)$ output by the PRG. The attack is based on the following simple observations:

- Let (x_1, x_2, x_3) be the first three bytes output by the 17-bit LFSR. The initial state s_2 of the second LFSR is easily obtained once both (z_1, z_2, z_3) and (x_1, x_2, x_3) are known by subtracting one from the other. More precisely, subtract the integer $2^{16}x_3 + 2^8 x_2 + x_1$ from the integer $2^{17} + 2^{16}z_3 + 2^8 z_2 + z_1$.

- The output (x_1, x_2, x_3) is determined by the 16-bit seed s_1.

With these two observations the attacker can recover the seed s by trying all possible 16-bit values for s_1. For each guess for s_1 compute the corresponding (x_1, x_2, x_3) output from the 17-bits LFSR. Subtract (x_1, x_2, x_3) from (z_1, z_2, z_3) to obtain a candidate seed s_2 for the second LFSR. Now, confirm that (s_1, s_2) are the correct secret seed s by running the PRG and comparing the resulting output to the given sequence $\bar{z}$. If the sequences do not match, try another guess for s_1. Once the attacker hits the correct value for s_1, the generated sequence will match the given $\bar{z}$ in which case the attacker found the secret seed $s = (s_1, s_2)$.

We just showed that the entire seed s can be found after an expected 2^{15} guesses for s_1. This is much faster than the naive 2^{40}-time exhaustive search attack.

3.9 Case study: cryptanalysis of the RC4 stream cipher

The RC4 stream cipher, designed by Ron Rivest in 1987, was historically used for securing Web traffic (in the SSL/TLS protocol) and wireless traffic (in the 802.11b WEP protocol). It is designed to operate on 8-bit processors with little internal memory. While RC4 is still in use, it has been

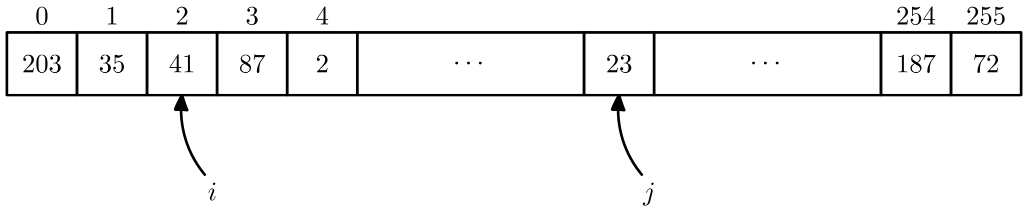


Figure 3.12: An example RC4 internal state

shown to be vulnerable to a number of significant attacks and should not be used in new projects. Our discussion of RC4 serves as an elegant example of stream cipher cryptanalysis.

At the heart of the RC4 cipher is a PRG, called the RC4 PRG. The PRG maintains an internal state consisting of an array S of 256 bytes plus two additional bytes i, j used as pointers into S. The array S contains all the numbers $0 \ldots 255$ and each number appears exactly once. Fig. 3.12 gives an example of an RC4 state.

The RC4 stream cipher key s is a seed for the PRG and is used to initialize the array S to a pseudo-random permutation of the numbers $0 \ldots 255$. Initialization is performed using the following **setup algorithm**:

input: string of bytes s

for $i \leftarrow 0$ to 255 do: $\quad S[i] \leftarrow i$

$j \leftarrow 0$
for $i \leftarrow 0$ to 255 do
$\quad k \leftarrow s\big[i \bmod |s|\big]$ $\quad$ // *extract one byte from seed*
$\quad j \leftarrow \big(\, j + S[i] + k \,\big) \bmod 256$
$\quad$ swap$(S[i], S[j])$

During the loop the index i runs linearly through the array while the index j jumps around. At each iteration the entry an index i is swapped with the entry at index j.

Once the array S is initialized, the PRG generates pseudo-random output one byte at a time using the following **stream generator**:

$i \leftarrow 0, \quad j \leftarrow 0$

repeat
$\quad i \leftarrow (i + 1) \bmod 256$
$\quad j \leftarrow (j + S[i]) \bmod 256$
$\quad$ swap$(S[i], S[j])$
$\quad$ output $\;S\big[\ (S[i] + S[j]) \bmod 256\ \big]$
forever

The procedure runs for as long as necessary. Again, the index i runs linearly through the array while the index j jumps around. Swapping $S[i]$ and $S[j]$ continuously shuffles the array S.

RC4 encryption speed. RC4 is well suited for software implementations. Other stream ciphers, such as Grain and Trivium, are designed for hardware and perform poorly when implemented in software. Table 3.1 provides running times for RC4 and a few other software stream ciphers.

cipher	speed[1](MB/sec)
RC4	126
SEAL	375
Salsa20	408
Sosemanuk	727

Table 3.1: Software stream cipher speeds (higher speed is better)

Modern processors operate on 64-bit words, making the 8-bit design of RC4 relatively slow on these architectures.

3.9.1 Security of RC4

At one point RC4 was believed to be a secure stream cipher and was widely deployed in applications. The cipher fell from grace after a number of attacks showed that its output is somewhat biased. We present two attacks that distinguish the output of RC4 from a random string. Throughout the section we let n denote the size of the array S. $n = 256$ for RC4.

Bias in the initial RC4 output. The RC4 setup algorithm initializes the array S to a permutation of $0 \ldots 255$ generated from the given random seed. For now, let us assume that the RC4 setup algorithm is perfect and generates a uniform permutation from the set of all 256! permutations. Mantin and Shamir [70] showed that, even assuming perfect initialization, the output of RC4 is biased.

Lemma 3.8 (Mantin-Shamir). *Suppose the array S is set to random permutation of $0 \ldots n-1$ and that i, j are set to 0. Then the probability that the second byte of the output of RC4 is equal to 0 is $2/n$.*

Proof idea. Let z_2 be the second byte output by RC4. Let P be the event that $S[2] = 0$ and $S[1] \neq 2$. The key observation is that when event P happens then $z_2 = 0$ with probability 1. See Fig. 3.13. However, when P does not happen then z_2 is uniformly distributed in $0 \ldots n-1$ and hence equal to 0 with probability $1/n$. Since $\Pr[P]$ is about $1/n$ we obtain (approximately) that

$$\begin{aligned}\Pr[z_2 = 0] &= \Pr\big[(z_2 = 0) \mid P\big] \cdot \Pr[P] + \Pr\big[(z_2 = 0) \mid \neg P\big] \cdot \Pr[\neg P] \\ &\approx 1 \cdot (1/n) + (1/n) \cdot (1 - 1/n) \approx 2/n \quad \Box\end{aligned}$$

The lemma shows that the probability that the second byte in the output of RC4 is 0 is twice what it should be. This leads to a simple distinguisher for the RC4 PRG. Given a string $x \in \{0 \ldots 255\}^\ell$, for $\ell \geq 2$, the distinguisher outputs 0 if the second byte of x is 0 and outputs 1 otherwise. By Lemma 3.8 this distinguisher has advantage approximately $1/n$, which is 0.39% for RC4.

The Mantin-Shamir distinguisher shows that the second byte of the RC4 output are biased. This was generalized by AlFardan et al. [2] who showed, by measuring the bias over many random keys, that there is bias in every one of the first 256 bytes of the output: the distribution on each

[1] Performance numbers were obtained using the Crypto++ 5.6.0 benchmarks running on a 1.83 GhZ Intel Core 2 processor.

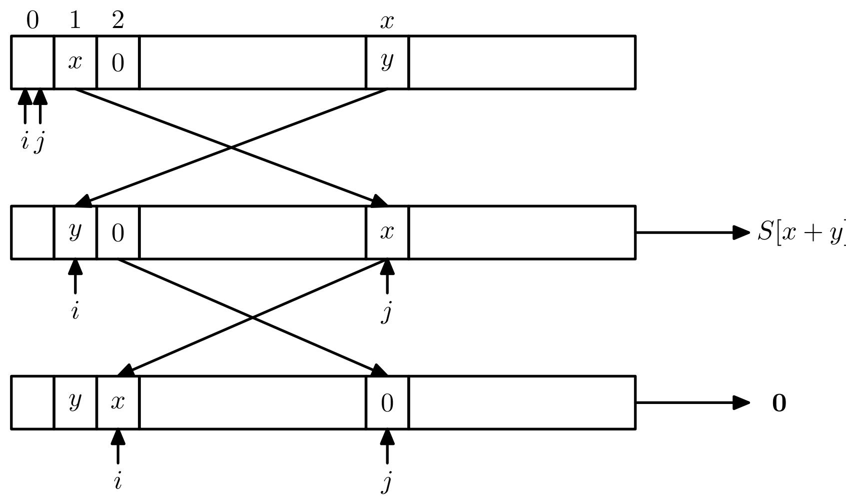


Figure 3.13: Proof of Lemma 3.8

byte is quite far from uniform. The bias is not as noticeable as in the second byte, but it is non-negligible and sufficient to attack the cipher. They show, for example, that given the encryption of a single plaintext encrypted under 2^{30} random keys, it is possible to recover the first 128 bytes of the plaintext with probability close to 1. This attack is easily carried out on the Web where a secret cookie is often embedded in the first few bytes of a message. This cookie is re-encrypted over and over with fresh keys every time the browser connects to a victim web server. Using Javascript the attacker can make the user's browser repeatedly re-connect to the target site giving the attacker the 2^{30} ciphertexts needed to mount the attack and expose the cookie.

In response, RSA Labs issued a recommendation suggesting that one discard the first 1024 bytes output by the RC4 stream generator and only use bytes 1025 and onwards. This defeats the initial key stream bias distinguishers, but does not defeat other attacks, which we discuss next.

Bias in the RC4 stream generator. Suppose the RC4 setup algorithm is modified so that the attack of the previous paragraph is ineffective. Fluhrer and McGrew [39] gave a direct attack on the stream generator. They argue that the number of times that the pair of bytes $(0, 0)$ appears in the RC4 output is larger than what it should be for a random sequence. This is sufficient to distinguish the output of RC4 from a random string.

Let $\mathrm{ST}_{\mathrm{RC4}}$ be the set of all possible internal states of RC4. Since there are $n!$ possible settings for the array S and n possible settings for each of i and j, the size of $\mathrm{ST}_{\mathrm{RC4}}$ is $n! \cdot n^2$. For $n = 256$, as used in RC4, the size of $\mathrm{ST}_{\mathrm{RC4}}$ is gigantic, namely about 10^{511}.

Lemma 3.9 (Fluhrer-McGrew). *Suppose RC4 is initialized with a random state T in ST_{RC4}. Let (z_1, z_2) be the first two bytes output by RC4 when started in state T. Then*

$$\begin{aligned} i \neq n-1 \quad &\Longrightarrow \quad \Pr[(z_1, z_2) = (0, 0)] \geq (1/n^2) \cdot \big(1 + (1/n)\big) \\ i \neq 0, 1 \quad &\Longrightarrow \quad \Pr[(z_1, z_2) = (0, 1)] \geq (1/n^2) \cdot \big(1 + (1/n)\big) \end{aligned}$$

A pair of consecutive outputs (z_1, z_2) is called a **digraph**. In a truly random string, the probability of all digraphs (x, y) is exactly $1/n^2$. The lemma shows that for RC4 the probability of $(0,0)$ is greater by $1/n^3$ from what it should be. The same holds for the digraph $(0,1)$. In fact, Fluhrer-McGrew identify several other anomalous digraphs, beyond those stated in Lemma 3.9.

The lemma suggests a simple distinguisher D between the output of RC4 and a random string. If the distinguisher finds more $(0,0)$ pairs in the given string than are likely to be in a random string it outputs 1, otherwise it outputs 0. More precisely, the distinguisher D works as follows:

input: string $x \in \{0 \dots n\}^\ell$
output: 0 or 1

let q be the number of times the pair $(0,0)$ appears in x
if $(q/\ell) - (1/n^2) > 1/(2n^3)$ output 0, else output 1

Using Theorem B.3 we can estimate D's advantage as a function of the input length ℓ. In particular, the distinguisher D achieves the following advantages:

$$\begin{array}{ll} \ell = 2^{14} \text{ bytes:} & \text{PRG}\mathsf{adv}[D, RC4] \geq 2^{-8} \\ \ell = 2^{34} \text{ bytes:} & \text{PRG}\mathsf{adv}[D, RC4] \geq 0.5 \end{array}$$

Using all the anomalous digraphs provided by Fluhrer and McGrew one can build a distinguisher that achieves advantage 0.8 using only $2^{30.6}$ bytes of output.

Related key attacks on RC4. Fluhrer, Mantin, and Shamir [38] showed that RC4 is insecure when used with related keys. We discuss this attack and its impact on the 802.11b WiFi protocol in Section 9.10, attack 2.

3.10 Generating random bits in practice

Random bits are needed in cryptography for many tasks, such as generating keys and other ephemeral values called nonces. Throughout the book we assume all parties have access to a good source of randomness, otherwise many desirable cryptographic goals are impossible. So far we used a PRG to stretch a short uniformly distributed secret seed to a long pseudorandom string. While a PRG is an important tool in generating random (or pseudorandom) bits it is only part of the story.

In practice, random bits are generated using a **random number generator**, or RNG. An RNG, like a PRG, outputs a sequence of random or pseudorandom bits. RNGs, however, have an additional interface that is used to continuously add entropy to the RNG's internal state, as shown in Fig. 3.14. The idea is that whenever the system has more random entropy to contribute to the RNG, this entropy is added into the RNG internal state. Whenever someone reads bits from the RNG, these bits are generated using the current internal state.

An example is the Linux RNG which is implemented as a device called `/dev/random`. Anyone can read data from the device to obtain random bits. To play with the `/dev/random` try typing `cat /dev/random` at a UNIX shell. You will see an endless sequence of random-looking characters. The UNIX RNG obtains its entropy from a number of hardware sources:

- keyboard events: inter-keypress timings provide entropy;
- mouse events: both interrupt timing and reported mouse positions are used;
- hardware interrupts: time between hardware interrupts is a good source of entropy;

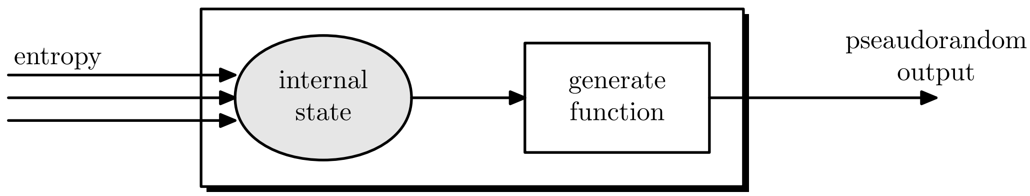


Figure 3.14: A Random Number Generator

These sources generate a continuous stream of randomness that is periodically XORed into the RNG internal state. Notice that keyboard input is not used as a source of entropy; only keypress timings are used. This ensures that user input is not leaked to other users in the system via the Linux RNG.

High entropy random generation. The entropy sources described above generate randomness at a relatively slow rate. To generate true random bits at a faster rate, Intel added a hardware random number generator to starting with the Ivy processor processor family in 2012. Output from the generator is read using the `RdRand` instruction that is intended to provide a fast uniform bit generator.

To reduce biases in the generator output, the raw bits are first passed through a function called a "conditioner" designed to ensure that the output is a sequence of uniformly distributed bits, assuming sufficient entropy is provided as input. We discuss this in more detail in Section 8.10 where we discuss the key deriviation problem.

The `RdRand` generator should not replace other entropy sources such as the four sources described above; it should only augment them as an *additional* entropy source for the RNG. This way, if the generator is defective it will not completely compromise the cryptographic application.

One difficulty with Intel's approach is that, over time, the hardware elements being sampled might stop producing randomness due to hardware glitch. For example, the sampled bits might always be '0' resulting in highly non-random output. To prevent this from happening the RNG output is constantly tested using a fixed set of statistical tests. If any of the tests reports "non-random" the generator is declared to be defective.

3.11 A broader perspective: computational indistinguishability

Our definition of security for a pseudo-random generator G formalized the intuitive idea that an adversary should not be able to effectively distinguish between $G(s)$ and r, where s is a randomly chosen seed, and r is a random element of the output space.

This idea generalizes quite naturally and usefully to other settings. Suppose P_0 and P_1 are probability distributions on some finite set $\mathcal{R}$. Our goal is to formally define the intuitive notion than an adversary cannot effectively distinguish between P_0 and P_1. As usual, this is done via an attack game. For $b = 0, 1$, we write $x \xleftarrow{R} P_b$ to denote the assignment to x of a value chosen at random from the set $\mathcal{R}$, according to the probability distribution P_b.

Attack Game 3.3 (Distinguishing P_0 from P_1). For given probability distributions P_0 and P_1 on a finite set $\mathcal{R}$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and

Experiment 1. For $b = 0, 1$, we define:

Experiment b:

- The challenger computes x as follows:

  $$x \xleftarrow{R} P_b$$

  and sends x to the adversary.
- Given x, the adversary computes and outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to P_0 and P_1 as

$$\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \square$$

Definition 3.4 (Computational indistinguishability). *Distributions P_0 and P_1 are called* ***computationally indistinguishable*** *if the value* $\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1]$ *is negligible for all efficient adversaries $\mathcal{A}$.*

Using this definition we can restate the definition of a secure PRG more simply: a PRG G defined over $(\mathcal{S}, \mathcal{R})$ is secure if and only if P_0 and P_1 are computationally indistinguishable, where P_1 is the uniform distribution on $\mathcal{R}$, and P_0 is distribution that assigns to each $r \in \mathcal{R}$ the value

$$P_0(r) := \frac{|\{s \in \mathcal{S} : G(s) = r\}|}{|\mathcal{S}|}.$$

Again, as discussed in Section 2.3.5, Attack Game 3.3 can be recast as a "bit guessing" game, where instead of having two separate experiments, the challenger chooses $b \in \{0, 1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$. In this game, we measure $\mathcal{A}$'s *bit-guessing advantage* $\text{Dist}\mathsf{adv}^*[\mathcal{A}, P_0, P_1]$ as $|\Pr[\hat{b} = b] - 1/2|$. The general result of Section 2.3.5 (namely, (2.13)) applies here as well:

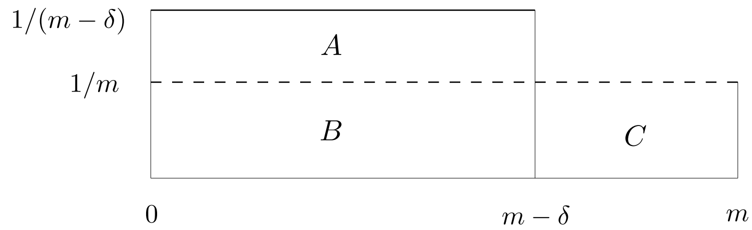
$$\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1] = 2 \cdot \text{Dist}\mathsf{adv}^*[\mathcal{A}, P_0, P_1]. \tag{3.14}$$

Typically, to prove that two distributions are computationally indistinguishable, we will have to make certain other computational assumptions. However, sometimes two distributions are so similar that no adversary can effectively distinguish between them, regardless of how much computing power the adversary may have. To make this notion of "similarity" precise, we introduce a useful tool, called **statistical distance**:

Definition 3.5. *Suppose P_0 and P_1 are probability distributions on a finite set $\mathcal{R}$. Then their* ***statistical distance*** *is defined as*

$$\Delta[P_0, P_1] := \frac{1}{2} \sum_{r \in \mathcal{R}} |P_0(r) - P_1(r)|.$$

Example 3.1. Suppose P_0 is the uniform distribution on $\{1, \ldots, m\}$, and P_1 is the uniform distribution on $\{1, \ldots, m - \delta\}$, where $\delta \in \{0, \ldots, m - 1\}$. Let us compute $\Delta[P_0, P_1]$. We could apply the definition directly; however, consider the following graph of P_0 and P_1:



The statistical distance between P_0 and P_1 is just 1/2 times the area of regions A and C in the diagram. Moreover, because probability distributions sum to 1, we must have

$$\text{area of } B + \text{area of } A = 1 = \text{area of } B + \text{area of } C,$$

and hence, the areas of region A and region C are the same. Therefore,

$$\Delta[P_0, P_1] = \text{area of } A = \text{area of } C = \delta/m. \quad \square$$

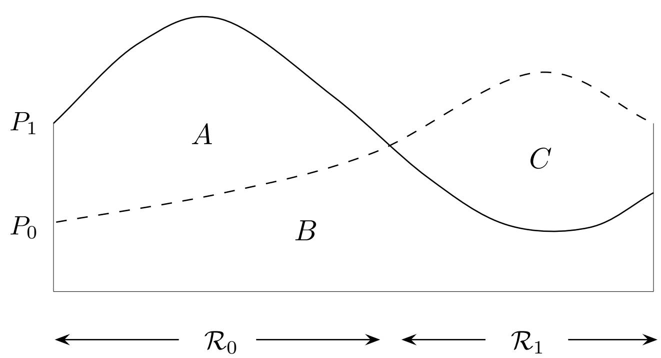
The following theorem allows us to make a connection between the notions of computational indistinguishability and statistical distance:

Theorem 3.10. *Let P_0 and P_1 be probability distributions on a finite set $\mathcal{R}$. Then we have*

$$\max_{\mathcal{R}' \subseteq \mathcal{R}} |P_0[\mathcal{R}'] - P_1[\mathcal{R}']| = \Delta[P_0, P_1],$$

where the maximum is taken over all subsets $\mathcal{R}'$ of $\mathcal{R}$.

Proof. Suppose we split the set $\mathcal{R}$ into two disjoint subsets: the set $\mathcal{R}_0$ consisting of those $r \in \mathcal{R}$ such that $P_0(r) < P_1(r)$, and the set $\mathcal{R}_1$ consisting of those $r \in \mathcal{R}$ such that $P_0(r) \geq P_1(r)$. Consider the following rough graph of the distributions of P_0 and P_1, where the elements of $\mathcal{R}_0$ are placed to the left of the elements of $\mathcal{R}_1$:



Now, as in Example 3.1,

$$\Delta[P_0, P_1] = \text{area of } A = \text{area of } C.$$

Observe that for every subset $\mathcal{R}'$ of $\mathcal{R}$, we have

$$P_0[\mathcal{R}'] - P_1[\mathcal{R}'] = \text{area of } C' - \text{area of } A',$$

where C' is the subregion of C that lies above $\mathcal{R}'$, and A' is the subregion of A that lies above $\mathcal{R}'$. It follows that $|P_0[\mathcal{R}'] - P_1[\mathcal{R}']|$ is maximized when $\mathcal{R}' = \mathcal{R}_0$ or $\mathcal{R}' = \mathcal{R}_1$, in which case it is equal to $\Delta[P_0, P_1]$. $\square$

The connection to computational indistinguishability is as follows:

Theorem 3.11. *Let P_0 and P_1 be probability distributions on a finite set $\mathcal{R}$. Then for every adversary $\mathcal{A}$, we have*

$$\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1] \leq \Delta[P_0, P_1].$$

Proof. Consider an adversary $\mathcal{A}$ that tries to distinguish P_0 from P_1, as in Attack Game 3.3.

First, we consider the case where $\mathcal{A}$ is deterministic. In this case, the output of $\mathcal{A}$ is a function $f(r)$ of the value $r \in \mathcal{R}$ presented to it by the challenger. Let $\mathcal{R}' := \{r \in \mathcal{R} : f(r) = 1\}$. If W_0 and W_1 are the events defined in Attack Game 3.3, then for $b = 0, 1$, we have

$$\Pr[W_b] = P_b[\mathcal{R}'].$$

By the previous theorem, we have

$$\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1] = |P_0[\mathcal{R}'] - P_1[\mathcal{R}']| \leq \Delta[P_0, P_1].$$

We now consider the case where $\mathcal{A}$ is probabilistic. We can view $\mathcal{A}$ as taking an auxiliary input t, representing its random choices. We view t as being chosen uniformly at random from some finite set $\mathcal{T}$. Thus, the output of $\mathcal{A}$ is a function $g(r, t)$ of the value $r \in \mathcal{R}$ presented to it by the challenger, and the value $t \in \mathcal{T}$ representing its random choices. For a given $t \in \mathcal{T}$, let $\mathcal{R}'_t := \{r \in \mathcal{R} : g(r, t) = 1\}$. Then, averaging over the random choice of t, we have

$$\Pr[W_b] = \frac{1}{|\mathcal{T}|} \sum_{t \in \mathcal{T}} P_b[\mathcal{R}'_t].$$

It follows that

$$\begin{aligned}
\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1] &= |\Pr[W_0] - \Pr[W_1]| \\
&= \frac{1}{|\mathcal{T}|} \Big| \sum_{t \in \mathcal{T}} (P_0[\mathcal{R}'_t] - P_1[\mathcal{R}'_t]) \Big| \\
&\leq \frac{1}{|\mathcal{T}|} \sum_{t \in \mathcal{T}} |P_0[\mathcal{R}'_t] - P_1[\mathcal{R}'_t]| \\
&\leq \frac{1}{|\mathcal{T}|} \sum_{t \in \mathcal{T}} \Delta[P_0, P_1] \\
&= \Delta[P_0, P_1]. \quad \square
\end{aligned}$$

As a consequence of this theorem, we see that if $\Delta[P_0, P_1]$ is negligible, then P_0 and P_1 are computationally indistinguishable.

One also defines the statistical distance between two random variables as the statistical distance between their corresponding distributions. That is, if $\mathbf{X}$ and $\mathbf{Y}$ are random variables taking values in a finite set $\mathcal{R}$, then their **statistical distance** is

$$\Delta[\mathbf{X}, \mathbf{Y}] := \frac{1}{2} \sum_{r \in \mathcal{R}} |\Pr[\mathbf{X} = r] - \Pr[\mathbf{Y} = r]|.$$

In this case, Theorem 3.10 says that

$$\max_{\mathcal{R}' \subseteq \mathcal{R}} \Big| \Pr[\mathbf{X} \in \mathcal{R}'] - \Pr[\mathbf{Y} \in \mathcal{R}'] \Big| = \Delta[\mathbf{X}, \mathbf{Y}],$$

where the maximum is taken over all subsets $\mathcal{R}'$ of $\mathcal{R}$.

Analogously, one can define distinguishing advantage with respect to random variables, rather than distributions. The advantage of working with random variables is that we can more conveniently work with distributions that are related to one another, as exemplified in the following theorem.

Theorem 3.12. *If $\mathcal{S}$ and $\mathcal{T}$ are finite sets, $\mathbf{X}$ and $\mathbf{Y}$ are random variables taking values in $\mathcal{S}$, and $f : \mathcal{S} \to \mathcal{T}$ is a function, then $\Delta[f(\mathbf{X}), f(\mathbf{Y})] \leq \Delta[\mathbf{X}, \mathbf{Y}]$.*

Proof. We have

$$\begin{aligned}
\Delta[f(\mathbf{X}), f(\mathbf{Y})] &= |\Pr[f(\mathbf{X}) \in \mathcal{T}'] - \Pr[f(\mathbf{Y}) \in \mathcal{T}']| \ \text{ for some } \mathcal{T}' \subseteq \mathcal{T} \\
&\qquad \text{(by Theorem 3.10)} \\
&= |\Pr[\mathbf{X} \in f^{-1}(\mathcal{T}')] - \Pr[\mathbf{Y} \in f^{-1}(\mathcal{T}')]| \\
&\leq \Delta[\mathbf{X}, \mathbf{Y}] \ \text{ (again by Theorem 3.10).} \quad \square
\end{aligned}$$

Example 3.2. Let $\mathbf{X}$ be uniformly distributed over the set $\{0, \ldots, m-1\}$, and let $\mathbf{Y}$ be uniformly distributed over the set $\{0, \ldots, N-1\}$, for $N \geq m$. Let $f(t) := t \bmod m$. We want to compute an upper bound on the statistical distance between $\mathbf{X}$ and $f(\mathbf{Y})$. We can do this as follows. Let $N = qm - r$, where $0 \leq r < m$, so that $q = \lceil N/m \rceil$. Also, let $\mathbf{Z}$ be uniformly distributed over $\{0, \ldots, qm-1\}$. Then $f(\mathbf{Z})$ is uniformly distributed over $\{0, \ldots, m-1\}$, since every element of $\{0, \ldots, m-1\}$ has the same number (namely, q) of pre-images under f which lie in the set $\{0, \ldots, qm-1\}$. Since statistical distance depends only on the distributions of the random variables, by the previous theorem, we have

$$\Delta[\mathbf{X}, f(\mathbf{Y})] = \Delta[f(\mathbf{Z}), f(\mathbf{Y})] \leq \Delta[\mathbf{Z}, \mathbf{Y}],$$

and as we saw in Example 3.1,

$$\Delta[\mathbf{Z}, \mathbf{Y}] = \frac{r}{qm} < \frac{1}{q} \leq \frac{m}{N}.$$

Therefore,

$$\Delta[\mathbf{X}, f(\mathbf{Y})] < \frac{m}{N}. \quad \square$$

Example 3.3. Suppose we want to generate a pseudo-random number in a given interval $\{0, \ldots, m-1\}$. However, suppose that we have at our disposal a PRG G that outputs L-bit strings. Of course, an L-bit string can be naturally viewed as a number in the range $\{0, \ldots, N-1\}$, where $N := 2^L$. Let us assume that $N \geq m$.

To generate a pseudo-random number in the interval $\{0, \ldots, m-1\}$, we can take the output of G, view it as a number in the interval $\{0, \ldots, N-1\}$, and reduce it modulo m. We will show that this procedure produces a number that is computationally indistinguishable from a truly random in the interval $\{0, \ldots, m-1\}$, assuming G is secure and m/N is negligible (e.g., $N \geq 2^{100} \cdot m$).

To this end, let P_0 be the distribution representing the output of G, reduced modulo m, let P_1 be the uniform distribution on $\{0, \ldots, m-1\}$, and let $\mathcal{A}$ be an adversary trying to distinguish P_0 from P_1, as in Attack Game 3.3.

Let Game 0 be Experiment 0 of Attack Game 3.3, in which $\mathcal{A}$ is presented with a random sample distributed according to P_0, and let W_0 be the event that $\mathcal{A}$ outputs 1 in this game.

Now define Game 1 to be the same as Game 0, except that we replace the output of G by a truly random value chosen from the interval $\{0, \ldots, N-1\}$. Let W_1 be the event that $\mathcal{A}$ outputs 1 in Game 1. One can easily construct an efficient adversary $\mathcal{B}$ that attacks G as in Attack Game 3.1, such that

$$\text{PRG}\mathsf{adv}[\mathcal{B}, G] = |\Pr[W_0] - \Pr[W_1]|.$$

The idea is that $\mathcal{B}$ takes its challenge value, reduces it modulo m, gives this value to $\mathcal{A}$, and outputs whatever $\mathcal{A}$ outputs.

Finally, we define Game 2 be Experiment 1 of Attack Game 3.3, in which $\mathcal{A}$ is presented with a random sample distributed according to P_1, the uniform distribution on $\{0, \ldots, m-1\}$. Let W_2 be the event that $\mathcal{A}$ outputs 1 in Game 2. If P is the distribution of the value presented to $\mathcal{A}$ in Game 1, then by Theorem 3.11, we have $|\Pr[W_1] - \Pr[W_2]| \leq \Delta[P, P_1]$; moreover, by Example 3.3, we have $\Delta[P, P_1] \leq m/N$.

Putting everything together, we see that

$$\begin{aligned}\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1] = |\Pr[W_0] - \Pr[W_2]| &\leq |\Pr[W_0] - \Pr[W_1]| + |\Pr[W_1] + \Pr[W_2]| \\ &\leq \text{PRG}\mathsf{adv}[\mathcal{B}, G] + \frac{m}{N},\end{aligned}$$

which, by assumption, is negligible. $\square$

3.11.1 Mathematical details

As usual, we fill in the mathematical details needed to interpret the definitions and results of this section from the point of view of asymptotic complexity theory.

In defining computational indistinguishability (Definition 3.4), one should consider two families of probability distributions $P_0 = \{P_{0,\lambda}\}_\lambda$ and $P_1 = \{P_{1,\lambda}\}_\lambda$, indexed by a security parameter λ. For each λ, the distributions $P_{0,\lambda}$ and $P_{1,\lambda}$ should take values in a finite set of bit strings $\mathcal{R}_\lambda$, where the strings in $\mathcal{R}_\lambda$ are bounded in length by a polynomial in λ. In Attack Game 3.3, the security parameter λ is an input to both the challenger and adversary, and in Experiment b, the challenger produces a sample, distributed according to $P_{b,\lambda}$. The advantage should properly be written $\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1](\lambda)$, which is a function of λ. Computationally indistinguishability means that this is a negligible function.

In some situations, it may be natural to introduce a probabilistically generated system parameter; however, from a technical perspective, this is not necessary, as such a system parameter can be incorporated in the distributions $P_{0,\lambda}$ and $P_{1,\lambda}$. One could also impose the requirement that $P_{0,\lambda}$ and $P_{1,\lambda}$ be efficiently sampleable; however, to keep the definition simple, we will not require this.

The definition of statistical distance (Definition 3.5) makes perfect sense from a non-asymptotic point of view, and does not require any modification or elaboration. Theorem 3.10 holds as stated, for specific distributions P_0 and P_1. Theorem 3.11 may be viewed asymptotically as stating that for all distribution families $P_0 = \{P_{0,\lambda}\}_\lambda$ and $P_1 = \{P_{1,\lambda}\}_\lambda$, for all adversaries (even computationally unbounded ones), and for all λ, we have

$$\text{Dist}\mathsf{adv}[\mathcal{A}, P_0, P_1](\lambda) \leq \Delta[P_{0,\lambda}, P_{1,\lambda}].$$

3.12 A fun application: coin flipping and commitments

Alice and Bob are going out on a date. Alice wants to see one movie and Bob wants to see another. They decide to flip a random coin to choose the movie. If the coin comes up "heads" they will go to Alice's choice; otherwise, they will go to Bob's choice. When Alice and Bob are in close proximity this is easy: one of them, say Bob, flips a coin and they both verify the result. When they are far apart and are speaking on the phone this is harder. Bob can flip a coin on his side and tell Alice the result, but Alice has no reason to believe the outcome. Bob could simply claim that the coin came up "tails" and Alice would have no way to verify this. Not a good way to start a date.

A simple solution to their problem makes use of a cryptographic primitive called **bit commitment**. It lets Bob commit to a bit $b \in \{0, 1\}$ of his choice. Later, Bob can open the commitment and convince Alice that b was the value he committed to. Committing to a bit b results in a **commitment string** c, that Bob sends to Alice, and an **opening string** s that Bob uses for opening the commitment later. A commitment scheme is secure if it satisfies the following two properties:

- **Hiding:** The commitment string c reveals no information about the committed bit b. More precisely, the distribution on c when committing to the bit 0 is indistinguishable from the distribution on c when committing to the bit 1. In the bit commitment scheme we present the binding property is based on the security of a given PRG G.

- **Binding:** Let c be a commitment string output by Bob. If Bob can open the commitment as some $b \in \{0, 1\}$ then he cannot open it as $\bar{b}$. This ensures that once Bob commits to a bit b he can open it as b and nothing else. In the commitment scheme we present the binding property holds unconditionally.

Coin flipping. Using a commitment scheme, Alice and Bob can generate a random bit $b \in \{0, 1\}$ so that no side can bias the result towards their preferred outcome, assuming the protocol terminates successfully. Such protocols are called **coin flipping protocols**. The resulting bit b determines what movie they go to.

Alice and Bob use the following simple coin flipping protocol:

Step 1: Bob chooses a random bit $b_0 \xleftarrow{R} \{0, 1\}$.
  Alice and Bob execute the commitment protocol by which Alice obtains
  a commitment c to b_0 and Bob obtains an opening string s.
Step 2: Alice chooses a random bit $b_1 \xleftarrow{R} \{0, 1\}$ and sends b_1 to Bob in the clear.
Step 3: Bob opens the commitment by revealing b_0 and s to Alice.
  Alice verifies that c is indeed a commitment to b_0 and aborts if verification fails.

Output: the resulting bit is $b := b_0 \oplus b_1$.

We argue that if the protocol terminates successfully and one side is honestly following the protocol then the other side cannot bias the result towards their preferred outcome. By the hiding property, Alice learns nothing about b_0 at the end of Step 1 and therefore her choice of bit b_1 is independent of the value of b_0. By the binding property, Bob can only open the commitment c in Step 3 to the bit b_0 he chose in Step 1. Because he chose b_0 before Alice chose b_1, Bob's choice of b_0 is independent of b_1. We conclude that the output bit b is the XOR of two independent bits. Therefore, if one side is honestly following the protocol, the other side cannot bias the resulting bit.

One issue with this protocol is that Bob learns the generated bit at the end of Step 2, before Alice learns the bit. In principle, if the outcome is not what Bob wants he could abort the protocol at the end of Step 2 and try to re-initiate the protocol hoping that the next run will go his way. More sophisticated coin flipping protocols avoid this problem, but at the cost of many more rounds of interaction (see, e.g., [77]).

Bit commitment from secure PRGs. It remains to construct a secure bit commitment scheme that lets Bob commit to his bit $b_0 \in \{0,1\}$. We do so using an elegant construction due to Naor [83].

Let $G : \mathcal{S} \to \mathcal{R}$ be a secure PRG where $|\mathcal{R}| \geq |\mathcal{S}|^3$ and $\mathcal{R} = \{0,1\}^n$ for some n. To commit to the bit b_0, Alice and Bob engage in the following protocol:

Bob commits to bit $b_0 \in \{0,1\}$:

Step 1: Alice chooses a random $r \in \mathcal{R}$ and sends r to Bob.

Step 2: Bob chooses a random $s \in \mathcal{S}$ and computes $c \leftarrow \mathrm{com}(s, r, b_0)$ where $\mathrm{com}(s, r, b_0)$ is the following function:

$$c = \mathrm{com}(s, r, b_0) := \begin{cases} G(s) & \text{if } b_0 = 0, \\ G(s) \oplus r & \text{if } b_0 = 1. \end{cases}$$

Bob outputs c as the commitment string and uses s as the opening string.

When it comes time to open the commitment Bob sends (b_0, s) to Alice. Alice accepts the opening if $c = \mathrm{com}(s, r, b_0)$ and rejects otherwise.

The hiding property follows directly from the security of the PRG: because the output $G(s)$ is computationally indistinguishable from a uniform random string in $\mathcal{R}$ it follows that $G(s) \oplus r$ is also computationally indistinguishable from a uniform random string in $\mathcal{R}$. Therefore, whether $b_0 = 0$ or $b_0 = 1$, the commitment string c is computationally indistinguishable from a uniform string in $\mathcal{R}$, as required.

The binding property holds unconditionally as long as $1/|\mathcal{S}|$ is negligible. The only way Bob can open a commitment $c \in \mathcal{R}$ as both 0 and 1 is if there exist two seeds $s_0, s_1 \in \mathcal{S}$ such that $c = G(s_0) = G(s_1) \oplus r$ which implies that $G(s_0) \oplus G(s_1) = r$. Let us say that $r \in \mathcal{R}$ is "bad" if there are seeds $s_0, s_1 \in \mathcal{S}$ such that $G(s_0) \oplus G(s_1) = r$. The number of pairs of seeds (s_0, s_1) is $|\mathcal{S}|^2$, and therefore the number of bad r is at most $|\mathcal{S}|^2$. It follows that the probability that Alice chooses a bad r is most $|\mathcal{S}|^2/|\mathcal{R}| < |\mathcal{S}|^2/|\mathcal{S}|^3 = 1/|\mathcal{S}|$ which is negligible. Therefore, the probability that Bob can open the commitment c as both 0 and 1 is negligible.

3.13 Notes

Citations to the literature to be added.

3.14 Exercises

3.1 (Semantic security for random messages). One can define a notion of semantic security for random messages. Here, one modifies Attack Game 2.1 so that instead of the adversary choosing the messages m_0, m_1, the challenger generates m_0, m_1 at random from the message space. Otherwise, the definition of advantage and security remains unchanged.

(a) Suppose that $\mathcal{E} = (E, D)$ is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{M} = \{0,1\}^L$. Assuming that $\mathcal{E}$ is semantically secure for random messages, show how to construct a new cipher $\mathcal{E}'$ that is secure in the ordinary sense. Your new cipher should be defined over $(\mathcal{K}', \mathcal{M}', \mathcal{C}')$, where $\mathcal{K}' = \mathcal{K}$ and $\mathcal{M}' = \mathcal{M}$.

(b) Give an example of a cipher that is semantically secure for random messages but that is not semantically secure in the ordinary sense.

3.2 (Encryption chain). Let $\mathcal{E} = (E, D)$ be a perfectly secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ where $\mathcal{K} = \mathcal{M}$. Let $\mathcal{E}' = (E', D')$ be a cipher where encryption is defined as $E'((k_1, k_2), m) := \big(E(k_1, k_2), E(k_2, m)\big)$. Show that $\mathcal{E}'$ is perfectly secure.

3.3 (Bit guessing definition of semantic security). This exercise develops an alternative characterization of semantic security. Let $\mathcal{E} = (E, D)$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Assume that one can efficiently generate messages from the message space $\mathcal{M}$ at random. We define an attack game between an adversary $\mathcal{A}$ and a challenger as follows. The adversary selects a message $m \in \mathcal{M}$ and sends m to the challenger. The challenger then computes:

$$b \overset{\text{R}}{\leftarrow} \{0,1\},\ k \overset{\text{R}}{\leftarrow} \mathcal{K},\ \ m_0 \leftarrow m,\ m_1 \overset{\text{R}}{\leftarrow} \mathcal{M},\ c \overset{\text{R}}{\leftarrow} E(k, m_b),$$

and sends the ciphertext c to $\mathcal{A}$, who then computes and outputs a bit $\hat{b}$. That is, the challenger encrypts either m or a random message, depending on b. We define $\mathcal{A}$'s advantage to be $|\Pr[\hat{b} = b] - 1/2|$, and we say the $\mathcal{E}$ is *real/random semantically secure* if this advantage is negligible for all efficient adversaries.

Show that $\mathcal{E}$ is real/random semantically secure if and only if it is semantically secure in the ordinary sense.

3.4 (Indistinguishability from random). In this exercise, we develop a notion of security for a cipher, called *psuedo-random ciphertext security*, which intuitively says that no efficient adversary can distinguish an encryption of a chosen message from a random ciphertext.

Let $\mathcal{E} = (E, D)$ be defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Assume that one can efficiently generate ciphertexts from the ciphertext space $\mathcal{C}$ at random. We define an attack game between an adversary $\mathcal{A}$ and a challenger as follows. The adversary selects a message $m \in \mathcal{M}$ and sends m to the challenger. The challenger then computes:

$$b \overset{\text{R}}{\leftarrow} \{0,1\},\ k \overset{\text{R}}{\leftarrow} \mathcal{K},\ \ c_0 \overset{\text{R}}{\leftarrow} E(k, m),\ \ c_1 \overset{\text{R}}{\leftarrow} \mathcal{C},\ \ \ c \overset{\text{R}}{\leftarrow} c_b$$

and sends the ciphertext c to $\mathcal{A}$, who then computes and outputs a bit $\hat{b}$. We define $\mathcal{A}$'s advantage to be $|\Pr[\hat{b} = b] - 1/2|$, and we say the $\mathcal{E}$ is *pseudo-random ciphertext secure* if this advantage is negligible for all efficient adversaries.

(a) Show that if a cipher is psuedo-random ciphertext secure, then it is semantically secure.

(b) Show that the one-time pad is psuedo-random ciphertext secure.

(c) Give an example of a cipher that is semantically secure, but not psuedo-random ciphertext secure.

3.5 (Small seed spaces are insecure). Suppose G is a PRG defined over $(\mathcal{S}, \mathcal{R})$ where $|\mathcal{R}| \geq 2|\mathcal{S}|$. Let us show that $|\mathcal{S}|$ must be super-poly. To do so, show that there is an adversary that achieves advantage at least $1/2$ in attacking the PRG G whose running is linear in $|\mathcal{S}|$.

3.6 (Another malleability example). Let us give another example illustrating the malleability of stream ciphers. Suppose you are told that the stream cipher encryption of the message "attack at dawn" is 6c73d5240a948c86981bc294814d (the plaintext letters are encoded as 8-bit ASCII and the given ciphertext is written in hex). What would be the stream cipher encryption of the message "attack at dusk" under the same key?

3.7 (Exercising the definition of a secure PRG). Suppose $G(s)$ is a secure PRG that outputs bit-strings in $\{0,1\}^n$. Which of are the following derived generators are secure?

(a) $G_1(s_1 \parallel s_2) := G(s_1) \wedge G(s_2)$ where $\wedge$ denotes bit-wise AND.

(b) $G_2(s_1 \parallel s_2) := G(s_1) \oplus G(s_2)$.

(c) $G_3(s) := G(s) \oplus 1^n$.

(d) $G_4(s) := G(s)[0\,..\,n-1]$.

(e) $G_5(s) := (G(s), G(s))$.

(f) $G_6(s_1 \parallel s_2) := (s_1, G(s_2))$.

3.8 (The converse of Theorem 3.1). In Section 3.2, we showed how to build a stream cipher from a PRG. In Theorem 3.1, we proved that this encryption scheme is semantically secure if the PRG is secure. Prove the converse: the PRG is secure if this encryption scheme is semantically secure.

3.9 (Predicting the next character). In Section 3.5, we showed that if one could effectively distinguish a random bit string from a pseudo-random bit string, then one could succeed in predicting the next bit of a pseudo-random bit string with probability significantly greater than 1/2 (where the position of the "next bit" was chosen at random). Generalize this from bit strings to strings over the alphabet $\{0,\dots,n-1\}$, for all $n \geq 2$, assuming that n is poly-bounded.

Hint: First generalize the distinguisher/predictor lemma (Lemma 3.5).

3.10 (Simple statistical distance calculations).

(a) Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random variables, each uniformly distributed over $\mathbb{Z}_p$, where p is prime. Calculate $\Delta[\ (\mathbf{X},\mathbf{Y}),\ (\mathbf{X},\mathbf{XY})\ ]$.

(b) Let $\mathbf{X}$ and $\mathbf{Y}$ be random variables, each taking values in the interval $[0,t]$. Show that $|E[\mathbf{X}] - E[\mathbf{Y}]| \leq t\Delta[\mathbf{X},\mathbf{Y}]$.

The following three exercises should be done together; they will be used in exercises in the following chapters.

3.11 (Distribution ratio). This exercise develops another way of comparing two probability distributions, which considers ratios of probabilities, rather than differences. Let $\mathbf{X}$ and $\mathbf{Y}$ be two random variables taking values on a finite set $\mathcal{R}$, and assume that $\Pr[\mathbf{X}=r] > 0$ for all $r \in \mathcal{R}$. Define

$$\rho[\mathbf{X},\mathbf{Y}] := \max\big\{\Pr[\mathbf{Y}=r]/\Pr[\mathbf{X}=r] : r \in \mathcal{R}\big\}$$

Show that for every subset $\mathcal{R}'$ of $\mathcal{R}$, we have $\Pr[\mathbf{Y} \in \mathcal{R}'] \leq \rho[\mathbf{X},\mathbf{Y}] \cdot \Pr[\mathbf{X} \in \mathcal{R}']$.

3.12 (A variant of Bernoulli's inequality). The following is a useful fact that will be used in the following exercise. Prove the following statement by induction on n: for any real numbers $x_1, \ldots, x_n$ in the interval $[0,1]$, we have

$$\prod_{i=1}^{n}(1-x_i) \geq 1 - \sum_{i=1}^{n} x_i.$$

3.13 (Sampling with and without replacement: distance and ratio). Let $\mathcal{X}$ be a finite set of size N, and let $Q \leq N$. Define random variables $\mathbf{X}$ and $\mathbf{Y}$, where $\mathbf{X}$ is uniformly distributed over all sequences of Q elements in $\mathcal{X}$, and $\mathbf{Y}$ is uniformly distributed over all sequences of Q *distinct* elements in $\mathcal{X}$. Let $\Delta[\mathbf{X}, \mathbf{Y}]$ be the statistical distance between $\mathbf{X}$ and $\mathbf{Y}$, and let $\rho[\mathbf{X}, \mathbf{Y}]$ be defined as in Exercise 3.11. Using the previous exercise, prove the following:

(a) $\Delta[\mathbf{X}, \mathbf{Y}] = 1 - \prod_{i=0}^{Q-1}(1 - i/N) \leq \dfrac{Q^2}{2N}$,

(b) $\rho[\mathbf{X}, \mathbf{Y}] = \dfrac{1}{\prod_{i=0}^{Q-1}(1-i/N)} \leq \dfrac{1}{1-\frac{Q^2}{2N}}$ (assuming $Q^2 < 2N$).

3.14 (Theorem 3.2 is tight). Let us show that the bounds in the parallel composition theorem, Theorem 3.2, are tight. Consider the following, rather silly PRG G_0, which "stretches" ℓ-bit strings to ℓ-bit strings, with ℓ even: for $s \in \{0,1\}^\ell$, we define

$G_0(s) :=$
  if $s[0\,..\,\ell/2-1] = 0^{\ell/2}$
    then output 0^ℓ
    else output s.

That is, if the first $\ell/2$ bits of s are zero, then $G_0(s)$ outputs the all-zero string, and otherwise, $G_0(s)$ outputs s.

Next, define the following PRG adversary $\mathcal{B}_0$ that attacks G_0:

> When the challenger presents $\mathcal{B}_0$ with $r \in \{0,1\}^\ell$, if r is of the form $0^{\ell/2} \parallel t$, for some $t \neq 0^{\ell/2}$, $\mathcal{B}_0$ outputs 1; otherwise, $\mathcal{B}_0$ outputs 0.

Now, let G_0' be the n-wise parallel composition of G_0. Using $\mathcal{B}_0$, we construct a PRG adversary $\mathcal{A}_0$ that attacks G_0':

> when the challenger presents $\mathcal{A}_0$ with the sequence of strings $(r_1, \ldots, r_n)$, $\mathcal{A}_0$ presents each r_i to $\mathcal{B}_0$, and outputs 1 if $\mathcal{B}_0$ ever outputs 1; otherwise, $\mathcal{A}_0$ outputs 0.

(a) Show that $\mathrm{PRG}\mathsf{adv}[\mathcal{B}_0, G_0] = 2^{-\ell/2} - 2^{-\ell}$.

(b) Show that $\mathrm{PRG}\mathsf{adv}[\mathcal{A}_0, G_0'] \geq n2^{-\ell/2} - n(n+1)2^{-\ell}$.

(c) Show that no adversary attacking G_0 has a better advantage than $\mathcal{B}_0$ (hint: make an argument based on statistical distance).

(d) Using parts (a)–(c), argue that Theorem 3.2 cannot be substantially improved; in particular, show that the following *cannot* be true:

> *There exists a constant $c < 1$ such that for every PRG G, poly-bounded n, and efficient adversary $\mathcal{A}$, there exists an efficient adversary $\mathcal{B}$ such that*
>
> $$\mathrm{PRG}\mathsf{adv}[\mathcal{A}, G'] \le cn \cdot \mathrm{PRG}\mathsf{adv}[\mathcal{B}, G],$$
>
> *where G' is the n-wise parallel composition of G.*

3.15 (A converse (of sorts) to Theorem 2.8). Let $\mathcal{E} = (E, D)$ be a semantically secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{M} = \{0, 1\}$. Show that for every efficient adversary $\mathcal{A}$ that receives an encryption of a random bit b, the probability that $\mathcal{A}$ correctly predicts b is at most $1/2 + \epsilon$, where ϵ is negligible.

Hint: Use Lemma 3.5.

3.16 (Previous-bit prediction). Suppose that $\mathcal{A}$ is an effective next-bit predictor. That is, suppose that $\mathcal{A}$ is an efficient adversary whose advantage in Attack Game 3.2 is non-negligible. Show how to use $\mathcal{A}$ to build an explicit, effective *previous-bit* predictor $\mathcal{B}$ that uses $\mathcal{A}$ as a black box. Here, one defines a *previous-bit prediction game* that is the same as Attack Game 3.2, except that the challenger sends $r[i+1 \mathinner{..} L-1]$ to the adversary. Also, express $\mathcal{B}$'s previous-bit prediction advantage in terms of $\mathcal{A}$'s next-bit prediction advantage.

3.17 (An insecure PRG based on linear algebra). Let A be a fixed $m \times n$ matrix with $m > n$ whose entries are all binary. Consider the following PRG $G : \{0,1\}^n \to \{0,1\}^m$ defined by

$$G(s) := A \cdot s \pmod 2$$

where $A \cdot s \bmod 2$ denotes a matrix-vector product where all elements of the resulting vector are reduced modulo 2. Show that this PRG is insecure no matter what matrix A is used.

3.18 (Generating an encryption key using a PRG). Let $G : \mathcal{S} \to \mathcal{R}$ a secure PRG. Let $\mathcal{E} = (E, D)$ be a semantically secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Assume $\mathcal{K} = \mathcal{R}$. Construct a new cipher $\mathcal{E}' = (E', D')$ defined over $(\mathcal{S}, \mathcal{M}, \mathcal{C})$, where $E'(s, m) := E(G(s), m)$ and $D'(s, c) := D(G(s), c)$. Show that $\mathcal{E}'$ is semantically secure.

3.19 (Nested PRG construction). Let $G_0 : \mathcal{S} \to \mathcal{R}_1$ and $G_1 : \mathcal{R}_1 \to \mathcal{R}_2$ be two secure PRGs. Show that $G(s) := G_1(G_0(s))$ mapping $\mathcal{S}$ to R_2 is a secure PRG.

3.20 (Self-nested PRG construction). Let G be a PRG that stretches n-bit strings to $2n$-bit strings. For $s \in \{0,1\}^n$, write $G(s) = G_0(s) \parallel G_1(s)$, so that $G_0(s)$ represents the first n bits of $G(s)$, and $G_1(s)$ represents the last n bits of $G(s)$. Define a new PRG G' that stretches n-bit strings to $4n$-bit strings, as follows: $G'(s) := G(G_0(s)) \parallel G(G_1(s))$. Show that if G is a secure PRG, then so is G'.

Hint: You can give a direct proof; alternatively, you can use the previous exercise together with Theorem 3.2.

Note: This construction is a special case of a more general construction discussed in Section 4.6.

3.21 (Bad seeds). Show that a secure PRG $G : \{0,1\}^n \to \mathcal{R}$ can become insecure if the seed is not uniformly random in $\mathcal{S}$.

(a) Consider the PRG $G' : \{0,1\}^{n+1} \to \mathcal{R} \times \{0,1\}$ defined as $G'(s_0 \parallel s_1) = (G(s_0), s_1)$. Show that G' is a secure PRG assuming G is secure.

(b) Show that G' becomes insecure if its random seed $s_0 \parallel s_1$ is chosen so that its last bit is always 0.

(c) Construct a secure PRG $G'' : \{0,1\}^{n+1} \to \mathcal{R} \times \{0,1\}$ that becomes insecure if its seed s is chosen to that the *parity* of the bits in s is always 0.

3.22 (Good intentions, bad idea). Let us show that a natural approach to strengthening a PRG is insecure. Let $m > n$ and let $G : \{0,1\}^n \to \{0,1\}^m$ be a PRG. Define a new generator $G'(s) := G(s) \oplus (0^{m-n} \parallel s)$ derived from G. Show that there is a secure PRG G for which G' is insecure.

Hint: Use the construction from part (a) of Exercise 3.21.

3.23 (Seed recovery attacks). Let G be a PRG defined over $(\mathcal{S}, \mathcal{R})$ where, $|\mathcal{S}|/|\mathcal{R}|$ is negligible, and suppose $\mathcal{A}$ is an adversary that given $G(s)$ outputs s with non-negligible probability. Show how to use $\mathcal{A}$ to construct a PRG adversary $\mathcal{B}$ that has non-negligible advantage in attacking G as a PRG. This shows that for a secure PRG it is intractable to recover the seed from the output.

3.24 (A PRG combiner). Suppose that G_1 and G_2 are PRG's defined over $(\mathcal{S}, \mathcal{R})$, where $\mathcal{R} = \{0,1\}^L$. Define a new PRG G' defined over $(\mathcal{S} \times \mathcal{S}, \mathcal{R})$, where $G'(s_1, s_2) = G_1(s_1) \oplus G_2(s_2)$. Show that if either G_1 or G_2 is secure (we may not know which one is secure), then G' is secure.

3.25 (A technical step in the proof of Lemma 3.5). This exercise develops a simple fact from probability that is helpful in understanding the proof of Lemma 3.5. Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random variables, taking values in S and T, respectively, where $\mathbf{Y}$ is uniformly distributed over T. Let $f : S \to \{0,1\}$ and $g : S \to T$ be functions. Show that the events $f(\mathbf{X}) = 1$ and $g(\mathbf{X}) = \mathbf{Y}$ are independent, and the probability of the latter is $1/|T|$.

Chapter 4

Block ciphers

This chapter continues the discussion begun in the previous chapter on achieving privacy against eavesdroppers. Here, we study another kind of cipher, called a **block cipher**. We also study the related concept of a **pseudo-random function**.

Block ciphers are the "work horse" of practical cryptography: not only can they can be used to build a stream cipher, but they can be used to build ciphers with stronger security properties (as we will explore in Chapter 5), as well as many other cryptographic primitives.

4.1 Block ciphers: basic definitions and properties

Functionally, a **block cipher** is a deterministic cipher $\mathcal{E} = (E, D)$ whose message space and ciphertext space are the same (finite) set $\mathcal{X}$. If the key space of $\mathcal{E}$ is $\mathcal{K}$, we say that $\mathcal{E}$ is a block cipher **defined over** $(\mathcal{K}, \mathcal{X})$. We call an element $x \in \mathcal{X}$ a **data block**, and refer to $\mathcal{X}$ as the **data block space** of $\mathcal{E}$.

For every fixed key $k \in \mathcal{K}$, we can define the function $f_k := E(k, \cdot)$; that is, $f_k : \mathcal{X} \to \mathcal{X}$ sends $x \in \mathcal{X}$ to $E(k, x) \in \mathcal{X}$. The usual correctness requirement for any cipher implies that for every fixed key k, the function f_k is one-to-one, and as $\mathcal{X}$ is finite, f_k must be onto as well. Thus, f_k is a permutation on $\mathcal{X}$, and $D(k, \cdot)$ is the inverse permutation f_k^{-1}.

Although syntactically a block cipher is just a special kind of cipher, the security property we shall expect for a block cipher is actually much stronger than semantic security: for a randomly chosen key k, the permutation $E(k, \cdot)$ should — for all practical purposes — "look like" a random permutation. This is a notion that we will soon make more precise.

One very important and popular block cipher is AES (the Advanced Encryption Standard). We will study the internal design of AES in more detail below, but for now, we just give a very high-level description. AES keys are 128-bit strings (although longer key sizes may be used, such as 192-bits or 256-bits). AES data blocks are 128-bit strings. See Fig. 4.1. AES was designed to be quite efficient: one evaluation of the encryption (or decryption) function takes just a few hundred cycles on a typical computer.

The definition of security for a block cipher is formulated as a kind of "black box test." The intuition is the following: an efficient adversary is given a "black box." Inside the box is a permutation f on $\mathcal{X}$, which is generated via one of two random processes:

- $f := E(k, \cdot)$, for a randomly chosen key k, or

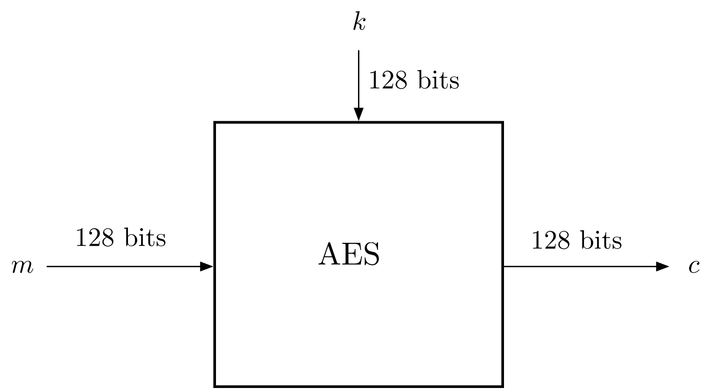


Figure 4.1: The block cipher AES

- f is a truly random permutation, chosen uniformly from among *all* permutations on $\mathcal{X}$.

The adversary cannot see inside the box, but he can "probe" it with questions: he can give the box a value $x \in \mathcal{X}$, and obtain the value $y := f(x) \in \mathcal{X}$. We allow the adversary to ask many such questions, and we quite liberally allow him to choose the questions in any way he likes; in particular, each question may even depend in some clever way on the answers to previous questions. Security means that the adversary should not be able to tell which type of function is inside the box — a randomly keyed block cipher, or a truly random permutation. Put another way, a secure block cipher should be **computationally indistinguishable** from a random permutation.

To make this definition more formal, let us introduce some notation:

$$\text{Perms}[\mathcal{X}]$$

denotes the set of *all* permutations on $\mathcal{X}$. Note that this is a very large set:

$$\big|\text{Perms}[\mathcal{X}]\big| = |\mathcal{X}|!.$$

For AES, with $|\mathcal{X}| = 2^{128}$, the number of permutations is about

$$\text{Perms}[\mathcal{X}] \approx 2^{2^{135}},$$

while the number of permutations defined by 128-bit AES keys is at most 2^{128}.

As usual, to define security, we introduce an attack game. Just like the attack game used to define a PRG, this attack game comprises two separate experiments. In both experiments, the adversary follows the same protocol; namely, it submits a sequence of queries $x_1, x_2, \ldots$ to the challenger; the challenger responds to query x_i with $f(x_i)$, where in the first experiment, $f := E(k, \cdot)$ for randomly chosen $k \in \mathcal{K}$, and while in the second experiment, f is randomly selected from $\text{Perms}[\mathcal{X}]$; throughout each experiment, the same f is used to answer all queries. When the adversary tires of querying the challenger, it outputs a bit.

Attack Game 4.1 (block cipher). For a given block cipher (E, D), defined over $(\mathcal{K}, \mathcal{X})$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define:

Experiment b:

- The challenger selects $f \in \text{Perms}[\mathcal{X}]$ as follows:

  if $b = 0$: $k \xleftarrow{\text{R}} \mathcal{K}$, $f \leftarrow E(k, \cdot)$;
  if $b = 1$: $f \xleftarrow{\text{R}} \text{Perms}[\mathcal{X}]$.

- The adversary submits a sequence of queries to the challenger.

  For $i = 1, 2, \ldots$, the ith query is a data block $x_i \in \mathcal{X}$.

  The challenger computes $y_i \leftarrow f(x_i) \in \mathcal{X}$, and gives y_i to the adversary.

- The adversary computes and outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\text{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := \big|\Pr[W_0] - \Pr[W_1]\big|.$$

Finally, we say that $\mathcal{A}$ is a **Q-query BC adversary** if $\mathcal{A}$ issues at most Q queries. □

Fig. 4.2 illustrates Attack Game 4.1.

Definition 4.1 (secure block cipher). *A block cipher $\mathcal{E}$ is* ***secure*** *if for all efficient adversaries $\mathcal{A}$, the value $\text{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.*

We stress that the queries made by the challenger in Attack Game 4.1 are allowed to be *adaptive*; that is, the adversary need not choose all its queries in advance; rather, it is allowed to concoct each query in some clever way that depends on the previous responses from the challenger (see Exercise 4.6).

As discussed in Section 2.3.5, Attack Game 4.1 can be recast as a "bit guessing" game, where instead of having two separate experiments, the challenger chooses $b \in \{0, 1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$. In this game, we measure $\mathcal{A}$'s *bit-guessing advantage* $\text{BC}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]$ as $|\Pr[\hat{b} = b] - 1/2|$. The general result of Section 2.3.5 (namely, (2.13)) applies here as well:

$$\text{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \text{BC}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]. \tag{4.1}$$

4.1.1 Some implications of security

Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{X})$. To exercise the definition of security a bit, we prove a couple of simple implications. For simplicity, we assume that $|\mathcal{X}|$ is large (i.e., super-poly).

A secure block cipher is unpredictable

We show that if $\mathcal{E}$ is secure in the sense of Definition 4.1, then it must be *unpredictable*, which means that every efficient adversary wins the following *prediction game* with negligible probability. In this game, the challenger chooses a random key k, and the adversary submits a sequence of queries $x_1, \ldots, x_Q$; in response to the ith query x_i, the challenger responds with $E(k, x_i)$. These queries are adaptive, in the sense that each query may depend on the previous responses. Finally, the adversary outputs a pair of values (x_{Q+1}, y), where $x_{Q+1} \notin \{x_1, \ldots, x_Q\}$. The adversary wins the game if $y = E(k, x_{Q+1})$.

To prove this implication, suppose that $\mathcal{E}$ is not unpredictable, which means there is an efficient adversary $\mathcal{A}$ that wins the above prediction game with non-negligible probability p. Then we can

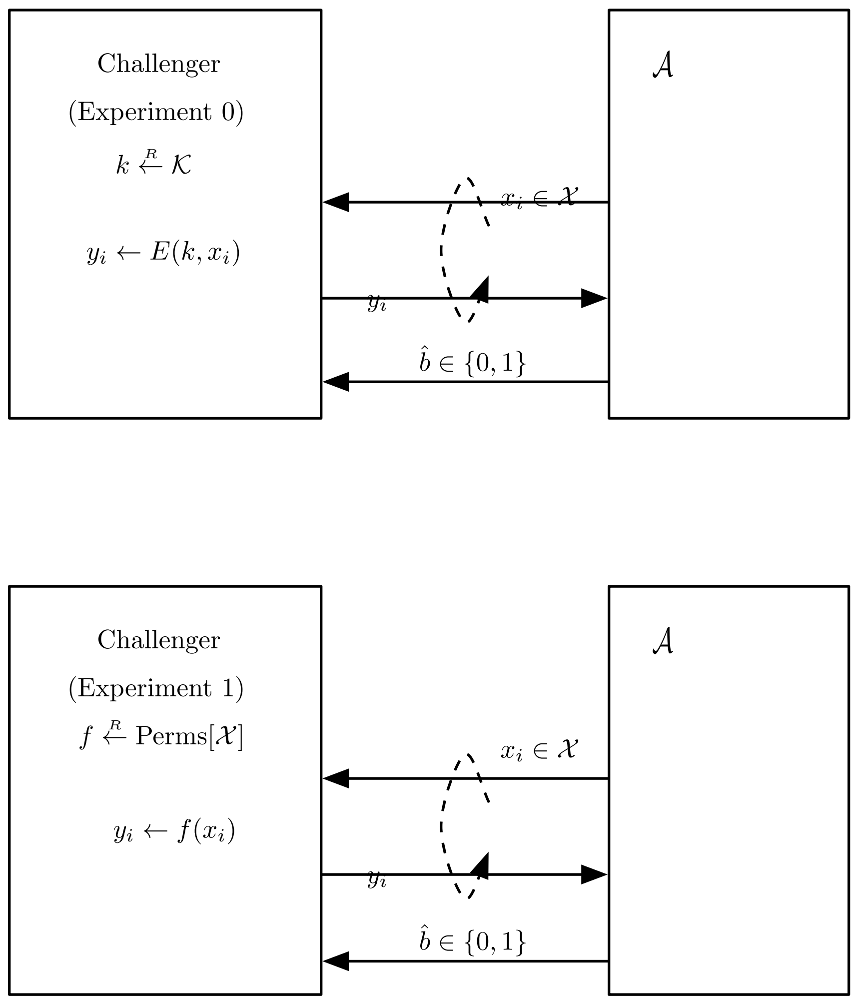


Figure 4.2: Attack Game 4.1

use $\mathcal{A}$ to break the security of $\mathcal{E}$ in the sense of Definition 4.1. To this end, we design an adversary $\mathcal{B}$ that plays Attack Game 4.1, and plays the role of challenger to $\mathcal{A}$ in the above prediction game. Whenever $\mathcal{A}$ makes a query x_i, adversary $\mathcal{B}$ passes x_i through to its own challenger, obtaining a response y_i, which it passes back to $\mathcal{A}$. Finally, when $\mathcal{A}$ outputs (x_{Q+1}, y), adversary $\mathcal{B}$ submits x_{Q+1} to its own challenger, obtaining y_{Q+1}, and outputs 1 if $y = y_{Q+1}$, and 0, otherwise.

On the one hand, if $\mathcal{B}$'s challenger is running Experiment 0, then $\mathcal{B}$ outputs 1 with probability p. On the other hand, if $\mathcal{B}$'s challenger running Experiment 1, then $\mathcal{B}$ outputs 1 with negligible probability ϵ (since we are assuming $|\mathcal{X}|$ is super-poly). This implies that $\mathcal{B}$'s advantage in Attack Game 4.1 is $|p - \epsilon|$, which is non-negligible.

Unpredictability implies security against key recovery

Next, we show that if $\mathcal{E}$ is unpredictable, then it is *secure against key recovery*, which means that every efficient adversary wins the following *key-recovery game* with negligible probability. In this game, the adversary interacts with the challenger exactly as in the prediction game, except that at the end, it outputs a candidate key $\hat{k} \in \mathcal{K}$, and wins the game if $\hat{k} = k$.

To prove this implication, suppose that $\mathcal{E}$ is not secure against key recovery, which means that there is an efficient adversary $\mathcal{A}$ that wins the key-recovery game with non-negligible probability p. Then we can use $\mathcal{A}$ to build an efficient adversary $\mathcal{B}$ that wins the prediction game with probability at least p. Adversary $\mathcal{B}$ simply runs $\mathcal{A}$'s attack, and when $\mathcal{A}$ outputs $\hat{k}$, adversary $\mathcal{B}$ chooses an arbitrary $x_{Q+1} \notin \{x_1, \ldots, x_Q\}$, computes $y \leftarrow E(\hat{k}, x_{Q+1})$, and outputs (x_{Q+1}, y).

It is easy to see that if $\mathcal{A}$ wins the key-recovery game, then $\mathcal{B}$ wins the prediction game.

Key space size and exhaustive-search attacks

Combining the above two implications, we conclude that if $\mathcal{E}$ is a secure block cipher, then it must be secure against key recovery. Moreover, if $\mathcal{E}$ is secure against key recovery, it must be the case that $|\mathcal{K}|$ is large.

One way to see this is as follows. An adversary can always win the key-recovery game with probability $1/|\mathcal{K}|$ by simply choosing $\hat{k}$ from $\mathcal{K}$ at random. If $|\mathcal{K}|$ is not super-poly, then $1/|\mathcal{K}|$ is non-negligible. Hence, when $|\mathcal{K}|$ is not super-poly this simple key guessing adversary wins the key-recovery game with non-negligible probability.

We can trade success probability for running time using a different attack, called an *exhaustive-search attack*. In this attack, our adversary makes a few, arbitrary queries $x_1, \ldots, x_Q$ in the key-recovery game, obtaining responses $y_1, \ldots, y_Q$. One can argue — heuristically, at least, assuming that $|\mathcal{X}| \geq |\mathcal{K}|$ and $|\mathcal{X}|$ is super-poly — that for fairly small values of Q ($Q = 2$, in fact), with all but negligible probability, only one key k satisfies

$$y_i = E(k, x_i) \quad \text{for} \quad i = 1, \ldots, Q. \tag{4.2}$$

So our adversary simply tries all possible keys to find one that satisfies (4.2). If there is only one such key, then the key that our adversary finds will be the key chosen by the challenger, and the adversary will win the game. Thus, our adversary wins the key-recovery game with all but negligible probability; however, its running time is linear in $|\mathcal{K}|$.

This time/advantage trade-off can be easily generalized. Indeed, consider an adversary that chooses t keys at random, testing if each such key satisfies (4.2). The running time of such an adversary is linear in t, and it wins the key-recovery game with probability $\approx t/|\mathcal{K}|$.

We describe a few real-world exhaustive search attacks in Section 4.2.2. We present a detailed treatment of exhaustive search in Section 4.7.2 where, in particular, we justify the heuristic assumption used above that with high probability there is at most one key satisfying (4.2).

So it is clear that if a block cipher has any chance of being secure, it must have a large key space, simply to avoid a key-recovery attack.

4.1.2 Efficient implementation of random permutations

Note that the challenger's protocol in Experiment 1 of Attack Game 4.1 is not very efficient: he is supposed to choose a *very* large random object. Indeed, just writing down an element of $\text{Perms}[\mathcal{X}]$ would require about $|\mathcal{X}| \log_2 |\mathcal{X}|$ bits. For AES, with $|\mathcal{X}| = 2^{128}$, this means about 10^{40} bits!

While this is not a problem from a purely definitional point of view, for both aesthetic and technical reasons, it would be nice to have a more efficient implementation. We can do this by using a "lazy" implementation of f. That is, the challenger represents the random permutation f by keeping track of input/output pairs (x_i, y_i). When the challenger receives the ith query x_i, he tests whether $x_i = x_j$ for some $j < i$; if so, he sets $y_i \leftarrow y_j$ (this ensures that the challenger implements a function); otherwise, he chooses y_i at random from the set $\mathcal{X} \setminus \{y_1, \ldots, y_{i-1}\}$ (this ensures that the function is a permutation); finally, he sends y_i to the adversary. We can write the logic of this implementation of the challenger as follows:

upon receiving the ith query $x_i \in \mathcal{X}$ from $\mathcal{A}$ do:
  if $x_i = x_j$ for some $j < i$
    then $y_i \leftarrow y_j$
    else $y_i \xleftarrow{R} \mathcal{X} \setminus \{y_1, \ldots, y_{i-1}\}$
  send y_i to $\mathcal{A}$.

To make this implementation as fast as possible, one would implement the test "if $x_i = x_j$ for some $j < i$" using an appropriate dictionary data structure (hash tables, digital search tries, balanced trees, etc.). Assuming random elements of $\mathcal{X}$ can be generated efficiently, one way to implement the step "$y_i \xleftarrow{R} \mathcal{X} \setminus \{y_1, \ldots, y_{i-1}\}$" is as follows:

repeat $y \xleftarrow{R} \mathcal{X}$ until $y \notin \{y_1, \ldots, y_{i-1}\}$
$y_i \leftarrow y$,

again, using appropriate dictionary data structure for the tests "$y \notin \{y_1, \ldots, y_{i-1}\}$." When $i < |\mathcal{X}|/2$ the loop will run for only two iterations in expectation.

One way to visualize this implementation is that the challenger in Experiment 1 is a "black box," but inside the box is a little **faithful gnome** whose job it is to maintain the table of input/output pairs which represents a random permutation f. See Fig. 4.3.

4.1.3 Strongly secure block ciphers

Note that in Attack Game 4.1, the decryption algorithm D was never used. One can in fact define a stronger notion of security by defining an attack game in which the adversary is allowed to make two types of queries to the challenger:

forward queries: the adversary sends a value $x_i \in \mathcal{X}$ to the challenger, who sends $y_i := f(x_i)$ to the adversary;

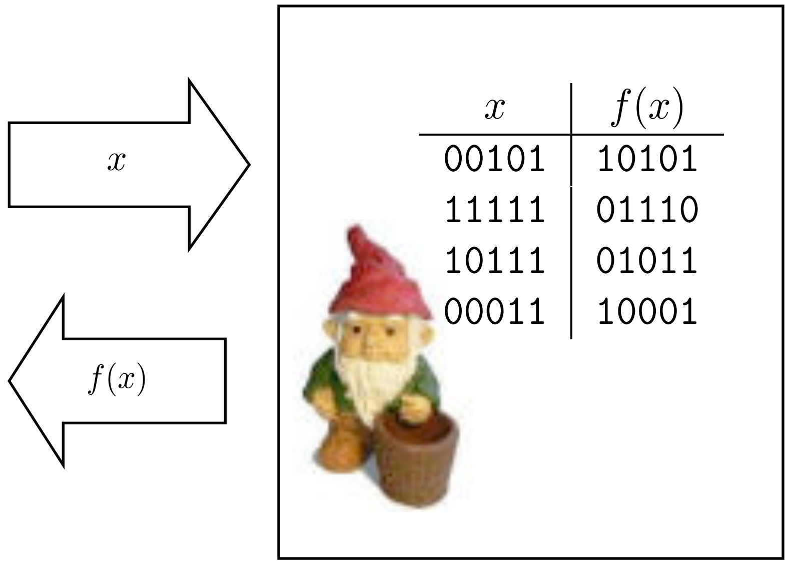


Figure 4.3: A faithful gnome implementing random permutation f

inverse queries: the adversary sends a value $y_i \in \mathcal{X}$ to the challenger, who sends $x_i := f^{-1}(y_i)$ to the adversary (in Experiment 0 in the attack game, this is done using algorithm D).

One then defines a corresponding advantage for this attack game. A block cipher is then called **strongly secure** if for all efficient adversaries, this advantage is negligible. We leave it to the reader to work out the details of this definition (see Exercise 4.9). We will not make use this notion in this text, other than an example application in a later chapter (Exercise 9.12).

4.1.4 Using a block cipher directly for encryption

Since a block cipher is a special kind of cipher, we can of course consider using it directly for encryption. The question is: is a secure block cipher also semantically secure?

The answer to this question is "yes," provided the message space is equal to the data block space. This will be implied by Theorem 4.1 below. However, data blocks for practical block ciphers are very short: as we mentioned, data blocks for AES are just 128-bits long. If we want to encrypt longer messages, a natural idea would be to break up a long message into a sequence of data blocks, and encrypt each data block separately. This use of a block cipher to encrypt long messages is called **electronic codebook mode**, or **ECB mode** for short.

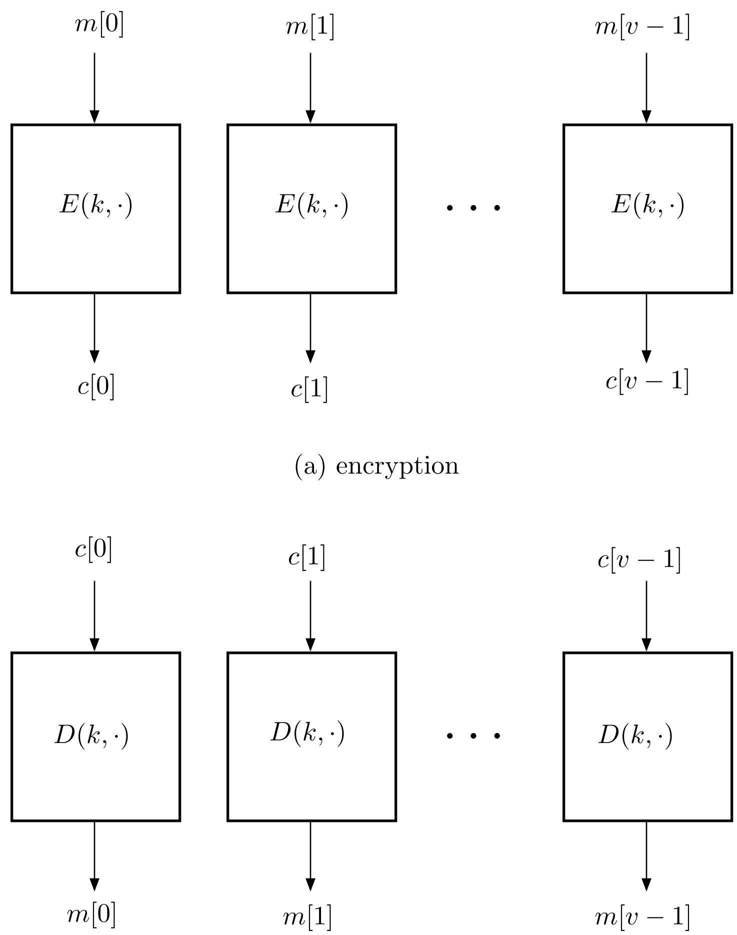
More precisely, suppose $\mathcal{E} = (E, D)$ is a block cipher defined over $(\mathcal{K}, \mathcal{X})$. For any poly-bounded $\ell \geq 1$, we can define a cipher $\mathcal{E}' = (E', D')$, defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{X}^{\leq \ell})$, as follows.

- For $k \in \mathcal{K}$ and $m \in \mathcal{X}^{\leq \ell}$, with $v := |m|$, we define

$$E'(k, m) := \big(E(k, m[0]), \ldots, E(k, m[v-1])\big).$$

- For $k \in \mathcal{K}$ and $c \in \mathcal{X}^{\leq \ell}$, with $v := |c|$, we define

$$D'(k, m) := \big(D(k, c[0]), \ldots, E(k, c[v-1])\big).$$



(a) encryption

(b) decryption

Figure 4.4: Encryption and decryption for ECB mode

Fig. 4.4 illustrates encryption and decryption. We call $\mathcal{E}'$ the **ℓ-wise ECB cipher derived from $\mathcal{E}$**.

The ECB cipher is very closely related to the substitution cipher discussed in Examples 2.3 and 2.6. The main difference is that instead of choosing a permutation at random from among all possible permutations on $\mathcal{X}$, we choose one from the much smaller set of permutations $\{E(k, \cdot) : k \in \mathcal{K}\}$. A less important difference is that in Example 2.3, we defined our substitution cipher to have a fixed length, rather than a variable length message space (this was really just an arbitrary choice — we could have defined the substitution cipher to have a variable length message space). Another difference is that in Example 2.3, we suggested an alphabet of size 27, while if we use a block cipher like AES with a 128-bit block size, the "alphabet" is much larger — it has 2^{128} elements. Despite these differences, some of the vulnerabilities discussed in Example 2.6 apply here as well. For example, an adversary can easily distinguish an encryption of two messages $m_0, m_1 \in \mathcal{X}^2$, where m_0 consists of two equal blocks (i.e., $m_0[0] = m_0[1]$) and m_1 consists of two unequal blocks (i.e., $m_1[0] \neq m_1[1]$). For this reason alone, the *ECB cipher does not satisfy our definition of semantic*

(a) plaintext

(b) plaintext encrypted in ECB mode using AES

Figure 4.5: Encrypting in ECB mode

security, and its use as an encryption scheme is strongly discouraged.

This ability to easily tell which plaintext blocks are the same is graphically illustrated in Fig. 4.5 (due to B. Preneel). Here, visual data is encrypted in ECB mode, with each data block encoding some small patch of pixels in the original data. Since identical patches of pixels get mapped to identical blocks of ciphertext, some patterns in the original picture are visible in the ciphertext.

Note, however, that some of the vulnerabilities discussed in Example 2.6 do not apply directly here. Suppose we are encrypting ASCII text. If the block size of the cipher is 128-bits, then each character of text will be typically encoded as a byte, with 16 characters packed into a data block. Therefore, an adversary will not be able to trivially locate positions where individual characters are repeated, as was the case in Example 2.6.

We close this section with a proof that ECB mode is in fact secure if the message space is restricted to sequences on *distinct* data blocks. This includes as a special case the encryption of single-block messages. It is also possible to encode longer messages as sequences of distinct data blocks. For example, suppose we are using AES, which has 128-bit data blocks. Then we could allocate, say, 32 bits out of each block as a counter, and use the remaining 96 bits for bits of the message. With such a strategy, we can encode any message of up to $2^{32} \cdot 96$ bits as a sequence of distinct data blocks. Of course, this strategy has the disadvantage that ciphertexts are 33% longer than plaintexts.

Theorem 4.1. *Let $\mathcal{E} = (E, D)$ be a block cipher. Let $\ell \geq 1$ be any poly-bounded value, and let $\mathcal{E}' = (E', D')$ be the ℓ-wise ECB cipher derived from $\mathcal{E}$, but with the message space restricted to all sequences of at most ℓ* distinct *data blocks. If $\mathcal{E}$ is a secure block cipher, then $\mathcal{E}'$ is a semantically secure cipher.*

In particular, for every $\mathcal{A}$ SS adversary that plays Attack Game 2.1 with respect to $\mathcal{E}'$, there exists a BC adversary $\mathcal{B}$ that plays Attack Game 4.1 with respect to $\mathcal{E}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}'] = 2 \cdot \mathrm{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}]. \tag{4.3}$$

Proof idea. The basic idea is that if an adversary is given an encryption of a message, which is a sequence of distinct data blocks, then what he sees is effectively just a sequence of random data blocks (sampled without replacement). □

Proof. If $\mathcal{E}$ is defined over $(\mathcal{K}, \mathcal{X})$, let $\mathcal{X}_*^{\leq \ell}$ denote the set of all sequences of at most ℓ distinct elements of $\mathcal{X}$.

Let $\mathcal{A}$ be an efficient adversary that attacks $\mathcal{E}'$ as in Attack Game 2.1. Our goal is to show that $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}']$ is negligible, assuming that $\mathcal{E}$ is a secure block cipher. It is more convenient to work with the bit-guessing version of the SS attack game. We prove:

$$\mathsf{SSadv}^*[\mathcal{A}, \mathcal{E}'] = \mathrm{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}] \tag{4.4}$$

for some efficient adversary $\mathcal{B}$. Then (4.3) follows from Theorem 2.10.

So consider the adversary $\mathcal{A}$'s attack of $\mathcal{E}'$ in the bit-guessing version of Attack Game 2.1. In this game, $\mathcal{A}$ presents the challenger with two messages m_0, m_1 of the same length; the challenger then chooses a random key k and a random bit b, and encrypts m_b under k, giving the resulting ciphertext c to $\mathcal{A}$; finally, $\mathcal{A}$ outputs a bit $\hat{b}$. The adversary $\mathcal{A}$ wins the game if $\hat{b} = b$.

The logic of the challenger in this game may be written as follows:

> upon receiving $m_0, m_1 \in \mathcal{X}_*^{\leq \ell}$, with $v := |m_0| = |m_1|$, do:
> $\quad b \xleftarrow{\text{R}} \{0, 1\}$
> $\quad k \xleftarrow{\text{R}} \mathcal{K}$
> $\quad c \leftarrow (E(k, m_b[0]), \ldots, E(k, m_b[v-1]))$
> $\quad$ send c to $\mathcal{A}$.

Let us call this **Game 0.** We will define two more games: Game 1 and Game 2. For $j = 0, 1, 2$, we define W_j to be the event that $\hat{b} = b$ in Game j. By definition, we have

$$\mathsf{SSadv}^*[\mathcal{A}, \mathcal{E}'] = |\Pr[W_0] - 1/2|. \tag{4.5}$$

Game 1. This is the same as Game 0, except the challenger uses a random $f \in \mathrm{Perms}[\mathcal{X}]$ in place of $E(k, \cdot)$. Our challenger now looks like this:

> upon receiving $m_0, m_1 \in \mathcal{X}_*^{\leq \ell}$, with $v := |m_0| = |m_1|$, do:
> $\quad b \xleftarrow{\text{R}} \{0, 1\}$
> $\quad f \xleftarrow{\text{R}} \mathrm{Perms}[\mathcal{X}]$
> $\quad c \leftarrow (f(m_b[0]), \ldots, f(m_b[v-1]))$
> $\quad$ send c to $\mathcal{A}$.

Intuitively, the fact that $\mathcal{E}$ is a secure block cipher implies that the adversary should not notice the switch. To prove this rigorously, we show how to build a BC adversary $\mathcal{B}$ that is an elementary wrapper around $\mathcal{A}$, such that

$$|\Pr[W_0] - \Pr[W_1]| = \mathrm{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}]. \tag{4.6}$$

The design of $\mathcal{B}$ follows directly from the logic of Games 0 and 1. Adversary $\mathcal{B}$ plays Attack Game 4.1 with respect to $\mathcal{E}$, and works as follows:

Let f be the function chosen by $\mathcal{B}$'s BC challenger in Attack Game 4.1. We let $\mathcal{B}$ play the role of challenger to $\mathcal{A}$, as follows:

upon receiving $m_0, m_1 \in \mathcal{X}_*^{\leq \ell}$ from $\mathcal{A}$, with $v := |m_0| = |m_1|$, do:
  $b \xleftarrow{R} \{0,1\}$
  $c \leftarrow (f(m_b[0]), \ldots, f(m_b[v-1]))$
  send c to $\mathcal{A}$.

Note that $\mathcal{B}$ computes the values $f(m_b[0]), \ldots, f(m_b[v-1])$ by querying its own BC challenger. Finally, when $\mathcal{A}$ outputs a bit $\hat{b}$, $\mathcal{B}$ outputs the bit $\delta(\hat{b}, b)$ (see (3.7)).

It should be clear that when $\mathcal{B}$ is in Experiment 0 of its attack game, it outputs 1 with probability $\Pr[W_0]$, while when $\mathcal{B}$ is in Experiment 1 of its attack game, it outputs 1 with probability $\Pr[W_1]$. The equation (4.6) now follows.

Game 2. We now rewrite the challenger in Game 1 so that it uses the "faithful gnome" implementation of a random permutation, discussed in Section 4.1.2. Each of the messages m_0 and m_1 is required to consist of distinct data blocks (our challenger does not have to verify this), and so our gnome's job is quite easy: it does not even have to look at the input data blocks, as these are guaranteed to be distinct; however, it still has to ensure that the output blocks it generates are distinct.

We can express the logic of our challenger as follows:

$y_0 \xleftarrow{R} \mathcal{X}$, $y_1 \xleftarrow{R} \mathcal{X} \setminus \{y_0\}$, …, $y_{\ell-1} \xleftarrow{R} \mathcal{X} \setminus \{y_0, \ldots, y_{\ell-2}\}$
upon receiving $m_0, m_1 \in \mathcal{X}_*^{\leq \ell}$, with $v := |m_0| = |m_1|$, do:
  $b \xleftarrow{R} \{0,1\}$
  $c \leftarrow (y_0, \ldots, y_{v-1})$
  send c to $\mathcal{A}$.

Since our gnome is faithful, we have

$$\Pr[W_1] = \Pr[W_2]. \tag{4.7}$$

Moreover, we claim that

$$\Pr[W_2] = 1/2. \tag{4.8}$$

This follows from the fact that in Game 2, the adversary's output $\hat{b}$ is a function of its own random choices, together with $y_0, \ldots, y_{\ell-1}$; since these values are (by definition) independent of b, it follows that $\hat{b}$ and b are independent. The equation (4.8) now follows.

Combining (4.5), (4.6), (4.7), and (4.8), yields (4.4), which completes the proof. □

4.1.5 Mathematical details

As usual, we address a few mathematical details that were glossed over above.

Since a block cipher is just a special kind of cipher, there is really nothing to say about the definition of a block cipher that was not already said in Section 2.4. As usual, Definition 4.1 needs to be properly interpreted. First, in Attack Game 4.1, it is to be understood that for each value of the security parameter λ, we get a different probability space, determined by the random choices of the challenger and the random choices of the adversary. Second, the challenger generates a system parameter Λ, and sends this to the adversary at the very start of the game. Third, the advantage $\mathsf{BCadv}[\mathcal{A}, \mathcal{E}]$ is a function of the security parameter λ, and security means that this function is a negligible function.

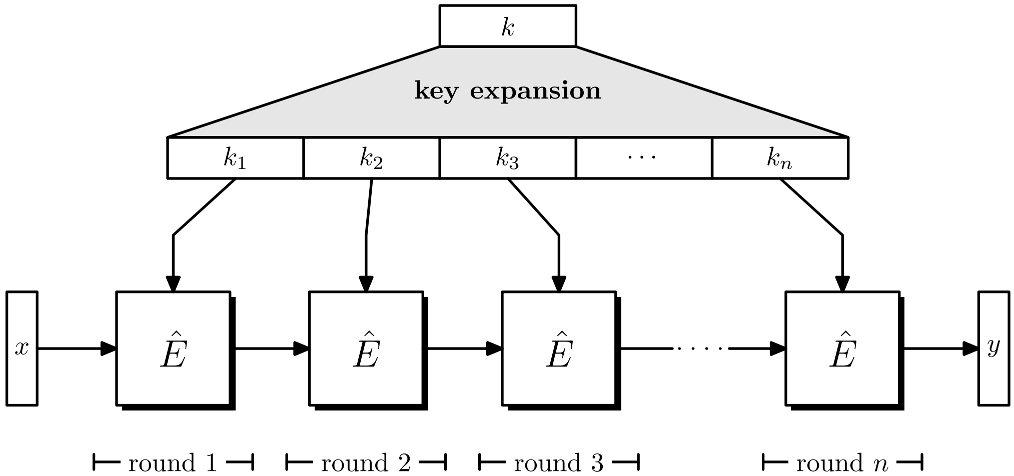


Figure 4.6: Encryption in a real-world block cipher

4.2 Constructing block ciphers in practice

Block ciphers are a basic primitive in cryptography from which many other systems are built. Virtually all block ciphers used in practice use the same basic framework called the **iterated cipher** paradigm. To construct an iterated block cipher the designer makes two choices:

- First, he picks a simple block cipher $\hat{\mathcal{E}} := (\hat{E}, \hat{D})$ that is clearly insecure on its own. We call $\hat{\mathcal{E}}$ the **round cipher**.

- Second, he picks a simple (not necessarily secure) PRG G that is used to expand the key k into d keys $k_1, \ldots, k_d$ for $\hat{\mathcal{E}}$. We call G the **key expansion function**.

Once these two choices are made, the iterated block cipher $\mathcal{E}$ is completely specified. The encryption algorithm $E(k, x)$ works as follows (see Fig. 4.6):

Algorithm $E(k, x)$:

- step 1. **key expansion**: use the key expansion function G to stretch the key k of $\mathcal{E}$ to d keys of $\hat{\mathcal{E}}$:

$$(k_1, \ldots, k_d) \leftarrow G(k)$$

- step 2. **iteration**: for $i = 1, \ldots, d$ apply $\hat{E}(k_i, \cdot)$, namely:

$$y \leftarrow \hat{E}(k_d,\ \hat{E}(k_{d-1}, \ldots, \hat{E}(k_2,\ \hat{E}(k_1,\ x)) \ldots))$$

Each application of $\hat{E}$ is called a **round** and the total number of rounds is d. The keys $k_1, \ldots, k_d$ are called **round keys**. The decryption algorithm $D(k, y)$ is identical except that the round keys are applied in reverse order. $D(k, y)$ is defined as:

$$x \leftarrow \hat{D}(k_1,\ \hat{D}(k_2, \ldots, \hat{D}(k_{d-1},\ \hat{D}(k_d,\ y)) \ldots))$$

	key size (bits)	block size (bits)	number of rounds	performance[1] (MB/sec)
DES	56	64	16	80
3DES	168	64	48	30
AES-128	128	128	10	163
AES-256	256	128	14	115

Table 4.1: Sample block ciphers

Table 4.1 lists a few common block ciphers and their parameters. We describe DES and AES in the next section.

Does iteration give a secure block cipher? Nobody knows. However, heuristic evidence suggests that security of a block cipher comes from iterating a simple cipher many times. Not all round ciphers will work. For example, iterating a linear function

$$\hat{E}(k, x) := k \cdot x \bmod q$$

will never result in a secure block cipher since the iterate of $\hat{E}$ is just another linear function. There is currently no way to classify which round ciphers will eventually result in a secure block cipher. Moreover, for a candidate round cipher $\hat{E}$ there is no rigorous methodology to gauge how many times it needs to be iterated before it becomes a secure block cipher. All we know is that certain functions, like linear functions, never lead to secure block ciphers, while simple non-linear functions appear to give a secure block cipher after a few iterations.

The challenge for the cryptographer is to come up with a fast round cipher that converges to a secure block cipher within a few rounds. Looking at Table 4.1 one is impressed that AES-128 uses a simple round cipher and yet seems to produce a secure block cipher after only ten rounds.

A word of caution. While this section explains the inner workings of several block ciphers, it does not teach how to design new block ciphers. In fact, one of the main take-away messages from this section is that readers should not design block ciphers on their own, but instead always use the standard ciphers described here. Block-cipher design is non-trivial and many years of analysis are needed before one gains confidence in a specific proposal. Furthermore, readers should not even implement block ciphers on their own since implementations of block-ciphers tend to be vulnerable to timing and power attacks, as discussed in Section 4.3.2. It is much safer to use one of the standard implementations freely available in crypto libraries such as OpenSSL. These implementations have gone through considerable analysis over the years and have been hardened to resist attack.

4.2.1 Case study: DES

The Data Encryption Standard (DES) was developed at IBM in response to a solicitation for proposals from the National Bureau of Standards (now the National Institute of Standards). It was published in the Federal Register in 1975 and was adopted as a standard for "unclassified" applications in 1977. The DES algorithm single-handedly jump started the field of cryptanalysis;

[1] OpenSSL 1.0.1e on Intel(R) Xeon(R) CPU E5-2698 v3 @ 2.30GHz (Haswell).

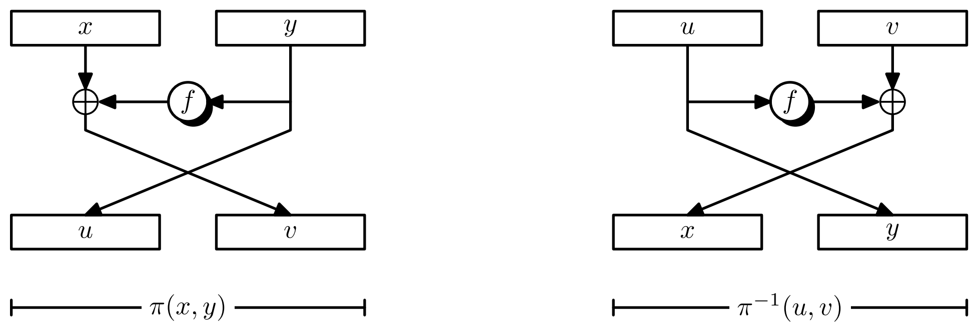


Figure 4.7: The Feistel permutation

everyone wanted to break it. Since inception, DES has undergone considerable analysis that lead to the development of many new tools for analyzing block ciphers.

The precursor to DES is an earlier IBM block cipher called Lucifer. Certain variants of Lucifer operated on 128-bit blocks using 128-bit keys. The National Bureau of Standards, however, asked for a block cipher that used shorter blocks (64 bits) and shorter keys (56 bits). In response, the IBM team designed a block cipher that met these requirements and eventually became DES. Setting the DES key size to 56 bits was widely criticized and lead to speculation that DES was deliberately made weak due to pressure from US intelligence agencies. In the coming chapters, we will see that reducing the block size to 64 bits also creates problems.

Due to its short key size, the DES algorithm is now considered insecure and should not be used. However, a strengthened version of DES called Triple-DES (3DES) was reaffirmed as a US standard in 1998. NIST has approved Triple-DES through the year 2030 for government use. In 2002 DES was superseded by a new and more efficient block cipher standard called AES that uses 128-bit (or longer) keys, and operates on 128-bit blocks.

The DES algorithm

The DES algorithm consists of 16 iterations of a simple round cipher. To describe DES it suffices to describe the DES round cipher and the DES key expansion function. We describe each in turn.

The Feistel permutation. One of the key innovations in DES, invented by Horst Feistel at IBM, builds a permutation from an arbitrary function. Let $f : \mathcal{X} \to \mathcal{X}$ be a function. We construct a permutations $\pi : \mathcal{X}^2 \to \mathcal{X}^2$ as follows (Fig. 4.7):

$$\pi(x, y) := \big(y,\ x \oplus f(y)\big)$$

To show that π is one-to-one we construct its inverse, which is given by:

$$\pi^{-1}(u, v) = \big(v \oplus f(u),\ u\big)$$

The function π is called a **Feistel permutation** and is used to build the DES round cipher. The composition of n Feistel permutations is called an **n-round Feistel network**. Block ciphers designed as a Feistel network are called **Feistel ciphers**. For DES, the function f takes 32-bit inputs and the resulting permutation π operates on 64-bit blocks.

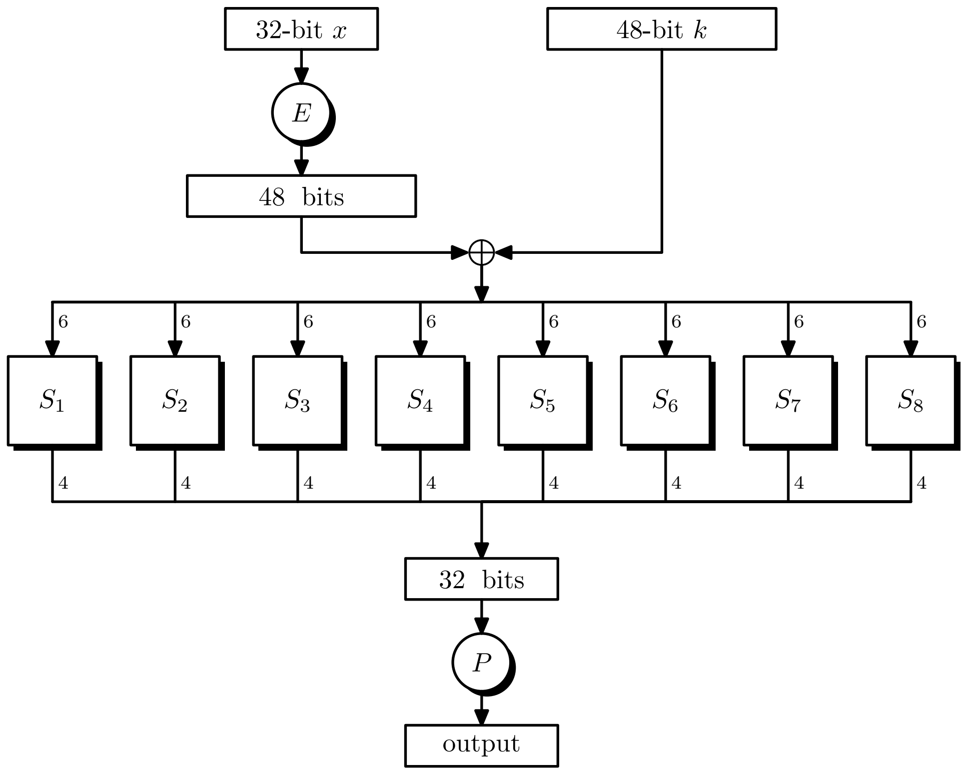


Figure 4.8: The DES round function $F(k, x)$

Note that the Feistel inverse function π^{-1} is almost identical to π. As a result the same hardware can be used for evaluating both π and π^{-1}. This in turn means that the encryption and decryption circuits can use the same hardware.

The DES round function $F(k, x)$. The DES encryption algorithm is a 16-round Feistel network where each round uses a different function $f : \mathcal{X} \to \mathcal{X}$. In round number i the function f is defined as

$$f(x) := F(k_i, x)$$

where k_i is a 48-bit key for round number i and F is a fixed function called the **DES round function**. The function F is the centerpiece of the DES algorithm and is shown in Fig. 4.8. F uses several auxiliary functions E, P, and $S_1, \ldots, S_8$ defined as follows:

- The function E expands a 32-bit input to a 48-bit output by rearranging and replicating the input bits. For example, E maps input bit number 1 to output bits 2 and 48; it maps input bit 2 to output bit number 3, and so on.

- The function P, called the **mixing permutation**, maps a 32-bit input to a 32-bit output by rearranging the bits of the input. For example, P maps input bit number 1 to output bit number 9; input bit number 2 to output number 15, and so on.

- At the heart of the DES algorithm are the functions $S_1, \dots, S_8$ called **S-boxes**. Each S-box S_i maps a 6-bit input to a 4-bit output by a lookup table. The DES standard lists these 8 look-up tables, where each table contains 64 entries.

Given these functions, the DES round function $F(k, x)$ works as follows:

> input: $k \in \{0,1\}^{48}$ and $x \in \{0,1\}^{32}$
> output: $y \in \{0,1\}^{32}$
>
> $F(k, x)$:
> $\quad t \leftarrow E(x) \oplus k \quad \in \{0,1\}^{48}$
> $\quad$ separate t into 8 groups of 6-bits each: $t := t_1 \parallel \cdots \parallel t_8$
> $\quad$ for $i = 1$ to 8 : $\quad s_i \leftarrow S_i(t_i)$
> $\quad s \leftarrow s_1 \parallel \cdots \parallel s_8 \quad \in \{0,1\}^{32}$
> $\quad y \leftarrow P(s) \quad \in \{0,1\}^{32}$
> $\quad$ output y

Except for the S-boxes, the DES round cipher is made up entirely of XORs and bit permutations. The eight S-boxes are the only components that introduce non-linearity into the design. IBM published the criteria used to design the S-boxes in 1994 [26], after the discovery of a powerful attack technique called "differential cryptanalysis" in the open literature. This IBM report makes it clear that the designers of DES knew in 1973 of attack techniques that would only become known in the open literature many years later. They designed DES to resist these attacks. The reason for keeping the S-box design criteria secret is explained in the following quote [26]:

> The design [of DES] took advantage of knowledge of certain cryptanalytic techniques, most prominently the technique of "differential cryptanalysis," which were not known in the published literature. After discussions with NSA, it was decided that disclosure of the design considerations would reveal the technique of differential cryptanalysis, a powerful technique that can be used against many ciphers. This in turn would weaken the competitive advantage of the United States enjoyed over other countries in the field of cryptography.

Once differential cryptanalysis became public there was no longer any reason to keep the design of DES secret. Due to the importance of the S-boxes we list a few of the criteria that went into their design, as explained in [26].

1. The size of the look-up tables, mapping 6-bits to 4-bits, was the largest that could be accommodated on a single chip using 1974 technology.

2. No output bit of an S-box should be close to a linear function of the input bits. That is, if we select any output bit and any subset of the 6 input bits, then the fraction of inputs for which this output bit equals the XOR of these input bits should be close to 1/2.

3. If we fix the leftmost and rightmost bits of the input to an S-box then the resulting 4-bit to 4-bit function is one-to-one. In particular, this implies that each S-box is a 4-to-1 map.

4. Changing one bit of the input to an S-box changes at least two bits of the output.

5. For each $\Delta \in \{0,1\}^6$, among the 64 pairs $x, y \in \{0,1\}^6$ such that $x \oplus y = \Delta$, the quantity $S_i(x) \oplus S_i(y)$ must not attain a single value more than eight times.

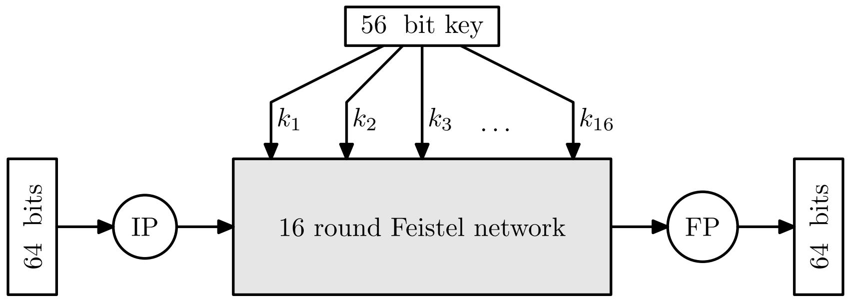


Figure 4.9: The complete DES circuit

These criteria were designed to make DES as strong as possible, given the 56-bit key-size constraints. It is now known that if the S-boxes were simply chosen at random, then with high probability the resulting DES cipher would be insecure. In particular, the secret key could be recovered after only several million queries to the challenger.

Beyond the S-boxes, the mixing permutation P also plays an important role. It ensures that the S-boxes do not always operate on the same group of 6 bits. Again, [26] lists a number of criteria used to choose the permutation P. If the permutation P was simply chosen at random then DES would be far less secure.

The key expansion function. The DES key expansion function G takes as input the 56-bit key k and outputs 16 keys $k_1, \ldots, k_{16}$, each 48-bits long. Each key k_i consists of 48 bits chosen from the 56-bit key, with each k_i using a different subset of bits from k.

The DES algorithm. The complete DES algorithm is shown in Fig. 4.9. It consists of 16 iterations of the DES round cipher plus initial and final permutations called IP and FP. These permutations simply rearrange the 64 incoming and outgoing bits. The permutation FP is the inverse of IP.

IP and FP have no cryptographic significance and were included for unknown reasons. Since bit permutations are slow in software, but fast in hardware, one theory is that IP and FP are intended to deliberately slow down software implementations of DES.

4.2.2 Exhaustive search on DES: the DES challenges

Recall that an exhaustive search attack on a block cipher (E, D) (Section 4.1.1) refers to the following attack: the adversary is given a small number of plaintext blocks $x_1, \ldots, x_Q \in \mathcal{X}$ and their encryption $y_1, \ldots, y_Q$ using a block cipher key k in $\mathcal{K}$. The adversary finds k by trying all possible keys $\underaccent{\sim}{k} \in \mathcal{K}$ until it finds a key that maps all the given plaintext blocks to the given ciphertext blocks. If enough ciphertext blocks are given, then k is the only such key, and it will be found by the adversary.

For block ciphers like DES and AES-128 *three* blocks are enough to ensure that with high probability there is a unique key mapping the given plaintext blocks to the given ciphertext blocks. We will see why in Section 4.7.2 where we discuss ideal ciphers and their properties. For now it suffices to know that given three plaintext/ciphertext blocks an attacker can use exhaustive search to find the secret key k.

In 1974, when DES was designed, an exhaustive search attack on a key space of size 2^{56} was believed to be infeasible. With improvements in computer hardware it was shown that a 56-bit is woefully inadequate.

To prove that exhaustive search on DES is feasible, RSA data security setup a sequence of challenges, called the **DES challenges**. The rules were simple: on a pre-announced date RSA data security posted three input/output pairs for DES. The first group to find the corresponding key wins ten thousand US dollars. To make the challenge more entertaining, the challenge consisted of n DES outputs $y_1, y_2, \ldots, y_n$ where the first three outputs, y_1, y_2, y_3, were the result of applying DES to the 24-byte plaintext message:

$$\underbrace{\texttt{The unkn}}_{x_1}\underbrace{\texttt{own mess}}_{x_2}\underbrace{\texttt{age is: }}_{x_3}$$

which consists of three DES blocks: each block is 8 bytes which is 64 bits, a single DES block. The goal was to find a DES key that maps x_i to y_i for all $i = 1, 2, 3$ and then use this key to decrypt the secret message encoded in $y_4 \ldots y_n$.

The first challenge was posted in January 1997. It was solved by the DESCHALL project in 96 days. The team used a distributed Internet search with the help of 78,000 volunteers who contributed idle cycles on their machines. The person whose machine found the secret-key received 40% of the prize money. Once decrypted, the secret message encoded in $y_4 \ldots y_n$ was "Strong cryptography makes the world a safer place."

A second challenge, posted in January 1998, was solved by the **distributed.net** project in only 41 days by conducting a similar Internet search, but on a larger scale.

In early 1998, the Electronic Frontiers Foundation (EFF) contracted Paul Kocher to construct a dedicated machine to do DES exhaustive key search. The machine, called **DeepCrack**, cost 250,000 US dollars and contained about 1900 dedicated DES chips housed in six cabinets. The chips worked in parallel, each searching through an assigned segment of the key space. When RSA data security posted the next challenge in July 1998, DeepCrack solved it in 56 hours and easily won the ten thousand dollar prize: not quite enough to cover the cost of the machine, but more than enough to make an important point about DES.

The final challenge was posted in January 1999. It was solved within 22 hours using a combined DeepCrack and *distributed.net* effort. This put the final nail in DES's coffin showing that a 56-bit secret key can be recovered in just a few hours.

To complete the story, in 2007 the COPACOBANA team built a cluster of off the shelf 120 FPGA boards at a total cost of about ten thousand US dollars. The cluster can search through the entire 2^{56} DES key space in about 12.8 days [50].

The conclusion from all this work is that a 56-bit key is way too short. The minimum safe key size these days is 128 bits.

Is AES-128 vulnerable to exhaustive search? Let us extrapolate the DES results to AES. While these estimates are inherently imprecise, they give some indication as to the complexity of exhaustive search on AES. The minimum AES key space size is 2^{128}. If scanning a space of size 2^{56} takes 22 hours then scanning a space of size 2^{128} will take time:

$$(22 \text{ hours}) \times 2^{128-56} \approx 1.18 \cdot 10^{20} \text{ years.}$$

Even allowing for a billion fold improvement in computing speed and computing resources and accounting for the fact that evaluating AES is faster than evaluating DES, the required time far exceeds our capabilities. It is fair to conclude that a brute-force exhaustive search attack on AES will never be practical. However, more sophisticated brute-force attacks on AES-128 exploiting time-space tradeoffs may come withing reach, as discussed in [13].

4.2.3 Strengthening ciphers against exhaustive search: the $3\mathcal{E}$ construction

The DES cipher has proved to be remarkably resilient to sophisticated attacks. Despite many years of analysis the most practical attack on DES is a brute force exhaustive search over the entire key space. Unfortunately, the 56-bit key space is too small.

A natural question is whether we can strengthen the cipher against exhaustive search without changing its inner structure. The simplest solution is to iterate the cipher several time using independent keys.

Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{X})$. We define the block cipher $3\mathcal{E} = (E_3, D_3)$ as

$$E_3(\ (k_1, k_2, k_3),\ x) := E\big(k_3,\ E(k_2,\ E(k_1, x))\big)$$

The $3\mathcal{E}$ block cipher takes keys in $\mathcal{K}^3$. For DES the $3\mathcal{E}$ block cipher, called **Triple-DES**, uses keys whose length is $3 \times 56 = 168$ bits.

Security. To analyze the security of $3\mathcal{E}$ we will need a framework called the *ideal cipher model* which we present at the end of this chapter. We analyze the security of $3\mathcal{E}$ in that section.

The Triple-DES standard. NIST approved Triple-DES for government use through the year 2030. Strictly speaking, the NIST version of Triple-DES is defined as

$$E_3(\ (k_1, k_2, k_3),\ x) := E\big(k_3,\ D(k_2,\ E(k_1, x))\big).$$

The reason for this is that setting $k_1 = k_2 = k_3$ reduces the NIST Triple-DES to ordinary DES and hence Triple-DES hardware can be used to implement single DES. This will not affect our discussion of security of Triple-DES. Another variant of Triple-DES is discussed in Exercise 4.5.

The $2\mathcal{E}$ construction is insecure

While Triple-DES is not vulnerable to exhaustive search, its performance is three times slower than single DES, as shown in Table 4.1.

Why not use Double-DES? Its key size is $2 \times 56 = 112$ bits, which is already sufficient to defeat exhaustive search. Its performance is much better then Triple-DES.

Unfortunately, Double-DES is no more secure than single DES. More generally, let $\mathcal{E} = (E, D)$ be a block cipher with key space $\mathcal{K}$. We show that the $2\mathcal{E} = (E_2, D_2)$ construction, defined as

$$E_2(\ (k_1, k_2),\ x) := E\big(k_2,\ E(k_1, x)\big)$$

is no more secure than $\mathcal{E}$. The attack strategy is called **meet in the middle**.

We are given Q plaintext blocks $x_1, \ldots, x_Q$ and their $2\mathcal{E}$ encryptions $y_i = E_2\big(\ (k_1, k_2),\ x_i\big)$ for $i = 1, \ldots, Q$. We show how to recover the secret key (k_1, k_2) in time proportional to $|\mathcal{K}|$, even though the key space has size $|\mathcal{K}|^2$. As with exhaustive search, a small number of plaintext/ciphertext pairs is sufficient to ensure that there is a unique key (k_1, k_2) with high probability. Ten pairs are more than enough to ensure uniqueness for block ciphers like Double-DES.

Theorem 4.2. *Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{X})$. There is an algorithm $\mathcal{A}_{\text{EX}}$ that takes as input Q plaintext/ciphertext pairs $(x_i, y_i) \in \mathcal{X}^2$ for $i = 1, \ldots, Q$ and outputs a key pair $(k_1, k_2) \in \mathcal{K}^2$ such that*

$$y_i = E_2\big(\ (k_1, k_2),\ x_i\big) \quad \textit{for all } i = 1, \ldots, Q. \tag{4.9}$$

Its running time is dominated by a total of $2Q \cdot |\mathcal{K}|$ evaluations of algorithms E and D.

Proof. Let $\bar{x} := (x_1, \ldots, x_Q)$ and $\bar{y} := (y_1, \ldots, y_Q)$. To simplify the notation let us write

$$\bar{y} = E_2\big((k_1, k_2), \bar{x}\big) = E(k_2,\ E(k_1, \bar{x}))$$

to capture the Q relations in (4.9). We can write this as

$$D(k_2, \bar{y}) = E(k_1, \bar{x}) \tag{4.10}$$

To find a pair (k_1, k_2) satisfying (4.10) the algorithm $\mathcal{A}_{\text{EX}}$ does the following:

step 1: construct a table T containing all pairs $\big(k_1,\ E(k_1, \bar{x})\ \big)$ for all $k_1 \in \mathcal{K}$
step 2: for all $k_2 \in \mathcal{K}$ do:
  $\bar{x} \leftarrow D(k_2, \bar{y})$
  table lookup: if T contains a pair $(\cdot,\ \bar{x})$ then
    let $(k_1, \bar{x})$ be that pair and output (k_1, k_2) and halt

This meet in the middle attack is depicted in Fig. 4.10. By construction, the pair (k_1, k_2) output by the algorithm must satisfy (4.10), a required.

Step 1 requires $Q \cdot |\mathcal{K}|$ evaluations of E. Step 2 similarly requires $Q \cdot |\mathcal{K}|$ evaluations of D. Therefore, the total number of evaluation of E and D is $2Q \cdot |\mathcal{K}|$. We assume that the time to insert and look-up elements in the data structure holding the table T is less than the time to evaluate algorithms E and D. □

As discussed above, for relatively small values of Q, with overwhelming probability there will be only one key pair satisfying (4.9), and this will be the output of Algorithm $\mathcal{A}_{\text{EX}}$ in Theorem 4.2.

The running time of algorithm $\mathcal{A}$ in Theorem 4.2 is about the same as the time to do exhaustive search on $\mathcal{E}$, suggesting that $2\mathcal{E}$ does not strengthen $\mathcal{E}$ against exhaustive search. The theorem, however, only considers the running time of $\mathcal{A}$. Notice that $\mathcal{A}$ must keep a large table in memory which can be difficult. To attack Double-DES, $\mathcal{A}$ would need to store a table of size 2^{56} where each table entry contains a DES key and short ciphertext. Overall this amounts to about 2^{60} bytes or about a million Terrabytes. While not impossible, obtaining sufficient storage can be difficult. Alternatively an attacker can trade-off storage space for running time — it is easy to modify $\mathcal{A}$ so that at any given time it only stores an ϵ fraction of the table at the cost of increasing the running time by a factor of $1/\epsilon$.

A meet in the middle attack on Triple-DES. A similar meet in the middle attack applies to the $3\mathcal{E}$ construction from the previous section. While $3\mathcal{E}$ has key space $\mathcal{K}^3$, the meet in the middle attack on $3\mathcal{E}$ runs in time about $|\mathcal{K}|^2$ and takes space $|\mathcal{K}|$. In the case of Triple-DES, the attack requires about $|\mathcal{K}|^2 = 2^{112}$ evaluations of DES which is too long to run in practice. Hence, Triple-DES resists this meet in the middle attack and is the reason why Triple-DES is used in practice.

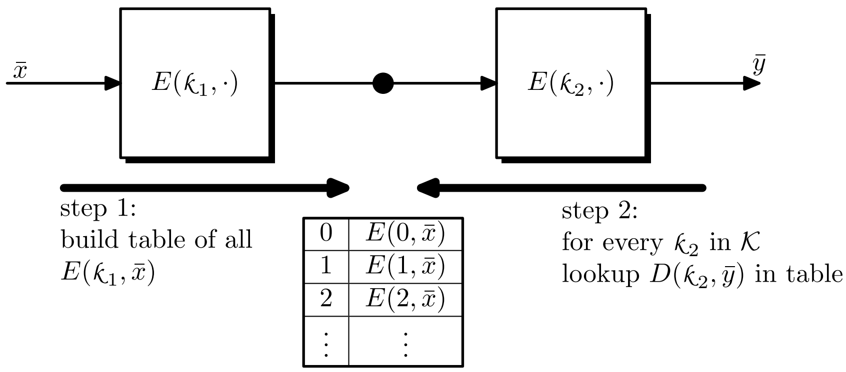


Figure 4.10: Meet in the middle attack on $2\mathcal{E}$

4.2.4 Case study: AES

Although Triple-DES is a NIST approved cipher, it has a number of significant drawbacks. First, Triple-DES is three times slower than DES and performs poorly when implemented in software. Second, the 64-bit block size is problematic for a number of important applications (i.e., applications in Chapter 6). By the mid-1990s it became apparent that a new federal block cipher standard is needed.

The AES process. In 1997 NIST put out a request for proposals for a new block cipher standard to be called the **Advanced Encryption Standard** or AES. The AES block cipher had to operate on 128-bit blocks and support three key sizes: 128, 192, and 256 bits. In September of 1997, NIST received 15 proposals, many of which were developed outside of the United Stated. After holding two open conferences to discuss the proposals, in 1999 NIST narrowed down the list to five candidates. A further round of intense cryptanalysis followed, culminating in the AES3 conference in April of 2000, at which a representative of each of the final five teams made a presentation arguing why their standard should be chosen as the AES. In October of 2000, NIST announced that **Rijndael**, a Belgian block cipher, had been selected as the AES cipher. The AES became an official standard in November of 2001 when it was published as a NIST standard in FIPS 197. This concluded a five year process to standardize a replacement to DES.

Rijndael was designed by Belgian cryptographers Joan Daemen and Vincent Rijmen [29]. AES is slightly different from the original Rijndael cipher. For example, Rijndael supports blocks of size 128, 192, or 256 bits while AES only supports 128-bit blocks.

The AES algorithm

Like many real-world block ciphers, AES is an iterated cipher that iterates a simple round cipher several times. The number of iterations depends on the size of the secret key:

cipher name	key-size (bits)	block-size (bits)	number of rounds
AES-128	128	128	10
AES-192	192	128	12
AES-256	256	128	14

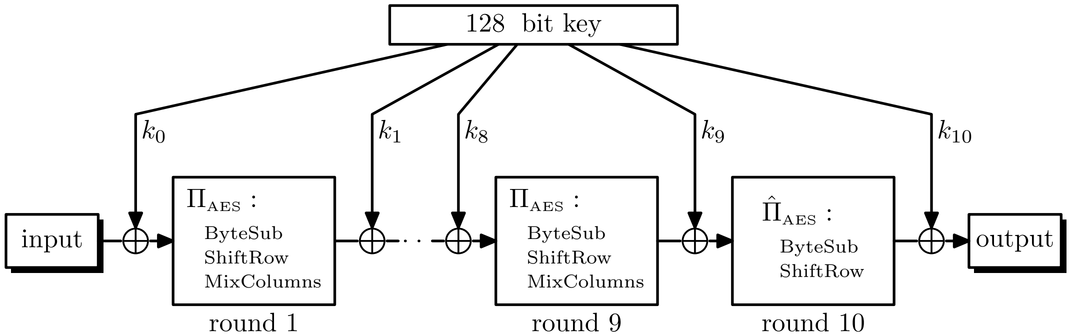


Figure 4.11: Schematic of the AES-128 block cipher

For example, the structure of the cipher AES-128 with its ten rounds is shown in Fig. 4.11. Here Π_{AES} is a fixed permutation (a one-to-one function) on $\{0,1\}^{128}$ that does not depend on the key. The last step of each round is to XOR the current round key with the output of Π_{AES}. This is repeated 9 times until in the last round a slightly modified permutation $\hat{\Pi}_{\text{AES}}$ is used. Inverting the AES algorithm is done by running the entire structure in the reverse direction. This is possible because every step is easily invertible.

Ciphers that follow the structure shown in Fig. 4.11 are called **alternating key ciphers**. They are also known as **iterated Even-Mansour ciphers**. They can be proven secure under certain "ideal" assumptions about the permutation Π_{AES} in each round. We present this analysis in Theorem 4.14 later in this chapter.

To complete the description of AES it suffices to describe the permutation Π_{AES}, and the AES key expansion PRG. We describe each in turn.

The AES round permutation. The permutation Π_{AES} is made up of a sequence of three invertible operations on the set $\{0,1\}^{128}$. The input 128-bits is organized as a 4×4 array of cells, where each cell is eight bits. The following three invertible operations are then carried out in sequence, one after the other, on this 4×4 array:

1. `SubBytes:` Let $S : \{0,1\}^8 \to \{0,1\}^8$ be a fixed permutation (a one-to-one function). This permutation is applied to each of the 16 cells, one cell at a time. The permutation S is specified in the AES standard as a hard-coded table of 256 entries. It is designed to have no fixed points, namely $S(x) \neq x$ for all $x \in \{0,1\}^8$, and no inverse fixed points, namely $S(x) \neq \bar{x}$ where $\bar{x}$ is the bit-wise complement of x. These requirements are needed to defeat certain attacks discussed in Section 4.3.1.

2. `ShiftRows:` This step performs a cyclic shift on the four rows of the input 4×4 array: the first row is unchanged, the second row is cyclically shifted one byte to the left, the third row is cyclically shifted two bytes, and the fourth row is cyclically shifted three bytes. In a diagram, this step performs the following transformation:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix} \Longrightarrow \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_5 & a_6 & a_7 & a_4 \\ a_{10} & a_{11} & a_8 & a_9 \\ a_{15} & a_{12} & a_{13} & a_{14} \end{pmatrix} \tag{4.11}$$

3. `MixColumns`: In this step the 4×4 array is treated as a matrix and this matrix is multiplied by a fixed matrix where arithmetic is interpreted in the finite field $\mathrm{GF}(2^8)$. Elements in the field $\mathrm{GF}(2^8)$ are represented as polynomials over $\mathrm{GF}(2)$ of degree less than eight where multiplication is done modulo the irreducible polynomial $x^8 + x^4 + x^3 + x + 1$. Specifically, the `MixColumns` transformation does:

$$\begin{pmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{pmatrix} \times \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_5 & a_6 & a_7 & a_4 \\ a_{10} & a_{11} & a_8 & a_9 \\ a_{15} & a_{12} & a_{13} & a_{14} \end{pmatrix} \Longrightarrow \begin{pmatrix} a'_0 & a'_1 & a'_2 & a'_3 \\ a'_5 & a'_6 & a'_7 & a'_4 \\ a'_{10} & a'_{11} & a'_8 & a'_9 \\ a'_{15} & a'_{12} & a'_{13} & a'_{14} \end{pmatrix} \tag{4.12}$$

   Here the scalars $01, 02, 03$ are interpreted as elements of $\mathrm{GF}(2^8)$ using their binary representation (e.g., 03 represents the element $x + 1$ in $\mathrm{GF}(2^8)$). This fixed matrix is invertible over $\mathrm{GF}(2^8)$ so that the entire transformation is invertible.

The permutation Π_{AES} used in the AES circuit of Fig. 4.11 is the sequential composition of the three permutation `SubBytes`, `ShiftRows`, and `MixColumns` in that order. In the very last round AES uses a slightly different function we call $\hat{\Pi}_{\text{AES}}$. This function is the same as Π_{AES} except that the MixColumns step is omitted. This omission is done so that the AES decryption circuit looks somewhat similar to the AES encryption circuit. Security implications of this omission are discussed in [34].

Because each step in Π_{AES} is easily invertible, the entire permutation Π_{AES} is easily invertible, as required for decryption.

Implementing AES using pre-computed tables. The AES round function is built from a permutation we called Π_{AES} defined as a sequence of three steps: `SubBytes`, `ShiftRows`, and `MixColumns`. The designers of AES did not intend for AES to be implemented that way on modern processors. Instead, they proposed an implementation of Π_{AES} the does all three steps at once using four fixed lookup tables called T_0, T_1, T_2, T_3.

To explain how this works, recall that Π_{AES} takes as input a 4×4 matrix $A = (a_i)_{i=0,\dots,15}$ and outputs a matrix $A' := \Pi_{\text{AES}}(A)$ of the same dimensions. Let us use $S[a]$ to denote the result of applying `SubBytes` to an input $a \in \{0,1\}^8$. Similarly, recall that the `MixColumns` step multiplies the current state by a fixed 4×4 matrix M. Let us use $M[i]$ to denote column number i of M, and $A'[i]$ to denote column number i of A'.

Now, looking at (4.12), we can write the four columns of the output of $\Pi_{\text{AES}}(A)$ as:

$$\begin{aligned} A'[0] &= M[0] \cdot S[a_0] + M[1] \cdot S[a_5] + M[2] \cdot S[a_{10}] + M[3] \cdot S[a_{15}] \\ A'[1] &= M[0] \cdot S[a_1] + M[1] \cdot S[a_6] + M[2] \cdot S[a_{11}] + M[3] \cdot S[a_{12}] \\ A'[2] &= M[0] \cdot S[a_2] + M[1] \cdot S[a_7] + M[2] \cdot S[a_8] + M[3] \cdot S[a_{13}] \\ A'[3] &= M[0] \cdot S[a_3] + M[1] \cdot S[a_4] + M[2] \cdot S[a_9] + M[3] \cdot S[a_{14}] \end{aligned} \tag{4.13}$$

where addition and multiplication is done in $\mathrm{GF}(2^8)$. Each column $M[i],\ i = 0,1,2,3$, is a vector of four bytes over $\mathrm{GF}(2^8)$, while the quantities $S[a_i]$ are 1-byte scalars in $\mathrm{GF}(2^8)$.

Every term in (4.13) can be evaluated quickly using a fixed pre-computed table. For $i = 0,1,2,3$ let us define a table T_i with 256 entries as follows:

$$\text{for } a \in \{0,1\}^8: \qquad T_i[a] := M[i] \cdot S[a] \quad \in \{0,1\}^{32}\ .$$

Plugging these tables into (4.13) gives a fast way to evaluate $\Pi_{\text{AES}}(A)$:

$$\begin{aligned}
A'[0] &= T_0[a_0] + T_1[a_5] + T_2[a_{10}] + T_3[a_{15}] \\
A'[1] &= T_0[a_1] + T_1[a_6] + T_2[a_{11}] + T_3[a_{12}] \\
A'[2] &= T_0[a_2] + T_1[a_7] + T_2[a_8] + T_3[a_{13}] \\
A'[3] &= T_0[a_3] + T_1[a_4] + T_2[a_9] + T_3[a_{14}]
\end{aligned}$$

The entire AES circuit written this way is a simple sequence of table lookups. Since each table T_i contains 256 entries, four bytes each, the total size of all four tables is 4KB. The circular structure of the matrix M makes it possible to compress the four tables to only 2KB with little impact on performance.

The one exception to (4.13) is the very last round of AES where the `MixColumns` step is omitted. To evaluate the last round we need a fifth 256-byte table S that only implements the `SubBytes` operation.

This optimization of AES is optional. Implementations in constrained environments where there is no room to store a 4KB table can choose to implement the three steps of Π_{AES} in code, which takes less than 4KB, but is not as fast. Thus AES can be adapted for both constrained and unconstrained environments.

As a word of caution, we note that a simplistic implementation of AES using this table lookup optimization is most likely vulnerable to cache timing attacks discussed in Section 4.3.2.

The AES-128 key expansion method. Looking back at Fig. 4.11 we see that key expansion for AES-128 needs to generate 11 rounds keys $k_0, \ldots, k_{10}$ where each round key is 128 bits. To do so, the 128-bit AES key is partitioned into four 32-bit words $w_{0,0}, w_{0,1}, w_{0,2}, w_{0,3}$ and these form the first round key k_0. The remaining ten round keys are generated sequentially: for $i = 1, \ldots, 10$, the 128-bit round key $k_i = (w_{i,0}, w_{i,1}, w_{i,2}, w_{i,3})$ is generated from the preceding round key $k_{i-1} = (w_{i-1,0}, w_{i-1,1}, w_{i-1,2}, w_{i-1,3})$ as follows:

$$\begin{aligned}
&w_{i,0} \leftarrow w_{i-1,0} \oplus g_i(w_{i-1,3}) \\
&w_{i,1} \leftarrow w_{i-1,1} \oplus w_{i,0} \\
&w_{i,2} \leftarrow w_{i-1,2} \oplus w_{i,1} \\
&w_{i,3} \leftarrow w_{i-1,3} \oplus w_{i,2} \quad .
\end{aligned}$$

Here the function $g_i : \{0,1\}^{32} \to \{0,1\}^{32}$ is a fixed function specified in the AES standard. It operates on its four byte input in three steps: (1) perform a one-byte left circular rotation on the 4-byte input, (2) apply `SubBytes` to each of the four bytes obtained, and (3) XOR the left most byte with a fixed round constant c_i. The round constants $c_1, \ldots, c_{10}$ are specified in the AES standard: round constant number i is the element x^{i-1} of the field $\text{GF}(2^8)$ treated as an 8-bit string.

The key expansion procedures for AES-192 and AES-256 are similar to those of AES-128. For AES-192 each iteration generates six 32-bit words (192 bits total) in a similar manner to AES-128, but only the first four 32-bit words (128 bits total) are used as the AES round key. For AES-256 each iteration generates eight 32-bit words (256 bits total) in a similar manner to AES-128, but only the first four 32-bit words (128 bits total) are used as the AES round key.

The AES key expansion method is intentionally designed to be invertible: given the last round key, one can work backwards to recover the full AES secret key k. The reason for this is to ensure that every AES-128 round key, on its own, has the same amount of entropy as the AES-128 secret key k. If AES-128 key expansion were not invertible then the last round key would not be uniform in $\{0,1\}^{128}$. Unfortunately, invertability also aids attacks: it is used in related key attacks and in side-channel attacks on AES, discussed next.

Security of AES. The AES algorithm withstood fairly sophisticated attempts at cryptanalysis lobbed at it. At the time of this writing, the best known attacks are as follows:

- **Key recovery:** Key recovery attacks refer to an adversary who is given multiple plaintext/ciphertext pairs and is able to recover the secret key from these pairs, as in an exhaustive search attack. The best known key recovery attack on AES-128 takes $2^{126.1}$ evaluations of AES [19]. This is about four times faster than exhaustive search and takes a prohibitively long time. Therefore this attack has little impact on the security of AES-128.

  The best known attack on AES-192 takes $2^{189.74}$ evaluation of AES which is again only about four times faster than exhaustive search. The best known attack on AES-256 takes $2^{254.42}$ evaluation of AES which is about three times faster than exhaustive search. None of these attacks impact the security of either AES variant.

- **Related key attacks:** In an ℓ-way related key attack the adversary is given ℓ lists of plaintext/ciphertext pairs: for $i = 1, \ldots, \ell$, list number i is generated using key k_i. The point is that all ℓ keys $k_1, \ldots, k_\ell$ must satisfy some fixed relation chosen by the adversary. The attacker's goal is to recover one of the keys, say k_1. In well-implemented cryptosystems, keys are always generated independently at random and are unlikely to satisfy the required relation. Therefore related key attacks do not typically affect correct crypto implementations.

  AES-256 is vulnerable to a related key attack that exploits its relatively simple key expansion mechanism [14]. The attack requires four related keys k_1, k_2, k_3, k_4 where the relation is a simple XOR relation: it requires that certain bits of the quantities $k_1 \oplus k_2$, $k_1 \oplus k_3$, and $k_2 \oplus k_4$ are set to specific values. Then given lists of plaintext/ciphertext pairs generated for each of the four keys, the attacker can recover the four keys in time $2^{99.5}$. This is far faster than the time it would take to mount an exhaustive search on AES-256. While the attack is quite interesting, it does not affect the security of AES-256 in well-implemented systems.

Hardware implementation of AES. At the time AES was standardized as a federal encryption standard most implementations were software based. The wide-spread adoption of AES in software products prompted all major processor vendors to extend their instruction set to add support for a hardware implementation of AES.

Intel, for example, added new instructions to its Xeon and Core families of processors called **AES-NI** (AES new instructions) that speed-up and simplify the process of using AES in software. The new instructions work as follows:

- `AESKEYGENASSIST`: runs the key expansion procedure to generate the AES round keys from the AES key.

- `AESENC`: runs one round of the AES encryption algorithm. The instruction is called as:

  ```
  AESENC xmm15, xmm1
  ```

  where the `xmm15` register holds the 128-bit data block and the `xmm1` register holds the 128-bit round key for that round. The resulting 128-bit data block is written to register `xmm15`.

Running this instruction nine times with the appropriate round keys loaded into registers `xmm1`, ..., `xmm9` executes the first nine rounds of AES encryption.

- `AESENCLAST`: invoked similar to `AESENC` to run last round of the AES algorithm. Recall that the last round function is different from the others: it omits the MixColumns step.
- `AESDEC` and `AESDECLAST`: runs one round of the AES decryption algorithm, analogous to the encryption instructions.

These AES-NI hardware instructions provide a significant speed-up over a heavily optimized software implementations of AES. Experiments by Emilia Käsper in 2009 show that on Intel Core 2 processors AES using the AES-NI instructions takes 1.35 cycles/byte (pipelined) while an optimized software implementation takes 7.59 cycles/byte.

In Intel's Skylake processors introduced in 2015 the `AESENC`, `AESDEC` and `AESENCLAST` instructions each take four cycles to complete. These instructions are fully pipelined so that a new instruction can be dispatched every cycle. In other words, Intel partitioned the execution of `AESENC` into a pipeline of four stages. Four AES blocks can be processed concurrently by different stages of the pipeline. While processing a single AES-128 block takes (4 cycles) $\times$ (10 rounds) $=$ 40 cycles (or 2.5 cycles/byte), processing four blocks in a pipeline takes only 44 cycles (or 0.69 cycles/byte). Hence, pipelining can speed up AES by almost a factor of four. As we will see in the next chapter, this plays an important role in choosing the exact method we use to encrypt long messages: it is best to choose an encryption method that can leverage the available parallelism to keep the pipeline busy.

Beyond speed, the hardware implementation of AES offers better security because it is resistant to the side-channel attacks discussed in the next section.

4.3 Sophisticated attacks on block ciphers

Widely deployed block ciphers like AES go through a lengthy selection process before they are standardized and continue to be subjected to cryptanalysis. In this section we survey some attack techniques that have been developed over the years.

In Section 4.3.1, we begin with attacks on the design of the cipher that may result in key compromise from observing plaintext/ciphertext pairs. Unlike brute-force exhaustive search attacks, these **algorithmic attacks** rely on clever analysis of the internal structure of a particular block cipher.

In Section 4.3.2, we consider a very different class of attacks, called **side-channel attacks**. In analyzing any cryptosystem, we consider scenarios in which an adversary interacts with the users of a cryptosystem. During the course of these interactions, the adversary collects information that may help it break the system. Throughout this book, we generally assume that this information is limited to the input/output behavior of the users (for example, plaintext/ciphertext pairs). However, this assumption ignores the fact that **computation is a physical process**. As we shall see, in some scenarios it is possible for the adversary to break a cryptosystem by measuring physical characteristics of the users' computations, for example, running time or power consumption.

Another class of attacks on the physical implementation of a cryptosystem is a **fault-injection attack**, which is discussed in Section 4.3.3. Finally, in Section 4.3.4, we consider another class of algorithmic attacks, in which the adversary can harness the laws of **quantum mechanics** to speed up its computations.

These clever attacks make two very important points:

1. Casual users of cryptography should only ever use standardized algorithms like AES, and not design their own block ciphers.

2. It is best to not implement algorithms on your own since, most likely the resulting implementations will be vulnerable to side-channel attacks; instead, it is better to use vetted implementations in widely used crypto libraries.

To further emphasize these points we encourage anyone who first learns about the inner-workings of AES to take the following entertaining pledge (originally due to Jeff Moser):

> I promise that once I see how simple AES really is, I will <u>not</u> implement it in production code even though it will be really fun. This agreement will remain in effect until I learn all about side-channel attacks and countermeasures to the point where I lose all interest in implementing AES myself.

4.3.1 Algorithmic attacks

Attacking the design of block ciphers is a vast field with many sophisticated techniques: linear cryptanalysis, differential cryptanalysis, slide attacks, boomerang attacks, and many others. We refer to [99] for a survey of the many elegant ideas that have been developed. Here we briefly describe a technique called *linear cryptanalysis* that has been used successfully against the DES block cipher. This technique, due to Matsui [72, 71], illustrates why designing efficient block-ciphers is so challenging. This method has been shown to not work against AES.

Linear cryptanalysis. Let (E, D) be a block cipher where data blocks and keys are bit strings. That is, $\mathcal{M} = \mathcal{C} = \{0,1\}^n$ and $\mathcal{K} = \{0,1\}^h$.

For a bit string $m \in \{0,1\}^n$ and a set of bit positions $S \subseteq \{0,\ldots,n-1\}$ we use $m[S]$ to denote the XOR of the bits in positions in S. That is, if $S = \{i_1,\ldots,i_\ell\}$ then $m[S] := m[i_1] \oplus \cdots \oplus m[i_\ell]$.

We say that the block cipher (E, D) has a **linear relation** if there exist sets of bit positions $S_0, S_1 \subseteq \{0,\ldots,n-1\}$ and $S_2 \subseteq \{0,\ldots,h-1\}$, such that *for all* keys $k \in \mathcal{K}$ and *for randomly chosen* $m \in \mathcal{M}$, we have

$$\Pr\Big[\; m[S_0] \oplus E(k,m)[S_1] = k[S_2] \;\Big] \geq \frac{1}{2} + \epsilon \tag{4.14}$$

for some non-negligible ϵ called the **bias**. For an "ideal" cipher the plaintext and ciphertext behave like independent strings so that the relation $m[S_0] \oplus E(k,m)[S_1] = k[S_2]$ in (4.14) holds with probability exactly $1/2$, and therefore $\epsilon = 0$. Surprisingly, the DES block cipher has a linear relation with a small, but non-negligible bias.

Let us see how a linear relation leads to an attack. Consider a cipher (E, D) that has a linear relation as in (4.14) for some non-negligible $\epsilon > 0$. We assume the linear relation is explicit so that the attacker knows the sets S_0, S_1 and S_2 used in the relation. Suppose that for some unknown secret key $k \in \mathcal{K}$ the attacker obtains many plaintext/ciphertext pairs (m_i, c_i) for $i = 1,\ldots,t$. We assume that the messages $m_1,\ldots,m_t$ are sampled uniformly and independently from $\mathcal{M}$ and that $c_i = E(k, m_i)$ for $i = 1,\ldots,t$. Using this information the attacker can learn one bit of information about the secret key k, namely the bit $k[S_2] \in \{0,1\}$ assuming sufficiently many plaintext/ciphertext pairs are given. The following lemma shows how.

Lemma 4.3. *Let (E, D) be a block cipher for which (4.14) holds. Let $m_1, \ldots, m_t$ be messages sampled uniformly and independently from the message space $\mathcal{M}$ and let $c_i := E(k, m_i)$ for $i = 1, \ldots, t$. Then*

$$\Pr\Big[\ k[S_2] = \text{Majority}_{i=1}^{t}(m_i[S_0] \oplus c_i[S_1])\ \Big] \geq 1 - e^{-t\epsilon^2/2}\ . \tag{4.15}$$

Here, Majority takes a majority vote on the given bits; for example, on input $(0, 0, 1)$, the majority is 0, and on input $(0, 1, 1)$, the majority is 1. The proof of the lemma is by a direct application of the Chernoff bound (Theorem **??**).

The bound in (4.15) shows that once the number of known plaintext/ciphertext pairs exceeds $4/\epsilon^2$, the output of the majority function equals $k[S_2]$ with more than 86% probability. Hence, the attacker can compute $k[S_2]$ from the given plaintext/ciphertext pairs and obtain one bit of information about the secret key. While this single key bit may not seem like much, it is a stepping stone towards a more powerful attack that can expose the entire key.

Linear cryptanalysis of DES. Matsui showed that 14-rounds of the DES block cipher has a linear relation where the bias is at least $\epsilon \geq 2^{-21}$. In fact, two linear relations are obtained: one by exploiting linearity in the DES encryption circuit and another from linearity in the DES decryption circuit. For a 64-bit plaintext m let m_{L} and m_{R} be the left and right 32-bits of m respectively. Similarly, for a 64-bit ciphertext c let c_{L} and c_{R} be the left and right 32-bits of c respectively. Then two linear relations for 14-rounds of DES are:

$$\begin{aligned} m_{\text{R}}[17, 18, 24] \oplus c_{\text{L}}[7, 18, 24, 29] \oplus c_{\text{R}}[15] &= k[S_{\text{e}}] \\ c_{\text{R}}[17, 18, 24] \oplus m_{\text{L}}[7, 18, 24, 29] \oplus m_{\text{R}}[15] &= k[S_{\text{d}}] \end{aligned} \tag{4.16}$$

for some bit positions $S_{\text{e}}, S_{\text{d}} \subseteq \{0, \ldots, 55\}$ in the 56-bit key k. Both relations have a bias of $\epsilon \geq 2^{-21}$ when applied to 14-rounds of DES.

These relations are extended to the entire 16-round DES by incorporating the first and last rounds of DES — rounds number 1 and 16 — into the relations. Let k_1 be the first round key and let k_{16} be the last round key. Then by definition of the DES round function we obtain from (4.16) the following relations on the entire 16-round DES circuit:

$$\Big(m_{\text{L}} \oplus F(k_1, m_{\text{R}})\Big)[17, 18, 24] \oplus c_{\text{R}}[7, 18, 24, 29] \oplus \Big(c_{\text{L}} \oplus F(k_{16}, c_{\text{R}})\Big)[15] = k[S'_{\text{e}}] \tag{4.17}$$

$$\Big(c_{\text{L}} \oplus F(k_{16}, c_{\text{R}})\Big)[17, 18, 24] \oplus m_{\text{R}}[7, 18, 24, 29] \oplus \Big(m_{\text{L}} \oplus F(k_1, m_{\text{R}})\Big)[15] = k[S'_{\text{d}}] \tag{4.18}$$

for appropriate bit positions $S'_{\text{e}}, S'_{\text{d}} \subseteq \{0, \ldots, 55\}$ in the 56-bit key.

Let us first focus on relation (4.17). Bits 17,18,24 of $F(k_1, m_{\text{R}})$ are the result of a single S-box and therefore they depend on only six bits of k_1. Similarly $F(k_{16}, c_{\text{R}})[15]$ depends on six bits of k_{16}. Hence, the left hand side of (4.17) depends on only 12 bits of the secret key k. Let us denote these 12 bits by $k^{(12)}$. We know that when the 12 bits are set to their correct value, the left hand side of (4.17), evaluated at a random plaintext/ciphertext pair, exhibits a bias of about 2^{-21} towards the bit $k[S'_{\text{e}}]$. When the 12 key bits of the key are set incorrectly one assumes that the bias in (4.17) is far less. As we will see, this has been verified experimentally.

This observation lets an attacker recover the 12 bits $k^{(12)}$ of the secret key k as follows. Given a list L of t plaintext/ciphertext pairs (e.g., $t = 2^{43}$) do:

- Step 1: for each of the 2^{12} candidates for the key bits $k^{(12)}$ compute the bias in (4.17). That is, evaluate the left hand side of (4.17) on all t plaintext/ciphertext pairs in L and let t_0 be the number of times that the expression evaluates to 0. The bias is computed as $\epsilon = |(t_0/t) - (1/2)|$. This produces a vector of 2^{12} biases, one for each candidate 12 bits for $k^{(12)}$.

- Step 2: sort the 2^{12} candidates by their bias, from largest to smallest. If the list L of given plaintext/ciphertext pairs is sufficiently large then the 12-bit candidate producing the highest bias is the most likely to be equal to $k^{(12)}$. This recovers 12 bits of the key. Once $k^{(12)}$ is known we can determine the bit $k[S'_{\mathrm{e}}]$ using Lemma 4.3, giving a total of 13 bits of k.

The relation (4.18) can be used to recover an additional 13 bits of the key k in exactly the same way. This gives the attacker a total 26 bits of the key. The remaining $56 - 26 = 30$ bits are recovered by exhaustive search.

Naively computing the biases in Step 1 takes time $2^{12} \times t$: for each candidate for $k^{(12)}$ one has to evaluate (4.17) on all t plaintext/ciphertext pairs in L. The following insight reduces the work to approximately time t. For a given pair (m, c), the left hand side of (4.17) can be computed from only thirteen bits of (m, c): six bits of m are needed to compute $F(k_1, m_{\mathrm{R}})[17, 18, 24]$, six bits of c are needed to compute $F(k_{16}, c_{\mathrm{R}})[15]$, and finally the single bit $m_{\mathrm{L}}[17, 18, 24] \oplus c_{\mathrm{R}}[7, 18, 24, 29] \oplus c_{\mathrm{L}}[15]$ is needed. These 13 bits are sufficient to evaluate the left hand side of (4.17) for any candidate key. Two plaintext/ciphertext pairs that agree on these 13 bits will always result in the same value for (4.17). We refer to these 13 bits as the *type* of the plaintext/ciphertext pair.

Before computing the biases in Step 1 we build a table of size 2^{13} that counts the number of plaintext/ciphertext pairs in L of each type. For $b \in \{0, 1\}^{13}$ table entry b is the number of plaintext/ciphertext pairs of type b. Constructing this table takes time t, but once the table is constructed computing all the biases in Step 1 can be done in time $2^{12} \times 2^{13} = 2^{25}$ which is much less than t. Therefore, the bulk of the work in Step 1 is counting the number of plaintext/ciphertext pairs of each type.

Matsui shows that given a list of 2^{43} plaintext/ciphertext pairs this attack succeeds with probability 85% using about 2^{43} evaluations of the DES circuit. Experimental results by Junod [61] show that with 2^{43} plaintext/ciphertext pairs, the correct 26 bits of the key are among the 2700 most likely candidates from Step 1 on average. In other words, the exhaustive search for the remaining 30 bits is carried out on average $2700 \approx 2^{11.4}$ times to recover the entire 56-bit key. Overall, the attack is dominated by the time to evaluate the DES circuit $2^{30} \times 2^{11.4} = 2^{41.4}$ times on average [61].

Lesson. Linear cryptanalysis of DES is possible because the fifth S-box, S_5, happens to be somewhat approximated by a linear function. The linearity of S_5 introduced a linear relation on the cipher that could be exploited to recover the secret key using 2^{41} DES evaluations, far less than the 2^{56} evaluations that would be needed in an exhaustive search. However, unlike exhaustive search, this attack requires a large number of plaintext/ciphertext pairs: the required 2^{43} pairs correspond to 64 *terabytes* of plaintext data. Nevertheless, this is a good illustration of how difficult it is to design secure block ciphers and why one should only use standardized and well-studied ciphers.

Linear cryptanalysis has been generalized over the years to allow for more complex non-linear relations among plaintext, ciphertext, and key bits. These generalizations have been used against other block ciphers such as LOKI91 and Q.

4.3.2 Side-channel attacks

Side-channel attacks do not attack the cryptosystem as a mathematical object. Instead, they exploit information inadvertently leaked by its physical implementation.

Consider an attacker who observes a cryptosystem as it operates on secret data, such as a secret key. The attacker can learn far more information than just the input/output behavior of the system. Two important examples are:

- **Timing side channel:** In a vulnerable implementation, the time it takes to encrypt a block of plaintext may depend on the value of the secret key. An attacker who measures encryption time can learn information about the key, as shown below.

- **Power side channel:** In a vulnerable implementation, the amount of power used by the hardware as it encrypts a block of plaintext can depend on the value of the secret key. An attacker who wants to extract a secret key from a device like a smartcard can measure the device's power usage as it operates and learn information about the key.

Many other side channels have been used to attack implementations: electromagnetic radiation emanating from a device as it encrypts, heat emanating from a device as it encrypts [79], and even sound [44].

Timing attacks

Timing attacks are a significant threat to crypto implementations. Timing information can be measured by a remote network attacker who interacts with a victim server and measures the server's response time to certain requests. For a vulnerable implementation, the response time can leak information about a secret key. Timing information can also be obtained by a local attacker on the same machine as the victim, for example, when a low-privilege process tries to extract a secret key from a high-privilege process. In this case, the attacker obtains very accurate timing measurements about its target. Timing attacks have been demonstrated in both the local and remote settings.

In this section, we describe a timing attack on AES that exploits memory caching behavior on the victim machine. We will assume that the adversary can accurately measure the victim's running time as it encrypts a block of plaintext with AES. The attack we present exploits timing variations due to caching in the machine's memory hierarchy.

Modern processors use a hierarchy of caches to speed up reads and writes to memory. The fastest layer, called the L1 cache, is relatively small (e.g. 64KB). Data is loaded into the L1 cache in blocks (called lines) of 64 bytes. Loading a line into L1 cache takes considerably more time than reading a line already in cache.

This cache-induced difference in timing leads to a devastating key recovery attack against the fast table-based implementation of AES presented on page 116. An implementation that ignores these caching effects will be easily broken by a timing attack.

Recall that the table-based implementation of AES uses four tables T_0, T_1, T_2, T_3 for all but the last round. The last round does not include the MixColumns step and evaluation of this last round uses an explicit S table instead of the tables T_0, T_1, T_2, T_3. Suppose that when each execution of AES begins, the S table is not in the L1 cache. The first time a table entry is read, that part of the table will be loaded into L1 cache. Consequently, this first read will be slow, but subsequent reads to the same entry will be much faster since the data is already cached. Since the S table is only used in the last round of AES no parts of the table will be loaded in cache prior to the last round.

Letting $A = (a_i)_{i=0,\dots,15}$ denote the 4×4 input to the last round, and letting $(w_i)_{i=0,\dots,15}$ denote the 4×4 last round key, the final AES output is computed as the 4×4 matrix:

$$C = (c_{i,j}) = \begin{pmatrix} S[a_0]+w_0 & S[a_1]+w_1 & S[a_2]+w_2 & S[a_3]+w_3 \\ S[a_5]+w_4 & S[a_6]+w_5 & S[a_7]+w_6 & S[a_4]+w_7 \\ S[a_{10}]+w_8 & S[a_{11}]+w_9 & S[a_8]+w_{10} & S[a_9]+w_{11} \\ S[a_{15}]+w_{12} & S[a_{12}]+w_{13} & S[a_{13}]+w_{14} & S[a_{14}]+w_{15} \end{pmatrix} \tag{4.19}$$

The attacker is given this final output C.

To mount the attack, consider two consecutive entries in the output matrix C, say $c_0 = S[a_0]+w_0$ and $c_1 = S[a_1]+w_1$. Subtracting one equation from the other we see that when $a_0 = a_1$ the following relation holds:

$$c_0 - c_1 = w_0 - w_1 \ .$$

Therefore, with $\Delta := w_0 - w_1$ we have that $c_0 - c_1 = \Delta$ whenever $a_0 = a_1$. Moreover, when $a_0 \neq a_1$ the structure of the S table ensures that $c_0 - c_1 \neq \Delta$.

The key insight is that whenever $a_0 = a_1$, reading $S[a_0]$ loads the a_0 entry of S into the L1 cache so that the second access to this entry via $S[a_1]$ is much faster. However, when $a_0 \neq a_1$ it is possible that both reads miss the L1 cache so that both are slow. Therefore, when $a_0 = a_1$ the expected running time of the entire AES cipher is slightly less than when $a_0 \neq a_1$.

The attacker's plan now is to run the victim AES implementation on many random input blocks and measure the running time. For each value of $\Delta \in \{0,1\}^8$ the attacker creates a list L_Δ of all output ciphertexts where $c_0 - c_1 = \Delta$. For each Δ-value it computes the average running time among all ciphertexts in L_Δ. Given enough samples, the lowest average running time is obtained for the Δ-value satisfying $\Delta = w_0 - w_1$. Hence, timing information reveals one linear relation about the last round key: $w_0 - w_1 = \Delta$.

Suppose the implementation evaluates the terms of (4.19) in some sequential order. Repeating the timing procedure above for different consecutive pairs c_i and c_{i+1} in C reveals the difference in $\mathrm{GF}(2^8)$ between every two consecutive bytes of the last round key. Then if the first byte of the last round key is known, all remaining bytes of the last round key can be computed from the known differences. Moreover, since key expansion in AES-128 is invertible, it is a simple matter to reconstruct the AES-128 secret key from the last round key.

To complete the attack, the attacker simply tries all 256 possible values for the first byte of last round key. For each candidate value the attacker obtains a candidate AES-128 key. This key can be tested by trying it out on a few known plaintext/ciphertext pairs. Once a correct AES-128 key is found, the attacker has obtained the desired key.

This attack, due to Bonneau and Mironov [23], works quite well in practice. Their experiments on a Pentium IV Xeon successfully recovered the AES secret key using about 2^{20} timing measurements of the encryption algorithm. The attack only takes a few minutes to run. We note that the Pentium IV Xeon uses 32-byte cache lines so that the S table is split across eight lines.

Mitigations. The simplest approach to defeat timing attacks on AES is to use the AES-NI instructions that implement AES in hardware. These instructions are faster than a software implementation and always take the same amount of time, independent of the key or input data.

On processors that do not have built-in AES instructions one is forced to use a software implementation. One approach to mitigate cache-timing attacks is to use a table-free implementation of AES. Several such implementations of AES using a technique called **bit-slicing** provide reasonable performance in software and are supposedly resistant to timing attacks.

Another approach is to pre-load the tables T_0, T_1, T_2, T_3 and S into $L1$ cache before every invocation of AES. This prevents the cache-based timing attack, but only if the tables are not evicted from $L1$ cache while AES is executing. Ensuring that the tables stay in $L1$ cache is non-trivial on a modern processor. Interrupts during AES execution can evict cache lines. Similarly, *hyperthreading* allows for multiple threads to execute concurrently on the same core. While one thread pre-loads the AES tables into $L1$ cache another thread executing concurrently can inadvertently evict them.

Yet another approach is to pad AES execution to the maximum possible time to prevent timing attacks, but this has a non-negligible impact on performance.

To conclude, we emphasize that the following mitigation does not work: adding a random number of instructions at the end of every AES execution to randomly pad the running time does not prevent the attack. The attacker can overcome this by simply obtaining more samples and averaging out the noise.

Power attacks on AES implementations

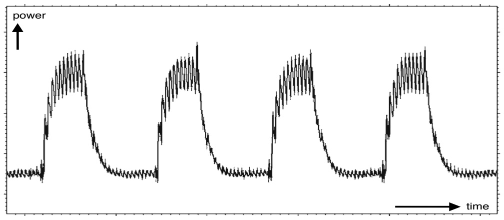
The amount of power consumed by a device as it operates can leak information about the inner-workings of the device, including secret keys stored on the device. Let us see how an attacker can use power measurements to quickly extract secret keys from a physical device.

As an example, consider a credit-card with an embedded chip where the chip contains a secret AES key. To make a purchase the user plugs the credit-card into a point-of-sale terminal. The terminal provides the card with the transaction details and the card authorizes the transaction using the secret embedded AES key. We leave the exact details for how this works to a later chapter.

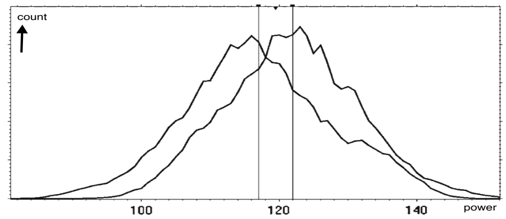
Since the embedded chip must draw power from the terminal (it has no internal power source) it is quite easy for the terminal to measure the amount of power consumed by the chip at any given time. In particular, an attacker can measure the amount of power consumed as the AES algorithm is evaluated. Fig. 4.12a shows a test device's power consumption as it evaluates the AES-128 algorithm four times (the x-axis is time and y-axis is power). Each hump is one run of AES and within each hump the ten rounds of AES-128 are clearly visible.

Simple power analysis. Suppose an implementation contains a branch instruction that depends on a bit of the secret key. Say, the branch is taken when the least significant bit of the key is '1' and not taken otherwise. Since taking a branch requires more power than not taking it, the power trace will show a spike at the branch point when the key bit is one and no spike otherwise. An attacker can simply look for a spike at the appropriate point in the power trace and learn that bit of the key. With multiple key-dependent branch instructions the entire secret key can be extracted. This works quite well against simple implementations of certain cryptosystems (such as RSA, which is covered in a later chapter).

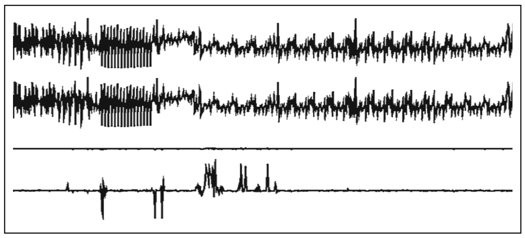
The attack of the previous paragraph, called **simple power analysis** (SPA), will not work on AES: during encryption the secret AES round keys are simply XORed into the cipher state. The power used by the XOR instruction only marginally depends on its operands and therefore



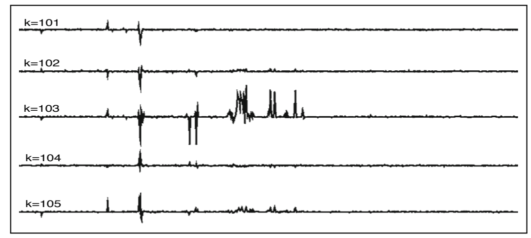
(a) Power used by four iterations of AES



(b) S-box LSB output is 0 vs. 1



(c) Power differential



(d) Differential for keys $k = 101, \ldots, 105$

Figure 4.12: AES differential power analysis (source: Kocher et al. [64])

the power used by the XOR reveals no useful information about the secret key. This resistance to simple power analysis was an attractive feature of AES.

Differential power analysis. Despite AES's resistance to SPA, a more sophisticated power analysis attack successfully extracts the AES secret key from simple implementations. Choose an AES key k at random and encrypt 4000 random plaintexts using the key k. For our test device the resulting 4000 power traces look quite different from each other indicating that the power trace is input dependent, the input being the random plaintext.

Next, consider the output of the first S-box in the first round. Call this output T. We hypothesize that the power consumed by the S-box lookup depends on the index being looked up. That is, we guess that the value of T is correlated with the power consumed by the table lookup instruction.

To test the hypothesis, let us split the 4000 traces into two piles according to the least significant bit of T: pile 1 contains traces where the LSB of T is 1 and pile 0 contains traces where the bit is 0. Consider the power consumed by traces in each pile at the moment in time when the card computes the output of the first S-box:

pile 1 (LSB = 1):	mean power 116.9 units, standard deviation 10.7
pile 0 (LSB = 0):	mean power 121.9 units, standard deviation 9.7

The two power distributions are shown in Fig. 4.12b. The distributions are close, but clearly different. Hence, with enough independent samples we can distinguish one distribution from the other.

To exploit this observation, consider Fig. 4.12c. The top line shows the power trace averaged over all traces in pile 1. The second line shows the power trace averaged over all traces in pile 0. The bottom line shows the difference between the two top traces, magnified by a factor of 15. The first spike in the bottom line is exactly at the time when the card computed the output of the first S-box. The size of the spike corresponds exactly to the difference in averages shown in Fig. 4.12b. This bottom line is called the **power differential**.

To attack a target device the attacker must first experiment with a clean device: the attacker loads a chosen secret key into the device and computes the power differential curve for the device as shown in Fig. 4.12c. Next, suppose the attacker obtains a device with an unknown embedded key. It can extract the key as follows:

first, measure the power trace for 4000 random plaintexts
next, for each candidate first byte $k \in \{0,1\}^8$ of the key do:
  split the 4000 samples into two piles according to the first bit of T
    (this is done using the current guess for k and the 4000 known plaintexts)
  if the resulting power differential curve matches the pre-computed curve:
    output k as the first byte of the key and stop

Fig. 4.12d shows this attack in action. When using the correct value for the first byte of the key ($k = 103$) we obtain the correct power differential curve. When the wrong guess is used ($k = 101, 102, 104, 105$) the power differential does not match the expected curve.

Iterating this procedure for all 16 bytes of the AES-128 key recovers the entire key.

Mitigations. A common defense against power analysis uses hardware tweaks. Conceptually, prior to executing AES the hardware draws a fixed amount of power to charge a capacitor and then runs the entire AES algorithm using power in the capacitor. Once AES is done the excess power left in the capacitor is discarded. The next application of AES again charges the capacitor and so on. This conceptual design (which takes some effort to implement correctly in practice) ensures that the device's power consumption is independent of secret keys embedded in the device.

Another mitigation approach concedes that some limited information about the secret key leaks every time the decryption algorithm runs. The goal is to then preemptively re-randomize the secret key after each invocation of the algorithm so that the attacker cannot combine the bits of information he learns from each execution. This approach is studied in an area called **leakage-resilient cryptography**.

4.3.3 Fault-injection attacks on AES

Another class of implementation attacks, called **fault injection attacks**, attempt to deliberately cause the hardware to introduce errors while running the cryptosystem. An attacker can exploit the malformed output to learn information about the secret key. Injecting faults can be done by over-clocking the target hardware, by heating it using a laser, or by directing electromagnetic interference at the target chip [60].

Fault injection attacks have been used to break vulnerable implementations of AES by causing the AES engine to malfunction during encryption of a plaintext block. The resulting malformed ciphertext can reveal information about the secret key [60]. Fault attacks are easiest to describe in the context of public-key systems and we will come back and discuss them in detail in Section **??** where we show how they result in a complete break of some implementations of RSA.

One defense against fault injection attacks is to always check the result of the computation. For example, an AES engine could check that the computed AES ciphertext correctly decrypts to the given input plaintext. If the check fails, the hardware outputs an error and discards the computed ciphertext. Unfortunately this slows down AES performance by a factor of two and is hardly done in practice.

4.3.4 Quantum exhaustive search attacks

All the attacks described so far work on classical computers available today. Our physical world, however, is governed by the laws of quantum mechanics. In theory, computers can be built to use these laws to solve problems in much less time than would be required on a classical computer. Although no one has yet succeeded in building quantum computers, it could be just be a matter of time before the first quantum computer is built.

Quantum computers have significant implications to cryptography because they can be used to speed up certain attacks and even completely break some systems. Consider again a block cipher (E, D) with key space $\mathcal{K}$. Recall that in a classical exhaustive search the attacker is given a few plaintext/ciphertext pairs created with some key $k \in \mathcal{K}$ and the attacker tries all keys until he finds a key that maps the given plaintexts to the given ciphertexts. On a classical computer this takes time proportional to $|\mathcal{K}|$.

Quantum exhaustive search. Surprisingly, on a quantum computer the same exhaustive search problem can be solved in time proportional to only $\sqrt{|\mathcal{K}|}$. For block ciphers like AES-128 this means that exhaustive search will only require about $\sqrt{2^{128}} = 2^{64}$ steps. Computations involving 2^{64} steps can already be done in a reasonable amount of time using classical computers and therefore one would expect that once quantum computers are built they will also be capable of carrying out this scale of computations. As a result, once quantum computers are built, AES-128 will be considered insecure.

The above discussion suggests that for a block cipher to resist a quantum exhaustive search attack its key space $|\mathcal{K}|$ must have at least 2^{256} keys, so that the time for quantum exhaustive search is on the order of 2^{128}. This threat of quantum computers is one reason why AES supports 256-bits keys. Of course, we have no guarantees that there is not a faster quantum algorithm for breaking the AES-256 block cipher, but at least quantum exhaustive search is out of the question.

Grover's algorithm. The algorithm for quantum exhaustive search is a special case of a more general result in quantum computing due to Lov Grover [49]. The result says the following: suppose we are given a function $f : \mathcal{K} \to \{0, 1\}$ defined as follows

$$f(k) = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

for some $k_0 \in \mathcal{K}$. The goal is to find k_0 given only "black-box" access to f, namely by only querying f at different inputs. On a classical computer it is clear that the best algorithm is to try all possible $k \in \mathcal{K}$ and this takes $|\mathcal{K}|$ queries to f in the worse case.

Grover's algorithm shows that k_0 can be found on a quantum computer in only $O\big(\sqrt{|\mathcal{K}|}\cdot\text{time}(f)\big)$ steps, where $\text{time}(f)$ is the time to evaluate $f(x)$. This is a very general result that holds for all functions f of the form shown in (4.20). This can be used to speed-up general hard optimization problems and is the "killer app" for quantum computers.

To break a block cipher like AES-128 given a few plaintext/ciphertext pairs we would define the function:

$$f_{\text{AES}}(k) = \begin{cases} 1 & \text{if } AES(k, \overline{m}) = \overline{c} \\ 0 & \text{otherwise} \end{cases}$$

where $\overline{m} = (m_0, \ldots, m_Q)$ and $\bar{c} = (c_0, \ldots, c_Q)$ are the given ciphertext blocks. Assuming enough block are given, there is a unique key $k_0 \in \mathcal{K}$ that satisfies $AES(k, \overline{m}) = \bar{c}$ and this key can be found in time proportional to $\sqrt{|\mathcal{K}|}$ using Grover's algorithm.

4.4 Pseudo-random functions: basic definitions and properties

While secure block ciphers are the building block of many cryptographic systems, a closely related concept, called a pseudo-random function (or PRF), turns out to be the right tool in many applications. PRFs are conceptually simpler objects than block ciphers and, as we shall see, they have a broad range of applications. PRFs and block ciphers are so closely related that we can use secure block ciphers as a stand in for secure pseudo-random functions (under certain assumptions). This is quite nice, because as we saw in the previous section, we have available to us a number of very practical, and plausibly secure block ciphers.

4.4.1 Definitions

A **pseudo-random function (PRF)** F is a deterministic algorithm that has two inputs: a key k and an **input data block** x; its output $y := F(k, x)$ is called an **output data block**. As usual, there are associated, finite spaces: the key space $\mathcal{K}$, in which k lies, the input space $\mathcal{X}$, in which x lies, and the output space $\mathcal{Y}$, in which y lies. We say that F is **defined over** $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$.

Intuitively, our notion of security for a pseudo-random function says that for a randomly chosen key k, the function $F(k, \cdot)$ should — for all practical purposes — "look like" a random function from $\mathcal{X}$ to $\mathcal{Y}$. To make this idea more precise, let us first introduce some notation:

$$\text{Funs}[\mathcal{X}, \mathcal{Y}]$$

denotes the set of *all* functions $f : \mathcal{X} \to \mathcal{Y}$. This is a very big set:

$$|\text{Funs}[\mathcal{X}, \mathcal{Y}]| = |\mathcal{Y}|^{|\mathcal{X}|}.$$

We also introduce an attack game:

Attack Game 4.2 (PRF). For a given PRF F, defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define:

Experiment b:

- The challenger selects $f \in \text{Funs}[\mathcal{X}, \mathcal{Y}]$ as follows:

  if $b = 0$: $k \xleftarrow{R} \mathcal{K}$, $f \leftarrow F(k, \cdot)$;
  if $b = 1$: $f \xleftarrow{R} \text{Funs}[\mathcal{X}, \mathcal{Y}]$.

- The adversary submits a sequence of queries to the challenger.

  For $i = 1, 2, \ldots$, the ith query is an input data block $x_i \in \mathcal{X}$.

  The challenger computes $y_i \leftarrow f(x_i) \in \mathcal{Y}$, and gives y_i to the adversary.

- The adversary computes and outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to F as

$$\text{PRF}\mathsf{adv}[\mathcal{A}, F] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \tag{4.21}$$

Finally, we say that $\mathcal{A}$ is a **Q-query PRF adversary** if $\mathcal{A}$ issues at most Q queries. □

Definition 4.2 (secure PRF). *A PRF F is **secure** if for all efficient adversaries $\mathcal{A}$, the value* $\text{PRF}\mathsf{adv}[\mathcal{A}, F]$ *is negligible.*

Again, we stress that the queries made by the challenger in Attack Game 4.2 are allowed to be *adaptive*: the adversary is allowed to concoct each query in a way that depends on the previous responses from the challenger (see Exercise 4.6).

As discussed in Section 2.3.5, Attack Game 4.2 can be recast as a "bit guessing" game, where instead of having two separate experiments, the challenger chooses $b \in \{0, 1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$. In this game, we measure $\mathcal{A}$'s *bit-guessing advantage* $\text{PRF}\mathsf{adv}^*[\mathcal{A}, F]$ as $|\Pr[\hat{b} = b] - 1/2|$. The general result of Section 2.3.5 (namely, (2.13)) applies here as well:

$$\text{PRF}\mathsf{adv}[\mathcal{A}, F] = 2 \cdot \text{PRF}\mathsf{adv}^*[\mathcal{A}, F]. \tag{4.22}$$

Weakly secure PRFs. For certain constructions that use PRFs it suffices that the PRF satisfy a weaker security property than Definition 4.2. We say that a PRF is *weakly secure* if no efficient adversary can distinguish the PRF from a random function when its queries are severely restricted: it can only query the function at *random* points in the domain. Restricting the adversary's queries to random inputs makes it potentially easier to build weakly secure PRFs. In Exercise 4.2 we examine natural PRF constructions that are weakly secure, but not fully secure.

We define weakly secure PRFs by slightly modifying Attack Game 4.2. Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$. We modify the way in which an adversary $\mathcal{A}$ interacts with the challenger: whenever the adversary queries the function, the challenger chooses a random $x \in \mathcal{X}$ and sends both x and $f(x)$ to the adversary. In other words, the adversary sees evaluations of the function f at *random* points in $\mathcal{X}$ and needs to decide whether the function is truly random or pseudorandom. We define the adversary's advantage in this game, denoted $\text{wPRF}\mathsf{adv}[\mathcal{A}, F]$, as in (4.21).

Definition 4.3 (weakly secure PRF). *A PRF F is **weakly secure** if for all efficient adversaries $\mathcal{A}$, the value* $\text{wPRF}\mathsf{adv}[\mathcal{A}, F]$ *is negligible.*

4.4.2 Efficient implementation of random functions

Just as in Section 4.1.2, we can implement the random function chosen from $\text{Funs}[\mathcal{X}, \mathcal{Y}]$ used by the challenger in Experiment 1 of Attack Game 4.2 by a **faithful gnome**. Just as in the block cipher case, the challenger keeps track of input/output pairs (x_i, y_i). When the challenger receives the ith query x_i, he tests whether $x_i = x_j$ for some $j < i$; if so, he sets $y_i \leftarrow y_j$ (this ensures that the challenger implements of function); otherwise, he chooses y_i at random from the set $\mathcal{Y}$; finally, he sends y_i to the adversary. We can write the logic of this implementation of the challenger as follows:

upon receiving the ith query $x_i \in \mathcal{X}$ from $\mathcal{A}$ do:
    if $x_i = x_j$ for some $j < i$
        then $y_i \leftarrow y_j$
        else $y_i \xleftarrow{R} \mathcal{Y}$
    send y_i to $\mathcal{A}$.

4.4.3 When is a secure block cipher a secure PRF?

In this section, we ask the question: when is a secure block cipher a secure PRF? In answering this question, we introduce a proof technique that is used heavily throughout cryptography.

Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{X})$, and let $N := |\mathcal{X}|$. We may naturally view E as a PRF, defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$. Now suppose that $\mathcal{E}$ is a secure block cipher; that is, no efficient adversary can effectively distinguish E from a random permutation. Does this imply that E is also a secure PRF? That is, does this imply that no efficient adversary can effectively distinguish E from a random function?

The answer to this question is "yes," provided N is super-poly. Before arguing this, let us argue that the answer is "no" when N is small.

Consider a PRF adversary playing Attack Game 4.2 with respect to E. Let f be the function chosen by the challenger: in Experiment 0, $f = E(k, \cdot)$ for random $k \in \mathcal{K}$, while in Experiment 1, f is randomly chosen from $\text{Funs}[\mathcal{X}, \mathcal{X}]$. Suppose that N is so small that an efficient adversary can afford to obtain the value of $f(x)$ for all $x \in \mathcal{X}$. Moreover, our adversary $\mathcal{A}$ outputs 1 if it sees that $f(x) = f(x')$ for two distinct values $x, x' \in \mathcal{X}$, and outputs 0 otherwise. Clearly, in Experiment 0, $\mathcal{A}$ outputs 1 with probability 0, since $E(k, \cdot)$ is a permutation. However, in Experiment 1, $\mathcal{A}$ outputs 1 with probability $1 - N!/N^N \geq 1/2$. Thus, $\text{PRF}\mathsf{adv}[\mathcal{A}, E] \geq 1/2$, and so E is not a secure PRF.

The above argument can be refined using the Birthday Paradox (see Section B.1). For any poly-bounded Q, we can define an efficient PRF adversary $\mathcal{A}$ that plays Attack Game 4.2 with respect to E, as follows. Adversary $\mathcal{A}$ simply makes Q distinct queries to its challenger, and outputs 1 iff it sees that $f(x) = f(x')$ for two distinct values $x, x' \in \mathcal{X}$ (from among the Q values given to the challenger). Again, in Experiment 0, $\mathcal{A}$ outputs 1 with probability 0; however, by Theorem B.1, in Experiment 1, $\mathcal{A}$ outputs 1 with probability at least $\min\big\{Q(Q-1)\big/4N, 0.63\big\}$. Thus, by making just $O(N^{1/2})$ queries, an adversary can easily see that a permutation does not behave like a random function.

It turns out that the "birthday attack" is about the best that any adversary can do, and when N is super-poly, this attack becomes infeasible:

Theorem 4.4 (PRF Switching Lemma). *Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{X})$, and let $N := |\mathcal{X}|$. Let $\mathcal{A}$ be an adversary that makes at most Q queries to its challenger. Then*

$$\Big|\text{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}] - \text{PRF}\mathsf{adv}[\mathcal{A}, E]\Big| \leq Q^2/2N.$$

Before proving this theorem, we derive the following simple corollary:

Corollary 4.5. *Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{X})$, and assume that $N := |\mathcal{X}|$ is super-poly. Then $\mathcal{E}$ is a secure block cipher if and only if E is a secure PRF.*

Proof. By definition, if $\mathcal{A}$ is an efficient adversary, the maximum number of queries Q it makes to its challenger is poly-bounded. Therefore, by Theorem 4.4, we have

$$\Big|\text{BC}\mathsf{adv}[\mathcal{A},\mathcal{E}] - \text{PRF}\mathsf{adv}[\mathcal{A},E]\Big| \le Q^2/2N$$

Since N is super-poly and Q is poly-bounded, the value $Q^2/2N$ is negligible (see Fact 2.6). It follows that $\text{BC}\mathsf{adv}[\mathcal{A},\mathcal{E}]$ is negligible if and only if $\text{PRF}\mathsf{adv}[\mathcal{A},E]$ is negligible. □

Actually, the proof of Theorem 4.4 has nothing to do with block ciphers and PRFs — it is really an argument concerning random permutations and random functions. Let us define a new attack game that tests an adversary's ability to distinguish a random permutation from a random function.

Attack Game 4.3 (permutation vs. function). For a given finite set $\mathcal{X}$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define:

Experiment b:

- The challenger selects $f \in \text{Funs}[\mathcal{X},\mathcal{X}]$ as follows:

  if $b = 0$: $f \xleftarrow{\text{R}} \text{Perms}[\mathcal{X}]$;
  if $b = 1$: $f \xleftarrow{\text{R}} \text{Funs}[\mathcal{X},\mathcal{X}]$.

- The adversary submits a sequence of queries to the challenger.

  For $i = 1, 2, \ldots$, the ith query is an input data block $x_i \in \mathcal{X}$.

  The challenger computes $y_i \leftarrow f(x_i) \in \mathcal{Y}$, and gives y_i to the adversary.

- The adversary computes and outputs a bit $\hat{b} \in \{0,1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to $\mathcal{X}$ as

$$\text{PF}\mathsf{adv}[\mathcal{A},\mathcal{X}] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \square$$

Theorem 4.6. *Let $\mathcal{X}$ be a finite set of size N. Let $\mathcal{A}$ be an adversary that makes at most Q queries to its challenger. Then*

$$\text{PF}\mathsf{adv}[\mathcal{A},\mathcal{X}] \le Q^2/2N.$$

We first show that the above theorem easily implies Theorem 4.4:

Proof of Theorem 4.4. Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K},\mathcal{X})$. Let $\mathcal{A}$ be an adversary that makes at most Q queries to its challenger. We define Games 0, 1, and 2, played between $\mathcal{A}$ and a challenger. For $j = 0, 1, 2$, we define p_j to be the probability that $\mathcal{A}$ outputs 1 in Game j. In each game, the challenger chooses a function $f : \mathcal{X} \to \mathcal{X}$ according to a particular distribution, and responds to each query $x \in \mathcal{X}$ made by $\mathcal{A}$ with the value $f(x)$.

Game 0: The challenger in this game chooses $f := E(k, \cdot)$, where $k \in \mathcal{K}$ is chosen at random.

Game 1: The challenger in this game chooses $f \in \text{Perms}[\mathcal{X}]$ at random.

Game 2: The challenger in this game chooses $f \in \text{Funs}[\mathcal{X},\mathcal{X}]$ at random.

Observe that by definition,

$$|p_1 - p_0| = \mathrm{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}],$$

$$|p_2 - p_0| = \mathrm{PRF}\mathsf{adv}[\mathcal{A}, E],$$

and that by Theorem 4.6,

$$|p_2 - p_1| = \mathrm{PF}\mathsf{adv}[\mathcal{A}, \mathcal{X}] \le Q^2/2N.$$

Putting these together, we get

$$\big|\mathrm{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}] - \mathrm{PRF}\mathsf{adv}[\mathcal{A}, E]\big| = \big||p_1 - p_0| - |p_2 - p_0|\big| \le |p_2 - p_1| \le Q^2/2N,$$

which proves the theorem. □

So it remains to prove Theorem 4.6. Before doing so, we state and prove a very simple, but extremely useful fact:

Theorem 4.7 (Difference Lemma). *Let Z, W_0, W_1 be events defined over some probability space. Suppose that $W_0 \wedge \bar{Z}$ occurs if and only if $W_1 \wedge \bar{Z}$ occurs. Then we have*

$$|\Pr[W_0] - \Pr[W_1]| \le \Pr[Z].$$

Proof. This is a simple calculation. We have

$$\begin{aligned} |\Pr[W_0] - \Pr[W_1]| &= |\Pr[W_0 \wedge Z] + \Pr[W_0 \wedge \bar{Z}] - \Pr[W_1 \wedge Z] - \Pr[W_1 \wedge \bar{Z}]| \\ &= |\Pr[W_0 \wedge Z] - \Pr[W_1 \wedge Z]| \\ &\le \Pr[Z]. \end{aligned}$$

The second equality follows from the assumption that $W_0 \wedge \bar{Z} \iff W_1 \wedge \bar{Z}$, and so in particular, $\Pr[W_0 \wedge \bar{Z}] = \Pr[W_1 \wedge \bar{Z}]$. The final inequality follows from the fact that both $\Pr[W_0 \wedge Z]$ and $\Pr[W_1 \wedge Z]$ are numbers between 0 and $\Pr[Z]$. □

In most of our applications of the Difference Lemma, W_0 will represent the event that a given adversary outputs 1 in some game against a certain challenger, while W_1 will be the event that the same adversary outputs 1 in a game played against a different challenger. To apply the Difference Lemma, we define these two games so that they both operate on the same underlying probability space. This means that we view the random choices made by both the adversary and the challenger as the same in both games — all that differs between the two games is the rule used by the challenger to compute its responses to the adversary's queries.

Proof of Theorem 4.6. Consider an adversary $\mathcal{A}$ that plays Attack Game 4.3 with respect to $\mathcal{X}$, where $N := |\mathcal{X}|$, and assume that $\mathcal{A}$ makes at most Q queries to the challenger. Consider Experiment 0 of this attack game. Using the "faithful gnome" idea discussed in Section 4.4.2, we can implement Experiment 0 by keeping track of input/output pairs (x_i, y_i); moreover, it will be convenient to choose initial "default" values z_i for y_i, where the values $z_1, \ldots, z_Q$ are chosen uniformly and independently at random from $\mathcal{X}$; these "default" values are over-ridden, if necessary, to ensure the challenger defines a random permutation. Here are the details:

$z_1, \ldots, z_Q \xleftarrow{\text{R}} \mathcal{X}$
upon receiving the ith query x_i from $\mathcal{A}$ do:
  if $x_i = x_j$ for some $j < i$ then
    $y_i \leftarrow y_j$
  else
    $y_i \leftarrow z_i$
(∗)   if $y_i \in \{y_1, \ldots, y_{i-1}\}$ then $y_i \xleftarrow{\text{R}} \mathcal{X} \setminus \{y_1, \ldots, y_{i-1}\}$
  send y_i to $\mathcal{A}$.

The line marked (∗) tests if the default value z_i needs to be over-ridden to ensure that no output is for two distinct inputs.

Let W_0 be the event that $\mathcal{A}$ outputs 1 in this game, which we call Game 0.

We now obtain a different game by modifying the above implementation of the challenger:

$z_1, \ldots, z_Q \xleftarrow{\text{R}} \mathcal{X}$
upon receiving the ith query x_i from $\mathcal{A}$ do:
  if $x_i = x_j$ for some $j < i$ then
    $y_i \leftarrow y_j$
  else
    $y_i \leftarrow z_i$
  send y_i to $\mathcal{A}$.

All we have done is dropped line marked (∗) in the original challenger: our "faithful gnome" becomes a "forgetful gnome," and simply forgets to make the output consistency check.

Let W_1 be the event that $\mathcal{A}$ outputs 1 in the game played against this modified challenger, which we call Game 1.

Observe that Game 1 is equivalent to Experiment 1 of Attack Game 4.3; in particular, $\Pr[W_1]$ is equal to the probability that $\mathcal{A}$ outputs 1 in Experiment 1 of Attack Game 4.3. Therefore, we have

$$\text{PF}\mathsf{adv}[\mathcal{A}, \mathcal{X}] = |\Pr[W_0] - \Pr[W_1]|.$$

We now want to apply the Difference Lemma. To do this, both games are understood to operate on the *same* underlying probability space. All of the random choices made by the adversary and challenger are the same in both games — all that differs is the rule used by the challenger to compute its responses. In particular, this means that the random choices made by $\mathcal{A}$, as well as the values $z_1, \ldots, z_Q$ chosen by the challenger, not only have identical distributions, but are *literally the same values* in both games.

Define Z to be the event that $z_i = z_j$ for some $i \neq j$. Now suppose we run Game 0 and Game 1, and event Z does not occur. This means that the z_i values are all distinct. Now, since the adversary's random choices are the same in both games, its first query in both games is the same, and therefore the challenger's response is the same in both games. The adversary's second query (which is a function of its random choices and the challenger's first response) is the same in both games. By the assumption that Z does not occur, the challenger's response is the same in both games. Continuing this argument, one sees that each of the adversary's queries and each of the challenger's responses are the same in both games, and therefore the adversary's output is the same in both games. Thus, if Z does not occur and the adversary outputs 1 in Game 0, then the adversary also outputs 1 in Game 1. Likewise, if Z does not occur and the adversary outputs 1 in Game 1, then the adversary outputs 1 in Game 0. More succinctly, we have $W_0 \wedge \bar{Z}$ occurs if and only if $W_1 \wedge \bar{Z}$ occurs. So the Difference Lemma applies, and we obtain

$$|\Pr[W_0] - \Pr[W_1]| \leq \Pr[Z].$$

It remains to bound $\Pr[Z]$. However, this follows from the union bound: for each pair (i, j) of distinct indices, $\Pr[z_i = z_j] = 1/N$, and as there are less than $Q^2/2$ such pairs, we have

$$\Pr[Z] \leq Q^2/2N.$$

That proves the theorem. $\Box$

While there are other strategies one might use to prove the previous theorem (see Exercise 4.24), the **forgetful gnome** technique that we used in the above proof is very useful and we will see it again many times in the sequel.

4.4.4 Constructing PRGs from PRFs

It is easy to construct a PRG from a PRF. Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, let $\ell \geq 1$ be a poly-bounded value, and let $x_1, \ldots, x_\ell$ be any fixed, distinct elements of $\mathcal{X}$ (this requires that $|\mathcal{X}| \geq \ell$). We define a PRG G with seed space $\mathcal{K}$ and output space $\mathcal{Y}^\ell$, as follows: for $k \in \mathcal{K}$,

$$G(k) := (F(k,\ x_1), \ldots, F(k,\ x_\ell)).$$

Theorem 4.8. *If F is a secure PRF, then the PRG G described above is a secure PRG.*

> *In particular, for very PRG adversary $\mathcal{A}$ that plays Attack Game 3.1 with respect to G, there is a PRF adversary $\mathcal{B}$ that plays Attack Game 4.2 with respect to F, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that*
>
> $$\mathrm{PRG}\mathsf{adv}[\mathcal{A}, G] = \mathrm{PRF}\mathsf{adv}[\mathcal{B}, F].$$

Proof. Let $\mathcal{A}$ be an efficient PRG adversary that plays Attack Game 3.1 with respect to G. We describe a corresponding PRF adversary $\mathcal{B}$ that plays Attack Game 4.2 with respect to F. Adversary $\mathcal{B}$ works as follows:

> $\mathcal{B}$ queries its challenger at $x_1, \ldots, x_\ell$, obtaining responses $y_1, \ldots, y_\ell$. Adversary $\mathcal{B}$ then plays the role of challenger to $\mathcal{A}$, giving $\mathcal{A}$ the value $(y_1, \ldots, y_\ell)$. Adversary $\mathcal{B}$ outputs whatever $\mathcal{A}$ outputs.

It is obvious from the construction that for $b = 0, 1$, the probability that $\mathcal{B}$ outputs 1 in Experiment b of Attack Game 4.2 with respect to F is precisely equal to the probability that $\mathcal{A}$ outputs 1 in Experiment b of Attack Game 3.1 with respect to G. The theorem then follows immediately. $\Box$

Deterministic counter mode

The above construction gives us another way to build a semantically secure cipher out of a secure block cipher. Suppose $\mathcal{E} = (E, D)$ is a block cipher defined over $(\mathcal{K}, \mathcal{X})$, where $\mathcal{X} = \{0,1\}^n$. Let $N := |\mathcal{X}| = 2^n$. Assume that N is super-poly and that $\mathcal{E}$ is a secure block cipher. Then by Theorem 4.4, the encryption function E is a secure PRF (defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$). We can then apply Theorem 4.8 to E to obtain a secure PRG, and finally apply Theorem 3.1 to this PRG to obtain a semantically secure stream cipher.

Let us consider this stream cipher in detail. This cipher $\mathcal{E}' = (E', D')$ has key space $\mathcal{K}$, and message and ciphertext space $\mathcal{X}^{\leq \ell}$, where ℓ is a poly-bounded value, and in particular, $\ell \leq N$. We can define $x_1, \ldots, x_\ell$ to be any convenient elements of $\mathcal{X}$; in particular, we can define x_i to be the n-bit binary encoding of $i-1$, which we denote $\langle i-1 \rangle_n$. Encryption and decryption for $\mathcal{E}'$ work as follows.

- For $k \in \mathcal{K}$ and $m \in \mathcal{X}^{\leq \ell}$, with $v := |m|$, we define
$$E'(k, m) := \big(E(k, \langle 0 \rangle_n) \oplus m[0], \ldots, E(k, \langle v-1 \rangle_n) \oplus m[v-1]\big).$$
- For $k \in \mathcal{K}$ and $c \in \mathcal{X}^{\leq \ell}$, with $v := |c|$, we define
$$D'(k, c) := \big(E(k, \langle 0 \rangle_n) \oplus c[0], \ldots, E(k, \langle v-1 \rangle_n) \oplus c[v-1]\big).$$

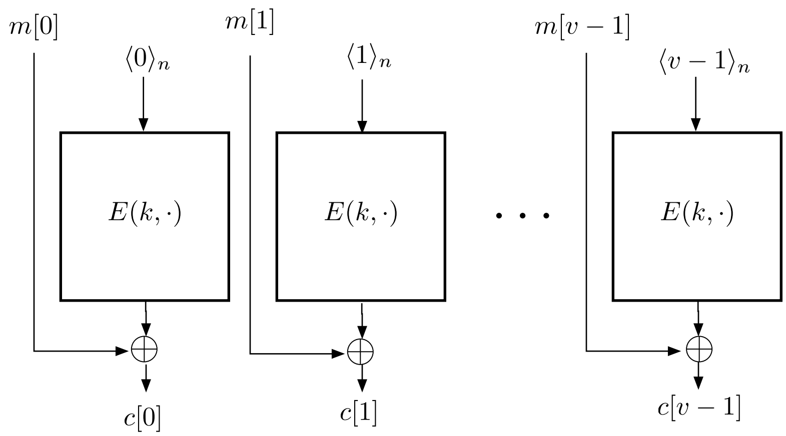
This mode of operation of operation of a block cipher is called **deterministic counter mode**. It is illustrated in Fig. 4.13. Notice that unlike ECB mode, the decryption algorithm D is never used. Putting together Theorems 4.4, 4.8, and 3.1, we see that cipher $\mathcal{E}'$ is semantically secure; in particular, for any efficient SS adversary $\mathcal{A}$, there exists an efficient BC adversary $\mathcal{B}$ such that
$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}'] \leq 2 \cdot \mathrm{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}] + \ell^2/N. \tag{4.23}$$

Clearly, deterministic counter mode has the advantage over ECB mode that it is semantically secure without making any restrictions on the message space. The only disadvantage is that security might degrade significantly for very long messages, because of the ℓ^2/N term in (4.23). Indeed, it is essential that $\ell^2/2N$ is very small. Consider the following attack on $\mathcal{E}'$. Set m_0 to be the message consisting of ℓ zero blocks, and set m_1 to be a message consisting of ℓ random blocks. If the challenger in Attack Game 2.1 encrypts m_0 using E', then the ciphertext will not contain any duplicate blocks. However, by the birthday paradox (see Theorem B.1), if the challenger encrypts m_1, the ciphertext will contain duplicate blocks with probability at least $\min\big\{\ell(\ell-1)\big/4N, 0.63\big\}$. So the adversary $\mathcal{A}$ that constructs m_0 and m_1 in this way, and outputs 1 if and only if the ciphertext contains duplicate blocks, has an advantage that grows quadratically in ℓ, and is non-negligible for $\ell \approx N^{1/2}$.

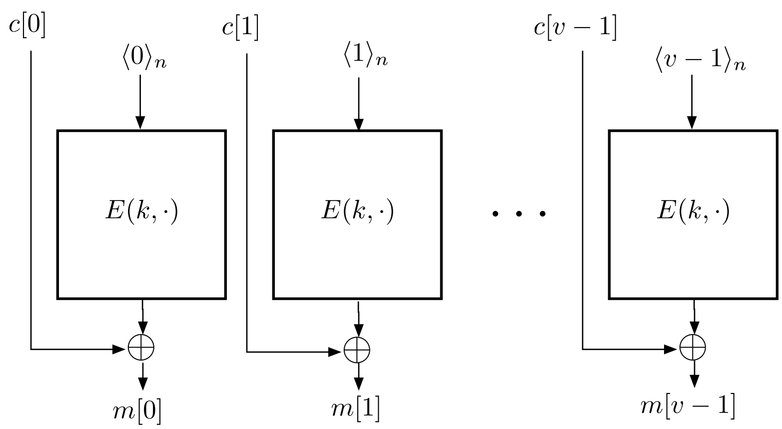
4.4.5 Mathematical details

As usual, we give a more mathematically precise definition of a PRF, using the terminology defined in Section 2.4.

Definition 4.4 (pseudo-random function). *A **pseudo-random function** consists of an algorithm F, along with three families of spaces with system parameterization P:*
$$\mathbf{K} = \{\mathcal{K}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \mathbf{X} = \{\mathcal{X}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \textit{and} \quad \mathbf{Y} = \{\mathcal{Y}_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$



(a) encryption



(b) decryption

Figure 4.13: Encryption and decryption for deterministic counter mode

such that

1. $\mathbf{K}$, $\mathbf{X}$, *and* $\mathbf{Y}$ *are efficiently recognizable.*

2. $\mathbf{K}$ *and* $\mathbf{Y}$ *are efficiently sampleable.*

3. *Algorithm* F *is a deterministic algorithm that on input* $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $k \in \mathcal{K}_{\lambda,\Lambda}$, *and* $x \in \mathcal{X}_{\lambda,\Lambda}$, *runs in time bounded by a polynomial in* λ, *and outputs an element of* $\mathcal{Y}_{\lambda,\Lambda}$.

As usual, in defining security, the attack game is parameterized by security and system parameters, and the advantage is a function of the security parameter.

4.5 Constructing block ciphers from PRFs

In this section, we show how to construct a secure block cipher from any secure PRF whose output space and input space is $\{0,1\}^n$, where 2^n is super-poly. The construction is called the Luby-Rackoff construction (after its inventors). The result itself is mainly of theoretical interest, as block ciphers that are used in practice have a more ad hoc design; however, the result is sometimes seen as a justification for the design of some practical block ciphers as Feistel networks (see Section 4.2.1).

Let F be a PRF, defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$, where $\mathcal{X} = \{0,1\}^n$. We describe a block cipher $\mathcal{E} = (E, D)$ whose key space is $\mathcal{K}^3$, and whose data block space is $\mathcal{X}^2$.

Given a key $(k_1, k_2, k_3) \in \mathcal{K}^3$ and a data block $(u, v) \in \mathcal{X}^2$, the encryption algorithm E runs as follows:

$w \leftarrow u \oplus F(k_1,\ v)$
$x \leftarrow v \oplus F(k_2,\ w)$
$y \leftarrow w \oplus F(k_3,\ x)$
output (x, y).

Given a key $(k_1, k_2, k_3) \in \mathcal{K}^3$ and an data block $(x, y) \in \mathcal{X}^2$, the decryption algorithm D runs as follows:

$w \leftarrow y \oplus F(k_3,\ x)$
$v \leftarrow x \oplus F(k_2,\ w)$
$u \leftarrow w \oplus F(k_1,\ v)$
output (u, v).

See Fig. 4.14 for an illustration of $\mathcal{E}$.

It is easy to see that $\mathcal{E}$ is a block cipher. It is useful to see algorithm E as consisting of 3 "rounds." For $k \in \mathcal{K}$, let us define the "round function"

$$\begin{aligned} \phi_k : \quad & \mathcal{X}^2 \to \mathcal{X}^2 \\ & (a, b) \mapsto (b, a \oplus F(k, b)). \end{aligned}$$

It is easy to see that for any fixed k, the function ϕ_k is a permutation on $\mathcal{X}^2$; indeed, if $\sigma(a, b) := (b, a)$, then

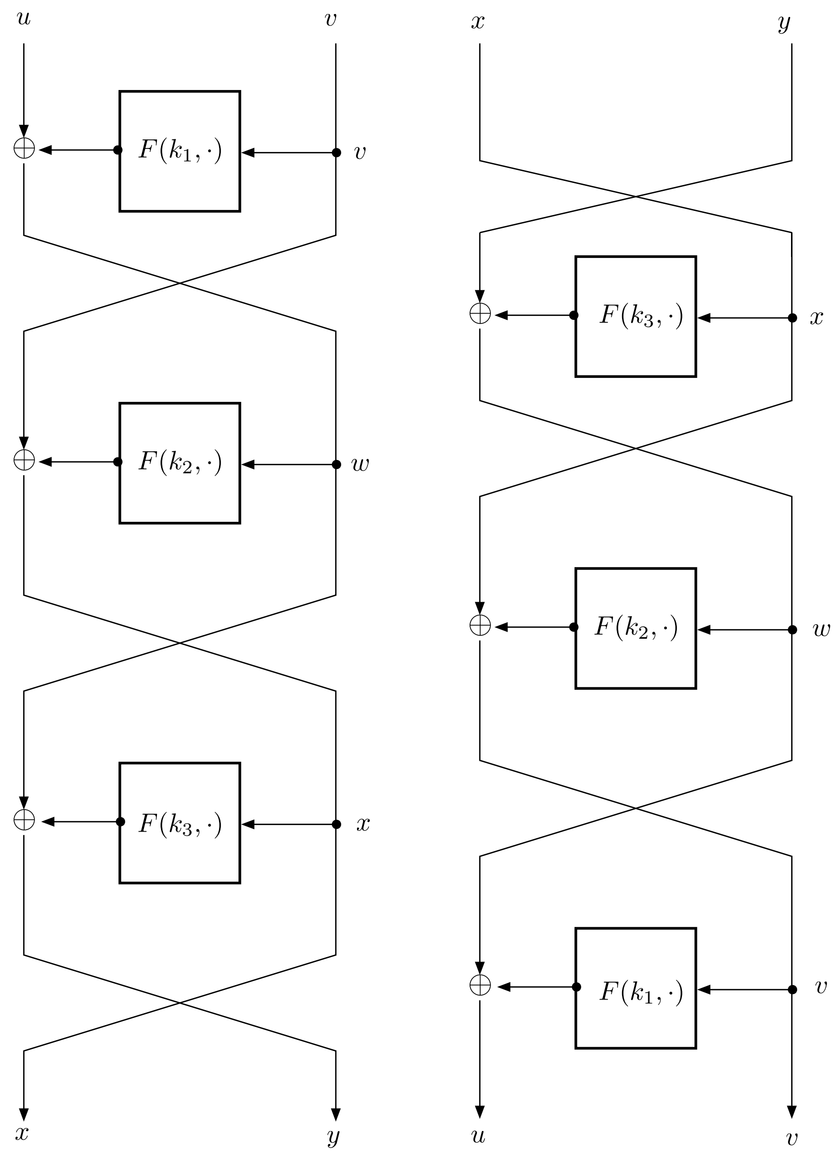
$$\phi_k^{-1} = \sigma \circ \phi_k \circ \sigma.$$

Moreover, we see that

$$E((k_1, k_2, k_3), \cdot) = \phi_{k_3} \circ \phi_{k_2} \circ \phi_{k_1}$$

and

$$D((k_1, k_2, k_3), \cdot) = \phi_{k_1}^{-1} \circ \phi_{k_2}^{-1} \circ \phi_{k_3}^{-1} = \sigma \circ \phi_{k_1} \circ \phi_{k_2} \circ \phi_{k_3} \circ \sigma.$$



(a) Encryption (b) Decryption

Figure 4.14: Encryption and decryption with Luby-Rackoff

Theorem 4.9. *If F is a secure PRF and $N := |\mathcal{X}| = 2^n$ is super-poly, then the Luby-Rackoff cipher $\mathcal{E} = (E, D)$ constructed from F is a secure block cipher.*

In particular, for every Q-query BC adversary $\mathcal{A}$ that attacks $\mathcal{E}$ as in Attack Game 4.1, there exists a PRF adversary $\mathcal{B}$ that plays Attack Game 4.2 with respect to F, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \le 3 \cdot \mathrm{PRF}\mathsf{adv}[\mathcal{B}, F] + \frac{Q^2}{N} + \frac{Q^2}{2N^2}.$$

Proof idea. By Corollary 4.5, and the assumption that N is super-poly, it suffices to show that E is a secure PRF. So we want to show that if an adversary is playing in Experiment 0 of Attack Game 4.2 with respect to E, the challenger's responses effectively "look like" completely random bit strings. We may assume that the adversary never makes the same query twice. Moreover, as F is a PRF, we can replace $F(k_1, \cdot)$, $F(k_2, \cdot)$, and $F(k_3, \cdot)$ by truly random functions, f_1, f_2, and f_3, and the adversary should hardly notice the difference.

So now, given a query (u_i, v_i), the challenger computes its response (x_i, y_i) as follows:

$$\begin{aligned} &w_i \leftarrow u_i \oplus f_1(v_i) \\ &x_i \leftarrow v_i \oplus f_2(w_i) \\ &y_i \leftarrow w_i \oplus f_3(x_i). \end{aligned}$$

A rough, intuitive argument goes like this. Suppose that no two w_i values are the same. Then all of the outputs of f_2 will be random and independent. From this, we can argue that the x_i's are also random and independent. Then from this, it will follow that except with negligible probability, the inputs to f_3 will be distinct. From this, we can conclude that the y_i's are essentially random and independent.

So we will be in good shape if we can show that all of the w_i's are distinct. But the w_i's are obtained indirectly from the random function f_1, and so with some care, one can indeed argue that the w_i will be distinct, except with negligible probability. □

Proof. Let $\mathcal{A}$ be an efficient BC adversary that plays Attack Game 4.1 with respect to $\mathcal{E}$, and which makes at most Q queries to its challenger. We want to show that $\mathrm{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible. To do this, we first show that $\mathrm{PRF}\mathsf{adv}[\mathcal{A}, E]$ is negligible, and the result will then follow from the PRF Switching Lemma (i.e., Theorem 4.4) and the assumption that N is super-poly.

To simplify things a bit, we replace $\mathcal{A}$ with an adversary $\mathcal{A}_0$ with the following properties:

- $\mathcal{A}_0$ always makes exactly Q queries to its challenger;
- $\mathcal{A}_0$ never makes the same query more than once;
- $\mathcal{A}_0$ is just as efficient as $\mathcal{A}$ (more precisely, $\mathcal{A}_0$ is an elementary wrapper around $\mathcal{A}$);
- $\mathrm{PRF}\mathsf{adv}[\mathcal{A}_0, E] = \mathrm{PRF}\mathsf{adv}[\mathcal{A}, E]$.

Adversary $\mathcal{A}_0$ simply runs the same protocol as $\mathcal{A}$; however, it keeps a table of query/response pairs so as to avoid making duplicate queries; moreover, it "pads" the execution of $\mathcal{A}$ if necessary, so as to make exactly Q queries.

The overall strategy of the proof is as follows. First, we define Game 0 to be the game played between $\mathcal{A}_0$ and the challenger of Experiment 0 of Attack Game 4.2 with respect to E. We then define several more games: Game 1, Game 2, and Game 3. Each of these games is played between $\mathcal{A}_0$ and a different challenger; moreover, the challenger in Game 3 is equivalent to the challenger of Experiment 1 of Attack Game 4.2. Also, for $j = 0, \ldots, 3$, we define W_j to be the event that $\mathcal{A}_0$ outputs 1 in Game j. We will show that for $j = 1, \ldots, 3$ that the value $|\Pr[W_j] - \Pr[W_{j-1}]|$ is negligible, from which it will follow that

$$|\Pr[W_3] - \Pr[W_0]| = \mathrm{PRF}\mathsf{adv}[\mathcal{A}_0, E]$$

is also negligible.

Game 0. Let us begin by giving a detailed description of the challenger in Game 0 that is convenient for our purposes:

$k_1, k_2, k_3 \xleftarrow{\mathrm{R}} \mathcal{K}$
upon receiving the ith query $(u_i, v_i) \in \mathcal{X}^2$ (for $i = 1, \ldots, Q$) do:
  $w_i \leftarrow u_i \oplus F(k_1,\ v_i)$
  $x_i \leftarrow v_i \oplus F(k_2,\ w_i)$
  $y_i \leftarrow w_i \oplus F(k_3,\ x_i)$
  send (x_i, y_i) to the adversary.

Recall that the adversary $\mathcal{A}_0$ is guaranteed to always make Q distinct queries $(u_1, v_1), \ldots, (u_Q, v_Q)$; that is, the (u_i, v_i) values are distinct *as pairs*, so that for $i \neq j$, we may have $u_i = u_j$ or $v_i = v_j$, but not both.

Game 1. We next play the "PRF card," replacing the three functions $F(k_1, \cdot), F(k_2, \cdot), F(k_3, \cdot)$ by truly random functions f_1, f_2, f_3. Intuitively, since F is a secure PRF, the adversary $\mathcal{A}_0$ should not notice the difference. Our challenger in Game 1 thus works as follows:

$f_1, f_2, f_3 \xleftarrow{\mathrm{R}} \mathrm{Funs}[\mathcal{X}, \mathcal{X}]$
upon receiving the ith query $(u_i, v_i) \in \mathcal{X}^2$ (for $i = 1, \ldots, Q$) do:
  $w_i \leftarrow u_i \oplus f_1(v_i)$
  $x_i \leftarrow v_i \oplus f_2(w_i)$
  $y_i \leftarrow w_i \oplus f_3(x_i)$
  send (x_i, y_i) to the adversary.

As discussed in Exercise 4.26, we can model the three PRFs $F(k_1, \cdot), F(k_2, \cdot), F(k_3, \cdot)$ as a single PRF F', called the 3-wise parallel composition of F: the PRF F' is defined over $(\mathcal{K}^3, \{1,2,3\} \times \mathcal{X}, \mathcal{X})$, and $F'((k_1, k_2, k_3), (s, x)) := F(k_s, x)$. We can easily construct an adversary $\mathcal{B}'$, just as efficient as $\mathcal{A}_0$, such that

$$|\Pr[W_1] - \Pr[W_0]| = \mathrm{PRF}\mathsf{adv}[\mathcal{B}', F']. \tag{4.24}$$

Adversary $\mathcal{B}'$ simply runs $\mathcal{A}_0$ and outputs whatever $\mathcal{A}_0$ outputs; when $\mathcal{A}_0$ queries its challenger with a pair (u_i, v_i), adversary $\mathcal{B}'$ computes the response (x_i, y_i) for $\mathcal{A}_0$ by computing

$w_i \leftarrow u_i \oplus f'(1, v_i)$
$x_i \leftarrow v_i \oplus f'(2, w_i)$
$y_i \leftarrow w_i \oplus f'(3, x_i)$.

Here, the f' denotes the function chosen by $\mathcal{B}'$'s challenger in Attack Game 4.2 with respect to F'. It is clear that $\mathcal{B}'$ outputs 1 with probability $\Pr[W_0]$ in Experiment 0 of that attack game, while it outputs 1 with probability $\Pr[W_1]$ in Experiment 1, from which (4.24) follows.

By Exercise 4.26, there exists an adversary $\mathcal{B}$, just as efficient as $\mathcal{B}'$, such that

$$\text{PRF}\mathsf{adv}[\mathcal{B}', F'] = 3 \cdot \text{PRF}\mathsf{adv}[\mathcal{B}, F]. \tag{4.25}$$

Game 2. We next make a purely conceptual change: we implement the random functions f_2 and f_3 using the "faithful gnome" idea discussed in Section 4.4.2. This is not done for efficiency, but rather, to set us up so as to be able to make (and easily analyze) a more substantive modification later, in Game 3. Our challenger in this game works as follows:

$f_1 \xleftarrow{\text{R}} \text{Funs}[\mathcal{X}, \mathcal{X}]$
$X_1, \ldots, X_Q \xleftarrow{\text{R}} \mathcal{X}$
$Y_1, \ldots, Y_Q \xleftarrow{\text{R}} \mathcal{X}$
upon receiving the ith query $(u_i, v_i) \in \mathcal{X}^2$ (for $i = 1, \ldots, Q$) do:
  $w_i \leftarrow u_i \oplus f_1(v_i)$
  $x'_i \leftarrow X_i$; $\boxed{\text{if } w_i = w_j \text{ for some } j < i \text{ then } x'_i \leftarrow x'_j;}$ $x_i \leftarrow v_i \oplus x'_i$
  $y'_i \leftarrow Y_i$; $\boxed{\text{if } x_i = x_j \text{ for some } j < i \text{ then } y'_i \leftarrow y'_j;}$ $y_i \leftarrow w_i \oplus y'_i$
  send (x_i, y_i) to the adversary.

The idea is that the value x'_i represents $f_2(w_i)$. By default, x'_i is equal to the random value X_i; however, the boxed code over-rides this default value if w_i is the same as w_j for some $j < i$. Similarly, the value y'_i represents $f_3(x_i)$. By default, y'_i is equal to the random value Y_i, and the boxed code over-rides the default if necessary.

Since the challenger in Game 2 completely equivalent to that of Game 1, we have

$$\Pr[W_2] = \Pr[W_1]. \tag{4.26}$$

Game 3. We now employ the "forgetful gnome" technique, which we already saw in the proof of Theorem 4.6. The idea is to simply eliminate the consistency checks made by the challenger in Game 2. Here is the logic of the challenger in Game 3:

$f_1 \xleftarrow{\text{R}} \text{Funs}[\mathcal{X}, \mathcal{X}]$
$X_1, \ldots, X_Q \xleftarrow{\text{R}} \mathcal{X}$
$Y_1, \ldots, Y_Q \xleftarrow{\text{R}} \mathcal{X}$
upon receiving the ith query $(u_i, v_i) \in \mathcal{X}^2$ (for $i = 1, \ldots, Q$) do:
  $w_i \leftarrow u_i \oplus f_1(v_i)$
  $x'_i \leftarrow X_i$; $x_i \leftarrow v_i \oplus x'_i$
  $y'_i \leftarrow Y_i$; $y_i \leftarrow w_i \oplus y'_i$
  send (x_i, y_i) to the adversary.

Note that this description is literally the same as the description of the challenger in Game 2, except that we have simply erased the underlined code in the latter.

For the purposes of analysis, we view Games 2 and 3 as operating on the same underlying probability space. This probability space is determined by

- the random choices made by the adversary, which we denote by *Coins*, and
- the random choices made by the challenger, namely, f_1, $X_1, \ldots, X_Q$, and $Y_1, \ldots, Y_Q$.

What differs between the two games is the rule that the challenger uses to compute its responses to the queries made by the adversary.

Claim 1: *in Game 3, the random variables Coins*, $f_1, x_1, y_1, \ldots, x_Q, y_Q$ *are mutually independent.* To prove this claim, observe that by construction, the random variables

$$Coins, \quad f_1, \quad X_1, \ldots, X_Q, \quad Y_1, \ldots, Y_Q$$

are mutually independent. Now condition on any fixed values of *Coins* and f_1. The first query (u_1, v_1) is now fixed, and hence so is w_1; however, in this conditional probability space, X_1 and Y_1 are still uniformly and independently distributed over $\mathcal{X}$, and so x_1 and y_1 are also uniformly and independently distributed. One continues the argument, conditioning on fixed values of x_1, y_1 (in addition to fixed values of *Coins* and f_1), observing that now u_2, v_2, and w_2 are also fixed, and that x_2 and y_2 are uniformly and independently distributed. It should be clear how the claim follows by induction.

Let Z_1 be the event that $w_i = w_j$ for some $i \neq j$ in Game 3. Let Z_2 be the event that $x_i = x_j$ for some $i \neq j$ in Game 3. Let $Z := Z_1 \vee Z_2$. Note that the event Z is defined in terms of the variables w_i and x_i values in Game 3. Indeed, the variables w_i and z_i may not be computed in the same way in Games 2 and 3, and so we have explicitly defined the event Z in terms of their values in Game 3. Nevertheless, it is straightforward to see that Games 2 and 3 proceed identically if Z does not occur. In particular:

Claim 2: *the event* $W_2 \wedge \bar{Z}$ *occurs if and only if the event* $W_3 \wedge \bar{Z}$ *occurs.* To prove this claim, consider any fixed values of the variables

$$Coins, \quad f_1, \quad X_1, \ldots, X_Q, \quad Y_1, \ldots, Y_Q$$

for which Z does not occur. It will suffice to show that the output of $\mathcal{A}_0$ is the same in both Games 2 and 3. Since the query (u_1, v_1) depends only on *Coins*, we see that the variables u_1, v_1, and hence also w_1, x_1, y_1 have the same values in both games. Since the query (u_2, v_2) depends only on *Coins* and (x_1, y_1), it follows that the variables u_2, v_2 and hence w_2 have the same values in both games; since Z does not occur, we see $w_2 \neq w_1$ and hence the variable x_2 has the same value in both games; again, since Z does not occur, it follows that $x_2 \neq x_1$, and hence the variable y_2 has the same value in both games. Continuing this argument, we see that for $i = 1, \ldots, Q$, the variables u_i, v_i, w_i, x_i, y_i have the same values in both games. Since the output of $\mathcal{A}_0$ is a function of these variables and *Coins*, the output is the same in both games. That proves the claim.

Claim 2, together with the Difference Lemma (i.e., Theorem 4.7) and the Union Bound, implies

$$|\Pr[W_3] - \Pr[W_2]| \leq \Pr[Z] \leq \Pr[Z_1] + \Pr[Z_2]. \tag{4.27}$$

By the fact that $x_1, \ldots, x_Q$ are mutually independent (see Claim 1), it is obvious that

$$\Pr[Z_2] \leq \frac{Q^2}{2} \cdot \frac{1}{N}, \tag{4.28}$$

since Z_2 is the union of less than $Q^2/2$ events, each of which occurs with probability $1/N$.

Let us now analyze the event Z_1. We claim that

$$\Pr[Z_1] \leq \frac{Q^2}{2} \cdot \frac{1}{N}. \tag{4.29}$$

To prove this, it suffices to prove it conditioned on any fixed values of $Coins, x_1, y_1, \ldots, x_Q, y_Q$. If these values are fixed, then so are $u_1, v_1, \ldots, u_Q, v_Q$. However, by independence (see Claim 1), the variable f_1 is still uniformly distributed over $\text{Funs}[\mathcal{X}, \mathcal{X}]$ in this conditional probability space. Now consider any fixed pair of indices i, j, with $i \neq j$. Suppose first that $v_i = v_j$. Then since $\mathcal{A}_0$ never makes the same query twice, we must have $u_i \neq u_j$, and it is easy to see that $w_i \neq w_j$ for any choice of f_1. Next suppose that $v_i \neq v_j$. Then the values $f_1(v_i)$ and $f_2(v_j)$ are uniformly and independently distributed over $\mathcal{X}$ in this conditional probability space, and

$$\Pr[f_1(v_i) \oplus f_1(v_j) = u_i \oplus u_j] = \frac{1}{N}$$

in this conditional probability space.

Thus, we have shown that in Game 3, for all pairs i, j with $i \neq j$,

$$\Pr[w_i = w_j] \leq \frac{1}{N}$$

The inequality (4.29) follows from the Union Bound.

As another consequence of Claim 1, we observe that Game 3 is equivalent to Experiment 1 of Attack Game 4.2 with respect to E. From this, together with (4.24), (4.25), (4.26), (4.27), (4.28), and (4.29), we conclude that

$$\text{PRF}\mathsf{adv}[\mathcal{A}_0, E] \leq 3 \cdot \text{PRF}\mathsf{adv}[\mathcal{B}, F] + \frac{Q^2}{N}.$$

Finally, applying Theorem 4.4 to the cipher $\mathcal{E}$, whose data block space has size N^2, we have

$$\text{BC}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq 3 \cdot \text{PRF}\mathsf{adv}[\mathcal{B}, F] + \frac{Q^2}{N} + \frac{Q^2}{2N^2}.$$

That concludes the proof of the theorem. $\Box$

4.6 The tree construction: from PRGs to PRFs

It turns out that given a suitable, secure PRG, one can construct a secure PRF with a technique called the **tree construction**. Combining this result with the Luby-Rackoff construction in Section 4.5, we see that from any secure PRG, we can construct a secure block cipher. While this result is of some theoretical interest, the construction is not very efficient, and is not really used in practice. However, we note that a simple generalization of this construction plays an important role in practical schemes for message authentication; we shall discuss this in Section 6.4.2.

Our starting point is a PRG G defined over $(\mathcal{S}, \mathcal{S}^2)$; that is, the seed space is a set $\mathcal{S}$, and the output space is the set $\mathcal{S}^2$ of all seed pairs. For example, G might stretch n-bit strings to $2n$-bit strings.[2] It will be convenient to write $G(s) = (G_0(s), G_1(s))$; that is, $G_0(s) \in \mathcal{S}$ denotes the first component of $G(s)$ and $G_1(s)$ denotes the second component of $G(s)$. From G, we shall build a PRF F with key space $\mathcal{S}$, input space $\{0,1\}^\ell$ (where ℓ is an arbitrary, poly-bounded value), and output space $\mathcal{S}$.

Let us first define the algorithm G^*, that takes as input $s \in \mathcal{S}$ and $x = (a_1, \ldots, a_n) \in \{0,1\}^*$, where $a_i \in \{0,1\}$ for $i = 1, \ldots, n$, and outputs an element $t \in \mathcal{S}$, computed as follows:

[2] Indeed, we could even start with a PRG that stretches n bit strings to $(n+1)$-bit strings, and then apply the n-wise sequential construction analyzed in Theorem 3.3 to obtain a suitable G.

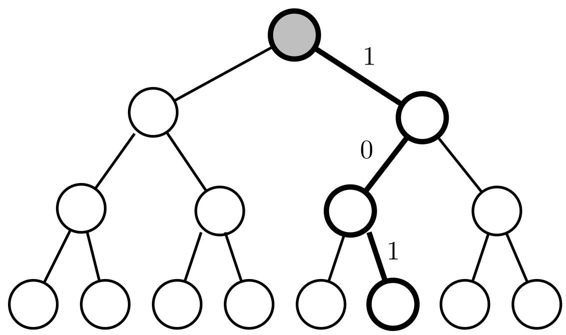


Figure 4.15: Evaluation tree for $\ell = 3$. The highlighted path corresponds to the input $x = 101$. The root is shaded to indicate it is assigned a random label. All other nodes are assigned derived labels.

$t \leftarrow s$
for $i \leftarrow 1$ to n do
    $t \leftarrow G_{a_i}(t)$
output t.

For $s \in \mathcal{S}$ and $x \in \{0,1\}^\ell$, we define

$$F(s,x) := G^*(s,x).$$

We shall call the PRF F derived from G in this way the **tree construction**.

It is useful to envision the bits of an input $x \in \{0,1\}^\ell$ as tracing out a path through a complete binary tree of height ℓ and with 2^ℓ leaves, which we call the **evaluation tree**: a bit value of 0 means branch left and a bit value of 1 means branch right. In this way, any node in the tree can be uniquely addressed by a bit string of length at most ℓ; strings of length $j \leq \ell$ address nodes at level j in the tree: the empty string addresses the root (which is at level 0), strings of length 1 address the children of the root (which are at level 1), etc. The nodes in the evaluation tree are labeled with elements of $\mathcal{S}$, using the following rule:

- the root of the tree is labeled with s;
- the label of any other node is **derived** from the label t of its parent as follows: if the node is a left child, its label is $G_0(t)$, and if the node is a right child, its label is $G_1(t)$.

The value of the $F(s,x)$ is then the label on the leaf addressed by x. See Fig. 4.15.

Theorem 4.10. *If G is a secure PRG, then the PRF F obtained from G using the tree construction is a secure PRF.*

In particular, for every PRF adversary $\mathcal{A}$ that plays Attack Game 4.2 with respect to F, and which makes at most Q queries to its challenger, there exists a PRG adversary $\mathcal{B}$ that plays Attack Game 3.1 with respect to G, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{PRF}\mathsf{adv}[\mathcal{A}, F] = \ell Q \cdot \text{PRG}\mathsf{adv}[\mathcal{B}, G].$$

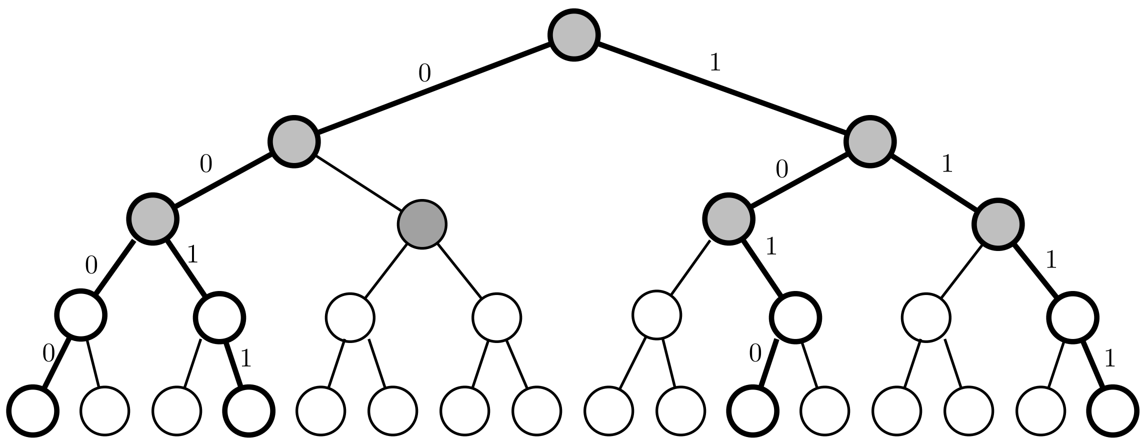


Figure 4.16: Evaluation tree for Hybrid 2 with $\ell = 4$. The shaded nodes are assigned random labels, while the unshaded nodes are assigned derived labels. The highlighted paths correspond to inputs 0000, 0011, 1010, and 1111.

Proof idea. The basic idea of the proof is a hybrid argument. We build a sequence of games, Hybrid 0, ..., Hybrid ℓ. Each of these games is played between a given PRF adversary, attacking F, and a challenger whose behavior is slightly different in each game. In Hybrid j, the challenger builds an evaluation tree whose nodes are labeled as follows:

- nodes at levels 0 through j are assigned random labels;
- the nodes at levels $j + 1$ through ℓ are assigned derived labels.

In response to a query $x \in \{0, 1\}^\ell$ in Hybrid j, the challenger sends to the adversary the label of the leaf addressed by x. See Fig. 4.16

Clearly, Hybrid 0 is equivalent to Experiment 0 of Attack Game 4.2, while Hybrid ℓ is equivalent to Experiment 1. Intuitively, under the assumption that G is a secure PRG, the adversary should not be able to tell the difference between Hybrids j and $j + 1$ for $j = 0, \ldots, \ell - 1$. In making this intuition rigorous, we have to be a bit careful: the evaluation tree is huge, and to build an efficient PRG adversary that attacks G, we cannot afford to write down the entire tree (or even one level of the tree). Instead, we use the fact that if the PRF adversary makes at most Q queries to its challenger (which is a poly-bounded value), then at any level j in the evaluation tree, the paths traced out by these Q queries touch at most Q nodes at level j (in Fig. 4.16, these would be the first, third, and fourth nodes at level 2 for the given inputs). The PRG adversary we construct will use a variation of the faithful gnome idea to effectively maintain the relevant random labels at level j, as needed. $\square$

Proof. Let $\mathcal{A}$ be an efficient adversary that plays Attack Game 4.2 with respect to F. Let us assume that $\mathcal{A}$ makes at most a poly-bounded number Q of queries to the challenger.

As discussed above, we define $\ell + 1$ hybrid games, Hybrid 0, ..., Hybrid ℓ, each played between $\mathcal{A}$ and a challenger. In Hybrid j, the challenger works as follows:

$f \xleftarrow{\text{R}} \text{Funs}[\{0,1\}^j, \mathcal{S}]$

upon receiving a query $x = (a_1, \ldots, a_\ell) \in \{0,1\}^\ell$ from $\mathcal{A}$ do:

  $u \leftarrow (a_1, \ldots, a_j)$, $v \leftarrow (a_{j+1}, \ldots, \ell)$

  $y \leftarrow G^*(f(u), v)$

  send y to $\mathcal{A}$.

Intuitively, for $u \in \{0,1\}^j$, $f(u)$ represents the random label at the node at level j addressed by u. Thus, each node at level j is assigned a random label, while nodes at levels $j+1$ through ℓ are assigned derived labels. Note that in our description of this game, we do not explicitly assign labels to nodes at levels 0 through $j-1$, as these labels do not affect any outputs.

For $j = 0, \ldots, \ell$, let p_j be the probability that $\mathcal{A}$ outputs 1 in Hybrid j. As Hybrid 0 is equivalent to Experiment 0 of Attack Game 4.2, and Hybrid ℓ is equivalent to Experiment 1, we have:

$$\text{PRF}\mathsf{adv}[\mathcal{A}, F] = |p_\ell - p_0|. \tag{4.30}$$

Let G' denote the Q-wise parallel composition of G, which we discussed in Section 3.4.1. G' takes as input $(s_1, \ldots, s_Q) \in \mathcal{S}^Q$ and outputs $(G(s_1), \ldots, G(s_Q)) \in (\mathcal{S}^2)^Q$. By Theorem 3.2, if G is a secure PRF, then so is G'.

We now build an efficient PRG adversary $\mathcal{B}'$ that attacks G', such that

$$\text{PRG}\mathsf{adv}[\mathcal{B}', G'] = \frac{1}{\ell} \cdot |p_\ell - p_0|. \tag{4.31}$$

We first give an overview of how $\mathcal{B}'$ works. In playing Attack Game 4.2 with respect to G', the challenger presents to $\mathcal{B}'$ a vector

$$\vec{r} = ((r_{10}, r_{11}), \ldots, (r_{Q0}, r_{Q1})) \in (\mathcal{S}^2)^Q. \tag{4.32}$$

In Experiment 0 of the attack game, $\vec{r} = G(\vec{s})$ for random $\vec{s} \in \mathcal{S}^Q$, while in Experiment 1, $\vec{r}$ is randomly chosen from $(\mathcal{S}^2)^Q$. To distinguish these two experiments, $\mathcal{B}'$ plays the role of challenger to $\mathcal{A}$ by choosing $\omega \in \{1, \ldots, \ell\}$ at random, and uses the elements of $\vec{r}$ to label nodes at level ω of the evaluation tree in a consistent fashion. To do this, $\mathcal{B}'$ maintains a lookup table, which allows it to associate with each prefix $u \in \{0,1\}^{\omega-1}$ of some query $x \in \{0,1\}^\ell$ an index p, so that the children of the node addressed by u are labeled by the seed pair (r_{p0}, r_{p1}). Finally, when $\mathcal{A}$ terminates and outputs a bit, $\mathcal{B}'$ outputs the same bit. As will be evident from the details of the construction of $\mathcal{B}'$, conditioned on $\omega = j$ for any fixed $j = 1, \ldots, \ell$, the probability that $\mathcal{B}'$ outputs 1 is:

- p_{j-1}, if $\mathcal{B}'$ is in Experiment 0 of its attack game, and
- p_j, if $\mathcal{B}'$ is in Experiment 1 of its attack game.

Then by the usual telescoping sum calculation, we get (4.31).

Now the details. We implement our lookup table as an associative array $Map : \{0,1\}^* \to \mathbb{Z}_{>0}$. Here is the logic for $\mathcal{B}'$:

upon receiving $\vec{r}$ as in (4.32) from its challenger, $\mathcal{B}'$ plays the role of challenger to $\mathcal{A}$, as follows:

$\omega \xleftarrow{\text{R}} \{1, \ldots, \ell\}$
initialize an empty associative array $Map : \{0,1\}^* \to \mathbb{Z}_{>0}$
$ctr \leftarrow 0$
upon receiving a query $x = (a_1, \ldots, a_\ell) \in \{0,1\}^\ell$ from $\mathcal{A}$ do:
  $u \leftarrow (a_1, \ldots, a_{\omega-1})$, $d \leftarrow a_\omega$, $v \leftarrow (a_{\omega+1}, \ldots, a_\ell)$
  if $u \notin \text{Domain}(Map)$ then
    $ctr \leftarrow ctr + 1$, $Map[u] \leftarrow ctr$
  $p \leftarrow Map[u]$, $y \leftarrow G^*(r_{pd}, v)$
  send y to $\mathcal{A}$.

Finally, $\mathcal{B}'$ outputs whatever $\mathcal{A}$ outputs.

For $b = 0, 1$, let W_b be the event that $\mathcal{B}'$ outputs 1 in Experiment b of Attack Game 4.2 with respect to G'. We claim that for any fixed $j = 1, \ldots, \ell$, we have

$$\Pr[W_0 \mid \omega = j] = p_{j-1} \quad \text{and} \quad \Pr[W_1 \mid \omega = j] = p_j.$$

Indeed, condition on $\omega = j$ for fixed j, and consider how $\mathcal{B}'$ labels nodes in the evaluation tree. On the one hand, when $\mathcal{B}'$ is in Experiment 1 of its attack game, it effectively assigns random labels to nodes at level j, and the lookup table ensures that this is done consistently. On the other hand, when $\mathcal{B}'$ is in Experiment 0 of its attack game, it effectively assigns pseudo-random labels to nodes at level j, which is the same as assigning random labels to the parents of these nodes at level $j-1$, and assigning derived labels at level j; again, the lookup table ensures a consistent labeling.

From the above claim, equation (4.31) now follows by a familiar, telescoping sum calculation:

$$\begin{aligned}
\text{PRG}\mathsf{adv}[\mathcal{B}', G'] &= |\Pr[W_1] - \Pr[W_0]| \\
& \frac{1}{\ell} \cdot \Big| \sum_{j=1}^{\ell} \Pr[W_1 \mid \omega = j] - \sum_{j=1}^{\ell} \Pr[W_0 \mid \omega = j] \Big| \\
&= \frac{1}{\ell} \cdot \Big| \sum_{j=1}^{\ell} p_j - \sum_{j=1}^{\ell} p_{j-1} \Big| \\
&= \frac{1}{\ell} \cdot |p_\ell - p_0|.
\end{aligned}$$

Finally, by Theorem 3.2, there exists an efficient PRG adversary $\mathcal{B}$ such that

$$\text{PRG}\mathsf{adv}[\mathcal{B}', G'] = Q \cdot \text{PRG}\mathsf{adv}[\mathcal{B}, G]. \tag{4.33}$$

The theorem now follows by combining equations (4.30), (4.31), and (4.33). $\Box$

4.6.1 Variable length tree construction

It is natural to consider how the tree construction works on *variable length* inputs. Again, let G be a PRG defined over $(\mathcal{S}, \mathcal{S}^2)$, and let G^* be as defined above. For any poly-bounded value ℓ we define the PRF $\tilde{F}$, with key space $\mathcal{S}$, input space $\{0,1\}^{\leq \ell}$, and output space $\mathcal{S}$, as follows: for $s \in \mathcal{S}$ and $x \in \{0,1\}^{\leq \ell}$, we define

$$\tilde{F}(s, x) = G^*(s, x).$$

Unfortunately, $\tilde{F}$ is not a secure PRF. The reason is that there is a trivial **extension attack**. Suppose $u, v \in \{0,1\}^{\leq \ell}$ such that u is a proper prefix of v; that is, $v = u \parallel w$ for some non-empty string w. Then given u and v, along with $y := \tilde{F}(s, u)$, we can easily compute $F(s, v)$ as $G^*(y, w)$. Of course, for a truly random function, we could not predict its value at v, given its value at u, and so it is easy to distinguish $\tilde{F}(s, \cdot)$ from a random function.

Even though $\tilde{F}$ is not a secure PRF, we can still say something interesting about it. We show that $\tilde{F}$ is a PRF against restricted set of adversaries called **prefix-free adversaries**.

Definition 4.5. *Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{Y})$. We say that a PRF adversary $\mathcal{A}$ playing Attack Game 4.2 with respect to F is a **prefix-free adversary** if all of its queries are non-empty strings over $\mathcal{X}$ of length at most ℓ, no one of which is a proper prefix of another.*[3] *We denote $\mathcal{A}$'s advantage in winning the game by $\mathrm{PRF^{pf}}\mathsf{adv}[\mathcal{A}, F]$. Further, let us say that F is a **prefix-free secure PRF** if $\mathrm{PRF^{pf}}\mathsf{adv}[\mathcal{A}, F]$ is negligible for all efficient, prefix-free adversaries $\mathcal{A}$.*

For example, if a prefix-free adversary issues a query for the sequence (a_1, a_2, a_3) then it cannot issue queries for (a_1) or for (a_1, a_2).

Theorem 4.11. *If G is a secure PRG, then the variable length tree construction $\tilde{F}$ derived from G is a prefix-free secure PRF.*

In particular, for every prefix-free adversary $\mathcal{A}$ that plays Attack Game 4.2 with respect to $\tilde{F}$, and which makes at most Q queries to its challenger, there exists a PRG adversary $\mathcal{B}$ that plays Attack Game 3.1 with respect to G, where $\mathcal{B}$ is an elementary wrapper $\mathcal{A}$, such that

$$\mathrm{PRF^{pf}}\mathsf{adv}[\mathcal{A}, \tilde{F}] = \ell Q \cdot \mathrm{PRG}\mathsf{adv}[\mathcal{B}, G].$$

Proof. The basic idea of the proof is exactly the same as that of Theorem 4.10. We sketch here the main ideas, highlighting the differences from that proof.

Let $\mathcal{A}$ be an efficient, prefix-free adversary that plays Attack Game 4.2 with respect to $\tilde{F}$. Assume that $\mathcal{A}$ makes at most Q queries to its challenger. Moreover, it will be convenient to assume that $\mathcal{A}$ never makes the same query twice. Thus, we are assuming that $\mathcal{A}$ never makes two queries, one of which is equal to, or is a prefix of, another. The challenger in Attack Game 4.2 will not enforce this assumption — we simply assume that $\mathcal{A}$ is playing by the rules.

As before, we view the evaluation of $\tilde{F}(s, \cdot)$ in terms of an evaluation tree: the root is labeled by s, and the labels on all other nodes are assigned derived labels. The only difference now is that inputs to $\tilde{F}(s, \cdot)$ may address *internal* nodes of the evaluation tree. However, the prefix-freeness restriction means that no input can address a node that is an ancestor of a node addressed by a different input.

We again define hybrid games, Hybrid 0, ..., Hybrid ℓ. In these games, the challenger uses an evaluation tree labeled in exactly the same way as in the proof of Theorem 4.10: in Hybrid j, nodes at levels 0 through j are assigned random labels, and nodes at other levels are assigned derived labels. The challenger responds to a query x by returning the label of the node in the tree addressed by x, which need not be a leaf. More formally, the challenger in Hybrid j works as follows:

[3] For sequences $x = (a_1 \ldots a_s)$ and $y = (b_1 \ldots b_t)$, if $s \leq t$ and $a_i = b_i$ for $i = 1, \ldots, s$, then we say that x is a **prefix** of y; moreover, if $s < t$, then we say x is a **proper prefix** of y.

$f \stackrel{R}{\leftarrow} \text{Funs}[\{0,1\}^{\leq j}, \mathcal{S}]$
upon receiving a query $x = (a_1, \ldots, a_n) \in \{0,1\}^{\leq \ell}$ from $\mathcal{A}$ do:
    if $n < j$ then
        then $y \leftarrow f(x)$
        else $u \leftarrow (a_1, \ldots, a_j)$, $v \leftarrow (a_{j+1}, \ldots, a_n)$, $y \leftarrow G^*(f(u), v)$
    send y to $\mathcal{A}$.

For $j = 0, \ldots, \ell$, define p_j to be the probability that $\mathcal{A}$ outputs 1 in Hybrid j. As the reader may easily verify, we have

$$\text{PRF}^{\text{pf}}\mathsf{adv}[\mathcal{A}, \tilde{F}] = |p_\ell - p_0|.$$

Next, we define an efficient PRG adversary $\mathcal{B}'$ that attacks the Q-wise parallel composition G' of G, such that

$$\text{PRG}\mathsf{adv}[\mathcal{B}', G'] = \frac{1}{\ell} \cdot |p_\ell - p_0|.$$

Adversary $\mathcal{B}'$ runs as follows:

upon receiving $\vec{r}$ as in (4.32) from its challenger, $\mathcal{B}'$ plays the role of challenger to $\mathcal{A}$, as follows:

$\omega \stackrel{R}{\leftarrow} \{1, \ldots, \ell\}$
initialize an empty associative array $Map : \{0,1\}^* \to \mathbb{Z}_{>0}$
$ctr \leftarrow 0$
upon receiving a query $x = (a_1, \ldots, a_n) \in \{0,1\}^{\leq \ell}$ from $\mathcal{A}$ do:
    if $n < \omega$ then
(∗) $y \stackrel{R}{\leftarrow} \mathcal{S}$
    else
        $u \leftarrow (a_1, \ldots, a_{\omega-1})$, $d \leftarrow a_\omega$, $v \leftarrow (a_{\omega+1}, \ldots, n)$
        if $u \notin \text{Domain}(Map)$ then
            $ctr \leftarrow ctr + 1$, $Map[u] \leftarrow ctr$
        $p \leftarrow Map[u]$, $y \leftarrow G^*(r_{pd}, v)$
    send y to $\mathcal{A}$.

Finally, $\mathcal{B}'$ outputs whatever $\mathcal{A}$ outputs.

For $b = 0, 1$, let W_b be the event that $\mathcal{B}'$ outputs 1 in Experiment b of Attack Game 4.2 with respect to G'. It is not too hard to see that for any fixed $j = 1, \ldots, \ell$, we have

$$\Pr[W_0 \mid \omega = j] = p_{j-1} \quad \text{and} \quad \Pr[W_1 \mid \omega = j] = p_j.$$

Indeed, condition on $\omega = j$ for fixed j, and consider how $\mathcal{B}'$ labels nodes in the evaluation tree. At the line marked (∗), $\mathcal{B}'$ assigns random labels to all nodes in the evaluation tree at levels 0 through $j - 1$, and the assumption that $\mathcal{A}$ never makes the same query twice guarantees that these labels are consistent (the same node does not receive two different labels at different times). Now, on the one hand, when $\mathcal{B}'$ is in Experiment 1 of its attack game, it effectively assigns random labels to nodes at level j as well, and the lookup table ensures that this is done consistently. On the other hand, when $\mathcal{B}'$ is in Experiment 0 of its attack game, it effectively assigns pseudo-random labels to nodes at level j, which is the same as assigning random labels to the parents of these nodes at level

$j-1$; the prefix-freeness assumption ensures that none of these parent nodes are inconsistently assigned random labels at the line marked $(*)$.

The rest of the proof goes through as in the proof of Theorem 4.10. □

4.7 The ideal cipher model

Block ciphers are used in a variety of cryptographic constructions. Sometimes it is impossible or difficult to prove a security theorem for some of these constructions under standard security assumptions. In these situations, a heuristic technique — called the **ideal cipher model** — is sometimes employed. Roughly speaking, in this model, the security analysis is done by treating the block cipher *as if* it were a family of random permutations. If $\mathcal{E} = (E, D)$ is a block cipher defined over $(\mathcal{K}, \mathcal{X})$, then the family of random permutations is $\{\Pi_k\}_{k \in \mathcal{K}}$, where each Π_k is a truly random permutation on $\mathcal{X}$, and the Π_k's collectively are mutually independent. These random permutations are much too large to write down and cannot be used in a real construction. Rather, they are used to *model* a construction based on a real block cipher, to obtain a *heuristic* security argument for a given construction. We stress the heuristic nature of the ideal cipher model: while a proof of security in this model is better than nothing, it does not rule out an attack by an adversary that exploits the design of a particular block cipher, even one that is secure in the sense of Definition 4.1.

4.7.1 Formal definitions

Suppose we have some type of cryptographic scheme $\mathcal{S}$ whose implementation makes use of a block cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{X})$. Moreover, suppose the scheme $\mathcal{S}$ evaluates E at various inputs $(k, a) \in \mathcal{K} \times \mathcal{X}$, and D at various inputs $(k, b) \in \mathcal{K} \times \mathcal{X}$, but does not look at the internal implementation of $\mathcal{E}$. In this case, we say that $\mathcal{S}$ **uses** $\mathcal{E}$ **as an oracle**.

We wish to analyze the security of $\mathcal{S}$. Let us assume that whatever security property we are interested in, say "property X," is modeled (as usual) as a game between a challenger (specific to property X) and an arbitrary adversary $\mathcal{A}$. Presumably, in responding to certain queries, the challenger computes various functions associated with the scheme $\mathcal{S}$, and these functions may in turn require the evaluation of E and/or D at certain points. This game defines an advantage $\mathsf{Xadv}[\mathcal{A}, \mathcal{S}]$, and security with respect to property X means that this advantage should be negligible for all efficient adversaries $\mathcal{A}$.

If we wish to analyze $\mathcal{S}$ in the ideal cipher model, then the attack game defining security is modified so that $\mathcal{E}$ is effectively replaced by a family of random permutations $\{\Pi_k\}_{k \in \mathcal{K}}$, as described above, to which both the adversary and the challenger have oracle access. More precisely, the game is modified as follows.

- At the beginning of the game, the challenger chooses $\Pi_k \in \text{Perms}[\mathcal{K}]$ at random, for each $k \in \mathcal{K}$.

- In addition to its standard queries, the adversary $\mathcal{A}$ may submit *ideal cipher queries*. There are two types of queries: Π-queries and Π^{-1}-queries.

  - For a Π-query, the adversary submits a pair $(k, a) \in \mathcal{K} \times \mathcal{X}$, to which the challenger responds with $\Pi_k(a)$.

– For a Π^{-1}-query, the adversary submits a pair $(\mathscr{k}, \mathscr{b}) \in \mathcal{K} \times \mathcal{X}$, to which the challenger responds with $\Pi_{\mathscr{k}}^{-1}(\mathscr{b})$.

The adversary may make any number of ideal cipher queries, arbitrarily interleaved with standard queries.

- In processing standard queries, the challenger performs its computations using $\Pi_{\mathscr{k}}(\mathscr{a})$ in place of $E(\mathscr{k}, \mathscr{a})$ and $\Pi_{\mathscr{k}}^{-1}(\mathscr{b})$ in place of $D(\mathscr{k}, \mathscr{b})$.

The adversary's advantage is defined using the same rule as before, but is denoted $\mathrm{X}^{\mathrm{ic}}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$ to emphasize that this is an advantage *in the ideal cipher model*. Security in the ideal cipher model means that $\mathrm{X}^{\mathrm{ic}}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$ should be negligible for all efficient adversaries $\mathcal{A}$.

It is important to understand the role of the ideal cipher queries. Essentially, they model the ability of an adversary to make "offline" evaluations of E and D.

Ideal permutation model. Some constructions, like Even-Mansour (discussed below), make use of a permutation $\pi : \mathcal{X} \to \mathcal{X}$, rather than a block cipher. In the security analysis, one might heuristically model π as a random permutation Π, to which all parties in the attack game have oracle access to Π and Π^{-1}. We call this the **ideal permutation model**. One can view this as a special case of the ideal cipher model by simply defining $\Pi = \Pi_{\mathscr{k}_0}$ for some fixed, publicly available key $\mathscr{k}_0 \in \mathcal{K}$.

4.7.2 Exhaustive search in the ideal cipher model

Let (E, D) be a block cipher defined over $(\mathcal{K}, \mathcal{X})$ and let k be some random secret key in $\mathcal{K}$. Suppose an adversary is able to intercept a small number of input/output pairs (x_i, y_i) generated using k:

$$y_i = E(k, x_i) \quad \text{for all } i = 1, \dots, Q.$$

The adversary can now recover k by trying all possible keys in $\mathscr{k} \in \mathcal{K}$ until a key $\mathscr{k}$ satisfying $y_i = E(\mathscr{k}, x_i)$ for all $i = 1, \dots, Q$ is found. For block ciphers used in practice it is likely that this $\mathscr{k}$ is equal to the secret key k used to generate the given pairs. This **exhaustive search** over the key space recovers the block-cipher secret-key in time $O(|\mathcal{K}|)$ using a small number of input/output pairs. We analyze the number of input/output pairs needed to mount a successful attack in Theorem 4.12 below.

Exhaustive search is the simplest example of a key-recovery attack. Since we will present a number of key-recovery attacks, let us first define the key-recovery attack game in more detail. We will primarily use the key-recovery game as means of presenting attacks.

Attack Game 4.4 (key-recovery). For a given block cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{X})$, and for a given adversary $\mathcal{A}$, define the following game:

- The challenger picks a random $k \xleftarrow{R} \mathcal{K}$.
- $\mathcal{A}$ queries the challenger several times. For $i = 1, 2, \dots,$ the ith query consists of a message $x_i \in \mathcal{M}$. The challenger, given x_i, computes $y_i \xleftarrow{R} E(k, x_i)$, and gives y_i to $\mathcal{A}$.
- Eventually $\mathcal{A}$ outputs an candidate key $\mathscr{k} \in \mathcal{K}$.

We say that $\mathcal{A}$ wins the game if $\mathfrak{k} = k$. We let $\mathrm{KR}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ denote the probability that $\mathcal{A}$ wins the game. $\square$

The key-recovery game extends naturally to the ideal cipher model, where $E(\mathfrak{k}, \mathfrak{a}) = \Pi_{\mathfrak{k}}(\mathfrak{a})$ and $D(\mathfrak{k}, \mathfrak{b}) = \Pi_{\mathfrak{k}}^{-1}(\mathfrak{b})$, and $\{\Pi_{\mathfrak{k}}\}_{\mathfrak{k}\in\mathcal{K}}$ is a family of independent random permutations. In this model, we allow the adversary to make arbitrary Π- and Π^{-1}-queries, in addition to its standard queries to $E(k, \cdot)$. We let $\mathrm{KR}^{\mathrm{ic}}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ denote the adversary's key-recovery advantage when $\mathcal{E}$ is an ideal cipher.

It is worth noting that security against key-recovery attacks does not imply security in the sense of indistinguishability (Definition 4.1). The simplest example is the constant block cipher $E(k, x) = x$ for which key-recovery is not possible (the adversary obtains no information about k), but the block cipher is easily distinguished from a random permutation.

Exhaustive search. The following theorem bounds the number of input/output pairs needed for exhaustive search, assuming the cipher is an ideal cipher. For real-world parameters, taking $Q = 3$ in the theorem is often sufficient to ensure success.

Theorem 4.12. *Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{X})$. Then there exists an adversary $\mathcal{A}_{\mathrm{EX}}$ that plays Attack Game 4.4 with respect to $\mathcal{E}$, modeled as an ideal cipher, making Q standard queries and $Q|\mathcal{K}|$ ideal cipher queries, such that*

$$\mathrm{KR}^{\mathrm{ic}}\mathsf{adv}[\mathcal{A}_{\mathrm{EX}}, \mathcal{E}] \geq (1 - \epsilon) \quad \textit{where} \quad \epsilon := \frac{|\mathcal{K}|}{(|\mathcal{X}| - Q)^Q} \tag{4.34}$$

Proof. In the ideal cipher model, we are modeling the block cipher $\mathcal{E} = (E, D)$ as a family $\{\Pi_{\mathfrak{k}}\}_{\mathfrak{k}\in\mathcal{K}}$ of random permutations on $\mathcal{X}$. In Attack Game 4.4, the challenger chooses $k \in \mathcal{K}$ at random. An adversary may make standard queries to obtain the value $E(k, x) = \Pi_k(x)$ at points $x \in \mathcal{X}$ of his choosing. An adversary may also make ideal cipher queries, obtaining the values $\Pi_{\mathfrak{k}}(\mathfrak{a})$ and $\Pi_{\mathfrak{k}}^{-1}(\mathfrak{b})$ for points $\mathfrak{k} \in \mathcal{K}$ and $\mathfrak{a}, \mathfrak{b} \in \mathcal{X}$ of his choosing. These ideal cipher queries correspond to "offline" evaluations of E and D.

Our adversary $\mathcal{A}_{\mathrm{EX}}$ works as follows:

let $\{x_1, \ldots, x_Q\}$ be an arbitrary set of distinct messages in $\mathcal{X}$
for $i = 1, \ldots, Q$ do:
    make a standard query to obtain $y_i := E(k, x_i) = \Pi_k(x_i)$
for each $\mathfrak{k} \in \mathcal{K}$ do:
    for $i = 1, \ldots, Q$ do:
        make an ideal cipher query to obtain $\mathfrak{b}_i := \Pi_{\mathfrak{k}}(x_i)$
    if $y_i = \mathfrak{b}_i$ for all $i = 1, \ldots, Q$ then
        output $\mathfrak{k}$ and terminate

Let k be the challenger's secret-key. We show that $\mathcal{A}_{\mathrm{EX}}$ outputs k with probability at least $1 - \epsilon$, with ϵ defined as in (4.34). Since $\mathcal{A}_{\mathrm{EX}}$ tries all keys, this amounts to showing that the probability that there is more than one key consistent with the given (x_i, y_i) pairs is at most ϵ. We shall show that this holds for every possible choice of k, so for the remainder of the proof, we shall view k as fixed. We shall also view $x_1, \ldots, x_Q$ as fixed, so all the probabilities are with respect to the random permutations $\Pi_{\mathfrak{k}}$ for $\mathfrak{k} \in \mathcal{K}$.

For each $\tilde{k} \in \mathcal{K}$, let $W_{\tilde{k}}$ be the event that $y_i = \Pi_{\tilde{k}}(x_i)$ for all $i = 1, \ldots, Q$. Note that by definition, W_k occurs with probability 1. Let W be the event that $W_{\tilde{k}}$ occurs for some $\tilde{k} \neq k$. We want to show that $\Pr[W] \leq \epsilon$.

Fix $\tilde{k} \neq k$. Since the permutation Π_k is chosen independently of the permutation $\Pi_{\tilde{k}}$, we know that

$$\Pr[W_{\tilde{k}}] = \frac{1}{|\mathcal{X}|} \cdot \frac{1}{|\mathcal{X}| - 1} \cdots \frac{1}{|\mathcal{X}| - Q + 1} \leq \left(\frac{1}{|\mathcal{X}| - Q}\right)^Q$$

As this holds for all $\tilde{k} \neq k$, the result follows from the union bound. $\square$

Security of the $3\mathcal{E}$ construction

The attack presented in Theorem 4.2 works equally well against the $3\mathcal{E}$ construction. The size of the key space is $|\mathcal{K}|^3$, but one obtains a "meet in the middle" key recovery algorithm that runs in time $O\big(|\mathcal{K}|^2 \cdot Q\big)$. For Triple-DES this algorithm requires more than $2^{2 \cdot 56}$ evaluations of Triple-DES, which is far beyond our computing power.

One wonders whether better attacks against $3\mathcal{E}$ exist. When $\mathcal{E}$ is an ideal cipher we can prove a lower bound on the amount of work needed to distinguish $3\mathcal{E}$ from a random permutation.

Theorem 4.13. *Let $\mathcal{E} = (E, D)$ be an ideal block cipher defined over $(\mathcal{K}, \mathcal{X})$, and consider an attack against the $3\mathcal{E}$ construction in the ideal cipher model. If $\mathcal{A}$ is an adversary that makes at most Q queries (including both standard and ideal cipher queries) in the ideal cipher variant of Attack Game 4.1, then*

$$\mathrm{BC^{ic}}\mathsf{adv}[\mathcal{A}, 3\mathcal{E}] \leq C_1 L \frac{Q^2}{|\mathcal{K}|^3} + C_2 \frac{Q^{2/3}}{|\mathcal{K}|^{2/3} |\mathcal{X}|^{1/3}} + C_3 \frac{1}{|\mathcal{K}|},$$

where $L := \max(|\mathcal{K}|/|\mathcal{X}|, \log_2 |\mathcal{X}|)$, and C_1, C_2, C_3 are constants (that do not depend on $\mathcal{A}$ or $\mathcal{E}$).

The statement of the theorem is easier to understand if we assume that $|\mathcal{K}| \leq |\mathcal{X}|$, as is the case with DES. In this case, the bound can be restated as

$$\mathrm{BC^{ic}}\mathsf{adv}[\mathcal{A}, 3\mathcal{E}] \leq C \log_2 |\mathcal{X}| \frac{Q^2}{|\mathcal{K}|^3},$$

for a constant C. Ignoring the $\log \mathcal{X}$ term, this says that an adversary must make roughly $|\mathcal{K}|^{1.5}$ queries to obtain a significant advantage (say, 1/4). Compare this to the meet-in-the-middle attack. To achieve a significant advantage, that adversary must make roughly $|\mathcal{K}|^2$ queries. Thus, meet-in-the-middle attack may not be the most powerful attack.

To conclude our discussion of Triple-DES, we note that the $3\mathcal{E}$ construction does not always strengthen the cipher. For example, if $\mathcal{E} = (E, D)$ is such that the set of $|\mathcal{K}|$ permutations $\{E(\tilde{k}, \cdot) : \tilde{k} \in \mathcal{K}\}$ is a group, then $3\mathcal{E}$ would be no more secure than $\mathcal{E}$. Indeed, in this case $\pi := E_3((k_1, k_2, k_3), \cdot)$ is identical to $E(k, \cdot)$ for some $k \in \mathcal{K}$. Consequently, distinguishing $3\mathcal{E}$ from a random permutation is no harder than doing so for $\mathcal{E}$. Of course, block ciphers used in practice are not groups (as far as we know).

4.7.3 The Even-Mansour block cipher and the $\mathcal{E}X$ construction

Let $\mathcal{X} = \{0,1\}^n$. Let $\pi : \mathcal{X} \to \mathcal{X}$ be a permutation and let π^{-1} be its inverse function. Even and Mansour defined the following simple block cipher $\mathcal{E}_{EM} = (E, D)$ defined over $(\mathcal{X}^2, \mathcal{X})$:

$$E\big((P_1, P_2),\ x\big) := \pi(x \oplus P_1) \oplus P_2 \qquad \text{and} \qquad D\big((P_1, P_2),\ y\big) := \pi^{-1}(y \oplus P_2) \oplus P_1 \tag{4.35}$$

How do we analyze the security of this block cipher? Clearly for some π's this construction is insecure, for example when π is the identity function. For what π is $\mathcal{E}_{EM}$ a secure block cipher?

The only way we know to analyze security of $\mathcal{E}_{EM}$ is by modeling π as a random permutation Π on the set $\mathcal{X}$ (i.e., in the ideal cipher model using a fixed key). We show in Theorem 4.14 below that in the ideal cipher model, for all adversaries $\mathcal{A}$:

$$\mathrm{BC^{ic}}\mathsf{adv}[\mathcal{E}_{EM}, \mathcal{A}] \leq \frac{2Q_{\mathrm{s}}Q_{\mathrm{ic}}}{|\mathcal{X}|} \tag{4.36}$$

where Q_{s} is the number of queries $\mathcal{A}$ makes to $\mathcal{E}_{EM}$ and Q_{ic} is the number of queries $\mathcal{A}$ makes to Π and Π^{-1}. Hence, the Even-Mansour block cipher is secure (in the ideal cipher model) whenever $|\mathcal{X}|$ is sufficiently large. Exercise 4.21 shows that the bound (4.36) is tight.

The Even-Mansour security theorem (Theorem 4.14) does not require the keys P_1 and P_2 to be independent. In fact, the bounds in (4.36) remain unchanged if we set $P_1 = P_2$ so that the key for $\mathcal{E}_{EM}$ is a single element of $\mathcal{X}$. However, we note that if one leaves out either of P_1 or P_2, the construction is completely insecure (see Exercise 4.20).

Iterated Even-Mansour and AES. Looking back at our description of AES (Fig. 4.11) one observes that the Even-Mansour cipher looks a lot like one round of AES where the round function Π_{AES} plays the role of π. Of course one round of AES is not a secure block cipher: the bound in (4.36) does not imply security because Π_{AES} is not a random permutation.

Suppose one replaces each occurrence of Π_{AES} in Fig. 4.11 by a different permutation: one function for each round of AES. The resulting structure, called **iterated Even-Mansour**, can be analyzed in the ideal cipher model and the resulting security bounds are better than those stated in (4.36).

These results suggest a theoretical justification for the AES structure in the ideal cipher model.

The $\mathcal{E}X$ construction and DESX. If we apply the Even-Mansour construction to a full-fledged block cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{X})$, we obtain a new block cipher called $\mathcal{E}X = (EX, DX)$ where

$$EX\big((k, P_1, P_2),\ x\big) := E(k,\ x \oplus P_1) \oplus P_2\ , \qquad DX\big((k, P_1, P_2),\ y\big) := D(k,\ y \oplus P_2) \oplus P_1. \tag{4.37}$$

This new cipher $\mathcal{E}X$ has a key space $\mathcal{K} \times \mathcal{X}^2$ which can be much larger than the key space for the underlying cipher $\mathcal{E}$.

Theorem 4.14 below shows that — in the ideal cipher model — this larger key space translates to better security: the maximum advantage against $\mathcal{E}X$ is much smaller than the maximum advantage against $\mathcal{E}$, whenever $|\mathcal{X}|$ is sufficiently large.

Applying $\mathcal{E}X$ to the DES block cipher gives an efficient method to immunize DES against exhaustive search attacks. With $P_1 = P_2$ we obtain a block cipher called **DESX** whose key size is $56 + 64 = 120$ bits: enough to resist exhaustive search. Theorem 4.14 shows that attacks in the ideal cipher model on the resulting cipher are impractical. Since evaluating DESX requires only one call to DES, the DESX block cipher is three times faster than the Triple-DES block cipher and this makes it seem as if DESX is the preferred way to strengthen DES. However, non black-box attacks like differential and linear cryptanalysis still apply to DESX where as they are ineffective against Triple-DES. Consequently, DESX should not be used in practice.

4.7.4 Proof of the Even-Mansour and $\mathcal{E}X$ theorems

We shall prove security of the Even-Mansour block cipher (4.35) in the ideal permutation model and of the $\mathcal{E}X$ construction (4.37) in the ideal cipher model.

We prove their security in a single theorem below. Taking a single-key block cipher (i.e., $|\mathcal{K}| = 1$) proves security of Even-Mansour in the ideal permutation model. Taking a block cipher with a larger key space proves security of $\mathcal{E}X$. Note that the pads P_1 and P_2 need not be independent and the theorem holds if we set $P_2 = P_1$.

Theorem 4.14. *Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{X})$. Let $\mathcal{E}X = (EX, DX)$ be the block cipher derived from $\mathcal{E}$ as in construction (4.37), where P_1 and P_2 are each uniformly distributed over a subset of $\mathcal{X}'$ of $\mathcal{X}$. If we model $\mathcal{E}$ as an ideal cipher, and if $\mathcal{A}$ is an adversary in Attack Game 4.1 for $\mathcal{E}X$ that makes at most $Q_{\rm s}$ standard queries (i.e., EX-queries) and $Q_{\rm ic}$ ideal cipher queries (i.e., Π- or Π^{-1}-queries), then we have*

$$\mathrm{BC^{ic}}\mathsf{adv}[\mathcal{A}, \mathcal{E}X] \leq \frac{2Q_{\rm s}Q_{\rm ic}}{|\mathcal{K}||\mathcal{X}'|}. \quad \Box \tag{4.38}$$

To understand the security benefit of the $\mathcal{E}X$ construction consider the following: modeling $\mathcal{E}$ as an ideal cipher gives $\mathrm{BC^{ic}}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq Q_{\rm ic}/|\mathcal{K}|$ for all $\mathcal{A}$. Hence, Theorem 4.14 shows that, in the ideal cipher model, applying $\mathcal{E}X$ to $\mathcal{E}$ shrinks the maximum advantage by a factor of $2Q_{\rm s}/|\mathcal{X}'|$.

The bounds in Theorem 4.14 are tight: there is an adversary $\mathcal{A}$ that achieves the advantage shown in (4.38); see Exercise 4.21. The advantage of this $\mathcal{A}$ is unchanged even when P_1 and P_2 are chosen independently. Therefore, we might as well always choose $P_2 = P_1$.

We also note that it is actually no harder to prove that $\mathcal{E}X$ is a *strongly secure* block cipher (see Section 4.1.3) in the ideal cipher model, with exactly the same security bounds as in Theorem 4.14.

Proof idea. The basic idea is to show that the ideal cipher queries and the standard queries do not interact with each other, except with probability as bounded in (4.38). Indeed, to make the two types of queries interact with each other, the adversary has to make

$$(\mathscr{k} = k \text{ and } \mathscr{a} = x \oplus P_1) \quad \text{or} \quad (\mathscr{k} = k \text{ and } \mathscr{b} = y \oplus P_2)$$

for some input/output pair (x, y) corresponding to a standard query and some input/output triple $(\mathscr{k}, \mathscr{a}, \mathscr{b})$ corresponding to an ideal cipher query. Essentially, the adversary will have to simultaneously guess the random key k as well as one of the random pads P_1 or P_2.

Assuming there are no such interactions, we can effectively realize all of the standard queries as $\Pi(x \oplus P_1) \oplus P_2$ using a random permutation Π that is independent of the random permutations used to realize the ideal cipher queries. But $\Pi'(x) := \Pi(x \oplus P_1) \oplus P_2$ is just a random permutation.

Before giving a rigorous proof of Theorem 4.14, we present a technical lemma, called the **Domain Separation Lemma**, that will greatly simplify the proof, and is useful in analyzing other constructions.

To motivate the lemma, consider the following two experiments. In the one experiment, called the "split experiment", an adversary has oracle access to two random permutations Π_1, Π_2 on a set $\mathcal{X}$. The adversary can make a series of queries, each of the form (μ, d, z), where $\mu \in \{1, 2\}$ specifies which of the two permutations to evaluate, $d \in \{\pm 1\}$ specifies the direction to evaluate the permutation, and $z \in \mathcal{X}$ the input to the permutation. On such a query, the challenger responds with $z' := \Pi_\mu^d(z)$. Another experiment, called the "coalesced experiment", is exactly the same as the split experiment, except that there is only a single permutation Π, and the challenger answers the query (μ, d, z) with $z' := \Pi^d(z)$, ignoring completely the index μ. The question is: under what condition can the adversary distinguish between these two experiments?

Obviously, if the adversary can submit a query $(1, +1, a)$ and a query $(2, +1, a)$, then in the split experiment, the results will almost certainly be different, while in the coalesced experiment, they will surely be the same. Another type of attack is possible as well: the adversary could make a query $(1, +1, a)$ obtaining b, and then submit the query $(2, -1, b)$, obtaining a'. In the split experiment, a and a' will almost certainly be different, while in the coalesced experiment, they will surely be the same. Besides these two examples, one could get two more examples which reverse the direction of all the queries. The Domain Separation Lemma will basically say that unless the adversary makes queries of one of these four types, he cannot distinguish between these two experiments.

Of course, the Domain Separation Lemma is only useful in contexts where the adversary is somehow constrained so that he cannot freely make queries of his choice. Indeed, we will only use it inside of the proof of a security theorem where the "adversary" in the Domain Separation Lemma comprises components of a challenger and an adversary in a more interesting attack game.

In the more general statement of the lemma, we replace Π_1 and Π_2 by a family of permutations of permutations $\{\Pi_\mu\}_{\mu \in U}$, and we replace Π by a family $\{\overline{\Pi}_\nu\}_{\nu \in V}$. We also introduce a function $f : U \to V$ that specifies how several permutations in the split experiment are collapsed into one permutation in the coalesced experiment: for each $\nu \in V$, all the permutations Π_μ in the split experiment for which $f(\mu) = \nu$ are collapsed into the single permutation $\overline{\Pi}_\nu$ in the coalesced experiment.

In the generalized version of the distinguishing game, if the adversary makes a query (μ, d, z), then in the split experiment, the challenger responds with $z' := \Pi_\mu^d(z)$, while in the coalesced experiment, the challenger responds with $z' := \Pi_{f(\mu)}^d(z)$. In the split experiment, we also keep track of the subset of the domains and ranges of the permutations that correspond to actual queries made by the adversary in the split experiment. That is, we build up sets $\mathrm{Dom}_\mu^{(d)}$ for each $\mu \in U$ and $d \in \pm 1$, so that $a \in \mathrm{Dom}_\mu^{(+1)}$ if and only if the adversary issues a query of the form $(\mu, +1, a)$ or a query of the form $(\mu, -1, b)$ that yields a. Similarly, $b \in \mathrm{Dom}_\mu^{(-1)}$ if and only if the adversary issues a query of the form $(\mu, -1, b)$ or a query of the form $(\mu, +1, a)$ that yields b. We call $\mathrm{Dom}_\mu^{(+1)}$ the **sampled domain** of Π_μ and $\mathrm{Dom}_\mu^{(-1)}$ the **sampled range** of Π_μ.

Attack Game 4.5 (domain separation). Let $U, V, \mathcal{X}$ be finite, nonempty sets, and let $f : U \to V$ be a function. For a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define:

Experiment b:

- For each $\mu \in U$, and each $\nu \in V$ the challenger sets $\Pi_\mu \xleftarrow{\text{R}} \text{Perms}[\mathcal{X}]$ and $\overline{\Pi}_\nu \xleftarrow{\text{R}} \text{Perms}[\mathcal{X}]$ Also, for each $\mu \in U$ and $d \in \{\pm 1\}$ the challenger sets $\mathrm{Dom}_\mu^{(d)} \leftarrow \emptyset$.

- The adversary submits a sequence of queries to the challenger.

For $i = 1, 2, \ldots$, the ith query is $(\mu_i, d_i, z_i) \in U \times \{\pm 1\} \times \mathcal{X}$.

If $b = 0$: the challenger sets $z'_i \leftarrow \overline{\Pi}^{d_i}_{f(\mu_i)}(z_i)$.

If $b = 1$: the challenger sets $z'_i \leftarrow \Pi^{d_i}_{\mu_i}(z_i)$; the challenger also adds the value z_i to the set $\mathrm{Dom}^{(d_i)}_{\mu_i}$, and adds the value z'_i to the set $\mathrm{Dom}^{(-d_i)}_{\mu_i}$.

In either case, the challenger then sends z'_i to the adversary.

- Finally, the adversary outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **domain separation distinguishing advantage** as $|\Pr[W_0] - \Pr[W_1]|$. We also define the **domain separation failure event** Z to be the event that *in Experiment 1*, at the end of the game we have $\mathrm{Dom}^{(d)}_{\mu} \cap \mathrm{Dom}^{(d)}_{\mu'} \neq \emptyset$ for some $d \in \{\pm 1\}$ and some pair of *distinct* indices $\mu, \mu' \in U$ with $f(\mu) = f(\mu')$. Finally, we define the **domain separation failure probability** to be $\Pr[Z]$. □

Experiment 1 is the above game is the split experiment and Experiment 0 is the coalesced experiment.

Theorem 4.15 (Domain Separation Lemma). *In Attack Game 4.5, an adversary's domain separation distinguishing advantage is bounded by the domain separation failure probability.*

In the applying the Domain Separation Lemma, we will typically analyze some attack game in which permutations start out as coalesced, and then force them to be separated. We can bound the impact of this change on the outcome of the attack by analyzing the domain separation failure probability in the attack game with the split permutations.

Before proving the Domain Separation Lemma, it is perhaps more instructive to see how it is used in the proof of Theorem 4.14.

Proof of Theorem 4.14. Let $\mathcal{A}$ be an adversary as in the statement of the theorem. For $b = 0, 1$ let p_b be the probability that $\mathcal{A}$ outputs 1 in Experiment b of the block cipher attack game in the ideal cipher model (Attack Game 4.1). So by definition we have

$$\mathrm{BC}^{\mathrm{ic}}\mathsf{adv}[\mathcal{A}, \mathcal{E}X] = |p_0 - p_1|. \tag{4.39}$$

We shall prove the theorem using a sequence of two games, applying the Domain Separation Lemma.

Game 0. We begin by describing Game 0, which corresponds to Experiment 0 of the block cipher attack game in the ideal cipher model. Recall that in this model, we have a family of random permutations, and the encryption function is implemented in terms of this family. Also recall that in addition to standard queries that probe the function $E_k(\cdot)$, the adversary may also probe the random permutations.

Initialize:
  for each $\Bbbk \in \mathcal{K}$, set $\Pi_{\Bbbk} \xleftarrow{\mathrm{R}} \mathrm{Perms}[\mathcal{X}]$
  $k \xleftarrow{\mathrm{R}} \mathcal{K}$, choose P_1, P_2

standard EX-query x:
1. $\quad a \leftarrow x \oplus P_1$
2. $\quad b \leftarrow \Pi_k(a)$
3. $\quad y \leftarrow b \oplus P_2$
4. $\quad$ return y

ideal cipher Π-query k, a:
1. $\quad b \leftarrow \Pi_k(a)$
2. $\quad$ return b

ideal cipher Π^{-1}-query k, b:
1. $\quad a \leftarrow \Pi_k^{-1}(b)$
2. $\quad$ return a

Let W_0 be the event that $\mathcal{A}$ outputs 1 at the end of Game 0. It should be clear from construction that

$$\Pr[W_0] = p_0. \tag{4.40}$$

Game 1. In this game, we apply the Domain Separation Lemma. The basic idea is that we will declare "by fiat" that the random permutations used in processing the standard queries are independent of the random permutations used in processing ideal cipher queries. Effectively, each permutation Π_k gets split into two independent permutations: $\Pi_{\mathrm{std},k}$, which is used by the challenger in responding to standard EX-queries, and $\Pi_{\mathrm{ic},k}$, which is used in responding to ideal cipher queries. In detail (changes from Game 0 are highlighted):

Initialize:
$\quad$ for each $k \in \mathcal{K}$, set $\boxed{\Pi_{\mathrm{std},k} \xleftarrow{R} \text{Perms}[\mathcal{X}] \text{ and } \Pi_{\mathrm{ic},k} \xleftarrow{R} \text{Perms}[\mathcal{X}]}$
$\quad k \xleftarrow{R} \mathcal{K}$, choose P_1, P_2

standard EX-query x:
1. $\quad a \leftarrow x \oplus P_1$
2. $\quad b \leftarrow \boxed{\Pi_{\mathrm{std},k}(a)}$ // *add a to sampled domain of $\Pi_{\mathrm{std},k}$, add b to sampled range of $\Pi_{\mathrm{std},k}$*
3. $\quad y \leftarrow b \oplus P_2$
4. $\quad$ return y

ideal cipher Π-query k, a:
1. $\quad b \leftarrow \boxed{\Pi_{\mathrm{ic},k}(a)}$ // *add a to sampled domain of $\Pi_{\mathrm{ic},k}$, add b to sampled range of $\Pi_{\mathrm{ic},k}$*
2. $\quad$ return b

ideal cipher Π^{-1}-query k, b:
1. $\quad a \leftarrow \boxed{\Pi_{\mathrm{ic},k}^{-1}(b)}$ // *add a to sampled domain of $\Pi_{\mathrm{ic},k}$, add b to sampled range of $\Pi_{\mathrm{ic},k}$*
2. $\quad$ return a

Let W_1 be the event that $\mathcal{A}$ outputs 1 at the end of Game 1. Let Z be the event that in Game 1 there exists $k \in \mathcal{K}$, such that the sampled domains of $\Pi_{\mathrm{ic},k}$ and $\Pi_{\mathrm{std},k}$ overlap or the sampled ranges of $\Pi_{\mathrm{ic},\mathpzc{k}}$ and $\Pi_{\mathrm{std},\mathpzc{k}}$ overlap. The Domain Separation Lemma says that

$$|\Pr[W_0] - \Pr[W_1]| \leq \Pr[Z]. \tag{4.41}$$

In applying the Domain Separation Lemma, the "coalescing function" f maps from $\{\mathrm{std}, \mathrm{ic}\} \times \mathcal{K}$ to $\mathcal{K}$, sending the pair $(\cdot, \mathpzc{k})$ to $\mathpzc{k}$. Observe that the challenger only makes queries to Π_k, where k is the secret key, and so such an overlap can occur only at $\mathpzc{k} = k$. Also observe that in Game 1, the random variables k, P_1, and P_2 are completely independent of the adversary's view.

So the event Z occurs if and only if for some input/output triple $(\mathpzc{k}, \mathpzc{a}, \mathpzc{b})$ triple arising from a Π- or Π^{-1}-query, and for some input/output pair (x, y) arising from an EX-query, we have

$$(\mathpzc{k} = k \text{ and } \mathpzc{a} = x \oplus P_1) \;\text{ or }\; (\mathpzc{k} = k \text{ and } \mathpzc{b} = y \oplus P_2). \tag{4.42}$$

Using the union bound, we can therefore bound $\Pr[Z]$ as a sum of probabilities of $2Q_{\mathrm{s}}Q_{\mathrm{ic}}$ events, each of the form $\mathpzc{k} = k$ and $\mathpzc{a} = x \oplus P_1$, or of the form $\mathpzc{k} = k$ and $\mathpzc{b} = y \oplus P_2$. By independence, since k is uniformly distributed over a set of size $|\mathcal{K}|$, and each of P_1 and P_2 is uniformly distributed over a set of size $|\mathcal{X}'|$, each such event occurs with probability at most $1/(|\mathcal{K}||\mathcal{X}'|)$. It follows that

$$\Pr[Z] \leq \frac{2Q_{\mathrm{s}}Q_{\mathrm{ic}}}{|\mathcal{K}||\mathcal{X}'|}. \tag{4.43}$$

Finally, observe that Game 1 is equivalent to Experiment 1 of the block cipher attack game in the ideal cipher model: the EX-queries present to the adversary the random permutation $\Pi'(x) := \Pi_{\mathrm{std},k}(x \oplus P_1) \oplus P_2$ and this permutation is independent of the random permutations used in the Π- and Π^{-1}-queries. Thus,

$$\Pr[W_1] = p_1. \tag{4.44}$$

The bound (4.38) now follows from (4.39), (4.40), (4.41), (4.43), and (4.44). This completes the proof of the theorem. □

Finally, we turn to the proof of the Domain Separation Lemma, which is a simple (if tedious) application of the Difference Lemma and the "forgetful gnome" technique.

Proof of Theorem 4.15. We define a sequence of games.

Game 0. This game will be equivalent to the coalesced experiment in Attack Game 4.5, but designed in a way that will facilitate the analysis.

In this game, the challenger maintains various sets Π of pairs $(\mathpzc{a}, \mathpzc{b})$. Each set Π represents a function that can be extended to a permutation on $\mathcal{X}$ that sends $\mathpzc{a}$ to $\mathpzc{b}$ for every $(\mathpzc{a}, \mathpzc{b})$ in Π. We call such a set Π a *partial permutation on* $\mathcal{X}$. Define

$$\mathrm{Domain}(\Pi) = \{\mathpzc{a} \in \mathcal{X} : (\mathpzc{a}, \mathpzc{b}) \in \Pi \text{ for some } \mathpzc{b} \in \mathcal{X}\}\ ,$$
$$\mathrm{Range}(\Pi) = \{\mathpzc{b} \in \mathcal{X} : (\mathpzc{a}, \mathpzc{b}) \in \Pi \text{ for some } \mathpzc{a} \in \mathcal{X}\}\ .$$

Also, for $\mathpzc{a} \in \mathrm{Domain}(\Pi)$, define $\Pi(\mathpzc{a})$ to be the unique $\mathpzc{b}$ such that $(\mathpzc{a}, \mathpzc{b}) \in \Pi$. Likewise, for $\mathpzc{b} \in \mathrm{Range}(\Pi)$, define $\Pi^{-1}(\mathpzc{b})$ to be the unique $\mathpzc{a}$ such that $(\mathpzc{a}, \mathpzc{b}) \in \Pi$.

Here is the logic of the challenger in Game 0:

Initialize:
for each $\nu \in V$, initialize the partial permutation $\overline{\Pi}_\nu \leftarrow \emptyset$

Process query $(\mu, +1, a)$:
1. if $a \in \text{Domain}(\overline{\Pi}_{f(\mu)})$ then $b \leftarrow \overline{\Pi}_{f(\mu)}(a)$, return b
2. $b \xleftarrow{R} \mathcal{X} \setminus \text{Range}(\overline{\Pi}_{f(\mu)})$
3. add (a, b) to $\overline{\Pi}_{f(\mu)}$
4. return b

Process query $(\mu, -1, b)$:
1. if $b \in \text{Range}(\overline{\Pi}_{f(\mu)})$ then $a \leftarrow \overline{\Pi}^{-1}_{f(\mu)}(b)$, return a
2. $a \xleftarrow{R} \mathcal{X} \setminus \text{Domain}(\overline{\Pi}_{f(\mu)})$
3. add (a, b) to Π_μ
4. return a

This game is clearly equivalent to the coalesced experiment in Attack Game 4.5. Let W_0 be the event that the adversary outputs 1 in this game.

Game 1. Now we modify this game to get an equivalent game, but it will facilitate the application of the Difference Lemma in moving to the next game. For $\mu, \mu' \in U$, let us write $\mu \sim \mu'$ if $f(\mu) = f(\mu')$. The is an equivalence relation on U, and we write $[\mu]$ for the equivalence class containing μ.

Here is the logic of the challenger in Game 1:

Initialize:
for each $\mu \in U$, initialize the partial permutation $\Pi_\mu \leftarrow \emptyset$

Process query $(\mu, +1, a)$:
1a. if $a \in \text{Domain}(\Pi_\mu)$ then $b \leftarrow \Pi_\mu(a)$, return b
* 1b. if $a \in \text{Domain}(\Pi_{\mu'})$ for some $\mu' \in [\mu]$ then $b \leftarrow \Pi_{\mu'}(a)$, return b
2a. $b \xleftarrow{R} \mathcal{X} \setminus \text{Range}(\Pi_\mu)$
* 2b. if $b \in \bigcup_{\mu' \in [\mu]} \text{Range}(\Pi_{\mu'})$ then $b \xleftarrow{R} \mathcal{X} \setminus \bigcup_{\mu' \in [\mu]} \text{Range}(\Pi_{\mu'})$
3. add (a, b) to Π_μ
4. return b

Process query $(\mu, -1, b)$:
1a. if $b \in \text{Range}(\Pi_\mu)$ then $a \leftarrow \Pi^{-1}_\mu(b)$, return a
* 1b. if $b \in \text{Range}(\Pi_{\mu'})$ for some $\mu' \in [\mu]$ then $a \leftarrow \Pi^{-1}_{\mu'}(b)$, return a
2a. $a \xleftarrow{R} \mathcal{X} \setminus \text{Domain}(\Pi_\mu)$
* 2b. if $a \in \bigcup_{\mu' \in [\mu]} \text{Domain}(\Pi_{\mu'})$ then $a \xleftarrow{R} \mathcal{X} \setminus \bigcup_{\mu' \in [\mu]} \text{Domain}(\Pi_{\mu'})$
3. add (a, b) to Π_μ
4. return a

Let W_1 be the event that the adversary outputs 1 in this game.

It is not hard to see that the challenger's behavior in this game is equivalent to that in Game 0, and so $\Pr[W_0] = \Pr[W_1]$. The idea is that for every $\nu \in f(U) \subseteq V$, the partial permutation $\overline{\Pi}_\nu$ in Game 0 is partitioned into a family of disjoint partial permutations $\{\Pi_\mu\}_{\mu \in f^{-1}(\nu)}$, so that

$$\overline{\Pi}_\nu = \bigcup_{\mu \in f^{-1}(\nu)} \Pi_\mu,$$

and

$$\begin{aligned}&\text{Domain}(\Pi_\mu) \cap \text{Domain}(\Pi_{\mu'}) = \emptyset \quad \text{and} \quad \text{Range}(\Pi_\mu) \cap \text{Range}(\Pi_{\mu'}) = \emptyset \\ &\text{for all } \mu, \mu' \in f^{-1}(\nu) \text{ with } \mu \neq \mu'.\end{aligned} \tag{4.45}$$

Game 2. Now we simply delete the lines marked with a "*" in Game 1. Let W_2 be the event that the adversary outputs 1 in this game.

It is clear that this game is equivalent to the split experiment in Attack Game 4.5, and so $|\Pr[W_2] - \Pr[W_1]|$ is equal to the adversary's advantage in Attack Game 4.5. We want to use the Difference Lemma to bound $|\Pr[W_2] - \Pr[W_1]|$. To make this entirely rigorous, one models both games as operating on the same underlying probability space: we define a collection of random variables representing the coins of the adversary, as well as the various random samples from different subsets of $\mathcal{X}$ made by the challenger. These random variables completely describe both Games 1 and 2: the only difference between the two games are the deterministic computation rules that determine the outcomes. Define Z be to be the event that at the end of Game 2, the condition (4.45) does not hold. One can verify that Games 1 and 2 proceed identically unless Z holds, so by the Difference Lemma, we have $|\Pr[W_2] - \Pr[W_1]| \leq \Pr[Z]$. Moreover, it is clear that $\Pr[Z]$ is precisely the failure probability in Attack Game 4.5. □

4.8 Fun application: comparing information without revealing it

In this section we describe an important application for PRFs called **sub-key derivation**. Alice and Bob have a shared key k for a PRF. They wish to generate a sequence of shared keys $k_1, k_2, \ldots$ so that key number i can be computed without having to compute all earlier keys. Naturally, they set $k_i := F(k, i)$ where F is a secure PRF whose input space is $\{1, 2, \ldots, B\}$ for some bound B. The generated sequence of keys is indistinguishable from random keys.

As a fun application of this, consider the following problem: Alice is on vacation at the Squaw valley ski resort and wants to know if her friend Bob is also there. If he is they could ski together. Alice could call Bob and ask him if he is on the slopes, but this would reveal to Bob where she is and Alice would rather not do that. Similarly, Bob values his privacy and does not want to tell Alice where he is, unless Alice happens to be close by.

Abstractly, this problem can be phrased as follows: Alice has a number $a \in \mathbb{Z}_p$ and Bob has a number $b \in \mathbb{Z}_p$ for some prime p. These numbers indicate their approximate positions on earth. Think of dividing the surface of the earth into p squares and the numbers a and b indicate what square Alice and Bob are currently at. If Bob is at the resort then $a = b$, otherwise $a \neq b$.

Alice wants to learn if $a = b$; however, if $a \neq b$ then Alice should learn nothing else about b. Bob should learn nothing at all about a.

In a later chapter we will see how to solve this exact problem. Here, we make the problem easier by allowing Alice and Bob to interact with a server, Sam, that will help Alice learn if $a = b$, but will itself learn nothing at all. The only assumption about Sam is that it does not collude with Alice or Bob, that is, it does not reveal private data that Alice or Bob send to it. Clearly, Alice and Bob could send a and b to Sam and he will tell Alice if $a = b$, but then Sam would learn both a and b. Our goal is that Sam learns nothing, not even if $a = b$.

To describe the basic protocol, suppose Alice and Bob have a shared secret key $(k_0, k_1) \in \mathbb{Z}_p^2$. Moreover, Alice and Bob each have a private channel to Sam. The protocol for comparing a and b is shown in Fig. 4.17. It begins with Bob choosing a random r in $\mathbb{Z}_p$ and sending (r, x_{b}) to Sam.

$$
\begin{array}{lcccl}
\text{Alice} & & \text{Server} & & \text{Bob} \\
\text{input: } a & & \text{Sam} & & \text{input: } b \\
 & & & \xleftarrow{\quad r, \quad x_{\mathrm{b}} \leftarrow r(b+k_0)+k_1 \quad} & r \xleftarrow{\mathrm{R}} \mathbb{Z}_p \\
 & \xrightarrow{\quad x_{\mathrm{a}} \leftarrow a + k_0 \quad} & & & \\
x + k_1 \stackrel{?}{=} 0 & \xleftarrow{\quad x \leftarrow r\, x_{\mathrm{a}} - x_{\mathrm{b}} \quad} & & &
\end{array}
$$

Figure 4.17: Comparing a and b without revealing them

Bob can do this whenever he wants, even before Alice initiates the protocol. When Alice wants to test equality, she sends x_{a} to Sam. Sam computes $x \leftarrow r\, x_{\mathrm{a}} - x_{\mathrm{b}}$ and sends x back to Alice. Now, observe that

$$x + k_1 = r(a - b)$$

so that $x + k_1 = 0$ when $a = b$ and $x + k_1$ is very likely to be non-zero otherwise (assuming p is sufficiently large so that $r \neq 0$ with high probability). This lets Alice learn if $a = b$.

What is revealed by this protocol? Clearly Bob learns nothing. Alice learns $r(a - b)$, but if $a \neq b$ this quantity is uniformly distributed in $\mathbb{Z}_p$. Therefore, when $a \neq b$ Alice just obtains a uniform element in $\mathbb{Z}_p$ and this reveals nothing beyond the fact that $a \neq b$. Sam sees $r, x_{\mathrm{a}}, x_{\mathrm{b}}$, but all three values are independent of a and b: x_{a} and x_{b} are one-time pad encryptions under keys k_0 and k_1, respectively. Therefore, Sam learns nothing. Notice that the only privacy assumption about Sam is that it does not reveal (r, x_{b}) to Alice or x_{a} to Bob.

The trouble, much like with the one-time pad, is that the shared key (k_0, k_1) can only be used for a *single* equality test, otherwise the protocol becomes insecure. If (k_0, k_1) is used to test if $a = b$ and later the same key (k_0, k_1) is used to test if $a' = b'$ then Alice and Sam learn information they are not supposed to. For example, Sam learns $a - a'$. Moreover, Alice can deduce $(a - b)/(a' - b')$ which reveals information about b and b' (e.g., if $a = a' = 0$ then Alice learns the ratio of b and b').

Sub-key derivation. What if Alice wants to repeatedly test proximity to Bob? The solution is to generate a new independent key (k_0, k_1) for each invocation of the protocol. We do so by deriving instance-specific sub-keys using a secure PRF.

Let F be a secure PRF defined over $(\mathcal{K},\ \{1, \dots, B\},\ \mathbb{Z}_p^2)$ and suppose that Alice and Bob share a long term key $k \in \mathcal{K}$. Bob maintains a counter cnt_{b} that is initially set to 0. Every time Bob sends his encrypted location (r, x_{b}) to Sam he increments cnt_{b} and derives sub-keys (k_0, k_1) from the long-term key k as:

$$(k_0, k_1) \leftarrow F(k,\ cnt_{\mathrm{b}}). \tag{4.46}$$

He sends $(r, x_{\mathrm{b}}, cnt_{\mathrm{b}})$ to Sam. Bob can do this whenever he wants, say every few minutes, or every time he moves to a new location.

Whenever Alice wants to test proximity to Bob she first asks Sam to send her the value of the counter in the latest message from Bob. She makes sure the counter value is larger than the previous value Sam sent her (to prevent a mischievous Sam or Bob from tricking Alice into re-using an old counter value). Alice then computes (k_0, k_1) herself using (4.46) and carries out the protocol with Sam in Fig. 4.17 using these keys.

Because F is a secure PRF, the sequence of derived sub-keys is indistinguishable from random independently sampled keys. This ensures that the repeated protocol reveals nothing about the tested values beyond equality. By using a PRF, Alice is able to quickly compute (k_0, k_1) for the latest value of cnt_{b}.

4.9 Notes

Citations to the literature to be added.

4.10 Exercises

4.1 (Exercising the definition of a secure PRF). Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, where $\mathcal{K} = \mathcal{X} = \mathcal{Y} = \{0,1\}^n$.

(a) Show that $F_1(k, x) = F(k, x) \parallel 0$ is not a secure PRF.

(b) Prove that $F_2\big(k,\ (x, y)\big) := F(k, x) \oplus F(k, y)$ is insecure.

(c) Prove that $F_3(k, x) := F(k, x) \oplus x$ is a secure PRF.

(d) Prove that $F_4\big((k_1, k_2), x\big) := F(k_1, x) \oplus F(k_2, x)$ is a secure PRF.

(e) Show that $F_5(k, x) := F(k, x) \parallel F(k, x \oplus 1^n)$ is insecure.

(f) Prove that $F_6(k, x) := F(F(k, 0^n), x)$ is a secure PRF.

(g) Show that $F_7(k, x) := F(F(k, 0^n), x) \parallel F(k, x)$ is insecure.

(h) Show that $F_8(k, x) := F(k, x) \parallel F\big(k, F(k, x)\big)$ is insecure.

4.2 (Weak PRFs). Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ where $\mathcal{Y} := \{0,1\}^n$ and $|\mathcal{X}|$ is super-poly. Define

$$F_2\big(k,\ (x, y)\big) := F(k, x) \oplus F(k, y).$$

We showed in Exercise 4.1 part (b) that F_2 is not a secure PRF.

(a) Show that F_2 is a weakly secure PRF (as in Definition 4.3), assuming F is weakly secure. In particular, for any Q-query weak PRF adversary $\mathcal{A}$ attacking F_2 (i.e., an adversary that only queries the function at random points in $\mathcal{X}$) there is a weak PRF adversary $\mathcal{B}$ attacking F, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{wPRF}\mathsf{adv}[\mathcal{A}, F_2] \le \mathrm{wPRF}\mathsf{adv}[\mathcal{B}, F] + (Q/|\mathcal{X}|)^4.$$

(b) Suppose F is a secure PRF. Show that F_2 is weakly secure even if we modify the weak PRF attack game and allow the adversary $\mathcal{A}$ to query F_2 at one chosen point in addition to the Q random points. A PRF that is secure in this sense is sufficient for a popular data integrity mechanism discussed in Section 7.4.

(c) Show that F_2 is no longer secure if we modify the weak PRF attack game and allow the adversary $\mathcal{A}$ to query F_2 at two chosen points in addition to the Q random points.

4.3 (Format preserving encryption). Suppose we are given a block cipher (E, D) operating on domain $\mathcal{X}$. We want a block cipher (E', D') that operates on a smaller domain $\mathcal{X}' \subseteq \mathcal{X}$. Define (E', D') as follows:

$$
\begin{array}{ll}
E'(k,x) := & y \leftarrow E(k,x) \\
& \text{while } y \notin \mathcal{X}' \text{ do: } y \leftarrow E(k,y) \\
& \text{output } y
\end{array}
$$

$D'(k, y)$ is defined analogously, applying $D(k, \cdot)$ until the result falls in $\mathcal{X}'$. Clearly (E', D') are defined on domain $\mathcal{X}'$.

(a) With $t := |\mathcal{X}|/|\mathcal{X}'|$, how many evaluations of E are needed in expectation to evaluate $E'(k, x)$ as a function of t? You answer shows that when t is small (e.g., $t \leq 2$) evaluating $E'(k, x)$ can be done efficiently.

(b) Show that if (E, D) is a secure block cipher with domain $\mathcal{X}$ then (E', D') is a secure block cipher with domain $\mathcal{X}'$. Try proving security by induction on $|\mathcal{X}| - |\mathcal{X}'|$.

Discussion: This exercise is used in the context of encrypted 16-digit credit card numbers where the ciphertext also must be a 16-digit number. This type of encryption, called **format preserving encryption**, amounts to constructing a block cipher whose domain size is exactly 10^{16}. This exercise shows that it suffices to construct a block cipher (E, D) with domain size 2^{54} which is the smallest power of 2 larger than 10^{16}. The procedure in the exercise can then be used to shrink the domain to size 10^{16}.

4.4 (Truncating PRFs). Let F be a PRF whose range is $\mathcal{Y} = \{0, 1\}^n$. For some $\ell < n$ consider the PRF F' with a range $\mathcal{Y}' = \{0, 1\}^\ell$ defined as: $F'(k, x) = x[0 \dots \ell - 1]$. That is, we truncate the output of $F(k, x)$ to the first ℓ bits. Show that if F is a secure PRF then so is F'.

4.5 (Two-key Triple-DES). Consider the following variant of the $3\mathcal{E}$ construction that uses only two keys: for a block cipher (E, D) with key space $\mathcal{K}$ define $3\mathcal{E}'$ as $E((k_1, k_2), m) := E(k_1, E(k_2, E(k_1, m)))$. Show that this block cipher can be defeated by a meet in the middle attack using $O(|\mathcal{K}|)$ evaluation of E and D and using $O(|\mathcal{K}|)$ encryption queries to the block cipher challenger. Further attacks on this method are discussed in [74, 68].

4.6 (adaptive vs non-adaptive security). This exercise develops an argument that shows that a PRF may be secure against every adversary that makes its queries non-adaptively, (i.e., all at once) but is insecure against adaptive adversaries (i.e., the kind allowed in Attack Game 4.2).

To be a bit more precise, we define the non-adaptive version of Attack Game 4.2 as follows. The adversary submits all at once the query $(x_1, \dots, x_Q)$ to the challenger, who responds with $(y_1, \dots, y_Q)$, where $y := f(x_i)$. The rest of the attack game is the same: in Experiment 0, $k \xleftarrow{\text{R}} \mathcal{K}$ and $f \xleftarrow{\text{R}} F(k, \cdot)$, while in Experiment 1, $f \xleftarrow{\text{R}} \text{Funs}[\mathcal{X}, \mathcal{Y}]$. Security against non-adaptive adversaries means that all efficient adversaries have only negligible advantage; advantage is defined as usual: $|\Pr[W_0] - \Pr[W_1]|$, where W_b is the event that the adversary outputs 1 in Experiment b.

Suppose F is a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$, where $N := |\mathcal{X}|$ is super-poly. We proceed to "sabotage" F, constructing a new PRF $\tilde{F}$ as follows. Let x' be some fixed element of $\mathcal{X}$. For $x = F(k, x')$ define $\tilde{F}(k, x) := x'$, and for all other x define $\tilde{F}(k, x) := F(k, x)$.

(a) Show that $\tilde{F}$ is not a secure PRF against adaptive adversaries.

(b) Show that $\tilde{F}$ is a secure PRF against non-adaptive adversaries.

(c) Show that a similar construction is possible for block ciphers: given a secure block cipher (E, D) defined over $(\mathcal{K}, \mathcal{X})$ where $|\mathcal{X}|$ is super-poly, construct a new, "sabotaged" block cipher $(\tilde{E}, \tilde{D})$ that is secure against non-adaptive adversaries, but insecure against adaptive adversaries.

4.7 (PRF security definition). This exercise develops an alternative characterization of PRF security for a PRF F defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$. As usual, we need to define an attack game between an adversary $\mathcal{A}$ and a challenger. Initially, the challenger generates

$$b \xleftarrow{\text{R}} \{0,1\}, \ k \xleftarrow{\text{R}} \mathcal{K}, \ y_1 \xleftarrow{\text{R}} \mathcal{Y}$$

Then $\mathcal{A}$ makes a series of queries to the challenger. There are two types of queries:

Encryption: In an *function query*, $\mathcal{A}$ submits an $x \in \mathcal{X}$ to the challenger, who responds with $y \leftarrow F(k, x)$. The adversary may make any (poly-bounded) number of function queries.

Test: In a *test query*, $\mathcal{A}$ submits an $x \in \mathcal{X}$ to the challenger, who computes $y_0 \leftarrow F(k, x)$ and responds with y_b. The adversary is allowed to make only a *single* test query (with any number of function queries before and after the test query).

At the end of the game, $\mathcal{A}$ outputs a bit $\hat{b} \in \{0, 1\}$. As usual, we define $\mathcal{A}$'s advantage in the above attack game to be $|\Pr[\hat{b} = b] - 1/2|$. We say that F is Alt-PRF secure if this advantage is negligible for all efficient adversaries. Show that F is a secure PRF if and only if F is Alt-PRF secure.

Discussion: This characterization shows that the value of a secure PRF at a point x_0 in $\mathcal{X}$ looks like a random element of $\mathcal{Y}$, even after seeing the value of the PRF at many other points of $\mathcal{X}$.

4.8 (Key malleable PRFs). Let F be a PRF defined over $(\{0,1\}^n, \{0,1\}^n, \mathcal{Y})$.

(a) We say that F is **XOR-malleable** if $F(k, \ x \oplus c) = F(k, x) \oplus c$ for all k, x, c in $\{0,1\}^n$.

(b) We say that F is **key XOR-malleable** if $F(k \oplus c, \ x) = F(k, x) \oplus c$ for all k, x, c in $\{0,1\}^n$.

Clearly an XOR-malleable PRF cannot be secure: malleability lets an attacker distinguish the PRF from a random function. Show that the same holds for a key XOR-malleable PRF.

Remark: In contrast, we note that there are secure PRFs where $F(k_1 \oplus k_2, \ x) = F(k_1, x) \oplus F(k_2, x)$. See Exercise 11.1 for an example, where the xor on the left is replaced by addition, and the xor on the right is replaced by multiplication.

4.9 (Strongly secure block ciphers). In Section 4.1.3 we sketched out the notion of a strongly secure block cipher.

(a) Write out the complete definition of a strongly secure block cipher as a game between a challenger and an adversary.

(b) Consider the following cipher $\mathcal{E}' = (E', D')$ built from a block cipher (E, D) defined over $(\mathcal{K}, \{0,1\}^n)$:

$$E'(k, m) := D(k, \ t \oplus E(k, m)\ ) \quad \text{and} \quad D'(k, c) := E(k, \ t \oplus D(k, m)\ )$$

where $t \in \{0,1\}^n$ is a fixed constant. For what values of t is this cipher $\mathcal{E}'$ semantically secure? Prove semantic security assuming the underlying block cipher is strongly secure.

4.10 (Meet-in-the-middle attacks). Let us study the security of the $4\mathcal{E}$ construction where a block cipher (E, D) is iterated four times using four different keys: $E_4(\ (k_1, k_2, k_3, k_4),\ m) = E\big(k_4,\ E(k_3,\ E(k_2,\ E(k_1, m)))\big)$ where (E, D) is a block cipher with key space $\mathcal{K}$.

(a) Show that there is a meet in the middle attack on $4\mathcal{E}$ that recovers the secret key in time $|\mathcal{K}|^2$ and memory space $|\mathcal{K}|^2$.

(b) Show that there is a meet in the middle attack on $4\mathcal{E}$ that recovers the secret key in time $|\mathcal{K}|^2$, but only uses memory space $|\mathcal{K}|$. If you get stuck see [32].

4.11 (Tweakable block ciphers). A tweakable block cipher is a block cipher whose encryption and decryption algorithm take an additional input t, called a "tweak", which is drawn from a "tweak space" $\mathcal{T}$. As usual, keys come from a key space $\mathcal{K}$, and data blocks from a data block space $\mathcal{X}$. The encryption and decryption functions operate as follows: for $k \in \mathcal{K}, x \in \mathcal{X}, t \in \mathcal{T}$, we have $y = E(k, x, t) \in \mathcal{X}$ and $x = D(k, y, t)$. So for each $k \in \mathcal{K}$ and $t \in \mathcal{T}$, $E(k, \cdot, t)$ defines a permutation on $\mathcal{X}$ and $D(k, \cdot, t)$ defines the inverse permutation. Unlike keys, tweaks are typically publicly known, and may even be adversarially chosen.

Security is defined by a game with two experiments. In both experiments, the challenger defines a family of permutations $\{\Pi_t\}_{t \in \mathcal{T}}$, where each Π_t is a permutation on $\mathcal{X}$. In Experiment 0, the challenger sets $k \xleftarrow{R} \mathcal{K}$, and

$$\Pi_t := E(k, \cdot, t) \text{ for all } t \in \mathcal{T}.$$

In Experiment 1, the challenger sets

$$\Pi_t \xleftarrow{R} \mathrm{Perms}[\mathcal{X}] \text{ for all } t \in \mathcal{T}.$$

Both experiments then proceed identically. The adversary issues a series of queries. Each query is one of two types:

forward query: the adversary sends $(x, t) \in \mathcal{X} \times \mathcal{T}$, and the challenger responds with $y := \Pi_t(x)$;

inverse queries: the adversary sends $(y, t) \in \mathcal{X} \times \mathcal{T}$, and the challenger responds with $x := \Pi_t^{-1}(y)$.

At the end of the game, the adversary outputs a bit. If p_b is the probability that the adversary outputs 1 in Experiment b, the adversary's advantage is defined to be $|p_0 - p_1|$. We say that (E, D) is a *secure* tweakable block cipher if every efficient adversary has negligible advantage.

This definition of security generalizes the notion of a *strongly* secure block cipher (see Section 4.1.3 and Exercise 4.9). In applications of tweakable block ciphers, this strong security notion is more appropriate (e.g., see Exercise 9.17).

(a) Prove security of the construction $\tilde{E}(k, m, t) := E(E(k, t), m)$ where (E, D) is a strongly secure block cipher defined over $(\mathcal{K}, \mathcal{K})$.

(b) Show that there is an attack on the construction from part (a) that achieves advantage $\geq 1/2$ and which makes $Q \approx \sqrt{|\mathcal{K}|}$ queries.

Hint: In addition to the $\approx \sqrt{|\mathcal{K}|}$ queries, your adversary should make an additional $\approx \sqrt{|\mathcal{K}|}$ "offline" evaluations of the cipher (E, D).

(c) Prove security of the construction

$$E'\big((k_0, k_1), m, t\big) := \big\{p \leftarrow F(k_0, t);\quad \text{output } p \oplus E(k_1,\ m \oplus p)\big\},$$

where (E, D) is a *strongly* secure block cipher and F is a secure PRF. In Exercise 7.10 we will see a more efficient variant of this construction.

Hint: Use the assumption that (E, D) is a strongly secure block cipher to replace $E(k_1, \cdot)$ in the challenger by a truly random permutation $\widetilde{\Pi}$; then, use the Domain Separation Lemma (see Theorem 4.15) to replace $\widetilde{\Pi}$ by a family of independent permutations $\{\widetilde{\Pi}_t\}_{t \in \mathcal{T}}$, and analyze the corresponding domain separation failure probability.

Discussion: Tweakable block ciphers are used in disk sector encryption where encryption must not expand the data: the ciphertext size is required to have the same size as the input. The sector number is used as the tweak to ensure that even if two sectors contain the same data, the resulting encrypted sectors are different. The construction in part (c) is usually more efficient than that in part (a), as the latter uses a different block cipher key with every evaluation, which can incur extra costs. See further discussion in Exercise 7.10.

4.12 (PRF combiners). We want to build a PRF F using two PRFs F_1 and F_2, so that if at some future time one of F_1 or F_2 is broken (but not both) then F is still secure. Put another way, we want to construct F from F_1 and F_2 such that F is secure if either F_1 or F_2 is secure.

Suppose F_1 and F_2 both have output spaces $\{0, 1\}^n$, and both have a common input space. Define

$$F(\ (k_1, k_2),\ x) := F_1(k_1, x) \oplus F_2(k_2, x).$$

Show that F is secure if either F_1 or F_2 is secure.

4.13 (Block cipher combiners). Continuing with Exercise 4.12, we want to build a block cipher $\mathcal{E} = (E, D)$ from two block ciphers $\mathcal{E}_1 = (E_1, D_1)$ and $\mathcal{E}_2 = (E_2, D_2)$ so that if at some future time one of $\mathcal{E}_1$ or $\mathcal{E}_2$ is broken (but not both) then $\mathcal{E}$ is still secure. Suppose both $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined over $(\mathcal{K}, \mathcal{X})$. Define $\mathcal{E}$ as:

$$E(\ (k_1, k_2),\ x) := E_1\big(k_1,\ E_2(k_2, x)\big) \qquad \text{and} \qquad D(\ (k_1, k_2),\ y) := D_2\big(k_2,\ D_1(k_1, y)\big).$$

(a) Show that $\mathcal{E}$ is secure if either $\mathcal{E}_1$ or $\mathcal{E}_2$ is secure.

(b) Show that this is not a secure combiner for PRFs. That is, $F(\ (k_1, k_2),\ x) := F_1\big(k_1,\ F_2(k_2, x)\big)$ need not be a secure PRF even if one of F_1 or F_2 is.

4.14 (Key leakage). Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, where $\mathcal{K} = \mathcal{X} = \mathcal{Y} = \{0, 1\}^n$.

(a) Let $\mathcal{K}_1 = \{0, 1\}^{n+1}$. Construct a new PRF F_1, defined over $(\mathcal{K}_1, \mathcal{X}, \mathcal{Y})$, with the following property: the PRF F_1 is secure; however, if the adversary learns the last bit of the key then the PRF is no longer secure. This shows that leaking even a *single* bit of the secret key can completely destroy the PRF security property.

Hint: Let $k_1 = k \parallel b$ where $k \in \{0, 1\}^n$ and $b \in \{0, 1\}$. Set $F_1(k_1, x)$ to be the same as $F(k, x)$ for all $x \neq 0^n$. Define $F_1(k_1, 0^n)$ so that F_1 is a secure PRF, but becomes easily distinguishable from a random function if the last bit of the secret key k_1 is known to the adversary.

(b) Construct a new PRF F_2, defined over $(\mathcal{K} \times \mathcal{K}, \mathcal{X}, \mathcal{Y})$, that remains secure if the attacker learns any *single* bit of the key. Your function F_2 may only call F once.

4.15 (Variants of Luby-Rackoff). Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$.

(a) Show that two-round Luby-Rackoff is not a secure block cipher.

(b) Show that three-round Luby-Rackoff is not a strongly secure block cipher.

4.16 (Insecure tree construction). In the tree construction for building a PRF from a PRG (Section 4.6), the secret key is used at the root of the tree and the input is used to trace a path through the tree. Show that a construction that does the opposite is not a secure PRF. That is, using the input as the root and using the key to trace through the tree is not a secure PRF.

4.17 (Truncated tree construction). Suppose we cut off the tree construction from Section 4.6 after only three levels of the tree, so that there are only eight leaves, as in Fig. 4.15. Give a direct proof, using a sequence of seven hybrids, that outputting the values at all eight leaves gives a secure PRG defined over $(\mathcal{S}, \mathcal{S}^8)$, assuming the underlying PRG is secure.

4.18 (Augmented tree construction). Suppose we are given a PRG G defined over $(\mathcal{K} \times \mathcal{S}, \mathcal{S}^2)$. Write $G(k,s) = (G_0(k,s), G_1(k,s))$. Let us define the PRF G^* with key space $\mathcal{K}^n \times \mathcal{S}$ and input space $\{0,1\}^n$ as follows:

$$
\begin{array}{l}
G^*\big((k_0, \ldots, k_{n-1}, s),\ x \in \{0,1\}^n\big) := \\
\qquad t \leftarrow s \\
\qquad \text{for } i \leftarrow 0 \text{ to } n-1 \text{ do} \\
\qquad\qquad b \leftarrow x[i] \\
\qquad\qquad t \leftarrow G_b(k_i, t) \\
\qquad \text{output } t.
\end{array}
$$

(a) Given an example secure PRG G for which G^* is insecure as a PRF.

(b) Show that G^* is a secure PRF if for every poly-bounded Q the following PRG is secure:

$$G'(k, s_0, \ldots, s_{Q-1}) := (G(k, s_0), \ldots, G(k, s_{Q-1}))\ .$$

4.19 (Synthesizers and parallel PRFs). For a secure PRG G defined over $(\mathcal{S}, \mathcal{R})$ we showed that $G^n(s_1, \ldots, s_n) := \big(G(s_1), \ldots, G(s_n)\big)$ is a secure PRG over $(\mathcal{S}^n, \mathcal{R}^n)$. The proof requires that the components $s_1, \ldots, s_n$ of the seed be chosen uniformly and independently over $\mathcal{S}^n$. A secure synthesizer is a PRG for which this holds even if $s_1, \ldots, s_n$ are not independent of one another. Specifically, a **synthesizer** is an efficient function $S : \mathcal{X}^2 \to \mathcal{X}$. The synthesizer is said to be *n-way secure* if

$$S^n(x_1, y_1, \ldots, x_n, y_n) := \big(\ S(x_i, y_j)\ \big)_{i,j=1,\ldots,n}\ \in \mathcal{X}^{(n^2)}$$

is a secure PRG defined over $(\mathcal{X}^{2n}, \mathcal{X}^{(n^2)})$. Here S is being evaluated at n^2 inputs that are not independent of one another and yet S^n is a secure PRG.

(a) Not every secure PRG is a secure synthesizer. Let G be a secure PRG over $(\mathcal{S}, \mathcal{R})$. Show that $S(x,y) := \big(G(x), y\big)$ is a secure PRG defined over $(\mathcal{S}^2,\ \mathcal{R} \times \mathcal{S})$, but is an insecure 2-way synthesizer.

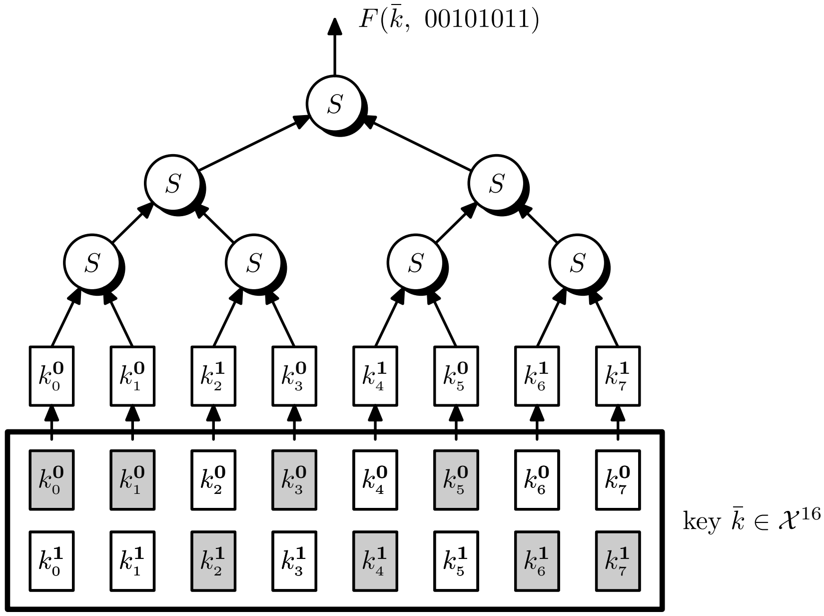


Figure 4.18: A PRF built from a synthesizer S. The PRF input in $\{0,1\}^n$ is used to select n components from the key $\bar{k} \in \mathcal{X}^{2n}$. The selected components, shown as shaded squares, are used as shown in the figure.

(b) A secure synthesizer lets us build a large domain PRF that can be evaluated quickly on a parallel computer. Show that if $S : \mathcal{X}^2 \to \mathcal{X}$ is a Q-way secure synthesizer, for poly-bounded Q, then the PRF in Fig. 4.18 is a secure PRF defined over $(\mathcal{X}^{2n},\ \{0,1\}^n,\ \mathcal{X})$. For simplicity, assume that n is a power of 2. Observe that the PRF can be evaluated in only $\log_2 n$ steps on a parallel computer.

4.20 (Insecure variants of Even-Mansour). In Section 4.7.3 we discussed the Even-Mansour block cipher (E, D) built from a permutation $\pi : \mathcal{X} \to \mathcal{X}$ where $\mathcal{X} = \{0,1\}^n$. Recall that $E\big((P_0, P_1),\ m\big) := \pi(m \oplus P_0) \oplus P_1$.

(a) Show that $E_1(P_0, m) := \pi(m \oplus P_0)$ is not a secure block cipher.

(b) Show that $E_2(P_1, m) := \pi(m) \oplus P_1$ is not a secure block cipher.

4.21 (Birthday attack on Even-Mansour). Let's show that the bounds in the Even-Mansour security theorem (Theorem 4.14) are tight. For $\mathcal{X} := \{0,1\}^n$, recall that the Even-Mansour block cipher (E, D), built from a permutation $\pi : \mathcal{X} \to \mathcal{X}$, is defined as: $E\big((k_0, k_1),\ m\big) := \pi(m \oplus k_0) \oplus k_1$. We show how to break this block cipher in time approximately $2^{n/2}$.

(a) Show that for all $a, m, \Delta \in \mathcal{X}$ and $\bar{k} := (k_0, k_1) \in \mathcal{X}^2$, whenever $a = m \oplus k_0$, we have

$$E\big(\bar{k},\ m\big) \oplus E\big(\bar{k},\ m \oplus \Delta\big) = \pi(a) \oplus \pi(a \oplus \Delta)$$

(b) Use part (a) to construct an adversary $\mathcal{A}$ that wins the block cipher security game against (E, D) with advantage close to 1, in the ideal cipher model. With $q := 2^{n/2}$ and some non-zero

$\Delta \in \mathcal{X}$, the adversary $\mathcal{A}$ queries the cipher at $2q$ random points $m_i,\ m_i \oplus \Delta \in \mathcal{X}$ and queries the permutation π at $2q$ random points $a_i,\ a_i \oplus \Delta \in \mathcal{X}$, for $i = 1, \ldots, q$.

4.22 (A variant of the Even-Mansour cipher). Let $\mathcal{M} := \{0,1\}^m$, $\mathcal{K} := \{0,1\}^n$, and $\mathcal{X} := \{0,1\}^{n+m}$. Consider the following cipher (E, D) defined over $(\mathcal{K}, \mathcal{M}, \mathcal{X})$ built from a permutation $\pi : \mathcal{X} \to \mathcal{X}$:

$$E(k, x) := (k \parallel 0^m) \oplus \pi(k \parallel x) \tag{4.47}$$

$D(k, c)$ is defined analogously. Show that if we model π as an ideal permutation Π, then for every block cipher adversary $\mathcal{A}$ attacking (E, D) we have

$$\mathrm{BC^{ic}}\mathsf{adv}[\mathcal{A}, E] \le \frac{2Q_{\mathrm{ic}}}{|\mathcal{K}|} \ . \tag{4.48}$$

Here Q_{ic} is the number of queries $\mathcal{A}$ makes to Π- and Π^{-1}-oracles.

4.23 (Analysis of Salsa and ChaCha). In this exercise we analyze the Salsa and ChaCha stream ciphers from Section 3.6 in the ideal permutation model. Let $\pi : \mathcal{X} \to \mathcal{X}$ be a permutation, where $\mathcal{X} = \{0,1\}^{n+m}$. Let $\mathcal{K} := \{0,1\}^n$ and define the PRF F, which is defined over $(\mathcal{K}, \{0,1\}^m, \mathcal{X})$, as

$$F(k, x) := (k \parallel x) \oplus \pi(k \parallel x) \ . \tag{4.49}$$

This PRF is an abstraction of the PRF underlying the Salsa and ChaCha stream ciphers. Use Exercise 4.22 to show that if we model π as an ideal permutation Π, then for every PRF adversary $\mathcal{A}$ attacking F we have

$$\mathrm{PRF^{ic}}\mathsf{adv}[\mathcal{A}, F] \le \frac{2Q_{\mathrm{ic}}}{|\mathcal{K}|} + \frac{Q_{\mathrm{F}}^2}{2|\mathcal{X}|} \tag{4.50}$$

where Q_{F} is the number of queries that $\mathcal{A}$ makes to an $F(k, \cdot)$ oracle and Q_{ic} is the number of queries $\mathcal{A}$ makes to Π- and Π^{-1}-oracles. In Salsa and ChaCha, Q_{F} is at most $|\mathcal{X}|^{1/4}$ so that $\frac{Q_{\mathrm{F}}^2}{2|\mathcal{X}|}$ is "negligible."

Discussion: The specific permutation π used in the Salsa and ChaCha stream ciphers is not quite an ideal permutation. For example, $\pi(0^{n+m}) = 0^{n+m}$. Hence, your analysis applies to the general framework, but not specifically to Salsa and ChaCha.

4.24 (Alternative proof of Theorem 4.6). Let $\mathbf{X}$ and $\mathbf{Y}$ be random variables as defined in Exercise 3.13. Consider an adversary $\mathcal{A}$ in Attack Game 4.3 that makes at most Q queries to its challenger. Show that $\mathrm{PF}\mathsf{adv}[\mathcal{A}, \mathcal{X}] \le \Delta[\mathbf{X}, \mathbf{Y}] \le Q^2/2N$.

4.25 (A one-sided switching lemma). Following up on the previous exercise, one can use part (b) of Exercise 3.13 to get a "one sided" version of Theorem 4.6, which can be useful in some settings. Consider an adversary $\mathcal{A}$ in Attack Game 4.3 that makes at most Q queries to its challenger. Let W_0 and W_1 be as defined in that game: W_0 is the event that $\mathcal{A}$ outputs 1 when probing a random permutation, and W_1 is the event that $\mathcal{A}$ outputs 1 when probing a random function. Assume $Q^2 < N$. Show that $\Pr[W_0] \le \rho[\mathbf{X}, \mathbf{Y}] \cdot \Pr[W_1] \le 2\Pr[W_1]$.

4.26 (Parallel composition of PRFs). Just as we can compose PRGs in parallel, while maintaining security (see Section 3.4.1), we can also compose PRFs in parallel, while maintaining security.

Suppose we have a PRF F, defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$. We want to model the situation where an adversary is given n black boxes (where $n \geq 1$ is poly-bounded): the boxes either contain $F(k_1, \cdot), \dots, F(k_n, \cdot)$, where the k_i are random (and independent) keys, or they contain $f_1, \dots, f_n$, where the f_i are random elements of Funs$[\mathcal{X}, \mathcal{Y}]$, and the adversary should not be able to tell the difference.

A convenient way to model this situation is to consider the **n-wise parallel composition of** F, which is a PRF F' whose key space is $\mathcal{K}^n$, whose input space is $\{1, \dots, n\} \times \mathcal{X}$, and whose output space is $\mathcal{Y}$. Given a key $k' = (k_1, \dots, k_n)$, and an input $x' = (s, x)$, with $s \in \{1, \dots, n\}$ and $x \in \mathcal{X}$, we define $F'(k', x') := F(k_s, x)$.

Show that if F is a secure PRF, then so is F'. In particular, show that for every PRF adversary $\mathcal{A}$, then exist a PRF adversary $\mathcal{B}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that $\mathrm{PRF}\mathsf{adv}[\mathcal{A}, F'] = n \cdot \mathrm{PRF}\mathsf{adv}[\mathcal{B}, F]$.

4.27 (Universal attacker on PRFs). Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ where $|\mathcal{K}| < |\mathcal{X}|$. Let $Q < |\mathcal{K}|$. Show that there is a PRF adversary $\mathcal{A}$ that runs in time proportional to Q, makes one query to the PRF challenger, and has advantage

$$\mathrm{PRF}\mathsf{adv}[\mathcal{A}, F] \geq \left| \frac{Q}{|\mathcal{K}|} - \frac{Q}{|\mathcal{X}|} \right|.$$

4.28 (Distributed PRFs). Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ where $\mathcal{Y} := \{0,1\}^n$. In Exercise 4.1 part (d) we showed that if F is secure then so is

$$F'\big((k_1, k_2), x\big) := F(k_1, x) \oplus F(k_2, x).$$

This F' has a useful property: the PRF key (k_1, k_2) can be split into two shares, k_1 and k_2. If Alice is given one share and Bob the other share, then both Alice and Bob are needed to evaluate the PRF, and neither can evaluate the PRF on its own. Moreover, the PRF can be evaluated distributively, that is, without re-constituting the key (k_1, k_2): to evaluate the PRF at a point x_0, Alice simply sends $F(k_1, x_0)$ to Bob.

(a) To show that Alice cannot evaluate F' by herself, show that F' is a secure PRF even if the adversary is given k_1. Argue that the same holds for k_2.

(b) Construct a PRF where the key can be split into three shares s_1, s_2, s_3 so that any *two* shares can be used evaluate the PRF distributively, but no single share is sufficient to evaluate the PRF on its own.

Hint: Consider the PRF $F''\big((k_1, k_2, k_3), x\big) := F(k_1, x) \oplus F(k_2, x) \oplus F(k_3, x)$ and show how to construct the shares s_1, s_2, s_3 from the keys k_1, k_2, k_3. Make sure to prove that the F'' is a secure PRF when the adversary is given a single share, namely s_i for some $i \in \{1, 2, 3\}$.

(c) Generalize the construction from part (b) to construct a PRF F''' supporting three-out-of-five sharing of the key: any three shares can be used to evaluate the PRF distributively, but no two shares can.

Hint: The key space for F''' is $\mathcal{K}^{10}$.

Chapter 5

Chosen Plaintext Attack

This chapter focuses on the problem of securely encrypting several messages in the presence of an adversary who eavesdrops, and who may even may influence the choice of some messages in order to glean information about other messages. This leads us to the notion of semantic security against a *chosen plaintext attack*.

5.1 Introduction

In Chapter 2, we focused on the problem of encrypting a single message. Now we consider the problem of encrypting several messages. To make things more concrete, suppose Alice wants to use a cipher to encrypt her files on some file server, while keeping her secret keys for the cipher stored securely on her USB memory stick.

One possible approach is for Alice to encrypt each individual file using a different key. This entails that for each file, she stores an encryption of that file on the file server, as well as a corresponding secret key on her memory stick. As we will explore in detail in Section 5.2, this approach will provide Alice with reasonable security, provided she uses a semantically secure cipher. Now, although a file may be several megabytes long, a key for any practical cipher is just a few bytes long. However, if Alice has many thousands of files to encrypt, she must store many thousands of keys on her memory stick, which may not have sufficient storage for all these keys.

As we see, the above approach, while secure, is not very space efficient, as it requires one key per file. Faced with this problem, Alice may simply decide to encrypt all her files with the same key. While more efficient, this approach may be insecure. Indeed, if Alice uses a cipher that provides only semantic security (as in Definition 2.3), this may not provide Alice with any meaningful security guarantee, and may very well expose her to a realistic attack.

For example, suppose Alice uses the stream cipher $\mathcal{E}$ discussed in Section 3.2. Here, Alice's key is a seed s for a PRG G, and viewing a file m as a bit string, Alice encrypts m by computing the ciphertext $c := m \oplus \Delta$, where Δ consists of the first $|m|$ bits of the "key stream" $G(s)$. But if Alice uses this same seed s to encrypt many files, an adversary can easily mount an attack. For example, if an adversary knows some of the bits of one file, he can directly compute the corresponding bits of the key stream, and hence obtain the corresponding bits of *any* file. How might an adversary know some bits of a given file? Well, certain files, like email messages, contain standard header information (see Example 2.6), and so if the adversary knows that a given ciphertext is an encryption of an email, he can get the bits of the key stream that correspond to the location of the bits in this standard header. To mount an even more devastating attack, the adversary may try something even more devious: he could simply send Alice a large email, say one megabyte in length; assuming that Alice's software automatically stores an encryption of this email on her server, when the adversary snoops her file server, he can recover a corresponding one megabyte chunk of the key stream, and now he decrypt any one megabyte file stored on Alice's server! This email may even be caught in Alice's spam filter, and never actually seen by Alice, although her encryption software may very well diligently encrypt this email along with everything else. This type of an attack is called a *chosen plaintext attack*, because the adversary forces Alice to give him the encryption of one or more plaintexts of his choice during his attack on the system.

Clearly, the stream cipher above is inadequate for the job. In fact, the stream cipher, as well as *any other deterministic cipher*, should not be used to encrypt multiple files with the same key. Why? Any deterministic cipher that is used to encrypt several files with the same key will suffer from an inherent weakness: an adversary will always be able to tell when two files are identical or not. Indeed, with a deterministic cipher, if the same key is used to encrypt the same message, the resulting ciphertext will always be the same (and conversely, for *any* cipher, if the same key is used to encrypt two different messages, the resulting ciphertexts must be different). While this type of attack is certainly not as dramatic as those discussed above, in which the adversary can read Alice's files almost at will, it is still a serious vulnerability. For example, while the discussion in Section 4.1.4 about ECB mode was technically about encrypting a single message consisting of many data blocks, it applies equally well to the problem of encrypting many single-block messages under the same key.

In fact, it *is* possible for Alice to use a cipher to securely encrypt all of her files under a single, short key, but she will need to use a cipher that is better suited to this task. In particular, because of the above inherent weakness of any deterministic cipher, she will have to use a *probabilistic* cipher, that is, a cipher that uses a probabilistic encryption algorithm, so that different encryptions of the same plaintext under the same key will (generally) produce different encryptions. For her task, she will want a cipher that achieves a level of security stronger than semantic security. The appropriate notion of security is called *semantic security against chosen plaintext attack.* In Section 5.3 and the sections following, we formally define this concept, look at some constructions based on semantically secure ciphers, PRFs, and block ciphers, and look at a few case studies of "real world" systems.

While the above discussion motivated the topics in this chapter using the example of the "file encryption" problem, one can also motivate these topics by considering the "secure network communication" problem. In this setting, one considers the situation where Alice and Bob share a secret key (or keys), and Alice wants to secretly transmit several of messages to Bob over an insecure network. Now, if Alice can conveniently concatenate all of her messages into one long message, then she can just use a stream cipher to encrypt the whole lot, and be done with it. However, for a variety of technical reasons, this may not be feasible: if she wants to be able to transmit the messages in an arbitrary order and at arbitrary times, then she is faced with a problem very similar to that of the "file encryption" problem. Again, if Alice and Bob want to use a single, short key, the right tool for the job is a cipher semantically secure against chosen plaintext attack.

We stress again that just like in Chapter 2, the techniques covered in this chapter *do not* provide any *data integrity*, nor do they address the problem of how two parties come to share a secret key to begin with. These issues are dealt with in coming chapters.

5.2 Security against multi-key attacks

Consider again the “file encryption” problem discussed in the introduction to this chapter. Suppose Alice chooses to encrypt each of her files under different, independently generated keys using a semantically secure cipher. Does semantic security imply a corresponding security property in this “multi-key” setting?

The answer to this question is “yes.” We begin by stating the natural security property corresponding to semantic security in the multi-key setting.

Attack Game 5.1 (multi-key semantic security). For a given cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define

Experiment b:

- The adversary submits a sequence of queries to the challenger.

  For $i = 1, 2, \ldots$, the ith query is a pair of messages, $m_{i0}, m_{i1} \in \mathcal{M}$, of the same length.

  The challenger computes $k_i \xleftarrow{\text{R}} \mathcal{K}$, $c_i \xleftarrow{\text{R}} E(k_i, m_{ib})$, and sends c_i to the adversary.

- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\text{MSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := |\Pr[W_0] - \Pr[W_1]|. \quad \Box$$

We stress that in the above attack game, the adversary's queries are *adaptively chosen*, in the sense that for each $i = 1, 2, \ldots$, the message pair (m_{i0}, m_{i1}) may be computed by the adversary in some way that depends somehow on the previous encryptions $c_1, \ldots, c_{i-1}$ output by the challenger.

Definition 5.1 (Multi-key semantic security). *A cipher $\mathcal{E}$ is called **multi-key semantically secure** if for all efficient adversaries $\mathcal{A}$, the value $\text{MSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.*

As discussed in Section 2.3.5, Attack Game 5.1 can be recast as a “bit guessing” game, where instead of having two separate experiments, the challenger chooses $b \in \{0, 1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$. In this game, we measure $\mathcal{A}$'s *bit-guessing advantage* $\text{MSS}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]$ as $|\Pr[\hat{b} = b] - 1/2|$, and as usual (by (2.13)), we have

$$\text{MSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \text{MSS}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]. \tag{5.1}$$

As the next theorem shows, semantic security implies multi-key semantic security.

Theorem 5.1. *If a cipher $\mathcal{E}$ is semantically secure, it is also multi-key semantically secure.*

In particular, for every MSS adversary $\mathcal{A}$ that attacks $\mathcal{E}$ as in Attack Game 5.1, and which makes at most Q queries to its challenger, there exists an SS adversary $\mathcal{B}$ that attacks $\mathcal{E}$ as in Attack Game 2.1, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{MSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = Q \cdot \text{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}].$$

Proof idea. The proof is a straightforward hybrid argument, which is a proof technique we introduced in the proofs of Theorem 3.2 and 3.3 (the reader is advised to review those proofs, if necessary). In Experiment 0 of the MSS attack game, the challenger is encrypting $m_{10}, m_{20}, \ldots, m_{Q0}$. Intuitively, since the key k_1 is only used to encrypt the first message, and $\mathcal{E}$ is semantically secure, if we modify the challenger so that it encrypts m_{11} instead of m_{10}, the adversary should not behave significantly differently. Similarly, we may modify the challenger so that it encrypts m_{21} instead of m_{20}, and the adversary should not notice the difference. If we continue in this way, making a total of Q modifications to the challenger, we end up in Experiment 1 of the MSS game, and the adversary should not notice the difference. $\Box$

Proof. Suppose $\mathcal{E} = (E, D)$ is defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$. Let $\mathcal{A}$ be an MSS adversary that plays Attack Game 5.1 with respect to $\mathcal{E}$, and which makes at most Q queries to its challenger in that game.

First, we introduce $Q + 1$ hybrid games, Hybrid 0, …, Hybrid Q, played between a challenger and $\mathcal{A}$. For $j = 0, 1, \ldots, Q$, when $\mathcal{A}$ makes its ith query (m_{i0}, m_{i1}), the challenger in Hybrid j computes its response c_i as follows:

$k_i \xleftarrow{R} \mathcal{K}$
if $i > j$ then $\quad c_i \xleftarrow{R} E(k_i, m_{i0}) \quad$ else $\quad c_i \xleftarrow{R} E(k_i, m_{i1})$.

Put another way, the challenger in Hybrid j encrypts

$$m_{11}, \ldots, m_{j1}, \quad m_{(j+1)0}, \ldots, m_{Q0},$$

generating different keys for each of these encryptions.

For $j = 0, 1, \ldots, Q$, let p_j denote the probability that $\mathcal{A}$ outputs 1 in Hybrid j. Observe that p_0 is equal to the probability that $\mathcal{A}$ outputs 1 in Experiment 0 of Attack Game 5.1 with respect to $\mathcal{E}$, while p_Q is equal to the probability that $\mathcal{A}$ outputs 1 in Experiment 1 of Attack Game 5.1 with respect to $\mathcal{E}$. Therefore, we have

$$\text{MSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = |p_Q - p_0|. \tag{5.2}$$

We next devise an SS adversary $\mathcal{B}$ that plays Attack Game 2.1 with respect to $\mathcal{E}$, as follows:

First, $\mathcal{B}$ chooses $\omega \in \{1, \ldots, Q\}$ at random.

Then, $\mathcal{B}$ plays the role of challenger to $\mathcal{A}$ — when $\mathcal{A}$ makes its ith query (m_{i0}, m_{i1}), $\mathcal{B}$ computes its response c_i as follows:

if $i > \omega$ then
  $k_i \xleftarrow{R} \mathcal{K}, \;\; c_i \xleftarrow{R} E(k_i, m_{i0})$
else if $i = \omega$ then
  $\mathcal{B}$ submits (m_{i0}, m_{i1}) to its own challenger
  c_i is set to the challenger's response
else // $i < \omega$
  $k_i \xleftarrow{R} \mathcal{K}, \;\; c_i \xleftarrow{R} E(k_i, m_{i1})$.

Finally, $\mathcal{B}$ outputs whatever $\mathcal{A}$ outputs.

Put another way, adversary $\mathcal{B}$ encrypts

$$m_{11}, \ldots, m_{(\omega-1)1},$$

generating its own keys for this purpose, submits $(m_{\omega 0}, m_{\omega 1})$ to its own encryption oracle, and encrypts

$$m_{(\omega+1)0}, \ldots, m_{Q0},$$

again, generating its own keys.

We claim that

$$\text{MSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = Q \cdot \text{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}]. \tag{5.3}$$

To prove this claim, for $b = 0, 1$, let W_b be the event that $\mathcal{B}$ outputs 1 in Experiment b of its attack game. If ω denotes the random number chosen by $\mathcal{B}$, then the key observation is that for $j = 1, \ldots, Q$, we have:

$$\Pr[W_0 \mid \omega = j] = p_{j-1} \quad \text{and} \quad \Pr[W_1 \mid \omega = j] = p_j.$$

Equation (5.3) now follows from this observation, together with (5.2), via the usual telescoping sum calculation:

$$\begin{aligned}
\text{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}] &= |\Pr[W_1] - \Pr[W_0]| \\
&= \frac{1}{Q} \cdot \left| \sum_{j=1}^{Q} \Pr[W_1 \mid \omega = j] - \sum_{j=1}^{Q} \Pr[W_0 \mid \omega = j] \right| \\
&= \frac{1}{Q} \cdot |p_Q - p_0| \\
&= \frac{1}{Q} \cdot \text{MSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}],
\end{aligned}$$

and the claim, and hence the theorem, is proved. □

Let us return now to the "file encryption" problem discussed in the introduction to this chapter. What this theorem says is that if Alice uses independent keys to encrypt each of her files with a semantically secure cipher, then an adversary who sees the ciphertexts stored on the file server will effectively learn nothing about Alice's files (except possibly some information about their lengths). Notice that this holds even if the adversary plays an active role in determining the contents of some of the files (e.g., by sending Alice an email, as discussed in the introduction).

5.3 Semantic security against chosen plaintext attack

Now we consider the problem that Alice faced in introduction of this chapter, where she wants to encrypt all of her files on her system using a single, and hopefully short, secret key. The right notion of security for this task is **semantic security against chosen plaintext attack**, or **CPA security** for short.

Attack Game 5.2 (CPA security). For a given cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define

Experiment b:

- The challenger selects $k \xleftarrow{\text{R}} \mathcal{K}$.

- The adversary submits a sequence of queries to the challenger.

  For $i = 1, 2, \ldots$, the ith query is a pair of messages, $m_{i0}, m_{i1} \in \mathcal{M}$, of the same length.

  The challenger computes $c_i \xleftarrow{R} E(k, m_{ib})$, and sends c_i to the adversary.

- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := |\Pr[W_0] - \Pr[W_1]|. \quad \Box$$

The only difference between the CPA attack game and the MSS Attack Game 5.1 is that in the CPA game, the same key is used for all encryptions, whereas in the MSS attack game, a different key is chosen for each encryption. In particular, the adversary's queries may adaptively chosen in the CPA game, just as in the MSS game.

Definition 5.2 (CPA security). *A cipher $\mathcal{E}$ is called* ***semantically secure against chosen plaintext attack****, or simply* ***CPA secure****, if for all efficient adversaries $\mathcal{A}$, the value* $\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

As in Section 2.3.5, Attack Game 5.2 can be recast as a "bit guessing" game, where instead of having two separate experiments, the challenger chooses $b \in \{0, 1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$; we define $\mathcal{A}$'s *bit-guessing advantage* as $\text{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}] := |\Pr[\hat{b} = b] - 1/2|$, and as usual (by (2.13)), we have

$$\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \text{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]. \tag{5.4}$$

Again, we return to the "file encryption" problem discussed in the introduction to this chapter. What this definition says is that if Alice uses just a single key to encrypt each of her files with a CPA secure cipher, then an adversary who sees the ciphertexts stored on the file server will effectively learn nothing about Alice's files (except possibly some information about their lengths). Again, notice that this holds even if the adversary plays an active role in determining the contents of some of the files.

Example 5.1. Just to exercise the definition a bit, let us show that no deterministic cipher can possibly satisfy the definition of CPA security. Suppose that $\mathcal{E} = (E, D)$ is a deterministic cipher. We construct a CPA adversary $\mathcal{A}$ as follows. Let m, m' be any two, distinct messages in the message space of $\mathcal{E}$. The adversary $\mathcal{A}$ makes two queries to its challenger: the first is (m, m'), and the second is (m, m). Suppose c_1 is the challenger's response to the first query and c_2 is the challenger's response to the second query. Adversary $\mathcal{A}$ outputs 1 if $c_1 = c_2$, and 0 otherwise.

Let us calculate $\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$. On then one hand, in Experiment 0 of Attack Game 5.2, the challenger encrypts m in responding to both queries, and so $c_1 = c_2$; hence, $\mathcal{A}$ outputs 1 with probability 1 in this experiment (this is precisely where we use the assumption that $\mathcal{E}$ is deterministic). On the other hand, in Experiment 1, the challenger encrypts m' and m, and so $c_1 \neq c_2$; hence, $\mathcal{A}$ outputs 1 with probability 0 in this experiment. It follows that $\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 1$.

The attack in this example can be generalized to show that not only must a CPA-secure cipher be probabilistic, but it must be very unlikely that two encryptions of the same message yield the same ciphertext — see Exercise 5.11. $\Box$

Remark 5.1. Analogous to Theorem 5.1, it is straightforward to show that if a cipher is CPA-secure, it is also CPA-secure in the multi-key setting. See Exercise 5.2. $\Box$

5.4 Building CPA secure ciphers

In this section, we describe a number of ways of building ciphers that are semantically secure against chosen plaintext attack. As we have already discussed in Example 5.1, any such cipher must be probabilistic. We begin in Section 5.4.1 with a generic construction that combines any semantically secure cipher with a pseudo-random function (PRF). The PRF is used to generate "one time" keys. Next, in Section 5.4.2, we develop a probabilistic variant of the *counter mode* cipher discussed in Section 4.4.4. While this scheme can be based on any PRF, in practice, the PRF is usually instantiated with a block cipher. Finally, in Section 5.4.3, we present a cipher that is constructed from a block cipher using a method called *cipher block chaining (CBC) mode.*

These last two constructions, counter mode and CBC mode, are called *modes of operation of a block cipher*. Another mode of operation we have already seen in Section 4.1.4 is *electronic codebook (ECB) mode*. However, because of the lack of security provided by this mode of operation, its is seldom used. There are other modes of operations that provide CPA security, which we develop in the exercises.

5.4.1 A generic hybrid construction

In this section, we show how to turn any semantically secure cipher $\mathcal{E} = (E, D)$ into a CPA secure cipher $\mathcal{E}'$ using an appropriate PRF F.

The basic idea is this. A key for $\mathcal{E}'$ is a key k' for F. To encrypt a single message m, a random input x for F is chosen, and a key k for $\mathcal{E}$ is derived by computing $k \leftarrow F(k', x)$. Then m is encrypted using this key k: $c \xleftarrow{R} E(k, m)$. The ciphertext is $c' := (x, c)$. Note that we need to include x as part of c' so that we can decrypt: the decryption algorithm first derives the key k by computing $k \leftarrow F(k', x)$, and then recovers m by computing $m \leftarrow D(k, c)$.

For all of this to work, the output space of F must match the key space of $\mathcal{E}$. Also, the input space of F must be super-poly, so that the chances of accidentally generating the same x value twice is negligible.

Now the details. Let $\mathcal{E} = (E, D)$ be a cipher, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Let F be a PRF defined over $(\mathcal{K}', \mathcal{X}, \mathcal{K})$; that is, the output space of F should be equal to the key space of $\mathcal{E}$. We define a new cipher $\mathcal{E}' = (E', D')$, defined over $(\mathcal{K}', \mathcal{M}, \mathcal{X} \times \mathcal{C})$, as follows:

- for $k' \in \mathcal{K}'$ and $m \in \mathcal{M}$, we define

$$E'(k', m) := \quad \begin{array}{l} x \xleftarrow{R} \mathcal{X}, \quad k \leftarrow F(k', x), \quad c \xleftarrow{R} E(k, m) \\ \text{output } (x, c); \end{array}$$

- for $k' \in \mathcal{K}'$ and $c' = (x, c) \in \mathcal{X} \times \mathcal{C}$, we define

$$D'(k', c') := \quad \begin{array}{l} k \leftarrow F(k', x), \quad m \leftarrow D(k, c) \\ \text{output } m. \end{array}$$

It is easy to verify that $\mathcal{E}'$ is indeed a cipher, and is our first example of a *probabilistic* cipher.

Example 5.2. Before proving CPA security of $\mathcal{E}'$ let us first see the construction in action. Suppose $\mathcal{E}$ is the one-time pad, namely $E(k, m) := k \oplus m$ where $\mathcal{K} = \mathcal{M} = \mathcal{C} = \{0, 1\}^L$. Applying the generic hybrid construction above to the one-time pad results in the following popular cipher $\mathcal{E}_0 = (E_0, D_0)$:

- for $k' \in \mathcal{K}'$ and $m \in \mathcal{M}$, define

$$E_0(k', m) := \quad x \xleftarrow{R} \mathcal{X}, \ \text{ output } (x, \ F(k', x) \oplus m)$$

- for $k' \in \mathcal{K}'$ and $c' = (x, c) \in \mathcal{X} \times \mathcal{C}$, define

$$D_0(k', c') := \ \text{output } F(k', x) \oplus c$$

CPA security of this cipher follows from the CPA security of the generic hybrid construction $\mathcal{E}'$ which is proved in Theorem 5.2 below. □

Theorem 5.2. *If F is a secure PRF, $\mathcal{E}$ is a semantically secure cipher, and $N := |\mathcal{X}|$ is super-poly, then the cipher $\mathcal{E}'$ described above is a CPA secure cipher.*

In particular, for every CPA adversary $\mathcal{A}$ that attacks $\mathcal{E}'$ as in the bit-guessing version of Attack Game 5.2, and which makes at most Q queries to its challenger, there exists a PRF adversary $\mathcal{B}_F$ that attacks F as in Attack Game 4.2, and an SS adversary $\mathcal{B}_\mathcal{E}$ that attacks $\mathcal{E}$ as in the bit-guessing version of Attack Game 2.1, where both $\mathcal{B}_F$ and $\mathcal{B}_\mathcal{E}$ are elementary wrappers around $\mathcal{A}$, such that

$$\mathrm{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}'] \leq \frac{Q^2}{N} + 2 \cdot \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F] + Q \cdot \mathrm{SS}\mathsf{adv}[\mathcal{B}_\mathcal{E}, \mathcal{E}]. \tag{5.5}$$

Proof idea. First, using the assumption that F is a PRF, we can effectively replace F by a truly random function. Second, using the assumption that N is super-poly, we argue that except with negligible probability, no two x-values are ever the same. But in this scenario, the challenger's keys are now all independently generated, and so the challenger is really playing the same role as the challenger in the Attack Game 5.1. The result then follows from Theorem 5.1. □

Proof. Let $\mathcal{A}$ be an efficient CPA adversary that attacks $\mathcal{E}'$ as in Attack Game 5.2. Assume that $\mathcal{A}$ makes at most Q queries to its challenger. Our goal is to show that $\mathrm{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}']$ is negligible, assuming that F is a secure PRF, that N is super-poly, and that $\mathcal{E}$ is semantically secure.

It is convenient to use the bit-guessing versions of the CPA and semantic security attack games. We prove:

$$\mathrm{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}'] \leq \frac{Q^2}{2N} + \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F] + Q \cdot \mathrm{SS}\mathsf{adv}^*[\mathcal{B}_\mathcal{E}, \mathcal{E}] \tag{5.6}$$

for efficient adversaries $\mathcal{B}_F$ and $\mathcal{B}_\mathcal{E}$. Then (5.5) follows from (5.4) and Theorem 2.10.

The basic strategy of the proof is as follows. First, we define Game 0 to be the game played between $\mathcal{A}$ and the challenger in the bit-guessing version of Attack Game 5.2 with respect to $\mathcal{E}'$. We then define several more games: Game 1, Game 2, and Game 3. Each of these games is played between $\mathcal{A}$ and a different challenger; moreover, as we shall see, Game 3 is equivalent to the bit-guessing version of Attack Game 5.1 with respect to $\mathcal{E}$. In each of these games, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. Also, for $j = 0, \ldots, 3$, we define W_j to be the event that $\hat{b} = b$ in Game j. We will show that for $j = 1, \ldots, 3$, the value $|\Pr[W_j] - \Pr[W_{j-1}]|$ is negligible; moreover, from the assumption that $\mathcal{E}$ is semantically secure, and from Theorem 5.1, it will follow that $|\Pr[W_3] - 1/2|$ is negligible; from this, it follows that $\mathrm{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}'] := |\Pr[W_0] - 1/2|$ is negligible.

Game 0. Let us begin by giving a detailed description of the challenger in Game 0 that is convenient for our purposes:

$b \xleftarrow{R} \{0,1\}$
$k' \xleftarrow{R} \mathcal{K}'$
for $i \leftarrow 1$ to Q do
  $x_i \xleftarrow{R} \mathcal{X}$
  $k_i \leftarrow F(k', x_i)$
upon receiving the ith query $(m_{i0}, m_{i1}) \in \mathcal{M}^2$:
  $c_i \xleftarrow{R} E(k_i, m_{ib})$
  send (x_i, c_i) to the adversary.

By construction, we have

$$\mathrm{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}'] = \big|\Pr[W_0] - 1/2\big|, \tag{5.7}$$

Game 1. Next, we play our "PRF card," replacing $F(k', \cdot)$ by a truly random function $f \in \mathrm{Funs}[\mathcal{X}, \mathcal{K}]$. The challenger in this game looks like this:

$b \xleftarrow{R} \{0,1\}$
$f \xleftarrow{R} \mathrm{Funs}[\mathcal{X}, \mathcal{K}]$
for $i \leftarrow 1$ to Q do
  $x_i \xleftarrow{R} \mathcal{X}$
  $\boxed{k_i \leftarrow f(x_i)}$
upon receiving the ith query $(m_{i0}, m_{i1}) \in \mathcal{M}^2$:
  $c_i \xleftarrow{R} E(k_i, m_{ib})$
  send (x_i, c_i) to the adversary.

We claim that

$$\big|\Pr[W_1] - \Pr[W_0]\big| = \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F], \tag{5.8}$$

where $\mathcal{B}_F$ is an efficient PRF adversary; moreover, since we are assuming that F is a secure PRF, it must be the case that $\mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F]$ is negligible.

The design of $\mathcal{B}_F$ is naturally suggested by the syntax of Games 0 and 1. If $f \in \mathrm{Funs}[\mathcal{X}, \mathcal{K}]$ denotes the function chosen by its challenger in Attack Game 4.2 with respect to F, adversary $\mathcal{B}_F$ runs as follows:

First, $\mathcal{B}_F$ makes the following computations:

$b \xleftarrow{R} \{0,1\}$
for $i \leftarrow 1$ to Q do
  $x_i \xleftarrow{R} \mathcal{X}$
  $k_i \xleftarrow{R} f(x_i)$.

Here, $\mathcal{B}_F$ obtains the value $f(x_i)$ by querying its own challenger with x_i.

Next, adversary $\mathcal{B}_F$ plays the role of challenger to $\mathcal{A}$; specifically, when $\mathcal{A}$ makes its ith query (m_{i0}, m_{i1}), adversary $\mathcal{B}_F$ computes

$$c_i \xleftarrow{R} E(k_i, m_{ib})$$

and sends (x_i, c_i) to $\mathcal{A}$.

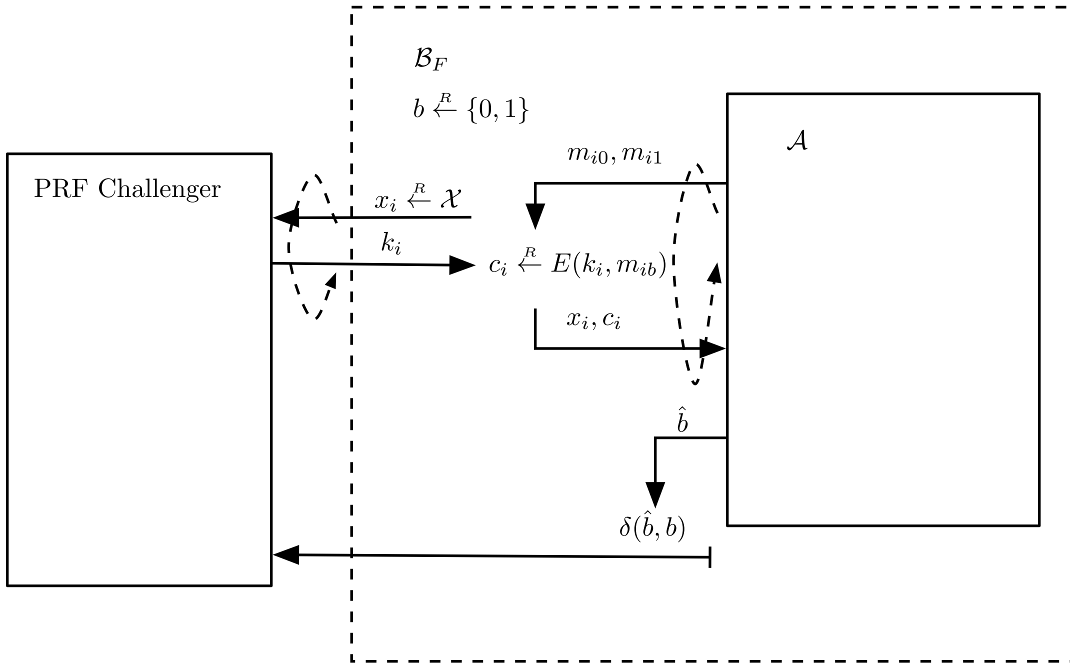


Figure 5.1: Adversary $\mathcal{B}_F$ in the proof of Theorem 5.2

Eventually, $\mathcal{A}$ halts and outputs a bit $\hat{b}$, at which time adversary $\mathcal{B}_F$ halts and outputs 1 if $\hat{b} = b$, and outputs 0 otherwise.

See Fig. 5.1 for a picture of adversary $\mathcal{B}_F$. As usual, $\delta(x, y)$ is defined to be 1 if $x = y$, and 0 otherwise.

Game 2. Next, we use our "faithful gnome" idea (see Section 4.4.2) to implement the random function f. Our "gnome" has to keep track of the inputs to f, and detect if the same input is used twice. In the following logic, our gnome uses a truly random key as the "default" value for k_i, but over-rides this default value if necessary, as indicated in the line marked $(*)$:

$b \xleftarrow{R} \{0,1\}$
for $i \leftarrow 1$ to Q do
  $x_i \xleftarrow{R} \mathcal{X}$
  $k_i \xleftarrow{R} \mathcal{K}$
$(*)$ if $x_i = x_j$ for some $j < i$ then $k_i \leftarrow k_j$
upon receiving the ith query $(m_{i0}, m_{i1}) \in \mathcal{M}^2$:
  $c_i \xleftarrow{R} E(k_i, m_{ib})$
  send (x_i, c_i) to the adversary.

As this is a faithful implementation of the random function f, we have

$$\Pr[W_2] = \Pr[W_1]. \tag{5.9}$$

Game 3. Next, we make our gnome "forgetful," simply dropping the line marked $(*)$ in the previous game:

$b \xleftarrow{R} \{0,1\}$
for $i \leftarrow 1$ to Q do
  $x_i \xleftarrow{R} \mathcal{X}$
  $k_i \xleftarrow{R} \mathcal{K}$
upon receiving the ith query $(m_{i0}, m_{i1}) \in \mathcal{M}^2$:
  $c_i \xleftarrow{R} E(k_i, m_{ib})$
  send (x_i, c_i) to the adversary.

To analyze the quantity $|\Pr[W_3] - \Pr[W_2]|$, we use the Difference Lemma (Theorem 4.7). To this end, we view Games 2 and 3 as operating on the same underlying probability space: the random choices made by the adversary and the challenger are identical in both games — all that differs is the rule used by the challenger to compute its responses. In particular, the variables x_i are identical in both games. Define Z to be the event that $x_i = x_j$ for some $i \neq j$. Clearly, Games 2 and 3 proceed identically unless Z occurs; in particular, $W_2 \wedge \bar{Z}$ occurs if and only if $W_3 \wedge \bar{Z}$ occurs. Applying the Difference Lemma, we therefore have

$$\big|\Pr[W_3] - \Pr[W_2]\big| \leq \Pr[Z]. \tag{5.10}$$

Moreover, it is easy to see that

$$\Pr[Z] \leq \frac{Q^2}{2N}, \tag{5.11}$$

since Z is the union of less that $Q^2/2$ events, each of which occurs with probability $1/N$.

Observe that in Game 3, independent encryption keys k_i are used to encrypt each message. So next, we play our "semantic security card," claiming that

$$|\Pr[W_3] - 1/2| = \mathrm{MSS}\mathsf{adv}^*[\bar{\mathcal{B}}_{\mathcal{E}}, \mathcal{E}], \tag{5.12}$$

where $\bar{\mathcal{B}}_{\mathcal{E}}$ is an efficient adversary that plays the bit-guessing version of Attack Game 5.1 with respect to $\mathcal{E}$, making at most Q queries to its challenger in that game.

The design of $\bar{\mathcal{B}}_{\mathcal{E}}$ is naturally suggested by the syntactic form of Game 3. It works as follows:

> Playing the role of challenger to $\mathcal{A}$, upon receiving the ith query (m_{i0}, m_{i1}) from $\mathcal{A}$, adversary $\bar{\mathcal{B}}_{\mathcal{E}}$ submits (m_{i0}, m_{i1}) to its own challenger, obtaining a ciphertext $c_i \in \mathcal{C}$; then $\bar{\mathcal{B}}_{\mathcal{E}}$ selects x_i at random from $\mathcal{X}$, and sends (x_i, c_i) to $\mathcal{A}$ in response to the latter's query.
>
> When $\mathcal{A}$ finally outputs a bit $\hat{b}$, $\bar{\mathcal{B}}_{\mathcal{E}}$ outputs this same bit.

See Fig. 5.2 for a picture of adversary $\bar{\mathcal{B}}_{\mathcal{E}}$.

It is evident from the construction (and (2.13)) that (5.12) holds. Moreover, by Theorem 5.1 and (5.1), we have

$$\mathrm{MSS}\mathsf{adv}^*[\bar{\mathcal{B}}_{\mathcal{E}}, \mathcal{E}] = Q \cdot \mathrm{SS}\mathsf{adv}^*[\mathcal{B}_{\mathcal{E}}, \mathcal{E}], \tag{5.13}$$

where $\mathcal{B}_{\mathcal{E}}$ is an efficient adversary playing the bit-guessing version of Attack Game 2.1 with respect to $\mathcal{E}$.

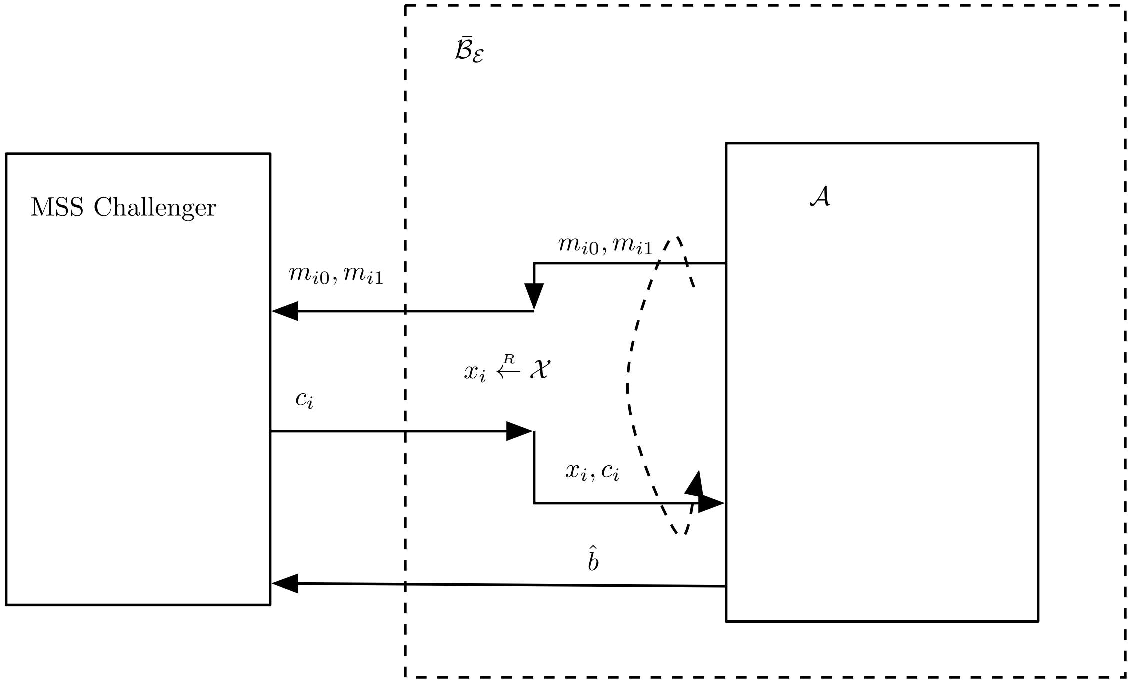


Figure 5.2: Adversary $\bar{\mathcal{B}}_{\mathcal{E}}$ in the proof of Theorem 5.2

Putting together (5.7) through (5.13), we obtain (5.6). Also, one can check that the running times of both $\mathcal{B}_F$ and $\mathcal{B}_{\mathcal{E}}$ are roughly the same as that of $\mathcal{A}$; indeed, they are elementary wrappers around $\mathcal{A}$, and (5.5) holds regardless of whether $\mathcal{A}$ is efficient. □

While the above proof was a bit long, we hope the reader agrees that it was in fact quite natural, and that all of the steps were fairly easy to follow. Also, this proof illustrates how one typically employs more than one security assumption in devising a security proof as a sequence of games.

Remark 5.2. We briefly mention that the hybrid construction $\mathcal{E}'$ in Theorem 5.2 is CPA secure even if the PRF F used in the construction is only weakly secure (as in Definition 4.3). To prove Theorem 5.2 under this weaker assumption observe that in both Games 0 and 1 the challenger only evaluates the PRF at *random* points in $\mathcal{X}$. Therefore, the adversary's advantage in distinguishing Games 0 and 1 is negligible even if F is only weakly secure. □

5.4.2 Randomized counter mode

We can build a CPA secure cipher directly out of a secure PRF, as follows. Suppose F is a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$. We shall assume that $\mathcal{X} = \{0, \ldots, N-1\}$, and that $\mathcal{Y} = \{0,1\}^n$.

For any poly-bounded $\ell \geq 1$, we define a cipher $\mathcal{E} = (E, D)$, with key space $\mathcal{K}$, message space $\mathcal{Y}^{\leq \ell}$, and ciphertext space $\mathcal{X} \times \mathcal{Y}^{\leq \ell}$, as follows:

- for $k \in \mathcal{K}$ and $m \in \mathcal{Y}^{\leq \ell}$, with $v := |m|$, we define

$$
\begin{aligned}
&E(k, m) := \\
&\qquad x \xleftarrow{\text{R}} \mathcal{X} \\
&\qquad \text{compute } c \in \mathcal{Y}^v \text{ as follows:} \\
&\qquad\qquad \text{for } j \leftarrow 0 \text{ to } v-1 \text{ do} \\
&\qquad\qquad\qquad c[j] \leftarrow F(k, x+j \bmod N) \oplus m[j] \\
&\qquad \text{output } (x, c);
\end{aligned}
$$

- for $k \in \mathcal{K}$ and $c' = (x, c) \in \mathcal{X} \times \mathcal{Y}^{\leq \ell}$, with $v := |c|$, we define

$$
\begin{aligned}
&D(k, c') := \\
&\qquad \text{compute } m \in \mathcal{Y}^v \text{ as follows:} \\
&\qquad\qquad \text{for } j \leftarrow 0 \text{ to } v-1 \text{ do} \\
&\qquad\qquad\qquad m[j] \leftarrow F(k, x+j \bmod N) \oplus c[j] \\
&\qquad \text{output } m.
\end{aligned}
$$

This cipher is much like the stream cipher one would get by building a PRG out of F using the construction in Section 4.4.4. The difference is that instead of using a fixed sequence of inputs to F to derive a key stream, we use a random starting point, which we then increment to obtain successive inputs to F. The x component of the ciphertext is typically called an **initial value**, or **IV** for short.

In practice, F is typically implemented using the encryption function of a block cipher, and $\mathcal{X} = \mathcal{Y} = \{0,1\}^n$, where we naturally view n-bit strings as numbers in the range $0, \ldots, 2^n - 1$. As it happens, the decryption function of the block cipher is not needed at all in this construction. See Fig. 5.3 for an illustration of this mode.

It is easy to verify that $\mathcal{E}$ is indeed a (probabilistic) cipher. Also, note that the message space of $\mathcal{E}$ is variable length, and that for the purposes of defining CPA security using Attack Game 5.2, the length of a message $m \in \mathcal{Y}^{\leq \ell}$ is its natural length $|m|$.

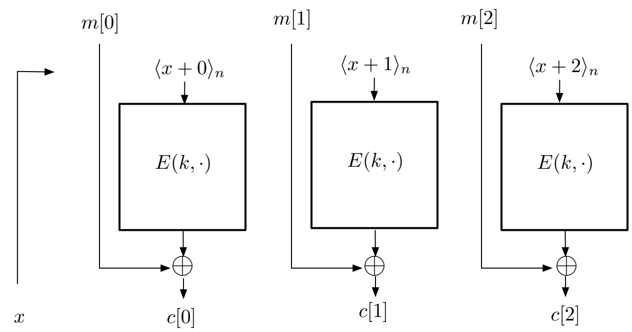
Theorem 5.3. *If F is a secure PRF and N is super-poly, then for any poly-bounded $\ell \geq 1$, the cipher $\mathcal{E}$ described above is a CPA secure cipher.*

In particular, for every CPA adversary $\mathcal{A}$ that attacks $\mathcal{E}$ as in Attack Game 5.2, and which makes at most Q queries to its challenger, there exists a PRF adversary $\mathcal{B}$ that attacks F as in Attack Game 4.2, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

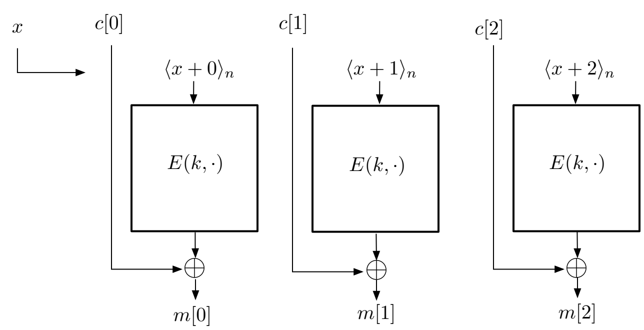
$$
\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq \frac{4Q^2\ell}{N} + 2 \cdot \text{PRF}\mathsf{adv}[\mathcal{B}, F]. \tag{5.14}
$$

Proof idea. Suppose we start with an adversary that plays the CPA attack game with respect to $\mathcal{E}$. First, using the assumption that F is a PRF, we can effectively replace F by a truly random function f. Second, using the assumption that N is super-poly, and the fact that each IV is chosen at random, we can argue that except with negligible probability, the challenger never evaluates f at the same point twice. But in this case, the challenger is effectively encrypting each message using an independent one-time pad, and so we can conclude that the adversary's advantage in the original CPA attack game is negligible. $\Box$

Proof. Let $\mathcal{A}$ be an efficient adversary that plays Attack Game 5.2 with respect to $\mathcal{E}$, and which makes at most Q queries to its challenger in that game. We want to show that $\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible, assuming that F is a secure PRF and that N is super-poly.



(a) encryption



(b) decryption

Figure 5.3: Randomizd counter mode ($v = 3$)

It is convenient to use the bit-guessing version of the CPA attack game, We prove:

$$\mathrm{CPA}\mathsf{adv}^*[\mathcal{A},\mathcal{E}] \le \frac{2Q^2\ell}{N} + \mathrm{PRF}\mathsf{adv}[\mathcal{B},F] \tag{5.15}$$

for an efficient adversary $\mathcal{B}$. Then (5.14) follows from (5.4).

The basic strategy of the proof is as follows. First, we define Game 0 to be the game played between $\mathcal{A}$ and the challenger in the bit-guessing version of Attack Game 5.2 with respect to $\mathcal{E}$. We then define several more games: Game 1, Game 2, and Game 3. Each of these games is played between $\mathcal{A}$ and a different challenger. In each of these games, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. Also, for $j = 0, \ldots, 3$, we define W_j to be the event that $\hat{b} = b$ in Game j. We will show that for $j = 1, \ldots, 3$, the value $|\Pr[W_j] - \Pr[W_{j-1}]|$ is negligible; moreover, it will be evident that $\Pr[W_3] = 1/2$, from which it will follow that $\mathrm{CPA}\mathsf{adv}^*[\mathcal{A},\mathcal{E}] := |\Pr[W_0] - 1/2|$ is negligible.

Game 0. We may describe the challenger in Game 0 as follows:

$b \xleftarrow{\text{R}} \{0,1\}$
$k \xleftarrow{\text{R}} \mathcal{K}$
for $i \leftarrow 1$ to Q do
    $x_i \xleftarrow{\text{R}} \mathcal{X}$
    for $j \leftarrow 0$ to $\ell - 1$ do
        $x'_{ij} \leftarrow x_i + j \bmod N$
        $y_{ij} \leftarrow F(k, x'_{ij})$
upon receiving the ith query (m_{i0}, m_{i1}), with $v_i := |m_{i0}| = |m_{i1}|$:
    compute $c_i \in \mathcal{Y}^{v_i}$ as follows:
        for $j \leftarrow 0$ to $v_i - 1$ do: $c_i[j] \leftarrow y_{ij} \oplus m_{ib}[j]$
    send (x_i, c_i) to the adversary.

By construction, we have we have

$$\mathrm{CPA}\mathsf{adv}^*[\mathcal{A},\mathcal{E}] = \big|\Pr[W_0] - 1/2\big|. \tag{5.16}$$

Game 1. Next, we play our "PRF card," replacing $F(k, \cdot)$ by a truly random function $f \in \mathrm{Funs}[\mathcal{X},\mathcal{Y}]$. The challenger in this game looks like this:

$b \xleftarrow{\text{R}} \{0,1\}$
$f \xleftarrow{\text{R}} \mathrm{Funs}[\mathcal{X},\mathcal{Y}]$
for $i \leftarrow 1$ to Q do
    $x_i \xleftarrow{\text{R}} \mathcal{X}$
    for $j \leftarrow 0$ to $\ell - 1$ do
        $x'_{ij} \leftarrow x_i + j \bmod N$
        $y_{ij} \leftarrow f(x'_{ij})$
...

We have left out part of the code for the challenger, as it will not change in any of our games. We claim that

$$\big|\Pr[W_1] - \Pr[W_0]\big| = \mathrm{PRF}\mathsf{adv}[\mathcal{B},F], \tag{5.17}$$

where $\mathcal{B}$ is an efficient adversary; moreover, since we are assuming that F is a secure PRF, it must be the case that $\mathrm{PRF}\mathsf{adv}[\mathcal{B},F]$ is negligible. This is hopefully (by now) a routine argument, and we leave the details of this to the reader.

Game 2. Next, we use our "faithful gnome" idea to implement the random function f. In describing the logic of our challenger in this game, we use the standard lexicographic ordering on pairs of indices (i, j); that is, $(i', j') < (i, j)$ if and only if

$$i' < i \quad \text{or} \quad i' = i \text{ and } j' < j.$$

In the following logic, our "gnome" uses a truly random value as the "default" value for each y_{ij}, but over-rides this default value if necessary, as indicated in the line marked $(*)$:

$b \xleftarrow{R} \{0, 1\}$
for $i \leftarrow 1$ to Q do
    $x_i \xleftarrow{R} \mathcal{X}$
    for $j \leftarrow 0$ to $\ell - 1$ do
        $x'_{ij} \leftarrow x_i + j \bmod N$
        $y_{ij} \xleftarrow{R} \mathcal{Y}$
$(*)$ if $x'_{ij} = x'_{i'j'}$ for some $(i', j') < (i, j)$ then $y_{ij} \leftarrow y_{i'j'}$
    ...

As this is a faithful implementation of the random function f, we have

$$\Pr[W_2] = \Pr[W_1]. \tag{5.18}$$

Game 3. Now we make our gnome "forgetful," dropping the line marked $(*)$ in the previous game:

$b \xleftarrow{R} \{0, 1\}$
for $i \leftarrow 1$ to Q do
    $x_i \xleftarrow{R} \mathcal{X}$
    for $j \leftarrow 0$ to $\ell - 1$ do
        $x'_{ij} \leftarrow x_i + j \bmod N$
        $y_{ij} \xleftarrow{R} \mathcal{Y}$
    ...

To analyze the quantity $|\Pr[W_3] - \Pr[W_2]|$, we use the Difference Lemma (Theorem 4.7). To this end, we view Games 2 and 3 as operating on the same underlying probability space: the random choices made by the adversary and the challenger are identical in both games — all that differs is the rule used by the challenger to compute its responses. In particular, the variables x'_{ij} are identical in both games. Define Z to be the event that $x'_{ij} = x'_{i'j'}$ for some $(i, j) \neq (i', j')$. Clearly, Games 2 and 3 proceed identically unless Z occurs; in particular, $W_2 \wedge \bar{Z}$ occurs if and only if $W_3 \wedge \bar{Z}$ occurs. Applying the Difference Lemma, we therefore have

$$\big|\Pr[W_3] - \Pr[W_2]\big| \leq \Pr[Z]. \tag{5.19}$$

We claim that

$$\Pr[Z] \leq \frac{2Q^2\ell}{N}. \tag{5.20}$$

To prove this claim, we may assume that $N \geq 2\ell$ (this should anyway generally hold, since we are assuming that ℓ is poly-bounded and N is super-poly). Observe that Z occurs if and only if

$$\{x_i, \ldots, x_i + \ell - 1\} \cap \{x_{i'}, \ldots, x_{i'} + \ell - 1\} \neq \emptyset$$

for some pair of indices i and i' with $i \neq i'$ (and arithmetic is done mod N). Consider any fixed such pair of indices. Conditioned on any fixed value of x_i, the value $x_{i'}$ is uniformly distributed over $\{0, \ldots, N-1\}$, and the intervals overlap if and only if

$$x_{i'} \in \{x_i + j : -\ell + 1 \leq j \leq \ell - 1\},$$

which happens with probability $(2\ell - 1)/N$. The inequality (5.20) now follows.

Finally, observe that in Game 3 the y_{ij} values are uniformly and independently distributed over $\mathcal{Y}$, and thus the challenger is essentially using independent one-time pads to encrypt. In particular, it is easy to see that the adversary's output in this game is independent of b. Therefore,

$$\Pr[W_3] = 1/2. \tag{5.21}$$

Putting together (5.16) through (5.21), we obtain (5.15), and the theorem follows. □

Remark 5.3. One can also view randomized counter mode as a special case of the generic hybrid construction in Section 5.4.1. See Exercise 5.5. □

Case study: AES counter mode

The IPsec protocol uses a particular variants of AES counter mode, as specified in RFC 3686. Recall that AES uses a 128 bit block. Rather than picking a random 128-bit IV for every message, RFC 3686 picks the IV as follows:

- The most significant 32 bits are chosen at random at the time that the secret key is generated and are fixed for the life of the key. The same 32 bit value is used for all messages encrypted using this key.
- The next 64 bits are chosen at random in $\{0, 1\}^{64}$.
- The least significant 32 bits are set to the number 1.

This resulting 128-bit IV is used as the initial value of the counter. When encryption a message the least significant 32 bits are incremented by one for every block of the message. Consequently, the maximum message length that can be encrypted is 2^{32} AES blocks or 2^{36} bytes.

With this choice of IV the decryptor knows the 32 most significant bits of the IV as well as the 32 least significant bits. Hence, only 64 bits of the IV need to be sent with the ciphertext.

The proof of Theorem 5.3 can be adapted to show that this method of choosing IVs is secure. The slight advantage of this method over picking a random 128-bit IV is that the resulting ciphertext is a little shorter. A random IV forces the encryptor to include all 128 bits in the ciphertext. With the method of RFC 3686 only 64 bits are needed, thus shrinking the ciphertext by 8 bytes.

5.4.3 CBC mode

An historically important encryption method is to use a block cipher in cipher block chaining (CBC) mode. This method is used in older versions of the TLS protocol (e.g., TLS 1.0). It is inferior to counter mode encryption as discussed in the next section.

Suppose $\mathcal{E} = (E, D)$ is a block cipher defined over $(\mathcal{K}, \mathcal{X})$, where $\mathcal{X} = \{0, 1\}^n$. Let $N := |\mathcal{X}| = 2^n$. For any poly-bounded $\ell \geq 1$, we define a cipher $\mathcal{E}' = (E', D')$, with key space $\mathcal{K}$, message space $\mathcal{X}^{\leq \ell}$, and ciphertext space $\mathcal{X}^{\leq \ell+1} \setminus \mathcal{X}^0$; that is, the ciphertext space consists of all nonempty sequences of at most $\ell + 1$ data blocks. Encryption and decryption are defined as follows:

- for $k \in \mathcal{K}$ and $m \in \mathcal{X}^{\leq \ell}$, with $v := |m|$, we define

$$
\begin{aligned}
&E'(k, m) := \\
&\qquad \text{compute } c \in \mathcal{X}^{v+1} \text{ as follows:} \\
&\qquad\qquad c[0] \xleftarrow{\text{R}} \mathcal{X} \\
&\qquad\qquad \text{for } j \leftarrow 0 \text{ to } v-1 \text{ do} \\
&\qquad\qquad\qquad c[j+1] \leftarrow E(k,\ c[j] \oplus m[j]) \\
&\qquad \text{output } c;
\end{aligned}
$$

- for $k \in \mathcal{K}$ and $c \in \mathcal{X}^{\leq \ell+1} \setminus \mathcal{X}^0$, with $v := |c| - 1$, we define

$$
\begin{aligned}
&D'(k, c) := \\
&\qquad \text{compute } m \in \mathcal{X}^{v} \text{ as follows:} \\
&\qquad\qquad \text{for } j \leftarrow 0 \text{ to } v-1 \text{ do} \\
&\qquad\qquad\qquad m[j] \leftarrow D(k,\ c[j+1]) \oplus c[j] \\
&\qquad \text{output } m.
\end{aligned}
$$

See Fig. 5.4 for an illustration of the encryption and decryption algorithm in the case $|m| = 3$. Here, the first component $c[0]$ of the ciphertext is also called an initial value, or IV. Note that unlike the counter mode construction in Section 5.4.2, in CBC mode, we must use a block cipher, as we actually need to use the decryption algorithm of the block cipher.

It is easy to verify that $\mathcal{E}'$ is indeed a (probabilistic) cipher. Also, note that the message space of $\mathcal{E}$ is variable length, and that for the purposes of defining CPA security using Attack Game 5.2, the length of a message $m \in \mathcal{X}^{\leq \ell}$ is its natural length $|m|$.

Theorem 5.4. *If $\mathcal{E} = (E, D)$ is a secure block cipher defined over $(\mathcal{K}, \mathcal{X})$, and $N := |\mathcal{X}|$ is super-poly, then for any poly-bounded $\ell \geq 1$, the cipher $\mathcal{E}'$ described above is a CPA secure cipher.*

In particular, for every CPA adversary $\mathcal{A}$ that attacks $\mathcal{E}'$ as in the bit-guessing version of Attack Game 5.2, and which makes at most Q queries to its challenger, there exists BC adversary $\mathcal{B}$ that attacks $\mathcal{E}$ as in Attack Game 4.1, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$
\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}'] \leq \frac{2Q^2\ell^2}{N} + 2 \cdot \text{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}]. \tag{5.22}
$$

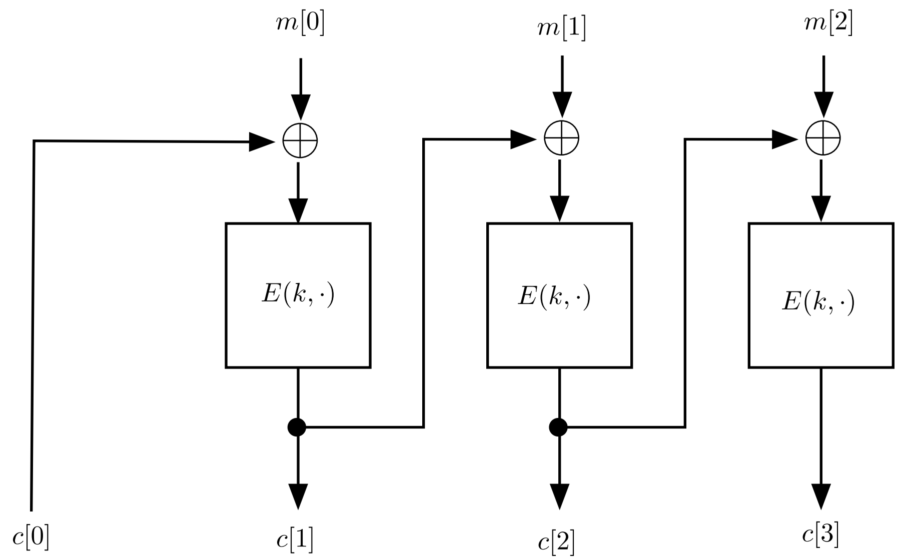
Proof idea. The basic idea of the proof is very similar to that of Theorem 5.3. We start with an adversary that plays the CPA attack game with respect to $\mathcal{E}'$. We then replace E by a truly random function f. Then we argue that except with negligible probability, the challenger never evaluates f at the same point twice. But then what the adversary sees is nothing but a bunch of random bits, and so learns nothing at all about the message being encrypted. $\square$

Proof. Let $\mathcal{A}$ be an efficient CPA adversary that attacks $\mathcal{E}'$ as in Attack Game 5.2. Assume that $\mathcal{A}$ makes at most Q queries to its challenger in that game. We want to show that $\text{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}']$ is negligible, assuming that $\mathcal{E}$ is a secure block cipher and that N is super-poly. Under these assumptions, by Corollary 4.5, the encryption function E is a secure PRF, defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$.

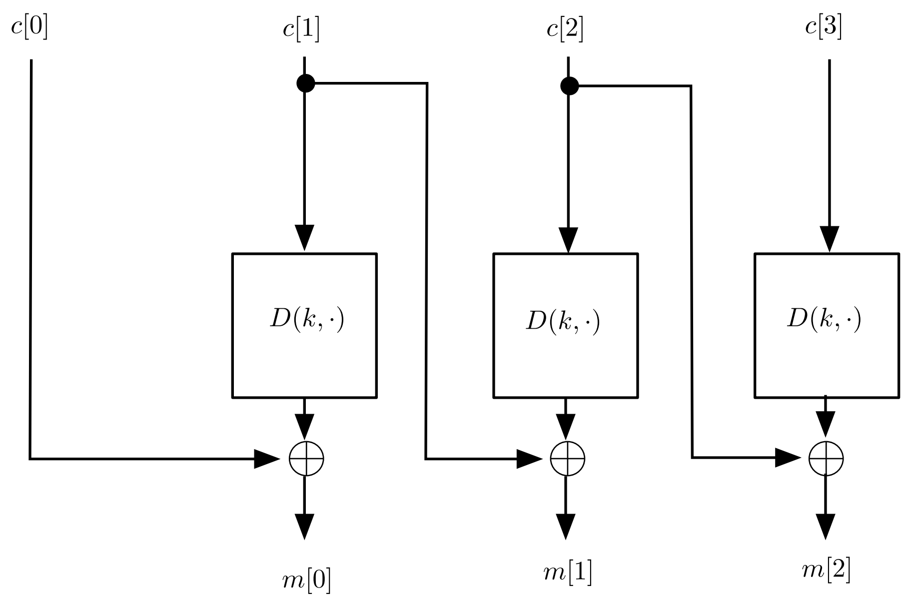
It is convenient to use the bit-guessing version of the CPA attack game, We prove:

$$
\text{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}'] \leq \frac{Q^2\ell^2}{N} + \text{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}] \tag{5.23}
$$

for an efficient adversary $\mathcal{B}$. Then (5.22) follows from (5.4).



(a) encryption



(b) decryption

Figure 5.4: Encryption and decryption for CBC mode with $\ell = 3$

As usual, we define a sequence of games: Game 0, Game 1, Game 2, Game 3. Each of these games is played between $\mathcal{A}$ and a challenger. The challenger in Game 0 is the one from the bit-guessing version of Attack Game 5.2 with respect to $\mathcal{E}'$. In each of these games, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. Also, for $j = 0, \ldots, 3$, we define W_j to be the event that $\hat{b} = b$ in Game j. We will show that for $j = 1, \ldots, 3$, the value $|\Pr[W_j] - \Pr[W_{j-1}]|$ is negligible; moreover, it will be evident that $\Pr[W_3] = 1/2$, from which it will follow that $|\Pr[W_0] - 1/2|$ is negligible.

Here we go!

Game 0. We may describe the challenger in Game 0 as follows:

$b \xleftarrow{R} \{0,1\}$, $k \xleftarrow{R} \mathcal{K}$
upon receiving the ith query (m_{i0}, m_{i1}), with $v_i := |m_{i0}| = |m_{i1}|$:
    compute $c_i \in \mathcal{X}^{v_i+1}$ as follows:
        $c_i[0] \xleftarrow{R} \mathcal{X}$
        for $j \leftarrow 0$ to $v_i - 1$ do
            $x_{ij} \leftarrow c_i[j] \oplus m_{ib}[j]$
            $c_i[j+1] \leftarrow E(k, x_{ij})$
    send c_i to the adversary.

By construction, we have

$$\mathrm{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}'] = \big|\Pr[W_0] - 1/2\big|. \tag{5.24}$$

Game 1. We now play the "PRF card," replacing $E(k, \cdot)$ by a truly random function $f \in \mathrm{Funs}[\mathcal{X}, \mathcal{X}]$. Our challenger in this game looks like this:

$b \xleftarrow{R} \{0,1\}$, $f \xleftarrow{R} \mathrm{Funs}[\mathcal{X}, \mathcal{X}]$
upon receiving the ith query (m_{i0}, m_{i1}), with $v_i := |m_{i0}| = |m_{i1}|$:
    compute $c_i \in \mathcal{X}^{v_i+1}$ as follows:
        $c_i[0] \xleftarrow{R} \mathcal{X}$
        for $j \leftarrow 0$ to $v_i - 1$ do
            $x_{ij} \leftarrow c_i[j] \oplus m_{ib}[j]$
            $c_i[j+1] \leftarrow f(x_{ij})$
    send c_i to the adversary.

We claim that

$$\big|\Pr[W_1] - \Pr[W_0]\big| = \mathrm{PRF}\mathsf{adv}[\mathcal{B}, E], \tag{5.25}$$

where $\mathcal{B}$ is an efficient adversary; moreover, since we are assuming that $\mathcal{E}$ is a secure block cipher, and that N is super-poly, it must be the case that $\mathrm{PRF}\mathsf{adv}[\mathcal{B}, E]$ is negligible. This is hopefully (by now) a routine argument, and we leave the details of this to the reader.

Game 2. The next step in this dance should by now be familiar: we implement f using a faithful gnome. We do so by introducing random variables y_{ij} which represent the "default" values for $c_i[j]$, which get over-ridden if necessary in the line marked $(*)$ below:

$b \xleftarrow{R} \{0,1\}$
set $y_{ij} \xleftarrow{R} \mathcal{X}$ for $i = 1, \ldots, Q$ and $j = 0, \ldots, \ell$
upon receiving the ith query (m_{i0}, m_{i1}), with $v_i := |m_{i0}| = |m_{i1}|$:
    compute $c_i \in \mathcal{X}^{v_i+1}$ as follows:
        $c_i[0] \leftarrow y_{i0}$
        for $j \leftarrow 0$ to $v_i - 1$ do
            $x_{ij} \leftarrow c_i[j] \oplus m_{ib}[j]$
            $c_i[j+1] \leftarrow y_{i(j+1)}$
$(*)$ if $x_{ij} = x_{i'j'}$ for some $(i', j') < (i, j)$ then $c_i[j+1] \leftarrow c_{i'}[j'+1]$
    send c_i to the adversary.

We clearly have

$$\Pr[W_2] = \Pr[W_1]. \tag{5.26}$$

Game 3. Now we make gnome forgetful, removing the check in the line marked $(*)$:

$b \xleftarrow{R} \{0,1\}$
set $y_{ij} \xleftarrow{R} \mathcal{X}$ for $i = 1, \ldots, Q$ and $j = 0, \ldots, \ell$
upon receiving the ith query (m_{i0}, m_{i1}), with $v_i := |m_{i0}| = |m_{i1}|$:
    compute $c_i \in \mathcal{X}^{v_i+1}$ as follows:
        $c_i[0] \leftarrow y_{i0}$
        for $j \leftarrow 0$ to $v_i - 1$ do
            $x_{ij} \leftarrow c_i[j] \oplus m_{ib}[j]$
            $c_i[j+1] \xleftarrow{R} y_{i(j+1)}$
    send c_i to the adversary.

To analyze the quantity $|\Pr[W_3] - \Pr[W_2]|$, we use the Difference Lemma (Theorem 4.7). To this end, we view Games 2 and 3 as operating on the same underlying probability space: the random choices made by the adversary and the challenger are identical in both games — all that differs is the rule used by the challenger to compute its responses.

We define Z to be the event that $x_{ij} = x_{i'j'}$ in Game 3. Note that the event Z is defined in terms of the x_{ij} values in Game 3. Indeed, the x_{ij} values may not be computed in the same way in Games 2 and 3, and so we have explicitly defined the event Z in terms of their values in Game 3. Nevertheless, it is clear that Games 2 and 3 proceed identically unless Z occurs; in particular, $W_2 \wedge \bar{Z}$ occurs if and only if $W_3 \wedge \bar{Z}$ occurs. Applying the Difference Lemma, we therefore have

$$\big|\Pr[W_3] - \Pr[W_2]\big| \leq \Pr[Z]. \tag{5.27}$$

We claim that

$$\Pr[Z] \leq \frac{Q^2 \ell^2}{2N}. \tag{5.28}$$

To prove this, let *Coins* denote the random choices made by $\mathcal{A}$. Observe that in Game 3, the values

$$Coins, \quad b, \quad y_{ij} \ (i = 1, \ldots Q, \ j = 0, \ldots, \ell)$$

are independently distributed.

Consider any fixed index $i = 1, \ldots, Q$. Let us condition on any fixed values of *Coins*, b, and $y_{i'j}$ for $i' = 1, \ldots, i-1$ and $j = 0, \ldots, \ell$. In this conditional probability space, the values of

m_{i0}, m_{i1}, and v_i are completely determined, as are the values $v_{i'}$ and $x_{i'j}$ for $i' = 1, \ldots, i-1$ and $j = 0, \ldots, v_{i'}-1$; however, the values of $y_{i0}, \ldots, y_{i\ell}$ are still uniformly and independently distributed over $\mathcal{X}$. Moreover, as $x_{ij} = y_{ij} \oplus m_{ib}[j]$ for $j = 0, \ldots, v_i - 1$, it follows that these x_{ij} values are also uniformly and independently distributed over $\mathcal{X}$. Thus, for any fixed index $j = 0, \ldots, v_i - 1$, and any fixed indices i' and j', with $(i', j') < (i, j)$, the probability that $x_{ij} = x_{i'j'}$ in this conditional probability space is $1/N$. The bound (5.28) now follows from an easy calculation.

Finally, we claim that

$$\Pr[W_3] = 1/2. \tag{5.29}$$

This follows from the fact that

$$\mathit{Coins}, \quad b, \quad y_{ij} \ (i = 1, \ldots Q, \ j = 0, \ldots, \ell)$$

are independently distributed, and the fact that the adversary's output $\hat{b}$ is a function of

$$\mathit{Coins}, \quad y_{ij} \ (i = 1, \ldots Q, \ j = 0, \ldots, \ell).$$

From this, we see that $\hat{b}$ and b are independent, and so (5.29) follows immediately.

Putting together (5.24) through (5.29), we have

$$\mathrm{CPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}'] \leq \frac{Q^2 \ell^2}{2N} + \mathrm{PRF}\mathsf{adv}[\mathcal{B}, E].$$

By Theorem 4.4, we have

$$\big|\mathrm{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}] - \mathrm{PRF}\mathsf{adv}[\mathcal{B}, E]\big| \leq \frac{Q^2 \ell^2}{2N},$$

and (5.23) follows, which proves the theorem. $\Box$

5.4.4 Case study: CBC padding in TLS 1.0

Let $\mathcal{E} = (E, D)$ be a block cipher with domain $\mathcal{X}$. Our description of CBC mode encryption using $\mathcal{E}$ assumes that messages to be encrypted are elements of $\mathcal{X}^{\leq \ell}$. When the domain is $\mathcal{X} = \{0,1\}^{128}$, as in the case of AES, this implies that the length of messages to be encrypted must be a multiple of 16 bytes. Since the length of messages in practice need not be a multiple of 16 we need a way augment CBC to handle messages whose length is not necessarily a multiple of the block size.

Suppose we wish to encrypt a v-byte message m using AES in CBC mode when v is not necessarily a multiple of 16. The first thing that comes to mind is to somehow pad the message m so that its length in bytes is a multiple of 16. Clearly the padding function needs to be invertible so that during decryption the padding can be removed.

The TLS 1.0 protocol defines the following padding function for encrypting a v-byte message with AES in CBC mode: let $p := 16 - (v \bmod 16)$, then append p bytes to the message m where the content of each byte is value $p - 1$. For example, consider the following two cases:

- if m is 29 bytes long then $p = 3$ and the pad consists of the three bytes "222" so that the padded message is 32 bytes long which is exactly two AES blocks.
- if the length of m is a multiple of the block size, say 32 bytes, then $p = 16$ and the pad consists of 16 bytes. The padded message is then 48 bytes long which is three AES blocks.

It may seem odd that when the message is a multiple of the block size we add a full dummy block at the end. This is necessary so that the decryption procedure can properly remove the pad. Indeed, it should be clear that this padding method is invertible for all input message lengths.

It is an easy fact to prove that every invertible padding scheme for CBC mode encryption built from a secure block cipher gives a CPA secure cipher for messages of arbitrary length.

Padding in CBC mode can be avoided using a method called **ciphertext stealing** as long as the plaintext is longer than a single block. The ciphertext stealing variant of CBC is the topic of Exercise 5.16. When encrypting messages whose length is less than a block, say single byte messages, there is still a need to pad.

5.4.5 Concrete parameters and a comparison of counter and CBC modes

We conclude this section with a comparison of the counter and CBC mode constructions. We assume that counter mode is implemented with a PRF F that maps n-bit blocks to n-bit blocks, and that CBC is implemented with an n-bit block cipher. In each case, the message space consists of sequences of at most ℓ n-bit data blocks. With the security theorems proved in this section, we have the following bounds:

$$\begin{aligned}\mathrm{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\mathrm{ctr}}] &\le \frac{4Q^2\ell}{2^n} + 2\cdot \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F],\\ \mathrm{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\mathrm{cbc}}] &\le \frac{2Q^2\ell^2}{2^n} + 2\cdot \mathrm{BC}\mathsf{adv}[\mathcal{B}_{\mathcal{E}}, \mathcal{E}].\end{aligned}$$

Here, $\mathcal{A}$ is any CPA adversary making at most Q queries to its challenger, ℓ is the maximum length (in data blocks) of any one message. For the purposes of this discussion, let us simply ignore the terms $\mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F]$ and $\mathrm{BC}\mathsf{adv}[\mathcal{B}_{\mathcal{E}}, \mathcal{E}]$.

One can immediately see that counter mode has a quantitative security advantage. To make things more concrete, suppose the block size is $n = 128$, and that each message is 1MB (2^{23} bits) so that $\ell = 2^{16}$ blocks. If we want to keep the adversary's advantage below 2^{-32}, then for counter mode, we can encrypt up to $Q = 2^{39.5}$ messages, while for CBC we can encrypt only up to 2^{32} messages. Once Q message are encrypted with a given key, a fresh key must be generated and used for subsequent messages. Therefore, with counter mode a single key can be used to securely encrypt many more messages as compared with CBC.

Counter mode has several other advantages over CBC:

- *Parallelism and pipelining.* Encryption and decryption for counter mode is trivial to parallelize, whereas encryption in CBC mode is inherently sequential (decryption in CBC mode is parallelizable). Modes that support parallelism greatly improve performance when the underlying hardware can execute many instructions in parallel as is often the case in modern processors. More importantly, consider a hardware implementation of a single block cipher round that supports pipelining, as in Intel's implementation of AES-128 (page 118). Pipelining enables multiple encryption instructions to execute at the same time. A parallel mode such as counter mode keeps the pipeline busy, where as in CBC encryption the pipeline is mostly unused due to the sequential nature of this mode. As a result, counter mode encryption on Intel's Haswell processors is about seven times faster than CBC mode encryption, assuming the plaintext data is already loaded into L1 cache.

- *Shorter ciphertext length.* For very short messages, counter mode ciphertexts are significantly shorter than CBC mode ciphertexts. Consider, for example, a one-byte plaintext (which arises naturally when encrypting individual key strokes as in SSH). A counter mode ciphertext need only be one block plus one byte: one block for the random IV plus one byte for the encrypted plaintext. In contrast, a CBC ciphertext is two full blocks. This results in 15 redundant bytes per CBC ciphertext assuming 128-bit blocks.

- *Encryption only.* CBC mode uses both algorithms E and D of the block cipher where as counter mode uses only algorithm E. This can reduce an implementation code size.

Remark 5.4. Both randomized counter mode and CBC require a random IV. Some crypto libraries actually leave it to the higher-level application to supply the IV. This can lead to problems if the higher-level applications do not take pains to ensure the IVs are sufficiently random. For example, for counter mode, it is necessary that the IVs are sufficiently spread out, so that the corresponding intervals do not overlap. In fact, this property is sufficient as well. In contrast, for CBC mode, more is required: it is essential that IVs be unpredictable — see Exercise 5.12.

Leaving it to the higher-level application to supply the IV is actually an example of *nonce-based encryption*, which we will explore in detail next, in Section 5.5. □

5.5 Nonce-based encryption

All of the CPA-secure encryption schemes we have seen so far suffer from *ciphertext expansion*: ciphertexts are longer than plaintexts. For example, the generic hybrid construction in Section 5.4.1 generates ciphertexts (x, c), where x belongs to the input space of some PRF and c encrypts the actual message; the counter mode construction in Section 5.4.2 generates ciphertexts of the essentially same form (x, c); similarly, the CBC mode construction in Section 5.4.3 includes the IV as a part of the ciphertext.

For very long messages, the expansion is not too bad. For example, with AES and counter mode or CBC mode, a 1MB message results is a ciphertext that is just 16 bytes longer, which may be a perfectly acceptable expansion rate. However, for messages of 16 bytes or less, ciphertexts are at least twice as long as plaintexts.

The bad news is, some amount of ciphertext expansion is inevitable for any CPA-secure encryption scheme (see Exercise 5.10). The good news is, in certain settings, one can get by without any ciphertext expansion. For example, suppose Alice and Bob are fully synchronized, so that Alice first sends an encryption m_1, then an encryption m_2, and so on, while Bob first decrypts the encryption of m_1, and then decrypts the encryption of m_2, and so on. For concreteness, assume Alice and Bob are using the generic hybrid construction of Section 5.4.1. Recall that the encryption of message m_i is (x_i, c_i), where $c_i := E(k_i, m_i)$ and $k_i := F(x_i)$. The essential property of the x_i's needed to ensure security was simply that they are distinct. When Alice and Bob are fully synchronized (i.e., ciphertexts sent by Alice reach Bob in-order), they simply have to agree on a fixed sequence $x_1, x_2, \ldots,$ of distinct elements in the input space of the PRF F. For example, x_i might simply be the binary encoding of i.

This mode of operation of an encryption scheme does not really fit into our definitional framework. Historically, there are two ways to modify the framework to allow for this type of operation. One approach is to allow for *stateful* encryption schemes, where both the encryption and decryption algorithms maintain some internal state that evolves with each application of the algorithm. In the example of the previous paragraph, the state would just consist of a counter that is incremented with each application of the algorithm. This approach requires encryptor and decryptor to be fully synchronized, which limits its applicability, and we shall not discuss it further.

The second, and more popular, approach is called *nonce-based encryption.* Instead of maintaining internal states, both the encryption and decryption algorithms take an additional input $\mathcal{N}$, called a *nonce*. The syntax for nonce-based encryption becomes

$$c = E(k, m, \mathcal{N}),$$

where $c \in \mathcal{C}$ is the ciphertext, $k \in \mathcal{K}$ is the key, $m \in \mathcal{M}$ is the message, and $\mathcal{N} \in \mathcal{N}$ is the nonce. Moreover, the encryption algorithm E is required to be deterministic. Likewise, the decryption syntax becomes

$$m = D(k, c, \mathcal{N}).$$

The intention is that a message encrypted with a particular nonce should be decrypted with the same nonce — it is up to the application using the encryption scheme to enforce this. More formally, the correctness requirement is that

$$D(k,\ E(k, m, \mathcal{N}),\ \mathcal{N}) = m$$

for all $k \in \mathcal{K}$, $m \in \mathcal{M}$, and $\mathcal{N} \in \mathcal{N}$. We say that such a nonce-based cipher $\mathcal{E} = (E, D)$ is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C}, \mathcal{N})$.

Intuitively, a nonce-based encryption scheme is CPA secure if it does not leak any useful information to an eavesdropper, assuming that *no nonce is used more than once* in the encryption process — again, it is up to the application using the scheme to enforce this. Note that this requirement on how nonces are used is very weak, much weaker than requiring that they are unpredictable, let alone randomly chosen.

We can readily formalize this notion of security by slightly tweaking our original definition of CPA security.

Attack Game 5.3 (nonce-based CPA security). For a given cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C}, \mathcal{N})$, and for a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define

Experiment b:

- The challenger selects $k \xleftarrow{\text{R}} \mathcal{K}$.
- The adversary submits a sequence of queries to the challenger.

  For $i = 1, 2, \ldots$, the ith query is a pair of messages, $m_{i0}, m_{i1} \in \mathcal{M}$, of the same length, and a nonce $\mathcal{N}_i \in \mathcal{N} \setminus \{\mathcal{N}_1, \ldots, \mathcal{N}_{i-1}\}$.

  The challenger computes $c_i \leftarrow E(k, m_{ib}, \mathcal{N}_i)$, and sends c_i to the adversary.
- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\text{nCPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := |\Pr[W_0] - \Pr[W_1]|. \quad \square$$

Note that in the above game, the nonces are completely under the adversary's control, subject only to the constraint that they are unique.

Definition 5.3 (nonce-based CPA security). *A nonce-based cipher $\mathcal{E}$ is called* ***semantically secure against chosen plaintext attack****, or simply* ***CPA secure****, if for all efficient adversaries $\mathcal{A}$, the value* $\text{nCPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

As usual, as in Section 2.3.5, Attack Game 5.3 can be recast as a "bit guessing" game, and we have

$$\text{nCPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \text{nCPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}], \tag{5.30}$$

where $\text{nCPA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}] := |\Pr[\hat{b} = b] - 1/2|$ in a version of Attack Game 5.3 where the challenger just chooses b at random.

5.5.1 Nonce-based generic hybrid encryption

Let us recast the generic hybrid construction in Section 5.4.1 as a nonce-based encryption scheme. As in that section, $\mathcal{E}$ is a cipher, which we shall now insist is deterministic, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and F is a PRF defined over $(\mathcal{K}', \mathcal{X}, \mathcal{K})$. We define the nonce-based cipher $\mathcal{E}'$, which is defined over $(\mathcal{K}', \mathcal{M}, \mathcal{C}, \mathcal{X})$, as follows:

- for $k' \in \mathcal{K}'$, $m \in \mathcal{M}$, and $x \in \mathcal{X}$, we define $E'(k', m, x) := E(k, m)$, where $k := F(k', x)$;
- for $k' \in \mathcal{K}'$, $c \in \mathcal{C}$, $x \in \mathcal{X}$, we define $D'(k', c) := D(k, c)$, where $k := F(k', x)$.

All we have done is to treat the value $x \in \mathcal{X}$ as a nonce; otherwise, the scheme is exactly the same as that defined in Section 5.4.1.

One can easily verify the correctness requirement for $\mathcal{E}'$. Moreover, one can easily adapt the proof of Theorem 5.2 to prove that the following:

Theorem 5.5. *If F is a secure PRF and $\mathcal{E}$ is a semantically secure cipher, then the cipher $\mathcal{E}'$ described above is a CPA secure cipher.*

> *In particular, for every nCPA adversary $\mathcal{A}$ that attacks $\mathcal{E}'$ as in the bit-guessing version of Attack Game 5.3, and which makes at most Q queries to its challenger, there exists a PRF adversary $\mathcal{B}_F$ that attacks F as in Attack Game 4.2, and an SS adversary $\mathcal{B}_\mathcal{E}$ that attacks $\mathcal{E}$ as in the bit-guessing version of Attack Game 2.1, where both $\mathcal{B}_F$ and $\mathcal{B}_\mathcal{E}$ are elementary wrappers around $\mathcal{A}$, such that*

$$\text{nCPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}'] \le 2 \cdot \text{PRF}\mathsf{adv}[\mathcal{B}_F, F] + Q \cdot \text{SS}\mathsf{adv}[\mathcal{B}_\mathcal{E}, \mathcal{E}]. \tag{5.31}$$

We leave the proof as an exercise for the reader. Note that the term $\frac{Q^2}{N}$ in (5.5), which represent the probability of a collision on the input to F, is missing from (5.31), simply because by definition, no collisions can occur.

5.5.2 Nonce-based Counter mode

Next, we recast the counter-mode cipher from Section 5.4.2 to the nonce-based encryption setting. Let us make a first attempt, by simply treating the value $x \in \mathcal{X}$ in that construction as a nonce.

Unfortunately, this scheme cannot satisfy the definition of nonce-based CPA security. The problem is, an attacker could choose two distinct nonces $x_1, x_2 \in \mathcal{X}$, such that the intervals

$\{x_1, \ldots, x_1 + \ell - 1\}$ and $\{x_2, \ldots, x_2 + \ell - 1\}$ overlap (again, arithmetic is done mod N). In this case, the security proof will break down; indeed, it is easy to mount a quite devastating attack, as discussed in Section 5.1, since that attacker can essentially force the encryptor to re-use some of the same bits of the "key stream".

Fortunately, the fix is easy. Let us assume that ℓ divides N (in practice, both ℓ and N will be powers of 2, so this is not an issue). Then we use as the nonce space $\{0, \ldots, N/\ell - 1\}$, and translate the nonce $\mathcal{N}$ to the PRF input $x := \mathcal{N}\ell$. It is easy to see that for any two distinct nonces $\mathcal{N}_1$ and $\mathcal{N}_2$, for $x_1 := \mathcal{N}_1\ell$ and $x_2 := \mathcal{N}_2\ell$, the intervals $\{x_1, \ldots, x_1 + \ell - 1\}$ and $\{x_2, \ldots, x_2 + \ell - 1\}$ do not overlap.

With $\mathcal{E}$ modified in this way, we can easily adapt the proof of Theorem 5.3 to prove the following:

Theorem 5.6. *If F is a secure PRF, then the nonce-based cipher $\mathcal{E}$ described above is CPA secure.*

In particular, for every nCPA adversary $\mathcal{A}$ that attacks $\mathcal{E}$ as in Attack Game 5.3, there exists a PRF adversary $\mathcal{B}$ that attacks F as in Attack Game 4.2, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{nCPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq 2 \cdot \mathrm{PRF}\mathsf{adv}[\mathcal{B}, F]. \tag{5.32}$$

We again leave the proof as an exercise for the reader.

5.5.3 Nonce-based CBC mode

Finally, we consider how to recast the CBC-mode encryption scheme in Section 5.4.3 as a nonce-based encryption scheme. As a first attempt, one might simply try to view the IV $c[0]$ as a nonce. Unfortunately, this does not yield a CPA secure nonce-based encryption scheme. In the nCPA attack game, the adversary could make two queries:

$(m_{10}, m_{11}, \mathcal{N}_1)$,
$(m_{20}, m_{21}, \mathcal{N}_2)$,

where

$$m_{10} = \mathcal{N}_1 \neq \mathcal{N}_2 = m_{20}, \; m_{11} = m_{21}.$$

Here, all messages are one-block messages. In Experiment 0 of the attack game, the resulting ciphertexts will be the same, whereas in Experiment 1, they will be different. Thus, we can perfectly distinguish between the two experiments.

Again, the fix is fairly straightforward. The idea is to map nonces to pseudo-random IV's by passing them through a PRF. So let us assume that we have a PRF F defined over $(\mathcal{K}', \mathcal{N}, \mathcal{X})$. Here, the key space $\mathcal{K}'$ and input space $\mathcal{N}$ of F may be arbitrary sets, but the output space $\mathcal{X}$ of F must match the block space of the underlying block cipher $\mathcal{E} = (E, D)$, which is defined over $(\mathcal{K}, \mathcal{X})$. In the nonce-based CBC scheme $\mathcal{E}'$, the key space is $\mathcal{K} \times \mathcal{K}'$, and in the encryption and decryption algorithms, the IV is computed from the nonce $\mathcal{N}$ and key k' as $c[0] := F(k', \mathcal{N})$.

With these modifications, we can now prove the following variant of Theorem 5.4:

Theorem 5.7. *If $\mathcal{E} = (E, D)$ is a secure block cipher defined over $(\mathcal{K}, \mathcal{X})$, and $N := |\mathcal{X}|$ is super-poly, and F is a secure PRF defined over $(\mathcal{K}', \mathcal{N}, \mathcal{X})$, then for any poly-bounded $\ell \geq 1$, the nonce-based cipher $\mathcal{E}'$ described above is CPA secure.*

In particular, for every nCPA adversary $\mathcal{A}$ that attacks $\mathcal{E}'$ as in the bit-guessing version of Attack Game 5.3, and which makes at most Q queries to its challenger, there exists BC adversary $\mathcal{B}$ that attacks $\mathcal{E}$ as in Attack Game 4.1, and a PRF adversary $\mathcal{B}_F$ that attacks F as in Attack Game 4.2, where $\mathcal{B}$ and $\mathcal{B}_F$ are elementary wrappers around $\mathcal{A}$, such that

$$\text{nCPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}'] \le \frac{2Q^2\ell^2}{N} + 2 \cdot \text{PRF}\mathsf{adv}[\mathcal{B}_F, F] + 2 \cdot \text{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}]. \tag{5.33}$$

Again, we leave the proof as an exercise for the reader. Note that in the above construction, we may use the underlying block cipher $\mathcal{E}$ for the PRF F; however, it is essential that independent keys k and k' are used (see Exercise 5.14).

5.6 A fun application: revocable broadcast encryption

Movie studios spend a lot of effort making blockbuster movies, and then sell the movies (on DVDs) to millions of customers who purchase them to watch at home. A customer should be able to watch movies on a stateless standalone movie player, that has no network connection.

The studios are worried about piracy, and do not want to send copyrighted digital content in the clear to millions of users. A simple solution could work as follows. Every authorized manufacturer is given a **device key** $k_\text{d} \in \mathcal{K}$, and it embeds this key in every device that it sells. If there are a hundred authorized device manufacturers, then there are a hundred device keys $k_\text{d}^{(1)}, \ldots, k_\text{d}^{(100)}$. A movie m is encrypted as:

$$c_\text{m} := \left\{ \begin{array}{l} k \xleftarrow{R} \mathcal{K} \\ \text{for } i = 1, \ldots, 100: \ \ c_i \xleftarrow{R} E(k_\text{d}^{(i)},\ k) \\ c \xleftarrow{R} E'(k, m) \\ \text{output } (c_1, \ldots, c_{100},\ c) \end{array} \right\}$$

where (E, D) is a CPA secure cipher, and (E', D') is semantically secure with key space $\mathcal{K}$. We analyze this construction in Exercise 5.4, where we show that it is CPA secure. We refer to $(c_1, \ldots, c_{100})$ as the ciphertext header, and refer to c is the body.

Now, every authorized device can decrypt the movie using its embedded device key. First, decrypt the appropriate ciphertext in the header, and then use the obtained key k to decrypt the body. This mechanism forms the basis of the **content scrambling system** (CSS) used to encrypted DVDs. We previously encountered CSS in Section 3.8.

The trouble with this scheme is that once a single device is comprised, and its device key k_d is extracted and published, then anyone can use this k_d to decrypt every movie ever published. There is no way to revoke k_d without breaking many consumer devices in the field. In fact, this is exactly how CSS was broken: the device key was extracted from an authorized player, and then used in a system called **DeCSS** to decrypt encrypted DVDs.

The lesson from CSS is that global unrevocable device keys are a bad idea. Once a single key is leaked, all security is lost. When the DVD format was updated to a new format called Blu-ray, the industry got a second chance to design the encryption scheme. In the new scheme, called the **Advanced Access Content System** (AACS), every device gets a random device key unique to that device. The system is designed to support billions of devices, each with its own key.

The goals of the system are twofold. First, every authorized device should be able to decrypt every Blu-ray disk. Second, whenever a device key is extracted and published, it should be possible

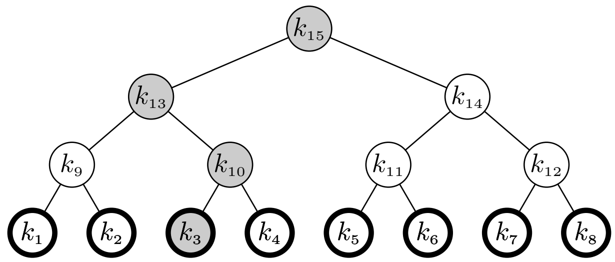


Figure 5.5: The tree of keys for $n = 8$ devices; shaded nodes are the keys embedded in device 3.

to revoke that key, so that this device key cannot be used to decrypt future Blu-ray disks, but without impacting any other devices in the field.

A revocable broadcast system. Suppose there are n devices in the system, where for simplicity, let us assume n is a power of two. We treat these n devices as the leaves of a complete binary tree, as shown in Fig. 5.5. Every internal node in the tree is assigned a random key in the key space $\mathcal{K}$. The keys embedded in device number $i \in \{1, \dots, n\}$ is the set of keys on the path from leaf number i to the root. This way, every device is given exactly $\log_2 n$ keys in $\mathcal{K}$.

When the system is first launched, and no device keys are yet revoked, all content is encrypted using the key at the root (key number 15 in Fig. 5.5). More precisely, we encrypt a movie m as:

$$c_{\mathrm{m}} := \left\{ \ k \xleftarrow{\mathrm{R}} \mathcal{K}, \ c_1 \xleftarrow{\mathrm{R}} E(k_{\mathrm{root}}, \ k), \ c \xleftarrow{\mathrm{R}} E'(k, m), \ \text{output } (c_1, c) \ \right\}$$

Because all devices have the root key k_{root}, all devices can decrypt.

Revoking devices. Now, suppose device number i is attacked, and all the keys stored on it are published. Then all future content will be encrypted using the keys associated with the siblings of the $\log_2 n$ nodes on the path from leaf i to the root. For example, when device number 3 in Fig. 5.5 is revoked, all future content is encrypted using the three keys k_4, k_9, k_{14} as

$$c_{\mathrm{m}} := \left\{ \begin{array}{l} k \xleftarrow{\mathrm{R}} \mathcal{K} \\ c_1 \xleftarrow{\mathrm{R}} E(k_4, \ k), \quad c_2 \xleftarrow{\mathrm{R}} E(k_9, \ k), \quad c_3 \xleftarrow{\mathrm{R}} E(k_{14}, \ k) \\ c \xleftarrow{\mathrm{R}} E'(k, m) \\ \text{output } (c_1, c_2, c_3, c) \end{array} \right\} \tag{5.34}$$

Again, (c_1, c_2, c_3) is the ciphertext header, and c is the ciphertext body. Observe that device number 3 cannot decrypt c_{m}, because it cannot decrypt any of the ciphertexts in the header. However, every other device can easily decrypt using one of the keys at its disposal. For example device number 6 can use k_{14} to decrypt c_3. In effect, changing the encryption scheme to encrypt as in (5.35) revokes device number 3, without impacting any other device. The cost to this is that the ciphertext header now contains $\log_2 n$ blocks, as opposed to a single block before the device was revoked.

More generally, suppose r devices have been compromised and need to be revoked. Let $S \subseteq \{1, \dots, n\}$ be the set of non-compromised devices, so that that $|S| = n - r$. New content will be encrypted using keys in the tree so that devices in S can decrypt, but all devices outside of S cannot. The set of keys that makes this possible is characterized by the following definition:

Definition 5.4. *Let T be a complete binary tree with n leaves, where n is a power of two. Let $S \subseteq \{1, \ldots, n\}$ be a set of leaves. We say that a set of nodes $W \subseteq \{1, \ldots, 2n-1\}$* ***covers*** *the set S if every leaf in S is a descendant of some node in W, and leaves outside of S are not. We use* $\text{cover}(S)$ *to denote the smallest set of nodes that covers S.*

Fig. 5.6 gives an example of a cover of the set of leaves $\{1, 2, 4, 5, 6\}$. The figure captures a settings where devices number 3, 7, and 8 are revoked. It should be clear that if we use keys in $\text{cover}(S)$ to encrypt a movie m, then devices in S can decrypt, but devices outside of S cannot. In particular, we encrypt m as follows:

$$c_{\text{m}} := \left\{ \begin{array}{l} k \xleftarrow{\text{R}} \mathcal{K} \\ \text{for } u \in \text{cover}(S) : c_u \xleftarrow{\text{R}} E(k_u,\ k) \\ c \xleftarrow{\text{R}} E'(k, m) \\ \text{output } (\{c_u\}_{u \in \text{cover}(S)},\ c) \end{array} \right\}. \tag{5.35}$$

The more devices are revoked, the larger the header of c_{m} becomes. The following theorem shows how big the header gets in the worst case. The proof is an induction argument that also suggests an efficient recursive algorithm to compute an optimal cover.

Theorem 5.8. *Let T be a complete binary tree with n leaves, where n is a power of two. For every $1 \le r \le n$, and every set S of $n - r$ leaves, we have*

$$|\text{cover}(S)| \le r \cdot \log_2(n/r)$$

Proof. We prove the theorem by induction on $\log_2 n$. For $n = 1$ the theorem is trivial. Now, assume the theorem holds for a tree with $n/2$ leaves, and let us prove it for a tree T with n leaves. The tree T is made up of a root node, and two disjoint sub-trees, T_1 and T_2, each with $n/2$ leaves. Let us split the set $S \subseteq \{1, \ldots, n\}$ in two: $S = S_1 \cup S_2$, where S_1 is contained in $\{1, \ldots, n/2\}$, and S_2 is contained in $\{n/2 + 1, \ldots, n\}$. That is, S_1 are the elements of S that are leaves in T_1, and S_2 are the elements of S that are leaves in T_2. Let $r_1 := (n/2) - |S_1|$ and $r_2 := (n/2) - |S_2|$. Then clearly $r = r_1 + r_2$.

First, suppose both r_1 and r_2 are greater than zero. By the induction hypothesis, we know that for $i = 1, 2$ we have $|\text{cover}(S_i)| \le r_i \log_2(n/2r_i)$. Therefore,

$$\begin{aligned} |\text{cover}(S)| = |\text{cover}(S_1)| + |\text{cover}(S_2)| &\le r_1 \log_2(n/2r_1) + r_2 \log_2(n/2r_2) \\ &= r \log_2(n/r) + \big(r \log_2 r - r_1 \log_2(2r_1) - r_2 \log_2(2r_2)\big) \le r \log_2(n/r), \end{aligned}$$

which is what we had to prove in the induction step. The last inequality follows from a simple fact about logarithms, namely that for all numbers $r_1 \ge 1$ and $r_2 \ge 1$, we have

$$(r_1 + r_2) \log_2(r_1 + r_2) \le r_1 \log_2(2r_1) + r_2 \log_2(2r_2).$$

Second, if $r_1 = 0$ then $r_2 = r \ge 1$, and the induction step follows from:

$$|\text{cover}(S)| = 1 + |\text{cover}(S_2)| \le 1 + r \log_2(n/2r) = 1 + r \log_2(n/r) - r \le r \log_2(n/r),$$

as required. The case $r_2 = 0$ follows similarly. This completes the induction step, and the proof. $\square$

Theorem 5.8 shows that r devices can be revoked at the cost of increasing the ciphertext header size to $O(r \log n)$ blocks. For moderate values of r this is not too big. Nevertheless, this general

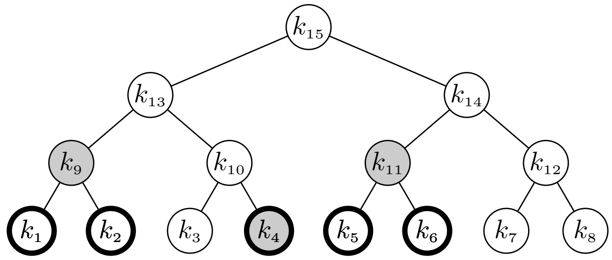


Figure 5.6: The three shaded nodes are the minimal cover for $\{1, 2, 4, 5, 6\}$.

approach can be improved [82, 51, 48]. The best system using this approach embeds $O(\log n)$ keys in every device, same as here, but the header size is only $O(r)$ blocks. The AACS system uses the subset-tree difference method [82], which has a worst case header of size $2r - 1$ blocks, but stores $\frac{1}{2}\log^2 n$ keys per device.

While AACS is a far better designed than CSS, it too has been attacked. In particular, the process of a revoking an AACS key is fairly involved and can take several months. For a while, it seemed that hackers could extract new device keys from unrevoked players faster than the industry could revoke them.

5.7 Notes

Citations to the literature to be added.

5.8 Exercises

5.1 (Double encryption). Let $\mathcal{E} = (E, D)$ be a cipher. Consider the cipher $\mathcal{E}_2 = (E_2, D_2)$, where $E_2(k, m) = E(k, E(k, m))$. One would expect that if encrypting a message once with E is secure then encrypting it twice as in E_2 should be no less secure. However, that is not always true.

(a) Show that there is a semantically secure cipher $\mathcal{E}$ such that $\mathcal{E}_2$ is not semantically secure.

(b) Prove that for every CPA secure ciphers $\mathcal{E}$, the cipher $\mathcal{E}_2$ is also CPA secure. That is, show that for every CPA adversary $\mathcal{A}$ attacking $\mathcal{E}_2$ there is a CPA adversary $\mathcal{B}$ attacking $\mathcal{E}$ with about the same advantage and running time.

5.2 (Multi-key CPA security). Generalize the definition of CPA security to the multi-key setting, analogous to Definition 5.1. In this attack game, the adversary gets to obtain encryptions of many messages under many keys. The game begins with the adversary outputting a number Q indicating the number of keys it wants to attack. The challenger chooses Q random keys. In every subsequent encryption query, the adversary submits a pair of messages and specifies under which of the Q keys it wants to encrypt; the challenger responds with an encryption of either the first or second message under the specified key (depending on whether the challenger is running Experiment 0 or 1). Flesh out all the details of this attack game, and prove, using a hybrid argument, that (single-key) CPA security implies multi-key CPA security. You should show that security degrades linearly in Q. That is, the advantage of any adversary $\mathcal{A}$ in breaking the multi-key

CPA security of a scheme is at most $Q \cdot \epsilon$, where ϵ is the advantage of an adversary $\mathcal{B}$ (which is an elementary wrapper around $\mathcal{A}$) in attacking the scheme's (single-key) CPA security.

5.3 (An alternate definition of CPA security). This exercise develops an alternative characterization of CPA security for a cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. As usual, we need to define an attack game between an adversary $\mathcal{A}$ and a challenger. Initially, the challenger generates

$$b \xleftarrow{\text{R}} \{0,1\}, \; k \xleftarrow{\text{R}} \mathcal{K}.$$

Then $\mathcal{A}$ makes a series of queries to the challenger. There are two types of queries:

Encryption: In an *encryption query*, $\mathcal{A}$ submits a message $m \in \mathcal{M}$ to the challenger, who responds with a ciphertext $c \xleftarrow{\text{R}} E(k, m)$. The adversary may make any (poly-bounded) number of encryption queries.

Test: In a *test query*, $\mathcal{A}$ submits a pair of messages $m_0, m_1 \in \mathcal{M}$ to the challenger, who responds with a ciphertext $c \xleftarrow{\text{R}} E(k, m_b)$. The adversary is allowed to make only a *single* test query (with any number of encryption queries before and after the test query).

At the end of the game, $\mathcal{A}$ outputs a bit $\hat{b} \in \{0,1\}$.

As usual, we define $\mathcal{A}$'s advantage in the above attack game to be $|\Pr[\hat{b} = b] - 1/2|$. We say that $\mathcal{E}$ is Alt-CPA secure if this advantage is negligible for all efficient adversaries.

Show that $\mathcal{E}$ is CPA secure if and only if $\mathcal{E}$ is Alt-CPA secure.

5.4 (Hybrid CPA construction). Let (E_0, D_0) be a semantically secure cipher defined over $(\mathcal{K}_0, \mathcal{M}, \mathcal{C}_0)$, and let (E_1, D_1) be a CPA secure cipher defined over $(\mathcal{K}, \mathcal{K}_0, \mathcal{C}_1)$.

(a) Define the following hybrid cipher (E, D) as:

$$E(k, m) := \big\{k_0 \xleftarrow{\text{R}} \mathcal{K}_0, \; c_1 \xleftarrow{\text{R}} E_1(k, k_0), \; c_0 \xleftarrow{\text{R}} E_0(k_0, m), \; \text{output } (c_1, c_0)\big\}$$
$$D\big(k, \; (c_1, c_0)\big) := \big\{k_0 \leftarrow D_1(k, c_1), \; m \leftarrow D_0(k_0, c_0), \; \text{output } m\big\}$$

Here c_1 is called the ciphertext header, and c_0 is called the ciphertext body. Prove that (E, D) is CPA secure.

(b) Suppose m is some large copyrighted content. A nice feature of (E, D) is that the content owner can make the long ciphertext body c_0 public for anyone to download at their leisure. Suppose both Alice and Bob take the time to download c_0. When later Alice, who has key k_a, pays for access to the content, the content owner can quickly grant her access by sending her the short ciphertext header $c_\text{a} \xleftarrow{\text{R}} E_1(k_\text{a}, k_0)$. Similarly, when Bob, who has key k_b, pays for access, the content owner grants him access by sending him the short header $c_\text{b} \xleftarrow{\text{R}} E_1(k_\text{b}, k_0)$. Now, an eavesdropper gets to see

$$E'\big((k_\text{a}, k_\text{b}), \; m\big) := (c_\text{a}, c_\text{b}, c_0)$$

Generalize your proof from part (a) to show that this cipher is also CPA secure.

5.5 (A simple proof of randomized counter mode security). As mentioned in Remark 5.3, we can view randomized counter mode as a special case of the generic hybrid construction in Section 5.4.1. To this end, let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, where $\mathcal{X} = \{0, \ldots, N-1\}$ and

$\mathcal{Y} = \{0,1\}^n$, where N is super-poly. For poly-bounded $\ell \geq 1$, consider the PRF F' defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y}^\ell)$ as follows:

$$F'(k, x) := \Big(\; F(k, x), \; F(k, x+1 \bmod N), \; \dots, \; F(k, x+\ell-1 \bmod N) \; \Big).$$

(a) Show that F' is a weakly secure PRF, as in Definition 4.3.

(b) Using part (a) and Remark 5.2, give a short proof that randomized counter mode is CPA secure.

5.6 (CPA security from a block cipher). Let $\mathcal{E} = (E, D)$ be a block cipher defined over $(\mathcal{K}, \mathcal{M} \times \mathcal{R})$. Consider the cipher $\mathcal{E}' = (E', D')$, where

$$E'(k, m) := \big\{ r \xleftarrow{\text{R}} \mathcal{R}, \; c \xleftarrow{\text{R}} E\big(k, \; (m, r)\big), \; \text{output } c \big\}$$

$$D'(k, c) := \big\{ (m, r') \leftarrow D(k, c), \; \text{output } m \big\}$$

This cipher is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{M} \times \mathcal{R})$. Show that if $\mathcal{E}$ is a secure block cipher, and $1/|\mathcal{R}|$ is negligible, then $\mathcal{E}'$ is CPA secure.

5.7 (pseudo-random ciphertext security). In Exercise 3.4, we developed a notion of security called pseudo-random ciphertext security. This notion naturally extends to multiple ciphertexts. For a cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, we define two experiments: in Experiment 0 the challenger first picks a random key $k \xleftarrow{\text{R}} \mathcal{K}$ and then the adversary submits a sequence of queries, where the ith query is a message $m_i \in \mathcal{M}$, to which the challenger responds with $E(k, m_i)$. Experiment 1 is the same as Experiment 0 except that the challenger responds to the adversary's queries with random, independent elements of $\mathcal{C}$. We say that $\mathcal{E}$ is psuedo-random multi-ciphertext secure if no efficient adversary can distinguish between these two experiments with a non-negligible advantage.

(a) Consider the counter-mode construction in Section 5.4.2, based on a PRF F defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, but with a fixed-length plaintext space $\mathcal{Y}^\ell$ and a corresponding fixed-length ciphertext space $\mathcal{X} \times \mathcal{Y}^\ell$. Under the assumptions that F is a secure PRF, $|\mathcal{X}|$ is super-poly, and ℓ is poly-bounded, show that this cipher is psuedo-random multi-ciphertext secure.

(b) Consider the CBC construction Section 5.4.3, based on a block cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{X})$, but with a fixed-length plaintext space $\mathcal{X}^\ell$ and corresponding fixed-length ciphertext space $\mathcal{X}^{\ell+1}$. Under the assumptions that $\mathcal{E}$ is a secure block cipher, $|\mathcal{X}|$ is super-poly, and ℓ is poly-bounded, show that this cipher is psuedo-random multi-ciphertext secure.

(c) Show that a psuedo-random multi-ciphertext secure cipher is also CPA secure.

(d) Give an example of a CPA secure cipher that is not psuedo-random multi-ciphertext secure.

5.8 (Deterministic CPA and SIV). We have seen that any cipher that is CPA secure must be probabilistic, since for a deterministic cipher, an adversary can always see if the same message is encrypted twice. We may define a relaxed notion of CPA security that says that this is the *only* thing the adversary can see. This is easily done by placing the following restriction on the adversary in Attack Game 5.2: for all indices i, j, we insist that $m_{i0} = m_{j0}$ if and only if $m_{i1} = m_{j1}$. We say that a cipher is **deterministic CPA secure** if every efficient adversary has negligible advantage in this restricted CPA attack game. In this exercise, we develop a general approach for building deterministic ciphers that are deterministic CPA secure.

Let $\mathcal{E} = (E, D)$ be a CPA-secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. We let $E(k, m; r)$ denote running algorithm $E(k, m)$ with randomness $r \xleftarrow{R} \mathcal{R}$ (for example, if $\mathcal{E}$ implements counter mode or CBC encryption then r is the random IV used by algorithm E). Let F be a secure PRF defined over $(\mathcal{K}', \mathcal{M}, \mathcal{R})$. Define the deterministic cipher $\mathcal{E}' = (E', D')$, defined over $(\mathcal{K} \times \mathcal{K}', \mathcal{M}, \mathcal{C})$ as follows:

$$\begin{aligned} E'\big((k, k'),\ m\big) \quad &:= E(k, m;\ F(k', m)), \\ D'\big((k, k'),\ c\big) \quad &:= D(k, c)\ . \end{aligned}$$

Show that $\mathcal{E}'$ is deterministic CPA secure. This construction is known as the **Synthetic IV** (or **SIV**) construction.

5.9 (Generic nonce-based encryption and nonce re-use resilience). In the previous exercise, we saw how we could generically convert a probabilistic CPA-secure cipher into a deterministic cipher that satisfies a somewhat weaker notion of security called deterministic CPA security.

(a) Show how to modify that construction so that we can convert any CPA-secure probabilistic cipher into a nonce-based CPA-secure cipher.

(b) Show how to combine the two approaches to get a cipher that is nonce-based CPA secure, but also satisfies the definition of deterministic CPA security if we drop the uniqueness requirement on nonces.

Discussion: This is an instance of a more general security property called **nonce re-use resilience**: the scheme provides full security if nonces are unique, and even if they are not, a weaker and still useful security guarantee is provided.

5.10 (Ciphertext expansion vs. security). Let $\mathcal{E} = (E, D)$ be an encryption scheme messages and ciphertexts are bit strings.

(a) Suppose that for all keys and all messages m, the encryption of m is the exact same length as m. Show that (E, D) cannot be semantically secure under a chosen plaintext attack.

(b) Suppose that for all keys and all messages m, the encryption of m is exactly ℓ bits longer than the length of m. Show an attacker that can win the CPA security game using $\approx 2^{\ell/2}$ queries and advantage $\approx 1/2$. You may assume the message space contains more than $\approx 2^{\ell/2}$ messages.

5.11 (Repeating ciphertexts). Let $\mathcal{E} = (E, D)$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Assume that there are at least two messages in $\mathcal{M}$, that all messages have the same length, and that we can efficiently generate messages in $\mathcal{M}$ uniformly at random. Show that if $\mathcal{E}$ is CPA secure, then it is infeasible for an adversary to make an encryptor generate the same ciphertext twice. The precise attack game is as follows. The challenger chooses $k \in \mathcal{K}$ at random and the adversary make a series of queries; the ith query is a message m_i, to which the challenger' responds with $c_i \xleftarrow{R} E(k, m_i)$. The adversary wins the game if any two c_i's are the same. Show that if $\mathcal{E}$ is CPA secure, then every efficient adversary wins this game with negligible probability. In particular, show that the advantage of any adversary $\mathcal{A}$ in winning the repeated-ciphertext attack game is at most 2ϵ, where ϵ is the advantage of an adversary $\mathcal{B}$ (which is an elementary wrapper around $\mathcal{A}$) that breaks the scheme's CPA security.

5.12 (Predictable IVs). Let us see why in CBC mode an unpredictable IV is necessary for CPA security. Suppose a defective implementation of CBC encrypts a sequence of messages by always using the last ciphertext block of the ith message as the IV for the $(i+1)$-st message. The TLS 1.0 protocol, used to protect Web traffic, implements CBC encryption this way. Construct an efficient adversary that wins the CPA game against this implementation with advantage close to 1. We note that the Web-based BEAST attack [35] exploits this defect to completely break CBC encryption in TLS 1.0.

5.13 (CBC encryption with small blocks is insecure). Suppose the block cipher used for CBC encryption has a block size of n bits. Construct an attacker that wins the CPA game against CBC that makes $\approx 2^{n/2}$ queries to its challenger and gains an advantage $\approx 1/2$. Your answer explains why CBC cannot be used with a block cipher that has a small block size (e.g. $n = 64$ bits). This is one reason why AES has a block size of 128 bits.

Discussion: This attack was used to show that 3DES is no longer secure for Internet use, due to its 64-bit block size [11].

5.14 (An insecure nonce-based CBC mode). Consider the nonce-based CBC scheme $\mathcal{E}'$ described in Section 5.5.3. Suppose that the nonce space $\mathcal{N}$ is equal to block space $\mathcal{X}$ of the underlying block cipher $\mathcal{E} = (E, D)$, and the PRF F is just the encryption algorithm E. If the two keys k and k' in the construction are chosen independently, the scheme is secure. Your task is to show that if only one key k is chosen, and other key k' is just set to k, then the scheme is insecure.

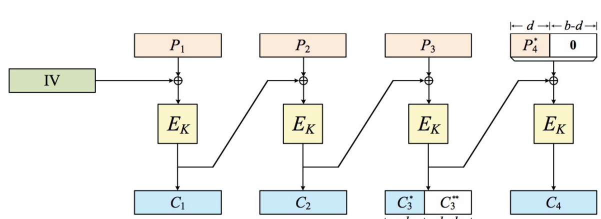
5.15 (Output feedback mode). Suppose F is a PRF defined over $(\mathcal{K}, \mathcal{X})$, and $\ell \geq 1$ is poly-bounded.

(a) Consider the following PRG $G : \mathcal{K} \to \mathcal{X}^\ell$. Let x_0 be an arbitrary, fixed element of $\mathcal{X}$. For $k \in \mathcal{K}$, let $G(k) := (x_1, \ldots, x_\ell)$, where $x_i := F(k, x_{i-1})$ for $i = 1, \ldots, \ell$. Show that G is a secure PRG, assuming F is a secure PRF and that $|\mathcal{X}|$ is super-poly.

(b) Next, assume that $\mathcal{X} = \{0,1\}^n$. We define a cipher $\mathcal{E} = (E, D)$, defined over $(\mathcal{K}, \mathcal{X}^\ell, \mathcal{X}^{\ell+1})$, as follows. Given a key $k \in \mathcal{K}$ and a message $(m_1, \ldots, m_\ell) \in \mathcal{X}^\ell$, the encryption algorithm E generates the ciphertext $(c_0, c_1, \ldots, c_\ell) \in \mathcal{X}^{\ell+1}$ as follows: it chooses $x_0 \in \mathcal{X}$ at random, and sets $c_0 = x_0$; it then computes $x_i = F(k, x_{i-1})$ and $c_i = m_i \oplus x_i$ for $i = 1, \ldots, \ell$. Describe the corresponding decryption algorithm D, and show that $\mathcal{E}$ is CPA secure, assuming F is a secure PRF and that $|\mathcal{X}|$ is super-poly.

Note: This construction is called **output feedback mode** (or **OFB**).

5.16 (CBC ciphertext stealing). One problem with CBC encryption is that messages need to be padded to a multiple of the block length and sometimes a dummy block needs to be added. The following figure describes a variant of CBC that eliminates the need to pad:



The method pads the last block with zeros if needed (a dummy block is never added), but the output ciphertext contains only the shaded parts of C_1, C_2, C_3, C_4. Note that, ignoring the IV, the ciphertext is the same length as the plaintext. This technique is called *ciphertext stealing*.

(a) Explain how decryption works.

(b) Can this method be used if the plaintext contains only one block?

5.17 (Single ciphertext block corruption in CBC mode). Let c be an ℓ block CBC-encrypted ciphertext, for some $\ell > 3$. Suppose that exactly one block of c is corrupted, and the result is decrypted using the CBC decryption algorithm. How many blocks of the decrypted plaintext are corrupted?

5.18 (The malleability of CBC mode). Let c be the CBC encryption of some message $m \in \mathcal{X}^\ell$, where $\mathcal{X} := \{0,1\}^n$. You do not know m. Let $\Delta \in \mathcal{X}$. Show how to modify the ciphertext c to obtain a new ciphertext c' that decrypts to m', where $m'[0] = m[0] \oplus \Delta$, and $m'[i] = m[i]$ for $i = 1, \dots, \ell - 1$. That is, by modifying c appropriately, you can flip bits of your choice in the first block of the decryption of c, without affecting any the other blocks.

5.19 (Online ciphers). In practice there is a strong desire to encrypt one block of plaintext at a time, outputting the corresponding block of ciphertext right away. This lets the system transmit ciphertext blocks as soon as they are ready without having to wait until the entire message is processed by the encryption algorithm.

(a) Define a CPA-like security game that captures this method of encryption. Instead of forcing the adversary to submit a complete pair of messages in every encryption query, the adversary should be allowed to issue a query indicating the beginning of a message, then repeatedly issue more queries containing message blocks, and finally issue a query indicating the end of a message. Responses to these queries will include all ciphertext blocks that can be computed given the information given.

(b) Show that randomized CBC encryption is not CPA secure in this model.

(c) Show that randomized counter mode is online CPA secure.

5.20 (Redundant bits do not harm CPA security). Let $\mathcal{E} = (E, D)$ be a CPA-secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Show that appending to a ciphertext additional data that is computed from the ciphertext does not damage CPA security. Specifically, let $g : \mathcal{C} \to \mathcal{Y}$ be some efficiently computable function. Show that the following modified cipher $\mathcal{E}' = (E', D')$ is CPA-secure:

$$\begin{aligned}
&E'(k, m) := \big\{c \leftarrow E(k, m),\ t \leftarrow g(c),\ \text{output } (c, t)\big\} \\
&D'\big(k,\ (c, t)\big) := D(k, c)
\end{aligned}$$

Chapter 6

Message integrity

In previous chapters we focused on security against an eavesdropping adversary. The adversary had the ability to eavesdrop on transmitted messages, but could not change messages en-route. We showed that chosen plaintext security is the natural security property needed to defend against such attacks.

In this chapter we turn our attention to active adversaries. We start with the basic question of *message integrity*: Bob receives a message m from Alice and wants to convince himself that the message was not modified en-route. We will design a mechanism that lets Alice compute a short message integrity tag t for the message m and send the pair (m, t) to Bob, as shown in Fig. 6.1. Upon receipt, Bob checks the tag t and rejects the message if the tag fails to verify. If the tag verifies then Bob is assured that the message was not modified in transmission.

We emphasize that in this chapter the message itself need not be secret. Unlike previous chapters, our goal here is not to conceal the message. Instead, we only focus on message integrity. In Chapter 9 we will discuss the more general question of simultaneously providing message secrecy and message integrity. There are many applications where message integrity is needed, but message secrecy is not. We give two examples.

Example 6.1. Consider the problem of delivering financial news or stock quotes over the Internet. Although the news items themselves are public information, it is vital that no third party modify the data on its way to the user. Here message secrecy is irrelevant, but message integrity is critical. Our constructions will ensure that if user Bob rejects all messages with an invalid message integrity tag then an attacker cannot inject modified content that will look legitimate. One caveat is that an attacker can still change the order in which news reports reach Bob. For example, Bob might see report number 2 before seeing report number 1. In some settings this may cause the user to take an incorrect action. To defend against this, the news service may wish to include a sequence number with each report so that the user's machine can buffer reports and ensure that the user always sees news items in the correct order. □

In this chapter we are only concerned with attacks that attempt to modify data. We do not consider Denial of Service (DoS) attacks, where the attacker delays or prevents news items from reaching the user. DoS attacks are often handled by ensuring that the network contains redundant paths from the sender to the receiver so that an attacker cannot block all paths. We will not discuss these issues here.

Example 6.2. Consider an application program — such as a word processor or mail client —

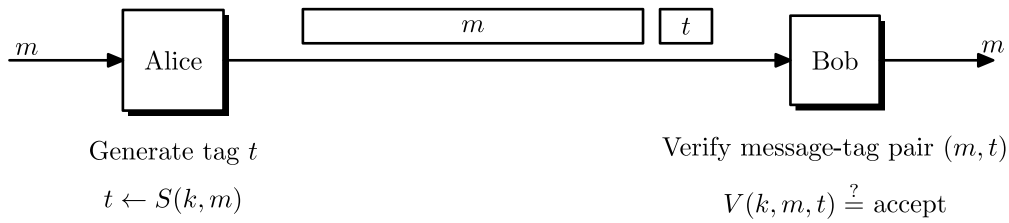


Figure 6.1: Short message integrity tag added to messages

stored on disk. Although the application code is not secret (it might even be in the public domain), its integrity is important. Before running the program the user wants to ensure that a virus did not modify the code stored on disk. To do so, when the program is first installed, the user computes a message integrity tag for the code and stores the tag on disk alongside the program. Then, every time, before starting the application the user can validate this message integrity tag. If the tag is valid, the user is assured that the code has not been modified since the tag was initially generated. Clearly a virus can overwrite both the application code and the integrity tag. Nevertheless, our constructions will ensure that no virus can fool the user into running unauthenticated code. As in our first example, the attacker can swap two authenticated programs — when the user starts application A he will instead be running application B. If both applications have a valid tag the system will not detect the swap. The standard defense against this is to include the program name in the executable file. That way, when an application is started the system can display to the user an authenticated application name. □

The question, then, is how to design a secure message integrity mechanism. We first argue the following basic principle:

> Providing message integrity between two communicating parties requires that the sending party has a secret key unknown to the adversary.

Without a secret key, ensuring message integrity is not possible: the adversary has enough information to compute tags for arbitrary messages of its choice — it knows how the message integrity algorithm works and needs no other information to compute tags. For this reason all cryptographic message integrity mechanisms require a secret key unknown to the adversary. In this chapter, we will assume that both sender and receiver will share the secret key; later in the book, this assumption will be relaxed.

We note that communication protocols not designed for security often use *keyless* integrity mechanisms. For example, the Ethernet protocol uses CRC32 as its message integrity algorithm. This algorithm, which is publicly available, outputs 32-bit tags embedded in every Ethernet frame. The TCP protocol uses a keyless 16-bit checksum which is embedded in every packet. We emphasize that these keyless integrity mechanisms are designed to detect *random* transmission errors, not malicious errors. The argument in the previous paragraph shows that an adversary can easily defeat these mechanisms and generate legitimate-looking traffic. For example, in the case of Ethernet, the adversary knows exactly how the CRC32 algorithm works and this lets him compute valid tags for arbitrary messages. He can then tamper with Ethernet traffic without being detected.

6.1 Definition of a message authentication code

We begin by defining what is a message integrity system based on a shared secret key between the sender and receiver. For historical reasons such systems are called Message Authentication Codes or MACs for short.

Definition 6.1. *A* ***MAC*** *system* $\mathcal{I} = (S, V)$ *is a pair of efficient algorithms,* S *and* V*, where* S *is called a* ***signing algorithm*** *and* V *is called a* ***verification algorithm****. Algorithm* S *is used to generate tags and algorithm* V *is used to verify tags.*

- *S is a probabilistic algorithm that is invoked as $t \xleftarrow{\text{R}} S(k, m)$, where k is a key, m is a message, and the output t is called a* ***tag****.*

- *V is a deterministic algorithm that is invoked as $r \leftarrow V(k, m, t)$, where k is a key, m is a message, t is a tag, and the output r us either* accept *or* reject*.*

- *We require that tags generated by S are always accepted by V; that is, the MAC must satisfy the following* ***correctness property****: for all keys k and all messages m,*

$$\Pr[V(k,\ m,\ S(k,\ m)\ ) = \mathsf{accept}] = 1.$$

As usual, we say that keys lie in some finite ***key space*** $\mathcal{K}$*, messages lie in a finite* ***message space*** $\mathcal{M}$*, and tags lie in some finite* ***tag space*** $\mathcal{T}$*. We say that $\mathcal{I} = (S, V)$ is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$.*

Fig. 6.1 illustrates how algorithms S and V are used for protecting network communications between two parties. Whenever algorithm V outputs accept for some message-tag pair (m, t), we say that t is a **valid tag** for m under key k, or that (m, t) is a **valid pair** under k. Naturally, we want MAC systems where tags are as short as possible so that the overhead of transmitting the tag is minimal.

We will explore a variety of MAC systems. The simplest type of system is one in which the signing algorithm S is deterministic, and the verification algorithm is defined as

$$V(k, m, t) = \begin{cases} \mathsf{accept} & \text{if } S(k, m) = t, \\ \mathsf{reject} & \text{otherwise.} \end{cases}$$

We shall call such a MAC system a **deterministic MAC system**. One property of a deterministic MAC system is that it has **unique tags**: for a given key k, and a given message m, there is a unique valid tag for m under k. Not all MAC systems we explore will have such a simple design: some have a randomized signing algorithm, so that for a given key k and message m, the output of $S(k, m)$ may be one of many possible valid tags, and the verification algorithm works some other way. As we shall see, such **randomized MAC systems** are not necessary to achieve security, but they can yield better efficiency/security trade-offs.

Secure MACs. Next, we turn to describing what it means for a MAC to be secure. To construct MACs that remain secure in a variety of applications we will insist on security in a very hostile environment. Since most real-world systems that use MACs operate in less hostile settings, our conservative security definitions will imply security for all these systems.

We first intuitively explain the definition and then motivate why this conservative definition makes sense. Suppose an adversary is attacking a MAC system $\mathcal{I} = (S, V)$. Let k be some

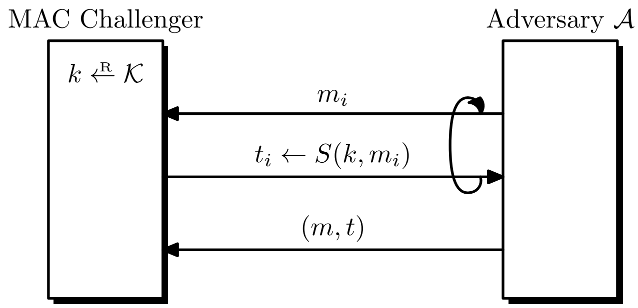


Figure 6.2: MAC attack game (Attack Game 6.1)

randomly chosen MAC key, which is unknown to the attacker. We allow the attacker to request tags $t := S(k, m)$ for arbitrary messages m of its choice. This attack, called a **chosen message attack**, enables the attacker to collect millions of valid message-tag pairs. Clearly we are giving the attacker considerable power — it is hard to imagine that a user would be foolish enough to sign arbitrary messages supplied by an attacker. Nevertheless, we will see that chosen message attacks come up in real world settings. We refer to message-tag pairs (m, t) that the adversary obtains using the chosen message attack as **signed pairs**.

Using the chosen message attack we ask the attacker to come up with an **existential MAC forgery**. That is, the attacker need only come up with some *new* valid message-tag pair (m, t). By "new", we mean a message-tag pair that is different from all of the signed pairs. The attacker is free to choose m arbitrarily; indeed, m need not have any special format or meaning and can be complete gibberish.

We say that a MAC system is secure if even an adversary who can mount a chosen message attack cannot create an existential forgery. This definition gives the adversary more power than it typically has in the real world and yet we ask it to do something that will normally be harmless; forging the MAC for a meaningless message seems to be of little use. Nevertheless, as we will see, this conservative definition is very natural and enables us to use MACs for lots of different applications.

More precisely, we define secure MACs using an attack game between a challenger and an adversary $\mathcal{A}$. The game is described below and in Fig. 6.2.

Attack Game 6.1 (MAC security). For a given MAC system $\mathcal{I} = (S, V)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger picks a random $k \xleftarrow{R} \mathcal{K}$.
- $\mathcal{A}$ queries the challenger several times. For $i = 1, 2, \ldots$, the ith *signing query* is a message $m_i \in \mathcal{M}$. Given m_i, the challenger computes a tag $t_i \xleftarrow{R} S(k, m_i)$, and then gives t_i to $\mathcal{A}$.
- Eventually $\mathcal{A}$ outputs a candidate forgery pair $(m, t) \in \mathcal{M} \times \mathcal{T}$ that is not among the signed pairs, i.e.,

$$(m, t) \notin \big\{(m_1, t_1), (m_2, t_2), \ldots\big\}.$$

We say that $\mathcal{A}$ wins the above game if (m, t) is a valid pair under k (i.e., $V(k, m, t) = \mathsf{accept}$). We define $\mathcal{A}$'s advantage with respect to $\mathcal{I}$, denoted MAC$\mathsf{adv}[\mathcal{A}, \mathcal{I}]$, as the probability that $\mathcal{A}$ wins the game. Finally, we say that $\mathcal{A}$ is a **Q-query MAC adversary** if $\mathcal{A}$ issues at most Q signing queries. $\square$

Definition 6.2. *We say that a MAC system $\mathcal{I}$ is secure if for all efficient adversaries $\mathcal{A}$, the value* $\text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}]$ *is negligible.*

In case the adversary wins Attack Game 6.1, the pair (m, t) it sends the challenger is called an **existential forgery**. MAC systems that satisfy Definition 6.2 are said to be **existentially unforgeable under a chosen message attack**.

In the case of a deterministic MAC system, the only way for $\mathcal{A}$ to win Attack Game 6.1 is to produce a valid message-tag pair (m, t) for some new message $m \notin \{m_1, m_2, \ldots\}$. Indeed, security in this case just means that S is *unpredictable*, in the sense described in Section 4.1.1; that is, given $S(k, m_1), S(k, m_2), \ldots$, it is hard to predict $S(k, m)$ for any $m \notin \{m_1, m_2, \ldots\}$.

In the case of a randomized MAC system, our security definition captures a stronger property. There may be many valid tags for a given message. Let m be some message and suppose the adversary requests one or more valid tags $t_1, t_2, \ldots$ for m. Can the adversary produce a new valid tag t' for m? (i.e. a tag satisfying $t' \notin \{t_1, t_2, \ldots\}$). Our definition says that a valid pair (m, t'), where t' is new, is a valid existential forgery. Therefore, for a MAC to be secure it must be difficult for an adversary to produce a new valid tag t' for a previously signed message m. This may seem like an odd thing to require of a MAC. If the adversary already has valid tags for m, why should we care if it can produce another one? As we will see in Chapter 9, our security definition, which prevents the adversary from producing new tags on signed messages, is necessary for the applications we have in mind.

Going back to the examples in the introduction, observe that existential unforgeability implies that an attacker cannot create a fake news report with a valid tag. Similarly, the attacker cannot tamper with a program on disk without invalidating the tag for the program. Note, however, that when using MACs to protect application code, users must provide their secret MAC key every time they want to run the application. This will quickly annoy most users. In Chapter 8 we will discuss a keyless method to protect public application code.

To exercise the definition of secure MACs let us first see a few consequences of it. Let $\mathcal{I} = (S, V)$ be a MAC defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, and let k be a random key in $\mathcal{K}$.

Example 6.3. Suppose m_1 and m_2 are almost identical messages. Say m_1 is a money transfer order for \$100 and m_2 is a transfer order for \$101. Clearly, an adversary who intercepts a valid tag for m_1 should not be able to deduce from it a valid tag for m_2. A MAC system that satisfies Definition 6.2 ensures this. To see why, suppose an adversary $\mathcal{A}$ can forge the tag for m_2 given the tag for m_1. Then $\mathcal{A}$ can win Attack Game 6.1: it uses the chosen message attack to request a tag for m_1, deduces a forged tag t_2 for m_2, and outputs (m_2, t_2) as a valid existential forgery. Clearly $\mathcal{A}$ wins Attack Game 6.1. Hence, existential unforgeability captures the fact that a tag for one message m_1 gives no useful information for producing a tag for another message m_2, even when m_2 is almost identical to m_1. $\square$

Example 6.4. Our definition of secure MACs gives the adversary the ability to obtain the tag for arbitrary messages. This may seem like giving the adversary too much power. In practice, however, there are many scenarios where chosen message attacks are feasible. The reason is that the MAC signer often does not know the source of the data being signed. For example, consider a backup system that dumps the contents of disk to backup tapes. Since backup integrity is important, the system computes an integrity tag on every disk block that it writes to tape. The tag is stored on tape along with the data block. Now, suppose an attacker writes data to a low security part of disk. The attacker's data will be backed up and the system will compute a tag over it. By examining the resulting backup tape the attacker obtains a tag on his chosen message. If the MAC system is secure against a chosen message attack then this does not help the attacker break the system. □

Remark 6.1. Just as we did for other security primitives, one can generalize the notion of a secure MAC to the multi-key setting, and prove that a secure MAC is also secure in the multi-key setting. See Exercise 6.3. □

6.1.1 Mathematical details

As usual, we give a more mathematically precise definition of a MAC, using the terminology defined in Section 2.4. This section may be safely skipped on first reading.

Definition 6.3 (MAC). *A* ***MAC*** *system is a pair of efficient algorithms, S and V, along with three families of spaces with system parameterization P:*

$$\mathbf{K} = \{\mathcal{K}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \mathbf{M} = \{\mathcal{M}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \textit{and} \quad \mathbf{T} = \{\mathcal{T}_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$

As usual, $\lambda \in \mathbb{Z}_{\geq 1}$ is a security parameter and $\Lambda \in \mathrm{Supp}(P(\lambda))$ is a domain parameter. We require that

1. $\mathbf{K}$, $\mathbf{M}$, *and* $\mathbf{T}$ *are efficiently recognizable.*
2. $\mathbf{K}$ *is efficiently sampleable.*
3. *Algorithm S is an efficient probabilistic algorithm that on input λ, Λ, k, m, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $k \in \mathcal{K}_{\lambda,\Lambda}$, and $m \in \mathcal{M}_{\lambda,\Lambda}$, outputs an element of $\mathcal{T}_{\lambda,\Lambda}$.*
4. *Algorithm V is an efficient deterministic algorithm that on input $\lambda, \Lambda, k, m, t$, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $k \in \mathcal{K}_{\lambda,\Lambda}$, $m \in \mathcal{M}_{\lambda,\Lambda}$, and $t \in \mathcal{T}_{\lambda,\Lambda}$, outputs either* accept *or* reject*.*

In defining security, we parameterize Attack Game 6.1 by the security parameter λ, which is given to both the adversary and the challenger. The advantage $\mathrm{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}]$ is then a function of λ. Definition 6.2 should be read as saying that $\mathrm{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}](\lambda)$ is a negligible function.

6.2 MAC verification queries do not help the attacker

In our definition of secure MACs (Attack Game 6.1) the adversary has no way of testing whether a given message-tag pair is valid. In fact, the adversary cannot even tell if it wins the game, since only the challenger has the secret key needed to run the verification algorithm. In real life, an attacker capable of mounting a chosen message attack can probably also test whether a given message-tag pair is valid. For example, the attacker could build a packet containing the message-tag pair in question and send this packet to the victim's machine. Then, by examining the machine's behavior the attacker can tell whether the packet was accepted or dropped, indicating whether the tag was valid or not.

Consequently, it makes sense to extend Attack Game 6.1 by giving the adversary the extra power to verify message-tag pairs. Of course, we continue to allow the adversary to request tags for arbitrary messages of his choice.

Attack Game 6.2 (MAC security with verification queries). For a given MAC system $\mathcal{I} = (S, V)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger picks a random $k \xleftarrow{R} \mathcal{K}$.
- $\mathcal{A}$ queries the challenger several times. Each query can be one of two types:
  - *Signing query:* for $i = 1, 2, \ldots$, the ith signing query consists of a message $m_i \in \mathcal{M}$. The challenger computes a tag $t_i \xleftarrow{R} S(k, m_i)$, and gives t_i to $\mathcal{A}$.
  - *Verification query:* for $j = 1, 2, \ldots$, the jth verification query consists of a message-tag pair $(\hat{m}_j, \hat{t}_j) \in \mathcal{M} \times \mathcal{T}$ that is not among the previously signed pairs, i.e.,
  $$(\hat{m}_j, \hat{t}_j) \notin \big\{(m_1, t_1), (m_2, t_2), \ldots\big\}.$$
  The challenger responds to $\mathcal{A}$ with $V(k, \hat{m}_j, \hat{t}_j)$.

We say that $\mathcal{A}$ wins the above game if the challenger ever responds to a verification query with accept. We define $\mathcal{A}$'s advantage with respect to $\mathcal{I}$, denoted $\text{MAC}^{\text{vq}}\mathsf{adv}[\mathcal{A}, \mathcal{I}]$, as the probability that $\mathcal{A}$ wins the game. □

The two definitions are equivalent. Attack Game 6.2 is essentially the same as the original Attack Game 6.1, except that $\mathcal{A}$ can issue MAC verification queries. We prove that this extra power does not help the adversary.

Theorem 6.1. *If $\mathcal{I}$ is a secure MAC system, then it is also secure in the presence of verification queries.*

In particular, for every MAC adversary $\mathcal{A}$ that attacks $\mathcal{I}$ as in Attack Game 6.2, and which makes at most Q_v verification queries and at most Q_s signing queries, there exists a Q_s-query MAC adversary $\mathcal{B}$ that attacks $\mathcal{I}$ as in Attack Game 6.1, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{MAC}^{\text{vq}}\mathsf{adv}[\mathcal{A}, \mathcal{I}] \leq \text{MAC}\mathsf{adv}[\mathcal{B}, \mathcal{I}] \cdot Q_\text{v}.$$

Proof idea. Let $\mathcal{A}$ be a MAC adversary that attacks $\mathcal{I}$ as in Attack Game 6.2, and which makes at most Q_v verification queries and at most Q_s signing queries. From adversary $\mathcal{A}$, we build an adversary $\mathcal{B}$ that attacks $\mathcal{I}$ as in Attack Game 6.1 and makes at most Q_s signing queries. Adversary $\mathcal{B}$ can easily answer $\mathcal{A}$'s signing queries by forwarding them to $\mathcal{B}$'s challenger and relaying the resulting tags back to $\mathcal{A}$.

The question is how to respond to $\mathcal{A}$'s verification queries. Note that $\mathcal{A}$ by definition, $\mathcal{A}$ only submits verification queries on message pairs that are not among the previously signed pairs. So $\mathcal{B}$ adopts a simple strategy: it responds with reject to all verification queries from $\mathcal{A}$. If $\mathcal{B}$ answers incorrectly, it has a forgery which would let it win Attack Game 6.1. Unfortunately, $\mathcal{B}$ does not know which of these verification queries is a forgery, so it simply guesses, choosing one at random. Since $\mathcal{A}$ makes at most Q_v verification queries, $\mathcal{B}$ will guess correctly with probability at least $1/Q_\text{v}$. This is the source of the Q_v factor in the error term. □

Proof. In more detail, adversary $\mathcal{B}$ plays the role of challenger to $\mathcal{A}$ in Attack Game 6.2, while at the same time, it plays the role of adversary in Attack Game 6.1, interacting with the MAC challenger in that game. The logic is as follows:

initialization:
  $\omega \xleftarrow{R} \{1, \ldots, Q_v\}$
upon receiving a signing query $m_i \in \mathcal{M}$ from $\mathcal{A}$ do:
  forward m_i to the MAC challenger, obtaining the tag t_i
  send t_i to $\mathcal{A}$
upon receiving a verification query $(\hat{m}_j, \hat{t}_j) \in \mathcal{M} \times \mathcal{T}$ from $\mathcal{A}$ do:
  if $j = \omega$
    then output $(\hat{m}_j, \hat{t}_j)$ as a candidate forgery pair and halt
    else send reject to $\mathcal{A}$

To rigorously justify the construction of adversary $\mathcal{B}$, we analyze the the behavior of $\mathcal{A}$ in three closely related games.

Game 0. This is the original attack game, as played between the challenger in Attack Game 6.2 and adversary $\mathcal{A}$. Here is the logic of the challenger in this game:

initialization:
  $k \xleftarrow{R} \mathcal{K}$
upon receiving a signing query $m_i \in \mathcal{M}$ from $\mathcal{A}$ do:
  $t_i \xleftarrow{R} S(k, m_i)$
  send t_i to $\mathcal{A}$
upon receiving a verification query $(\hat{m}_j, \hat{t}_j) \in \mathcal{M} \times \mathcal{T}$ from $\mathcal{A}$ do:
  $r_j \leftarrow V(k, \hat{m}_j, \hat{t}_j)$
(∗)   send r_j to $\mathcal{A}$

Let W_0 be the event that in Game 0, $r_j =$ accept for some j. Evidently,

$$\Pr[W_0] = \mathrm{MAC}^{\mathrm{vq}}\mathsf{adv}[\mathcal{A}, \mathcal{I}]. \tag{6.1}$$

Game 1. This is the same as Game 1, except that the line marked (∗) above is changed to:

  send reject to $\mathcal{A}$

That is, when responding to a verification query, the challenger always responds to $\mathcal{A}$ with reject. We also define W_1 to be the event that in Game 1, $r_j =$ accept for some j. Even though the challenger does not notify $\mathcal{A}$ that W_1 occurs, both Games 0 and 1 proceed identically until this event happens, and so events W_0 and W_1 are really the same; therefore,

$$\Pr[W_1] = \Pr[W_0]. \tag{6.2}$$

Also note that in Game 1, although the r_j values are used to define the winning condition, they are not used for any other purpose, and so do not influence the attack in any way.

Game 2. This is the same as Game 1, except that at the beginning of the game, the challenger chooses $\omega \xleftarrow{R} \{1, \ldots, Q_v\}$. We define W_2 to be the event that in Game 2, $r_\omega =$ accept. Since the choice of ω is independent of the attack itself, we have

$$\Pr[W_2] \geq \Pr[W_1]/Q_v. \tag{6.3}$$

Evidently, by construction, we have

$$\Pr[W_2] = \text{MAC}\mathsf{adv}[\mathcal{B}, \mathcal{I}]. \tag{6.4}$$

The theorem now follows from (6.1)–(6.3). $\square$

In summary, we showed that Attack Game 6.2, which gives the adversary more power, is equivalent to Attack Game 6.1 used in defining secure MACs. The reduction introduces a factor of Q_{v} in the error term. Throughout the book we will make use of both attack games:

- When constructing secure MACs it easier to use Attack Game 6.1 which restricts the adversary to signing queries only. This makes it easier to prove security since we only have to worry about one type of query. We will use this attack game throughout the chapter.
- When using secure MACs to build higher level systems (such as authenticated encryption) it is more convenient to assume that the MAC is secure with respect to the stronger adversary described in Attack Game 6.2.

We also point out that if we had used a weaker notion of security, in which the adversary only wins by presenting a valid tag on a new message (rather than new valid message-tag pair), then the analogs of Attack Game 6.1 and Attack Game 6.2 are *not* equivalent (see Exercise 6.7).

6.3 Constructing MACs from PRFs

We now turn to constructing secure MACs using the tools at our disposal. In previous chapters we used pseudo random functions (PRFs) to build various encryption systems. We gave examples of practical PRFs such as AES (while AES is a block cipher it can be viewed as a PRF thanks to the PRF switching lemma, Theorem 4.4). Here we show that any secure PRF can be directly used to build a secure MAC.

Recall that a PRF is an algorithm F that takes two inputs, a key k and an input data block x, and outputs a value $y := F(k, x)$. As usual, we say that F is defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, where keys are in $\mathcal{K}$, inputs are in $\mathcal{X}$, and outputs are in $\mathcal{Y}$. For a PRF F we define the **deterministic MAC system $\mathcal{I} = (S, V)$ derived from F** as:

$$S(k,\ m) := F(k,\ m);$$

$$V(k,\ m,\ t) := \begin{cases} \mathsf{accept} & \text{if } F(k, m) = t, \\ \mathsf{reject} & \text{otherwise.} \end{cases}$$

As already discussed, any PRF with a large (i.e., super-poly) output space is unpredictable (see Section 4.1.1), and therefore, as discussed in Section 6.1, the above construction yields a secure MAC. For completeness, we state this as a theorem:

Theorem 6.2. *Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, where $|\mathcal{Y}|$ is super-poly. Then the deterministic MAC system $\mathcal{I}$ derived from F is a secure MAC.*

In particular, for every Q-query MAC adversary $\mathcal{A}$ that attacks $\mathcal{I}$ as in Attack Game 6.1, there exists a $(Q + 1)$-query PRF adversary $\mathcal{B}$ that attacks F as in Attack Game 4.2, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}] \leq \text{PRF}\mathsf{adv}[\mathcal{B}, F] + 1/|\mathcal{Y}|$$

Proof idea. Let $\mathcal{A}$ be an efficient MAC adversary. We derive an upper bound on MAC$\mathsf{adv}[\mathcal{A}, \mathcal{I}]$ by bounding $\mathcal{A}$'s ability to generate forged message-tag pairs. As usual, replacing the underlying secure PRF F with a truly random function f in Funs$[\mathcal{X}, \mathcal{Y}]$ does not change $\mathcal{A}$'s advantage much. But now that the adversary $\mathcal{A}$ is interacting with a truly random function it is faced with a hopeless task: using the chosen message attack it obtains the value of f at a few points of his choice. He then needs to guess the value of $f(m) \in \mathcal{Y}$ at some new point m. But since f is a truly random function, $\mathcal{A}$ has no information about $f(m)$, and therefore has little chance of guessing $f(m)$ correctly. □

Proof. We make this intuition rigorous by letting $\mathcal{A}$ interact with two closely related challengers.

Game 0. As usual, we begin by reviewing the challenger in the MAC Attack Game 6.1 as it applies to $\mathcal{I}$. We implement the challenger in this game as follows:

(∗) $k \xleftarrow{R} \mathcal{K}$, $f \leftarrow F(k, \cdot)$
    upon receiving the ith signing query $m_i \in \mathcal{M}$ (for $i = 1, 2, \ldots$) do:
        $t_i \leftarrow f(m_i)$
        send t_i to the adversary

At the end of the game, the adversary outputs a message-tag pair (m, t). We define W_0 to be the event that the condition

$$t = f(m) \qquad \text{and} \qquad m \notin \{m_1, m_2, \ldots\} \tag{6.5}$$

holds in Game 0. Clearly, $\Pr[W_0] = \text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}]$.

Game 1. We next play the usual "PRF card," replacing the function $F(k, \cdot)$ by a truly random function f in Funs$[\mathcal{X}, \mathcal{Y}]$. Intuitively, since F is a secure PRF, the adversary $\mathcal{A}$ should not notice the difference. Our challenger in Game 1 is the same as in Game 0 except that we change line (*) as follows:

(∗) $f \xleftarrow{R} \text{Funs}[\mathcal{X}, \mathcal{Y}]$

We define W_1 to be the event that condition (6.5) holds in Game 1. It should be clear how to design the corresponding PRF adversary $\mathcal{B}$ such that:

$$|\Pr[W_1] - \Pr[W_0]| = \text{PRF}\mathsf{adv}[\mathcal{B}, F].$$

Next, we directly bound $\Pr[W_1]$. The adversary $\mathcal{A}$ sees the values of f at various points $m_1, m_2, \ldots$ and is then required to guess the value of f at some new point m. But since f is a truly random function, the value $f(m)$ is independent of its value at all other points. Hence, since $m \notin \{m_1, m_2, \ldots\}$, adversary $\mathcal{A}$ will guess $f(m)$ with probability $1/|\mathcal{Y}|$. Therefore, $\Pr[W_1] \leq 1/|\mathcal{Y}|$. Putting it all together, we obtain

$$\text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}] = \Pr[W_0] \leq |\Pr[W_0] - \Pr[W_1]| + \Pr[W_1] \leq \text{PRF}\mathsf{adv}[\mathcal{B}, F] + \frac{1}{|\mathcal{Y}|}$$

as required. □

Concrete tag lengths. The theorem shows that to ensure MAC$\mathsf{adv}[\mathcal{A}, \mathcal{I}] < 2^{-128}$ we need a PRF whose output space $\mathcal{Y}$ satisfies $|\mathcal{Y}| > 2^{128}$. If the output space $\mathcal{Y}$ is $\{0, 1\}^n$ for some n, then the resulting tags must be at least 128 bits long.

6.4 Prefix-free PRFs for long messages

In the previous section we saw that any secure PRF is also a secure MAC. However, the concrete examples of PRFs from Chapter 4 only take short inputs and can therefore only be used to provide integrity for very short messages. For example, viewing AES as a PRF gives a MAC for 128-bit messages. Clearly, we want to build MACs for much longer messages.

All the MAC constructions in this chapter follow the same paradigm: they start from a PRF for short inputs (like AES) and produce a PRF, and therefore a MAC, for much longer inputs. Hence, our goal for the remainder of the chapter is the following:

given a secure PRF on short inputs construct a secure PRF on long inputs.

We solve this problem in three steps:

- First, in this section we construct *prefix-free secure* PRFs for long inputs. More precisely, given a secure PRF that operates on single-block (e.g., 128-bit) inputs, we construct a prefix-free secure PRF that operates on variable-length sequences of blocks. Recall that a prefix-free secure PRF (Definition 4.5) is only secure in a limited sense: we only require that *prefix-free adversaries* cannot distinguish the PRF from a random function. A prefix-free PRF adversary issues queries that are non-empty sequences of blocks, and no query can be a proper prefix of another.
- Second, in the next few sections we show how to convert prefix-free secure PRFs for long inputs into fully secure PRFs for long inputs. Thus, by the end of these sections we will have several secure PRFs, and therefore secure MACs, that operate on long inputs.
- Third, in Section 6.8 we show how to convert a PRF that operates on messages that are strings of blocks into a PRF that operates on strings of *bits*.

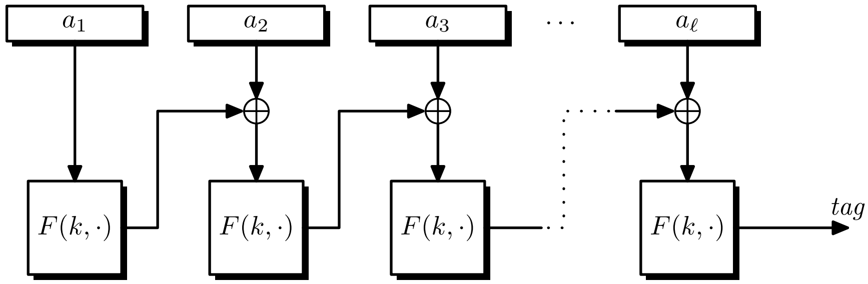
Prefix-free PRFs. We begin with two classic constructions for prefix-free secure PRFs. The **CBC** construction is shown in Fig. 6.3a. The **cascade** construction is shown in Fig. 6.3b. We show that when the underlying F is a secure PRF, both CBC and cascade are prefix-free secure PRFs.

6.4.1 The CBC prefix-free secure PRF

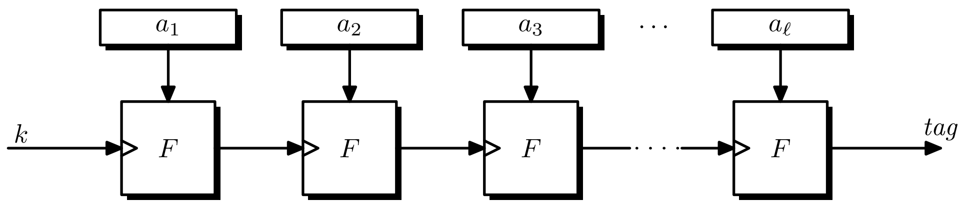
Let F be a PRF that maps n-bit inputs to n-bit outputs. In symbols, F is defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$ where $\mathcal{X} = \{0,1\}^n$. For any poly-bounded value ℓ, we build a new PRF, denoted F_{CBC}, that maps messages in $\mathcal{X}^{\leq \ell}$ to outputs in $\mathcal{X}$. The function F_{CBC}, described in Fig. 6.3a, works as follows:

input: $k \in \mathcal{K}$ and $m = (a_1, \ldots, a_v) \in \mathcal{X}^{\leq \ell}$ for some $v \in \{0, \ldots, \ell\}$
output: a tag in $\mathcal{X}$

$t \leftarrow 0^n$
for $i \leftarrow 1$ to v do:
  $t \leftarrow F(k,\ \ a_i \oplus t\ )$
output t



(a) The CBC construction $F_{\text{CBC}}(k, m)$



(b) The cascade construction $F^*(k, m)$

Figure 6.3: Two prefix-free secure PRFs

F_{CBC} is similar to CBC mode encryption from Fig. 5.4, but with two important differences. First, F_{CBC} does not output any intermediate values along the CBC chain. Second, F_{CBC} uses a fixed IV, namely 0^n, where as CBC mode encryption uses a random IV per message.

The following theorem shows that F_{CBC} is a prefix-free secure PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{X})$.

Theorem 6.3. *Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$ where $\mathcal{X} = \{0,1\}^n$ and $|\mathcal{X}| = 2^n$ is super-poly. Then for any poly-bounded value ℓ, we have that F_{CBC} is a prefix-free secure PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{X})$.*

In particular, for every prefix-free PRF adversary $\mathcal{A}$ that attacks F_{CBC} as in Attack Game 4.2, and issues at most Q queries, there exists a PRF adversary $\mathcal{B}$ that attacks F as in Attack Game 4.2, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{PRF}^{\text{pf}}\mathsf{adv}[\mathcal{A}, F_{\text{CBC}}] \leq \text{PRF}\mathsf{adv}[\mathcal{B}, F] + \frac{(Q\ell)^2}{2|\mathcal{X}|}. \tag{6.6}$$

Exercise 6.6 develops an attack on fixed-length F_{CBC} that demonstrates that security degrades quadratically in Q. This shows that the quadratic dependence on Q in (6.6) is necessary. A more difficult proof of security shows that security only degrades linearly in ℓ (see Section 6.13). In particular, the error term in (6.6) can be reduced to an expression dominated by $O(Q^2\ell/|\mathcal{X}|)$

Proof idea. We represent the adversary's queries in a rooted tree, where edges in the tree are labeled by message blocks (i.e., elements of $\mathcal{X}$). A query for $F_{\text{CBC}}(k, m)$, where $m = (a_1, \ldots, a_v) \in \mathcal{X}^v$ and $1 \leq v \leq \ell$, defines a path in the tree, starting at the root, as follows:

$$root \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{a_3} \cdots \xrightarrow{a_v} p_v. \tag{6.7}$$

Thus, two messages m and m' correspond to paths in the tree which both start at the root; these two paths may share a common initial subpath corresponding to the longest common prefix of m and m'.

With each node p in this tree, we associate a value $\gamma_p \in \mathcal{X}$ which represents the computed value in the CBC chain. More precisely, we define $\gamma_{root} := 0^n$, and for any non-root node q with parent p, if the corresponding edge in the tree is $p \xrightarrow{a} q$, then $\gamma_q := F(k, \gamma_p \oplus a)$. With these conventions, we see that if a message m traces out a path as in (6.7), then $\gamma_{p_v} = F_{\text{CBC}}(k, m)$.

The crux of the proof is to argue that if F behaves like a random function, then for every pair of distinct edges in the tree, say $p \xrightarrow{a} q$ and $p' \xrightarrow{a'} q'$, we have $\gamma_p \oplus a \neq \gamma_{p'} \oplus a'$ with overwhelming probability. To prove that there are no collisions of this type, the prefix-freeness restriction is critical, as it guarantees that the adversary never sees γ_p and $\gamma_{p'}$, and hence a and a' are independent of these values. Once we have established that there are no collisions of these types, it will follow that all values associated with non-root nodes are random and independent, and this holds in particular for the values associated with the leaves, which represent the outputs of F_{CBC} seen by the adversary. Therefore, the adversary cannot distinguish F_{CBC} from a random function. $\Box$

Proof. We make this intuition rigorous by letting $\mathcal{A}$ interact with three closely related challengers in three games. For $j = 0, 1, 2, 3$, we let W_j be the event that $\mathcal{A}$ outputs 1 at the end of Game j.

Game 0. This is Experiment 0 of Attack Game 4.2.

Game 1. We next play the usual "PRF card," replacing the function $F(k, \cdot)$ by a truly random function f in Funs$[\mathcal{X}, \mathcal{X}]$. Clearly, we have

$$\left|\Pr[W_1] - \Pr[W_0]\right| = \text{PRF}\mathsf{adv}[\mathcal{B}, F] \tag{6.8}$$

for an efficient adversary $\mathcal{B}$.

Game 2. We now make a purely conceptual change, implementing the random function f as a "faithful gnome" (as in Section 4.4.2). However, it will be convenient for us to do this is a particular way, using the "query tree" discussed above.

To this end, first let $B := Q\ell$, which represents an upper bound on how many points at which f will evaluated. Our challenger first prepares random values

$$\beta_i \xleftarrow{\text{R}} \mathcal{X} \qquad (i = 1, \ldots, B).$$

These will be the only random values used by our challenger.

As the adversary makes queries, our challenger will dynamically build up the query tree. Initially, the tree contains only the root. Whenever the adversary makes a query, the challenger traces out the corresponding path in the existing query tree; at some point, this path will extend beyond the existing query tree, and our challenger adds the necessary nodes and edges so that the query tree grows to include the new path.

Our challenger must also compute the values γ_p associated with each node. Initially, $\gamma_{root} = 0^n$. When adding a new edge $p \xrightarrow{a} q$ to the tree, if this is the ith edge being added (for $i = 1, \ldots, B$), our challenger does the following:

$$\begin{aligned} &\gamma_q \leftarrow \beta_i \\ (*) \qquad &\text{if } \exists \text{ another edge } p' \xrightarrow{a'} q' \text{ with } \gamma_{p'} \oplus a' = \gamma_p \oplus a \text{ then } \gamma_q \leftarrow \gamma_{q'} \end{aligned}$$

The idea is that we use the next unused value in our prepared list $\beta_1, \ldots, \beta_B$ as the "default" value for γ_q. The line marked $(*)$ performs the necessary consistency check, which ensures that our gnome is indeed faithful.

Because this change is purely conceptual, we have

$$\Pr[W_2] = \Pr[W_1]. \tag{6.9}$$

Game 3. Next, we make our gnome forgetful, by removing the consistency check marked $(*)$ in the logic in Game 2.

To analyze the effect of this change, let Z be the event that *in Game 3*, for some distinct pair of edges $p \xrightarrow{a} q$ and $p' \xrightarrow{a'} q'$, we have $\gamma_{p'} \oplus a' = \gamma_p \oplus a$.

Now, the only randomly chosen values in Games 2 and 3 are the random choices of the adversary, *Coins*, and the list of values $\beta_1, \ldots, \beta_B$. Observe that for any fixed choice of values *Coins*, $\beta_1, \ldots, \beta_B$, if Z does not occur, then in fact Games 2 and 3 proceed identically. Therefore, we may apply the Difference Lemma (Theorem 4.7), obtaining

$$\big|\Pr[W_3] - \Pr[W_2]\big| \leq \Pr[Z]. \tag{6.10}$$

We next bound $\Pr[Z]$. Consider two distinct edges $p \xrightarrow{a} q$ and $p' \xrightarrow{a'} q'$. We want to bound the probability that $\gamma_{p'} \oplus a' = \gamma_p \oplus a$, which is equivalent to

$$\gamma_{p'} \oplus \gamma_p = a' \oplus a. \tag{6.11}$$

There are two cases to consider.

Case 1: $p = p'$. Since the edges are distinct, we must have $a' \neq a$, and hence (6.11) holds with probability 0.

Case 2: $p \neq p'$. The requirement that the adversary's queries are prefix free implies that in Game 3, the adversary never sees — or learns anything about — the values γ_p and $\gamma_{p'}$. One of p or p' could be the root, but not both. It follows that the value $\gamma_p \oplus \gamma_{p'}$ is uniformly distributed over $\mathcal{X}$ and is independent of $a \oplus a'$. From this, it follows that (6.11) holds with probability $1/|\mathcal{X}|$.

By the union bound, it follows that

$$\Pr[Z] \leq \frac{B^2}{2|\mathcal{X}|}. \tag{6.12}$$

Combining (6.8), (6.9), (6.10), and (6.12), we obtain

$$\mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{A}, F_{\mathrm{CBC}}] = \big|\Pr[W_3] - \Pr[W_0]\big| \leq \mathrm{PRF}\mathsf{adv}[\mathcal{B}, F] + \frac{B^2}{2|\mathcal{X}|}. \tag{6.13}$$

Moreover, Game 3 corresponds exactly to Experiment 1 of Attack Game 4.2, from which the theorem follows. □

6.4.2 The cascade prefix-free secure PRF

Let F be a PRF that takes keys in $\mathcal{K}$ and produces outputs in $\mathcal{K}$. In symbols, F is defined over $(\mathcal{K}, \mathcal{X}, \mathcal{K})$. For any poly-bounded value ℓ, we build a new PRF F^*, called the **cascade of** F, that maps messages in $\mathcal{X}^{\leq \ell}$ to outputs in $\mathcal{K}$. The function F^*, illustrated in Fig. 6.3b, works as follows:

input: $k \in \mathcal{K}$ and $m = (a_1, \ldots, a_v) \in \mathcal{X}^{\leq \ell}$ for some $v \in \{0, \ldots, \ell\}$
output: a tag in $\mathcal{K}$

$t \leftarrow k$
for $i \leftarrow 1$ to v do:
$\quad t \leftarrow F(t,\ a_i)$
output t

The following theorem shows that F^* is a prefix-free secure PRF.

Theorem 6.4. *Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{K})$. Then for any poly-bounded value ℓ, the cascade F^* of F is a prefix-free secure PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{K})$.*

In particular, for every prefix-free PRF adversary $\mathcal{A}$ that attacks F^* as in Attack Game 4.2, and issues at most Q queries, there exists a PRF adversary $\mathcal{B}$ that attacks F as in Attack Game 4.2, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{PRF^{pf}}\mathsf{adv}[\mathcal{A}, F^*] \leq Q\ell \cdot \mathrm{PRF}\mathsf{adv}[\mathcal{B}, F]. \tag{6.14}$$

Exercise 6.6 develops an attack on fixed-length F^* that demonstrates that security degrades quadratically in Q. This is disturbing as it appears to contradict the linear dependence on Q in (6.14). However, rest assured there is no contradiction here. The adversary $\mathcal{A}$ from Exercise 6.6, which uses $\ell = 3$, has advantage about 1/2 when Q is about $\sqrt{|\mathcal{K}|}$. Plugging $\mathcal{A}$ into the proof of Theorem 6.4 we obtain a PRF adversary $\mathcal{B}$ that attacks the PRF F making about Q queries to gain an advantage about $1/Q$. Note that $1/Q \approx Q/|\mathcal{K}|$ when Q is close to $\sqrt{|\mathcal{K}|}$. There is nothing surprising about this adversary $\mathcal{B}$: it is essentially the universal PRF attacker from Exercise 4.27. Hence, (6.14) is consistent with the attack from Exercise 6.6. Another way to view this is that the quadratic dependence on Q is already present in (6.14) because there is an implicit factor of Q hiding in the quantity $\mathrm{PRF}\mathsf{adv}[\mathcal{B}, F]$.

The proof of Theorem 6.4 is similar to the proof that the variable-length tree construction in Section 4.6 is a prefix-free secure PRF (Theorem 4.11). Let us briefly explain how to extend the proof of Theorem 4.11 to prove Theorem 6.4.

Relation to the tree construction. The cascade construction is a generalization of the variable-length tree construction of Section 4.6. Recall that the tree construction builds a secure PRF from a secure PRG that maps a seed to a pair of seeds. It is easy to see that when F is a PRF defined over $(\mathcal{K}, \{0,1\}, \mathcal{K})$ then Theorem 6.4 is an immediate corollary of Theorem 4.11: simply define the PRG G mapping $k \in \mathcal{K}$ to $G(k) := (F(k,0), F(k,1)) \in \mathcal{K}^2$, and observe that cascade applied to F is the same as the variable-length tree construction applied to G.

The proof of Theorem 4.11 generalizes easily to prove Theorem 6.4 for any PRF. For example, suppose that F is defined over $(\mathcal{K}, \{0,1,2\}, \mathcal{K})$. This corresponds to a PRG G mapping $k \in \mathcal{K}$ to $G(k) := (F(k,0), F(k,1), F(k,2)) \in \mathcal{K}^3$. The cascade construction construction applied to F can be viewed as a ternary tree, instead of a binary tree, and the proof of Theorem 4.11 carries over with no essential changes.

But why stop at width three? We can make the tree as wide as we wish. The cascade construction using a PRF F defined over $(\mathcal{K}, \mathcal{X}, \mathcal{K})$ corresponds to a tree of width $|\mathcal{X}|$. Again, the proof of Theorem 4.11 carries over with no essential changes. We leave the details as an exercise for the interested reader (Exercise 4.26 may be convenient here).

Comparing the CBC and cascade PRFs. Note that CBC uses a fixed key k for all applications of F while cascade uses a different key in each round. Since block ciphers are typically optimized to encrypt many blocks using the same key, the constant re-keying in cascade may result in worse performance than CBC. Hence, CBC is the more natural choice when using an off the shelf block cipher like AES.

An advantage of cascade is that there is no additive error term in Theorem 6.4. Consequently, the cascade construction remains secure even if the underlying PRF has a small domain $\mathcal{X}$. CBC, in contrast, is secure only when $\mathcal{X}$ is large. As a result, cascade can be used to convert a PRG into a PRF for large inputs while CBC cannot.

6.4.3 Extension attacks: CBC and cascade are insecure MACs

We show that the MACs derived from CBC and cascade are insecure. This will imply that CBC and cascade are not secure PRFs. All we showed in the previous section is that CBC and cascade are *prefix-free* secure PRFs.

Extension attack on cascade. Given $F^*(k, m)$ for some message m in $\mathcal{X}^{\leq \ell}$, anyone can compute

$$t' := F^*(k, \quad m \parallel m') \tag{6.15}$$

for any $m' \in \mathcal{X}^*$, without knowledge of k. Once $F^*(k, m)$ is known, anyone can continue evaluating the chain using blocks of the message m' and obtain t'. We refer to this as the **extension property** of cascade.

The extension property immediately implies that the MAC derived from F^* is terribly insecure. The forger can request the MAC on message m and then deduce the MAC on $m \parallel m'$ for any m' of his choice. It follows, by Theorem 6.2, that F^* is not a secure PRF.

An attack on CBC. We describe a simple MAC forger on the MAC derived from CBC. The forger works as follows:

1. pick an arbitrary $a_1 \in \mathcal{X}$;
2. request the tag t on the one-block message (a_1);
3. define $a_2 := a_1 \oplus t$ and output t as a MAC forgery for the two-block message $(a_1, a_2) \in \mathcal{X}^2$.

Observe that $t = F(k, a_1)$ and $a_1 = F(k, a_1) \oplus a_2$. By definition of CBC we have:

$$F_{\text{CBC}}\big(k,\ (a_1, a_2)\ \big) = F\big(k,\ \ F(k,\ a_1) \oplus a_2\ \big) = F\big(k,\ a_1\big) = t.$$

Hence, $\big((a_1, a_2),\ t\big)$ is an existential forgery for the MAC derived from CBC. Consequently, F_{CBC} cannot be a secure PRF. Note that the attack on the cascade MAC is far more devastating than on the CBC MAC. But in any case, these attacks show that neither CBC nor cascade should be used directly as MACs.

6.5 From prefix-free secure PRF to fully secure PRF (method 1): encrypted PRF

We show how to convert the prefix-free secure PRFs F_{CBC} and F^* into secure PRFs, which will give us secure MACs for variable length inputs. More generally, we show how to convert a prefix-free secure PRF PF to a secure PRF. We present three methods:

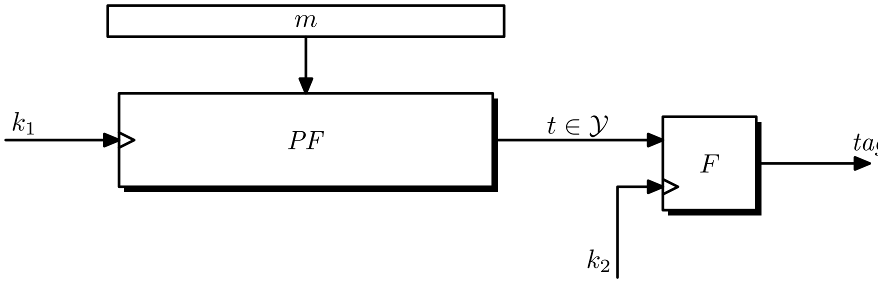


Figure 6.4: The encrypted PRF construction $EF(k, m)$

- Encrypted PRF: encrypt the short output of PF with another PRF.
- Prefix-free encoding: encode the input to PF so that no input is a prefix of another.
- CMAC: a more efficient prefix-free encoding using randomization.

In this section we discuss the encrypted PRF method. The construction is straightforward. Let PF be a PRF mapping $\mathcal{X}^{\leq \ell}$ to $\mathcal{Y}$ and let F be a PRF mapping $\mathcal{Y}$ to $\mathcal{T}$. Define

$$EF\big((k_1, k_2),\ m\big) := F\big(k_2,\ PF(k_1,\ m)\big) \tag{6.16}$$

The construction is shown in Fig. 6.4.

We claim that when PF is either CBC or cascade then EF is a secure PRF. More generally, we show that EF is secure whenever PF is an *extendable PRF*, defined as follows:

Definition 6.4. *Let PF be a PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{Y})$. We say that PF is an* ***extendable PRF*** *if for all $k \in \mathcal{K}$, $x, y \in \mathcal{X}^{\leq \ell - 1}$, and $a \in \mathcal{X}$ we have:*

$$\textit{if} \quad PF(k, x) = PF(k, y) \quad \textit{then} \quad PF(k,\ x \parallel a) = PF(k,\ y \parallel a).$$

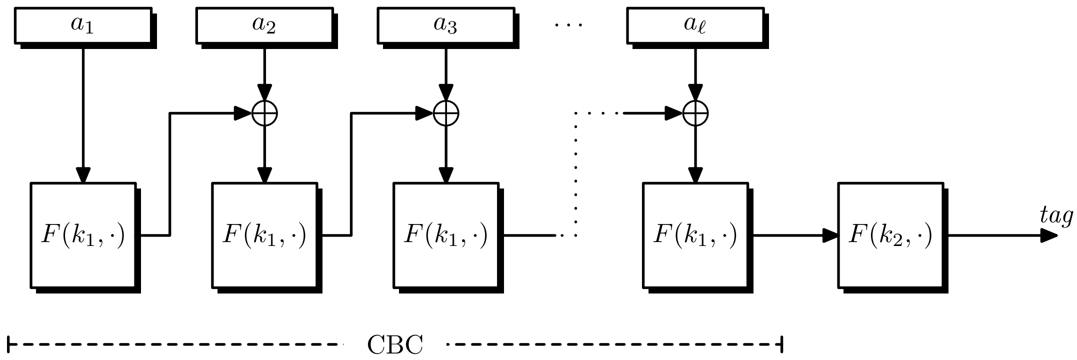
It is easy to see that both CBC and cascade are extendable PRFs. The next theorem shows that when PF is an extendable, prefix-free secure PRF then EF is a secure PRF.

Theorem 6.5. *Let PF be an extendable and prefix-free secure PRF defined over $(\mathcal{K}_1, \mathcal{X}^{\leq \ell+1}, \mathcal{Y})$, where $|\mathcal{Y}|$ is super-poly and ℓ is poly-bounded. Let F be a secure PRF defined over $(\mathcal{K}_2, \mathcal{Y}, \mathcal{T})$. Then EF, as defined in (6.16), is a secure PRF defined over $(\mathcal{K}_1 \times \mathcal{K}_2, \mathcal{X}^{\leq \ell}, \mathcal{T})$.*

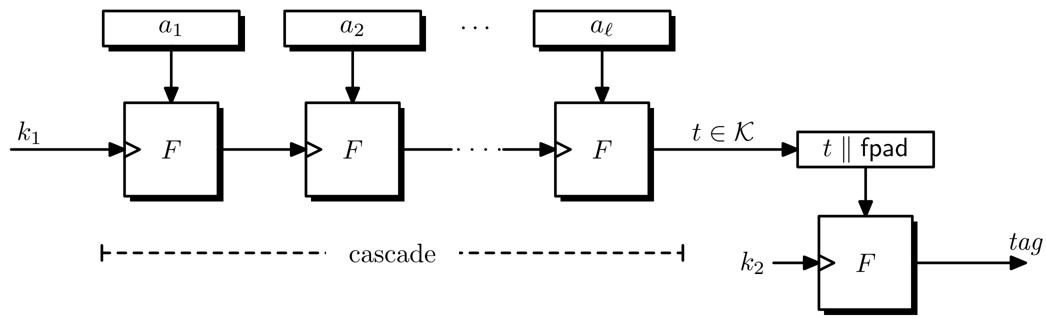
> *In particular, for every PRF adversary $\mathcal{A}$ that attacks EF as in Attack Game 4.2, and issues at most Q queries, there exist a PRF adversary $\mathcal{B}_1$ attacking F as in Attack Game 4.2, and a prefix-free PRF adversary $\mathcal{B}_2$ attacking PF as in Attack Game 4.2, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are elementary wrappers around $\mathcal{A}$, such that*

$$\mathrm{PRF}\mathsf{adv}[\mathcal{A}, EF] \leq \mathrm{PRF}\mathsf{adv}[\mathcal{B}_1, F] + \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}_2, PF] + \frac{Q^2}{2|\mathcal{Y}|}. \tag{6.17}$$

We prove Theorem 6.5 in the next chapter (Section 7.3.1) after we develop the necessary tools. Note that to make EF a secure PRF on inputs of length up to ℓ, this theorem requires that PF is prefix-free secure on inputs of length $\ell + 1$.



(a) The ECBC construction ECBC(k, m) (encrypted CBC)



(b) The NMAC construction NMAC(k, m) (encrypted cascade)

Figure 6.5: Secure PRF constructions for variable length inputs

The bound in (6.17) is tight. Although not entirely necessary, let us assume that $\mathcal{Y} = \mathcal{T}$, that F is a block cipher, and that $|\mathcal{X}|$ is not too small. These assumptions will greatly simplify the argument. We exhibit an attack that breaks *EF* with constant probability after $Q \approx \sqrt{|\mathcal{Y}|}$ queries. Our attack will, in fact, break *EF* as a MAC. The adversary picks Q random inputs $x_1, \ldots, x_Q \in \mathcal{X}^2$ and queries its MAC challenger at all Q inputs to obtain $t_1, \ldots, t_Q \in \mathcal{T}$. By the birthday paradox (Corollary B.2), for any fixed key k_1, with constant probability there will be distinct indices i, j such that $x_i \neq x_j$ and $PF(k_1, x_i) = PF(k_1, x_j)$. On the one hand, if such a collision occurs, we will detect it, because $t_i = t_j$ for such a pair of indices. On the other hand, if $t_i = t_j$ for some pair of indices i, j, then our assumption that F is a block cipher guarantees that $PF(k_1, x_i) = PF(k_1, x_j)$. Now, assuming that $x_i \neq x_j$ and $PF(k_1, x_i) = PF(k_1, x_j)$, and since PF is extendable, we know that for all $a \in \mathcal{X}$, we have $PF\big(k_1, (x_i \parallel a)\big) = PF\big(k_1, (x_j \parallel a)\big)$. Therefore, our adversary can obtain the MAC tag t for $x_i \parallel a$, and this tag t will also be a valid tag for $x_j \parallel a$. This attack easily generalizes to show the necessity of the term $Q^2/(2|\mathcal{Y}|)$ in (6.17).

6.5.1 ECBC and NMAC: MACs for variable length inputs

Figures 6.5a and 6.5b show the result of applying the *EF* construction (6.16) to CBC and cascade.

The Encrypted-CBC PRF

Applying *EF* to CBC results in a classic PRF (and hence a MAC) called **encrypted-CBC** or **ECBC** for short. This MAC is standardized by ANSI (see Section 6.9) and is used in the banking industry. The ECBC PRF uses the same underlying PRF F for both CBC and the final encryption. Consequently, ECBC is defined over $(\mathcal{K}^2,\ \mathcal{X}^{\leq \ell},\ \mathcal{X})$.

Theorem 6.6 (ECBC security). *Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$. Suppose $\mathcal{X}$ is super-poly, and let ℓ be a poly-bounded length parameter. Then* ECBC *is a secure PRF defined over $(\mathcal{K}^2, \mathcal{X}^{\leq \ell}, \mathcal{X})$.*

> *In particular, for every PRF adversary $\mathcal{A}$ that attacks* ECBC *as in Attack Game 4.2, and issues at most Q queries, there exist PRF adversaries $\mathcal{B}_1, \mathcal{B}_2$ that attack F as in Attack Game 4.2, and which are elementary wrappers around $\mathcal{A}$, such that*

$$\mathrm{PRF}\mathsf{adv}[\mathcal{A}, \mathrm{ECBC}] \leq \mathrm{PRF}\mathsf{adv}[\mathcal{B}_1, F] + \mathrm{PRF}\mathsf{adv}[\mathcal{B}_2, F] + \frac{(Q(\ell+1))^2 + Q^2}{2|\mathcal{X}|}. \tag{6.18}$$

Proof. CBC is clearly extendable and is a prefix-free secure PRF by Theorem 6.3. Hence, if the underlying PRF F is secure, then ECBC is a secure PRF by Theorem 6.5. □

The argument given after Theorem 6.5 shows that there is an attacker that after $Q \approx \sqrt{|\mathcal{X}|}$ queries breaks this PRF with constant advantage. Recall that for 3DES we have $\mathcal{X} = \{0,1\}^{64}$. Hence, after about a billion queries (or more precisely, 2^{32} queries) an attacker can break the ECBC-3DES MAC with constant probability.

The NMAC PRF

Applying *EF* to cascade results in a PRF (and hence a MAC) called **Nested MAC** or **NMAC** for short. A variant of this MAC is standardized by the IETF (see Section 8.7.2) and is widely used in Internet protocols.

We wish to use the same underlying PRF F for the cascade construction and for the final encryption. Unfortunately, the output of cascade is in $\mathcal{K}$ while the message input to F is in $\mathcal{X}$. To solve this problem we need to embed the output of cascade into $\mathcal{X}$. More precisely, we assume that $|\mathcal{K}| \leq |\mathcal{X}|$ and that there is an efficiently computable one-to-one function g that maps $\mathcal{K}$ into $\mathcal{X}$. For example, suppose $\mathcal{K} := \{0,1\}^{\kappa}$ and $\mathcal{X} := \{0,1\}^n$ where $\kappa \leq n$. Define $g(t) := t \parallel \mathsf{fpad}$ where fpad is a fixed pad of length $n - \kappa$ bits. This fpad can be as simple as a string of 0s. With this translation, all of NMAC can be built from a single secure PRF F, as shown in Fig. 6.5b.

Theorem 6.7 (NMAC security). *Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{K})$, where $\mathcal{K}$ can be embedded into $\mathcal{X}$. Then* NMAC *is a secure PRF defined over $(\mathcal{K}^2, \mathcal{X}^{\leq \ell}, \mathcal{K})$.*

> *In particular, for every PRF adversary $\mathcal{A}$ that attacks* NMAC *as in Attack Game 4.2, and issues at most Q queries, there exist PRF adversaries $\mathcal{B}_1, \mathcal{B}_2$ that attack F as in Attack Game 4.2, and which are elementary wrappers around $\mathcal{A}$, such that*

$$\mathrm{PRF}\mathsf{adv}[\mathcal{A}, \mathrm{NMAC}] \leq (Q(\ell+1)) \cdot \mathrm{PRF}\mathsf{adv}[\mathcal{B}_1, F] + \mathrm{PRF}\mathsf{adv}[\mathcal{B}_2, F] + \frac{Q^2}{2|\mathcal{K}|}. \tag{6.19}$$

Proof. NMAC is clearly extendable and is a prefix-free secure PRF by Theorem 6.4. Hence, if the underlying PRF F is secure, then NMAC is a secure PRF by Theorem 6.5. □

ECBC and NMAC are streaming MACs. Both ECBC and NMAC can be used to authenticate variable size messages in $\mathcal{X}^{\leq \ell}$. Moreover, there is no need for the message length to be known ahead of time. A MAC that has this property is said to be a **streaming MAC**. This property enables applications to feed message blocks to the MAC one block at a time and at some arbitrary point decide that the message is complete. This is important for applications like streaming video, where the message length may not be known ahead of time.

In contrast, some MAC systems require that the message length be prepended to the message body (see Section 6.6). Such MACs are harder to use in practice since they require applications to determine the message length before starting the MAC calculations.

6.6 From prefix-free secure PRF to fully secure PRF (method 2): prefix-free encodings

Another approach to converting a prefix-free secure PRF into a secure PRF is to encode the input to the PRF so that no encoded input is a prefix of another. We use the following terminology:

- We say that a set $S \subseteq \mathcal{X}^{\leq \ell}$ is a **prefix-free set** if no element in S is a proper prefix of any other. For example, if (x_1, x_2, x_3) belongs to a prefix-free set S, then neither x_1 nor (x_1, x_2) are in S.

- Let $\mathcal{X}_{>0}^{\leq \ell}$ denote the set of all non-empty strings over $\mathcal{X}$ of length at most ℓ. We say that a function $pf : \mathcal{M} \to \mathcal{X}_{>0}^{\leq \ell}$ is a **prefix-free encoding** if pf is injective (i.e., one-to-one) and the image of pf in is a prefix-free set.

Let PF be a prefix-free secure PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{Y})$ and $pf : \mathcal{M} \to \mathcal{X}_{>0}^{\leq \ell}$ be a prefix-free encoding. Define the derived PRF F as

$$F(k, m) := PF(k, pf(m)).$$

Then F is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{Y})$. We obtain the following trivial theorem.

Theorem 6.8. *If PF is a prefix-free secure PRF and pf is a prefix-free encoding then F is a secure PRF.*

6.6.1 Prefix free encodings

To construct PRFs using Theorem 6.8 we describe two prefix-free encodings $pf : \mathcal{M} \to \mathcal{X}^{\leq \ell}$. We assume that $\mathcal{X} = \{0, 1\}^n$ for some n.

Method 1: prepend length. Set $\mathcal{M} := \mathcal{X}^{\leq \ell - 1}$ and let $m = (a_1, \ldots, a_v) \in \mathcal{M}$. Define

$$pf(m) := (\langle v \rangle, a_1, \ldots, a_v) \quad \in \mathcal{X}_{>0}^{\leq \ell}$$

where $\langle v \rangle \in \mathcal{X}$ is the binary representation of v, the length of m. We assume that $\ell < 2^n$ so that the message length can be encoded as an n-bit binary string.

We argue that pf is a prefix-free encoding. Clearly pf is injective. To see that the image of pf is a prefix-free set let $pf(x)$ and $pf(y)$ be two elements in the image of pf. If $pf(x)$ and $pf(y)$ contain the same number of blocks, then neither is a proper prefix of the other. Otherwise, $pf(x)$

and $pf(y)$ contain a different number of blocks and must therefore differ in the first block. But then, again, neither is a proper prefix of the other. Hence, pf is a prefix-free encoding.

This prefix-free encoding is not often used in practice since the resulting MAC is not a streaming MAC: an application using this MAC must commit to the length of the message to MAC ahead of time. This is undesirable for streaming applications such as streaming video where the length of packets may not be known ahead of time.

Method 2: stop bits. Let $\bar{\mathcal{X}} := \{0,1\}^{n-1}$ and let $\mathcal{M} = \bar{\mathcal{X}}_{>0}^{\leq \ell}$. For $m = (a_1, \ldots, a_v) \in \mathcal{M}$, define

$$pf(m) := \big((a_1 \parallel 0),\ (a_2 \parallel 0),\ \ldots,\ (a_{v-1} \parallel 0),\ (a_v \parallel 1)\big) \quad \in \mathcal{X}_{>0}^{\leq \ell}$$

Clearly pf is injective. To see that the image of pf is a prefix-free set let $pf(x)$ and $pf(y)$ be two elements in the image of pf. Let v be the number of blocks in $pf(x)$. If $pf(y)$ contains v or fewer blocks then $pf(x)$ is not a proper prefix of $pf(y)$. If $pf(y)$ contains more than v blocks then block number v in $pf(y)$ ends in 0, but block number v in $pf(x)$ ends in 1. Hence, $pf(x)$ and $pf(y)$ differ in block v and therefore $pf(x)$ is not a proper prefix of $pf(y)$.

The MAC resulting from this prefix-free encoding is a streaming MAC. This encoding, however, increases the length of the message to MAC by v bits. When computing the MAC on a long message using either CBC or cascade, this encoding will result in additional evaluations of the underlying PRF (e.g. AES). In contrast, the encrypted PRF method of Section 6.5 only adds one additional application of the underlying PRF. For example, to MAC a megabyte message (2^{20} bytes) using ECBC-AES and pf one would need an additional 511 evaluations of AES beyond what is needed for the encrypted PRF method. In practice, things are even worse. Since computers prefer byte-aligned data, one would most likely need to append an entire byte to every block, rather than just a bit. Then to MAC a megabyte message using ECBC-AES and pf would result in 4096 additional evaluations of AES over the encrypted PRF method — an overhead of about 6%.

6.7 From prefix-free secure PRF to fully secure PRF (method 3): CMAC

Both prefix free encoding methods from the previous section are problematic. The first resulted in a non-streaming MAC. The second required more evaluations of the underlying PRF for long messages. We can do better by randomizing the prefix free encoding. We build a streaming secure PRF that introduces no overhead beyond the underlying prefix-free secure PRF. The resulting MACs, shown in Fig. 6.6, are superior to those obtained from encrypted PRFs and deterministic encodings. This approach is used in a NIST MAC standard called CMAC and described in Section 6.10.

First, we introduce some convenient notation:

Definition 6.5. *For two strings $x, y \in \mathcal{X}^{\leq \ell}$, let us write $x \sim y$ if x is a prefix of y or y is a prefix of x.*

Definition 6.6. *Let ϵ be a real number, with $0 \leq \epsilon \leq 1$. A* ***randomized ϵ-prefix-free*** *encoding is a function $rpf : \mathcal{K} \times \mathcal{M} \to \mathcal{X}_{>0}^{\leq \ell}$ such that for all $m_0, m_1 \in \mathcal{M}$ with $m_0 \neq m_1$, we have*

$$\Pr\big[rpf(k, m_0) \sim rpf(k, m_1)\big] \leq \epsilon,$$

where the probability is over the random choice of k in $\mathcal{K}$.

Note that the image of $rpf(k, \cdot)$ need not be a prefix-free set. However, without knowledge of k it is difficult to find messages $m_0, m_1 \in \mathcal{M}$ such that $rpf(k, m_0)$ is a proper prefix of $rpf(k, m_1)$ (or vice versa). The function $rpf(k, \cdot)$ need not even be injective.

A simple rpf**.** Let $\mathcal{K} := \mathcal{X}$ and $\mathcal{M} := \mathcal{X}^{\leq \ell}_{>0}$. Define

$$rpf(k,\ (a_1, \ldots, a_v)) := \big(a_1, \ldots, a_{v-1}, (a_v \oplus k)\big) \in \mathcal{X}^{\leq \ell}_{>0}$$

It is easy to see that rpf is a randomized $(1/|\mathcal{X}|)$-prefix-free encoding. Let $m_0, m_1 \in \mathcal{M}$ with $m_0 \neq m_1$. Suppose that $|m_0| = |m_1|$. Then it is clear that for all choices of k, $rpf(k, m_0)$ and $rpf(k, m_1)$ are distinct strings of the same length, and so neither is a prefix of the other. Next, suppose that $|m_0| < |m_1|$. If $v := |rpf(k, m_0)|$, then clearly $rpf(k, m_0)$ is a proper prefix of $rpf(k, m_1)$ if and only if

$$m_0[v-1] \oplus k = m_1[v-1].$$

But this holds with probability $1/|\mathcal{X}|$ over the random choice of k, as required. Finally, the case $|m_0| > |m_1|$ is handled by a symmetric argument.

Using rpf**.** Let PF be a prefix-free secure PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{Y})$ and $rpf : \mathcal{K}_1 \times \mathcal{M} \to \mathcal{X}^{\leq \ell}_{>0}$ be a randomized prefix-free encoding. Define the derived PRF F as

$$F\big((k, k_1),\ m\big) := PF\big(k, rpf(k_1, m)\big). \tag{6.20}$$

Then F is defined over $(\mathcal{K} \times \mathcal{K}_1, \mathcal{M}, \mathcal{Y})$. We obtain the following theorem, which is analogous to Theorem 6.8.

Theorem 6.9. *If PF is a prefix-free secure PRF, ϵ is negligible, and rpf a randomized ϵ-prefix-free encoding, then F defined in (6.20) is a secure PRF.*

> *In particular, for every PRF adversary $\mathcal{A}$ that attacks F as in Attack Game 4.2, and issues at most Q queries, there exist prefix-free PRF adversaries $\mathcal{B}_1$ and $\mathcal{B}_2$ that attack PF as in Attack Game 4.2, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are elementary wrappers around $\mathcal{A}$, such that*
>
> $$\mathrm{PRF}\mathsf{adv}[\mathcal{A}, F] \leq \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}_1, PF] + \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}_2, PF] + Q^2\epsilon/2. \tag{6.21}$$

Proof idea. If the adversary's set of inputs to F give rise to a prefix-free set of inputs to PF, then the adversary sees just some random looking outputs. Moreover, if the adversary sees random outputs, it obtains no information about the rpf key k_1, which ensures that the set of inputs to PF is indeed prefix free (with overwhelming probability). Unfortunately, this argument is circular. However, we will see in the detailed proof how to break this circularity. □

Proof. Without loss of generality, we assume that $\mathcal{A}$ never issues the same query twice. We structure the proof as a sequence of three games. For $j = 0, 1, 2$, we let W_j be the event that $\mathcal{A}$ outputs 1 at the end of Game j.

Game 0. The challenger in Experiment 0 of the PRF Attack Game 4.2 with respect to F works as follows.

$k \xleftarrow{R} \mathcal{K}, \quad k_1 \xleftarrow{R} \mathcal{K}_1$

upon receiving a signing query $m_i \in \mathcal{M}$ (for $i = 1, 2, \ldots$) do:
    $x_i \leftarrow rpf(k_1, m_i) \in \mathcal{X}_{>0}^{\leq \ell}$
    $y_i \leftarrow PF(k, x_i)$
    send y_i to $\mathcal{A}$

Game 1. We change the challenger in Game 0 to ensure that all queries to PF are prefix free. Recall the notation $x \sim y$, which means that x is a prefix of y or y is a prefix of x.

$k \xleftarrow{R} \mathcal{K}, \quad k_1 \xleftarrow{R} \mathcal{K}_1, \quad r_1, \ldots, r_Q \xleftarrow{R} \mathcal{Y}$

upon receiving a signing query $m_i \in \mathcal{M}$ (for $i = 1, 2, \ldots$) do:
    $x_i \leftarrow rpf(k_1, m_i) \in \mathcal{X}_{>0}^{\leq \ell}$
(1)   if $x_i \sim x_j$ for some $j < i$
        then $y_i \leftarrow r_i$
(2)   else $y_i \leftarrow PF(k, x_i)$
    send y_i to $\mathcal{A}$

Let Z_1 be the event that the condition on line (1) holds at some point during Game 1. Clearly, Games 1 and 2 proceed identically until event Z_1 occurs; in particular, $W_0 \wedge \bar{Z}_1$ occurs if and only if $W_1 \wedge \bar{Z}_1$ occurs. Applying the Difference Lemma (Theorem 4.7), we obtain

$$\big|\Pr[W_1] - \Pr[W_0]\big| \leq \Pr[Z_1]. \tag{6.22}$$

Unfortunately, we are not quite in a position to bound $\Pr[Z_1]$ at this point. At this stage in the analysis, we cannot say that the evaluations of PF at line (2) do not leak some information about k_1 that could help $\mathcal{A}$ make Z_1 happen. This is the circularity problem we alluded to above. To overcome this problem, we will delay the analysis of Z_1 to the next game.

Game 2. Now we play the usual "PRF card," replacing the function $PF(k, \cdot)$ by a truly random function. This is justified, since by construction, in Game 1, the set of inputs to $PF(k, \cdot)$ is prefix-free. To implement this change, we may simply replace the line marked (2) by

(2)   else $y_i \leftarrow r_i$

After making this change, we see that y_i gets assigned the random value r_i, regardless of whether the condition on line (1) holds or not.

Now, let Z_2 be the event that the condition on line (1) holds at some point during Game 2. It is not hard to see that

$$|\Pr[Z_1] - \Pr[Z_2]| \leq \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}_1, F] \tag{6.23}$$

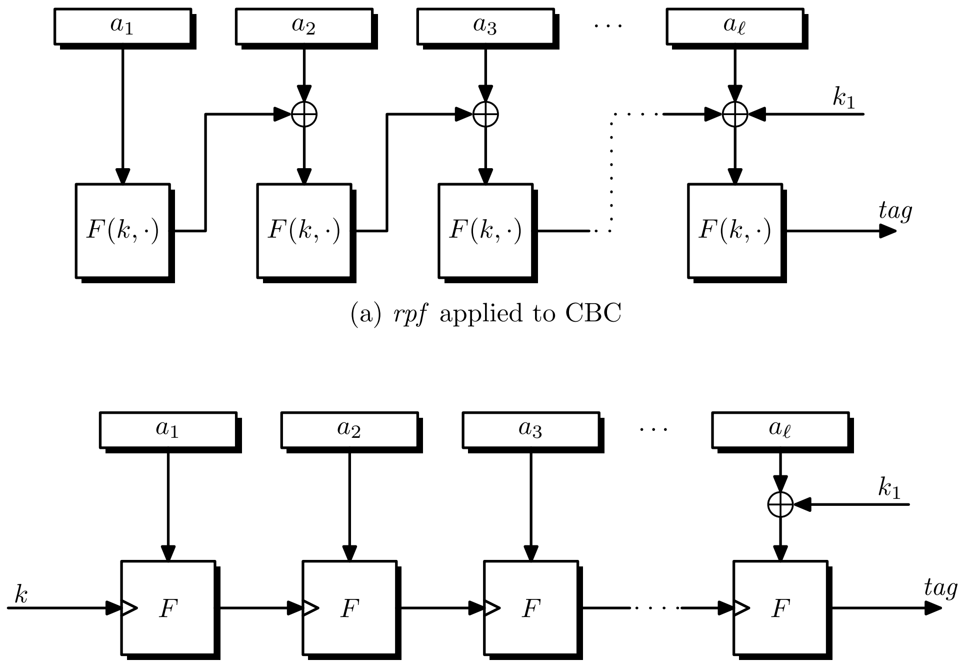
and

$$|\Pr[W_1] - \Pr[W_2]| \leq \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}_2, F] \tag{6.24}$$

for efficient prefix-free PRF adversaries $\mathcal{B}_1$ and $\mathcal{B}_2$. These two adversaries are basically the same, except that $\mathcal{B}_1$ outputs 1 if the condition on line (1) holds, while $\mathcal{B}_2$ ouputs whatever $\mathcal{A}$ outputs.

Moreover, in Game 2, the value of k_1 is clearly independent of $\mathcal{A}$'s queries, and so by making use of the ϵ-prefix-free property of rpf, and the union bound we have

$$\Pr[Z_2] \leq Q^2\epsilon/2 \tag{6.25}$$



(a) *rpf* applied to CBC

(b) *rpf* applied to cascade

Figure 6.6: Secure PRFs using random prefix-free encodings

Finally, Game 2 perfectly emulates for $\mathcal{A}$ a random function in Funs$[\mathcal{M}, \mathcal{Y}]$. Game 2 is therefore identical to Experiment 1 of the PRF Attack Game 4.2 with respect to F, and hence

$$|\Pr[W_0] - \Pr[W_2]| = \mathrm{PRF}\mathsf{adv}[\mathcal{A}, F]. \tag{6.26}$$

Now combining (6.22)–(6.26) proves the theorem. □

6.8 Converting a block-wise PRF to bit-wise PRF

So far we constructed a number of PRFs for variable length inputs in $\mathcal{X}^{\leq \ell}$. Typically $\mathcal{X} = \{0,1\}^n$ where n is the block size of the underlying PRF from which CBC or cascade are built (e.g., $n = 128$ for AES). All our MACs so far are designed to authenticate messages whose length is a multiple of n bits.

In this section we show how to convert these PRFs into PRFs for messages of arbitrary bit length. That is, given a PRF for messages in $\mathcal{X}^{\leq \ell}$ we construct a PRF for messages in $\{0,1\}^{\leq n\ell}$.

Let F be a PRF taking inputs in $\mathcal{X}^{\leq \ell+1}$. Let $inj : \{0,1\}^{\leq n\ell} \to \mathcal{X}^{\leq \ell+1}$ be an injective (i.e., one-to-one) function. Define the derived PRF F_{bit} as

$$F_{\mathrm{bit}}(k, x) := F(k,\ inj(x)).$$

Then we obtain the following trivial theorem.

Theorem 6.10. *If F is a secure PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell+1}, \mathcal{Y})$ then F_{bit} is a secure PRF defined over $(\mathcal{K},\ \{0,1\}^{\leq n\ell},\ \mathcal{Y})$.*

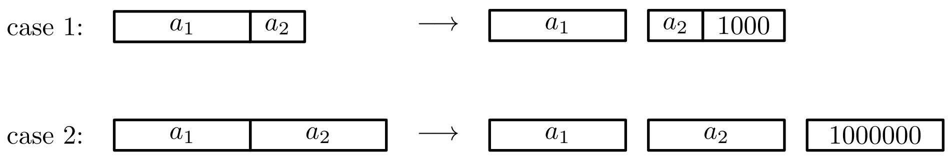


Figure 6.7: An injective function $inj : \{0,1\}^{\leq n\ell} \to \mathcal{X}^{\leq \ell+1}$

An injective function. For $\mathcal{X} := \{0,1\}^n$, a standard example of an injective inj from $\{0,1\}^{\leq n\ell}$ to $\mathcal{X}^{\leq \ell+1}$ works as follows. If the input message length is not a multiple of n then inj appends $100\ldots00$ to pad the message so its length is the next multiple of n. If the given message length is a multiple of n then inj appends an entire n-bit block $(1 \parallel 0^{n-1})$. Fig. 6.7 describes this in a picture. More precisely, the function works as follows:

input: $m \in \{0,1\}^{\leq n\ell}$
$u \leftarrow |m| \bmod n, \quad m' \leftarrow m \parallel 1 \parallel 0^{n-u-1}$
output m' as a sequence of n-bit message blocks

To see that inj is injective we show that it is invertible. Given $y \leftarrow inj(m)$ scan y from right to left and remove all the 0s until and including the first 1. The remaining string is m.

A common mistake is to pad the given message to a multiple of a block size using an all-0 pad. This pad is not injective and results in an insecure MAC: for any message m whose length is not a multiple of the block length, the MAC on m is also a valid MAC for $m \parallel 0$. Consequently, the MAC is vulnerable to existential forgery.

Injective functions must expand. When we feed an n-bit single block message into inj, the function adds a "dummy" block and outputs a two-block message. This is unfortunate for applications that MAC many single block messages. When using CBC or cascade, the dummy block forces the signer and verifier to evaluate the underlying PRF twice for each message, even though all messages are one block long. Consequently, inj forces all parties to work twice as hard as necessary.

It is natural to look for injective functions from $\{0,1\}^{\leq n\ell}$ to $\mathcal{X}^{\leq \ell}$ that never add dummy blocks. Unfortunately, there are no such functions simply because the set $\{0,1\}^{\leq n\ell}$ is larger than the set $\mathcal{X}^{\leq \ell}$. Hence, all injective functions must occasionally add a "dummy" block to the output.

The CMAC construction described in Section 6.10 provides an elegant solution to this problem. CMAC avoids adding dummy blocks by using a *randomized* injective function.

6.9 Case study: ANSI CBC-MAC

When building a MAC from a PRF, implementors often shorten the final tag by only outputting the w most significant bits of the PRF output. Exercise 4.4 shows that truncating a secure PRF has no effect on its security as a PRF. Truncation, however, affects the derived MAC. Theorem 6.2 shows that the smaller w is the less secure the MAC becomes. In particular, the theorem adds a $1/2^w$ error in the concrete security bounds.

Two ANSI standards (ANSI X9.9 and ANSI X9.19) and two ISO standards (ISO 8731-1 and ISO/IEC 9797) specify variants of ECBC for message authentication using DES as the underlying

PRF. These standards truncate the final 64-bit output of the ECBC-DES and use only the leftmost w bits of the output, where $w = 32, 48$, or 64 bits. This reduces the tag length at the cost of reduced security.

Both ANSI CBC-MAC standards specify a padding scheme to be used for messages whose length is not a multiple of the DES or AES block size. The padding scheme is identical to the function *inj* described in Section 6.8. The same padding scheme is used when signing a message and when verifying a message-tag pair.

6.10 Case study: CMAC

Cipher-based MAC — CMAC — is a variant of ECBC adopted by the National Institute of Standards (NIST) in 2005. It is based on a proposal due to Black and Rogaway and an extension due to Iwata and Kurosawa. CMAC improves over ECBC used in the ANSI standard in two ways. First, CMAC uses a randomized prefix-free encoding to convert a prefix-free secure PRF to a secure PRF. This saves the final encryption used in ECBC. Second, CMAC uses a "two key" method to avoid appending a dummy message block when the input message length is a multiple of the underlying PRF block size.

CMAC is the best approach to building a bit-wise secure PRF from the CBC prefix-free secure PRF. It should be used in place of the ANSI method. In Exercise 6.14 we show that the CMAC construction applies equally well to cascade.

The CMAC bit-wise PRF. The CMAC algorithm consists of two steps. First, a sub-key generation algorithm is used to derive three keys k_0, k_1, k_2 from the MAC key k. Then the three keys k_0, k_1, k_2 are used to compute the MAC.

Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$ where $\mathcal{X} = \{0,1\}^n$. The NIST standard uses AES as the PRF F. The CMAC signing algorithm is given in Table 6.1 and is illustrated in Fig. 6.8. The figure on the left is used when the message length is a multiple of the block size n. The figure on the right is used otherwise. The standard allows for truncating the final output to w bits by only outputting the w most significant bits of the final value t.

Security. The CMAC algorithm described in Fig. 6.8 can be analyzed using the randomized prefix-free encoding paradigm. In effect, CMAC converts the CBC prefix-free secure PRF directly into a *bit-wise* secure PRF using a randomized prefix-free encoding $rpf : \mathcal{K} \times \mathcal{M} \to \mathcal{X}_{>0}^{\leq \ell}$ where $\mathcal{K} := \mathcal{X}^2$ and $\mathcal{M} := \{0,1\}^{\leq n\ell}$. The encoding rpf is defined as follows:

> input: $m \in \mathcal{M}$ and $(k_1, k_2) \in \mathcal{X}^2$
>
> if $|m|$ is not a positive multiple of n then
> $u \leftarrow |m| \bmod n$
> partition m into a sequence of bit strings $a_1, \ldots, a_v \in \mathcal{X}$,
>   so that $m = a_1 \parallel \cdots \parallel a_v$ and $a_1, \ldots, a_{v-1}$ are n-bit strings
>
> if $|m|$ is a positive multiple of n
>   then output $\big(a_1,\ \ldots,\ a_{v-1},\ (a_v \oplus k_1)\big)$
>   else output $\big(a_1,\ \ldots,\ a_{v-1},\ ((a_v \parallel 1 \parallel 0^{n-u-1}) \oplus k_2)\big)$

The argument that rpf is a randomized 2^{-n}-prefix-free encoding is similar to the one is Section 6.7. Hence, CMAC fits the randomized prefix-free encoding paradigm and its security follows from

input: Key $k \in \mathcal{K}$ and $m \in \{0,1\}^*$
output: tag $t \in \{0,1\}^w$ for some $w \leq n$

<u>Setup:</u>

Run a sub-key generation algorithm
to generate keys $k_0, k_1, k_2 \in \mathcal{X}$ from $k \in \mathcal{K}$
$\ell \leftarrow \text{length}(m)$
$u \leftarrow \max(1, \lceil \ell/n \rceil)$
Break m into consecutive n-bit blocks so that
$m = a_1 \parallel a_2 \parallel \cdots \parallel a_{u-1} \parallel a_u^*$ where $a_1, \ldots, a_{u-1} \in \{0,1\}^n$.
$(*)$ If $\text{length}(a_u^*) = n$
then $a_u = k_1 \oplus a_u^*$
else $a_u = k_2 \oplus (a_u^* \parallel 1 \parallel 0^j)$ where $j = nu - \ell - 1$

<u>CBC:</u>

$t \leftarrow 0^n$
for $i \leftarrow 1$ to u do:
$t \leftarrow F(k_0, \;\; t \oplus a_i)$
Output $t[0 \ldots w-1]$ // *Output w most significant bits of t.*

Table 6.1: CMAC signing algorithm

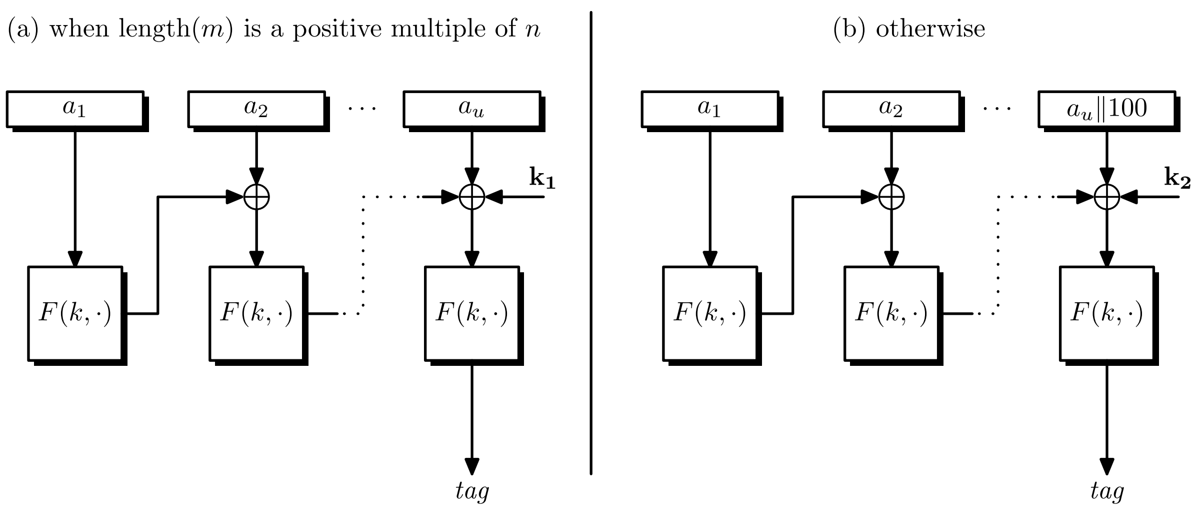


Figure 6.8: CMAC signing algorithm

Theorem 6.9. The keys k_1, k_2 are used to resolve collisions between a message whose length is a positive multiple of n and a message that has been padded to make it a positive multiple of n. This is essential for the analysis of the CMAC *rpf*.

Sub-key generation. The sub-key generation algorithm generates the keys (k_0, k_1, k_2) from k. It uses a fixed mask string R_n that depends on the block size of F. For example, for a 128-bit block size, the standard specifies $R_{128} := 0^{120}10000111$. For a bit string X we denote by $X << 1$ the bit string that results from discarding the leftmost bit X and appending a 0-bit on the right. The sub-key generation algorithm works as follows:

input: key $k \in \mathcal{K}$
output: keys $k_0, k_1, k_2 \in \mathcal{X}$

$k_0 \leftarrow k$
$L \leftarrow F(k, 0^n)$
(1) if $\text{msb}(L) = 0$ then $k_1 \leftarrow (L << 1)$ else $k_1 \leftarrow (L << 1) \oplus R_n$
(2) if $\text{msb}(k_1) = 0$ then $k_2 \leftarrow (k_1 << 1)$ else $k_2 \leftarrow (k_1 << 1) \oplus R_n$
output k_0, k_1, k_2.

where $\text{msb}(L)$ refers to the most significant bit of L. The lines marked (1) and (2) may look a bit mysterious, but in effect, they simply multiply L by x and by x^2 (respectively) in the finite field $\text{GF}(2^n)$. For a 128-bit block size the defining polynomial for $\text{GF}(2^{128})$ corresponding to R_{128} is $g(X) := X^{128} + X^7 + X^2 + X + 1$. Exercise 6.16 explores insecure variants of sub-key generation.

The three keys (k_0, k_1, k_2) output by the sub-key generation algorithm can be used for authenticating multiple messages. Hence, its running time is amortized across many messages.

Clearly the keys k_0, k_1, and k_2 are not independent. If they were, or if they were derived as, say, $k_i := F(k, \alpha_i)$ for constants $\alpha_0, \alpha_1, \alpha_2$, the security of CMAC would follow directly from the arguments made here and our general framework. Nevertheless, a more intricate analysis allows one to prove that CMAC is indeed secure [58].

6.11 PMAC: a parallel MAC

The MACs we developed so far, ECBC, CMAC, and NMAC, are inherently sequential: block number i cannot be processed before block number $i-1$ is finished. This makes it difficult to exploit hardware parallelism or pipelining to speed up MAC generation and verification. In this section we construct a secure MAC that is well suited for a parallel architecture. The best construction is called PMAC. We present PMAC_0 which is a little easier to describe.

Let F_1 be a PRF defined over $(\mathcal{K}_1, \mathbb{Z}_p, \mathcal{Y})$, where p is a prime and $\mathcal{Y} := \{0,1\}^n$. Let F_2 be a PRF defined over $(\mathcal{K}_2, \mathcal{Y}, \mathcal{Z})$.

We build a new PRF, called PMAC_0, which takes as input a key and a message in $\mathbb{Z}_p^{\leq \ell}$ for some ℓ. It outputs a value in $\mathcal{Z}$. A key for PMAC_0 consists of $k \in \mathbb{Z}_p$, $k_1 \in \mathcal{K}_2$, and $k_2 \in \mathcal{K}_2$. The PMAC_0 construction works as follows:

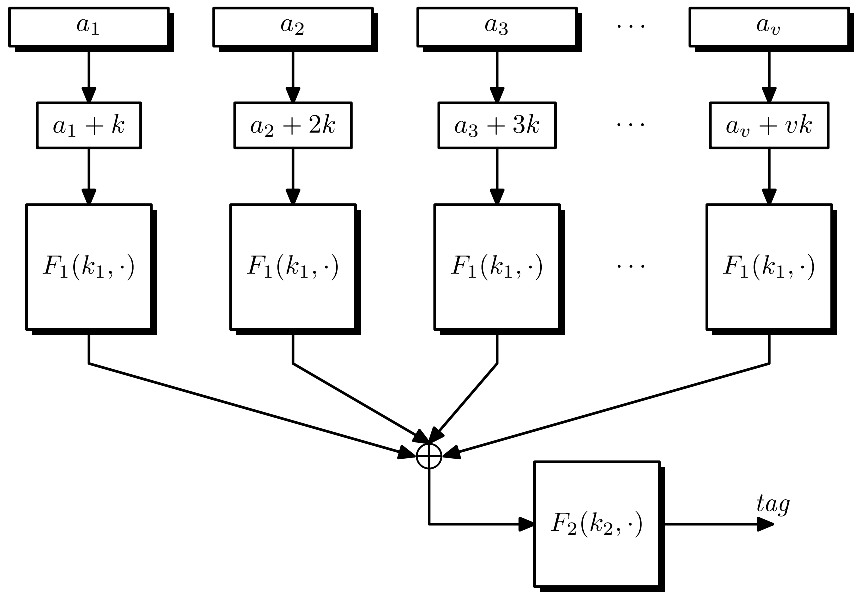


Figure 6.9: PMAC_0 construction

input: $m = (a_1, \ldots, a_v) \in \mathbb{Z}_p^v$ for some $0 \le v \le \ell$, and
key $\vec{k} = (k, k_1, k_2)$ where $k \in \mathbb{Z}_p$, $k_1 \in \mathcal{K}_1$, and $k_2 \in \mathcal{K}_2$
output: tag in $\mathcal{Z}$

$\text{PMAC}_0(\vec{k}, m)$:
  $t \leftarrow 0^n \in \mathcal{Y}$, $mask \leftarrow 0 \in \mathbb{Z}_p$
  for $i \leftarrow 1$ to v do:
    $mask \leftarrow mask + k$ // $mask = i \cdot k \in \mathbb{Z}_p$
    $r \leftarrow a_i + mask$ // $r = a_i + i \cdot k \in \mathbb{Z}_p$
    $t \leftarrow t \oplus F_1\big(k_1, r\big)$
  output $F_2(k_2,\ t)$

The main loop adds the masks $k, 2k, 3k, \ldots$ to message blocks prior to evaluating the PRF F_1. On a sequential machine this requires two additions modulo p per iteration. On a parallel machine each processor can independently compute $a_i + ik$ and then apply F_1. See Fig. 6.9.

PMAC_0 is a secure PRF and hence gives a secure MAC for large messages. The proof will follow easily from Theorem 7.7 developed in the next chapter. For now we state the theorem and delay its proof to Section 7.3.3.

Theorem 6.11. *If F_1 and F_2 are secure PRFs, and $|\mathcal{Y}|$ and the prime p are super-poly, then* PMAC_0 *is a secure PRF for any poly-bounded ℓ.*

In particular, for every PRF adversary $\mathcal{A}$ that attacks PMAC_0 *as in Attack Game 4.2, and issues at most Q queries, there exist PRF adversaries $\mathcal{B}_1$ and $\mathcal{B}_2$, which are elementary wrappers around $\mathcal{A}$, such that*

$$\text{PRF}\mathsf{adv}[\mathcal{A}, \text{PMAC}_0] \le \text{PRF}\mathsf{adv}[\mathcal{B}_1, F_1] + \text{PRF}\mathsf{adv}[\mathcal{B}_2, F_2] + \frac{Q^2}{2|\mathcal{Y}|} + \frac{Q^2\ell^2}{2p}. \tag{6.27}$$

When using PMAC_0, the input message must be partitioned into blocks, where each block is an element of $\mathbb{Z}_p$. In practice, that is inconvenient. It is much easier to break the message into blocks, where each block is an n-bit string in $\{0,1\}^n$, for some n. A better parallel MAC construction, presented next, does exactly that by using the finite field $\text{GF}(2^n)$ instead of $\mathbb{Z}_p$. This is a good illustration for why $\text{GF}(2^n)$ is so useful in cryptography. We often need to work in a field for security reasons, but a prime finite field like $\mathbb{Z}_p$ is inconvenient to use in practice. Instead, we use $\text{GF}(2^n)$ where arithmetic operations are much faster. $\text{GF}(2^n)$ also lets us naturally operate on n-bit blocks.

PMAC: better than PMAC_0**.** Although PMAC_0 is well suited for a parallel architecture, there is room for improvement. Better implementations of the PMAC_0 approach are available. Examples include PMAC [16] and XECB [47], both of which are parallelizable. PMAC, for example, provides the following improvements over PMAC_0:

- PMAC uses arithmetic in the finite field $\text{GF}(2^n)$ instead of in $\mathbb{Z}_p$. Elements of $\text{GF}(2^n)$ can be represented as n-bit strings, and addition in $\text{GF}(2^n)$ is just a bit-wise XOR. Because of this, PMAC just uses $F_1 = F_2 = F$, where F is a PRF defined over $(\mathcal{K}, \mathcal{Y}, \mathcal{Y})$, and the input space of PMAC consists of sequences of elements of $\mathcal{Y} = \{0,1\}^n$, rather than elements of $\mathbb{Z}_p$.

- The PMAC mask for block i is defined as $\gamma_i \cdot k$ where $\gamma_1, \gamma_2, \ldots$ are fixed constants in $\text{GF}(2^n)$ and multiplication is defined in $\text{GF}(2^n)$. The γ_i's are specially chosen so that computing $\gamma_{i+1} \cdot k$ from $\gamma_i \cdot k$ is very cheap.

- PMAC derives the key k as $k \leftarrow F(k_1, 0^n)$ and sets $k_2 \leftarrow k_1$. Hence PMAC uses a shorter secret key than PMAC_0.

- PMAC uses a trick to save one application of F.

- PMAC uses a variant of the CMAC *rpf* to provide a bit-wise PRF.

The end result is that PMAC is as efficient as ECBC and NMAC on a sequential machine, but has much better performance on a parallel or pipelined architecture. PMAC is the best PRF construction in this chapter; it works well on a variety of computer architectures and is efficient for both long and short messages.

PMAC_0 **is incremental.** Suppose Bob computes the tag t for some long message m. Some time later he changes one block in m and wants to recompute the tag of this new message m'. When using CBC-MAC the tag t is of no help — Bob must recompute the tag for m' from scratch. With PMAC_0 we can do much better. Suppose the PRF F_2 used in the construction of PMAC_0 is the encryption algorithm of a block cipher such as AES, and let D be the corresponding decryption algorithm. Let m' be the result of changing block number i of m from a_i to a'_i. Then the tag $t' := \text{PMAC}_0(k, m')$ for m' can be easily derived from the tag $t := \text{PMAC}_0(k, m)$ for m as follows:

$$
\begin{aligned}
&t_1 \leftarrow D(k_2, t) \\
&t_2 \leftarrow t_1 \;\oplus\; F_1(k_1,\ a_i + ik) \;\oplus\; F_1(k_1,\ a'_i + ik) \\
&t' \leftarrow F_2(k_2, t_2)
\end{aligned}
$$

Hence, given the tag on some long message m (as well as the MAC secret key) it is easy to derive tags for local edits of m. MACs that have this property are said to be **incremental**. We just showed that the PMAC_0, implemented using a block cipher, is incremental.

6.12 A fun application: searching on encrypted data

To be written.

6.13 Notes

Citations to the literature to be added.

6.14 Exercises

6.1 (The 802.11b insecure MAC). Consider the following MAC (a variant of this was used for WiFi encryption in 802.11b WEP). Let F be a PRF defined over $(\mathcal{K}, \mathcal{R}, \mathcal{X})$ where $\mathcal{X} := \{0,1\}^{32}$. Let CRC32 be a simple and popular error-detecting code meant to detect random errors; CRC32(m) takes inputs $m \in \{0,1\}^{\leq \ell}$ and always outputs a 32-bit string. For this exercise, the only fact you need to know is that $\text{CRC32}(m_1) \oplus \text{CRC32}(m_2) = \text{CRC32}(m_1 \oplus m_2)$. Define the following MAC system (S, V):

$$S(k,m) := \{\ r \xleftarrow{\text{R}} \mathcal{R},\ t \leftarrow F(k,r) \oplus \text{CRC32}(m),\ \text{output } (r,t)\ \}$$
$$V(k,m,(r,t)) := \{\ \textsf{accept} \text{ if } t = F(k,r) \oplus \text{CRC32}(m) \text{ and } \textsf{reject} \text{ otherwise}\}$$

Show that this MAC system is insecure.

6.2 (Tighter bounds with verification queries). Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, and let $\mathcal{I}$ be the MAC system derived from F, as discussed in Section 6.3. Let $\mathcal{A}$ be an adversary that attacks $\mathcal{I}$ as in Attack Game 6.2, and which makes at most Q_v verification queries and at most Q_s signing queries. Theorem 6.1 says that there exists a Q_s-query MAC adversary $\mathcal{B}$ that attacks $\mathcal{I}$ as in Attack Game 6.1, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that $\text{MAC}^{\text{vq}}\textsf{adv}[\mathcal{A}, \mathcal{I}] \leq \text{MAC}\textsf{adv}[\mathcal{B}, \mathcal{I}] \cdot Q_\text{v}$. Theorem 6.2 says that there exists a $(Q_\text{s}+1)$-query PRF adversary $\mathcal{B}'$ that attacks F as in Attack Game 4.2, where $\mathcal{B}'$ is an elementary wrapper around $\mathcal{B}$, such that $\text{MAC}\textsf{adv}[\mathcal{B}, \mathcal{I}] \leq \text{PRF}\textsf{adv}[\mathcal{B}', F] + 1/|\mathcal{Y}|$. Putting these two statements together, we get

$$\text{MAC}^{\text{vq}}\textsf{adv}[\mathcal{A}, \mathcal{I}] \leq (\text{PRF}\textsf{adv}[\mathcal{B}', F] + 1/|\mathcal{Y}|) \cdot Q_\text{v}$$

This bound is not the best possible. Give a direct analysis that shows that there exists a $(Q_\text{s}+Q_\text{v})$-query PRF adversary $\mathcal{B}''$, where $\mathcal{B}''$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{MAC}^{\text{vq}}\textsf{adv}[\mathcal{A}, \mathcal{I}] \leq \text{PRF}\textsf{adv}[\mathcal{B}'', F] + Q_\text{v}/|\mathcal{Y}|.$$

6.3 (Multi-key MAC security). Just as we did for semantically secure encryption in Exercise 5.2, we can extend the definition of a secure MAC from the single-key setting to the multi-key setting. In this exercise, you will show that security in the single-key setting implies security in the multi-key setting.

(a) Show how to generalize Attack Game 6.2 so that an attacker can submit both signing queries and verification queries with respect to several MAC keys $k_1, \ldots, k_Q$. At the beginning of the game the adversary outputs a number Q indicating the number of keys it wants to attack and the challenger chooses Q random keys $k_1, \ldots, k_Q$. Subsequently, every query from the attacker includes an index $j \in \{1, \ldots, Q\}$. The challenger uses the key k_j to respond to the query.

(b) Show that every efficient adversary $\mathcal{A}$ that wins your multi-key MAC attack game with probability ϵ can be transformed into an efficient adversary $\mathcal{B}$ that wins Attack Game 6.2 with probability ϵ/Q.

Hint: This is *not* done using a hybrid argument, but rather a "guessing" argument, somewhat analogous to that used in the proof of Theorem 6.1. Adversary $\mathcal{B}$ plays the role of challenger to adversary $\mathcal{A}$. Once $\mathcal{A}$ outputs a number Q, $\mathcal{B}$ chooses Q random keys $k_1, \ldots, k_Q$ and a random index $\omega \in \{1, \ldots, Q\}$. When $\mathcal{A}$ issues a query for key number $j \neq \omega$, adversary $\mathcal{B}$ uses its key k_j to answer the query. When $\mathcal{A}$ issues a query for the key k_ω, adversary $\mathcal{B}$ answers the query by querying its MAC challenger. If $\mathcal{A}$ outputs a forgery under key k_ω then $\mathcal{B}$ wins the MAC forgery game. Show that $\mathcal{B}$ wins Attack Game 6.2 with probability ϵ/Q. We call this style of argument "plug-and-pray:" $\mathcal{B}$ "plugs" the key he is challenged on at a random index ω and "prays" that $\mathcal{A}$ uses the key at index ω to form his existential forgery.

6.4 (Multicast MACs). Consider a scenario in which Alice wants to broadcast the same message to n users, $U_1, \ldots, U_n$. She wants the users to be able to authenticate that the message came from her, but she is not concerned about message secrecy. More generally, Alice may wish to broadcast a series of messages, but for this exercise, let us focus on just a single message.

(a) In the most trivial solution, Alice shares a MAC key k_i with each user U_i. When she broadcasts a message m, she appends tags $t_1, \ldots, t_n$ to the message, where t_i is a valid tag for m under key k_i. Using its shared key k_i, every user U_i can verify m's authenticity by verifying that t_i is a valid tag for m under k_i.

Assuming the MAC is secure, show that this broadcast authentication scheme is secure *even if users collude*. For example, users $U_1, \ldots, U_{n-1}$ may collude, sharing their keys $k_1, \ldots, k_{n-1}$ among each other, to try to make user U_n accept a message that is not authentic.

(b) While the above broadcast authentication scheme is secure, even in the presence of collisions, it is not very efficient; the number of keys and tags grows linearly in n.

Here is a more efficient scheme, but with a weaker security guarantee. We illustrate it with $n = 6$. The goal is to get by with $\ell < 6$ keys and tags. We will use just $\ell = 4$ keys, $k_1, \ldots, k_4$. Alice stores all four of these keys. There are $6 = \binom{4}{2}$ subsets of $\{1, \ldots, 4\}$ of size 2. Let us number these subsets $J_1, \ldots, J_6$. For each user U_i, if $J_i = \{v, w\}$, then this user stores keys k_v and k_w.

When Alice broadcasts a message m, she appends tags $t_1, \ldots, t_4$, corresponding to keys $k_1, \ldots, k_4$. Each user U_i verifies tags t_u and t_v, using its keys k_u, k_v, where $J_i = \{v, w\}$ as above.

Assuming the MAC is secure, show that this broadcast authentication scheme is secure *provided no two users collude*. For example, using the keys that he has, user U_1 may attempt to trick user U_6 into accepting an inauthentic message, but users U_1 and U_2 may not collude and share their keys in such an attempt.

(c) Show that the scheme presented in part (b) is completely insecure if two users are allowed to collude.

6.5 (MAC combiners). We want to build a MAC system $\mathcal{I}$ using two MAC systems $\mathcal{I}_1 = (S_1, V_1)$ and $\mathcal{I}_2 = (S_2, V_2)$, so that if at some time one of $\mathcal{I}_1$ or $\mathcal{I}_2$ is broken (but not both) then $\mathcal{I}$ is still secure. Put another way, we want to construct $\mathcal{I}$ from $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}$ is secure if either $\mathcal{I}_1$ or $\mathcal{I}_2$ is secure.

(a) Define $\mathcal{I} = (S, V)$, where

$$S(\ (k_1, k_2),\ m) := (\ S_1(k_1, m),\ S_2(k_2, m)\ ),$$

and V is defined in the obvious way: on input $(k, m, (t_1, t_2))$, V accepts iff both $V_1(k_1, m, t_1)$ and $V_2(k_2, m, t_2)$ accept. Show that $\mathcal{I}$ is secure if either $\mathcal{I}_1$ or $\mathcal{I}_2$ is secure.

(b) Suppose that $\mathcal{I}_1$ and $\mathcal{I}_2$ are deterministic MAC systems (see the definition on page 211), and that both have tag space $\{0,1\}^n$. Define the deterministic MAC system $\mathcal{I} = (S, V)$, where

$$S(\ (k_1, k_2),\ m) := S_1(k_1, m) \oplus S_2(k_2, m).$$

Show that $\mathcal{I}$ is secure if either $\mathcal{I}_1$ or $\mathcal{I}_2$ is secure.

6.6 (Concrete attacks on CBC and cascade). We develop attacks on F_{CBC} and F^* as prefix-free PRFs to show that for both security degrades quadratically with number of queries Q that the attacker makes. For simplicity, let us develop the attack when inputs are exactly three blocks long.

(a) Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{X})$ where $\mathcal{X} = \{0,1\}^n$, where $|\mathcal{X}|$ is super-poly. Consider the F_{CBC} prefix-free PRF with input space $\mathcal{X}^3$. Suppose an adversary queries the challenger at points $(x_1, y_1, z),\ (x_2, y_2, z),\ \ldots (x_Q, y_Q, z)$, where the x_i's, the y_i's, and z are chosen randomly from $\mathcal{X}$. Show that if $Q \approx \sqrt{|\mathcal{X}|}$, the adversary can predict the PRF at a new point in $\mathcal{X}^3$ with probability at least 1/2.

(b) Show that a similar attack applies to the three-block cascade F^* prefix-free PRF built from a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{K})$. Assume $\mathcal{X} = \mathcal{K}$ and $|\mathcal{K}|$ is super-poly. After making $Q \approx \sqrt{|\mathcal{K}|}$ queries in $\mathcal{X}^3$, your adversary should be able to predict the PRF at a new point in $\mathcal{X}^3$ with probability at least 1/2.

6.7 (Weakly secure MACs). It is natural to define a weaker notion of security for a MAC in which we make it harder for the adversary to win; specifically, in order to win, the adversary must submit a valid tag on a new message. One can modify the winning condition in Attack Games 6.1 and 6.2 to reflect this weaker security notion. In Attack Game 6.1, this means that to win, in addition to being a valid pair, the adversary's candidate forgery pair (m, t) must satisfy the constraint that m is not among the signing queries. In Attack Game 6.2, this means that the adversary wins if the challenger ever responds to a verification query $(\hat{m}_j, \hat{t}_j)$ with accept, where $\hat{m}_j$ is not among the signing queries made prior to this verification query. These two modified attack games correspond to notions of security that we call *weak security without verification queries* and *weak security with verification queries*. Unfortunately, the analog of Theorem 6.1 does not hold relative to these weak security notions. In this exercise, you are to show this by giving an explicit counter-example. Assume the existence of a secure PRF (defined over any convenient input, output, and key spaces, of your choosing). Show how to "sabotage" this PRF to obtain a MAC that is weakly secure without verification queries but is not weakly secure with verification queries.

6.8 (Fixing CBC: a bad idea). We showed that CBC is a prefix-free secure PRF but not a secure PRF. We showed that prepending the length of the message makes CBC a secure PRF. Show that appending the length of the message prior to applying CBC does not make CBC a secure PRF.

6.9 (Fixing CBC: a really bad idea). To avoid extension attacks on CBC, one might be tempted to define a CBC-MAC with a *randomized* IV. This is a MAC with a probabilistic signing algorithm that on input $k \in \mathcal{K}$ and $(x_1, \ldots, x_v) \in \mathcal{X}^{\leq \ell}$, works as follows: choose $IV \in \mathcal{X}$ at random; output (IV, t), where $t := F_{\text{CBC}}(x_1 \oplus IV, x_2, \ldots, x_v)$. On input $(k, (x_1, \ldots, x_v), (IV, t))$, the verification algorithms tests if $t = F_{\text{CBC}}(x_1 \oplus IV, x_2, \ldots, x_v)$. Show that this MAC is completely insecure, and is not even a prefix-free secure PRF.

6.10 (Truncated CBC). Prove that truncating the output of CBC gives a secure PRF for variable length messages. More specifically, if CBC is instantiated with a block cipher that operates on n-bit blocks, and we truncate the output of CBC to $w < n$ bits, then this truncated version is a secure PRF on variable length inputs, provided $1/2^{n-w}$ is negligible.

Hint: Adapt the proof of Theorem 6.3.

6.11 (Truncated cascade). In the previous exercise, we saw that truncating the output of the CBC construction yields a secure PRF. In this exercise, you are to show that the same does *not* hold for the cascade construction, by giving an explicit counter-example. For your counter-example, you may assume a secure PRF F' (defined over any convenient input, output, and key spaces, of your choosing). Using F', construct another PRF F, such that (a) F is a secure PRF, but (b) the corresponding truncated version of F^* is not a secure PRF.

6.12 (Truncated cascade in the ideal cipher model). In the previous exercise, we saw that the truncated cascade may not be secure when instantiated with certain PRFs. However, in your counter-example, that PRF was constructed precisely to make cascade fail — intuitively, for "typical" PRFs, one would not expect this to happen. To substantiate this intuition, this exercise asks you prove that in the *ideal cipher model* (see Section 4.7), the cascade construction is a secure PRF. More precisely, if we model F as the encryption function of an *ideal cipher*, then the truncated version of F^* is a secure PRF. Here, you may assume that F operates on n-bit blocks and n-bit keys, and that the output of F^* is truncated to w bits, where $1/2^{n-w}$ is negligible.

6.13 (Non-adaptive attacks on CBC and cascade). This exercise examines whether variable length CBC and cascade are secure PRFs against *non-adaptive* adversaries, i.e., adversaries that make their queries all at once (see Exercise 4.6).

(a) Show that CBC is a secure PRF against non-adaptive adversaries, assuming the underlying function F is a PRF.

Hint: Adapt the proof of Theorem 6.3.

(b) Give a non-adaptive attack that breaks the security of cascade as a PRF, regardless of the choice of F.

6.14 (Generalized CMAC).

(a) Show that the CMAC *rpf* (Section 6.10) is a randomized 2^{-n}-prefix-free encoding.

(b) Use the CMAC *rpf* to convert cascade into a bit-wise secure PRF.

6.15 (A simple randomized prefix-free encoding). Show that appending a random message block gives a randomized prefix-free encoding. That is, the following function

$$rpf(k, m) = m \parallel k$$

is a randomized $1/|\mathcal{X}|$-prefix-free encoding. Here, $m \in \mathcal{X}^{\leq \ell}$ and $k \in \mathcal{X}$.

6.16 (An insecure variant of CMAC). Show that CMAC is insecure as a PRF if the sub-key generation algorithm outputs k_0 and k_2 as in the current algorithm, but sets $k_1 \leftarrow L$.

6.17 (Domain extension). This exercise explores some simple ideas for extending the domain of a MAC system that do not work. Let $\mathcal{I} = (S, V)$ be a deterministic MAC (see the definition on page 211), defined over $(\mathcal{K}, \mathcal{M}, \{0,1\}^n)$. Each of the following are signing algorithms for deterministic MACs with message space $\mathcal{M}^2$. You are to show that each of the resulting MACs are insecure.

(a) $S_1(k, (a_1, a_2)) = S(k, a_1) \parallel S(k, a_2)$,

(b) $S_2(k, (a_1, a_2)) = S(k, a_1) \oplus S(k, a_2)$,

(c) $S_3((k_1, k_2), (a_1, a_2)) = S(k_1, a_1) \parallel S(k_2, a_2)$,

(d) $S_4((k_1, k_2), (a_1, a_2)) = S(k_1, a_1) \oplus S(k_2, a_2)$.

6.18 (Integrity for database records). Let (S, V) be a secure MAC defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$. Consider a database containing records $m_1, \ldots, m_n \in \mathcal{M}$. To provide integrity for the data the data owner generates a random secret key $k \in \mathcal{K}$ and stores $t_i \leftarrow S(k, m_i)$ alongside record m_i for every $i = 1, \ldots, n$. This does not ensure integrity because an attacker can remove a record from the database or duplicate an old record without being detected. To prevent addition or removal of records the data owner generates another secret key $k' \in \mathcal{K}$ and computes $t \leftarrow S\big(k',\ (t_1, \ldots, t_n)\big)$ (we are assuming that $\mathcal{T}^n \subseteq \mathcal{M}$). She stores (k, k', t) on her own machine, away from the database.

(a) Show that updating a single record in the database can be done efficiently. That is, explain what needs to be done to recompute the tag t when a single record m_j in the database is replaced by an updated record m'_j.

(b) Does this approach ensure database integrity? Suppose the MAC (S, V) is built from a secure PRF F defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$ where $|\mathcal{T}|$ is super-poly. Show that the following PRF F_n is a secure PRF on message space $\mathcal{M}^n$

$$F_n\big((k, k'), (m_1, \ldots, m_n)\big) := F\big(k',\ \big(F(k, m_1), \ldots, F(k, m_n)\big)\big)\,.$$

6.19 (Timing attacks). Let (S, V) be a deterministic MAC system where tags $\mathcal{T}$ are n-bytes long. The verification algorithm $V(k, m, t)$ is implemented as follows: it first computes $t' \leftarrow S(k, m)$ and then does:

```
for i ← 0 to n − 1 do:
    if t[i] ≠ t'[i] output reject and exit
output accept
```

(a) Show that this implementation is vulnerable to a timing attack. An attacker who can submit arbitrary queries to algorithm V and accurately measure V's response time can forge a valid tag on every message m of its choice with at most $256 \cdot n$ queries to V.

(b) How would you implement V to prevent the timing attack from part (a)?

Chapter 7

Message integrity from universal hashing

In the previous chapter we showed how to build secure MACs from secure PRFs. In particular, we discussed the ECBC, NMAC, and PMAC constructions. We stated security theorems for these MACs, but delayed their proofs to this chapter.

In this chapter we describe a general paradigm for constructing MACs using hash functions. By a **hash function** we generally mean a function H that maps inputs in some large set $\mathcal{M}$ to short outputs in $\mathcal{T}$. Elements in $\mathcal{T}$ are often called **message digests** or just digests. Keyed hash functions, used throughout this chapter, also take as input a key k.

At a high level, MACs constructed from hash functions work in two steps. First, we use the hash function to hash the message m to a short digest t. Second, we apply a PRF to the digest t, as shown in Fig. 7.1.

As we will see, ECBC, NMAC, and PMAC_0 are instances of this "hash-then-PRF" paradigm. For example, for ECBC (described in Fig. 6.5a), the CBC function acts as a hash function that hashes long input messages into short digests. The final application of the PRF using the key k_2 is the final PRF step. The hash-then-PRF paradigm will enable us to directly and quite easily deduce the security of ECBC, NMAC, and PMAC_0.

The hash-then-PRF paradigm is very general and enables us to build new MACs out of a wide variety of hash functions. Some of these hash functions are very fast, and yield MACs that are more efficient than those discussed in the previous chapter.

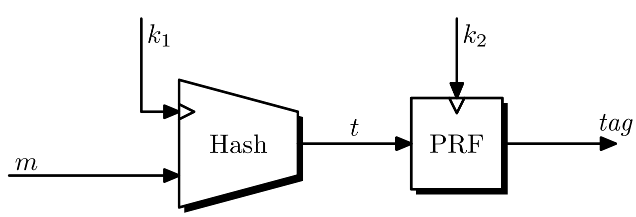


Figure 7.1: The hash-then-PRF paradigm

7.1 Universal hash functions (UHFs)

We begin our discussion by defining a **keyed hash function** — a widely used tool in cryptography. A keyed hash function H takes two inputs: a key k and a message m. It outputs a short digest $t := H(k, m)$. The key k can be thought of as a hash function selector: for every k we obtain a specific function $H(k, \cdot)$ from messages to digests. More precisely, keyed hash functions are defined as follows:

Definition 7.1 (Keyed hash functions). *A* ***keyed hash function*** *H is a deterministic algorithm that takes two inputs, a* ***key*** *k and a* ***message*** *m; its output $t := H(k, x)$ is called a* ***digest****. As usual, there are associated spaces: the keyspace $\mathcal{K}$, in which k lies, a message space $\mathcal{M}$, in which m lies, and the digest space $\mathcal{T}$, in which t lies. We say that the hash function H is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$.*

We note that the output digest $t \in \mathcal{T}$ can be much shorter than the input message m. Typically digests will have some fixed size, say 128 or 256 bits, independent of the input message length. A hash function $H(k, \cdot)$ can map gigabyte long messages into just 256-bit digests.

We say that two messages $m_0, m_1 \in \mathcal{M}$ form a **collision for H under key** $k \in \mathcal{K}$ if

$$H(k,\ m_0) = H(k,\ m_1) \qquad \text{and} \qquad m_0 \neq m_1.$$

Since the digest space $\mathcal{T}$ is typically much smaller than the message space $\mathcal{M}$, many such collisions exist. However, a general property we shall desire in a hash function is that it is hard to actually *find* a collision. As we shall eventually see, there are a number of ways to formulate this "collision resistance" property. These formulations differ in subtle ways in how much information about the key an adversary gets in trying to find a collision. In this chapter, we focus on the weakest formulation of this collision resistance property, in which the adversary must find a collision with *no information about the key at all.* On the one hand, this property is weak enough that we can actually build very efficient hash functions that satisfy this property without making any assumptions at all on the computational power of the adversary. On the other hand, this property is strong enough to ensure that the hash-then-PRF paradigm yields a secure MAC.

Hash functions that satisfy this very weak collision resistance property are called **universal hash functions**, or **UHFs**. Universal hash functions are used in various branches of computer science, most notably for the construction of efficient hash tables. UHFs are also widely used in cryptography. Before we can analyze the security of the hash-then-PRF paradigm, we first give a more formal definition of UHFs. As usual, to make this intuitive notion more precise, we define an attack game.

Attack Game 7.1 (universal hash function). For a keyed hash function H defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, and a given adversary $\mathcal{A}$, the attack game runs as follows.

- The challenger picks a random $k \xleftarrow{R} \mathcal{K}$ and keeps k to itself.
- $\mathcal{A}$ outputs two distinct messages $m_0, m_1 \in \mathcal{M}$.

We say that $\mathcal{A}$ wins the above game if $H(k, m_0) = H(k, m_1)$. We define $\mathcal{A}$'s advantage with respect to H, denoted $\text{UHF}\mathsf{adv}[\mathcal{A}, H]$, as the probability that $\mathcal{A}$ wins the game. □

We now define several different notions of UHF, which depend on the power of the adversary and its advantage in the above attack game.

Definition 7.2. *Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$,*

- *We say that H is an ϵ-**bounded universal hash function**, or ϵ-**UHF**, if* $\text{UHF}\mathsf{adv}[\mathcal{A}, H] \le \epsilon$ *for all adversaries $\mathcal{A}$ (even inefficient ones).*
- *We say that H is a **statistical UHF** if it is an ϵ-UHF for some negligible ϵ.*
- *We say that H is a **computational UHF** if* $\text{UHF}\mathsf{adv}[\mathcal{A}, H]$ *is negligible for all efficient adversaries $\mathcal{A}$.*

Statistical UHFs are secure against all adversaries, efficient or not: no adversary can win Attack Game 7.1 against a statistical UHF with non-negligible advantage. The main reason that we consider computationally unbounded adversaries is that we *can*: unlike most other security notions we discuss in this text, good UHFs are something we know how to build without any computational restrictions on the adversary. Note that every statistical UHF is also a computational UHF, but the converse is not true.

If H is a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, an alternative characterization of the ϵ-UHF property is the following (see Exercise 7.3):

for every pair of distinct messages $m_0, m_1 \in \mathcal{M}$ we have $\Pr[H(k, m_0) = H(k, m_1)] \le \epsilon$, where the probability is over the random choice of $k \in \mathcal{K}$. (7.1)

7.1.1 Multi-query UHFs

It will be convenient to consider a generalization of a computational UHF. Here the adversary wins if he can output a list of distinct messages so that some pair of messages in the list is a collision for $H(k, \cdot)$. The point is that although the adversary may not know exactly which pair of messages in his list cause the collision, he still wins the game. In more detail, a multi-query UHF is defined using the following game:

Attack Game 7.2 (multi-query UHF). For a keyed hash function H over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, and a given adversary $\mathcal{A}$, the attack game runs as follows.

- The challenger picks a random $k \xleftarrow{\text{R}} \mathcal{K}$ and keeps k to itself.
- $\mathcal{A}$ outputs distinct messages $m_1, \ldots, m_s \in \mathcal{M}$.

We say that $\mathcal{A}$ wins the above game if there are indices $i \ne j$ such that $H(k, m_i) = H(k, m_j)$. We define $\mathcal{A}$'s advantage with respect to H, denoted $\text{MUHF}\mathsf{adv}[\mathcal{A}, H]$, as the probability that $\mathcal{A}$ wins the game. We call $\mathcal{A}$ a **Q-query UHF adversary** if it always outputs a list of size $s \le Q$. □

Definition 7.3. *We say that a hash function H over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$ is a **multi-query UHF** if for all efficient adversaries $\mathcal{A}$, the quantity* $\text{MUHF}\mathsf{adv}[\mathcal{A}, H]$ *is negligible.*

Lemma 7.1 below shows that any UHF is also a multi-query UHF. However, for particular constructions, we can sometimes get better security bounds.

Lemma 7.1. *If H is a computational UHF, then it is also a multi-query UHF.*

In particular, for every Q-query UHF adversary $\mathcal{A}$, there exists a UHF adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\text{MUHF}\mathsf{adv}[\mathcal{A}, H] \le (Q^2/2) \cdot \text{UHF}\mathsf{adv}[\mathcal{B}, H]. \tag{7.2}$$

Proof. The UHF adversary $\mathcal{B}$ runs $\mathcal{A}$ and obtains $s \le Q$ distinct messages $m_1, \ldots, m_s$. It randomly picks a random pair of distinct indices i and j from $\{1, \ldots, s\}$, and outputs m_i and m_j. The list generated by $\mathcal{A}$ contains a collision for $H(k, \cdot)$ with probability MUHF$\mathsf{adv}[\mathcal{A}, H]$ and $\mathcal{B}$ will choose a colliding pair with probability at least $2/Q^2$. Hence, UHF$\mathsf{adv}[\mathcal{B}, H]$ is at least MUHF$\mathsf{adv}[\mathcal{A}, H] \cdot (2/Q^2)$, as required. □

7.1.2 Mathematical details

As usual, we give a more mathematically precise definition of a UHF using the terminology defined in Section 2.4.

Definition 7.4 (Keyed hash functions). *A* ***keyed hash function*** *is an efficient algorithm H, along with three families of spaces with system parameterization P:*

$$\mathbf{K} = \{\mathcal{K}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \mathbf{M} = \{\mathcal{M}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \textit{and} \quad \mathbf{T} = \{\mathcal{T}_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$

such that

1. $\mathbf{K}$*,* $\mathbf{M}$*, and* $\mathbf{T}$ *are efficiently recognizable.*
2. $\mathbf{K}$ *and* $\mathbf{T}$ *are efficiently sampleable.*
3. *Algorithm H is an efficient deterministic algorithm that on input $\lambda \in \mathbb{Z}_{\ge 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $k \in \mathcal{K}_{\lambda,\Lambda}$, and $m \in \mathcal{M}_{\lambda,\Lambda}$, outputs an element of $\mathcal{T}_{\lambda,\Lambda}$.*

In defining UHFs we parameterize Attack Game 7.1 by the security parameter λ. The advantage UHF$\mathsf{adv}[\mathcal{A}, H]$ is then a function of λ.

The information-theoretic property (7.1) is the more traditional approach in the literature in defining ϵ-UHFs for individual hash functions with no security or system parameters; in our asymptotic setting, if property (7.1) holds for each setting of the security and system parameters, then our definition of an ϵ-UHF will certainly be satisfied.

7.2 Constructing UHFs

The challenge in constructing good universal hash functions (UHFs) is to construct a function that achieves a small collision probability using a short key. Preferably, the size of the key should not depend on the length of the message being hashed. We give three constructions. The first is an elegant construction of a *statistical* UHF using modular arithmetic and polynomials. Our second construction is based on the CBC and cascade functions defined in Section 6.4. We show that both are *computational* UHFs. The third construction is based on PMAC$_0$ from Section 6.11.

7.2.1 Construction 1: UHFs using polynomials

We start with a UHF construction using polynomials modulo a prime. Let ℓ be a (poly-bounded) length parameter and let p be a prime. We define a hash function H_{poly} that hashes a message $m \in \mathbb{Z}_p^{\le \ell}$ to a single element $t \in \mathbb{Z}_p$. The key space is $\mathcal{K} := \mathbb{Z}_p$.

Let m be a message, so $m = (a_1, a_2, \ldots, a_v) \in \mathbb{Z}_p^{\le \ell}$ for some $0 \le v \le \ell$. Let $k \in \mathbb{Z}_p$ be a key. The hash function $H_{\mathrm{poly}}(k, m)$ is defined as follows:

$$H_{\mathrm{poly}}\big(k,\ (a_1, \ldots, a_v)\big) := k^v + a_1 k^{v-1} + a_2 k^{v-2} + \cdots + a_{v-1} k + a_v \in \mathbb{Z}_p \tag{7.3}$$

That is, we use $(1, a_1, a_2, \ldots, a_v)$ as the vector of coefficients of a polynomial $f(X)$ of degree v and then evaluate $f(X)$ at a secret point k.

A very useful feature of this hash function is that it can be evaluated without knowing the length of the message ahead of time. One can feed message blocks into the hash as they become available. When the message ends we obtain the final hash. We do so using Horner's method for polynomial evaluation:

Input: $m = (a_1, a_2, \ldots, a_v) \in \mathbb{Z}_p^{\leq \ell}$ and key $k \in \mathbb{Z}_p$
Output: $t := H_{\mathrm{poly}}(k, m)$
1. Set $t \leftarrow 1$
2. For $i \leftarrow 1$ to v:
3. $\quad t \leftarrow t \cdot k + a_i \ \in \mathbb{Z}_p$
4. Output t

It is not difficult to show that this algorithm produces the same value as defined in (7.3). Observe that a long message can be processed one block at a time using little additional space. Every iteration takes one multiplication and one addition.

On a machine that has several multiplication units, say four units, we can use a 4-way parallel version of Horner's method to utilize all the available units and speed up the evaluation of H_{poly}. Assuming the length of m is a multiple of 4, simply replace lines (2) and (3) above with the following

2. For $i \leftarrow 1$ to v incrementing i by 4 at every iteration:
3. $\quad t \leftarrow t \cdot k^4 + a_i \cdot k^3 + a_{i+1} \cdot k^2 + a_{i+2} \cdot k + a_{i+3} \ \in \mathbb{Z}_p$

One can precompute the values k^2, k^3, k^4 in $\mathbb{Z}_p$. Then at every iteration we process four blocks of the message using four multiplications that can all be done in parallel.

Security as a UHF. Next we show that H_{poly} is an (ℓ/p)-UHF. If p is super-poly, this implies that ℓ/p is negligible, which means that H_{poly} is a statistical UHF.

Lemma 7.2. *The function H_{poly} over $(\mathbb{Z}_p,\ (\mathbb{Z}_p)^{\leq \ell},\ \mathbb{Z}_p)$ defined in (7.3) is an (ℓ/p)-UHF.*

Proof. Consider two distinct messages $m_0 = (a_1, \ldots, a_u)$ and $m_1 = (b_1, \ldots, b_v)$ in $(\mathbb{Z}_p)^{\leq \ell}$. We show that $\Pr[H_{\mathrm{poly}}(k, m_0) = H_{\mathrm{poly}}(k, m_1)] \leq \ell/p$, where the probability is over the random choice of key k in $\mathbb{Z}_p$. Define the two polynomials:

$$\begin{aligned} f(X) &:= X^u + a_1 X^{u-1} + a_2 X^{u-2} + \cdots + a_{u-1} X + a_u \\ g(X) &:= X^v + b_1 X^{v-1} + b_2 X^{v-2} + \cdots + b_{v-1} X + b_v \end{aligned} \tag{7.4}$$

in $\mathbb{Z}_p[X]$. Then, by definition of H_{poly} we need to show that

$$\Pr[f(k) = g(k)] \leq \ell/p$$

where k is uniform in $\mathbb{Z}_p$. In other words, we need to bound the number of points $k \in \mathbb{Z}_p$ for which $f(k) - g(k) = 0$. Since the messages m_0 and m_1 are distinct we know that $f(X) - g(X)$ is a nonzero polynomial. Furthermore, its degree is at most ℓ and therefore it has at most ℓ roots in $\mathbb{Z}_p$. It follows that there are at most ℓ values of $k \in \mathbb{Z}_p$ for which $f(k) = g(k)$ and therefore, for a random $k \in \mathbb{Z}_p$ we have $\Pr[f(k) = g(k)] \leq \ell/p$ as required. □

Why the leading term k^v in $H_{\text{poly}}(k,m)$? The definition of $H_{\text{poly}}(k,m)$ in (7.3) includes a leading term k^v. This term ensures that the function is a statistical UHF for variable size inputs. If instead we defined $H_{\text{fpoly}}(k,m)$ without this term, namely

$$H_{\text{fpoly}}\big(k,\ (a_1,\dots,a_v)\big) := a_1k^{v-1} + a_2k^{v-2} + \cdots + a_{v-1}k + a_v \in \mathbb{Z}_p, \tag{7.5}$$

then the result would not be a UHF for variable size inputs. For example, the two messages $m_0 = (a_1,a_2) \in \mathbb{Z}_p^2$ and $m_1 = (0,a_1,a_2) \in \mathbb{Z}_p^3$ are a collision for H_{fpoly} under all keys $k \in \mathbb{Z}_p$. Nevertheless, in Exercise 7.16 we show that H_{fpoly} is a statistical UHF if we restrict its input space to messages of fixed length, i.e., $\mathcal{M} := \mathbb{Z}_p^{\ell}$ for some ℓ. Specifically, H_{fpoly} is an $(\ell-1)/p$-UHF. In contrast, the function H_{poly} defined in (7.3) is a statistical UHF for the input space $\mathbb{Z}_p^{\leq \ell}$ containing messages of varying lengths.

Remark 7.1. The function H_{poly} takes inputs in $\mathbb{Z}_p^{\leq \ell}$ and outputs values in $\mathbb{Z}_p$. This can be difficult to work with: we prefer to work with functions that operate on blocks of n-bits for some n. We can adapt the definition of H_{poly} in (7.3) so that instead of working in $\mathbb{Z}_p$, arithmetic is done in the finite field $\text{GF}(2^n)$. This version of H_{poly} is an $\ell/2^n$-UHF using the exact same analysis as in Lemma 7.2. It outputs values in $\text{GF}(2^n)$. In Exercise 7.1 we show that simply defining H_{poly} modulo 2^n (i.e., working in $\mathbb{Z}_{2^n}$) is a completely insecure UHF. □

Caution in using UHFs. UHFs can be brittle — an adversary who learns the value of the function at a few points can completely recover the secret key. For example, the value of $H_{\text{poly}}(k,\cdot)$ at a single point completely exposes the secret key $k \in \mathbb{Z}_p$. Indeed, if $m = (a_1)$, since $H_{\text{poly}}(k,m) = k + a_1$ an adversary who has both m and $H_{\text{poly}}(k,m)$ immediately obtains $k \in \mathbb{Z}_p$. Consequently, in all our applications of UHFs we will always hide values of the UHF from the adversary, either by encrypting them or by other means.

Mathematical details. The definition of H_{poly} requires a prime p. So far we simply assumed that p is a public value picked at the beginning of time and fixed forever. In the formal UHF framework (Section 7.1.2) the prime p is a system parameter, denoted by Λ. It is generated by a *system parameter generation algorithm* P that takes the security parameter λ as input and outputs some prime p.

More precisely, let $L : \mathbb{Z} \to \mathbb{Z}$ be some function that maps the security parameter to the desired bit length of the prime. Then the formal description of H_{poly} includes a description of an algorithm P that takes the security parameter λ as input and outputs a prime p of length $L(\lambda)$ bits. Specifically, $\Lambda := p$ and

$$\mathcal{K}_{\lambda,p} = \mathbb{Z}_p, \quad \mathcal{M}_{\lambda,p} = \mathbb{Z}_p^{\leq \ell(\lambda)}, \quad \text{and} \quad \mathcal{T}_{\lambda,p} = \mathbb{Z}_p,$$

where $\ell : \mathbb{Z} \to \mathbb{Z}^{\geq 0}$ is poly-bounded. By Lemma 7.2 we know that

$$\text{UHF}\mathsf{adv}[\mathcal{A}, H_{\text{poly}}](\lambda) \leq \ell(\lambda)/2^{L(\lambda)}$$

which is a negligible function of λ provided $2^{L(\lambda)}$ is super-poly.

7.2.2 Construction 2: CBC and cascade are computational UHFs

Next we show that the CBC and cascade constructions defined in Section 6.4 are computational UHFs. More generally, we show that any prefix-free secure PRF that is also extendable is a computational UHF. Recall that a PRF F over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{Y})$ is extendable if for all $k \in \mathcal{K},\ x, y \in \mathcal{X}^{\leq \ell - 1}$, and $a \in \mathcal{X}$ we have:

$$\text{if} \quad F(k, x) = F(k, y) \quad \text{then} \quad F(k,\ x \parallel a) = F(k,\ y \parallel a).$$

In the previous chapter we showed that both CBC and cascade are prefix-free secure PRFs and that both are extendable.

Theorem 7.3. *Let PF be an extendable and prefix-free secure PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell+1}, \mathcal{Y})$ where $|\mathcal{Y}|$ is super-poly and $|\mathcal{X}| > 1$. Then PF is a computational UHF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{Y})$.*

In particular, for every UHF adversary $\mathcal{A}$ that plays Attack Game 7.1 with respect to PF, there exists a prefix-free PRF adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{UHF}\mathsf{adv}[\mathcal{A}, PF] \leq \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}, PF] + \frac{1}{|\mathcal{Y}|}. \tag{7.6}$$

Moreover, $\mathcal{B}$ makes only two queries to PF.

Proof. Let $\mathcal{A}$ be a UHF adversary attacking PF. We build a prefix-free PRF adversary $\mathcal{B}$ attacking PF. $\mathcal{B}$ plays the adversary in the PRF Attack Game 4.2. Its goal is to distinguish between Experiment 0 where it queries a function $f \leftarrow PF(k, \cdot)$ for a random $k \in \mathcal{K}$, and Experiment 1 where it queries a random function $f \xleftarrow{\text{R}} \mathrm{Funs}[\mathcal{X}^{\leq \ell+1}, \mathcal{Y}]$.

We first give some intuition as to how $\mathcal{B}$ works. $\mathcal{B}$ starts by running the UHF adversary $\mathcal{A}$ to obtain two distinct messages $m_0, m_1 \in \mathcal{X}^{\leq \ell}$. By the definition of $\mathcal{A}$, we know that in Experiment 0 we have

$$\Pr\big[f(m_0) = f(m_1)\big] = \mathrm{UHF}\mathsf{adv}[\mathcal{A}, PF]$$

while in Experiment 1, since f is a random function and $m_0 \neq m_1$, we have

$$\Pr\big[f(m_0) = f(m_1)\big] = 1/|\mathcal{Y}|.$$

Hence, if $\mathcal{B}$ could query f at m_0 and m_1 it could distinguish between the two experiments with advantage $\big|\mathrm{UHF}\mathsf{adv}[\mathcal{A}, PF] - 1/|\mathcal{Y}|\big|$, which would prove the theorem.

Unfortunately, this design for $\mathcal{B}$ does not quite work: m_0 might be a proper prefix of m_1, in which case $\mathcal{B}$ is not allowed to query f at both m_0 and m_1, since $\mathcal{B}$ is supposed to be a prefix-free adversary. However, the extendability property provides a simple solution: we extend both m_0 and m_1 by a single block $a \in \mathcal{X}$ so that $m_0 \parallel a$ is no longer a proper prefix of $m_1 \parallel a$. If $m_0 = (a_1, \ldots, a_u)$ and $m_1 = (b_1, \ldots, b_v)$, then any $a \neq b_{u+1}$ will do the trick. Moreover, by the extension property we know that

$$PF(k,\ m_0) = PF(k,\ m_1) \qquad \Longrightarrow \qquad PF(k,\ m_0 \parallel a) = PF(k,\ m_1 \parallel a).$$

Since $m_0 \parallel a$ is no longer a proper prefix of $m_1 \parallel a$, our $\mathcal{B}$ is free to query f at both inputs and obtain the desired advantage in distinguishing Experiment 0 from Experiment 1.

In more detail, adversary $\mathcal{B}$ works as follows:

run $\mathcal{A}$ to obtain two distinct messages m_0, m_1 in $\mathcal{X}^{\leq \ell}$, where

$m_0 = (a_1, \ldots, a_u)$ and $m_1 = (b_1, \ldots, b_v)$

assume $u \leq v$ (otherwise, swap the two messages)

if m_0 is a proper prefix of m_1

choose some $a \in \mathcal{X}$ such that $a \neq a_{u+1}$

$m_0' \leftarrow m_0 \parallel a$ and $m_1' \leftarrow m_1 \parallel a$

else

$m_0' \leftarrow m_0$ and $m_1' \leftarrow m_1$

// At this point we know that m_0' is not a proper prefix of m_1' nor vice versa.

query f at m_0' and m_1' and obtain $t_0 := f(m_0')$ and $t_1 := f(m_1')$

if $t_0 = t_1$ output 1; otherwise output 0

Observe that $\mathcal{B}$ is a prefix-free PRF adversary that only makes two queries to f, as required. Now, for $b = 0, 1$ let p_b be the probability that $\mathcal{B}$ outputs 1 in Experiment b. Then in Experiment 0, we know that

$$p_0 := \Pr\left[f(m_0') = f(m_1')\right] \geq \Pr\left[f(m_0) = f(m_1)\right] = \mathrm{UHF}\mathsf{adv}[\mathcal{A}, PF]. \tag{7.7}$$

In Experiment 1, we know that

$$p_1 := \Pr\left[f(m_0') = f(m_1')\right] = 1/|\mathcal{Y}|. \tag{7.8}$$

Therefore, by (7.7) and (7.8):

$$\mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}, PF] = |p_0 - p_1| \geq p_0 - p_1 \geq \mathrm{UHF}\mathsf{adv}[\mathcal{A}, PF] - 1/|\mathcal{Y}|,$$

from which (7.6) follows. □

***PF* as a multi-query UHF.** Lemma 7.1 shows that PF is also a multi-query UHF. However, a direct proof of this fact gives a better security bound.

Theorem 7.4. *Let PF be an extendable and prefix-free secure PRF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell+1}, \mathcal{Y})$, where $|\mathcal{X}|$ and $|\mathcal{Y}|$ are super-poly and ℓ is poly-bounded. Then PF is a multi-query UHF defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{Y})$.*

In particular, if $|\mathcal{X}| > \ell Q$, then for every Q-query UHF adversary $\mathcal{A}$, there exists a Q-query prefix-free PRF adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{MUHF}\mathsf{adv}[\mathcal{A}, PF] \leq \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}, PF] + \frac{Q^2}{2|\mathcal{Y}|}. \tag{7.9}$$

Proof. The proof is similar to the proof of Theorem 7.3. Adversary $\mathcal{B}$ begins by running the Q-query UHF adversary $\mathcal{A}$ to obtain distinct messages $m_1, \ldots, m_s$ in $\mathcal{X}^{\leq \ell}$, where $s \leq Q$. Next, $\mathcal{B}$ finds an $a \in \mathcal{X}$ such that a is not equal to any of the message blocks in $m_1, \ldots, m_s$. Since $|\mathcal{X}|$ is super-poly, we may assume it is larger than ℓQ, and therefore this a must exist. Let $m_i' := m_i \parallel a$ for $i = 1, \ldots, s$. Then, by definition of a, the set $\{m_1', \ldots, m_s'\}$ is a prefix-free set. The prefix-free adversary $\mathcal{B}$ now queries the challenger at $m_1', \ldots, m_s'$ and obtains $t_1, \ldots, t_s$ in response. $\mathcal{B}$ outputs 1 if there exist $i \neq j$ such that $t_j = t_j$ and outputs 0 otherwise.

To analyze the advantage of $\mathcal{B}$ we let p_b be the probability that $\mathcal{B}$ outputs 1 in PRF Experiment b, for $b = 0, 1$. As in (7.7), the extension property implies that

$$p_0 \geq \mathrm{MUHF}\mathsf{adv}[\mathcal{A}, PF].$$

In Experiment 1 the union bound implies that

$$p_1 \leq \frac{Q(Q-1)}{2|\mathcal{Y}|}.$$

Therefore,

$$\mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}, PF] = |p_0 - p_1| \geq p_0 - p_1 \geq \mathrm{MUHF}\mathsf{adv}[\mathcal{A}, PF] - \frac{Q^2}{2|\mathcal{Y}|}$$

from which (7.9) follows. □

Applications of Theorems 7.3 and 7.4. Applying Theorem 7.4 to CBC and cascade proves that both are computational UHFs. We state the resulting error bounds in the following corollary, which follows from the bounds in the CBC theorem (Theorem 6.3) and the cascade theorem (Theorem 6.4).[1]

Corollary 7.5. *Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$. Then the CBC construction F_{CBC} (assuming $\mathcal{Y} = \mathcal{X}$ is super-poly size) and the cascade construction F^* (assuming $\mathcal{Y} = \mathcal{K}$), which take inputs in $\mathcal{X}^{\leq \ell}$, for poly-bounded ℓ are computational UHFs.*

In particular, for every Q-query UHF adversary $\mathcal{A}$, there exist prefix-free PRF adversaries $\mathcal{B}_1, \mathcal{B}_2$, which are elementary wrappers around $\mathcal{A}$, such that

$$\mathrm{MUHF}\mathsf{adv}[\mathcal{A}, F_{\text{CBC}}] \leq \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}_1, F] + \frac{Q^2(\ell+1)^2 + Q^2}{2|\mathcal{Y}|} \quad \textit{and} \tag{7.10}$$

$$\mathrm{MUHF}\mathsf{adv}[\mathcal{A}, F^*] \leq Q(\ell+1) \cdot \mathrm{PRF}^{\mathrm{pf}}\mathsf{adv}[\mathcal{B}_2, F] + \frac{Q^2}{2|\mathcal{Y}|}. \tag{7.11}$$

Setting $Q := 2$ in (7.10)–(7.11) gives the error bounds on F_{CBC} and F^* as UHFs.

7.2.3 Construction 3: a parallel UHF from a small PRF

The CBC and cascade constructions yield efficient UHFs from small domain PRFs, but they are inherently sequential: they cannot take advantage of hardware parallelism. Fortunately, constructing a UHF from a small domain PRF that is suitable for a parallel architecture is not difficult. An example called XOR-hash, denoted $F^{\oplus}$, is shown in Fig. 7.2. XOR-hash is defined over $(\mathcal{K}, \mathcal{X}^{\leq \ell}, \mathcal{Y})$, where $\mathcal{Y} = \{0,1\}^n$, and is built from a PRF F defined over $(\mathcal{K}, \mathcal{X} \times \{1, \ldots, \ell\}, \mathcal{Y})$. The XOR-hash works as follows:

input: $k \in \mathcal{K}$ and $m = (a_1, \ldots, a_v) \in \mathcal{X}^{\leq \ell}$ for some $0 \leq v \leq \ell$
output: a tag in $\mathcal{Y}$

    $t \leftarrow 0^n$
    for $i = 1$ to v do:
        $t \leftarrow t \oplus F(k,\ (a_i, i)\ )$
    output t

[1] Note that Theorem 7.4 compels us to apply Theorems 6.3 and 6.4 using $\ell + 1$ in place of ℓ.

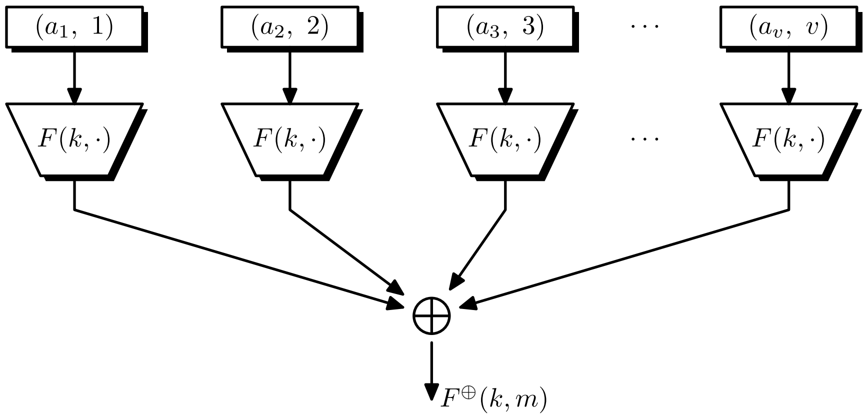


Figure 7.2: A parallel UHF from a small PRF

Evaluating $F^{\oplus}$ can easily be done in parallel. The following theorem shows that $F^{\oplus}$ is a computational UHF. Note that unlike our previous UHF constructions, security does not depend on the length of the input message. In the next section we will use $F^{\oplus}$ to construct a secure MAC suitable for parallel architectures.

Theorem 7.6. *Let F be a secure PRF and assume $|\mathcal{Y}|$ is super-poly. Then $F^{\oplus}$ is a computational UHF.*

In particular, for every UHF adversary $\mathcal{A}$, there exists a PRF adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{UHF}\mathsf{adv}[\mathcal{A}, F^{\oplus}] \leq \mathrm{PRF}\mathsf{adv}[\mathcal{B}, F] + \frac{1}{|\mathcal{Y}|}. \tag{7.12}$$

Proof. The proof is a sequence of two games.

Game 0. The challenger in this game computes:

$k \xleftarrow{\mathrm{R}} \mathcal{K},\ f \leftarrow F(k, \cdot)$

The adversary $\mathcal{A}$ outputs two distinct messages U, V in $\mathcal{X}^{\leq \ell}$. Let $u := |U|$ and $v := |V|$. We define W_0 to be the event that the condition

$$\bigoplus_{i=0}^{u-1} f(U[i], i) = \bigoplus_{j=0}^{v-1} f(V[j], j) \tag{7.13}$$

holds in Game 0. Clearly, we have

$$\Pr[W_0] = \mathrm{UHF}\mathsf{adv}[\mathcal{A}, F^{\oplus}]. \tag{7.14}$$

Game 1. We play the “PRF card” and replace the challenger’s computation by

$f \xleftarrow{\mathrm{R}} \mathrm{Funs}[\mathcal{X} \times \{1, \ldots, \ell\},\ \mathcal{Y}]$

We define W_1 to be the event that the condition (7.13) holds in Game 1.

As usual, there is a PRF adversary $\mathcal{B}$ such that

$$\left|\Pr[W_0] - \Pr[W_1]\right| \leq \mathrm{PRF}\mathsf{adv}[\mathcal{B}, F] \tag{7.15}$$

The crux of the proof is in bounding $\Pr[W_1]$, namely bounding the probability that (7.13) holds for the messages U, V. Assume $u \geq v$, swapping U and V if necessary. It is easy to see that since U and V are distinct, there must be an index i^* such that the pair $(U[i^*], i^*)$ on the left side of (7.13) does not appear among the pairs $(V[j], j)$ on the right side of (7.13): if $u > v$ then $i^* = u - 1$ does the job; otherwise, if $u = v$, then there must exist some i^* such that $U[i^*] \neq V[i^*]$, and this i^* does the job.

We can re-write (7.13) as

$$f(U[i^*], i^*) = \bigoplus_{i \neq i^*} f(U[i], i) \quad \oplus \quad \bigoplus_{j} f(V[j], j). \tag{7.16}$$

Since the left and right sides of (7.16) are independent, and the left side is uniformly distributed over $\mathcal{Y}$, equality holds with probability $1/|\mathcal{Y}|$. It follows that

$$\Pr[W_1] = 1/|\mathcal{Y}| \tag{7.17}$$

The proof of the theorem follows from (7.14), (7.15), and (7.17). $\square$

In Exercise 7.27 we generalize Theorem 7.6 to derive bounds for $F^{\oplus}$ as a multi-query UHF.

7.3 PRF(UHF) composition: constructing MACs using UHFs

We now proceed to show that the hash-then-PRF paradigm yields a secure PRF provided the hash is a computational UHF. ECBC, NMAC, and PMAC_0 can all be viewed as instances of this construction and their security follows quite easily from the security of the hash-then-PRF paradigm.

Let H be a keyed hash function defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{X})$ and let F be a PRF defined over $(\mathcal{K}_F, \mathcal{X}, \mathcal{T})$. As usual, we assume $\mathcal{M}$ contains much longer messages than $\mathcal{X}$, so that H hashes long inputs to short digests. We build a new PRF, denoted F', by composing the hash function H with the PRF F, as shown in Fig. 7.3. More precisely, F' is defined as follows:

$$F'\big((k_1, k_2),\ m\big) := F(k_2,\ H(k_1,\ m)\ ) \tag{7.18}$$

We refer to F' as the **composition of F and H**. It takes inputs in $\mathcal{M}$ and outputs values in $\mathcal{T}$ using a key (k_1, k_2) in $\mathcal{K}_H \times \mathcal{K}_F$. Thus, we obtain a PRF with the same output space as the underlying F, but taking much longer inputs. The following theorem shows that F' is a secure PRF.

Theorem 7.7 (PRF(UHF) composition). *Suppose H is a computational UHF and F is a secure PRF. Then F' defined in (7.18) is a secure PRF.*

> *In particular, suppose $\mathcal{A}$ is a PRF adversary that plays Attack Game 4.2 with respect to F' and issues at most Q queries. Then there exist a PRF adversary $\mathcal{B}_F$ and a UHF adversary $\mathcal{B}_H$, which are elementary wrappers around $\mathcal{A}$, such that*

$$\mathrm{PRF}\mathsf{adv}[\mathcal{A}, F'] \leq \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F] + (Q^2/2) \cdot \mathrm{UHF}\mathsf{adv}[\mathcal{B}_H, H]. \tag{7.19}$$

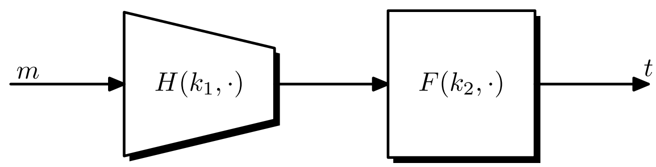


Figure 7.3: PRF(UHF) composition: MAC signing

More generally, there exists a Q-query UHF adversary $\mathcal{B}'_H$, which is an elementary wrapper around $\mathcal{A}$ such that

$$\mathrm{PRF}\mathsf{adv}[\mathcal{A}, F'] \leq \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F] + \mathrm{MUHF}\mathsf{adv}[\mathcal{B}'_H, H]. \tag{7.20}$$

To understand why H needs to be a UHF let us suppose for a minute that it is not. In particular, suppose it was easy to find distinct $m_0, m_1 \in \mathcal{M}$ such that $H(k_1, m_0) = H(k_1, m_1)$, without knowledge of k_1. This collision on H implies that $F'((k_1, k_2),\ m_0) = F'((k_1, k_2),\ m_1)$. But then F' is clearly not a secure PRF: the adversary could ask for $t_0 := F'((k_1, k_2),\ m_0)$ and $t_1 := F'((k_1, k_2),\ m_1)$ and then output 1 only if $t_0 = t_1$. When interacting with F' the adversary would always output 1, but for a random function he would most often output 0. Thus, the adversary successfully distinguishes F' from a random function. This argument shows that for F' to be a PRF it must be difficult to find collisions for H without knowledge of k_1. In other words, for F' to be a PRF the hash function H must be a UHF. Theorem 7.7 shows that this condition is sufficient.

Remark 7.2. The bound in Theorem 7.7 is tight. Consider the UHF H_{poly} discussed in Section 7.2.1. For concreteness, let us assume that $\ell = 2$, so the message space for H_{poly} is $\mathbb{Z}_p^2$, the output space is $\mathbb{Z}_p$, and the collision probability is $\epsilon = 1/p$. In Exercise 7.26, you are asked to show that for any fixed hash key k_1, among $\sqrt{p}$ random inputs to $H_{\mathrm{poly}}(k_1, \cdot)$, the probability of a collision is bounded from below by a constant; moreover, for any such collision, one can efficiently recover the key k_1. Now consider the MAC obtained from PRF(UHF) composition using H_{poly}. If the adversary ever finds two messages m_0, m_1 that cause an internal collision (i.e., a collision on H_{poly}) he can recover the secret H_{poly} key and then break the MAC. This shows that the term $(Q^2/2)\epsilon$ that appears in (7.19) cannot be substantially improved upon. □

Proof of Theorem 7.7. We now prove that the composition of F and H is a secure PRF.

Proof idea. Let $\mathcal{A}$ be an efficient PRF adversary that plays Attack Game 4.2 with respect to F'. We derive an upper bound on $\mathrm{PRF}\mathsf{adv}[\mathcal{A}, F']$. That is, we bound $\mathcal{A}$'s ability to distinguish F' from a truly random function in $\mathrm{Funs}[\mathcal{M}, \mathcal{X}]$. As usual, we first observe that replacing the underlying secure PRF F with a truly random function f does not change $\mathcal{A}$'s advantage much. Next, we will show that, since f is a random function, the only way $\mathcal{A}$ can distinguish $F' := f(H(k_1, m))$ from a truly random function is if he can find two inputs m_0, m_1 such that $H(k_1, m_0) = H(k_1, m_1)$. But since H is a computational UHF, $\mathcal{A}$ cannot find collisions for $H(k_1, \cdot)$. Consequently, F' cannot be distinguished from a random function. □

Proof. We prove the bound in (7.20). Equation (7.19) follows from (7.20) by Lemma 7.1. We let $\mathcal{A}$ interact with closely related challengers in three games. For $j = 0, 1, 2$, we define W_j to be the event that $\mathcal{A}$ outputs 1 at the end of Game j.

Game 0. The Game 0 challenger is identical to the challenger in Experiment 0 of the PRF Attack Game 4.2 with respect to F'. Without loss of generality we assume that $\mathcal{A}$'s queries to F' are all distinct. The challenger works as follows:

$$
\begin{array}{l}
k_1 \xleftarrow{\text{R}} \mathcal{K}_H, \;\; k_2 \xleftarrow{\text{R}} \mathcal{K}_F \\
\text{upon receiving the } i\text{th query } m_i \in \mathcal{M} \text{ (for } i = 1, 2, \ldots) \text{ do:} \\
\qquad x_i \leftarrow H(k_1,\ m_i) \\
\qquad t_i \leftarrow F(k_2,\ x_i) \\
\qquad \text{send } t_i \text{ to the adversary}
\end{array}
$$

Note that since $\mathcal{A}$ is guaranteed to make distinct queries, all the m_i values are distinct.

Game 1. Now we play the usual "PRF card," replacing the function $F(k_2, \cdot)$ by a truly random function f in Funs$[\mathcal{X}, \mathcal{T}]$, which we implement as a faithful gnome (as in Section 4.4.2). The Game 1 challenger works as follows:

$$
\begin{array}{ll}
& k_1 \xleftarrow{\text{R}} \mathcal{K}_H, \quad t'_1, \ldots, t'_Q \xleftarrow{\text{R}} \mathcal{T} \\
& \text{upon receiving the } i\text{th query } m_i \in \mathcal{M} \text{ (for } i = 1, 2, \ldots) \text{ do:} \\
& \qquad x_i \leftarrow H(k_1,\ m_i) \\
& \qquad t_i \leftarrow t'_i \\
(*) & \qquad \text{if } x_i = x_j \text{ for some } j < i \text{ then } t_i \leftarrow t_j \\
& \qquad \text{send } t_i \text{ to the adversary}
\end{array}
$$

For $i = 1, \ldots, Q$, the value t'_i is chosen in advance to be the default, random value for $t_i = f(x_i)$. Although the messages are distinct, their hash values might not be. The line marked with a $(*)$ ensures that the challenger emulates a function in Funs$[\mathcal{X}, \mathcal{T}]$ — if two hash values collide, the challenger's response to both queries is the same. As usual, one can easily show that there is a PRF adversary $\mathcal{B}_F$ whose running time is about the same as that of $\mathcal{A}$ such that:

$$
\bigl|\Pr[W_1] - \Pr[W_0]\bigr| = \text{PRF}\mathsf{adv}[\mathcal{B}_F, F] \tag{7.21}
$$

Game 2. Next, we make our gnome forgetful, by removing the line marked $(*)$.

We show that $\mathcal{A}$ cannot distinguish Games 1 and 2 using the fact that $\mathcal{A}$ cannot find collisions for H. Formally, we analyze the quantity $|\Pr[W_2] - \Pr[W_1]|$ using the Difference Lemma (Theorem 4.7). Let Z be the event that in Game 2 we have $x_i = x_j$ for some $i \neq j$. Event Z is essentially the winning condition in the multi-query UHF game (Attack Game 7.2) with respect to H. In particular, there is a Q-query UHF adversary $\mathcal{B}'_H$ that wins Attack Game 7.2 with probability equal to $\Pr[Z]$. Adversary $\mathcal{B}'_H$ simply emulates the challenger in Game 2 until $\mathcal{A}$ terminates and then outputs the queries $m_1, m_2, \ldots$ from $\mathcal{A}$ as its final list. This works, because in Game 2, the challenger does not really need the hash key k_1: it simply responds to each query with a random element of $\mathcal{T}$. Thus, adversary $\mathcal{B}'_H$ can easily emulate the challenger in Game 2 without knowledge of k_1. By definition of Z, we have MUHF$\mathsf{adv}[\mathcal{B}'_H, H] = \Pr[Z]$.

Clearly, Games 1 and 2 proceed identically unless event Z occurs; in particular, $W_2 \wedge \bar{Z}$ occurs if and only if $W_1 \wedge \bar{Z}$ occurs. Applying the Difference Lemma, we obtain

$$
\bigl|\Pr[W_2] - \Pr[W_1]\bigr| \leq \Pr[Z] = \text{MUHF}\mathsf{adv}[\mathcal{B}'_H, H]. \tag{7.22}
$$

Finishing the proof. The Game 2 challenger emulates for $\mathcal{A}$ a random function in Funs$[\mathcal{M}, \mathcal{T}]$

and is therefore identical to an Experiment 1 PRF challenger with respect to F'. We obtain

$$\begin{aligned}\text{PRF}\mathsf{adv}[\mathcal{A}, F'] = &\big|\Pr[W_2] - \Pr[W_0]\big| \leq \\ &\big|\Pr[W_2] - \Pr[W_1]\big| + \big|\Pr[W_1] - \Pr[W_0]\big| = \\ &\text{PRF}\mathsf{adv}[\mathcal{B}_F, F] + \text{MUHF}\mathsf{adv}[\mathcal{B}'_H, H]\end{aligned}$$

which proves (7.20), as required. □

7.3.1 Using PRF(UHF) composition: ECBC and NMAC security

Using Theorem 7.7 we can quickly prove security of many MAC constructions. It suffices to show that the MAC signing algorithm can be described as the composition of a PRF with a UHF. We begin by showing that ECBC and NMAC can be described this way and give more examples in the next two sub-sections.

Security of ECBC and NMAC follows directly from PRF(UHF) composition. The proof for both schemes runs as follows:

- First, we proved that CBC and cascade are prefix-free secure PRFs (Theorems 6.3 and 6.4). We observed that both are extendable.
- Next, we showed that any extendable prefix-free secure PRF is also a computational UHF (Theorem 7.3). In particular, CBC and cascade are computational UHFs.
- Finally, we proved that the composition of a computational UHF and a PRF is a secure PRF (Theorem 7.7). Hence, ECBC and NMAC are secure PRFs.

More generally, the encrypted PRF construction (Theorem 6.5) is an instance of PRF(UHF) composition and hence its proof follows from Theorem 7.7. The concrete bounds in the ECBC and NMAC theorems (Theorems 6.6 and 6.7) are obtained by plugging (7.10) and (7.11), respectively, into (7.20).

One can simplify the proof of ECBC and NMAC security by directly proving that CBC and cascade are computational UHFs. We proved that they are prefix-free secure PRFs, which is more than we need. However, this stronger result enabled us to construct other secure MACs such as CMAC (see Section 6.7).

7.3.2 Using PRF(UHF) composition with polynomial UHFs

Of course, one can use the PRF(UHF) construction with a polynomial-based UHF, such as H_{poly}. Depending on the underlying hardware, this construction can be much faster than either ECBC, NMAC, or PMAC_0 especially for very long messages.

Recall that H_{poly} hashes messages in $\mathbb{Z}_p^{\leq \ell}$ to digests in $\mathbb{Z}_p$, where p is a prime. Now, we may very well want to use for our PRF a block cipher, like AES, that takes as input an n-bit block.

To make this work, we have to somehow make an adjustment so that the digest space of the hash is equal to input space of the PRF. One way to do this is to choose the prime p so that it is just a little bit smaller that 2^n, so that we can encode hash digests as inputs to the PRF. This approach works; however, it has the drawback that we have to view the input to the hash as a sequence of elements of $\mathbb{Z}_p$. So, for example, with $n = 128$ as in AES, we could choose a 128-bit prime, but then the input to the hash would have to be broken up into, say, 120-bit (i.e., 15 byte) blocks. It would be even more convenient if we could also process the input to the hash directly as a sequence of n-bit blocks. Part (d) of Exercise 7.23 shows how this can be done, using a prime that is just a little bit *bigger* than 2^n. Yet another approach is that instead of basing the hash on arithmetic modulo a prime p, we instead base it on arithmetic in the finite field $\mathrm{GF}(2^n)$, as discussed in Remark 7.1.

7.3.3 Using PRF(UHF) composition: PMAC_0 security

Next we show that the PMAC_0 construction discussed in Section 6.11 is an instance of PRF(UHF) composition. Recall that PMAC_0 is built out of two PRFs, F_1, which is defined over $(\mathcal{K}_1, \mathbb{Z}_p, \mathcal{Y})$, and F_2, which is defined over $(\mathcal{K}_2, \mathcal{Y}, \mathcal{Z})$, where $\mathcal{Y} := \{0,1\}^n$.

The reader should review the PMAC_0 construction, especially Fig. 6.9. One can see that PMAC_0 is the composition of the PRF F_2 with a certain keyed hash function $\widehat{H}$, which is everything else in Fig. 6.9.

The goal now is to show that $\widehat{H}$ is a computational UHF. To do this, we observe that $\widehat{H}$ can be viewed as an instance of the XOR-hash construction in Section 7.2.3, applied to the PRF F' defined over $(\mathbb{Z}_p \times \mathcal{K}_1, \mathbb{Z}_p \times \{1, \ldots, \ell\}, \mathcal{Y})$ as follows:

$$F'((k, k_1), (a, i)) := F_1(k_1, a + i \cdot k).$$

So it suffices to show that F' is a secure PRF. But it turns out we can view F' itself as an instance of PRF(UHF) composition. Namely, it is the composition of the PRF F_1 with the keyed hash function H defined over $(\mathbb{Z}_p, \mathbb{Z}_p \times \{1, \ldots, \ell\}, \mathbb{Z}_p)$ as $H(k, (a, i)) := a + i \cdot k$. However, H is just a special case of case of H_{fpoly} (see Section 7.2.1). In particular, by the result of Exercise 7.16, H is a $1/p$-UHF.

The security of PMAC_0 follows from the above observations. The concrete security bound (6.27) in Theorem 6.11 follows from the concrete security bound (7.20) in Theorem 7.7 and the more refined analysis of XOR-hash in Exercise 7.27.

In the design of PMAC_0, we assumed the input space of F_1 was equal to $\mathbb{Z}_p$. While this simplifies the analysis, it makes it harder to work with in practice. Just as in Section 7.3.2 above, we would prefer to work with a PRF defined in terms of a block cipher, like AES, which takes as input an n-bit block. One can apply the same techniques discussed Section 7.3.2 to get a variant of PMAC_0 whose input space consists of sequences of n-bit blocks, rather than sequences of elements of $\mathbb{Z}_p$. For example, see Exercise 7.25.

7.4 The Carter-Wegman MAC

In this section we present a different paradigm for constructing secure MAC systems that offers different tradeoffs compared to PRF(UHF) composition.

Recall that in PRF(UHF) composition the adversary's advantage in breaking the MAC after seeing Q signed messages grows as $\epsilon \cdot Q^2/2$ when using an ϵ-UHF. Therefore to ensure security when many messages need to be signed the ϵ-UHF must have a sufficiently small ϵ so that $\epsilon \cdot Q^2/2$ is small. This can hurt the performance of an ϵ-UHF like H_{poly} where the smaller ϵ the slower the hash function. As an example, suppose that after signing $Q := 2^{32}$ messages the adversary's advantage in breaking the MAC should be no more than 2^{-64} then ϵ must be at most $1/2^{127}$.

Our second MAC paradigm, called a Carter-Wegman MAC, maintains the same level of security as PRF(UHF) composition, but does so with a much larger value of ϵ. With the parameters in the

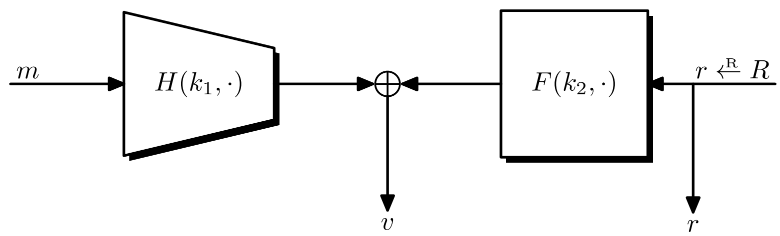


Figure 7.4: Carter-Wegman MAC signing algorithm

example above, ϵ need only be $1/2^{64}$ and this can improve the speed of the hash function, especially for long messages. The downside is that the resulting tags are longer than those generated by a PRF(UHF) composition MAC of comparable security. In Exercise 7.5 we explore a different randomized MAC construction that achieves the same security as Carter-Wegman with the same ϵ, but with shorter tags.

The Carter-Wegman MAC is our first example of a randomized MAC system. The signing algorithm is randomized and there are many valid tags for every message.

To describe the Carter-Wegman MAC first fix some large integer N and set $\mathcal{T} := \mathbb{Z}_N$, the group of size N where addition is defined "modulo N." We use a hash function H and a PRF F that output values in $\mathbb{Z}_N$:

- H is a keyed hash function defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{T})$,
- F is a PRF defined over $(\mathcal{K}_F, \mathcal{R}, \mathcal{T})$.

The Carter-Wegman MAC, denoted $\mathcal{I}_{\text{CW}}$, takes inputs in $\mathcal{M}$ and outputs tags in $\mathcal{R} \times \mathcal{T}$. It uses keys in $\mathcal{K}_H \times \mathcal{K}_F$. The **Carter-Wegman MAC derived from F and H** works as follows (see also Fig. 7.4):

- For key (k_1, k_2) and message m we define

$$
\begin{aligned}
&S\big(\ (k_1, k_2),\ m\ \big) := \\
&\qquad r \xleftarrow{R} \mathcal{R} \\
&\qquad v \leftarrow H(k_1, m) + F(k_2, r) \quad \in \mathbb{Z}_N \quad // \quad \textit{addition modulo } N \\
&\qquad \text{output } (r, v)
\end{aligned}
$$

- For key (k_1, k_2), message m, and tag (r, v) we define

$$
\begin{aligned}
&V\big(\ (k_1, k_2),\ m,\ (r, v)\ \big) := \\
&\qquad v^* \leftarrow H(k_1, m) + F(k_2, r) \quad \in \mathbb{Z}_N \quad // \quad \textit{addition modulo } N \\
&\qquad \text{if } v = v^* \text{ output } \mathsf{accept}\text{; otherwise output } \mathsf{reject}
\end{aligned}
$$

The Carter-Wegman signing algorithm uses a randomizer $r \in \mathcal{R}$. As we will see, the set $\mathcal{R}$ needs to be sufficiently large so that the probability that two tags use the same randomizer is negligible.

An encrypted UHF MAC. The Carter-Wegman MAC can be described as an encryption of the output of a hash function. Indeed, let $\mathcal{E} = (E, D)$ be the cipher

$$
E(k, m) := \big\{ r \xleftarrow{R} \mathcal{R},\ \text{output } (r,\ m + F\big(k, r\big)) \big\} \qquad \text{and} \qquad D\big(k, (r, c)\big) := c - F(k, r)
$$

where F is a PRF defined over $(\mathcal{K}_F, \mathcal{R}, \mathcal{T})$. This cipher is CPA secure when F is a secure PRF as shown in Example 5.2. Then the Carter-Wegman MAC can be written as:

$$S\big((k_1, k_2),\ m\big) := E(k_2,\ H(k_1, m))$$

$$V\big((k_1, k_2),\ m,\ t\big) := \begin{cases} \mathsf{accept} & \text{if } D(k_2, t) = H(k_1, m), \\ \mathsf{reject} & \text{otherwise.} \end{cases}$$

which we call the **encrypted UHF MAC system derived from $\mathcal{E}$ and H.**

Why encrypt the output of a hash function? Recall that in the PRF(UHF) composition MAC, if the adversary finds two messages m_1, m_2 that collide on the hash function (i.e., $H(k_1, m_1) = H(k_1, m_2)$) then the MAC for m_1 is the same as the MAC for m_2. Therefore, by requesting the tags for many messages the adversary can identify messages m_1 and m_2 that collide on the hash function (assuming collisions on the PRF are unlikely). A collision m_1, m_2 on the UHF can reveal information about the hash function key k_1 that may completely break the MAC. To prevent this we must use an ϵ-UHF with a sufficiently small ϵ to ensure that with high probability the adversary will never find a hash function collision. In contrast, by encrypting the output of the hash function with a CPA secure cipher we prevent the adversary from learning when a hash function collision occurred: the tags for m_1 and m_2 are different, with high probability, even if $H(k_1, m_1) = H(k_1, m_2)$. This lets us maintain security with a much smaller ϵ.

The trouble is that the encrypted UHF MAC is not generally secure even when (E, D) is CPA secure and H is an ϵ-UHF. For example, we show in Remark 7.5 below that the Carter-Wegman MAC is insecure when the hash function H is instantiated with H_{poly}. To obtain a secure Carter-Wegman MAC we strengthen the hash function H and require that it satisfy a stronger property called difference unpredictability defined below. Exercise 9.16 explores other aspects of the encrypted UHF MAC.

Security of the Carter-Wegman MAC. To prove security of $\mathcal{I}_{\text{CW}}$ we need the hash function H to satisfy a stronger property than universality (UHF). We refer to this stronger property as **difference unpredictability**. Roughly speaking, it means that for any two distinct messages, it is hard to predict the difference (in $\mathbb{Z}_N$) of their hashes. As usual, a game:

Attack Game 7.3 (difference unpredicability). For a keyed hash function H defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, where $\mathcal{T} = \mathbb{Z}_N$, and a given adversary $\mathcal{A}$, the attack game runs as follows.

- The challenger picks a random $k \xleftarrow{R} \mathcal{K}$ and keeps k to itself.
- $\mathcal{A}$ outputs two distinct messages $m_0, m_1 \in \mathcal{M}$ and a value $\delta \in \mathcal{T}$.

We say that $\mathcal{A}$ wins the game if $H(k, m_1) - H(k, m_0) = \delta$. We define $\mathcal{A}$'s advantage with respect to H, denoted $\text{DUF}\mathsf{adv}[\mathcal{A}, H]$, as the probability that $\mathcal{A}$ wins the game. □

Definition 7.5. *Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$,*

- *We say that H is an ϵ-**bounded difference unpredictable function**, or ϵ-**DUF**, if $\text{DUF}\mathsf{adv}[\mathcal{A}, H] \le \epsilon$ for all adversaries $\mathcal{A}$ (even inefficient ones).*
- *We say that H is a **statistical DUF** if it is an ϵ-DUF for some negligible ϵ.*
- *We say that H is a **computational DUF** if $\text{DUF}\mathsf{adv}[\mathcal{A}, H]$ is negligible for all efficient adversaries $\mathcal{A}$.*

Remark 7.3. Note that as we have defined a DUF, the digest space $\mathcal{T}$ must be of the form $\mathbb{Z}_N$ for some integer N. We did this to keep things simple. More generally, one can define a notion of difference unpredictability for a keyed hash function whose digest space comes equipped with an appropriate difference operator (in the language of abstract algebra, $\mathcal{T}$ should be an *abelian group*). Besides $\mathbb{Z}_N$, another popular digest space is the set of all n-bit strings, $\{0,1\}^n$, with the XOR used as the difference operator. In this setting, we use the terms **ϵ-XOR-DUF** and statistical/computational **XOR-DUF** to correspond to the terms ϵ-DUF and statistical/computational DUF. □

When H is a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, an alternative characterization of the ϵ-DUF property is the following:

> *for every pair of distinct messages $m_0, m_1 \in \mathcal{M}$, and every $\delta \in \mathcal{T}$, the following inequality holds:* $\Pr[H(k, m_1) - H(k, m_0) = \delta] \leq \epsilon$. *Here, the probability is over the random choice of $k \in \mathcal{K}$.*

Clearly if H is an ϵ-DUF then H is also an ϵ-UHF: a UHF adversary can be converted into a DUF adversary that wins with the same probability (just set $\delta = 0$).

We give a simple example of a statistical DUF that is very similar to the hash function H_{poly} defined in equation (7.3). Recall that H_{poly} is a UHF defined over $(\mathbb{Z}_p,\ (\mathbb{Z}_p)^{\leq \ell},\ \mathbb{Z}_p)$. It is clearly not a DUF: for $a \in \mathbb{Z}_p$ set $m_0 := (a)$ and $m_1 := (a+1)$ so that both m_0 and m_1 are tuples over $\mathbb{Z}_p$ of length 1. Then for every key k, we have

$$H_{\text{poly}}(k, m_1) - H_{\text{poly}}(k, m_0) = (k + a + 1) - (k + a) = 1$$

which lets the attacker win the DUF game.

A simple modification to H_{poly} yields a good DUF. For a message $m = (a_1, a_2, \ldots, a_v) \in \mathbb{Z}_p^{\leq \ell}$ and key $k \in \mathbb{Z}_p$ define a new hash function $H_{\text{xpoly}}(k, m)$ as:

$$H_{\text{xpoly}}(k, m) := k \cdot H_{\text{poly}}(k, m) = k^{v+1} + a_1 k^v + a_2 k^{v-1} + \cdots + a_v k \in \mathbb{Z}_p. \tag{7.23}$$

Lemma 7.8. *The function H_{xpoly} over $(\mathbb{Z}_p,\ (\mathbb{Z}_p)^{\leq \ell},\ \mathbb{Z}_p)$ defined in (7.23) is an $(\ell+1)/p$-DUF.*

Proof. Consider two distinct messages $m_0 = (a_1, \ldots, a_u)$ and $m_1 = (b_1, \ldots, b_v)$ in $(\mathbb{Z}_p)^{\leq \ell}$ and an arbitrary value $\delta \in \mathbb{Z}_p$. We want to show that $\Pr[H_{\text{xpoly}}(k, m_1) - H_{\text{xpoly}}(k, m_0) = \delta] \leq (\ell+1)/p$, where the probability is over the random choice of key k in $\mathbb{Z}_p$. Just as in the proof of Lemma 7.2, the inputs m_0 and m_1 define two polynomials $f(X)$ and $g(X)$ in $\mathbb{Z}_p[X]$, as in (7.4). However, $H_{\text{xpoly}}(k, m_1) - H_{\text{xpoly}}(k, m_0) = \delta$ holds if and only if k is root of the polynomial $X(g(X) - f(X)) - \delta$, which is a nonzero polynomial of degree at most $\ell + 1$, and so has at most $\ell + 1$ roots in $\mathbb{Z}_p$. Thus, the chances of choosing such a k is at most $(\ell+1)/p$. □

Remark 7.4. We can modify H_{xpoly} to operate on n-bit blocks by doing all arithmetic in the finite field $\text{GF}(2^n)$ instead of $\mathbb{Z}_p$. The exact same analysis as in Lemma 7.8 shows that the resulting hash function is an $(\ell+1)/2^n$-XOR-DUF. □

We now turn to the security analysis of the Carter-Wegman construction.

Theorem 7.9 (Carter-Wegman security). *Let F be a secure PRF defined over $(\mathcal{K}_F, \mathcal{R}, \mathcal{T})$ where $|\mathcal{R}|$ is super-poly. Let H be an computational DUF defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{T})$. Then the Carter-Wegman MAC $\mathcal{I}_{\text{CW}}$ derived from F and H is a secure MAC.*

In particular, for every MAC adversary $\mathcal{A}$ that attacks $\mathcal{I}_{\text{CW}}$ as in Attack Game 6.1, there exist a PRF adversary $\mathcal{B}_F$ and a DUF adversary $\mathcal{B}_H$, which are elementary wrappers around $\mathcal{A}$, such that

$$\text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}_{\text{CW}}] \leq \text{PRF}\mathsf{adv}[\mathcal{B}_F, F] + \text{DUF}\mathsf{adv}[\mathcal{B}_H, H] + \frac{Q^2}{2|\mathcal{R}|} + \frac{1}{|\mathcal{T}|}. \tag{7.24}$$

Remark 7.5. To understand why H needs to be a DUF, let us suppose for a minute that it is not. In particular, suppose it was easy to find distinct $m_0, m_1 \in \mathcal{M}$ and $\delta \in \mathcal{T}$ such that $H(k_1, m_1) = H(k_1, m_0) + \delta$, without knowledge of k_1. The adversary could then ask for the tag on the message m_0 and obtain (r, v) where $v = H(k_1, m_0) + F(k_2, r)$. Since

$$v = H(k_1, m_0) + F(k_2, r) \quad \Longrightarrow \quad v + \delta = H(k_1, m_1) + F(k_2, r),$$

the tag $(r, v + \delta)$ is a valid tag for m_1. Therefore, $\big(m_1,\ (r, v + \delta)\big)$ is an existential forgery on $\mathcal{I}_{\text{CW}}$. This shows that the Carter-Wegman MAC is easily broken when the hash function H is instantiated with H_{poly}. $\Box$

Remark 7.6. We also note that the term $Q^2/2|R|$ in (7.24) corresponds to the probability that two signing queries generate the same randomizer. In fact, if such a collision occurs, Carter-Wegman may be completely broken for certain DUFs (including H_{xpoly}) — see Exercises 7.13 and 7.14. $\Box$

Proof idea. Let $\mathcal{A}$ be an efficient MAC adversary that plays Attack Game 6.1 with respect to $\mathcal{I}_{\text{CW}}$. We derive an upper bound on $\text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}_{\text{CW}}]$. As usual, we first replace the underlying secure PRF F with a truly random function $f \in \text{Funs}[\mathcal{R}, \mathcal{T}]$ and argue that this doesn't change the adversary's advantage much. We then show that only three things can happen that enable the adversary to generate a forged message-tag pair and that the probability for each of those is small:

1. The challenger might get unlucky and choose the same randomizer $r \in \mathcal{R}$ to respond to two separate signing queries. This happens with probability at most $Q^2/(2|\mathcal{R}|)$.

2. The adversary might output a MAC forgery $\big(m, (r, v)\big)$ where $r \in \mathcal{R}$ is a fresh randomizer that was never used to respond to $\mathcal{A}$'s signing queries. Then $f(r)$ is independent of $\mathcal{A}$'s view and therefore the equality $v = H(k_1, m) + f(r)$ will hold with probability at most $1/|\mathcal{T}|$.

3. Finally, the adversary could output a MAC forgery $\big(m, (r, v)\big)$ where $r = r_j$ for some uniquely determined signed message-tag pair $(m_j, (r_j, v_j))$. But then

   $$v_j = H(k_1, m_j) + f(r_j) \quad \text{and} \quad v = H(k_1, m) + f(r_j).$$

   By subtracting the right equality from the left, the $f(r_j)$ term cancels, and we obtain

   $$v_j - v = H(k_1, m_j) - H(k_1, m).$$

   But since H is an computational DUF, the adversary can find such a relation with only negligible probability.

$\square$

Proof. We make the intuitive argument above rigorous by considering $\mathcal{A}$'s behavior in three closely related games. For $j = 0, 1, 2$, we define W_j to be the event that $\mathcal{A}$ wins Game j. Game 0 will be identical to the original MAC attack game with respect to $\mathcal{I}$. We then slightly modify each game in turn and argue that the attacker will not detect these modifications. Finally, we argue that $\Pr[W_3]$ is negligible, which will prove that $\Pr[W_0]$ is negligible, as required.

Game 0. We begin by reviewing the challenger in the MAC Attack Game 6.1 with respect to $\mathcal{I}_{\mathrm{CW}}$. We implement the challenger in this game as follows:

Initialization:
    $k_1 \xleftarrow{R} \mathcal{K}_H, \quad k_2 \xleftarrow{R} \mathcal{K}_F$
    $r_1, \ldots, r_Q \xleftarrow{R} \mathcal{R}$ // *prepare randomizers needed for the game*
upon receiving the ith signing query $m_i \in \mathcal{M}$ (for $i = 1, 2, \ldots$) do:
    $v_i \leftarrow H(k_1, m_i) + F(k_2, r_i) \in \mathcal{T}$
    send (r_i, v_i) to the adversary

At the end of the game, $\mathcal{A}$ outputs a message-tag pair $(m, (r, v))$ that is not among the signed message-tag pairs produced by the challenger. The winning condition in this game is defined to be the result of the following subroutine:

    if $v = H(k_1, m) + F(k_2, r)$
        then return win
        else return lose

Then, by construction

$$\mathrm{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}_{\mathrm{CW}}] = \Pr[W_0]. \tag{7.25}$$

Game 1. We next play the usual "PRF card," replacing the function $F(k_2, \cdot)$ by a truly random function f in $\mathrm{Funs}[\mathcal{R}, \mathcal{T}]$, which we implement as a faithful gnome (as in Section 4.4.2). Our challenger in Game 1 thus works as follows:

Initialization:
    $k_1 \xleftarrow{R} \mathcal{K}_H$
    $r_1, \ldots, r_Q \xleftarrow{R} \mathcal{R}$ // *prepare randomizers needed for the game*
    $u'_0, u'_1, \ldots, u'_Q \xleftarrow{R} \mathcal{T}$ // *prepare default f outputs*
upon receiving the ith signing query $m_i \in \mathcal{M}$ (for $i = 1, 2, \ldots$) do:
    $u_i \leftarrow u'_i$
(1)   if $r_i = r_j$ for some $j < i$ then $u_i \leftarrow u_j$
    $v_i \leftarrow H(k_1, m_i) + u_i \in \mathcal{T}$
    send (r_i, v_i) to the adversary

Suppose $\mathcal{A}$ makes exactly $s \leq Q$ signing queries before outputting its forgery attempt $(m, (r, v))$. The subroutine for the winning condition becomes:

$$
\begin{array}{ll}
(2) & \text{if } r = r_j \text{ for some } j = 1, \ldots, s \\
& \quad \text{then } u \leftarrow u_j \\
& \quad \text{else } \ u \leftarrow u'_0 \\
& \text{if } v = H(k_1, m) + u \\
& \quad \text{then return } \mathsf{win} \\
& \quad \text{else } \ \text{return } \mathsf{lose}.
\end{array}
$$

For $i = 1, \ldots, Q$, the value u'_i is chosen in advance to be the default, random value for $u_i = f(r_i)$. The tests at the lines marked (1) and (2) ensure that our gnome is faithful, i.e., that we emulate a function in Funs$[\mathcal{R}, \mathcal{T}]$. At (2), if the value $u = f(r)$ has already been defined, we use that value; otherwise, we use the fresh random value u'_0 for u.

As usual, one can show that there is a PRF adversary $\mathcal{B}_F$, just as efficient as $\mathcal{A}$, such that:

$$
\big|\Pr[W_1] - \Pr[W_0]\big| = \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F] \tag{7.26}
$$

Game 2. We make our gnome forgetful. We do this by deleting the line marked (1) in the challenger. In addition, we insert the following special test before the line marked (2) in the winning subroutine:

$$
\text{if } r_i = r_j \text{ for some } 1 \le i < j \le s \text{ then return } \mathsf{lose}
$$

Let Z to be the event that $r_i = r_j$ for some $1 \le i < j \le Q$. By the union bound we know that $\Pr[Z] \le Q^2/(2|\mathcal{R}|)$. Moreover, if Z does not happen, then Games 1 and 2 proceed identically. Therefore, by the Difference Lemma (Theorem 4.7), we obtain

$$
\big|\Pr[W_2] - \Pr[W_1]\big| \le \Pr[Z] \le Q^2/(2|\mathcal{R}|) \tag{7.27}
$$

To bound $\Pr[W_2]$, we decompose W_2 into two events:

- W'_2: $\mathcal{A}$ wins in Game 2 and $r = r_j$ for some $j = 1, \ldots, s$;
- W''_2: $\mathcal{A}$ wins in Game 2 and $r \ne r_j$ for all $j = 1, \ldots, s$.

Thus, we have $W_2 = W'_2 \cup W''_2$, and it suffices to analyze these events separately, since

$$
\Pr[W_2] \le \Pr[W'_2] + \Pr[W''_2]. \tag{7.28}
$$

Consider W''_2 first. If this happens, then $u = u'_0$ and $v = u + H(k_1, m)$; that is, $u'_0 = v - H(k_1, m)$. But since u'_0 and $v - H(k_1, m)$ are independent, this happens with probability $1/|\mathcal{T}|$. So we have

$$
\Pr[W''_2] \le 1/|\mathcal{T}|. \tag{7.29}
$$

Next, consider W'_2. Our goal here is to show that

$$
\Pr[W'_2] \le \mathrm{DUF}\mathsf{adv}[\mathcal{B}_H, H] \tag{7.30}
$$

for a DUF adversary $\mathcal{B}_H$ that is just as efficient as $\mathcal{A}$. To this end, consider what happens if $\mathcal{A}$ wins in Game 2 and $r = r_j$ for some $j = 1, \ldots, s$. Since $\mathcal{A}$ wins, and because of the special test that we added above the line marked (2), the values $r_1, \ldots, r_s$ are distinct, and so there can be only one such index j, and $u = u_j$. Therefore, we have the following two equalities:

$$
v_j = H(k_1, m_j) + u_j \qquad \text{and} \qquad v = H(k_1, m) + u_j;
$$

subtracting, we obtain

$$v_j - v = H(k_1, m_j) - H(k_1, m). \tag{7.31}$$

We claim that $m \neq m_j$. Indeed, if $m = m_j$, then (7.31) would imply $v = v_j$, which would imply $(m, (r, v)) = (m_j, (r_j, v_j))$; however, this is impossible, since we require that $\mathcal{A}$ does not submit a previously signed pair as a forgery attempt.

So, if W_2' occurs, we have $m \neq m_j$ and the equality (7.31) holds. But observe that in Game 2, the challenger's responses are completely independent of k_1, and so we can easily convert $\mathcal{A}$ into a DUF adversary $\mathcal{B}_H$ that succeeds with probability at least $\Pr[W_2']$ in Attack Game 7.3. Adversary $\mathcal{B}_H$ works as follows: it interacts with $\mathcal{A}$, simulating the challenger in Game 2 by simply responding to each signing query with a random pair $(r_i, v_i) \in \mathcal{R} \times \mathcal{T}$; when $\mathcal{A}$ outputs its forgery attempt $(m, (r, v))$, $\mathcal{B}_H$ determines if $r = r_j$ and $m \neq m_j$ for some $j = 1, \ldots, s$; if so, $\mathcal{B}_H$ outputs the triple $(m_j, m, v_j - v)$. The bound (7.30) is now clear.

The theorem follows from (7.25)–(7.30). $\Box$

7.4.1 Using Carter-Wegman with polynomial UHFs

If we want to use the Carter-Wegman construction with a polynomial-based DUF, such as H_{xpoly}, then we have make an adjustment so that the digest space of the hash function is equal to the *output* space of the PRF. Again, the issue is that our example H_{xpoly} has outputs in $\mathbb{Z}_p$, while for typical implementations, the PRF will have outputs that are n-bit blocks.

Similarly to what we did in Section 7.3.2, we can choose p to be a prime that is just a little bit bigger than 2^n. This also allows us to view the inputs to the hash as n-bit blocks. Part (b) of Exercise 7.23 shows how this can be done. One can also use a prime p that is a bit smaller than 2^n (see part (a) of Exercise 7.22), although this is less convenient, because inputs to the hash will have to broken up into blocks of size less than n. Alternatively, we can use a variant of H_{xpoly} where all arithmetic is done in the finite field $\text{GF}(2^n)$, as discussed in Remark 7.4.

7.5 Nonce-based MACs

In the Carter-Wegman construction in Section 7.4, the only essential property we need for these randomizers are that they are distinct. Similar to what we did in Section 5.5, we can study nonce-based MACs: not only can this approach reduce the size of the tag, it can also improve security.

A **nonce-based MAC** is similar to an ordinary MAC and consists of a pair of *deterministic* algorithms S and V for signing and verifying tags. However, these algorithms take an additional input $\mathcal{N}$ called a nonce that lies in a **nonce-space** $\mathcal{N}$. Algorithms S and V work as follows:

- S takes as input a key $k \in \mathcal{K}$, a message $m \in \mathcal{M}$, and a nonce $\mathcal{N} \in \mathcal{N}$. It outputs a tag $t \in \mathcal{T}$.
- V takes as input four values $k, m, t, \mathcal{N}$, where k is a key, m is a message, t is a tag, and $\mathcal{N}$ is a nonce. It outputs either accept or reject.

We say that the nonce-based MAC is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T}, \mathcal{N})$. As usual, we require that tags generated by S are always accepted by V, *as long as both are given the same nonce*. The MAC must satisfy the following **correctness property**: for all keys k, all messages m, and all nonces $\mathcal{N} \in \mathcal{N}$:

$$\Pr\left[V(k,\ m,\ S(k,\ m,\ \mathcal{N}),\ \mathcal{N}\ ) = \mathsf{accept}\right] = 1.$$

Just as in Section 5.5, in order to guarantee security, the sender should avoid using the same nonce twice (on the same key). If the sender can maintain state then a nonce can be implemented using a simple counter. Alternatively, nonces can be chosen at random, so long as the nonce space is large enough to ensure that the probability of generating the same nonce twice is negligible.

7.5.1 Secure nonce-based MACs

Nonce-based MACs must be existentially unforgeable under a chosen message attack when the adversary chooses the nonces. The adversary, however, must never request a tag using a previously used nonce. This captures the idea that nonces can be chosen arbitrarily, as long as they are never reused. Nonce-based MAC security is defined using the following game.

Attack Game 7.4 (nonce-based MAC security). For a given nonce-based MAC system $\mathcal{I} = (S, V)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T}, \mathcal{N})$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger picks a random $k \xleftarrow{R} \mathcal{K}$.
- $\mathcal{A}$ queries the challenger several times. For $i = 1, 2, \ldots,$ the ith signing query consists of a pair $(m_i, \mathcal{N}_i)$ where $m_i \in \mathcal{M}$ and $\mathcal{N}_i \in \mathcal{N}$. We require that $\mathcal{N}_i \neq \mathcal{N}_j$ for all $j < i$. The challenger computes $t_i \xleftarrow{R} S(k, m_i, \mathcal{N}_i)$, and gives t_i to $\mathcal{A}$.
- Eventually $\mathcal{A}$ sends outputs a candidate forgery triple $(m, t, \mathcal{N}) \in \mathcal{M} \times \mathcal{T} \times \mathcal{N}$, where

$$(m, t, \mathcal{N}) \notin \{(m_1, t_1, \mathcal{N}_1), (m_2, t_1, \mathcal{N}_2), \ldots\}.$$

We say that $\mathcal{A}$ wins the game if $V(k, m, t, \mathcal{N}) = \mathsf{accept}$. We define $\mathcal{A}$'s advantage with respect to $\mathcal{I}$, denoted nMAC$\mathsf{adv}[\mathcal{A}, \mathcal{I}]$, as the probability that $\mathcal{A}$ wins the game. □

Definition 7.6. *We say that a nonce-based MAC system $\mathcal{I}$ is secure if for all efficient adversaries $\mathcal{A}$, the value* nMAC$\mathsf{adv}[\mathcal{A}, \mathcal{I}]$ *is negligible.*

Nonce-based Carter-Wegman MAC. The Carter-Wegman MAC (Section 7.4) can be recast as a nonce-based MAC: We simply view the randomizer $r \in \mathcal{R}$ as a nonce, supplied as an input to the signing algorithm, rather than a randomly generated value that is a part of the tag. Using the notation of Section 7.4, the MAC system is then

$$S\big((k_1, k_2),\ m,\ \mathcal{N}\big) := H(k_1, m) + F(k_2, \mathcal{N})$$

$$V\big((k_1, k_2),\ m,\ t,\ \mathcal{N}\big) := \begin{cases} \mathsf{accept} & \text{if } t = S\big((k_1, k_2),\ m,\ \mathcal{N}\big) \\ \mathsf{reject} & \text{otherwise} \end{cases}$$

We obtain the following security theorem, which is the nonce-based analogue of Theorem 7.9. The proof is essentially the same as the proof of Theorem 7.9.

Theorem 7.10. *With the notation of Theorem 7.9 we obtain the following bounds*

$$\text{nMAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}_{\text{CW}}] \leq \text{PRF}\mathsf{adv}[\mathcal{B}_F, F] + \text{DUF}\mathsf{adv}[\mathcal{B}_H, H] + \frac{1}{|\mathcal{T}|}.$$

This bound is much tighter than (7.24): the Q^2-term is gone. Of course, it is gone because we insist that the same nonce is never used twice. If nonces are, in fact, generated by the signer at random, then the Q^2-term returns; however, if the signer implements the nonce as a counter, then we avoid the Q^2-term — the only requirement is that the signer does not sign more than $|\mathcal{R}|$ values. See also Exercise 7.12 for a subtle point regarding the implementation of F.

Analogous to the discussion in Remark 7.6, when using nonce-based Carter-Wegman it is vital that the nonce is never re-used for different messages. If this happens, Carter-Wegman may be completely broken — see Exercises 7.13 and 7.14.

7.6 Unconditionally secure one-time MACs

In Chapter 2 we saw that the one-time pad gives unconditional security as long as the key is only used to encrypt a single message. Even algorithms that run in exponential time cannot break the semantic security of the one-time pad. Unfortunately, security is lost entirely if the key is used more than once.

In this section we ask the analogous question for MACs: can we build a "one-time MAC" that is unconditionally secure if the key is only used to provide integrity for a single message?

We can model one-time MACs using the standard MAC Attack Game 6.1 used to define MAC security. To capture the one-time nature of the MAC we allow the adversary to issue *only one* signing query. We denote the adversary's advantage in this restricted game by $\mathrm{MAC}_1\mathsf{adv}[\mathcal{A},\mathcal{I}]$. This game captures the fact that the adversary sees only one message-tag pair and then tries to create an existential forgery using this pair.

Unconditional security means that $\mathrm{MAC}_1\mathsf{adv}[\mathcal{A},\mathcal{I}]$ is negligible for all adversaries $\mathcal{A}$, even computationally unbounded ones. In this section, we show how to implement efficient and unconditionally secure one-time MACs using hash functions.

7.6.1 Pairwise unpredictable functions

Let H be a keyed hash function defined over $(\mathcal{K},\mathcal{M},\mathcal{T})$. Intuitively, H is a **pairwise unpredictable function** if the following holds for a randomly chosen key $k \in \mathcal{K}$: given the value $H(k,m_0)$, it is hard to predict $H(k,m_1)$ for any $m_1 \neq m_0$. As usual, we make this definition rigorous using an attack game.

Attack Game 7.5 (pairwise unpredicability). For a keyed hash function H defined over $(\mathcal{K},\mathcal{M},\mathcal{T})$, and a given adversary $\mathcal{A}$, the attack game runs as follows.

- The challenger picks a random $k \xleftarrow{\mathrm{R}} \mathcal{K}$ and keeps k to itself.
- $\mathcal{A}$ sends a message $m_0 \in \mathcal{M}$ to the challenger, who responds with $t_0 = H(k,m_0)$.
- $\mathcal{A}$ outputs $(m_1,t_1) \in \mathcal{M} \times \mathcal{T}$, where $m_1 \neq m_0$.

We say that $\mathcal{A}$ wins the game if $t_1 = H(k,m_1)$. We define $\mathcal{A}$'s advantage with respect to H, denoted $\mathrm{PUF}\mathsf{adv}[\mathcal{A},H]$, as the probability that $\mathcal{A}$ wins the game. □

Definition 7.7. *We say that H is an ϵ-**bounded pairwise unpredictable function**, or ϵ-**PUF** for short, if* $\mathrm{PUF}\mathsf{adv}[\mathcal{A},H] \leq \epsilon$ *for all adversaries $\mathcal{A}$ (even inefficient ones).*

It should be clear that if H is an ϵ-PUF, then H is also an ϵ-UHF; if, in addition, $\mathcal{T}$ is of the form $\mathbb{Z}_N$ (or is an abelian group as in Remark 7.3), then H is an ϵ-DUF.

7.6.2 Building unpredictable functions

So far we know that any ϵ-PUF is also an ϵ-DUF. The converse is not true (see Exercise 7.28). Nevertheless, we show that any ϵ-DUF can be tweaked so that it becomes an ϵ-PUF. This tweak increases the key size.

Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$, where $\mathcal{T} = \mathbb{Z}_N$ for some N. We build a new hash function H' derived from H with the same input and output space as H. The key space, however, is $\mathcal{K} \times \mathcal{T}$. The function H' is defined as follows:

$$H'\big((k_1, k_2),\ m\big) = H(k_1, m) + k_2 \quad \in \mathcal{T} \tag{7.32}$$

Lemma 7.11. *If H is an ϵ-DUF, then H' is an ϵ-PUF.*

Proof. Let $\mathcal{A}$ attack H' as a PUF. In response to its query m_0, adversary $\mathcal{A}$ receives $t_0 := H(k_1, m_0) + k_2$. Observe that t_0 is uniformly distributed over $\mathcal{T}$, and is independent of k_1. Moreover, if $\mathcal{A}$'s prediction t_1 of $H(k_1, m_1) + k_2$ is correct, then $t_1 - t_0$ correctly predicts the difference $H(k_1, m_1) - H(k_1, m_0)$.

So we can define a DUF adversary $\mathcal{B}$ as follows: it runs $\mathcal{A}$, and when $\mathcal{A}$ submits its query m_0, $\mathcal{B}$ responds with a random $t_0 \in \mathcal{T}$; when $\mathcal{A}$ outputs (m_1, t_1), adversary $\mathcal{B}$ outputs $(m_0, m_1, t_1 - t_0)$. It is clear that

$$\mathrm{PUF}\mathsf{adv}[\mathcal{A}, H] \leq \mathrm{DUF}\mathsf{adv}[\mathcal{B}, H] \leq \epsilon. \quad \Box$$

In particular, Lemma 7.11 shows how to convert the function H_{xpoly}, defined in (7.23), into a an $(\ell + 1)/p$-PUF. We obtain the following keyed hash function defined over $(\mathbb{Z}_p^2, \mathbb{Z}_p^{\leq \ell}, \mathbb{Z}_p)$:

$$H'_{\text{xpoly}}((k_1, k_2), (a_1, \ldots, a_v)) := k_1^{v+1} + a_1 k_1^v + \cdots + a_v k_1 + k_2. \tag{7.33}$$

7.6.3 From PUFs to unconditionally secure one-time MACs

We now return to the problem of building unconditionally secure one-time MACs. In fact, PUFs are just the right tool for the job.

Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$. We can use H to define the MAC system $\mathcal{I} = (S, V)$ **derived from** H:

$$S(k,\ m) := H(k,\ m);$$

$$V(k,\ m,\ t) := \begin{cases} \mathsf{accept} & \text{if } H(k, m) = t, \\ \mathsf{reject} & \text{otherwise.} \end{cases}$$

The following theorem shows that PUFs are the MAC analogue of the one-time pad, since both provide unconditional security for one time use. The proof is immediate from the definitions.

Theorem 7.12. *Let H be an ϵ-PUF and let $\mathcal{I}$ be the MAC system derived from H. Then for all adversaries $\mathcal{A}$ (even inefficient ones), we have* $\mathrm{MAC}_1\mathsf{adv}[\mathcal{A}, \mathcal{I}] \leq \epsilon$.

The PUF construction in Section 7.6.2 is very similar to the Carter-Wegman MAC. The only difference is that the PRF is replaced by a truly random pad k_2. Hence, Theorem 7.12 shows that the Carter-Wegman MAC with a truly random pad is an unconditionally secure one-time MAC.

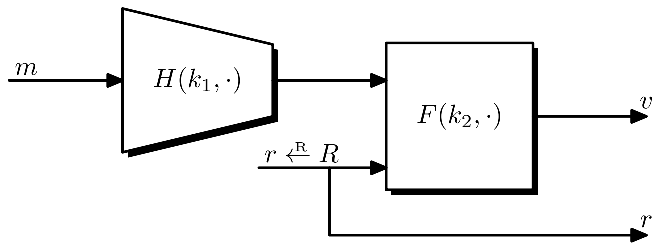


Figure 7.5: Randomized PRF(UHF) composition: MAC signing

7.7 A fun application: timing attacks

To be written.

7.8 Notes

Citations to the literature to be added.

7.9 Exercises

7.1 (Using H_{poly} with power-of-2 modulus). We can adapt the definition of H_{poly} in (7.3) so that instead of working in $\mathbb{Z}_p$ we work in $\mathbb{Z}_{2^n}$ (i.e., work modulo 2^n). Show that this version of H_{poly} is not a good UHF, and in particular an attacker can find two messages m_0, m_1 each of length two blocks that are guaranteed to collide.

7.2 (Non-adaptively secure PRFs are computational UHFs). Show that if F is a secure PRF against non-adaptive adversaries (see Exercise 4.6), and the size of the output space of F is super-poly, then F is a computational UHF.

Note: Using the result of Exercise 6.13, this gives another proof that CBC is a computational UHF.

7.3 (On the alternative characterization of the ϵ-UHF property). Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$. Suppose that for some pair of distinct messages m_0 and m_1, we have $\Pr[H(k, m_0) = H(k, m_1)] > \epsilon$, where the probability is over the random choice of $k \in \mathcal{K}$. Give an adversary $\mathcal{A}$ that wins Attack Game 7.1 with probability greater than ϵ. Your adversary is not allowed to just have the values m_0 and m_1 "hardwired" into its code, but it may be *very* inefficient.

7.4 (MAC(UHF) composition is insecure). The PRF(UHF) composition shows that a UHF can extend the input domain of a specific type of MAC, namely a MAC that is itself a PRF. Show that this construction cannot be extended to arbitrary MACs. That is, exhibit a secure MAC $\mathcal{I} = (S, V)$ and a computational UHF H for which the MAC(UHF) composition $\mathcal{I}' = (S', V')$ where $S'((k_1, k_2), m) = S(k_2, H(k_1, m))$ is insecure. In your design, you may assume the existence of a secure PRF defined over any convenient spaces. Then show how to "sabotage" this PRF so that it remains a secure MAC, but the MAC(UHF) composition becomes insecure.

7.5 (Randomized PRF(UHF) composition). In this exercise we develop a randomized variant of PRF(UHF) composition that provides better security with little impact on the running time. Let H be a keyed hash function defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{X})$ and let F be a PRF defined over $(\mathcal{K}_F,\ \mathcal{R} \times \mathcal{X},\ \mathcal{T})$. Define the **randomized PRF(UHF) system** $\mathcal{I} = (S, V)$ as follows: for key (k_1, k_2) and message $m \in \mathcal{M}$ define

$$S\big((k_1,k_2),\ m\big) := \big\{r \xleftarrow{\text{R}} \mathcal{R},\ x \leftarrow H(k_1,m),\ v \leftarrow F\big(k_2,(r,x)\big),\ \text{output } (r,v)\big\} \qquad \text{(see Fig. 7.5)}$$

$$V\big((k_1,k_2),\ m,\ (r,v)\big) := \begin{cases} \mathsf{accept} & \text{if } x \leftarrow H(k_1,m),\ v = F\big(k_2,(r,x)\big) \\ \mathsf{reject} & \text{otherwise.} \end{cases}$$

This MAC is defined over $(\mathcal{K}_F \times \mathcal{K}_H,\ \mathcal{M},\ \mathcal{R} \times \mathcal{T})$. The tag size is a little larger than in deterministic PRF(UHF) composition, but signing and verification time is about the same.

(a) Suppose $\mathcal{A}$ is a MAC adversary that plays Attack Game 6.1 with respect to $\mathcal{I}$ and issues at most Q queries. Show that there exists a PRF adversary $\mathcal{B}_F$ and UHF adversaries $\mathcal{B}_H$ and $\mathcal{B}'_H$, which are elementary wrappers around $\mathcal{A}$, such that

$$\begin{aligned} \text{MAC}\mathsf{adv}[\mathcal{A},\mathcal{I}] \le \text{PRF}\mathsf{adv}[\mathcal{B}_F, F] + \text{UHF}\mathsf{adv}[\mathcal{B}_H, H] + \frac{Q^2}{2|\mathcal{R}|}\text{UHF}\mathsf{adv}[\mathcal{B}'_H, H] \\ + \frac{Q^2}{2|\mathcal{R}||\mathcal{T}|} + \frac{1}{|\mathcal{T}|}. \end{aligned} \tag{7.34}$$

Discussion: When H is an ϵ-UHF let us set $\epsilon = 1/|\mathcal{T}|$ and $|\mathcal{R}| = Q^2/2$ so that the right most four terms in (7.34) are all equal. Then (7.34) becomes simply

$$\text{MAC}\mathsf{adv}[\mathcal{A},\mathcal{I}] \le \text{PRF}\mathsf{adv}[\mathcal{B}_F, F] + 4\epsilon. \tag{7.35}$$

Comparing to deterministic PRF(UHF) composition, the error term $\epsilon \cdot Q^2/2$ in (7.19) is far worse than in (7.35). This means that for the same parameters, randomized PRF(UHF) composition security is preserved for far many more queries than for deterministic PRF(UHF) composition.

In the Carter-Wegman MAC to get an error bound as in (7.35) we must set $|\mathcal{R}|$ to $|Q|^2/\epsilon$ in (7.24). In randomized PRF(UHF) composition we only need $|\mathcal{R}| = |Q|^2$ and therefore tags in randomized PRF(UHF) are shorter than in Carter-Wegman for the same security and the same ϵ.

(b) Rephrase the MAC system $\mathcal{I}$ as a nonce-based MAC system (as in Section 7.5). What are the concrete security bounds for this system?

Observe that if the nonce is accidentally re-used, or even always set to the same value, then the MAC system $\mathcal{I}$ still provides some security: security degrades to the security of deterministic PRF(UHF) composition. We refer to this as **nonce re-use resistance**.

7.6 (One-key PRF(UHF) composition). This exercise analyzes a one-key variant of the PRF(UHF) construction. Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ and let H be a keyed hash function defined over $(\mathcal{Y}, \mathcal{M}, \mathcal{X})$; in particular, the output space of F is equal to the key space of

H, and the output space of H is equal to the input space of F. Let $x_0 \in \mathcal{X}$ be a public constant. Consider the PRF F' defined over $(\mathcal{K}, \mathcal{M}, \mathcal{Y})$ as follows:

$$F'(k, m) := F(k, H(k_0, m)), \quad \text{where} \quad k_0 := F(k, x_0).$$

This is the same as the usual PRF(UHF) composition, except that we use a single key k and use F to derive the key k_0 for H.

(a) Show that F' is a secure PRF assuming that F is a PRF, that H is a computational UHF, and that H satisfies a certain *preimage resistance* property, defined by the following game.

In this game, the adversary computes a message M and the challenger (independently) chooses a random hash key $k_0 \in \mathcal{K}$. The adversary wins the game if $H(k_0, M) = x_0$, where $x_0 \in \mathcal{X}$ is a constant, as above. We say that H is preimage resistant if every efficient adversary wins this game with only negligible probability.

Hint: Modify the proof of Theorem 7.7.

(b) Show that the cascade construction is preimage resistant, assuming the underlying PRF is a secure PRF.

Hint: This follows almost immediately from the fact that the cascade is a prefix-free PRF.

7.7 (XOR-DUFs). In Remark 7.3 we adapted the definition of DUF to a hash function whose digest space $\mathcal{T}$ is the set of all n-bit strings, $\{0,1\}^n$, with the XOR used as the difference operator.

(a) Show that the XOR-hash $F^{\oplus}$ defined in Section 7.2.3 is a computational XOR-DUF.

(b) Show that the CBC construction F_{CBC} defined in Section 6.4.1 is a computational XOR-DUF.

Hint: Use the fact that F_{CBC} is a prefix-free secure PRF (or, alternatively, the result of Exercise 6.13).

7.8 (Luby-Rackoff with an XOR-DUF). Show that the Luby-Rackoff construction (see Section 4.5) remains secure if the first round function $F(k_1, \cdot)$ is replaced by a computational XOR-DUF.

7.9 (Nonce-based CBC cipher with an XOR-DUF). Show that in the nonce-based CBC cipher (Section 5.5.3) the PRF that is applied to the nonce can be replaced by an XOR-DUF.

7.10 (Tweakable block ciphers). Continuing with Exercise 4.11, show that in the construction from part (c) the PRF can be replaced by an XOR-DUF. That is, prove that the following construction is a strongly secure tweakable block cipher:

$$E'\big((k_0, k_1), m, t\big) := \big\{p \leftarrow h(k_0, t); \;\; \text{output } p \oplus E(k_1, \; m \oplus p) \;\big\}$$
$$D'\big((k_0, k_1), c, t\big) := \big\{p \leftarrow h(k_0, t); \;\; \text{output } p \oplus D(k_1, \; c \oplus p) \;\big\}$$

Here (E, D) is a strongly secure block cipher defined over $(\mathcal{K}_0, \mathcal{X})$ and h is an XOR-DUF defined over $(\mathcal{K}_1, \mathcal{T}, \mathcal{X})$ where $\mathcal{X} := \{0,1\}^n$.

Discussion: **XTS mode**, used in disk encryption systems, is based on this tweakable block cipher. The tweak in XTS is a combination of i, the disk sector number, and j, the position of the block within the sector. The XOR-DUF used in XTS is defined as $h\big(k_0, (i,j)\big) := E(k_0, i) \cdot \alpha^j \; \in \mathrm{GF}(2^n)$ where α is a fixed primitive element of $\mathrm{GF}(2^n)$. XTS uses ciphertext stealing (Exercise 5.16) to handle sectors whose bit length is not a multiple of n.

7.11 (Carter-Wegman with verification queries: concrete security). Consider the security of the Carter-Wegman construction (Section 7.4) in an attack with verification queries (Section 6.2). Show that following concrete security result: for every MAC adversary $\mathcal{A}$ that attacks $\mathcal{I}_{\mathrm{CW}}$ as in Attack Game 6.2, and which makes at most Q_{v} verification queries and at most Q_{s} signing queries, there exist a PRF adversary $\mathcal{B}_F$ and a DUF adversary $\mathcal{B}_H$, which are elementary wrappers around $\mathcal{A}$, such that

$$\mathrm{MAC^{vq}}\mathsf{adv}[\mathcal{A}, \mathcal{I}_{\mathrm{CW}}] \le \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F] + Q_{\mathrm{v}} \cdot \mathrm{DUF}\mathsf{adv}[\mathcal{B}_H, H] + \frac{Q_{\mathrm{s}}^2}{2|\mathcal{R}|} + \frac{Q_{\mathrm{v}}}{|\mathcal{T}|}.$$

7.12 (Nonce-based Carter-Wegman: improved security bounds). In Section 7.5, we studied a nonce-based version of the Carter-Wegman MAC. In particular, in Theorem 7.10, we derived the security bound

$$\mathrm{nMAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}_{\mathrm{CW}}] \le \mathrm{PRF}\mathsf{adv}[\mathcal{B}_F, F] + \mathrm{DUF}\mathsf{adv}[\mathcal{B}_H, H] + \frac{1}{|\mathcal{T}|},$$

and rejoiced in the fact that there were no Q^2-terms in this bound, where Q is a bound on the number of signing queries. Unfortunately, a common implementation of F is to use the encryption function of a block cipher $\mathcal{E}$ defined over $(\mathcal{K}, \mathcal{X})$, so $\mathcal{R} = \mathcal{X} = \mathcal{T} = \mathbb{Z}_N$. A straightforward application of the PRF switching lemma (see Theorem 4.4) gives us the security bound

$$\mathrm{nMAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}_{\mathrm{CW}}] \le \mathrm{BC}\mathsf{adv}[\mathcal{B}_{\mathcal{E}}, \mathcal{E}] + \frac{Q^2}{2N} + \mathrm{DUF}\mathsf{adv}[\mathcal{B}_H, H] + \frac{1}{N},$$

and a Q^2-term has returned! In particular, when $Q^2 \approx N$, this bound is entirely useless. However, one can obtain a better bound. Using the result of Exercise 4.25, show that assuming $Q^2 < N$, we have the following security bound:

$$\mathrm{nMAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}_{\mathrm{CW}}] \le \mathrm{BC}\mathsf{adv}[\mathcal{B}_{\mathcal{E}}, \mathcal{E}] + 2 \cdot \left(\mathrm{DUF}\mathsf{adv}[\mathcal{B}_H, H] + \frac{1}{N}\right).$$

7.13 (Carter-Wegman MAC falls apart under nonce re-use). Suppose that when using a nonce-based MAC, an implementation error causes the system to re-use a nonce more than once. Let us show that the nonce-based Carter-Wegman MAC falls apart if this ever happens.

(a) Consider the nonce-based Carter-Wegman MAC built from the hash function H_{xpoly}. Show that if the adversary obtains the tag on some one-block message m_1 using nonce $\mathcal{N}$ and the tag on a different one-block message m_2 using *the same* nonce $\mathcal{N}$, then the MAC system becomes insecure: the adversary can forge the MAC an any message of his choice with non-negligible probability.

(b) Consider the nonce-based Carter-Wegman MAC with an arbitrary hash function. Suppose that an adversary is free to re-use nonces at will. Show how to create an existential forgery.

Note: These attacks also apply to the randomized version of Carter-Wegman, if the signer is unlucky enough to generate the same randomizer $r \in \mathcal{R}$ more than once. Also, the attack in part (a) can be extended to work even if the messages are not single-block messages by using efficient algorithms for finding roots of polynomials over finite fields.

7.14 (Encrypted Carter-Wegman). Continuing with the previous exercise, we show how to make Carter-Wegman resistant to nonce re-use by encrypting the tag. To make things more concrete, suppose that H is an ϵ-DUF defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{X})$, where $\mathcal{X} = \mathbb{Z}_N$, and $\mathcal{E} = (E, D)$ is a secure block cipher defined over $(\mathcal{K}_\mathcal{E}, \mathcal{X})$. The encrypted Carter-Wegman nonce-based MAC system $\mathcal{I} = (S, V)$ has key space $\mathcal{K}_H \times \mathcal{K}_\mathcal{E}^2$, message space $\mathcal{M}$, tag space $\mathcal{X}$, nonce space $\mathcal{X}$, and is defined as follows:

- For key (k_1, k_2, k_3), message m, and nonce $\mathcal{N}$, we define

$$S((k_1, k_2, k_3), m, \mathcal{N}) := E(k_3,\ H(k_1, m) + E(k_2, \mathcal{N})\ )$$

- For key (k_1, k_2, k_3), message m, tag v, and nonce $\mathcal{N}$, we define

$$\begin{aligned}&V((k_1, k_2, k_3), m, v, \mathcal{N}) :=\\ &\qquad v^* \leftarrow E(k_3,\ H(k_1, m) + E(k_2, \mathcal{N})\ )\\ &\qquad \text{if } v = v^* \text{ output } \mathsf{accept}\text{; otherwise output } \mathsf{reject}\end{aligned}$$

(a) Show that assuming no nonces get re-used, this scheme is just as secure as Carter-Wegman. In particular, using the result of Exercise 7.12, show that for every adversary $\mathcal{A}$ that makes at most Q signing queries, where $Q^2 < N$, the probability that $\mathcal{A}$ produces an existential forgery is at most $\mathrm{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}] + 2(\epsilon + 1/N)$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$.

(b) Now suppose an adversary can re-use nonces at will. Show that for every such adversary $\mathcal{A}$ that makes at most Q signing queries, where $Q^2 < N$, the probability that $\mathcal{A}$ produces an existential forgery is at most $\mathrm{BC}\mathsf{adv}[\mathcal{B}, \mathcal{E}] + (Q + 1)^2\epsilon + 2/N$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$. Thus, while nonce re-use degrades security, it is not catastrophic.

Hint: Theorem 7.7 and Exercises 4.25 and 7.21 may be helpful.

7.15 (Composing UHFs). Let H_1 be a keyed hash function defined over $(\mathcal{K}_1, \mathcal{X}, \mathcal{Y})$. Let H_2 be a keyed hash function defined over $(\mathcal{K}_2, \mathcal{Y}, \mathcal{Z})$. Let H be the keyed hash function defined over $(\mathcal{K}_1 \times \mathcal{K}_2, \mathcal{X}, \mathcal{Z})$ as $H((k_1, k_2), x) := H_2(k_2, H(k_1, x))$.

(a) Show that if H_1 is an ϵ_1-UHF and H_2 is an ϵ_2-UHF, then H is an $(\epsilon_1 + \epsilon_2)$-UHF.

(b) Show that if H_1 is an ϵ_1-UHF and H_2 is an ϵ_2-DUF, then H is an $(\epsilon_1 + \epsilon_2)$-DUF.

7.16 (Variations on H_{poly}). Show that if p is prime and the input space is $\mathbb{Z}_p^\ell$ for some fixed (poly-bounded) value ℓ, then

(a) the function H_{fpoly} defined in (7.5) is an $(\ell - 1)/p$-UHF.

(b) the function H_{fxpoly} defined as

$$H_{\mathrm{fxpoly}}(k, (a_1, \ldots, a_\ell)) := k \cdot H_{\mathrm{fpoly}}(k, (a_1, \ldots, a_\ell)) = a_1 k^\ell + a_2 k^{v-1} + \cdots + a_\ell k \in \mathbb{Z}_p$$

is an (ℓ/p)-DUF.

7.17 (A DUF from an ideal permutation). Let $\pi : \mathcal{X} \to \mathcal{X}$ be an permutation where $\mathcal{X} := \{0,1\}^n$. Define $H : \mathcal{X} \times \mathcal{X}^{\leq \ell} \to \mathcal{X}$ as the following keyed hash function:

$$\begin{aligned}H(k, (a_1, \ldots, a_v)) :=\quad & h \leftarrow k\\ & \text{for } i \leftarrow 1 \text{ to } v \text{ do:}\quad h \leftarrow \pi(a_i \oplus h)\\ & \text{output } h\end{aligned}$$

Assuming 2^n is super-poly, show that H is a computational XOR-DUF (see Remark 7.3) in the ideal permutation model, where we model π as a random permutation Π (see Section 4.7).

We outline here one possible proof approach. The first idea is to use the same strategy that was used in the analysis of CBC in the proof of Theorem 6.3; indeed, one can see that the two constructions process message blocks in a very similar way. The second idea is to use the Domain Separation Lemma (Theorem 4.15) to streamline the proof.

Consider two games:

0. The original attack game: adversary makes a series of ideal permutation queries, which evaluate Π and Π^{-1} on points of the adversary's choice. Then the adversary submits two distinct messages m_0, m_1 to the challenger, along with a value δ, and hopes that $H(k, m_0) \oplus H(k, m_1) = \delta$.

1. Use the Domain Separation Lemma to split Π into many independent permutations. One is Π_{ip}, which is used to evaluate the ideal permutation queries. The others are of the form $\Pi_{\text{std},\alpha}$ for $\alpha \in \mathcal{X}_{>0}^{\leq \ell}$. These are used to perform the evaluations $H(k, m_0)$, $H(k, m_1)$: in the evaluation of $H(k, (a_1, \ldots, a_s))$, in the ith loop iteration in the hash algorithm, we use the permutation $\Pi_{\text{std},\alpha}$, where $\alpha = (a_1, \ldots, a_i)$. Now one just has to analyze the probability of separation failure.

Note that H is certainly not a secure PRF, even if we restrict ourselves to non-adaptive or prefix-free adversaries: given $H(k, m)$ for any message m, we can efficiently compute the key k.

7.18 (Optimal collision probability with shorter hash keys). For positive integer d, let $I_d := \{0, \ldots, d-1\}$ and $I_d^* := \{1, \ldots, d-1\}$.

(a) Let N be a positive integer and p be a prime. Consider the keyed hash function H defined over $(I_p \times I_p^*, I_p, I_N)$ as follows: $H((k_0, k_1), a) := ((k_0 + ak_1) \bmod p) \bmod N$. Show that H is a $1/N$-UHF.

(b) While the construction in part (a) gives a UHF with "optimal" collision probability, the key space is unfortunately larger than the message space. Using the result of part (a), along with part (a) of Exercise 7.15 and the result of Exercise 7.16, you are to design a hash function with "nearly optimal" collision probability, but with much smaller keys.

    Let N and ℓ be positive integers. Let α be a number with $0 < \alpha < 1$. Design a $(1 + \alpha)/N$-UHF with message space $\{0,1\}^\ell$ and output space I_N, where keys bit strings of length $O(\log(N\ell/\alpha))$.

7.19 (Inner product hash). Let p be a prime.

(a) Consider the keyed hash function H defined over $(\mathbb{Z}_p^\ell, \mathbb{Z}_p^\ell, \mathbb{Z}_p)$ as follows:

$$H((k_1, \ldots, k_\ell), (a_1, \ldots, a_\ell)) := a_1k_1 + \cdots + a_\ell k_\ell.$$

    Show that H is a $1/p$-DUF.

(b) Since multiplications can be much more expensive than additions, the following variant of the hash function in part (a) is sometimes preferable. Assume ℓ is even, and consider the keyed hash function H' defined over $(\mathbb{Z}_p^\ell, \mathbb{Z}_p^\ell, \mathbb{Z}_p)$ as follows:

$$H'((k_1, \dots, k_\ell), (a_1, \dots, a_\ell)) := \sum_{i=1}^{\ell/2} (a_{2i-1} + k_{2i-1})(a_{2i} + k_{2i}).$$

Show that H' is also a $1/p$-DUF.

(c) Although both H and H' are ϵ-DUFs with "optimal" ϵ values, the keys are unfortunately very large. Using a similar approach to part (b) of the previous exercise, design a $(1+\alpha)/p$-DUF with message space $\{0,1\}^\ell$ and output space $\mathbb{Z}_p$, where keys bit strings of length $O(\log(p\ell/\alpha))$.

7.20 (Division-free hash). This exercise develops a hash function that does not require and division or mod operations, which can be expensive. It can be implemented just using shifts and adds. For positive integer d, let $I_d := \{0, \dots, d-1\}$. Let n be a positive integer and set $N := 2^n$.

(a) Consider the keyed hash function H defined over $(I_{N^2}^\ell, I_N^\ell, \mathbb{Z}_N)$ as follows:

$$H((k_1, \dots, k_\ell), (a_1, \dots, a_\ell)) := [t]_N \in \mathbb{Z}_N, \quad \text{where } \ t := \big\lfloor \big(\ \big(\sum_i a_i k_i \big) \bmod N^2 \ \big) \ / \ N \big\rfloor.$$

Show that H is a $2/N$-DUF. Below in Exercise 7.30 we will see a minor variant of H that satisfies a stronger property, and in particular, is a $1/N$-DUF.

(b) Analogous to part (b) in the previous exercise, assume ℓ is even, and consider the keyed hash function H defined over $(I_{N^2}^\ell, I_N^\ell, \mathbb{Z}_N)$ as follows:

$$H'((k_1, \dots, k_\ell), (a_1, \dots, a_\ell)) := [t]_N \in \mathbb{Z}_N,$$

where

$$t := \big\lfloor \big(\ \big(\sum_{i=1}^{\ell/2} (a_{2i-1} + k_{2i-1})(a_{2i} + k_{2i}) \big) \bmod N^2 \ \big) \ / \ N \big\rfloor.$$

Show that H' is a $2/N$-DUF.

7.21 (DUF to UHF conversion). Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathbb{Z}_N)$. We construct a new keyed hash function H', defined over $(\mathcal{K}, \mathcal{M} \times \mathbb{Z}_N, \mathbb{Z}_N)$ as follows: $H'(k, (m, x)) := H(k, m) + x$. Show that if H is an ϵ-DUF, then H' is an ϵ-UHF.

7.22 (DUF modulus switching). We will be working with DUFs with digest spaces $\mathbb{Z}_m$ for various m, and so to make things clearer, we will work with digest spaces that are plain old sets of integers, and state explicitly the modulus m, as in "an ϵ-DUF modulo m". For positive integer d, let $I_d := \{0, \dots, d-1\}$.

Let p and N be integers greater than 1. Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, I_p)$. Let H' be the keyed hash function defined over $(\mathcal{K}, \mathcal{M}, I_N)$ as follows: $H'(k, m) := H(k, m) \bmod N$.

(a) Show that if $p \le N/2$ and H is an ϵ-DUF modulo p, then H' is an ϵ-DUF modulo N.

(b) Suppose that $p \ge N$ and H is an ϵ-DUF modulo p. Show that H' is an ϵ'-DUF modulo N for $\epsilon' = 2(p/N + 1)\epsilon$. In particular, if $\epsilon = \alpha/p$, we can take $\epsilon' = 4\alpha/N$.

7.23 (More flexible output spaces). As in the previous exercise, we work with DUFs whose digest spaces are plain old sets of integers, but we explicitly state the modulus m. Again, for positive integer d, we let $I_d := \{0, \dots, d-1\}$.

Let $1 < N \le p$, where p is prime.

(a) H^*_{fxpoly} is the keyed hash function defined over (I_p, I_N^ℓ, I_N) as follows:

$$H^*_{\text{fxpoly}}(k, (a_1, \dots, a_\ell)) := \Big((a_1 k^\ell + \cdots + a_\ell k) \bmod p\Big) \bmod N.$$

Show that H^*_{fxpoly} is a $4\ell/N$-DUF modulo N.

(b) H^*_{xpoly} is the keyed hash function defined over $(I_p, I_N^{\le \ell}, I_N)$ as follows:

$$H^*_{\text{xpoly}}(k, (a_1, \dots, a_v)) := \Big((k^{v+1} + a_1 k^v + \cdots + a_v k) \bmod p\Big) \bmod N.$$

Show that H^*_{xpoly} is a $4(\ell+1)/N$-DUF modulo N.

(c) H^*_{fpoly} is the keyed hash function defined over (I_p, I_N^ℓ, I_N) as follows:

$$H^*_{\text{fpoly}}(k, (a_1, \dots, a_\ell)) := \Big(\big((a_1 k^{\ell-1} + \cdots + a_{\ell-1} k) \bmod p\big) + a_\ell\Big) \bmod N.$$

Show that H^*_{fpoly} is a $4(\ell-1)/N$-UHF.

(d) H^*_{poly} is the keyed hash function is defined over $(I_p, I_N^{\le \ell}, I_N)$ as follows:

$$H^*_{\text{poly}}(k, (a_1, \dots, a_v)) := \Big(\big((k^v + a_1 k^{v-1} + \cdots + a_{v-1} k) \bmod p\big) + a_v\Big) \bmod N.$$

for $v > 0$, and for zero-length messages, it is defined to be the constant 1. Show that H^*_{poly} is a $4\ell/N$-UHF.

Hint: All of these results follow easily from the previous two exercises, except that the analysis in part (d) requires that zero-length messages are treated separately.

7.24 (Be careful: reducing at the wrong time can be dangerous). With notation as in the previous exercise, show that if $(3/2)N \le p < 2N$, the keyed hash function H defined over (I_p, I_N^2, I_N) as

$$H(k, (a, b)) := ((ak + b) \bmod p) \bmod N$$

is not a $(1/3)$-UHF. Contrast this function with that in part (c) of the previous exercise with $\ell = 2$.

7.25 (A PMAC$_0$ ***alternative).*** Again, for positive integer d, let $I_d := \{0, \dots, d-1\}$. Let $N = 2^n$ and let p be a prime with $N/4 < p < N/2$. Let H be the hash function defined over $(I_{N/4}, I_N \times I_{N/4}, I_N)$ as follows:

$$H(k, (a, i)) := (((i \cdot k) \bmod p) + a) \bmod N.$$

(a) Show that H is a $4/N$-UHF.

Hint: Use Exercise 7.21 and part (a) of Exercise 7.22.

(b) Show how to use H to modify PMAC_0 so that the message space is $\mathcal{Y}^{\leq \ell}$ (where $\mathcal{Y} = \{0,1\}^n$ and $\ell < N/4$), and the PRF F_1 is defined over $(\mathcal{K}_1, \mathcal{Y}, \mathcal{Y})$. Analyze the security of your construction, giving a concrete security bound.

7.26 (Collision lower-bounds for H_{poly}). Consider the function $H_{\text{poly}}(k, m)$ defined in (7.3) using a prime p and assume $\ell = 2$.

(a) Show that for all sufficiently large p, the following holds: for any fixed $k \in \mathbb{Z}_p$, among $\lfloor \sqrt{p} \rfloor$ random inputs to $H_{\text{poly}}(k, \cdot)$, the probability of a collision is bounded from below by a constant.

Hint: Use the birthday paradox (Appendix B.1).

(b) Show that given any collision for H_{poly} under key k, we can efficiently compute k. That is, give an efficient algorithm that takes two inputs $m, m' \in \mathbb{Z}_p^2$, and that outputs $\hat{k} \in \mathbb{Z}_p$, and satisfies the following property: for every $k \in \mathbb{Z}_p$, if $H(k, m) = H(k, m')$, then $\hat{k} = k$.

7.27 (XOR-hash analysis). Generalize Theorem 7.6 to show that for every Q-query UHF adversary $\mathcal{A}$, there exists a PRF adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\text{MUHF}\mathsf{adv}[\mathcal{A}, F^{\oplus}] \leq \text{PRF}\mathsf{adv}[\mathcal{B}, F] + \frac{Q^2}{2|\mathcal{Y}|}.$$

Moreover, $\mathcal{B}$ makes at most $Q\ell$ queries to F.

7.28 (H_{xpoly} is not a good PUF). Show that H_{xpoly} defined in (7.23) is not a good PUF by exhibiting an adversary that wins Attack Game 7.5 with probability 1.

7.29 (Converting a one-time MAC to a MAC). Suppose $\mathcal{I} = (S, V)$ is a (possibly randomized) MAC defined over $(\mathcal{K}_1, \mathcal{M}, \mathcal{T})$, where $\mathcal{T} = \{0,1\}^n$, that is one-time secure (see Section 7.6). Further suppose that F is a secure PRF defined over $(\mathcal{K}_2, \mathcal{R}, \mathcal{T})$, where $|\mathcal{R}|$ is super-poly. Consider the MAC $\mathcal{I}' = (S', V')$ defined over $(\mathcal{K}_1 \times \mathcal{K}_2, \mathcal{M}, \mathcal{R} \times \mathcal{T})$ as follows:

$$\begin{aligned} S'((k_1, k_2), m) &:= \left\{ \; r \xleftarrow{\text{R}} \mathcal{R}; \; t \xleftarrow{\text{R}} S(k_1, m); \; t' \leftarrow F(k_2, r) \oplus t; \quad \text{output } (r, t') \; \right\} \\ V'((k_1, k_2), m, (r, t')) &:= \left\{ \; t \leftarrow F(k_2, r) \oplus t'; \quad \text{output } V(k_1, m, t) \; \right\} \end{aligned}$$

Show that $\mathcal{I}'$ is a secure (many time) MAC.

7.30 (Pairwise independent functions). In this exercise, we develop the notion of a PRF that is unconditionally secure, provided the adversary can make at most *two* queries. We say that a PRF F defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ is an **ϵ-almost pairwise independent function**, or **ϵ-APIF**, if the following holds: for all adversaries $\mathcal{A}$ (even inefficient ones) that make at most 2 queries in Attack Game 4.2, we have $\text{PRF}\mathsf{adv}[\mathcal{A}, F] \leq \epsilon$. If $\epsilon = 0$, we call F a **pairwise independent function**, or **PIF**.

(a) Suppose that $|\mathcal{X}| > 1$ and that for all $x_0, x_1 \in \mathcal{X}$ with $x_0 \neq x_1$, and all $y_0, y_1 \in \mathcal{Y}$, we have

$$\Pr[F(k, x_0) = y_0 \quad \wedge \quad F(k, x_1) = y_1] = \frac{1}{|Y|^2},$$

where the probability is over the random choice of $k \in \mathcal{K}$. Show that F is a PIF.

(b) Consider the function H' built from H in (7.32). Show that if H is a $1/N$-DUF, then H' is a PIF.

(c) For positive integer d, let $I_d := \{0, \ldots, d-1\}$. Let n be a positive integer and set $N := 2^n$. Consider the keyed hash function H defined over $(I_{N^2}^{\ell+1}, I_N^\ell, I_N)$ as follows:

$$H((k_0, k_1, \ldots, k_\ell), (a_1, \ldots, a_\ell)) := \big\lfloor \big(\, \big(k_0 + \sum_i a_i k_i\big) \bmod N^2 \, \big) \, / \, N \big\rfloor.$$

Show that H is a PIF. Note: on a typical computer, if n is not too large, this can be implemented very easily with just integer multiplications, additions, and shifts.

(d) Show that in the PRF(UHF) composition, if H is an ϵ_1-UHF and F is an ϵ_2-APIF, then the composition F' is an $(\epsilon_1 + \epsilon_2)$-APIF.

(e) Show that any ϵ-APIF is an $(\epsilon + 1/|\mathcal{Y}|)$-PUF.

(f) Using an appropriate APIF, show how to construct a probabilistic cipher that is unconditionally CPA secure provided the adversary can make at most two queries in Attack Game 5.2.

Chapter 8

Message integrity from collision resistant hashing

In the previous chapter we discussed universal hash functions (UHFs) and showed how they can be used to construct MACs. Recall that UHFs are *keyed* hash functions for which finding collisions is difficult, as long as the key is kept secret.

In this chapter we study *keyless* hash functions for which finding collisions is difficult. Informally, a keyless function is an efficiently computable function whose description is fully public. There are no secret keys and anyone can evaluate the function. Let H be a keyless hash function from some large message space $\mathcal{M}$ into a small digest space $\mathcal{T}$. As in the previous chapter, we say that two messages $m_0, m_1 \in \mathcal{M}$ are a **collision** for the function H if

$$H(m_0) = H(m_1) \quad \text{and} \quad m_0 \neq m_1.$$

Informally, we say that the function H is **collision resistant** if finding a collision for H is difficult. Since the digest space $\mathcal{T}$ is much smaller than $\mathcal{M}$, we know that many such collisions exist. Nevertheless, if H is collision resistant, actually finding a pair m_0, m_1 that collide should be difficult. We give a precise definition in the next section.

In this chapter we will construct collision resistant functions and present several applications. To give an example of a collision resistant function we mention a US federal standard called the Secure Hash Algorithm Standard or SHA for short. The SHA standard describes a number of hash functions that offer varying degrees of collision resistance. For example, **SHA256** is a function that hashes long messages into 256-bit digests. It is believed that finding collisions for SHA256 is difficult.

Collision resistant hash functions have many applications. We briefly mention two such applications here and give the details later on in the chapter. Many other applications are described throughout the book.

Extending cryptographic primitives. An important application for collision resistance is its ability to extend primitives built for short inputs to primitives for much longer inputs. We give a MAC construction as an example. Suppose we are given a MAC system $\mathcal{I} = (S, V)$ that only authenticates short messages, say messages that are 256 bits long. We want to extend the domain of the MAC so that it can authenticate much longer inputs. Collision resistant hashing gives a very simple solution. To compute a MAC for some long message m we first hash m and then apply S to

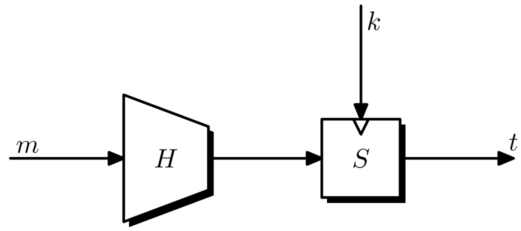


Figure 8.1: Hash-then-MAC construction

the resulting short digest, as described in Fig. 8.1. In other words, we define a new MAC system $\mathcal{I} = (S', V')$ where $S'(k,\ m) := S(k,\ H(m))$. MAC verification works analogously by first hashing the message and then verifying the tag of the digest.

Clearly this hash-then-MAC construction would be insecure if it were easy to find collisions for H. If an adversary could find two long messages m_0 and m_1 such that $H(m_0) = H(m_1)$ then he could forge tags using a chosen message attack. Suppose m_0 is an innocuous message while m_1 is evil, say a virus infected program. The adversary would ask for the tag on the message m_0 and obtain a tag t in response. Then the pair (m_0, t) is a valid message-tag pair, but so is the pair (m_1, t). Hence, the adversary is able to forge a tag for m_1, which breaks the MAC. Even worse, the valid tag may fool a user into running the virus. This argument shows that collision resistance is necessary for this hash-then-MAC construction to be secure. Later on in the chapter we prove that collision resistance is, in fact, sufficient to prove security.

The hash-then-MAC construction looks similar to the PRF(UHF) composition discussed in the previous chapter (Section 7.3). These two methods build similar looking MACs from very different building blocks. The main difference is that a collision resistant hash can extend the input domain of any MAC. On the other hand, a UHF can only extend the domain of a very specific type of MAC, namely a PRF. This is illustrated further in Exercise 7.4. Another difference is that the secret key in the hash-then-MAC method is exactly the same as in the underlying MAC. The PRF(UHF) method, in contrast, extends the secret key of the underlying PRF by adding a UHF secret key.

The hash-then-MAC construction performs better than PRF(UHF) when we wish to compute the tag for a single message m under multiple keys $k_1, \ldots, k_n$. That is, we wish to compute $S'(k_i, m)$ for all $i = 1, \ldots, n$. This comes up, for example, when providing integrity for a file on disk that is readable by multiple users. The file header contains one integrity tag per user so that each user can verify integrity using its own MAC key. With the hash-then-MAC construction it suffices to compute $H(m)$ once and then quickly derive the n tags from this single hash. With a PRF(UHF) MAC, the UHF depends on the key k_i and consequently we will need to rehash the entire message n times, once for each user. See also Exercise 6.4 for more on this problem.

File integrity. Another application for collision resistance is file integrity also discussed in the introduction of Chapter 6. Consider a set of n critical files that change infrequently, such as certain operating system files. We want a method to verify that these files are not modified by some malicious code or malware. To do so we need a small amount of read-only memory, namely memory that the malware can read, but cannot modify. Read-only memory can be implemented, for example, using a small USB disk that has a physical switch flipped to the "read-only" position. We place a hash of each of the n critical files in the read-only memory so that this storage area only

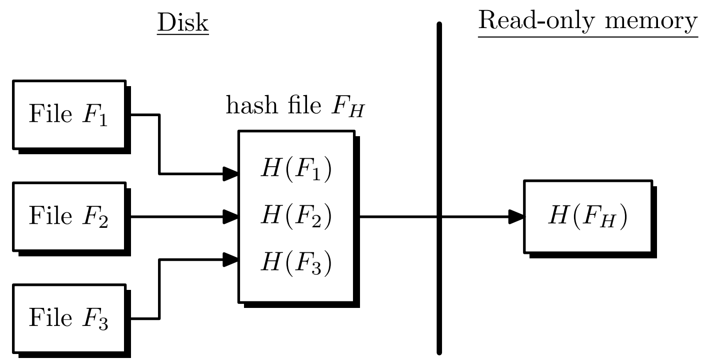


Figure 8.2: File integrity using small read-only memory

contains n short hashes. We can then check integrity of a file F by rehashing F and comparing the resulting hash to the one stored in read-only memory. If a mismatch is found, the system declares that file F is corrupt. The *TripWire* malware protection system [63] uses this mechanism to protect critical system files.

What property should the hash function H satisfy for this integrity mechanism to be secure? Let F be a file protected by this system. Since the malware cannot alter the contents of the read-only storage, its only avenue for modifying F without being detected is to find another file F' such that $H(F) = H(F')$. Replacing F by F' would not be caught by this hashing system. However, finding such an F' will be difficult if H is collision resistant. Collision resistance, thus, implies that the malware cannot change F without being detected by the hash.

This system stores all file hashes in read-only memory. When there are many files to protect the amount of read-only memory needed could become large. We can greatly reduce the size of read-only memory by viewing the entire set of file hashes as just another file stored on disk and denoted F_H. We store the hash of F_H in read-only memory, as described in Fig. 8.2. Then read-only memory contains a single hash value. To verify file integrity of some file F we first verify integrity of the file F_H by hashing the contents of F_H and comparing the result to the value in read-only memory. Then we verify integrity of F by hashing F and comparing the result with the corresponding hash stored in F_H. We describe a more efficient solution using authentication trees in Section 8.9.

In the introduction to Chapter 6 we proposed a MAC-based file integrity system. The system stored a tag of every file along with the file. We also needed a small amount of *secret storage* to store the user's secret MAC key. This key was used every time file integrity was verified. In comparison, when using collision resistant hashing there are no secrets and there is no need for secret storage. Instead, we need a small amount of read-only storage for storing file hashes. Generally speaking, read-only storage is much easier to build than secret storage. Hence, collision resistance seems more appropriate for this particular application. In Chapter 13 we will develop an even better solution to this problem, using digital signatures, that does not need read-only storage or online secret storage.

Security without collision resistance. By extending the input to the hash function with a few random bits we can prove security for both applications above using a weaker notion of collision resistance called **target collision resistance** or TCR for short. We show in Section 8.11.2 how to use TCR for both file integrity and for extending cryptographic primitives. The downside is that the resulting tags are longer than the ones obtained from collision resistant hashing. Hence, although in principle it is often possible to avoid relying on collision resistance, the resulting systems are not as efficient.

8.1 Definition of collision resistant hashing

A **(keyless) hash function** $H : \mathcal{M} \to \mathcal{T}$ is an efficiently computable function from some (large) message space $\mathcal{M}$ into a (small) digest space $\mathcal{T}$. We say that H is defined over $(\mathcal{M}, \mathcal{T})$. We define collision resistance of H using the following (degenerate) game:

Attack Game 8.1 (Collision Resistance). For a given hash function H over $(\mathcal{M}, \mathcal{T})$ and adversary $\mathcal{A}$, the adversary takes no input and outputs two messages m_0 and m_1 in $\mathcal{M}$.

We say that $\mathcal{A}$ wins the game if the pair m_0, m_1 is a collision for H, namely $m_0 \neq m_1$ and $H(m_0) = H(m_1)$. We define $\mathcal{A}$'s advantage with respect to H, denoted $\mathrm{CR}\mathsf{adv}[\mathcal{A}, H]$, as the probability that $\mathcal{A}$ wins the game. Adversary $\mathcal{A}$ is called a **collision finder**. □

Definition 8.1. *We say that a hash function H over $(\mathcal{M}, \mathcal{T})$ is* ***collision resistant*** *if for all efficient adversaries $\mathcal{A}$, the quantity $\mathrm{CR}\mathsf{adv}[\mathcal{A}, H]$ is negligible.*

At first glance, it may seem that collision resistant functions cannot exist. The problem is this: since $|\mathcal{M}| > |\mathcal{T}|$ there must exist inputs m_0 and m_1 in $\mathcal{M}$ that collide, namely $H(m_0) = H(m_1)$. An adversary $\mathcal{A}$ that simply prints m_0 and m_1 and exits is an efficient adversary that breaks the collision resistance of H. We may not be able to write the explicit program code for $\mathcal{A}$ (since we do not know m_0, m_1), but this $\mathcal{A}$ certainly exists. Consequently, for any hash function H defined over $(\mathcal{M}, \mathcal{T})$ there *exists* some efficient adversary $\mathcal{A}_H$ that breaks the collision resistance of H. Hence, it appears that no function H can satisfy Definition 8.1.

The way out of this is that, formally speaking, our hash functions are parameterized by a system parameter: each choice of a system parameter describes a different function H, and so we cannot simply "hardwire" a fixed collision into an adversary: an effective adversary must be able to efficiently compute a collision *as a function of the system parameter*. This is discussed in more depth in the Mathematical details section below.[1]

8.1.1 Mathematical details

As usual, we give a more mathematically precise definition of a collision resistant hash function using the terminology defined in Section 2.4.

Definition 8.2 (Keyless hash functions). *A* ***(keyless) hash function*** *is an efficient algorithm H, along with two families of spaces with system parameterization P:*

$$\mathbf{M} = \{\mathcal{M}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \textit{and} \quad \mathbf{T} = \{\mathcal{T}_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$

such that

1. $\mathbf{M}$*, and* $\mathbf{T}$ *are efficiently recognizable.*

[1]Some authors deal with this issue by have H take as input a randomly chosen key k, and giving k to the adversary at the beginning of this attack game. By viewing k as a system parameter, this approach is really the same as ours.

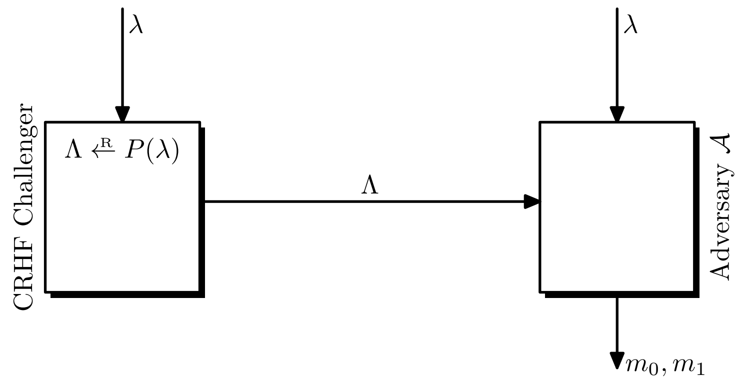


Figure 8.3: Asymptotic version of Attack Game 8.1

2. *Algorithm H is an efficient deterministic algorithm that on input $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \text{Supp}(P(\lambda))$, and $m \in \mathcal{M}_{\lambda,\Lambda}$, outputs an element of $\mathcal{T}_{\lambda,\Lambda}$.*

In defining collision resistance we parameterize Attack Game 8.1 by the security parameter λ. The asymptotic game is shown in Fig. 8.3. The advantage $\text{CR}\mathsf{adv}[\mathcal{A}, H]$ is then a function of λ. Definition 8.1 should be read as saying that $\text{CR}\mathsf{adv}[\mathcal{A}, H](\lambda)$ is a negligible function.

It should be noted that the security and system parameters are artifacts of the formal framework that are needed to make sense of Definition 8.1. In the real world, however, these parameters are picked when the hash function is designed, and are ignored from that point onward. SHA256, for example, does not take either a security parameter or a system parameter as input.

8.2 Building a MAC for large messages

To exercise the definition of collision resistance, we begin with an easy application described in the introduction — extending the message space of a MAC. Suppose we are given a secure MAC $\mathcal{I} = (S, V)$ for short messages. Our goal is to build a new secure MAC $\mathcal{I}'$ for much longer messages. We do so using a collision resistant hash function: $\mathcal{I}'$ computes a tag for a long message m by first hashing m to a short digest and then applying $\mathcal{I}$ to the digest, as shown in Fig. 8.1.

More precisely, let H be a hash function that hashes long messages in $\mathcal{M}$ to short digests in $\mathcal{T}_H$. Suppose $\mathcal{I}$ is defined over $(\mathcal{K}, \mathcal{T}_H, \mathcal{T})$. Define $\mathcal{I}' = (S', V')$ for long messages as follows:

$$S'(k, m) := S(k,\ H(m)\ ) \quad \text{and} \quad V'(k, m) := V(k,\ H(m)\ ) \tag{8.1}$$

Then $\mathcal{I}'$ authenticates long messages in $\mathcal{M}$. The following easy theorem shows that $\mathcal{I}'$ is secure, assuming H is collision resistant.

Theorem 8.1. *Suppose the MAC system $\mathcal{I}$ is a secure MAC and the hash function H is collision resistant. Then the derived MAC system $\mathcal{I}' = (S', V')$ defined in (8.1) is a secure MAC.*

In particular, suppose $\mathcal{A}$ is a MAC adversary attacking $\mathcal{I}'$ (as in Attack Game 6.1). Then there exist a MAC adversary $\mathcal{B}_{\mathcal{I}}$ and an efficient collision finder $\mathcal{B}_H$, which are elementary wrappers around $\mathcal{A}$, such that

$$\text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}'] \leq \text{MAC}\mathsf{adv}[\mathcal{B}_{\mathcal{I}}, \mathcal{I}] + \text{CR}\mathsf{adv}[\mathcal{B}_H, H].$$

It is clear that collision resistance of H is essential for the security of $\mathcal{I}'$. Indeed, if an adversary can find a collision m_0, m_1 on H, then he can win the MAC attack game as follows: submit m_0 to the MAC challenger for signing, obtaining a tag $t_0 := S(k, H(m_0))$, and then output the message-tag pair (m_1, t_0). Since $H(m_0) = H(m_1)$, the tag t_0 must be a valid tag on the message m_1.

Proof idea. Our goal is to show that no efficient adversary can win the MAC Attack Game 6.1 for our new MAC system $\mathcal{I}'$. An adversary $\mathcal{A}$ in this game asks the challenger to MAC a few long messages $m_1, m_2, \ldots \in \mathcal{M}$ and then tries to invent a new valid message-MAC pair (m, t). If $\mathcal{A}$ is able to produce a valid forgery (m, t) then one of two things must happen:

1. either m collides with some query m_i from $\mathcal{A}$, so that $H(m) = H(m_i)$ and $m \neq m_i$;
2. or m does not collide under H with any of $\mathcal{A}$'s queries $m_1, m_2, \ldots \in \mathcal{M}$.

It should be intuitively clear that if $\mathcal{A}$ produces forgeries of the first type then $\mathcal{A}$ can be used to break the collision resistance of H since m and m_i are a valid collision for H. On the other hand, if $\mathcal{A}$ produces forgeries of the second type then $\mathcal{A}$ can be used to break the MAC system $\mathcal{I}$: the pair $(H(m),\ t)$ is a valid MAC forgery for $\mathcal{I}$. Thus, if $\mathcal{A}$ wins the MAC attack game for $\mathcal{I}'$ we break one of our assumptions. □

Proof. We make this intuition rigorous. Let $m_1, m_2, \ldots \in \mathcal{M}$ be $\mathcal{A}$'s queries during the MAC attack game and let $(m, t) \in \mathcal{M} \times \mathcal{T}$ be the adversary's output, which we assume is not among the signed pairs. We define three events:

- Let X be the event that adversary $\mathcal{A}$ wins the MAC Attack Game 6.1 with respect to $\mathcal{I}'$.
- Let Y denote the event that some m_i collides with m under H, that is, for some i we have $H(m) = H(m_i)$ and $m \neq m_i$.
- Let Z denote the event that $\mathcal{A}$ wins Attack Game 6.1 on $\mathcal{I}'$ and event Y did not occur.

Using events Y and Z we can rewrite $\mathcal{A}$'s advantage in winning Attack Game 6.1 as follows:

$$\mathrm{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}'] \;=\; \Pr[X] \;\leq\; \Pr[X \wedge \neg Y] + \Pr[Y] \;=\; \Pr[Z] + \Pr[Y] \tag{8.2}$$

To prove the theorem we construct a collision finder $\mathcal{B}_H$ and a MAC adversary $\mathcal{B}_\mathcal{I}$ such that

$$\Pr[Y] = \mathrm{CR}\mathsf{adv}[\mathcal{B}_H, H] \quad \text{and} \quad \Pr[Z] = \mathrm{MAC}\mathsf{adv}[\mathcal{B}_\mathcal{I}, \mathcal{I}].$$

Both adversaries are straight-forward.

Adversary $\mathcal{B}_H$ plays the role of challenger to $\mathcal{A}$ in the MAC attack game, as follows:

Initialization:
    $k \xleftarrow{R} \mathcal{K}$
Upon receiving a signing query $m_i \in \mathcal{M}$ from $\mathcal{A}$ do:
    $t_i \xleftarrow{R} S(k,\ H(m_i)\ )$
    Send t_i to $\mathcal{A}$
Upon receiving the final message-tag pair (m, t) from $\mathcal{A}$ do:
    if $H(m) = H(m_i)$ and $m \neq m_i$ for some i
        then output the pair (m, m_i)

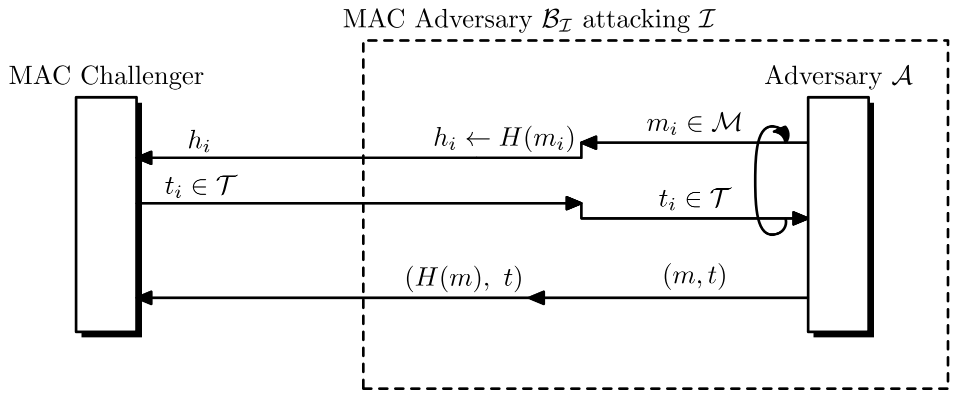


Figure 8.4: Adversary $\mathcal{B}_\mathcal{I}$ in the proof of Theorem 8.1

Algorithm $\mathcal{B}_H$ responds to $\mathcal{A}$'s signature queries exactly as in a real MAC attack game. Therefore, event Y happens during the interaction with $\mathcal{B}_H$ with the same probability that it happens in a real MAC attack game. Clearly when event Y happens, $\mathcal{A}_H$ succeeds in finding a collision for H. Hence, $\text{CR}\mathsf{adv}[\mathcal{B}_H, H] = \Pr[Y]$ as required.

MAC adversary $\mathcal{B}_\mathcal{I}$ is just as simple and is shown in Fig. 8.4. When $\mathcal{A}$ outputs the final message-tag pair (m,t) adversary $\mathcal{B}_\mathcal{I}$ outputs $(H(m),t)$. When event Z happens we know that $V'(k,m,t)$ outputs accept and the pair (m,t) is not equal to any of $(m_1,t_1),\ (m_2,t_2),\ldots \in \mathcal{M}\times\mathcal{T}$. Furthermore, since event Y does not happen, we know that $(H(m),t)$ is not equal to any of $(H(m_1),t_1),\ (H(m_2),t_2),\ldots \in \mathcal{T}_H \times \mathcal{T}$. It follows that $(H(m),\ t)$ is a valid existential forgery for $\mathcal{I}$. Hence, $\mathcal{B}_\mathcal{I}$ succeeds in creating an existential forgery with the same probability that event Z happens. In other words, $\text{MAC}\mathsf{adv}[\mathcal{B}_\mathcal{I}, \mathcal{I}] = \Pr[Z]$, as required. The proof now follows from (8.2). $\square$

8.3 Birthday attacks on collision resistant hash functions

Cryptographic hash functions are most useful when the output digest size is small. The challenge is to design hash functions whose output is as short as possible and yet finding collisions is difficult. It should be intuitively clear that the shorter the digest, the easier it is for an attacker to find collisions. To illustrate this, consider a hash function H that outputs ℓ-bit digests for some small ℓ. Clearly, by hashing $2^\ell + 1$ distinct messages the attacker will find two messages that hash to the same digest and will thus break collision resistance of H. This brute-force attack will break the collision resistance of any hash function. Hence, for instance, hash functions that output 16-bit digests cannot be collision resistant — a collision can always be found using only $2^{16} + 1 = 65537$ evaluations of the hash.

Birthday attacks. A far more devastating attack can be built using the birthday paradox discussed in Section B.1 in the appendix. Let H be a hash function defined over $(\mathcal{M}, \mathcal{T})$ and set $N := |\mathcal{T}|$. For standard hash functions N is quite large, for example $N = 2^{256}$ for SHA256. Throughout this section we will assume that the size of $\mathcal{M}$ is at least $100N$. This basically means that messages being hashed are slightly longer than the output digest. We describe a general collision finder that finds collisions for H after an expected $O(\sqrt{N})$ evaluations of H. For comparison, the brute-force attack above took $O(N)$ evaluations. This more efficient collision finder forces us to use much larger digests.

The birthday collision finder for H works as follows: it chooses $s \approx \sqrt{N}$ random and independent messages, $m_1, \ldots, m_s \xleftarrow{R} \mathcal{M}$, and looks for a collision among these s messages. We will show that the birthday paradox implies that a collision is likely to exist among these messages. More precisely, the birthday collision finder works as follows:

Algorithm `BirthdayAttack`:

1. Set $s \leftarrow \lceil 2\sqrt{N}\, \rceil + 1$
2. Generate s uniform random messages $m_1, \ldots, m_s$ in $\mathcal{M}$
3. Compute $x_i \leftarrow H(m_i)$ for all $i = 1, \ldots, s$
4. Look for distinct $i, j \in \{1, \ldots, s\}$ such that $H(m_i) = H(m_j)$
5. If such i, j exist and $m_i \neq m_j$ then
6.   output the pair (m_i, m_j)

We argue that when the adversary picks $s := \left\lceil 2\sqrt{N} \right\rceil + 1$ random messages in $\mathcal{M}$, then with probability at least $1/2$, there will exist distinct i, j such that $H(m_i) = H(m_j)$ and $m_i \neq m_j$. This means that the algorithm will output a collision with probability at least $1/2$.

Lemma 8.2. *Let $m_1, \ldots, m_s$ be the random messages sampled in Step 2. Assume $|\mathcal{M}| \geq 100N$. Then with probability at least $1/2$ there exists i, j in $\{1, \ldots, s\}$ such that $H(m_i) = H(m_j)$ and $m_i \neq m_j$.*

Proof. For $i = 1, \ldots, s$ let $x_i := H(m_i)$. First, we argue that two of the x_i values will collide with probability at least $3/4$. If the x_i were uniformly distributed in $\mathcal{T}$ then this would follow immediately from part (i) of Theorem B.1. Indeed, if the x_i were independent and uniform in $\mathcal{T}$ a collision among the x_i will occur with probability at least $1 - e^{-s(s-1)/2N} \geq 1 - e^{-2} \geq 3/4$.

However, in reality, the function $H(\cdot)$ might bias the output distribution. Even though the m_i are sampled uniformly from $\mathcal{M}$, the resulting x_i may not be uniform in $\mathcal{T}$. As a simple example, consider a hash function $H(\cdot)$ that only outputs digests in a certain small subset of $\mathcal{T}$. The resulting x_i would certainly not be uniform in $\mathcal{T}$. Fortunately (for the attacker) Corollary B.2 shows that non-uniform x_i only increase the probability of collision. Since the x_i are independent and identically distributed the corollary implies that a collision among the x_i will occur with probability at least $1 - e^{-s(s-1)/2N} \geq 3/4$ as required.

Next, we argue that a collision among the x_i is very likely to lead to a collision on $H(\cdot)$. Suppose $x_i = x_j$ for some distinct i, j in $\{1, \ldots, s\}$. Since $x_i = H(m_i)$ and $x_j = H(m_j)$, the pair m_i, m_j is a candidate for a collision on $H(\cdot)$. We just need to argue that $m_i \neq m_j$. We do so by arguing that all the $m_1, \ldots, m_s$ are distinct with probability at least $4/5$. This follows directly from part (ii) of Theorem B.1. Recall that $\mathcal{M}$ is greater than $100N$. Since $m_1, m_2, \ldots$ are uniform and independent in $\mathcal{M}$, and $s < |\mathcal{M}|/2$, part (ii) of Theorem B.1 implies that the probability of collision among these m_i is at most $1 - e^{-s(s-1)/100N} \leq 1/5$. Therefore, the probability that no collision occurs is at least $4/5$.

In summary, for the algorithm to discover a collision for $H(\cdot)$ it is sufficient that both a collision occurs on the x_i values and no collision occurs on the m_i values. This happens with probability at least $3/4 - 1/5 > 1/2$, as required. $\square$

Variations. Algorithm `BirthdayAttack` requires $O(\sqrt{N})$ memory space, which can be quite large: larger than the size of commercially available disk farms. However, a modified birthday collision finder, described in Exercise 8.7, will find a collision with an expected $4\sqrt{N}$ evaluations of the hash function and *constant* memory space.

The birthday attack is likely to fail if one makes fewer than $\sqrt{N}$ queries to $H(\cdot)$. Suppose we only make $s = \epsilon\sqrt{N}$ queries to $H(\cdot)$, for some small $\epsilon \in [0,1]$. For simplicity we assume that $H(\cdot)$ outputs digests distributed uniformly in $\mathcal{T}$. Then part (ii) of Theorem B.1 shows that the probability of finding a collision degrades exponentially to approximately $1 - e^{-(\epsilon^2)} \approx \epsilon^2$.

Put differently, if after evaluating the hash function s times an adversary should obtain a collision with probability at most δ, then we need the digest space $\mathcal{T}$ to satisfy $|\mathcal{T}| \geq s^2/\delta$. For example, if after 2^{80} evaluations of H a collision should be found with probability at most 2^{-80} then the digest size must be at least 240 bits. Cryptographic hash functions such as SHA256 output a 256-bit digest. Other hash functions, such as SHA384 and SHA512, output even longer digests, namely, 384 and 512 bits respectively.

8.4 The Merkle-Damgård paradigm

We now turn to constructing collision resistant hash functions. Many practical constructions follow the Merkle-Damgård paradigm: start from a collision resistant hash function that hashes short messages and build from it a collision resistant hash function that hashes much longer messages. This paradigm reduces the problem of constructing collision resistant hashing to the problem of constructing collision resistance for short messages, which we address in the next section.

Let $h : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be a hash function. We shall assume that $\mathcal{Y}$ is of the form $\{0,1\}^\ell$ for some ℓ. While it is not necessary, typically $\mathcal{X}$ is of the form $\{0,1\}^n$ for some n. The **Merkle-Damgård function derived from** h, denoted H_{MD} and shown in Fig. 8.5, is a hash function defined over $(\{0,1\}^{\leq L},\ \mathcal{X})$ that works as follows (the pad PB is defined below):

> input: $M \in \{0,1\}^{\leq L}$
> output: a tag in $\mathcal{X}$
> $\hat{M} \leftarrow M \parallel \text{PB}$ // *pad with PB to ensure that the length of M is a multiple of ℓ bits*
> partition $\hat{M}$ into consecutive ℓ-bit blocks so that
> $$\hat{M} = m_1 \parallel m_2 \parallel \cdots \parallel m_s \quad \text{where} \quad m_1, \dots, m_s \in \{0,1\}^\ell$$
> $t_0 \leftarrow \text{IV} \in \mathcal{X}$
> for $i = 1$ to s do:
> $\quad t_i \leftarrow h(t_{i-1}, m_i)$
> output t_s

The function SHA256 is a Merkle-Damgård function where $\ell = 512$ and $n = 256$.

Before proving collision resistance of H_{MD} let us first introduce some terminology for the various elements in Fig. 8.5:

- The hash function h is called the **compression function** of H.
- The constant IV is called the **initial value** and is fixed to some pre-specified value. One could take $\text{IV} = 0^n$, but usually the IV is set to some complicated string. For example, SHA256

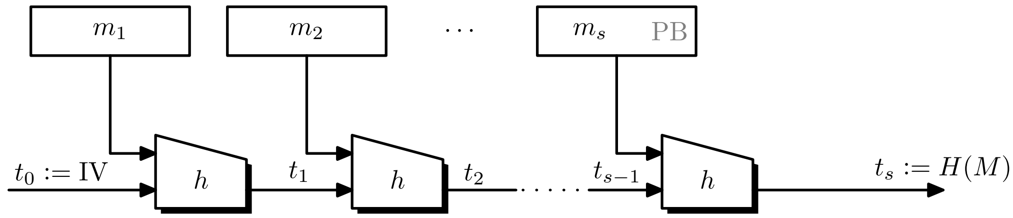


Figure 8.5: The Merkle-Damgård iterated hash function

uses a 256-bit IV whose value in hex is

$$\mathrm{IV} := \texttt{6A09E667 BB67AE85 3C6EF372 A54FF53A 510E527F 9B05688C 1F83D9AB 5BE0CD19}.$$

- The variables $m_1, \ldots, m_s$ are called message blocks.
- The variables $t_0, t_1, \ldots, t_s \in \mathcal{X}$ are called **chaining variables**.
- The string PB is called the **padding block**. It is appended to the message to ensure that the message length is a multiple of ℓ bits.

The padding block PB must contain an encoding of the input message length. We will use this in the proof of security below. A standard format for PB is as follows:

$$\mathrm{PB} := \boxed{100\ldots00 \parallel \langle s \rangle}$$

where $\langle s \rangle$ is a fixed-length bit string that encodes, in binary, the number of ℓ-bit blocks in M. Typically this field is 64-bits which means that messages to be hashed are less than 2^{64} blocks long. The '$100\ldots00$' string is a variable length pad used to ensure that the total message length, including PB, is a multiple of ℓ. The variable length string '$100\ldots00$' starts with a '1' to identify the position where the pad ends and the message begins. If the message length is such that there is no space for PB in the last block (for example, if the message length happens to be a multiple of ℓ), then an additional block is added just for the padding block.

Security of Merkle-Damgård. Next we prove that the Merkle-Damgård function is collision resistant, assuming the compression function is.

Theorem 8.3 (Merkle-Damgård). *Let L be a poly-bounded length parameter and let h be a collision resistant hash function defined over $(\mathcal{X} \times \mathcal{Y},\ \mathcal{X})$. Then the Merkle-Damgård hash function H_{MD} derived from h, defined over $(\{0,1\}^{\leq L}, \mathcal{X})$, is collision resistant.*

In particular, for every collision finder $\mathcal{A}$ attacking H_{MD} (as in Attack Game 8.1) there exists a collision finder $\mathcal{B}$ attacking h, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{CR}\mathsf{adv}[\mathcal{A}, H_{\mathrm{MD}}] = \mathrm{CR}\mathsf{adv}[\mathcal{B}, h].$$

Proof. The collision finder $\mathcal{B}$ for finding h-collisions works as follows: it first runs $\mathcal{A}$ to obtain two distinct messages M and M' in $\{0,1\}^{\leq L}$ such that $H_{\mathrm{MD}}(M) = H_{\mathrm{MD}}(M')$. We show that $\mathcal{B}$ can use

M and M' to find an h-collision. To do so, $\mathcal{B}$ scans M and M' starting from the last block and works its way backwards. To simplify the notation, we assume that M and M' already contain the appropriate padding block PB in their last block.

Let $M = m_1 m_2 \dots m_u$ be the u blocks of M and let $M' = m'_1 m'_2 \dots m'_v$ be the v blocks of M'. We let $t_0, t_1, \dots, t_u \in \mathcal{X}$ be the chaining values for M and $t'_0, t'_1, \dots, t'_s \in \mathcal{X}$ be the chaining values for M'. The very last application of h gives the final output digest and since $H_{\text{MD}}(M) = H_{\text{MD}}(M')$ we know that

$$h(t_{u-1}, m_u) = h(t'_{v-1}, m'_v).$$

If either $t_{u-1} \neq t'_{v-1}$ or $m_u \neq m'_v$ then the pair of inputs (t_{u-1}, m_u) and (t'_{v-1}, m'_v) is an h-collision. $\mathcal{B}$ outputs this collision and terminates.

Otherwise, $t_{u-1} = t'_{v-1}$ and $m_u = m'_v$. Recall that the padding blocks are contained in m_u and m'_v and these padding blocks contain an encoding of u and v. Therefore, since $m_u = m'_v$ we deduce that $u = v$ so that M and M' must contain the same number of blocks.

At this point we know that $u = v$, $m_u = m'_u$, and $t_{u-1} = t'_{u-1}$. We now consider the second-to-last block. Since $t_{u-1} = t'_{u-1}$ we know that

$$h(t_{u-2}, m_{u-1}) = h(t'_{u-2}, m'_{u-1}).$$

As before, if either $t_{u-2} \neq t'_{u-2}$ or $m_{u-1} \neq m'_{u-1}$ then $\mathcal{B}$ just found an h-collision. It outputs this collision and terminates.

Otherwise, we know that $t_{u-2} = t'_{u-2}$ and $m_{u-1} = m'_{u-1}$ and $m_u = m'_u$. We now consider the third block from the end. As before, we either find an h-collision or deduce that $m_{u-2} = m'_{u-2}$ and $t_{u-3} = t'_{u-3}$. We keep iterating this process moving from right to left one block at a time. At the ith block one of two things happens. Either the pair of messages (t_{i-1}, m_i) and (t'_{i-1}, m'_i) is an h-collision, in which case $\mathcal{B}$ outputs this collision and terminates. Or we deduce that $t_{i-1} = t'_{i-1}$ and $m_j = m'_j$ for all $j = i, i+1, \dots, u$.

Suppose this process continues all the way to the first block and we still did not find an h-collision. Then at this point we know that $m_i = m'_i$ for $i = 1, \dots, u$. But this implies that $M = M'$ contradicting the fact that M and M' were a collision for H_{MD}. Hence, since $M \neq M'$, the process of scanning blocks of M and M' from right to left must produce an h-collision. We conclude that $\mathcal{B}$ breaks the collision resistance of h as required.

In summary, we showed that whenever $\mathcal{A}$ outputs an H_{MD}-collision, $\mathcal{B}$ outputs an h-collision. Hence, $\text{CR}\mathsf{adv}[\mathcal{A}, H_{\text{MD}}] = \text{CR}\mathsf{adv}[\mathcal{B}, h]$ as required. □

Variations. Note that the Merkle-Damgård construction is inherently sequential — the ith block cannot be hashed before hashing all previous blocks. This makes it difficult to take advantage of hardware parallelism when available. In Exercise 8.8 we investigate a different hash construction that is better suited for a multi-processor machine.

The Merkle-Damgård theorem (Theorem 8.3) shows that collision resistance of the compression function is sufficient to ensure collision resistance of the iterated function. This condition, however, is not necessary. Black, Rogaway, and Shrimpton [17] give several examples of compression functions that are clearly not collision resistant, and yet the resulting iterated Merkle-Damgård functions are collision resistant.

8.4.1 Joux's attack

We briefly describe a cute attack that applies specifically to Merkle-Damgård hash functions. Let H_1 and H_2 be Merkle-Damgård hash functions that output tags in $\mathcal{X} := \{0,1\}^n$. Define $H_{12}(M) := H_1(M) \parallel H_2(M) \in \{0,1\}^{2n}$. One would expect that finding a collision for H_{12} should take time at least $\Omega(2^n)$. Indeed, this would be the case if H_1 and H_2 were independent random functions.

We show that when H_1 and H_2 are Merkle-Damgård functions we can find collisions for H in time approximately $n2^{n/2}$ which is far less than 2^n. This attack illustrates that our intuition about random functions may lead to incorrect conclusions when applied to a Merkle-Damgård function.

We say that an s-collision for a hash function H is a set of messages $M_1, \ldots, M_s \in \mathcal{M}$ such that $H(M_1) = \ldots = H(M_s)$. Joux showed how to find an s-collision for a Merkle-Damgård function in time $O((\log_2 s)|\mathcal{X}|^{1/2})$. Using Joux's method we can find a $2^{n/2}$-collision $M_1, \ldots, M_{2^{n/2}}$ for H_1 in time $O(n2^{n/2})$. Then, by the birthday paradox it is likely that two of these messages, say M_i, M_j, are also a collision for H_2. This pair M_i, M_j is a collision for both H_1 and H_2 and therefore a collision for H_{12}. It was found in time $O(n2^{n/2})$, as promised.

Finding s-collisions. To find an s-collision, let H be a Merkle-Damgård function over $(\mathcal{M}, \mathcal{X})$ built from a compression function h. We find an s-collision $M_1, \ldots, M_s \in \mathcal{M}$ where each message M_i contains $\log_2 s$ blocks. For simplicity, assume that s is a power of 2 so that $\log_2 s$ is an integer. As usual, we let t_0 denote the Initial Value (IV) used in the Merkle-Damgård construction.

The plan is to use the birthday attack $\log_2 s$ times on the compression function h. We first spend time $2^{n/2}$ to find two distinct blocks m_0, m'_0 such that (t_0, m_0) and (t_0, m'_0) collide under h. Let $t_1 := h(t_0, m_0)$. Next we spend another $2^{n/2}$ time to find two distinct blocks m_1, m'_1 such that (t_1, m_1) and (t_1, m'_1) collide under h. Again, we let $t_2 := h(t_1, m_1)$ and repeat. We iterate this process $b := \log_2 s$ times until we have b pairs of blocks:

$$(m_i, m'_i) \quad \text{for } i = 0, 1, \ldots b-1 \quad \text{that satisfy} \quad h(t_i, m_i) = h(t_i, m'_i).$$

Now, consider the message $M = m_0 m_1 \ldots m_{b-1}$. The main point is that replacing any block m_i in this message by m'_i will not change the chaining value t_{i+1} and therefore the value of $H(M)$ will not change. Consequently, we can replace any subset of $m_0, \ldots, m_{b-1}$ by the corresponding blocks in $m'_0, \ldots, m'_{b-1}$ without changing $H(M)$. As a result we obtain $s = 2^b$ messages

$$\begin{array}{c} m_0 m_1 \ldots m_{b-1} \\ m'_0 m_1 \ldots m_{b-1} \\ m_0 m'_1 \ldots m_{b-1} \\ m'_0 m'_1 \ldots m_{b-1} \\ \vdots \\ m'_0 m'_1 \ldots m'_{b-1} \end{array}$$

that all hash to same value under H. In summary, we found a 2^b-collision in time $O(b2^{n/2})$. As explained above, this lets us find collisions for $H(M) := H_1(M) \parallel H_2(M)$ in time $O(n2^{n/2})$.

8.5 Building Compression Functions

The Merkle-Damgård paradigm shows that to construct a collision resistant hash function for long messages it suffices to construct a collision resistant compression function h for short blocks. In

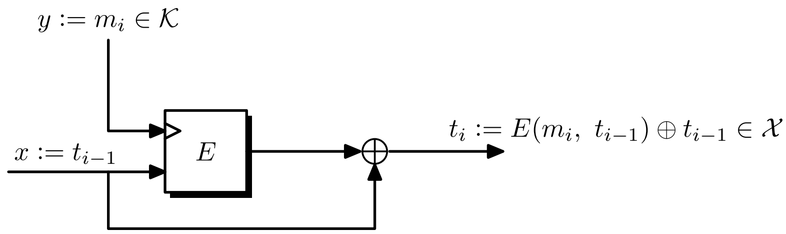


Figure 8.6: The Davies-Meyer compression function

this section we describe a few candidate compression functions. These constructions fall into two categories:

- Compression functions built from a block cipher. The most widely used method is called Davies-Meyer. The SHA family of cryptographic hash functions all use Davies-Meyer.

- Compression functions using number theoretic primitives. These are elegant constructions with clean proofs of security. Unfortunately, they are generally far less efficient than the first method.

8.5.1 A simple but inefficient compression function

We start with a compression function built using modular arithmetic. Let p be a large prime such that $q := (p-1)/2$ is also prime. Let x and y be suitably chosen integers in the range $[1, q]$. Consider the following simple compression function that takes as input two integers in $[1, q]$ and outputs an integer in $[1, q]$:

$$H(a, b) = \text{abs}(x^a y^b \bmod p), \quad \text{where} \quad \text{abs}(z) := \begin{cases} z & \text{if } z \leq q, \\ p - z & \text{if } z > q. \end{cases} \tag{8.3}$$

We will show later in Exercise 10.18 that this function is collision resistant assuming a certain standard number theoretic problem is hard. Applying the Merkle-Damgård paradigm to this function gives a collision resistant hash function for arbitrary size inputs. Although this is an elegant collision resistant hash with a clean security proof, it is far less efficient than functions derived from the Davies-Meyer construction and, as a result, is hardly ever used in practice.

8.5.2 Davies-Meyer compression functions

In Chapter 4 we spent the effort to build secure block ciphers like AES. It is natural to ask whether we can leverage these constructions to build fast compression functions. The Davies-Meyer method enables us to do just that, but security can only be shown in the ideal cipher model.

Let $\mathcal{E} = (E, D)$ be a block cipher over $(\mathcal{K}, \mathcal{X})$ where $\mathcal{X} = \{0,1\}^n$. The **Davies-Meyer compression function derived from** E maps inputs in $\mathcal{X} \times \mathcal{K}$ to outputs in $\mathcal{X}$. The function is defined as follows:

$$h_{\text{DM}}(x, y) := E(y,\ x) \oplus x$$

and is illustrated in Fig. 8.6. In symbols, h_{DM} is defined over $(\mathcal{X} \times \mathcal{K},\ \mathcal{X})$.

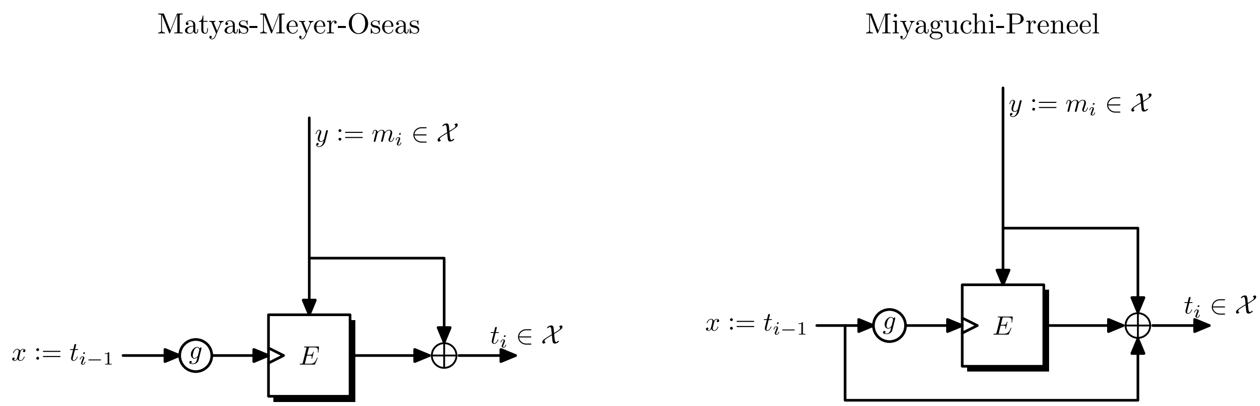


Figure 8.7: Other block cipher compression functions

When plugging this compression function into the Merkle-Damgård paradigm the inputs are a chaining variable $x := t_{i-1} \in \mathcal{X}$ and a message block $y := m_i \in \mathcal{K}$. The output is the next chaining variable $t_i := E(m_i,\ t_{i-1}) \oplus t_{i-1} \in \mathcal{X}$. Note that the message block is used as the block cipher key which seems a bit odd since the adversary has full control over the message. Nevertheless, we will show that h_{DM} is collision resistant and therefore the resulting Merkle-Damgård function is collision resistant.

When using h_{DM} in Merkle-Damgård the block cipher key (m_i) changes from one message block to the next, which is an unusual way of using a block cipher. Common block ciphers are optimized to encrypt long messages with a fixed key; changing the block cipher key on every block can slow down the cipher. Consequently, using Davies-Meyer with an off-the-shelf block cipher such as AES will result in a relatively slow hash function. Instead, one uses a custom block cipher specifically designed for rapid key changes.

Another reason to not use an off-the-shelf block cipher in Davies-Meyer is that the block size may be too short, for example 128 bits for AES. An AES-based compression function would produce a 128-bit output which is much too short for collision resistance: a collision could be found with only 2^{64} evaluations of the function. In addition, off-the-shelf block ciphers use relatively short keys, say 128 bits long. This would result in Merkle-Damgård processing only 128 message bits per round. Typical ciphers used in Merkle-Damgård hash functions use longer keys (typically, 512-bits or even 1024-bits long) so that many more message bits are processed in every round.

Davies-Meyer variants. The Davies-Meyer construction is not unique. Many other similar methods can convert a block cipher into a collision resistant compression function. For example, one could use

Matyas-Meyer-Oseas:	$h_1(x,y) := E(x,\ y) \oplus y$
Miyaguchi-Preneel:	$h_2(x,y) := E(x,\ y) \oplus y \oplus x$
Or even:	$h_3(x,y) := E(x \oplus y,\ y) \oplus y$

or many other such variants. Preneel et al. [89] give twelve different variants that can be shown to be collision resistant.

The Matyas-Meyer-Oseas function h_1 is similar to Davies-Meyer, but reverses the roles of the chaining variable and the message block — in h_1 the chaining variable is used as the block cipher key. The function h_1 maps elements in $(\mathcal{K} \times \mathcal{X})$ to $\mathcal{X}$. Therefore, to use h_1 in Merkle-Damgård we need an auxiliary encoding function $g : \mathcal{X} \to \mathcal{K}$ that maps the chaining variable $t_{i-1} \in \mathcal{X}$ to an element in $\mathcal{K}$, as shown in Fig. 8.7. The same is true for the Miyaguchi-Preneel function h_2. The Davies-Meyer function does not need such an encoding function. We note that the Miyaguchi-Preneel function has a minor security advantage over Davies-Meyer, as discussed in Exercise 8.14.

Many other natural variants of Davies-Meyer are totally insecure. For example, for the following functions

$$h_4(x, y) := E(y,\ x) \oplus y$$
$$h_5(x, y) := E(x,\ x \oplus y) \oplus x$$

we can find collisions in constant time (see Exercise 8.10).

8.5.3 Collision resistance of Davies-Meyer

We cannot prove that Davies-Meyer is collision resistant by assuming a standard complexity assumption about the block cipher. Simply assuming that $\mathcal{E} = (E, D)$ is a secure block cipher is insufficient for proving that h_{DM} is collision resistant. Instead, we have to model the block cipher as an *ideal cipher*.

We introduced the ideal cipher model back in Section 4.7. Recall that this is a heuristic technique in which we treat the block cipher as if it were a family of random permutations. If $\mathcal{E} = (E, D)$ is a block cipher with key space $\mathcal{K}$ and data block space $\mathcal{X}$, then the family of random permutations is $\{\Pi_k\}_{k \in \mathcal{K}}$, where each Π_k is a truly random permutation on $\mathcal{X}$, and the Π_k's collectively are mutually independent.

Attack Game 8.1 can be adapted to the ideal cipher model, so that before the adversary outputs a collision, it may make a series of Π-queries and Π^{-1}-queries to its challenger.

- For a Π-query, the adversary submits a pair $(k, a) \in \mathcal{K} \times \mathcal{X}$, to which the challenger responds with $b := \Pi_k(a)$.
- For a Π^{-1}-query, the adversary submits a pair $(k, b) \in \mathcal{K} \times \mathcal{X}$, to which the challenger responds with $a := \Pi_k^{-1}(b)$.

After making these queries, the adversary attempts to output a collision, which in the case of Davies-Meyer, means $(x, y) \neq (x', y')$ such that

$$\Pi_y(x) \oplus x = \Pi_{y'}(x') \oplus x'.$$

The adversary $\mathcal{A}$'s advantage in finding a collision for h_{DM} in the ideal cipher model is denoted $\text{CR}^{\text{ic}}\mathsf{adv}[\mathcal{A}, h_{\text{DM}}]$, and security in the ideal cipher model means that this advantage is negligible for all efficient adversaries $\mathcal{A}$.

Theorem 8.4 (Davies-Meyer). *Let h_{DM} be the Davies-Meyer hash function derived from a block cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{X})$, where $|\mathcal{X}|$ is large. Then h_{DM} is collision resistant in the ideal cipher model.*

In particular, every collision finding adversary $\mathcal{A}$ that issues at most q ideal-cipher queries will satisfy

$$\text{CR}^{\text{ic}}\mathsf{adv}[\mathcal{A}, h_{\text{DM}}] \leq (q + 1)(q + 2)/|\mathcal{X}|.$$

The theorem shows that Davies-Meyer is an optimal compression function: the adversary must issue $q = \Omega(\sqrt{|\mathcal{X}|})$ queries (and hence must run for at least that amount of time) if he is to find a collision for h_{DM} with constant probability. No compression function can have higher security due to the birthday attack.

Proof. Let $\mathcal{A}$ be a collision finder for h_{DM} that makes at most a total of q ideal cipher queries. We shall assume that $\mathcal{A}$ is "reasonable": before $\mathcal{A}$ outputs its collision attempt $(x, y), (x', y')$, it makes corresponding ideal cipher queries: for (x, y), either a Π-query on (y, x) or a Π^{-1}-query on $(y, \cdot)$ that yields x, and similarly for (x', y'). If $\mathcal{A}$ is not already reasonable, we can make it so by increasing total number of queries to at most $q' := q + 2$. So we will assume $\mathcal{A}$ is reasonable and makes at most q' ideal cipher queries from now on.

For $i = 1, \ldots, q'$, the ith ideal cipher query defines a triple (k_i, a_i, b_i): for a Π-query (k_i, a_i), we set $b_i := \Pi_{k_i}(a_i)$, and for a Π^{-1}-query (k_i, b_i), we set $a_i := \Pi_{k_i}^{-1}(b_i)$. We assume that $\mathcal{A}$ makes no extraneous queries, so that no triples repeat.

If the adversary outputs a collision, then by our reasonableness assumption, for some distinct pair of indices $i, j = 1, \ldots, q'$, we have $a_i \oplus b_i = a_j \oplus b_j$. Let us call this event Z. So we have

$$\mathrm{CR^{ic}}\mathsf{adv}[\mathcal{A}, h_{\mathrm{DM}}] \leq \Pr[Z].$$

Our goal is to show

$$\Pr[Z] \leq \frac{q'(q'-1)}{2^n}, \tag{8.4}$$

where $|\mathcal{X}| = 2^n$.

Consider any fixed indices $i < j$. Conditioned on any fixed values of the adversary's coins and the first $j - 1$ triples, one of a_j and b_j is completely fixed, while the other is uniformly distributed over a set of size at least $|\mathcal{X}| - j + 1$. Therefore,

$$\Pr[a_i \oplus b_i = a_j \oplus b_j] \leq \frac{1}{2^n - j + 1}.$$

So by the union bound, we have

$$\Pr[Z] \leq \sum_{j=1}^{q'} \sum_{i=1}^{j-1} \Pr[a_i \oplus b_i = a_j \oplus b_j] \leq \sum_{j=1}^{q'} \frac{j-1}{2^n - j + 1} \leq \sum_{j=1}^{q'} \frac{j-1}{2^n - q'} = \frac{q'(q'-1)}{2(2^n - q')}. \tag{8.5}$$

For $q' \leq 2^{n-1}$ this bound simplifies to $\Pr[Z] \leq q'(q'-1)/2^n$. For $q' > 2^{n-1}$ the bound holds trivially. Therefore, (8.4) holds for all q'. $\square$

8.6 Case study: SHA256

The Secure Hash Algorithm (SHA) was published by NIST in 1993 [FIPS 180] as part of the design specification of the Digital Signature Standard (DSS). This hash function, often called **SHA-0**, outputs 160-bit digests. Two years later, in 1995, NIST updated the standard [FIPS 180-1] by adding one extra instruction to the compression function. The resulting function is called **SHA-1**. NIST gave no explanation for this change, but it was later found that this extra instruction is crucial for collision resistance. SHA-1 became the de-facto standard for collision resistant hashing and is very widely deployed.

Name	year	digest size	message block size	Speed[2] MB/sec	best known attack time
SHA-0	1993	160	512		2^{39}
SHA-1	1995	160	512	153	2^{63}
SHA224	2004	224	512		
SHA256	2002	256	512	111	
SHA384	2002	384	1024		
SHA512	2002	512	1024	99	
MD4	1990	128	512		2^{1}
MD5	1992	128	512	255	2^{30}
Whirpool	2000	512	512	57	

Table 8.1: Merkle-Damgård collision resistant hash functions

The birthday attack can find collisions for SHA-1 using an expected 2^{80} evaluations of the function. In 2002 NIST added [FIPS 180-2] two new hash functions to the SHA family: **SHA256** and **SHA512**. They output larger digests (256 and 512-bit digests respectively) and therefore provide better protection against the birthday attack. NIST also approved SHA224 and SHA384 which are obtained from SHA256 and SHA512 respectively by truncating the output to 224 and 384 bits. These and a few other proposed hash functions are summarized in Table 8.1.

The years 2004–5 were bad years for collision resistant hash functions. A number of new attacks showed how to find collisions for a variety of hash functions. In particular, Wang, Yao, and Yao [103] presented a collision finder for SHA-1 that uses 2^{63} evaluations of the function — far less than the birthday attack. As a result SHA-1 is no longer considered collision resistant. The current recommended practice is to use SHA256 which we describe here.

The SHA256 function. SHA256 is a Merkle-Damgård hash function using a Davies-Meyer compression function h. This h takes as input a 256-bit chaining variable t and a 512-bit message block m. It outputs a 256-bit chaining variable.

We first describe the SHA256 Merkle-Damgård chain. Recall that the padding block PB in our description of Merkle-Damgård contained a 64-bit encoding of the number of *blocks* in the message being hashed. The same is true for SHA256 with the minor difference that PB encodes the number of *bits* in the message. Hence, SHA256 can hash messages that are at most $2^{64} - 1$ bits long. The Merkle-Damgård Initial Value (IV) in SHA256 is set to:

$$\text{IV} := \text{6A09E667 BB67AE85 3C6EF372 A54FF53A 510E527F 9B05688C 1F83D9AB 5BE0CD19} \in \{0,1\}^{256}$$

written in base 16.

Clearly the output of SHA256 can be truncated to obtain shorter digests at the cost of reduced security. This is, in fact, how the SHA224 hash function works — it is identical to SHA256 with two exceptions: (1) SHA224 uses a different initialization vector IV, and (2) SHA224 truncates the output of SHA256 to its left most 224 bits.

[2]Performance numbers were provided by Wei Dai using the Crypto++ 5.6.0 benchmarks running on a 1.83 GhZ Intel Core 2 processor. Higher numbers are better.

Next, we describe the SHA256 Davies-Meyer compression function h. It is built from a block cipher which we denote by E_{SHA256}. However, instead of using XOR as in Davies-Meyer, SHA256 uses addition modulo 2^{32}. That is, let

$$x_0, x_1, \ldots, x_7 \in \{0,1\}^{32} \qquad \text{and} \qquad y_0, y_1, \ldots, y_7 \in \{0,1\}^{32}$$

and set

$$x := x_0 \parallel \cdots \parallel x_7 \ \in \{0,1\}^{256} \qquad \text{and} \qquad y := y_0 \parallel \cdots \parallel y_7 \ \in \{0,1\}^{256}.$$

Define: $\quad x \boxplus y := (x_0 + y_0) \parallel \cdots \parallel (x_7 + y_7) \quad \in \{0,1\}^{256}$ $\quad$ where all additions are modulo 2^{32}. Then the SHA256 compression function h is defined as:

$$h(t, m) \ := \ E_{\text{SHA256}}(m, t) \ \boxplus \ t \quad \in \{0,1\}^{256}.$$

Our ideal cipher analysis of Davies-Meyer (Theorem 8.4) applies equally well to this modified function.

The SHA256 block cipher. To complete the description of SHA256 it remains to describe the block cipher E_{SHA256}. The algorithm makes use of a few auxiliary functions defined in Table 8.2. Here, SHR and ROTR denote the standard shift-right and rotate-right functions.

The cipher E_{SHA256} takes as input a 512-bit key k and a 256-bit message t. We first break both the key and the message into 32-bit words. That is, write:

$$k := k_0 \parallel k_1 \parallel \cdots \parallel k_{15} \quad \in \{0,1\}^{512}$$
$$t := t_0 \parallel t_1 \parallel \cdots \parallel t_7 \quad \in \{0,1\}^{256}$$

where each k_i and t_i is in $\{0,1\}^{32}$.

The code for E_{SHA256} is shown in Table 8.3. It iterates the same round function 64 times. In each round the cipher uses a round key $W_i \in \{0,1\}^{32}$ defined recursively during the key setup step. One cipher round, shown in Fig. 8.8, looks like two adjoined Feistel rounds. The cipher uses 64 fixed constants $K_0, K_1, \ldots, K_{63} \in \{0,1\}^{32}$ whose values are specified in the SHA256 standard. For example, $K_0 := 428A2F98$ and $K_1 := 71374491$, written base 16.

Interestingly, NIST never gave the block cipher E_{SHA256} an official name. The cipher was given the unofficial name **SHACAL-2** by Handschuh and Naccache (submission to NESSIE, 2000). Similarly, the block cipher underlying SHA-1 is called SHACAL-1. The SHACAL-2 block cipher is identical to E_{SHA256} with the only difference that it can encrypt using keys shorter than 512 bits. Given a key $k \in \{0,1\}^{\leq 512}$ the SHACAL-2 cipher appends zeros to the key to get a 512-bit key. It then applies E_{SHA256} to the given 256-bit message block. Decryption in SHACAL-2 is similar to encryption. This cipher is well suited for applications where SHA256 is already implemented, thus reducing the overall size of the crypto code.

8.6.1 Other Merkle-Damgård hash functions

MD4 and MD5. Two cryptographic hash functions designed by Rivest in 1990–1 [90, 91]. Both are Merkle-Damgård hash functions that output a 128-bit digest. They are quite similar, although MD5 uses a stronger compression function than MD4. Collisions for both hash functions can be found efficiently as described in Table 8.1. Consequently, these hash functions should no longer be used.

For x, y, z in $\{0,1\}^{32}$ define:

$$\begin{array}{rcll}
\mathrm{SHR}^n(x) & := & (x >> n) & \text{(Shift Right)} \\
\mathrm{ROTR}^n(x) & := & (x >> n) \vee (x << 32 - n) & \text{(Rotate Right)} \\
\mathrm{Ch}(x,y,z) & := & (x \wedge y) \oplus (\neg x \wedge z) & \\
\mathrm{Maj}(x,y,z) & := & (x \wedge y) \oplus (x \wedge z) \oplus (y \wedge z) & \\
\Sigma_0(x) & := & \mathrm{ROTR}^2(x) \oplus \mathrm{ROTR}^{13}(x) \oplus \mathrm{ROTR}^{22}(x) & \\
\Sigma_1(x) & := & \mathrm{ROTR}^6(x) \oplus \mathrm{ROTR}^{11}(x) \oplus \mathrm{ROTR}^{25}(x) & \\
\sigma_0(x) & := & \mathrm{ROTR}^7(x) \oplus \mathrm{ROTR}^{18}(x) \oplus \mathrm{SHR}^3(x) & \\
\sigma_1(x) & := & \mathrm{ROTR}^{17}(x) \oplus \mathrm{ROTR}^{19}(x) \oplus \mathrm{SHR}^{10}(x) &
\end{array}$$

Table 8.2: Functions used in the SHA256 block cipher

Input: plaintext $t = t_0 \parallel \cdots \parallel t_7 \in \{0,1\}^{256}$ and
key $k = k_0 \parallel k_1 \parallel \cdots \parallel k_{15} \in \{0,1\}^{512}$

Output: ciphertext in $\{0,1\}^{256}$.

// *Here all additions are modulo* 2^{32}.

// *The algorithm uses constants* $K_0, K_1, \ldots, K_{63} \in \{0,1\}^{32}$

Key setup: Construct 64 round keys $W_0, \ldots, W_{63} \in \{0,1\}^{32}$:

$$\begin{cases} \text{for } i = 0,1,\ldots,15 & \text{set } \; W_i \leftarrow k_i, \\ \text{for } i = 16,17,\ldots,63 & \text{set } \; W_i \leftarrow \sigma_1(W_{i-2}) + W_{i-7} + \sigma_0(W_{i-15}) + W_{i-16} \end{cases}$$

64 Rounds:

$\big(a_0,\ b_0,\ c_0,\ d_0,\ e_0,\ f_0,\ g_0,\ h_0\big) \;\leftarrow\; \big(t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7\big)$

for $i = 0$ to 63 do:

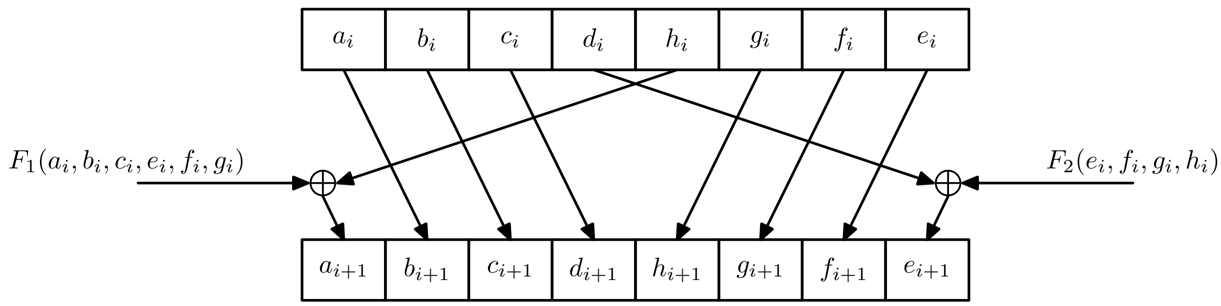
$T_1 \leftarrow h_i + \Sigma_1(e_i) + \mathrm{Ch}(e_i, f_i, g_i) + K_i + W_i$

$T_2 \leftarrow \Sigma_0(a_i) + \mathrm{Maj}(a_i, b_i, c_i)$

$\big(a_{i+1},\ b_{i+1},\ c_{i+1},\ d_{i+1},\ e_{i+1},\ f_{i+1},\ g_{i+1},\ h_{i+1}\big) \;\leftarrow$
$\big(T_1 + T_2,\ a_i,\ b_i,\ c_i,\ d_i + T_1,\ e_i,\ f_i,\ g_i\big)$

Output: $a_{64} \parallel b_{64} \parallel c_{64} \parallel d_{64} \parallel e_{64} \parallel f_{64} \parallel g_{64} \parallel h_{64} \in \{0,1\}^{256}$

Table 8.3: The SHA256 block cipher



$$F_1(a, b, c, e, f, g) := \Sigma_1(e) + \mathrm{Ch}(e, f, g) + \Sigma_0(a) + \mathrm{Maj}(a, b, c) + K_i + W_i$$

$$F_2(e, f, g, h) := h + \Sigma_1(e) + \mathrm{Ch}(e, f, g) + K_i + W_i$$

Figure 8.8: One round of the SHA256 block cipher

Whirpool. Whirlpool was designed by Barreto and Rijmen in 2000 and was adopted as an ISO/IEC standard in 2004. Whirpool is a Merkle-Damgård hash function. Its compression function uses the Miyaguchi-Preneel method (Fig. 8.7) with a block cipher called W. This block cipher is very similar to AES, but has a 512-bit block size. The resulting hash output is 512-bits.

Others. Many other Merkle-Damgård hash functions were proposed in the literature. Some examples include Tiger/192 [12] and RIPEMD-160 to name a few.

8.7 Case study: HMAC

In this section, we return to our problem of building a secure MAC that works on long messages. Merkle-Damgård hash functions such as SHA1 and SHA256 are very widely deployed. Most Crypto libraries include an implementation of multiple Merkle-Damgård functions. Furthermore, these implementations are very fast: one can typically hash a very long message with SHA256 much faster than one can apply, say, CBC-MAC with AES to the same message.

Of course, one might use the hash-then-MAC construction analyzed in Section 8.2. Recall that in this construction, we combine a secure MAC system $\mathcal{I} = (S, V)$ and a collision resistant hash function H, so that the resulting signing algorithm signs a message m by first hashing m using H to get a short digest $H(m)$, and then signs $H(m)$ using S to obtain the MAC tag $t = S(k, H(m))$. As we saw in Theorem 8.1 the resulting construction is secure. However, this construction is not very widely deployed. Why?

First of all, as discussed after the statement of Theorem 8.1, if one can find collisions in H, then the hash-then-MAC construction is completely broken. A collision-finding attack, such as a birthday attack (Section 8.3), or a more sophisticated attack, can be carried out entirely *offline*, that is, without the need to interact with any users of the system. In contrast, *online* attacks require many interactions between the adversary and honest users of the system. In general, offline attacks are considered especially dangerous since an adversary can invest huge computing resources over an extended period of time: in an attack on hash-then-MAC, an attacker could spend months quietly computing on many machines to find a collision on H, without arousing any suspicions.

Another reason not to use the hash-then-MAC construction directly is that we need both a hash function H and a MAC system $\mathcal{I}$. So an implementation might need software and/or hardware to execute both, say, SHA256 for the hash and CBC-MAC with AES for the MAC. All other things being equal, it would be nice to simply use one algorithm as the basis for a MAC.

This leads us to the following problem: how to take a *keyless* Merkle-Damgård hash function, such as SHA256, and use it somehow to implement a *keyed* function that is a secure MAC, or even better, a secure PRF. Moreover, we would like to be able to prove the security of this construction under an assumption that is (qualitatively, at least) weaker than collision resistance; in particular, the construction should not be susceptible to an offline collision-finding attack on the underlying compression function.

Assume that H is a Merkle-Damgård hash built from a compression function $h : \{0,1\}^n \times \{0,1\}^\ell \to \{0,1\}^n$. A few simple approaches come to mind.

Prepend the key: $F_{\text{pre}}(k, M) := H(k \parallel M)$. This is completely insecure, because of the following *extension attack*: given $F_{\text{pre}}(k, M)$, one can easily compute $F_{\text{pre}}(k, M \parallel \text{PB} \parallel M')$ for any M'. Here, PB is the Merkle-Damgård padding block for the message $k \parallel M$. Aside from this extension attack, the construction is secure, under reasonable assumptions (see Exercise 8.17).

Append the key: $F_{\text{post}}(k, M) := H(M \parallel k)$. This is somewhat similar to the hash-then-MAC construction, and relies on the collision resistance of h. Indeed, it is vulnerable to an offline collision-finding attack: assuming we find two distinct ℓ-bit strings M_0 and M_1 such that $h(\text{IV}, M_0) = h(\text{IV}, M_1)$, then we have $F_{\text{post}}(k, M_0) = F_{\text{post}}(k, M_1)$. For these reasons, this construction does not solve our problem. However, under the right assumptions (including the collision resistance of h, of course), we can still get a security proof (see Exercise 8.18).

Envelope method: $F_{\text{env}}(k, M) := H(k \parallel M \parallel k)$. Under reasonable pseudorandomness assumptions on h, and certain formatting assumptions (that k is an ℓ-bit string and M is padded out to a bit string whose length is a multiple of ℓ), this can be proven to be a secure PRF. See Exercise 8.16.

Two-key nest: $F_{\text{nest}}((k_1, k_2), M) := H(k_2 \parallel H(k_1 \parallel M))$. Under reasonable pseudorandomness assumptions on h, and certain formatting assumptions (that k_1 and k_2 are ℓ-bit strings), this can also be proven to be a secure PRF.

The two-key nest is very closely related to a classic MAC construction known as **HMAC**. HMAC is the most widely deployed MAC on the Internet. It is used in SSL, TLS, IPsec, SSH, and a host of other security protocols. TLS and IPsec also use HMAC as a means for deriving session keys during session setup. We will give a security analysis of the two-key nest, and then discuss its relation to HMAC.

8.7.1 Security of two-key nest

We will now show that the two-key nest is indeed a secure PRF, under appropriate psuedorandomness assumptions on h. Let us start by "opening up" the definition of $F_{\text{nest}}((k_1, k_2), M)$, using the fact that H is a Merkle-Damgård hash built from h. See Fig. 8.9. The reader should study this figure carefully. We are assuming that the keys k_1 and k_2 are ℓ-bit strings, so they each occupy one full message block. The input to the inner evaluation of H is the padded string $k_1 \parallel M \parallel \text{PB}_\text{i}$,

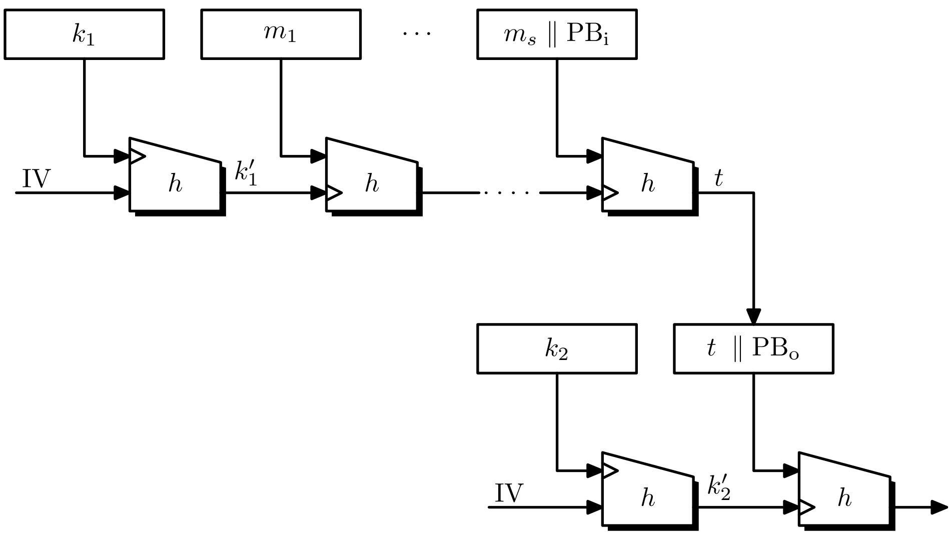


Figure 8.9: The two-key nest

which is broken into ℓ-bit blocks as shown. The output of the inner evaluation of H is the n-bit string t. The input to the outer evaluation of H is the padded string $k_2 \parallel t \parallel \mathrm{PB_o}$. We shall assume that n is significantly smaller than ℓ, so that $t \parallel \mathrm{PB_o}$ is a single ℓ-bit block, as shown in the figure.

We now state the pseudorandomness assumptions we need. We define the following two PRFs h_{bot} and h_{top} derived from h:

$$h_{\text{bot}}(k, m) := h(k, m) \qquad \text{and} \qquad h_{\text{top}}(k, m) := h(m, k). \tag{8.6}$$

For the PRF h_{bot}, the PRF key k is viewed as the first input to h, i.e., the n-bit chaining variable input, which is the *bottom* input to the h-boxes in Fig. 8.9. For the PRF h_{top}, the PRF key k is viewed as the second input to h, i.e., the ℓ-bit message block input, which is the *top* input to the h-boxes in the figure. To make the figure easier to understand, we have decorated the h-box inputs with a $>$ symbol, which indicates which input is to be viewed as a PRF key. Indeed, the reader will observe that we will treat the two evaluations of h that appear within the dotted boxes as evaluations of the PRF h_{top}, so that the values labeled k_1' and k_2' in the figure are computed as $k_1' \leftarrow h_{\text{top}}(k_1, \mathrm{IV})$ and $k_2' \leftarrow h_{\text{top}}(k_2, \mathrm{IV})$. All of the other evaluations of h in the figure will be treated as evaluations of h_{bot}.

Our assumption will be that h_{bot} and h_{top} are both secure PRFs. Later, we will use the ideal cipher model to justify this assumption for the Davies-Meyer compression function (see Section 8.7.3).

We will now sketch a proof of the following result:

If h_{bot} and h_{top} are secure PRFs, then so is the two-key nest.

The first observation is that the keys k_1 and k_2 are only used to derive k_1' and k_2' as $k_1' = h_{\text{top}}(k_1, \mathrm{IV})$ and $k_2' = h_{\text{top}}(k_2, \mathrm{IV})$. The assumption that h_{top} is a secure PRF means that in the PRF attack game, we can effectively replace k_1' and k_2' by truly random n-bit strings. The resulting construction drawn in Fig. 8.10. All we have done here is to throw away all of the elements in Fig. 8.9 that are within the dotted boxes. The function in this new construction takes as input

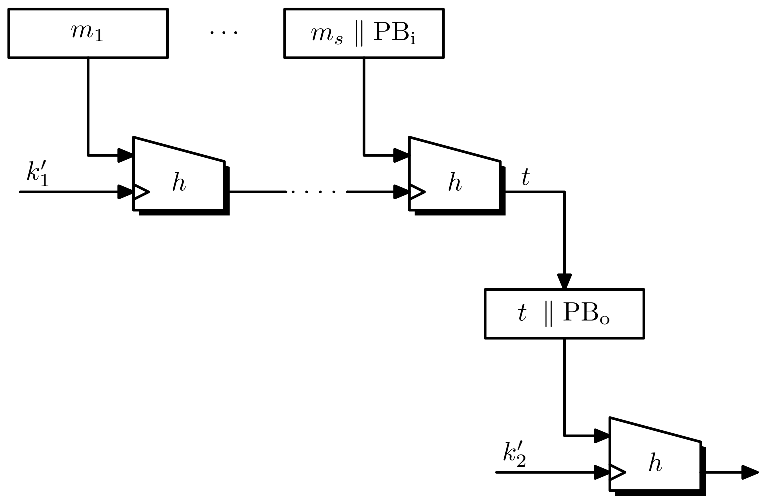


Figure 8.10: A bit-wise version of NMAC

the two keys k_1' and k_2' and a message M. By the above observations, it suffices to prove that the construction in Fig. 8.10 is a secure PRF.

Hopefully (without reading the caption), the reader will recognize the construction in Fig. 8.10 as none other than NMAC applied to h_{bot}, which we introduced in Section 6.5.1 (in particular, take a look at Fig. 6.5b). Actually, the construction in Fig. 8.10 is a bit-wise version of NMAC, obtained from the block-wise version via padding (as discussed in Section 6.8). Thus, security for the two-key nest now follows directly from the NMAC security theorem (Theorem 6.7) and the assumption that h_{bot} is a secure PRF.

8.7.2 The HMAC standard

The HMAC standard is exactly the same as the two-key nest (Fig. 8.9), but with one important difference: the keys k_1 and k_2 are not independent, but rather, are derived in a somewhat *ad hoc* way from a single key k.

To describe this in more detail, we first observe that HMAC itself is somewhat byte oriented, so all strings are byte strings. Message blocks for the underlying Merkle-Damgård hash are assumed to be B bytes (rather than ℓ bits). A key k for HMAC is a byte string of arbitrary length. To derive the keys k_1 and k_2, which are byte strings of length B, we first make k exactly B bytes long: if the length of k is less than or equal to B, we pad it out with zero bytes; otherwise, we replace it with $H(k)$ padded with zero bytes. Then we compute

$$k_1 \leftarrow k \oplus \mathsf{ipad} \quad \text{and} \quad k_2 \leftarrow k \oplus \mathsf{opad},$$

where ipad and opad ("i" and "o" stand for "inner" and "outer") are B-byte constant strings, defined as follows:

$$\begin{aligned} \mathsf{ipad} &= \text{the byte 0x36 repeated } B \text{ times} \\ \mathsf{opad} &= \text{the byte 0x5C repeated } B \text{ times} \end{aligned}$$

HMAC implemented using a hash function H is denoted HMAC-H. The most common HMACs used in practice are HMAC-SHA1 and HMAC-SHA256. The HMAC standard also allows the output of HMAC to be truncated. For example, when truncating the output of SHA1 to 80 bits, the HMAC function is denoted HMAC-SHA1-80. Implementations of TLS 1.0, for example, are required to support HMAC-SHA1-96.

Security of HMAC. Since the keys k'_1, k'_2 are related — their XOR is equal to $\mathsf{opad} \oplus \mathsf{ipad}$ — the security proof we gave for the two-key nest no longer applies: under the stated assumptions, we cannot justify the claim that the derived keys k'_1, k'_2 are indistinguishable from random. One solution is to make a stronger assumption about the compression function h – one needs to assume that h_{top} remains a PRF under a related key attack (as defined by Bellare and Kohno [6]). If h is itself a Davies-Meyer compression function, then this stronger assumption can be justified in the ideal cipher model.

8.7.3 Davies-Meyer is a secure PRF in the ideal cipher model

It remains to justify our assumption that the PRFs h_{bot} and h_{top} derived from h in (8.6) are secure. Suppose the compression function h is a Davies-Meyer function, that is $h(x, y) := E(y,\ x) \oplus x$ for some block cipher $\mathcal{E} = (E, D)$. Then

- $h_{\text{bot}}(k, m) := h(k, m) = E(m, k) \oplus k$ is a PRF defined over$(\mathcal{X}, \mathcal{K}, \mathcal{X})$, and
- $h_{\text{top}}(k, m) := h(m, k) = E(k, m) \oplus m$ is a PRF defined over$(\mathcal{K}, \mathcal{X}, \mathcal{X})$

When $\mathcal{E}$ is a secure block cipher, the fact that h_{top} is a secure PRF is trivial (see Exercise 4.1 part (c)). The fact that h_{bot} is a secure PRF is a bit surprising — the message m given as input to h_{bot} is used as the key for E. But m is chosen by the adversary and hence E is evaluated with a key that is completely under the control of the adversary. As a result, even though E is a secure block cipher, there is no security guarantee for h_{bot}. Nevertheless, we can prove that h_{bot} is a secure PRF, but this requires the ideal cipher model. Just assuming that $\mathcal{E}$ is a secure block cipher is insufficient.

If necessary, the reader should review the basic concepts regarding the ideal cipher model, which was introduced in Section 4.7. We also used the ideal cipher model earlier in this chapter (see Section 8.5.3).

In the ideal cipher model, we heuristically model a block cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{X})$ as a family of random permutations $\{\Pi_{\mathscr{k}}\}_{\mathscr{k} \in \mathcal{K}}$. We adapt the PRF Attack Game 4.2 to work in the ideal cipher model. The challenger, in addition to answering standard queries, also answers Π-queries and Π^{-1}-queries: a Π-query is a pair $(\mathscr{k}, \mathscr{a})$ to which the challenger responds with $\mathscr{b} := \Pi_{\mathscr{k}}(\mathscr{a})$; a Π^{-1}-query is a pair $(\mathscr{k}, \mathscr{b})$ to which is the challenger responds with $\mathscr{a} := \Pi_{\mathscr{k}}^{-1}(\mathscr{b})$. For a standard query m, the challenger responds with $v := f(m)$: in Experiment 0 of the attack game, f is $F(k, \cdot)$, where F is a PRF and k is a randomly chosen key; in Experiment 1, f is a truly random function. Moreover, in Experiment 0, F is evaluated using the random permutations in the role of E and D used in the construction of F. For our PRF $h_{\text{bot}}(k, m) = E(m, k) \oplus k = \Pi_m(k) \oplus k$.

For an adversary $\mathcal{A}$, we define $\mathrm{PRF^{ic}}\mathsf{adv}[\mathcal{A}, F]$ to be the advantage in the modified PRF attack game, and security in the ideal cipher model means that this advantage is negligible for all efficient adversaries.

Theorem 8.5 (Security of h_{bot}). *Let $\mathcal{E} = (E, D)$ be a block cipher over $(\mathcal{K}, \mathcal{X})$, where $|\mathcal{X}|$ is large. Then $h_{\text{bot}}(k, m) := E(m, k) \oplus k$ is a secure PRF in the ideal cipher model.*

In particular, for every PRF adversary $\mathcal{A}$ attacking h_{bot} and making at most a total of Q_{ic} ideal cipher queries, we have

$$\text{PRF}^{\text{ic}}\mathsf{adv}[\mathcal{A}, h_{\text{bot}}] \leq \frac{2Q_{\text{ic}}}{|\mathcal{X}|}.$$

The bound in the theorem is fairly tight, as brute-force key search gets very close to this bound.

Proof. The proof will mirror the analysis of the Evan-Mansour/$\mathcal{E}X$ constructions (see Theorem 4.14 in Section 4.7.4), and in particular, will make use of the Domain Separation Lemma (see Theorem 4.15, also in Section 4.7.4).

Let $\mathcal{A}$ be an adversary as in the statement of the theorem. Let p_b be the probability that $\mathcal{A}$ outputs 1 in Experiment b of Attack Game 4.2, for $b = 0, 1$. So by definition we have

$$\text{PRF}^{\text{ic}}\mathsf{adv}[\mathcal{A}, h_{\text{bot}}] = |p_0 - p_1|. \tag{8.7}$$

We shall prove the theorem using a sequence of two games, applying the Domain Separation Lemma.

Game 0. The game will correspond to Experiment 0 of the PRF attack game in the idea cipher model. We can write the logic of the challenger as follows:

Initialize:
    for each $\mathscr{k} \in \mathcal{K}$, set $\Pi_{\mathscr{k}} \xleftarrow{\text{R}} \text{Perms}[\mathcal{X}]$
    $k \xleftarrow{\text{R}} \mathcal{X}$

standard h_{bot}-query m:
1. $c \leftarrow \Pi_m(k)$
2. $v \leftarrow c \oplus k$
3. return v

The challenger in Game 0 processes ideal cipher queries *exactly as in Game 0 of the proof of Theorem 4.14*:

ideal cipher Π-query $\mathscr{k}, \mathscr{a}$:
1. $\mathscr{b} \leftarrow \Pi_{\mathscr{k}}(\mathscr{a})$
2. return $\mathscr{b}$

ideal cipher Π^{-1}-query $\mathscr{k}, \mathscr{b}$:
1. $\mathscr{a} \leftarrow \Pi_{\mathscr{k}}^{-1}(\mathscr{b})$
2. return $\mathscr{a}$

Let W_0 be the event that $\mathcal{A}$ outputs 1 at the end of Game 0. It should be clear from construction that

$$\Pr[W_0] = p_0. \tag{8.8}$$

Game 1. Just as in the proof of Theorem 4.14, we declare "by fiat" that standard queries and ideal cipher queries are processed using independent random permutations. In detail (changed from Game 0 are highlighted):

Initialize:

  for each $\mathcal{k} \in \mathcal{K}$, set $\Pi_{\text{std},\mathcal{k}} \xleftarrow{\text{R}} \text{Perms}[\mathcal{X}]$ and $\Pi_{\text{ic},\mathcal{k}} \xleftarrow{\text{R}} \text{Perms}[\mathcal{X}]$

  $k \xleftarrow{\text{R}} \mathcal{X}$

standard h_{bot}-query m:

1. $c \leftarrow \Pi_{\text{std},m}(k)$ // *add k to sampled domain of $\Pi_{\text{std},m}$, add c to sampled range of $\Pi_{\text{std},m}$*
2. $v \leftarrow c \oplus k$
3. return v

The challenger in Game 1 processes ideal cipher queries *exactly as in Game 1 of the proof of Theorem 4.14*:

ideal cipher Π-query $\mathcal{k}, \mathcal{a}$:

1. $\mathcal{b} \leftarrow \Pi_{\text{ic},\mathcal{k}}(\mathcal{a})$ // *add $\mathcal{a}$ to sampled domain of $\Pi_{\text{ic},\mathcal{k}}$, add $\mathcal{b}$ to sampled range of $\Pi_{\text{ic},\mathcal{k}}$*
2. return $\mathcal{b}$

ideal cipher Π^{-1}-query $\mathcal{k}, \mathcal{b}$:

1. $\mathcal{a} \leftarrow \Pi^{-1}_{\text{ic},\mathcal{k}}(\mathcal{b})$ // *add $\mathcal{a}$ to sampled domain of $\Pi_{\text{ic},\mathcal{k}}$, add $\mathcal{b}$ to sampled range of $\Pi_{\text{ic},\mathcal{k}}$*
2. return $\mathcal{a}$

Let W_1 be the event that $\mathcal{A}$ outputs 1 at the end of Game 1. Consider an input/output pair (m, v) for a standard query in Game 2. Observe that k is the only item ever added to the sampled domain of $\Pi_{\text{std},m}(k)$, and $c = v \oplus k$ is the only item ever added to the sampled range of $\Pi_{\text{std},m}(k)$. In particular, c is generated at random and k remains perfectly hidden (i.e., is independent of the adversary's view).

Thus, from the adversary's point of view, the standard queries behave identically to a random function, and the ideal cipher queries behave like ideal cipher queries for an *independent* ideal cipher. In particular, we have

$$\Pr[W_1] = p_1. \tag{8.9}$$

Finally, we use the Domain Separation Lemma to analyze $|\Pr[W_0] - \Pr[W_1]|$. The domain separation failure event Z is the event that in Game 1, the sampled domain of one of the $\Pi_{\text{std},m}$'s overlaps with the sampled domain of one of the $\Pi_{\text{ic},\mathcal{k}}$'s, or the sampled range of one of the $\Pi_{\text{std},m}$'s overlaps with the sampled range of one of the $\Pi_{\text{ic},\mathcal{k}}$'s. The Domain Separation Lemma tells us that

$$|\Pr[W_0] - \Pr[W_1]| \leq \Pr[Z]. \tag{8.10}$$

If Z occurs, then for some input/output triple $(\mathcal{k}, \mathcal{a}, \mathcal{b})$ corresponding to an ideal cipher query, $\mathcal{k} = m$ was the input to a standard query with output v, and either

(i) $\mathcal{a} = k$, or

(ii) $\mathcal{b} = v \oplus k$.

For any fixed triple $(\mathcal{k}, \mathcal{a}, \mathcal{b})$, by the independence of k, conditions (i) and (ii) each hold with probability $1/|\mathcal{X}|$, and so by the union bound

$$\Pr[Z] \leq \frac{2Q_{\text{ic}}}{|\mathcal{X}|}. \tag{8.11}$$

The theorem now follows from (8.7)–(8.11). □

8.8 The Sponge Construction and SHA3

For many years, essentially all collision resistant hash functions were based on the Merkle-Damgård paradigm. Recently, however, an alternative paradigm has emerged, called the **sponge construction**. Like Merkle-Damgård, it is a simple iterative construction built from a more primitive function; however, instead of a compression function $h : \{0,1\}^{n+\ell} \to \{0,1\}^n$, a permutation $\pi : \{0,1\}^n \to \{0,1\}^n$ is used. We stress that unlike a block cipher, the function π has no key. There are two other high-level differences between the sponge and Merkle-Damgård that we should point out:

- On the negative side, it is not known how to reduce the collision resistance of the sponge to a concrete security property of π. The only known analysis of the sponge is in the ideal permutation model, where we (heuristically) model π as a truly random permutation Π.

- On the positive side, the sponge is designed to be used flexibly and securely in a variety of applications where collision resistance is not the main property we need. For example, in Section 8.7, we looked at several possible ways to convert a hash function H into a PRF F. We saw, in particular, that the intuitive idea of simply prepending the key, defining $F_{\text{pre}}(k, M) := H(k \parallel M)$, does not work when H instantiated with a Merkle-Damgård hash. The sponge avoids these problems: it allows one to hash variable length inputs to variable length outputs, and if we model π as a random permutation, then one can argue that for all intents and purposes, the sponge is a random function (we will discuss this in more detail in Section 8.10). In particular, the construction F_{pre} is secure when H is instantiated with a sponge hash.

A new hash standard, called SHA3, is based on the sponge construction. After giving a description and analysis of the general sponge construction, we discuss some of the particulars of SHA3.

8.8.1 The sponge construction

We now describe the sponge construction. In addition specifying a permutation $\pi : \{0,1\}^n \to \{0,1\}^n$, we need to specify two positive integers numbers r and c such that $n = r + c$. The number r is called the **rate** of the sponge: larger rate values lead to faster evaluation. The number c is called the **capacity** of the sponge: larger capacity values lead to better security bounds. Thus, different choices of r and c lead to different speed/security trade-offs.

The sponge allows variable length inputs. To hash a long message $M \in \{0,1\}^{\leq L}$, we first append a padding string to M to make its length a multiple of r, and then break the padded M into a sequence of r-bit blocks $m_1, \ldots, m_s$. The requirements of the padding procedure are minimal: it just needs to be injective. Just adding a string of the form 10^* suffices, although in SHA3 a pad of the form 10^*1 is used: this latter padding has the effect of encoding the rate in the last block and helps to analyze security in applications that use the same sponge with different rates; however, we will not explore these use cases here. Note that an entire dummy block may need to be added if the length of M is already at or near a multiple of r.

The sponge allows variable length outputs. So in addition to a message $M \in \{0,1\}^{\leq L}$ as above, it takes as input a positive integer v, which specifies the number of output bits.

Here is how the sponge works:

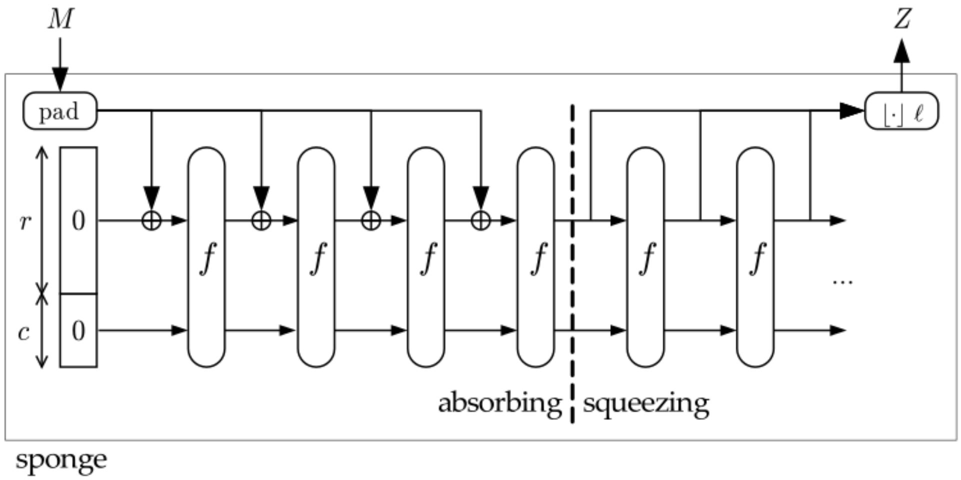


Figure 8.11: The sponge construction

Input: $M \in \{0,1\}^{\leq L}$ and $\ell > 0$
Output: a tag $h \in \{0,1\}^v$
// *Absorbing stage*
Pad M and break into r-bit blocks $m_1, \ldots, m_s$
$h \leftarrow 0^n$
for $i \leftarrow 1$ to s do
$m'_i \leftarrow m_i \parallel 0^c \;\in \{0,1\}^n$
$h \leftarrow \pi(h \oplus m'_i)$
// *Squeezing stage*
$z \leftarrow h[0 \mathinner{..} r-1]$
for $i \leftarrow 1$ to $\lceil v/r \rceil$ do
$h \leftarrow \pi(h)$
$z \leftarrow z \parallel (h[0 \mathinner{..} r-1])$
output $z[0 \mathinner{..} v-1]$

The diagram in Fig. 8.11 may help to clarify the algorithm. The sponge runs in two stages: the "absorbing stage" where the message blocks get "mixed in" to a chaining variable h, and a "squeezing stage" where the output is "pulled out" of the chaining variable. Note that input blocks and output blocks are r-bit strings, so that the remaining c bits of the chaining variable cannot be directly tampered with or seen by an attacker. This is what gives the sponge its security, and is the reason why c must be large. Indeed, if the sponge has small capacity, it is easy to find collisions (see Exercise 8.20).

In the SHA3 standard, the sponge construction is intended to be used as a collision resistant hash, and the output length is fixed to a value $v \leq r$, and so the squeezing stage simply outputs the first v bits of the output h of the absorbing stage. We will now prove that this version of the sponge is collision resistant in the ideal permutation model, assuming 2^c and 2^v are both super-poly.

Theorem 8.6. *Let H be the hash function obtained from a permutation $\pi : \{0,1\}^n \to \{0,1\}^n$, with capacity c, rate r (so $n = r + c$), and output length $v \leq r$. In the ideal permutation model, where*

π is modeled as a random permutation Π, the hash function H is collision resistant, assuming 2^v and 2^c are super-poly.

In particular, for every collision finding adversary $\mathcal{A}$, if the number of ideal-permutation queries plus the number of r-bit blocks in the output messages of $\mathcal{A}$ is bounded by q, then

$$\mathrm{CR^{ic}}\mathsf{adv}[\mathcal{A}, H] \le \frac{q(q-1)}{2^v} + \frac{q(q+1)}{2^c}.$$

Proof. As in the proof of Theorem 8.4, we assume our collision-finding adversary is "reasonable", in the sense that it makes ideal permutation queries corresponding to its output. We can easily convert an arbitrary adversary into a reasonable one by forcing the adversary evaluate the hash function on its output messages if it has not done so already. As we have defined it, q will be an upper bound on the total number of ideal permutation queries made by our reasonable adversary. So from now on, we assume a reasonable adversary $\mathcal{A}$ that makes at most q queries, and we bound the probability that such $\mathcal{A}$ finds anything during its queries that can be "assembled" into a collision (we make this more precise below).

We also assume that *no queries are redundant*. This means that if the adversary makes a Π-query on a yielding $b = \Pi(a)$, then the adversary never makes a Π^{-1}-query on b, and never makes another Π-query on a; similarly, if the adversary makes a Π^{-1}-query on b yielding $a = \Pi^{-1}(b)$, then the adversary never makes a Π-query on a, and never makes another Π^{-1}-query on b. Of course, there is no need for the adversary to make such redundant queries, which is why we exclude them; moreover, doing so greatly simplifies the "bookkeeping" in the proof.

It helps to visualize the adversary's attack as building up a directed graph G. The nodes in G consist of the set of all 2^n bit strings of length n. The graph G starts out with no edges, and every query that $\mathcal{A}$ makes adds an edge to the graph: an edge $a \to b$ is added if $\mathcal{A}$ makes a Π-query on a that yields b or a Π^{-1}-query on b that yields a. Notice that if we have an edge $a \to b$, then $\Pi(a) = b$, regardless of whether that edge was added via a Π-query or a Π^{-1}-query. We say that an edge added via a Π-query is a **forward edge**, and one added via a Π^{-1}-query is a **back edge**.

Note that the assumption that the adversary makes no redundant queries means that an edge gets added only once to the graph, and its classification is uniquely determined by the type of query that added the edge.

We next define a notion of special type of path in the graph that corresponds to sponge evaluation. For an n-bit string z, let $R(z)$ be the first r bits of z and $C(z)$ be the last c bits of z. We refer to $R(z)$ as the **R-part of** z and $C(z)$ as the **C-part of** z. For $s \ge 1$, a **C-path of length** s is a sequence of $2s$ nodes

$$a_0, b_1, a_1, b_2, a_2, \ldots, b_{s-1}, a_{s-1}, b_s,$$

where

- $C(a_0) = 0^c$ and for $i = 1, \ldots, s-1$, we have $C(b_i) = C(a_i)$, and
- G contains edges $a_{i-1} \to b_i$ for $i = 1, \ldots, s$.

For such a path p, the **message** of p is defined as $(m_0, \ldots, m_{s-1})$, where

$$m_0 := R(a_0) \quad \text{and} \quad m_i := R(b_i) \oplus R(a_i) \;\; \text{for} \;\; i = 1, \ldots, s-1.$$

and the **result** of p is defined to be $m_s := R(\mathfrak{b}_s)$. Such a C-path p corresponds to evaluating the sponge at the message $(m_0, \ldots, m_{s-1})$ and obtaining the (untruncated) output m_s. Let us write such a path as

$$m_0|\mathfrak{a}_0 \longrightarrow \mathfrak{b}_1|m_1|\mathfrak{a}_1 \longrightarrow \cdots \longrightarrow \mathfrak{b}_{s-2}|m_{s-2}|\mathfrak{a}_{s-2} \longrightarrow \mathfrak{b}_{s-1}|m_{s-1}|\mathfrak{a}_{s-1} \longrightarrow \mathfrak{b}_s|m_s. \tag{8.12}$$

The following diagram illustrates a C-path of length 3.

$$\begin{array}{ccccccc}
\mathfrak{a}_0 & \longrightarrow & \mathfrak{b}_1 & & & & \\
{\scriptstyle m_0 = R(\mathfrak{a}_0)} & & \mathfrak{a}_1 & \longrightarrow & \mathfrak{b}_2 & & \\
{\scriptstyle 0^c = C(\mathfrak{a}_0)} & & {\scriptstyle m_1 = R(\mathfrak{b}_1) \oplus R(\mathfrak{a}_1)} & & \mathfrak{a}_2 & \longrightarrow & \mathfrak{b}_3 \\
& & {\scriptstyle C(\mathfrak{b}_1) = C(\mathfrak{a}_1)} & & {\scriptstyle m_2 = R(\mathfrak{b}_2) \oplus R(\mathfrak{a}_2)} & & {\scriptstyle m_3 = R(\mathfrak{b}_3)} \\
& & & & {\scriptstyle C(\mathfrak{b}_2) = C(\mathfrak{a}_2)} & &
\end{array}$$

The path has message (m_0, m_1, m_2) and result m_3. Using the notation in (8.12), we write this path as

$$m_0|\mathfrak{a}_0 \longrightarrow \mathfrak{b}_1|m_1|\mathfrak{a}_1 \longrightarrow \mathfrak{b}_2|m_2|\mathfrak{a}_2 \longrightarrow \mathfrak{b}_3|m_3.$$

We can now state what a collision looks like in terms of the graph G. It is a pair of C-paths on different messages but whose results agree on their first v bits (recall $v \leq r$). Let us call such a pair of paths *colliding*.

To analyze the probability of finding a pair of colliding paths, it will be convenient to define another notion. Let p and p' be two C-paths on different messages whose final edges are $\mathfrak{a}_{s-1} \to \mathfrak{b}_s$ and $\mathfrak{a}'_{t-1} \to \mathfrak{b}'_t$. Let us call such a pair of paths *problematic* if

(i) $\mathfrak{a}_{s-1} = \mathfrak{a}'_{t-1}$, or

(ii) one of the edges in p or p' are back edges.

Let W be the event that $\mathcal{A}$ finds a pair of colliding paths. Let Z be the event that $\mathcal{A}$ finds a pair of problematic paths. Then we have

$$\Pr[W] \leq \Pr[Z] + \Pr[W \text{ and not } Z]. \tag{8.13}$$

First, we bound $\Pr[W \text{ and not } Z]$. For an n-bit string z, let $V(z)$ be the first v bits of z, and we refer to $V(z)$ as the **V-part of z**. Suppose $\mathcal{A}$ is able to find a pair of colliding paths that is not problematic. By definition, the final edges on these two paths correspond to Π-queries on distinct inputs that yield outputs whose V-parts agree. That is, if W and not Z occurs, then it must be the case that at some point $\mathcal{A}$ issued two Π-queries on distinct inputs $\mathfrak{a}$ and $\mathfrak{a}'$, yielding outputs $\mathfrak{b}$ and $\mathfrak{b}'$ such that $V(\mathfrak{b}) = V(\mathfrak{b}')$. We can use the union bound: for each pair of indices $i < j$, let X_{ij} be the event that the ith query is a Π-query on some value, say $\mathfrak{a}$, yielding $\mathfrak{b} = \Pi(\mathfrak{a})$, and the j-th query is also a Π-query on some other value $\mathfrak{a}' \neq \mathfrak{a}$, yielding $\mathfrak{b}' = \Pi(\mathfrak{a}')$ such that $V(\mathfrak{b}) = V(\mathfrak{b}')$. If we fix i and j, fix the coins of $\mathcal{A}$, and fix the outputs of all queries made prior to the jth query, then the values $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{a}'$ are all fixed, but the value $\mathfrak{b}'$ is uniformly distributed over a set of size at least $2^n - j + 1$. To get $V(\mathfrak{b}) = V(\mathfrak{b}')$, the value of $\mathfrak{b}'$ must be equal to one of the 2^{n-v} strings whose first v bits agree with that of $\mathfrak{b}$, and so we have

$$\Pr[X_{ij}] \leq \frac{2^{n-v}}{2^n - j + 1}.$$

A simple calculation like that done in (8.5) in the proof of Theorem 8.4 yields

$$\Pr[W \text{ and not } Z] \le \frac{q(q-1)}{2^v}. \tag{8.14}$$

Second, we bound $\Pr[Z]$, the probability that $\mathcal{A}$ finds a pair of problematic paths. The technical heart of the of the analysis is the following:

Main Claim: *If Z occurs, then one of the following occurs:*

(E1) some query yields an output whose C-part is 0^c, or

(E2) two different queries yield outputs whose C-parts are equal.

Just to be clear, (E1) means $\mathcal{A}$ made a query of the form:

> (i) a Π^{-1} query on some value $\mathscr{b}$ such that $C(\Pi^{-1}(\mathscr{b})) = 0^c$, or (ii) a Π query on some value $\mathscr{a}$ such that $C(\Pi(\mathscr{a})) = 0^c$,

and (E2) means $\mathcal{A}$ made pair of queries of the form:

> (i) a Π-query on some value $\mathscr{a}$ and a Π^{-1} query on some value $\mathscr{b}$, such that $C(\Pi(\mathscr{a})) = C(\Pi^{-1}(\mathscr{b}))$, or (ii) Π-queries on two distinct values $\mathscr{a}$ and $\mathscr{a}'$ such that $C(\Pi(\mathscr{a})) = C(\Pi(\mathscr{a}'))$.

First, suppose $\mathcal{A}$ is able to find a problematic pair of paths, and one of the paths contain a back edge. So at the end of the execution, there exists a C-path containing one or more back edges. Let p be such a path of shortest length, and write it as in (8.12). We observe that the last edge in p is a back edge, and all other edges (if any) in p are forward edges. Indeed, if this is not the case, then we can delete this edge from p, obtaining a shorter C-path containing a back edge, contradicting the assumption that p is a shortest path of this type. From this observation, we see that either:

- $s = 1$ and (E1) occurs with the Π^{-1} query on $\mathscr{b}_1$, or
- $s > 1$ and (E2) occurs with the Π^{-1} query on $\mathscr{b}_s$ and the Π-query on $\mathscr{a}_{s-2}$.

Second, suppose $\mathcal{A}$ is able to find a problematic pair of paths, neither of which contains any back edges. Let us call these paths p and p'. The argument in this case somewhat resembles the "backwards walk" in the Merkle-Damgård analysis. Write p as in (8.12) and write p' as

$$m'_0|\mathscr{a}'_0 \longrightarrow \mathscr{b}'_1|m'_1|\mathscr{a}'_1 \longrightarrow \cdots \longrightarrow \mathscr{b}'_{t-2}|m'_{t-2}|\mathscr{a}'_{t-2} \longrightarrow \mathscr{b}'_{t-1}|m'_{t-1}|\mathscr{a}'_{t-1} \longrightarrow \mathscr{b}'_t|m'_t.$$

We are assuming that $(m_0, \ldots, m_{s-1}) \neq (m'_0, \ldots, m'_{t-1})$ but $\mathscr{a}_{s-1} = \mathscr{a}'_{t-1}$, and that none of these edges are back edges. Let us also assume that we choose the paths so that they are shortest, in the sense that $s+t$ is minimal among all C-paths of this type. Also, let us assume that $s \le t$ (swapping if necessary). There are a few cases:

1. $s = 1$ and $t = 1$. This case is impossible, since in this case the paths are just $m_0|\mathscr{a}_0 \to \mathscr{b}_1|m_1$ and $m'_0|\mathscr{a}'_0 \to \mathscr{b}'_1|m'_1$, and we cannot have both $m_0 \neq m'_0$ and $\mathscr{a}_0 = \mathscr{a}'_0$.
2. $s = 1$ and $t \ge 2$. In this case, we have $\mathscr{a}_0 = \mathscr{b}'_{t-1}$, and so (E1) occurs on the Π-query on $\mathscr{a}'_{t-2}$.

3. $s \geq 2$ and $t \geq 2$. Consider the penultimate edges, which are forward edges:

$$\mathfrak{a}_{s-2} \rightarrow \mathfrak{b}_{s-1} | m_{s-1} | \mathfrak{a}_{s-1}$$

   and

$$\mathfrak{a}'_{t-2} \rightarrow \mathfrak{b}'_{t-1} | m'_{t-1} | \mathfrak{a}'_{t-1}.$$

   We are assuming $\mathfrak{a}_{s-1} = \mathfrak{a}'_{t-1}$. Therefore, the C-parts of $\mathfrak{b}_{s-1}$ and $\mathfrak{b}'_{t-1}$ are equal and their R-parts differ by $m_{s-1} \oplus m'_{t-1}$. There are two subcases:

   (a) $m_{s-1} = m'_{t-1}$. We argue that this case is impossible. Indeed, in this case, we have $\mathfrak{b}_{s-1} = \mathfrak{b}'_{t-1}$, and therefore $\mathfrak{a}_{s-2} = \mathfrak{a}'_{t-2}$, while the truncated messages $(m_0, \ldots, m_{s-2})$ and $(m'_1, \ldots, m'_{t-2})$ differ. Thus, we can simply throw away the last edge in each of the two paths, obtaining a shorter pair of paths that contradicts the minimality of $s + t$.

   (b) $m_{s-1} \neq m'_{t-1}$. In this case, we know: the C-parts of $\mathfrak{b}_{s-1}$ and $\mathfrak{b}'_{t-1}$ are the same, but their R-parts differ, and therefore, $\mathfrak{a}_{s-1} \neq \mathfrak{a}'_{t-2}$. Thus, (E2) occurs on the Π-queries on $\mathfrak{a}_{s-2}$ and $\mathfrak{a}'_{t-2}$.

That proves the Main Claim. We can now turn to the problem of bounding the probability that either (E1) or (E2) occurs. This is really just the same type of calculation we did at least twice already, once above in obtaining (8.13), and earlier in the proof of Theorem 8.4. The only difference from (8.13) is that we are now counting collisions on the C-parts, and we have a new type of "collision" to count, namely, "hitting 0^c" as in (E1). We leave it to the reader to verify:

$$\Pr[Z] \leq \frac{q(q+1)}{2^c}. \tag{8.15}$$

The theorem now follows from (8.13)–(8.15). □

8.8.2 Case study: SHA3, SHAKE256, and SHAKE512

The NIST standard for SHA3 specifies a family of sponge-based hash functions. At the heart of these hash functions is a permutation called Keccak, which maps 1600-bit strings to 1600-bit strings. We denote by Keccak[c] the sponge derived from Keccak with capacity c, and using the 10*1 padding rule. This is a function that takes two inputs: a message m and output length v. Here, the input m is an arbitrary bit string and the output of Keccak[c](m, v) is a v-bit string.

We will not describe the internal workings of the Keccak permutation; they can be found in the SHA3 standard. We just describe the different parameter choices that are standardized. The standard specifies four hash functions whose output lengths are fixed, and two hash functions with variable length outputs.

Here are the four fixed-length output hash functions:

- SHA3-224(m) = Keccak[448]$(m \parallel 01, 224)$;
- SHA3-256(m) = Keccak[512]$(m \parallel 01, 256)$;
- SHA3-384(m) = Keccak[768]$(m \parallel 01, 384)$;
- SHA3-512(m) = Keccak[1024]$(m \parallel 01, 512)$.

Note the two extra padding bits that are appended to the message. Note that in each case, the capacity c is equal to twice the output length v. Thus, as the output length grows, the security provided by the capacity grows as well, and the rate — and, therefore, the hashing speed — decreases.

Here are the two variable-length output hash functions:

- $\text{SHAKE128}(m, v) = \text{Keccak}[256](m \parallel 1111, v)$;
- $\text{SHAKE256}(m, v) = \text{Keccak}[512](m \parallel 1111, v)$.

Note the four extra padding bits that are appended to the message. The only difference between these two is the capacity size, which affects the speed and security. The various padding bits and the 10*1 padding rule ensure that these six functions behave independently.

8.9 Merkle trees: using collision resistance to prove database membership

To be written.

8.10 Key derivation and the random oracle model

Although hash functions like SHA256 were initially designed to provide collision resistance, we have already seen in Section 8.7 that practitioners are often tempted to use them to solve other problems. Intuitively, hash functions like SHA256 are designed to "thoroughly scramble" their inputs, and so this approach seems to make some sense. Indeed, in Section 8.7, we looked at the problem of taking an unkeyed hash function and turning it into a keyed function that is a secure PRF, and found that it was indeed possible to give a security analysis under reasonable assumptions.

In this section, we study another problem, called **key derivation**. Roughly speaking, the problem is this: we start with some secret data, and we want to convert it into an n-bit string that we can use as the key to some cryptographic primitive, like AES. Now, the secret data may be random in some sense — at the very least, somewhat hard to guess — but it may not look anything at all like a uniformly distributed, random, n-bit string. So how do we get from such a secret s to a cryptographic key t? Hashing, of course. In practice, one takes a hash function H, such as SHA256 (or, as we will ultimately recommend, some function built out of SHA256), and computes $t \leftarrow H(s)$.

Along the way, we will also introduce the *random oracle model*, which is a heuristic tool that is useful not only for analyzing the key derivation problem, but a host of other problems as well.

8.10.1 The key derivation problem

Let us look at the key derivation problem in more detail. Again, at a high level, the problem is to convert some discreet data that is hard to guess into an n-bit string we can use directly as a key to some standard cryptographic primitive, such as AES. The solution in all cases will be to hash the secret to obtain the key. We begin with some motivating examples.

- The secret might be a password. While such a password might be somewhat hard to guess, it could be dangerous to use such a password directly as an AES key. Even if the password were uniformly distributed over a large dictionary (already a suspect assumption), the distribution of its encoding as a bit string is certainly not. It could very well that a significant fraction of passwords correspond to "weak keys" for AES that make it vulnerable to attack. Recall that AES was designed to be used with a random bit string as the key, so how it behaves on passwords is another matter entirely.

- The secret could be the log of various types of system events on a running computer (e.g., the time of various interrupts such as those caused by key presses or mouse movements). Again, it might be difficult for an attacker who is outside the computer system to accurately predict the contents of such a log. However, using the log directly as an AES key is problematic: it is likely far too long, and far from uniformly distributed.

- The secret could be a cryptographic key which as been partially compromised. Imagine that a user has a 128-bit key, but that 64 of the bits have been leaked to the adversary. The key is still fairly difficult to guess, but it is still not uniformly distributed from the adversary's point of view, and so should not be used directly as an AES key.

- Later, we will see examples of number-theoretic transformations that are widely used in public-key cryptography. Looking ahead a bit, we will see that for a large, composite modulus N, if x is chosen at random modulo N, and an adversary is given $y := x^3 \bmod N$, it is hard to compute x. We can view x as the secret, and similarly to the previous example, we can view y as information that is leaked to the adversary. Even though the value of y completely determines x in an information-theoretic sense, it is still widely believed to be hard to compute. Therefore, we might want to treat x as secret data in exactly the same way as in the previous examples. Many of the same issues arise here, not the least of which is that x is typically much longer (typically, thousands of bits long) than an AES key.

As already mentioned, the solution that is adopted in practice is simply to hash the secret s using a hash function H to obtain the key $t \leftarrow H(s)$.

Let us now give a formal definition of the security property we are after.

We assume the secret s is sampled according to some fixed (and publicly known) probability distribution P. We assume any such secret data can be encoded as an element of some finite set $\mathcal{S}$. Further, we model the fact that some partial information about s could be leaked by introducing a function I, so that an adversary trying to guess s knows the side information $I(s)$.

Attack Game 8.2 (Guessing advantage). Let P be a probability distribution defined on a finite set $\mathcal{S}$ and let I be a function defined in $\mathcal{S}$. For a given adversary $\mathcal{A}$, the attack game runs as follows:

- the challenger chooses s at random according to P and sends $I(s)$ to $\mathcal{A}$;

- the adversary outputs a guess $\hat{s}$ for s, and wins the game if $\hat{s} = s$.

The probability that $\mathcal{A}$ wins this game is called its **guessing advantage**, and is denoted $\mathrm{Guess}\mathsf{adv}[\mathcal{A}, P, I]$. $\Box$

In the first example above, we might simplistically model s as being a password that is uniformly distributed over (the encodings of) some dictionary D of words. In this case, there is no side information given to the adversary, and the guessing advantage is $1/|D|$, regardless of the computational power of the adversary.

In the second example above, it seems very hard to give a meaningful and reliable estimate of the guessing advantage.

In the third example above, s is uniformly distributed over $\{0,1\}^{128}$, and $I(s)$ is (say) the first 64-bits of s. Clearly, any adversary, no matter how powerful, has guessing advantage no greater than 2^{-64}.

In the fourth example above, s is the number x and $I(s)$ is the number y. Since y completely determines x, it is possible to recover s from $I(s)$ by brute-force search. There are smarter and faster algorithms as well, but there is no known efficient algorithm to do this. So for all *efficient* adversaries, the guessing advantage appears to be *negligible*.

Now suppose we use a hash function $H : \mathcal{S} \to \mathcal{T}$ to derive the key t from s. Intuitively, we want t to "look random". To formalize this intuitive notion, we use the concept of computational indistinguishability from Section 3.11. So formally, the property that we want is that if s is sampled according to P and t is chosen at random from $\mathcal{T}$, the two distributions $(I(s), H(s))$ and $(I(s), t)$ are computationally indistinguishable. For an adversary $\mathcal{A}$, let $\text{Dist}\mathsf{adv}[\mathcal{A}, P, I, H]$ be the adversary's advantage in Attack Game 3.3 for these two distributions.

The type of theorem we would like to be able to prove would say, roughly speaking, if H satisfies some specific property, and perhaps some constraints are placed on P and I, then $\text{Dist}\mathsf{adv}[\mathcal{A}, P, I, H]$ is not too much larger than $\text{Guess}\mathsf{adv}[\mathcal{A}, P, I]$. In fact, in certain situations it *is* possible prove such a theorem. We will discuss this result later, in Section 8.10.4 — for now, we will simply say that this rigorous approach is not widely used in practice, for a number of reasons. Instead, we will examine in greater detail the heuristic approach of using an "off the shelf" hash function like SHA256 to derive keys.

Sub-key derivation. Before moving on, we consider the following, related problem: what to do with the key t derived from s. In some applications, we might use t directly as, say, and AES key. In other applications, however, we might need several keys: for example, an encryption key and a MAC key, or two different encryption keys for bi-directional secure communications (so Alice has one key for sending encrypting messages to Bob, and Bob uses a different key for sending encrypted messages to Alice). So once we have derived a single key t that "for all intents and purposes" behaves like a random bit string, we wish to derive several sub-keys. We call this the **sub-key derivation problem** to distinguish it from the key derivation problem. For the sub-key derivation problem, we assume that we start with a truly random key t — it is not, but when t is computationally indistinguishable from a truly random key, this assumption is justified.

Fortunately, for sub-key derivation, we already have all the tools we need at our disposal. Indeed, we can derive sub-keys from t using either a PRG or a PRF. For example, in the above example, if Alice and Bob have a shared key t, derived from a secret s, they can use a PRF F as follows:

- derive a MAC key $k_{\text{mac}} \xleftarrow{\text{R}} F(t, \texttt{"MAC-KEY"})$;
- derive an Alice-to-Bob encryption key $k_{\text{AB}} \xleftarrow{\text{R}} F(t, \texttt{"AB-KEY"})$;
- derive a Bob-to-Alice encryption key $k_{\text{BA}} \xleftarrow{\text{R}} F(t, \texttt{"BA-KEY"})$.

Assuming F is a secure PRF, then the keys k_{mac}, k_{AB}, and k_{BA} behave, for all intents and purposes, as independent random keys. To implement F, we can even use a hash-based PRF, like HMAC, so we can do everything we need — key derivation and sub-key derivation — using a single "off the shelf" hash function like SHA256.

So once we have solved the key derivation problem, we can use well-established tools to solve the sub-key derivation problem. Unfortunately, the practice of using "off the shelf" hash functions for key derivation is not very well understood or analyzed. Nevertheless, there are some useful heuristic models to explore.

8.10.2 Random oracles: a useful heuristic

We now introduce a heuristic that we can use to model the use of hash functions in a variety of applications, including key derivation. As we will see later in the text, this has become a popular heuristic that is used to justify numerous cryptographic constructions.

The idea is that we simply model a hash function H *as if* it were a truly random function $\mathcal{O}$. If H maps $\mathcal{M}$ to $\mathcal{T}$, then $\mathcal{O}$ is chosen uniformly at random from the set $\text{Funs}[\mathcal{M}, \mathcal{T}]$. We can translate any attack game into its random oracle version: the challenger uses $\mathcal{O}$ in place of H for all its computations, and in addition, the adversary is allowed to obtain the value of $\mathcal{O}$ at arbitrary input points of his choosing. The function $\mathcal{O}$ is called a **random oracle** and security in this setting is said to hold in the **random oracle model**. The function $\mathcal{O}$ is too large to write down and cannot be used in a real construction. Instead, we only use $\mathcal{O}$ as a means for carrying out a heuristic security analysis of the proposed system that actually uses H.

This approach to analyzing constructions using hash function is analogous to the *ideal cipher model* introduced in Section 4.7, where we replace a block cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{X})$ by a family of random permutations $\{\Pi_k\}_{k \in \mathcal{K}}$.

As we said, the random oracle model is used quite a bit in modern cryptography, and it would be nice to be able to use an "off the shelf" hash function H, and model it as a random oracle. However, if we want a truly general purpose tool, we have to be a bit careful, especially if we want to model H as a random oracle taking *variable length inputs*. The basic rule of thumb is that Merkle-Damgård hashes should not be used *directly* as general purpose random oracles. We will discuss in Section 8.10.3 how to safely (but again, heuristically) use Merkle-Damgård hashes as general purpose random oracles, and we will also see that the sponge construction (see Section 8.8) can be used directly "as is".

We stress that even though security results in the random oracle are rigorous, mathematical theorems, they are still only heuristic results that do not guarantee any security for systems built with any specific hash function. They do, however, rule out "generic attacks" on systems that *would* work if the hash function *were* a random oracle. So, while such results do not rule out all attacks, they do rule out generic attacks, which is better than saying nothing at all about the security of the system. Indeed, in the real world, given a choice between two systems, $\mathcal{S}_1$ and $\mathcal{S}_2$, where $\mathcal{S}_1$ comes with a security proof in the random oracle model, and $\mathcal{S}_2$ comes with a real security proof but is twice as slow as $\mathcal{S}_1$, most practitioners would (quite reasonably) choose $\mathcal{S}_1$ over $\mathcal{S}_2$.

Defining security in the random oracle model. Suppose we have some type of cryptographic scheme $\mathcal{S}$ whose implementation makes use of a subroutine for computing a hash function H defined over $(\mathcal{M}, \mathcal{T})$. The scheme $\mathcal{S}$ evaluates H at arbitrary points of its choice, but does not look at the internal implementation of H. We say that $\mathcal{S}$ **uses** H **as an oracle**. For example, $F_{\text{pre}}(k, x) := H(k \parallel x)$, which we briefly considered in Section 8.7, is a PRF that uses the hash function H as an oracle.

We wish to analyze the security of $\mathcal{S}$. Let us assume that whatever security property we are interested in, say "property X," is modeled (as usual) as a game between a challenger (specific to property X) and an arbitrary adversary $\mathcal{A}$. Presumably, in responding to certain queries, the challenger computes various functions associated with the scheme $\mathcal{S}$, and these functions may in turn require the evaluation of H at certain points. This game defines an advantage $\text{X}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$, and security with respect to property X means that this advantage should be negligible for all efficient adversaries $\mathcal{A}$.

If we wish to analyze $\mathcal{S}$ in the random oracle model, then the attack game defining security is modified so that H is effectively replaced by a *random function* $\mathcal{O} \in \text{Funs}[\mathcal{M}, \mathcal{T}]$, to which both the adversary and the challenger have oracle access. More precisely, the game is modified as follows.

- At the beginning of the game, the challenger chooses $\mathcal{O} \in \text{Funs}[\mathcal{M}, \mathcal{T}]$ at random.
- In addition to its standard queries, the adversary $\mathcal{A}$ may submit *random oracle queries*: it gives $m \in \mathcal{M}$ to the challenger, who responds with $t = \mathcal{O}(m)$. The adversary may make any number of random oracle queries, arbitrarily interleaved with standard queries.
- In processing standard queries, the challenger performs its computations using $\mathcal{O}$ in place of H.

The adversary's advantage is defined using the same rule as before, but is denoted $\text{X}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$ to emphasize that this is an advantage *in the random oracle model*. Security *in the random oracle model* means that $\text{X}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$ should be negligible for all efficient adversaries $\mathcal{A}$.

A simple example: PRFs in the random oracle model. We illustrate how to apply the random oracle framework to construct secure PRFs. In particular, we will show that F_{pre} is a secure PRF in the random oracle model. We first adapt the standard PRF security game to obtain a PRF security game in the random oracle model. To make things a bit clearer, if we have a PRF F that uses a hash function H as an oracle, we denote by $F^{\mathcal{O}}$ the function that uses the random oracle $\mathcal{O}$ in place of H.

Attack Game 8.3 (PRF in the random oracle model). Let F be a PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ that uses a hash function H defined over $(\mathcal{M}, \mathcal{T})$ as an oracle. For a given adversary $\mathcal{A}$, we define two experiments, Experiment 0 and Experiment 1. For $b = 0, 1$, we define:

Experiment b:

- $\mathcal{O} \xleftarrow{\text{R}} \text{Funs}[\mathcal{M}, \mathcal{T}]$.
- The challenger selects $f \in \text{Funs}[\mathcal{X}, \mathcal{Y}]$ as follows:

  if $b = 0$: $k \xleftarrow{\text{R}} \mathcal{K}$, $f \leftarrow F^{\mathcal{O}}(k, \cdot)$;
  if $b = 1$: $f \xleftarrow{\text{R}} \text{Funs}[\mathcal{X}, \mathcal{Y}]$.

- The adversary submits a sequence of queries to the challenger.
  - F-query: respond to a query $x \in \mathcal{X}$ with $y = f(x) \in \mathcal{Y}$.
  - $\mathcal{O}$-query: respond to a query $m \in \mathcal{M}$ with $t = \mathcal{O}(m) \in \mathcal{T}$.
- The adversary computes and outputs a bit $\hat{b} \in \{0, 1\}$.

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to F as

$$\mathrm{PRF^{ro}}\mathsf{adv}[\mathcal{A}, F] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \square$$

Definition 8.3. *We say that a PRF F is secure in the random oracle model if for all efficient adversaries $\mathcal{A}$, the value $\mathrm{PRF^{ro}}\mathsf{adv}[\mathcal{A}, F]$ is negligible.*

Consider again the PRF $F_{\mathrm{pre}}(k, x) := H(k \parallel x)$. Let us assume that F_{pre} is defined over $(\mathcal{K}, \mathcal{X}, \mathcal{T})$, where $\mathcal{K} = \{0,1\}^\kappa$ and $\mathcal{X} = \{0,1\}^{\leq L}$, and that H is defined over $(\mathcal{M}, \mathcal{T})$, where M includes all bit strings of length at most $\kappa + L$.

We will show that this is a secure PRF in the random oracle model. But wait! We already argued in Section 8.7 that F_{pre} is completely insecure when H is a Merkle-Damgård hash. This seems to be a contradiction. The problem is that, as already mentioned, it is not safe to use a Merkle-Damgård hash directly as a random oracle. We will see how to fix this problem in Section 8.10.3.

Theorem 8.7. *If $\mathcal{K}$ is large then F_{pre} is a secure PRF when H is modeled as a random oracle.*

In particular, if $\mathcal{A}$ is a random oracle PRF adversary, as in Attack Game 8.3, that makes at most Q_{ro} oracle queries, then

$$\mathrm{PRF^{ro}}\mathsf{adv}[\mathcal{A}, F_{\mathrm{pre}}] \leq Q_{\mathrm{ro}}/|\mathcal{K}|$$

Note that Theorem 8.7 is unconditional, in the sense that the only constraint on $\mathcal{A}$ is on the number of oracle queries: it does not depend on any complexity assumptions.

Proof idea. Once H is replaced with $\mathcal{O}$, the adversary has to distinguish $\mathcal{O}(k \parallel \cdot)$ from a random function in $\mathrm{Funs}[\mathcal{X}, \mathcal{T}]$, without the key k. Since $\mathcal{O}(k \parallel \cdot)$ is a random function in $\mathrm{Funs}[\mathcal{X}, \mathcal{T}]$, the only hope the adversary has is to somehow use the information returned from queries to $\mathcal{O}$. We say that an $\mathcal{O}$-query $k' \parallel x'$ is relevant if $k' = k$. It should be clear that queries to $\mathcal{O}$ that are not relevant cannot help distinguish $\mathcal{O}(k \parallel \cdot)$ from random since the returned values are independent of the function $\mathcal{O}(k \parallel \cdot)$. Moreover, the probability that after Q_{ro} queries the adversary succeeds in issuing a relevant query is at most $Q_{\mathrm{ro}}/|\mathcal{K}|$. $\square$

Proof. To make this proof idea rigorous we let $\mathcal{A}$ interact with two PRF challengers. For $j = 0, 1$, let W_j to be the event that $\mathcal{A}$ outputs 1 in Game j.

Game 0. We write the challenger in Game 0 so that it is equivalent to Experiment 0 of Attack Game 8.3, but will be more convenient for us to analyze. We assume the adversary never makes the same F_{pre}-query twice. Also, we use an associative array $Map : \mathcal{M} \to \mathcal{T}$ to build up the random oracle on the fly, using the "faithful gnome" idea we have used so often. Here is our challenger:

Initialization:
  initialize the empty associative array $Map : \mathcal{M} \to \mathcal{T}$
  $k \xleftarrow{R} \mathcal{K}$
Upon receiving an F_{pre}-query on $x \in \{0,1\}^{\leq L}$ do:
  $t \xleftarrow{R} \mathcal{T}$
(1) if $(k \parallel x) \in \text{Domain}(Map)$ then $t \leftarrow Map[k \parallel x]$
(2) $Map[k \parallel x] \leftarrow t$
  send t to $\mathcal{A}$
Upon receiving an $\mathcal{O}$-query $m \in \mathcal{M}$ do:
  $t \xleftarrow{R} \mathcal{T}$
  if $m \in \text{Domain}(Map)$ then $t \leftarrow Map[m]$
  $Map[m] \leftarrow t$
  send t to $\mathcal{A}$

It should be clear that this challenger is equivalent to that in Experiment 0 of Attack Game 8.3. In Game 0, whenever the challenger needs to sample the random oracle at some input (in processing either an F_{pre}-query or an $\mathcal{O}$-query), it generates a random "default output", overriding that default if it turns out the oracle has already been sampled at that input; in either case, the associative array records the input/output pair.

Game 1. We make our gnome "forgetful": we modify Game 0 by deleting the lines marked (1) and (2) in that game. Observe now that in Game 1, the challenger does not use Map or k in responding to F_{pre}-queries: it just returns a random value. So it is clear (by the assumption that $\mathcal{A}$ never makes the same F_{pre}-query twice) that Game 1 is equivalent to Experiment 1 of Attack Game 8.3, and hence

$$\text{PRF}^{\text{ro}}\mathsf{adv}[\mathcal{A}, F_{\text{pre}}] = |\Pr[W_1] - \Pr[W_0]|.$$

Let Z be the event that *in Game 1*, the adversary makes an $\mathcal{O}$-query at a point $m = (k \parallel \hat{x})$. It is clear that both games result in the same outcome unless Z occurs, so by the by Difference Lemma, we have

$$|\Pr[W_1] - \Pr[W_0]| \leq \Pr[Z].$$

Since the key k is completely independent of $\mathcal{A}$'s view in Game 1, each $\mathcal{O}$-query hits the key with probability $1/|\mathcal{K}|$, and so a simple application of the union bound yields

$$\Pr[Z] \leq Q_{\text{ro}}/|\mathcal{K}|.$$

That completes the proof. $\square$

Key derivation in the random oracle model. Let us now return to the key derivation problem introduced in Section 8.10.1. Again, we have a secret s sampled from some distribution P, and information $I(s)$ is leaked to the adversary. We want to argue that if H is modeled as a random oracle, then the adversary's advantage in distinguishing $(I(s), H(s))$ from $(I(s), t)$, where t is truly random, is not too much more than the adversary's advantage in guessing the secret s with only $I(s)$ (and not $H(s)$).

To model H as a random oracle $\mathcal{O}$, we convert the computational indistinguishability Attack Game 3.3 to the random oracle model, so that the attacker is now trying to distinguish

$(I(s), \mathcal{O}(s))$ from $(I(s), t)$, given oracle access to $\mathcal{O}$. The corresponding advantage is denoted $\text{Dist}^{\text{ro}}\mathsf{adv}[\mathcal{A}, P, I, H]$.

Before stating our security theorem, it is convenient to generalize Attack Game 8.2 to allow the adversary to output a list of guesses $\hat{s}_1, \ldots, \hat{s}_Q$, where and the adversary is said to win the game if $\hat{s}_i = s$ for some $i = 1, \ldots, Q$. An adversary $\mathcal{A}$'s probability of winning in this game is called his **list guessing advantage**, denoted $\text{ListGuess}\mathsf{adv}[\mathcal{A}, P, I]$.

Clearly, if an adversary $\mathcal{A}$ can win the above list guessing game with probability ϵ, we can convert him into an adversary that wins the singleton guessing game with probability ϵ/Q: we simply run $\mathcal{A}$ to obtain a list $\hat{s}_1, \ldots, \hat{s}_Q$, choose $i = 1, \ldots, Q$ at random, and output $\hat{s}_i$. However, sometimes we can do better than this: using the partial information $I(s)$ may allow us to rule out some of the $\hat{s}_i$'s, and in some situations, we may be able to identify the correct $\hat{s}_i$ uniquely. This depends on the application.

Theorem 8.8. *If H is modeled as a random oracle, then for every distinguishing adversary $\mathcal{A}$ that makes at most Q_{ro} random oracle queries, there exists a list guessing adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that*

$$\text{Dist}^{\text{ro}}\mathsf{adv}[\mathcal{A}, P, I, H] \leq \text{ListGuess}\mathsf{adv}[\mathcal{B}, P, I]$$

and $\mathcal{B}$ outputs a list of size at most Q_{ro}. In particular, there exists a guessing adversary $\mathcal{B}'$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\text{Dist}^{\text{ro}}\mathsf{adv}[\mathcal{A}, P, I, H] \leq Q_{\text{ro}} \cdot \text{Guess}\mathsf{adv}[\mathcal{B}', P, I].$$

Proof. The proof is almost identical to that of Theorem 8.7. We define two games, and for $j = 0, 1$, let W_j to be the event that $\mathcal{A}$ outputs 1 in Game j.

Game 0. We write the challenger in Game 0 so that it is equivalent to Experiment 0 of the $(I(s), H(s))$ vs $(H(s), t)$ distinguishing game. We build up the random oracle on the fly with an associative array $Map : \mathcal{S} \to \mathcal{T}$. Here is our challenger:

Initialization:
  initialize the empty associative array $Map : \mathcal{S} \to \mathcal{T}$
  generate s according to P
  $t \xleftarrow{\text{R}} \mathcal{T}$
(∗) $Map[s] \leftarrow t$
  send $(I(s), t)$ to $\mathcal{A}$
Upon receiving an $\mathcal{O}$-query $\hat{s} \in \mathcal{S}$ do:
  $\hat{t} \xleftarrow{\text{R}} \mathcal{T}$
  if $\hat{s} \in \text{Domain}(Map)$ then $\hat{t} \leftarrow Map[\hat{s}]$
  $Map[\hat{s}] \leftarrow \hat{t}$
  send $\hat{t}$ to $\mathcal{A}$

Game 1. We delete the line marked (∗). This game is equivalent to Experiment 1 of this distinguishing game, as the value t is now truly independent of the random oracle. Moreover, both games result in the same outcome unless the adversary $\mathcal{A}$ in Game 1 makes an $\mathcal{O}$-query at the point s. So our list guessing adversary $\mathcal{B}$ simply takes the value $I(s)$ that it receives from its own challenger, and plays the role of challenger to $\mathcal{A}$ as in Game 1. At the end of the game, $\mathcal{B}$ simply outputs $\text{Domain}(Map)$ — the list of points at which $\mathcal{A}$ made $\mathcal{O}$-queries. The essential points are:

our $\mathcal{B}$ can play this role with no knowledge of s besides $I(s)$, and it records all of the $\mathcal{O}$-queries made by $\mathcal{A}$. So by the Difference Lemma, we have

$$\mathrm{Dist}^{\mathrm{ro}}\mathsf{adv}[\mathcal{A}] = |\Pr[W_0] - \Pr[W_1]| \leq \mathrm{ListGuess}\mathsf{adv}[\mathcal{B}]. \quad \Box$$

8.10.3 Random oracles: safe modes of operation

We have already seen that $F_{\mathrm{pre}}(k, x) := H(k \parallel x)$ is secure in the random oracle model, and yet we know that it is completely insecure if H is a Merkle-Damgård hash. The problem is that a Merkle-Damgård construction has a very simple, iterative structure which exposes it to "extension attacks". While this structure is not a problem from the point of view of collision resistance, it shows that grabbing a hash function "off the shelf" and using it as if it were a random oracle is a dangerous move.

In this section, we discuss how to safely use a Merkle-Damgård hash as a random oracle. We will also see that the sponge construction (see Section 8.8) is already safe to use "as is"; in fact, the sponge was designed exactly for this purpose: to provide a variable-length input and variable-length output hash function that could be used directly as a random oracle.

Suppose H is a Merkle-Damgård hash built from a compression function $h : \{0,1\}^n \times \{0,1\}^\ell \to \{0,1\}^n$. One recommended mode of operation is to safe HMAC with a zero key:

$$\mathrm{HMAC}_0(m) := \mathrm{HMAC}(0^\ell, m) = H(\mathsf{opad} \parallel H(\mathsf{ipad} \parallel m)).$$

While this construction foils the obvious extension attacks, why should we have any confidence at all that HMAC_0 is safe to use as a general purpose random oracle? We can only give heuristic evidence. Essentially, what we want to argue is that there are no inherent structural weaknesses in HMAC_0 that give rise to a generic attack that treats the underlying compression function itself as a random oracle — or perhaps, more realistically, as a Davies-Meyer construction based on an ideal cipher.

So basically, we want to show that using certain modes of operation, we can build a "big" random oracle out of a "small" random oracle — or out of an ideal cipher or even ideal permutation. This is undoubtedly a rather quixotic task — using heuristics to justify heuristics — but we shall sketch the basic ideas.

The mathematical tool used to carry out such a task is called **indifferentiability**. We shall present a somewhat simplified version of this notion here. Suppose we are trying to build a "big" random oracle $\mathcal{O}$ out of a smaller primitive ρ, where ρ could be a random oracle on a small domain, or an ideal cipher, or an ideal permutation. Let us denote by $F[\rho]$ a particular construction for a random oracle based on the ideal primitive ρ.

Now consider a generic attack game defined by some challenger $\mathbf{C}$ and adversary $\mathcal{A}$. Let us write the interaction between $\mathbf{C}$ and $\mathcal{A}$ as $\langle \mathbf{C}, \mathcal{A} \rangle$. We assume that the interaction results in an output bit. All of our security definitions are modeled in terms of games of this form.

In the random oracle version of the attack game, with the big random oracle $\mathcal{O}$, we would give both the challenger and adversary oracle access to the random function $\mathcal{O}$, and we denote the interaction $\langle \mathbf{C}^{\mathcal{O}}, \mathcal{A}^{\mathcal{O}} \rangle$. However, if we are using the construction $F[\rho]$ to implement the big random oracle, then while the challenger accesses ρ only via the construction F, the adversary is allowed to directly query ρ. We denote this interaction as $\langle \mathbf{C}^{F[\rho]}, \mathcal{A}^{\rho} \rangle$.

For example, in the HMAC_0 construction, the compression function h is modeled as a random oracle ρ, or if h itself is built via Davies-Meyer, then the underlying block cipher is modeled as an ideal cipher ρ. In either case, $F[\rho]$ corresponds to the HMAC_0 construction itself. Note the asymmetry: in any attack game, the challenger only accesses ρ indirectly via $F[\rho]$ (HMAC_0 in this case), while the adversary can access ρ itself (the compression function h or the underlying block cipher).

We say that $F[\rho]$ is **indifferentiable** from $\mathcal{O}$ if the following holds:

> *for every efficient challenger* $\mathbf{C}$ *and efficient adversary* $\mathcal{A}$*, there exists an efficient adversary* $\mathcal{B}$*, which is an elementary wrapper around* $\mathcal{A}$*, such that*
>
> $$\left|\Pr[\langle \mathbf{C}^{F[\rho]}, \mathcal{A}^{\rho}\rangle \text{ outputs } 1] - \Pr[\langle \mathbf{C}^{\mathcal{O}}, \mathcal{B}^{\mathcal{O}}\rangle \text{ outputs } 1]\right|$$
>
> *is negligible.*

It should be clear from the definition that if we prove security of any cryptographic scheme in the random oracle model for the big random oracle $\mathcal{O}$, the scheme remains secure if we implement $\mathcal{O}$ using $F[\rho]$: if an adversary $\mathcal{A}$ could break the scheme with $F[\rho]$, then the adversary $\mathcal{B}$ above would break the scheme with $\mathcal{O}$.

Some safe modes. The HMAC_0 construction can be proven to be indifferentiable from a random oracle on variable length inputs, if we either model the compression function h itself as a random oracle, or if h is built via Davies-Meyer and we model the underlying block cipher as an ideal cipher.

One problem with using HMAC_0 as a random oracle is that its *output* is fairly short. Fortunately, it is fairly easy to use HMAC_0 to get a random oracle with longer outputs. Here is how. Suppose HMAC_0 has an n-bit output, and we need a random oracle with, say, $N > n$ bits of output. Set $q := \lceil N/n \rceil$. Let $e_0, e_1, \ldots, e_q$ be fixed-length encodings of the integers $0, 1, \ldots, q$. Our new hash function H' works as follows. On input m, we compute $t \leftarrow \mathrm{HMAC}_0(e_0 \parallel m)$. Then, for $i = 1, \ldots, q$, we compute $t_i \leftarrow \mathrm{HMAC}_0(e_i \parallel t)$. Finally, we output the first N bits of $t_1 \parallel t_2 \parallel \cdots \parallel t_q$. One can show that H' is indifferentiable from a random oracle with N-bit outputs. This result holds if we replace HMAC_0 with any hash function that is itself indifferentiable from a random oracle with n-bit outputs. Also note that when applied to long inputs, H' is quite efficient: it only needs to evaluate HMAC_0 once on a long input.

The sponge construction has been proven to be indifferentiable from a random oracle on variable length inputs, if we model the underlying permutation as an ideal permutation (assuming 2^c, where c is the capacity is super-poly.) This includes the standardized implementations SHA3 (for fixed length outputs) and the SHAKE variants (for variable length outputs), discussed in Section 8.8.2. The special padding rules used in the SHA3 and SHAKE specifications ensure that all of the variants act as independent random oracles.

Sometimes, we need random oracles whose output should be uniformly distributed over some specialized set. For example, we may want the output to be uniformly distributed over the set $S = \{0, \ldots, d-1\}$ for some positive integer d. To realize this, we can use a hash function H with an n-bit output, which we can view as an n-bit binary encoding of a number, and define $H'(m) := H(m) \bmod d$. If H is indifferentiable from a random oracle with n-bit outputs, and $2^n/d$ is super-poly, then the hash function H' is indifferentiable from a random oracle with outputs in S.

8.10.4 The leftover hash lemma

We now return to the key derivation problem. Under the right circumstances, we can solve the key derivation problem with no heuristics and no computational assumptions whatsoever. Moreover, the solution is a surprising and elegant application of universal hash functions (see Section 7.1). The result, known as the **leftover hash lemma**, says that if we use an ϵ-UHF to hash a secret that can be guessed with probability at most γ, then provided ϵ and γ are sufficiently small, the output of the hash is statistically indistinguishable from a truly random value. Recall that a UHF has a key, which we normally think of as a secret key; however, in this result, the key may be made public — indeed, it could be viewed as a public, system parameter that is generated once and for all, and used over and over again.

Our goal here is to simply state the result, and to indicate when and where it can (and cannot) be used. To state the result, we will need to use the notion of the statistical distance between two random variables, which we introduced in Section 3.11. Also, if $\mathbf{s}$ is a random variable taking values in a set $\mathcal{S}$, we define the **guessing probability of s** to be $\max_{x\in\mathcal{S}} \Pr[\mathbf{s} = x]$.

Theorem 8.9 (Leftover Hash Lemma). *Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{S}, \mathcal{T})$. Assume that H is a $(1 + \alpha)/N$-UHF, where $N := |\mathcal{T}|$. Let $\mathbf{k}, \mathbf{s}_1, \ldots, \mathbf{s}_m$ be mutually independent random variables, where $\mathbf{k}$ is uniformly distributed over $\mathcal{K}$, and each $\mathbf{s}_i$ has guessing probability at most γ. Let δ be the statistical difference between*

$$(\mathbf{k}, H(\mathbf{k}, \mathbf{s}_1), \ldots, H(\mathbf{k}, \mathbf{s}_m))$$

and the uniform distribution on $\mathcal{K} \times \mathcal{T}^m$. Then we have

$$\delta \leq \frac{1}{2} m \sqrt{N\gamma + \alpha}.$$

Let us look at what the lemma says when $m = 1$. We have a secret s that can be guessed with probability at most γ, given whatever side information $I(s)$ is known about s. To apply the lemma, the bound γ on the guessing probability must hold for all adversaries, even computationally unbounded ones. We then hash s using a random hash key k. It is essential that s (given $I(s)$) and k are independent — although we have not discussed the possibility here, there are potential use cases where the distribution of s or the function I can be somehow biased by an adversary in a way that depends on k, which is assumed public and known to the adversary. Therefore, to apply the lemma, we must ensure that s (given $I(s)$) and k are truly independent. If all of these conditions are met, then the lemma says that for any adversary $\mathcal{A}$, even a computationally unbounded one, its advantage in distinguishing $(k, I(s), H(k, s))$ from $(k, I(s), t)$, where t is a truly random element of $\mathcal{T}$, is bounded by δ, as in the lemma.

Now let us plug in some realistic numbers. If we want the output to be used as an AES key, we need $N = 2^{128}$. We know how to build $(1/N)$-UHFs, so we can take $\alpha = 0$ (see Exercise 7.18 — with α non-zero, but still quite small, one can get by with significantly shorter hash keys). If we want $\delta \leq 2^{-64}$, we will need the guessing probability γ to be about 2^{-256}.

So in addition to all the conditions listed above, we really need an extremely small guessing probability for the lemma to be applicable. None of the examples discussed in Section 8.10.1 meet these requirements: the guessing probabilities are either not small enough, or do not hold unconditionally against unbounded adversaries, or can only be heuristically estimated. So the practical applicability to the Leftover Hash Lemma is limited — but when it does apply, it can be a very powerful tool. Also, we remark that by using the lemma with $m > 1$, under the right conditions, we can model the situation where the same hash key is used to derive many keys from many independent secrets with small guessing probability. The distinguishing probability grows linearly with the number of derivations, which is not surprising.

Because of these practical limitations, it is more typical to use cryptographic hash functions, modeled as random oracles, for key derivation, rather than UHFs. Indeed, if one uses a UHF and any of the assumptions discussed above turns out to be wrong, this could easily lead to a catastrophic security breach. Using cryptographic hash functions, while only heuristically secure for key derivation, are also more forgiving.

8.10.5 Case study: HKDF

HKDF is a key derivation function specified in RFC 5869, and is deployed in many standards.

HKDF is specified in terms of the HMAC construction (see Section 8.7). So it uses the function $\text{HMAC}(k, m)$, where k and m are variable length byte strings, which itself is implemented in terms of a Merkle-Damgård hash H, such as SHA256.

The input to HKDF consists of a secret s, an optional salt value *salt* (discussed below), an optional *info* field (also discussed below), and an output length parameter L. The parameters s, *salt*, and *info* are variable length byte strings.

The execution of HKDF consists of two stages, called *extract* (which corresponds to what we called key derivation), and *expand* (which corresponds to what we called sub-key derivation).

In the extract stage, HKDF uses *salt* and s to compute

$$t \leftarrow \text{HMAC}(\mathit{salt}, s).$$

Using the intermediate key t, along with *info*, the expand (or sub-key derivation) stage computes L bytes of output data, as follows:

> $q \leftarrow \lceil L/\mathit{HashLen} \rceil$ // *HashLen is the output length (in bytes) of* H
> initialize z_0 to the empty string
> for $i \leftarrow 1$ to q do:
> $z_i \leftarrow \text{HMAC}(t, z_{i-1} \parallel \mathit{info} \parallel \mathit{Octet}(i))$ // $\mathit{Octet}(i)$ *is a single byte whose value is* i
> output the first L octets of $z_1 \parallel \ldots \parallel z_q$

When *salt* is empty, the extract stage of HKDF is the same as what we called HMAC_0 in Section 8.10.3. As discussed there, HMAC_0 can heuristically be viewed as a random oracle, and so we can use the analysis in Section 8.10.2 to show that this is a secure key derivation procedure in the random oracle model. This, if s is hard to guess, then t is indistinguishable from random.

Users of HKDF have the option of providing non-zero salt. The salt plays a role akin to the random hash key used in the Leftover Hash Lemma (see Section 8.10.4); in particular, it need not be secret, and may be reused. However, it is important that the salt value is independent of the secret s and cannot be manipulated by an adversary. The idea is that under these circumstances, the output of the extract stage of HKDF seems more likely to be indistinguishable from random, without relying on the full power of the random oracle model. Unfortunately, the known security proofs apply to limited settings, so in the general case, this is still somewhat heuristic.

The expand stage is just a simple application of HMAC as a PRF to derive sub-keys, as we discussed at the end of Section 8.10.1. The *info* parameter may be used to "name" the derived sub-keys, ensuring the independence of keys used for different purposes. Since the output length of the underlying hash is fixed, a simple iterative scheme is used to generate longer outputs. This stage can be analyzed rigorously under the assumption that the intermediate key t is indistinguishable from random, and that HMAC is a secure PRF — and we already know that HMAC is a secure PRF, under reasonable assumptions about the compression function of H.

8.11 Security without collision resistance

Theorem 8.1 shows how to extend the domain of a MAC using a collision resistant hash. It is natural to ask whether MAC domain extension is possible without relying on collision resistant functions. In this section we show that a weaker property called second preimage resistance is sufficient.

8.11.1 Second preimage resistance

We start by defining two classic security properties for non-keyed hash functions. Let H be a hash function defined over $(\mathcal{M}, \mathcal{T})$.

- We say that H is **one-way** if given $t := H(m)$ as input, for a random $m \in \mathcal{M}$, it is difficult to find an $m' \in \mathcal{M}$ such that $H(m') = t$. Such an m' is called an inverse of t. In other words, H is one-way if it is easy to compute but difficult to invert.

- We say that H is **2nd-preimage resistant** if given a random $m \in \mathcal{M}$ as input, it is difficult to find a different $m' \in \mathcal{M}$ such that $H(m) = H(m')$. In other words, it is difficult to find an m' that collides with a given m.

- For completeness, recall that a hash function is collision resistant if it is difficult to find two distinct messages $m, m' \in \mathcal{M}$ such that $H(m) = H(m')$.

Definition 8.4. *Let H be a hash function defined over $(\mathcal{M}, \mathcal{T})$. We define the advantage* $\mathrm{OW}\mathsf{adv}[\mathcal{A}, H]$ *of an adversary $\mathcal{A}$ in defeating the one-wayness of H as the probability of winning the following game:*

- *the challenger chooses $m \in \mathcal{M}$ at random and sends $t := H(m)$ to $\mathcal{A}$;*

- *the adversary $\mathcal{A}$ outputs $m' \in \mathcal{M}$, and wins if $H(m') = t$.*

*H is **one-way** if* $\mathrm{OW}\mathsf{adv}[\mathcal{A}, H]$ *is negligible for every efficient adversary $\mathcal{A}$.*

Similarly, we define the advantage $\mathrm{SPR}\mathsf{adv}[\mathcal{A}, H]$ *of an adversary $\mathcal{A}$ in defeating the 2nd-preimage resistance of H as the probability of winning the following game:*

- *the challenger chooses $m \in \mathcal{M}$ at random and sends m to $\mathcal{A}$;*

- *the adversary $\mathcal{A}$ outputs $m' \in \mathcal{M}$, and wins if $H(m') = H(m)$ and $m' \neq m$.*

*H is **2nd-preimage resistant** if* $\mathrm{SPR}\mathsf{adv}[\mathcal{A}, H]$ *is negligible for every efficient adversary $\mathcal{A}$.*

We mention some trivial relations between these notions when $\mathcal{M}$ is at least twice the size of $\mathcal{T}$. Under this condition we have the following implications:

$$H \text{ is collision resistant} \quad \Rightarrow \quad H \text{ is 2nd-preimage resistant} \quad \Rightarrow \quad H \text{ is one-way}$$

as shown in Exercise 8.22. The converse is not true. A hash function can be 2nd-preimage resistant, but not collision resistant. For example, SHA-1 is believed to be 2nd-preimage resistant even though SHA-1 is not collision resistant. Similarly, a hash function can be one-way, but not be 2nd-preimage resistant. For example, the function $h(x) := x^2 \bmod N$ for a large odd composite N is believed to be one-way. In other words, it is believed that given $x^2 \bmod N$ it is difficult to find x (as long as the factorization of N is unknown). However, this function H is trivially not 2nd-preimage resistant: given $x \in \{1, \ldots, N\}$ as input, the value $-x$ is a second preimage since $x^2 \bmod N = (-x)^2 \bmod N$.

Our goal for this section is to show that 2nd-preimage resistance is sufficient for extending the domain of a MAC and for providing file integrity. To give some intuition, consider the file integrity problem (which we discussed at the very beginning of this chapter). Our goal is to ensure that malware cannot modify a file without being detected. Recall that we hash all critical files on disk using a hash function H and store the resulting hashes in read-only memory. For a file F it should be difficult for the malware to find an F' such that $H(F') = H(F)$. Clearly, if H is collision resistant then finding such an F' is difficult. It would seem, however, that 2nd-preimage resistance of H is sufficient. To see why, consider malware trying to modify a specific file F without being detected. The malware is given F as input and must come up with a 2nd-preimage of F, namely an F' such that $H(F') = H(F)$. If H is 2nd-preimage resistant the malware cannot find such an F' and it would seem that 2nd-preimage resistance is sufficient for file integrity. Unfortunately, this argument doesn't quite work. Our definition of 2nd-preimage resistance says that finding a 2nd-preimage for a *random* F in $\mathcal{M}$ is difficult. But files on disk are not random bit strings — it may be difficult to find a 2nd-preimage for a random file, but it may be quite easy to find a 2nd-preimage for a specific file on disk.

The solution is to randomize the data before hashing it. To do so we first convert the hash function to a *keyed* hash function. We then require that the resulting keyed function satisfy a property called *target collision resistance* which we now define.

8.11.2 Randomized hash functions: target collision resistance

At the beginning of the chapter we mentioned two applications for collision resistance: extending the domain of a MAC and protecting file integrity. In this section we describe solutions to these problems that rely on a weaker security property than collision resistance. The resulting systems, although more likely to be secure, are not as efficient as the ones obtained from collision resistance.

Target collision resistance. Let H be a *keyed* hash function. We define what it means for H to be **target collision resistant**, or TCR for short, using the following attack game, also shown in Fig. 8.12.

Attack Game 8.4 (Target collision resistance). For a given keyed hash function H over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$ and adversary $\mathcal{A}$, the attack game runs as follows:

- $\mathcal{A}$ sends a message $m_0 \in \mathcal{M}$ to the challenger.
- The challenger picks a random $k \xleftarrow{R} \mathcal{K}$ and sends k to $\mathcal{A}$.
- $\mathcal{A}$ sends a second message $m_1 \in \mathcal{M}$ to the challenger.

The adversary is said to win the game if $m_0 \neq m_1$ and $H(k, m_0) = H(k, m_1)$. We define $\mathcal{A}$'s advantage with respect to H, denoted $\mathrm{TCR}\mathsf{adv}[\mathcal{A}, H]$, as the probability that $\mathcal{A}$ wins the game. □

Definition 8.5. *We say that a keyed hash function H over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$ is target collision resistant if* $\mathrm{TCR}\mathsf{adv}[\mathcal{A}, H]$ *is negligible.*

Casting the definition in our formal mathematical framework is done exactly as for universal hash functions (Section 7.1.2).

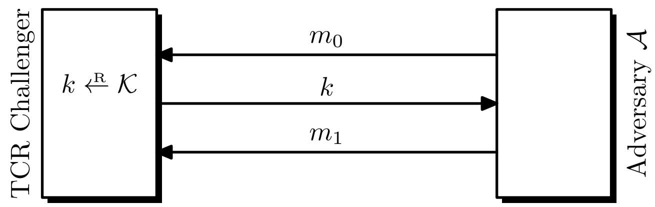


Figure 8.12: TCR Attack Game

We note that one can view a collision resistant hash H over $(\mathcal{M}, \mathcal{T})$ as a TCR function with an empty key. More precisely, let $\mathcal{K}$ be a set of size one containing only the empty word. We can define a keyed hash function H' over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$ as $H'(k, m) := H(m)$. It is not difficult to see that if H is collision resistant then H' is TCR. Thus, a collision resistant function can be viewed as the ultimate TCR hash — its key is the shortest possible.

8.11.3 TCR from 2nd-preimage resistance

We show how to build a keyed TCR hash function from a keyless 2nd-preimage resistant function such as SHA-1. Let H, defined over $(\mathcal{M}, \mathcal{T})$, be a 2nd-preimage resistant function. We construct a keyed TCR function H_{tcr} defined over $(\mathcal{M}, \mathcal{M}, \mathcal{T})$ as follows:

$$H_{\text{tcr}}(k, m) = H(k \oplus m) \tag{8.16}$$

Note that the length of the key k is equal to the length of the message being hashed. This is a problem for the applications we have in mind. As a result, we will only use this construction as a TCR hash for short messages. First we prove that the construction is secure.

Theorem 8.10. *Suppose H is 2nd-preimage resistant then H_{tcr} is TCR.*

In particular, for every TCR adversary $\mathcal{A}$ attacking H_{tcr} as in Attack Game 8.4, there exists a 2nd-preimage finder $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\text{TCR}\mathsf{adv}[\mathcal{A}, H_{tcr}] \leq \text{SPR}\mathsf{adv}[\mathcal{B}, H].$$

Proof. The proof is a simple direct reduction. Adversary $\mathcal{B}$ emulates the challenger in Attack Game 8.4 and works as follows:

Input: Random $m \in \mathcal{M}$
Output: $m' \in \mathcal{M}$ such that $m \neq m'$ and $H(m) = H(m')$

1. Run $\mathcal{A}$ and obtain an $m_0 \in \mathcal{M}$ from $\mathcal{A}$
2. $k \leftarrow m \oplus m_0$
3. Send k as the hash key to $\mathcal{A}$
4. $\mathcal{A}$ responds with an $m_1 \in \mathcal{M}$
5. Output $m' := m_1 \oplus k$

We show that $\text{SPR}\mathsf{adv}[\mathcal{B}, H] = \text{TCR}\mathsf{adv}[\mathcal{A}, H_{\text{tcr}}]$. First, denote by W the event that in step (4) the messages m_0, m_1 output by $\mathcal{A}$ are distinct and $H_{\text{tcr}}(k, m_0) = H_{\text{tcr}}(k, m_1)$.

The input m given to $\mathcal{B}$ is uniformly distributed in $\mathcal{M}$. Therefore, the key k given to $\mathcal{A}$ in step (2) is uniformly distributed in $\mathcal{M}$ and independent of $\mathcal{A}$'s current view, as required in Attack Game 8.4. It follows that $\mathcal{B}$ perfectly emulates the challenger in Attack Game 8.4 and consequently $\Pr[W] = \mathrm{TCR}\mathsf{adv}[\mathcal{A}, H_{\mathrm{tcr}}]$.

By definition of H_{tcr}, we also have the following:

$$\begin{aligned} H_{\mathrm{tcr}}(k, m_0) &= H((m \oplus m_0) \oplus m_0) = H(m) \\ H_{\mathrm{tcr}}(k, m_1) &= H(m_1 \oplus k) = H(m') \end{aligned} \tag{8.17}$$

Now, suppose event W happens. Then $H_{\mathrm{tcr}}(k, m_0) = H_{\mathrm{tcr}}(k, m_1)$ and therefore, by (8.17), we know that $H(m) = H(m')$. Second, we deduce that $m \neq m'$ which follows since $m_0 \neq m_1$ and $m' = m \oplus (m_1 \oplus m_0)$. Hence, when event W occurs, $\mathcal{B}$ outputs a 2nd-preimage of m. It now follows that:

$$\mathrm{SPR}\mathsf{adv}[\mathcal{B}, H] \geq \Pr[W] = \mathrm{TCR}\mathsf{adv}[\mathcal{A}, H_{\mathrm{tcr}}]$$

as required. $\square$

Target collision resistance for long inputs. The function H_{tcr} in (8.16) shows that a 2nd-preimage resistant function directly gives a TCR function. If we assume that the SHA256 compression function h is 2nd-preimage resistant (a weaker assumption than assuming that h is collision resistant) then, by Theorem 8.10 we obtain a TCR hash for inputs of length $512 + 265 = 768$ bits. The length of the required key is also 768 bits.

We will often need TCR functions for much longer inputs. Using the SHA256 compression function we already know how to build a TCR hash for short inputs using a short key. Thus, let us assume that we have a TCR function h defined over $(\mathcal{K},\ \mathcal{T} \times \mathcal{M},\ \mathcal{T})$ where $\mathcal{M} := \{0,1\}^{\ell}$ for some small ℓ, say $\ell = 512$. We build a new TCR hash for much larger inputs. Let $L \in \mathbb{Z}^{>0}$ be a power of 2. We build a **derived TCR hash** H that hashes messages in $\{0,1\}^{\leq \ell L}$ using keys in $(\mathcal{K} \times \mathcal{T}^{1+\log_2 L})$. Note that the length of the keys is *logarithmic* in the length of the message, which is much better than (8.16).

To describe the function H we need an auxiliary function $\nu : \mathbb{Z}^{>0} \to \mathbb{Z}^{>0}$ defined as:

$$\nu(x) := \text{largest } n \in \mathbb{Z}^{>0} \text{ such that } 2^n \text{ divides } x.$$

Thus, $\nu(x)$ counts the number of least significant bits of x that are zero. For example, $\nu(x) = 0$ if x is odd and $\nu(x) = n$ if $x = 2^n$. Note that $\nu(x) \leq 7$ for more than 99% of the integers.

The derived TCR hash H is similar to Merkle-Damgård. It uses the same padding block PB as in Merkle-Damgård and a fixed initial value IV. The derived TCR hash H is defined as follows (see Fig. 8.13):

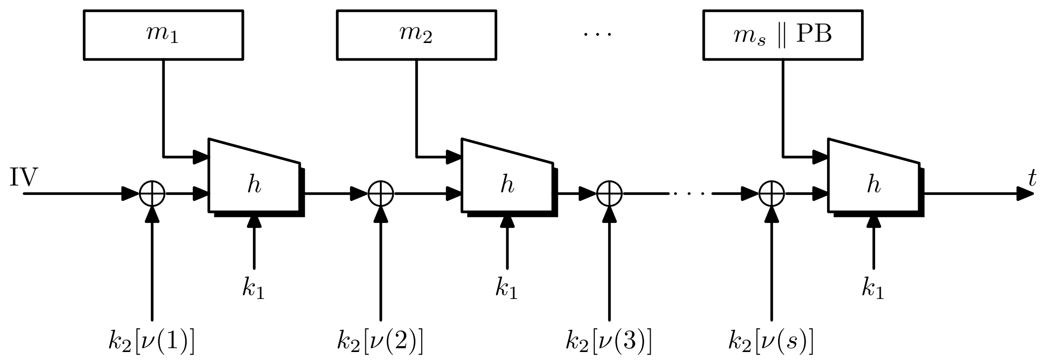


Figure 8.13: Extending the domain of a TCR hash

Input: Message $M \in \{0,1\}^{\leq \ell L}$ and key $(k_1, k_2) \in \mathcal{K} \times \mathcal{T}^{1+\log_2 L}$
Output: $t \in \mathcal{T}$

$M \leftarrow M \parallel \text{PB}$
Break M into consecutive ℓ-bit blocks so that

$$M = m_1 \parallel m_2 \parallel \cdots \parallel m_s \quad \text{where} \quad m_1, \ldots, m_s \in \{0,1\}^\ell$$

$t_0 \leftarrow \text{IV}$
for $i = 1$ to s do:
  $u \leftarrow k_2[\nu(i)] \oplus t_{i-1} \quad \in \mathcal{T}$
  $t_i \leftarrow h(k_1,\ (u, m_i)\ ) \quad \in \mathcal{T}$
Output t_s

We note that directly using Merkle-Damgård to extend the domain of a TCR hash does not work. Plugging $h(k_1, \cdot)$ directly into Merkle-Damgård can fail to give a TCR hash.

Security of the derived hash. The following theorem shows that the derived hash H is TCR assuming the underlying hash h is. We refer to [96, 76] for the proof of this theorem.

Theorem 8.11. *Suppose h is a TCR hash function that hashes messages in $(\mathcal{T} \times \{0,1\}^\ell)$. Then, for any bounded L, the derived function H is a TCR hash for messages in $\{0,1\}^{\leq \ell L}$.*

In particular, suppose $\mathcal{A}$ is a TCR adversary attacking H (as in Attack Game 8.4). Then there exists a TCR adversary $\mathcal{B}$ (whose running times are about the same as that of $\mathcal{A}$) such that

$$\text{TCR}\mathsf{adv}[\mathcal{A}, H] \leq L \cdot \text{TCR}\mathsf{adv}[\mathcal{B}, h].$$

As in Merkle-Damgård this construction is inherently sequential. A tree-based construction similar to Exercise 8.8 gives a TCR hash using logarithmic size keys that is more suitable for a parallel machine. We refer to [7] for the details.

8.11.4 Using target collision resistance

We now know how to build a TCR function for large inputs from a small 2nd-preimage resistant function. We show how to use such TCR functions to extend the domain for a MAC and to ensure file integrity. We start with file integrity.

File integrity

Let H be a TCR hash defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$. We use H to protect integrity of files $F_1, F_2, \ldots \in \mathcal{M}$ using a small amount of read-only memory. The idea is to pick a random key r_i in $\mathcal{K}$ for every file F_i and then store the pair $(r_i,\ H(r_i, F_i)\ )$ in read-only memory. Note that we are using a little more read-only memory than in the system based on collision resistance. To verify integrity of file F_i we simply recompute $H(r_i, F_i)$ and compare to the hash stored in read-only memory.

Why is this mechanism secure? Consider malware targeting a specific file F. We store in read-only memory the key r and $t := H(r, F)$. To modify F without being detected the malware must come up with a new file F' such that $t = H(r, F')$. In other words, the malware is given as input the file F along with a random key $r \in \mathcal{K}$ and must produce a new F' such that $H(r, F) = H(r, F')$. The adversary (the malware writer in this case) chooses which file F to attack. But this is precisely the TCR Attack Game 8.4 — the adversary chooses an F, gets a random key r, and must output a new F' that collides with F under r. Hence, if H is TCR the malware cannot modify F without being detected.

In summary, we can provide file integrity using a small amount of read-only memory and by relying only on 2nd-preimage resistance. The cost, in comparison to the system based on collision resistance, is that we need a little more read-only memory to store the key r. In particular, using the TCR construction from the previous section, the amount of additional read-only memory needed is logarithmic in the size of the files being protected. Using a recursive construction (see Exercise 8.24) we can reduce the additional read-only memory used to a small constant, but still non-zero.

Extending the domain of a MAC

Let H be a TCR hash defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{T})$. Let $\mathcal{I} = (S, V)$ be a MAC for authenticating short messages in $\mathcal{K}_H \times \mathcal{T}$ using keys in $\mathcal{K}$. We assume that $\mathcal{M}$ is much larger than $\mathcal{T}$. We build a new MAC $\mathcal{I}' = (S', V')$ for authenticating messages in $\mathcal{M}$ using keys in $\mathcal{K}$ as follows:

$$
\begin{array}{ll}
S'(k,\ m) := & V'\big(k,\ m,\ (t, r)\big) := \\
\qquad r \xleftarrow{\text{R}} \mathcal{K}_H & \qquad h \leftarrow H(r, m) \\
\qquad h \leftarrow H(r, m) & \qquad \text{Output } V(k,\ (r, h),\ t) \\
\qquad t \leftarrow S\big(k,\ (r, h)\big) & \\
\qquad \text{Output } (t, r) &
\end{array}
\tag{8.18}
$$

Note the MAC signing is randomized — we pick a random TCR key r, include r in the input to the signing algorithm S, and output r as part of the final tag. As a result, tags produced by this MAC are longer than tags produced from extending MACs using a collision resistance hash (as in Section 8.2). Using the construction from the previous section, the length of r is logarithmic in the size of the message being authenticated. This extra logarithmic size key is included in every tag. On the plus side, this construction only relies on H being TCR which is a much weaker property than collision resistance and hence much more likely to hold for H.

The following theorem proves security of the construction in (8.18) above. The theorem is the analog of Theorem 8.1 and its proof is similar. Note however, that the error bounds are not as tight as the bounds in Theorem 8.1.

Theorem 8.12. *Suppose the MAC system $\mathcal{I}$ is a secure MAC and the hash function H is TCR. Then the derived MAC system $\mathcal{I}' = (S', V')$ defined in (8.18) is a secure MAC.*

In particular, for every MAC adversary $\mathcal{A}$ attacking $\mathcal{I}'$ (as in Attack Game 6.1) that issues at most Q signing queries, there exist an efficient MAC adversary $\mathcal{B}_{\mathcal{I}}$ and an efficient TCR adversary $\mathcal{B}_H$, which are elementary wrappers around $\mathcal{A}$, such that

$$\text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}'] \leq \text{MAC}\mathsf{adv}[\mathcal{B}_{\mathcal{I}}, \mathcal{I}] + Q \cdot \text{TCR}\mathsf{adv}[\mathcal{B}_H, H].$$

Proof idea. Our goal is to show that no efficient MAC adversary can successfully attack $\mathcal{I}'$. Such an adversary $\mathcal{A}$ asks the challenger to sign a few long messages $m_1, m_2, \ldots \in \mathcal{M}$ and gets back tags (t_i, r_i) for $i = 1, 2, \ldots$. It then tries to invent a new valid message-MAC pair $(m,\ (t, r))$. If $\mathcal{A}$ is able to produce a valid forgery $(m,\ (t, r))$ then one of two things must happen:

1. either $(r, H(r, m))$ is equal to $(r_i, H(r_i, m_i))$ for some i;
2. or not.

It is not difficult to see that forgeries of the second type can be used to attack the underlying MAC $\mathcal{I}$. We show that forgeries of the first type can be used to break the target collision resistance of H. Indeed, if $(r, H(r, m)) = (r_i, H(r_i, m_i))$ then $r = r_i$ and therefore $H(r, m) = H(r, m_i)$. Thus m_i and m collide under the random key r. We will show that this lets us build an adversary $\mathcal{B}_H$ that wins the TCR game when attacking H. Unfortunately, $\mathcal{B}_H$ must guess ahead of time which of $\mathcal{A}$'s queries to use as m_i. Since there are Q queries to choose from, $\mathcal{B}_H$ will guess correctly with probability $1/Q$. This is the reason for the extra factor of Q in the error term. □

Proof. Let X be the event that adversary $\mathcal{A}$ wins the MAC Attack Game 6.1 with respect to $\mathcal{I}'$. Let $m_1, m_2, \ldots \in \mathcal{M}$ be $\mathcal{A}$'s queries during the game and let $(t_1, r_1),\ (t_2, r_2), \ldots$ be the challenger's responses. Furthermore, let $(m,\ (t, r))$ be the adversary's final output. We define two additional events:

- Let Y denote the event that for some $i = 1, 2, \ldots$ we have that $(r, H(r, m)) = (r_i, H(r, m_i))$ and $m \neq m_i$.
- Let Z denote the event that $\mathcal{A}$ wins Attack Game 6.1 on $\mathcal{I}'$ and event Y did not occur.

Then

$$\text{MAC}\mathsf{adv}[\mathcal{A}, \mathcal{I}'] \;=\; \Pr[X] \;\leq\; \Pr[X \wedge \neg Y] + \Pr[Y] \;=\; \Pr[Z] + \Pr[Y] \tag{8.19}$$

To prove the theorem we construct a TCR adversary $\mathcal{B}_H$ and a MAC adversary $\mathcal{B}_{\mathcal{I}}$ such that

$$\Pr[Y] \leq Q \cdot \text{TCR}\mathsf{adv}[\mathcal{B}_H, H] \quad \text{and} \quad \Pr[Z] = \text{MAC}\mathsf{adv}[\mathcal{B}_{\mathcal{I}}, \mathcal{I}].$$

Adversary $\mathcal{B}_{\mathcal{I}}$ is essentially the same as in the proof of Theorem 8.1. Here we only describe the TCR adversary $\mathcal{B}_H$, which emulates a MAC challenger for $\mathcal{A}$ as follows:

$k \xleftarrow{R} \mathcal{K}$
$u \xleftarrow{R} \{1, 2, \ldots, Q\}$
Run algorithm $\mathcal{A}$

Upon receiving the ith signing query $m_i \in \mathcal{M}$ from $\mathcal{A}$ do:
  If $i \neq u$ then
    $r_i \xleftarrow{R} \mathcal{K}_H$
  Else // $i = u$: *for query number* u *get* r_i *from the TCR challenger*
    $\mathcal{B}_H$ sends $\hat{m}_0 := m_i$ to its TCR challenger
    $\mathcal{B}_h$ receives a random key $\hat{r} \in \mathcal{K}$ from its challenger
    $r_i \leftarrow \hat{r}$

  $h \leftarrow H(r_i, m_i)$
  $t \leftarrow S(k,\ (r_i, h)\ )$
  Send (t, r) to $\mathcal{A}$

Upon receiving the final message-tag pair $(m,\ (t, r)\ )$ from $\mathcal{A}$ do:
  $\mathcal{B}_H$ sends $\hat{m}_1 := m$ to its challenger

Algorithm $\mathcal{B}_H$ responds to $\mathcal{A}$'s signature queries exactly as in a real MAC attack game. Therefore, event Y happens during the interaction with $\mathcal{B}_H$ with the same probability that it happens in a real MAC attack game. Now, when event Y happens there exists a $j \in \{1, 2, \ldots\}$ such that $(r, H(r, m)) = (r_j, H(r_j, m_j))$ and $m \neq m_j$. Suppose that furthermore $j = u$. Then $r = r_j = \hat{r}$ and therefore $H(\hat{r}, m) = H(\hat{r}, m_u)$. Hence, if event Y happens and $j = u$ then $\mathcal{B}_H$ wins the TCR attack game. In symbols,

$$\text{TCR}\mathsf{adv}[\mathcal{B}_H, H] = \Pr[Y \wedge (j = u)].$$

Notice that u is independent of $\mathcal{A}$'s view — it is only used for choosing which random key r_i is from $\mathcal{B}_H$'s challenger, but no matter what u is, the key r_i given to $\mathcal{A}$ is always uniformly random. Hence, event Y is independent of the event $j = u$. For the same reason, if the adversary makes a total of w queries then $\Pr[j = u] = 1/w \geq 1/Q$. In summary,

$$\text{TCR}\mathsf{adv}[\mathcal{B}_H, H] = \Pr[Y \wedge (j = u)] = \Pr[Y] \cdot \Pr[j = u] \geq \Pr[Y]/Q$$

as required. □

8.12 A fun application: an efficient commitment scheme

To be written.

8.13 Another fun application: proofs of work

To be written.

8.14 Notes

Citations to the literature to be added.

8.15 Exercises

8.1 (Truncating a CRHF is dangerous). Let H be a collision resistant hash function defined over $(\mathcal{M}, \{0,1\}^n)$. Use H to construct a hash function H' over $(\mathcal{M}, \{0,1\}^n)$ that is also collision resistant, but if one truncates the output of H' by one bit then H' is no longer collision resistant. That is, H' is collision resistant, but $H''(x) := H'(x)[0 \mathinner{..} n-2]$ is not.

8.2 (CRHF combiners). We want to build a CRHF H using two CRHFs H_1 and H_2, so that if at some future time one of H_1 or H_2 is broken (but not both) then H is still secure.

(a) Suppose H_1 and H_2 are defined over $(\mathcal{M}, \mathcal{T})$. Let $H(m) := \big(H_1(m), H_2(m)\big)$. Show that H is a secure CRHF if either H_1 or H_2 is secure.

(b) Show that $H'(x) = H_1(H_2(x))$ need not be a secure CRHF even if one of H_1 or H_2 is secure.

8.3 (Extending the domain of a PRF with a CRHF). Suppose F is a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ and H is a collision resistant hash defined over $(\mathcal{M}, \mathcal{X})$. Show that $F'(k, m) = F(k, H(m))$ is a secure PRF. This shows that H can be used to extend the domain of a PRF.

8.4 (Hash-then-encrypt MAC). Let H be a collision resistant hash defined over $(\mathcal{M}, \mathcal{X})$ and let $\mathcal{E} = (E, D)$ be a secure block cipher defined over $(\mathcal{K}, \mathcal{X})$. Show that the encrypted-hash MAC system (S, V) defined by $S(k, m) := E(k, H(m))$ is a secure MAC.

Hint: Use Theorem 8.1.

8.5 (Finding many collisions). Let H be a hash function defined over $(\mathcal{M}, \mathcal{T})$ where $N := |\mathcal{T}|$ and $|\mathcal{M}| \gg N$. We showed that $O(\sqrt{N})$ evaluations of H are sufficient to find a collision for H with probability 1/2. Show that $O\left(\sqrt{sN}\right)$ evaluations of H are sufficient to find s collisions $(x_0^{(1)}, x_1^{(1)}), \ldots, (x_0^{(s)}, x_1^{(s)})$ for H with probability at least 1/2. Therefore, finding a million collisions is only about a thousand times harder than finding a single collision.

8.6 (Finding multi-collisions). Continuing with Exercise 8.5, we say that an s-collision for H is a set of s distinct points $x_1, \ldots, x_s$ in $\mathcal{M}$ such that $H(x_1) = \cdots = H(x_s)$. Show that for each constant value of s, $O\left(N^{(s-1)/s}\right)$ evaluations of H are sufficient to find an s-collision for H, with probability at least 1/2.

8.7 (Collision finding in constant space). Let H be a hash function defined over $(\mathcal{M}, \mathcal{T})$ where $N := |\mathcal{M}|$. In Section 8.3 we developed a method to find an H collision with constant probability using $O(\sqrt{N})$ evaluations of H. However, the method required $O(\sqrt{N})$ memory space. In this exercise we develop a *constant-memory* collision finding method that runs in about the same time. More precisely, the method only needs memory to store two hash values in $\mathcal{T}$. You may assume that $H : \mathcal{M} \to \mathcal{T}$ is a random function chosen uniformly from Funs$[\mathcal{M}, \mathcal{T}]$ and $\mathcal{T} \subseteq \mathcal{M}$. A collision should be produced with probability at least 1/2.

(a) Let $x_0 \xleftarrow{\text{R}} \mathcal{M}$ and define $H^{(i)}(x_0)$ to be the ith iterate of H starting at x_0. For example, $H^{(3)}(x_0) = H(H(H(x_0)))$.

  (i) Let i be the smallest positive integer satisfying $H^{(i)}(x_0) = H^{(2i)}(x_0)$.

  (ii) Let j be the smallest positive integer satisfying $H^{(j)}(x_0) = H^{(j+i)}(x_0)$. Notice that $j \leq i$.

Show that $H^{(j-1)}(x_0)$ and $H^{(j+i-1)}(x_0)$ are an H collision with probability at least 3/4.

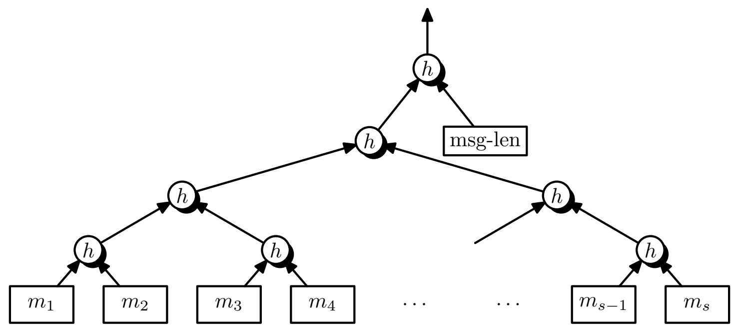


Figure 8.14: Tree-based Merkle-Damgård

(b) Show that i from part (a) satisfies $i = O(\sqrt{N})$ with probability at least 3/4 and that it can be found using $O(\sqrt{N})$ evaluations of H. Once i is found, finding j takes another $O(\sqrt{N})$ evaluations, as required. The entire process only needs to store two elements in T at any given time.

8.8 (A parallel Merkle-Damgård). The Merkle-Damgård construction in Section 8.4 gives a *sequential* method for extending the domain of a secure CRHF. The tree construction in Fig. 8.14 is a parallelizable approach. Prove that the resulting hash function is collision resistant, assuming h is collision resistant. Here h is a compression function $h : \mathcal{X}^2 \to \mathcal{X}$, and we assume the message length can be encoded as an element of $\mathcal{X}$.

8.9 (Secure variants of Davies-Meyer). Prove that the h_1, h_2, and h_3 variants of Davies-Meyer defined on page 292 are collision resistant in the ideal cipher model.

8.10 (Insecure variants of Davies-Meyer). Show that the h_4 and h_5 variants of Davies-Meyer defined on page 293 are not collision resistant.

8.11 (An insecure instantiation of Davies-Meyer). Let's show that Davies-Meyer may not be collision resistant when instantiated with a real-world block cipher. Let (E, D) be a block cipher defined over $(\mathcal{K}, \mathcal{X})$ where $\mathcal{K} = \mathcal{X} = \{0,1\}^n$. For $y \in \mathcal{X}$ let $\overline{y}$ denote the bit-wise complement of y.

(a) Suppose that $E(\overline{k}, \overline{x}) = \overline{E(k, x)}$ for all keys $k \in \mathcal{K}$ and all $x \in \mathcal{X}$. The DES block cipher has precisely this property. Show that the Davies-Meyer construction, $h(k, x) := E(k, x) \oplus x$, is not collision resistant when instantiated with algorithm E.

(b) Suppose (E, D) is an Even-Mansour cipher, $E(k, x) := \pi(x \oplus k) \oplus k$, where $\pi : \mathcal{X} \to \mathcal{X}$ is a fixed public permutation. Show that the Davies-Meyer construction instantiated with algorithm E is not collision resistant.

Hint: Show that this Even-Mansour cipher satisfies the property from part (a).

8.12 (Merkle-Damgård without length encoding). Suppose that in the Merkle-Damgård construction, we drop the requirement that the padding block encodes the message length. Let h be the compression function, let H be the resulting hash function, and let IV be the prescribed initial value.

(a) Show that H is collision resistant, assuming h is collision resistant and that *it is hard to find a preimage of IV under h.*

(b) Show that if h is a Davies-Meyer compression function, and we model the underlying block cipher as an ideal cipher, then for any fixed IV, it is hard to find a preimage of IV under h.

8.13 (2nd-preimage resistance of Merkle-Damgård). Let H be a Merkle-Damgård hash built out of a Davies-Meyer compression function $h : \{0,1\}^n \times \{0,1\}^\ell \to \{0,1\}^n$. Consider the attack game characterizing 2nd-preimage resistance in Definition 8.4. Let us assume that the initial, random message in that attack game consists of s blocks. We shall model the underlying block cipher used in the Davies-Meyer construction as an ideal cipher, and adapt the attack game to work in the ideal cipher model. Show that for every adversary $\mathcal{A}$ that makes at most Q ideal-cipher queries, we have

$$\mathrm{SPR}^{\mathrm{ic}}\mathsf{adv}[\mathcal{A}, H] \le \frac{(Q+s)s}{2^{n-1}}.$$

Discussion: This bound for finding second preimages is significantly better than the bound for finding arbitrary collisions. Unfortunately, we have to resort to the ideal cipher model to prove it.

8.14 (Fixed points). We consider the Davies-Meyer and Miyaguchi-Preneel compression functions defined in Section 8.5.2.

(a) Show that for a Davies-Meyer compression function it is easy to find a pair (t, m) such that $h_{\mathrm{DM}}(t, m) = t$. Such a pair is called a **fixed point** for h_{DM}.

(b) Show that in the ideal cipher model it is difficult to find fixed points for the Miyaguchi-Preneel compression function.

The next exercise gives an application for fixed points.

8.15 (Finding second preimages in Merkle-Damgård). In this exercise, we develop a second preimage attack on Merkle-Damgård that roughly matches the security bounds in Exercise 8.13. Let H_{MD} be a Merkle-Damgård hash built out of a Davies-Meyer compression function $h : \{0,1\}^n \times \{0,1\}^\ell \to \{0,1\}^n$. Recall that H_{MD} pads a given message with a padding block that encodes the message length. We will also consider the hash function H, which is the same as H_{MD}, but which uses a padding block that *does not* encode the message length. Throughout this exercise, we model the underlying block cipher in the Davies-Meyer construction as an ideal cipher. For concreteness, assume $\ell = 2n$.

(a) Let $s \approx 2^{n/2}$. You are given a message M that consists of s random ℓ-bit blocks. Show that by making $O(s)$ ideal cipher queries, with probability $1/2$ you can find a message $M' \ne M$ such that $H(M') = H(M)$. Here, the probability is over the random choice of M, the random permutations defining the ideal cipher, and the random choices made by your attack.

Hint: Repeatedly choose random blocks x in $\{0,1\}^\ell$ until $h(\mathrm{IV}, x)$ is the same as one of the s chaining variables obtained when computing $H(M)$. Use this x to construct the second preimage M'.

(b) Repeat part (a) for H_{MD}.

Hint: The attack in part (a) will likely find a second preimage M' that is shorter than M; because of length encoding, this will not be a second preimage under H_{MD}; nevertheless, show

how to use fixed points (see previous exercise) to modify M' so that it has the same length as M.

Discussion: Let H be a hash function with an n-bit output. If H is a *random* function then breaking second preimage resistance takes about 2^n time. This exercise shows that for Merkle-Damgård functions, breaking second preimage resistance can be done much faster, taking only about $2^{n/2}$ time.

8.16 (The envelope method is a secure PRF). Consider the envelope method for building a PRF from a hash function discussed in Section 8.7: $F_{\text{env}}(k, M) := H(k \parallel M \parallel k)$. Here, we assume that H is a Merkle-Damgård hash built from a compression function $h : \{0,1\}^n \times \{0,1\}^\ell \to \{0,1\}^n$. Assume that the keys for F_{env} are ℓ-bit strings. Furthermore, assume that the message M a bit string whose length is an even multiple of ℓ (we can always pad the message, if necessary). Under the assumption that both h_{top} and h_{bot} are secure PRFs, show that F_{env} is a secure PRF.

Hint: Use the result of Exercise 7.6; also, first consider a simplified setting where H does not append the usual Merkle-Damgård padding block to the inputs $k \parallel M \parallel k$ (this padding block does not really help in this setting, but it does not hurt either — it just complicates the analysis).

8.17 (The key-prepending method revisited). Consider the key-prepending method for building a PRF from a hash function discussed in Section 8.7: $F_{\text{pre}}(k, M) := H(k \parallel M)$. Here, we assume that H is a Merkle-Damgård hash built from a compression function $h : \{0,1\}^n \times \{0,1\}^\ell \to \{0,1\}^n$. Assume that the keys for F_{pre} are ℓ-bit strings. Under the assumption that both h_{top} and h_{bot} are secure PRFs, show that F_{pre} is a prefix-free secure PRF.

8.18 (The key-appending method revisted). Consider the following variant of the key-appending method for building a PRF from a hash function discussed in Section 8.7: $F'_{\text{post}}(k, M) := H(M \parallel \text{PB} \parallel k)$. Here, we assume that H is a Merkle-Damgård hash built from a compression function $h : \{0,1\}^n \times \{0,1\}^\ell \to \{0,1\}^n$. Also, PB is the standard Merkle-Damgård padding for M, which encodes the length of M. Assume that the keys for F'_{post} are ℓ-bit strings. Under the assumption that h is collision resistant and h_{top} is a secure PRF, show that F'_{post} is a secure PRF.

8.19 (Dual PRFs). The security analysis of HMAC assumes that the underlying compression function is a secure PRF when either input is used as the key. A PRF with this property is said to be a **dual PRF**. Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ where $\mathcal{Y} = \{0,1\}^n$ for some n. We wish to build a new PRF $\hat{F}$ that is a dual PRF. This $\hat{F}$ can be used as a building block for HMAC.

(a) Suppose $\mathcal{K} = \mathcal{X}$. Show that the most natural construction $\hat{F}(x, y) := F(x, y) \oplus F(y, x)$ is insecure: there exists a secure PRF F for which $\hat{F}$ is not a dual PRF.

Hint: Start from a secure PRF F' and the "sabotage" it to get the required F.

(b) Let G be a PRG defined over $(\mathcal{S},\ \mathcal{K} \times \mathcal{X})$. Let $G_0 : \mathcal{S} \to \mathcal{K}$ be the left output of G and let $G_1 : \mathcal{S} \to \mathcal{X}$ be the right output of G. Let $\hat{F}$ be the following PRF defined over $(\mathcal{S}, \mathcal{S}, \mathcal{Y})$:

$$\hat{F}(x, y) := F\Big(G_0(x),\ G_1(y)\Big) \ \oplus \ F\Big(G_0(y),\ G_1(x)\Big).$$

Prove that $\hat{F}$ is a dual PRF assuming G is a secure PRG and that G_1 is collision resistant.

8.20 (Sponge with low capacity is insecure). Let H be a sponge hash with rate r and capacity c, built from a permutation $\pi : \{0,1\}^n \to \{0,1\}^n$, where $n = r + c$ (see Section 8.8).

Assume $r \geq 2c$. Show how to find a collision for H with probability at least $1/2$ in time $O(2^{c/2})$. The colliding messages can be $2r$ bits each.

8.21 (Sponge as a PRF). Let H be a sponge hash with rate r and capacity c, built from a permutation $\pi : \{0,1\}^n \to \{0,1\}^n$, where $n = r + c$ (see Section 8.8). Consider again the PRF built from H by pre-pending the key: $F_{\text{pre}}(k, M) := H(k \parallel M)$. Assume that the key is r bits and the output of F_{pre} is also r bits. Prove that in the ideal permutation model, where π is replaced by a random permutation Π, this construction yields a secure PRF, assuming 2^r and 2^c are super-poly.

Note: This follows immediately from the fact that H is indifferentiable from a random oracle (see Section 8.10.3) and Theorem 8.7. However, you are to give a *direct proof* of this fact.

Hint: Use the same domain splitting strategy as outlined in Exercise 7.17.

8.22 (Relations among definitions). Let H be a hash function over $(\mathcal{M}, \mathcal{T})$ where $|\mathcal{M}| \geq 2|\mathcal{T}|$. We say that an element $m \in \mathcal{M}$ has a second preimage if there exists a different $m' \in \mathcal{M}$ such that $H(m) = H(m')$.

(a) Show that at least half the elements of $\mathcal{M}$ have a second preimage.

(b) Use part (a) to show that a 2nd-preimage hash must be one-way.

(c) Show that a collision resistant hash must be 2nd-preimage resistant.

8.23 (From TCR to 2nd-preimage resistance). Let H be a TCR hash defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$. Choose a random $r \in \mathcal{M}$. Prove that $f(x) := H(r, x) \parallel r$ is 2nd-preimage resistant, where r is treated as a system parameter.

8.24 (File integrity: reducing read-only memory). The file integrity construction in Section 8.11.4 uses additional read-only memory proportional to $\log |F|$ where $|F|$ is the size of the file F being protected.

(a) By first hashing the file F and then hashing the key r, show how to reduce the amount of additional read-only memory used to $O(\log\log |F|)$. This requires storing additional $O(\log |F|)$ bits on disk.

(b) Generalize your solution from part (a) to show how to reduce read-only overhead to constant size independent of $|F|$. The extra information stored on disk is still of size $O(\log |F|)$.

8.25 (Strong 2nd-preimage resistance). Let H be a hash function defined over $(\mathcal{X} \times \mathcal{Y}, \mathcal{T})$ where $\mathcal{X} := \{0,1\}^n$. We say that H is **strong 2nd-preimage resistant**, or simply **strong-SPR**, if no efficient adversary, given a random x in $\mathcal{X}$ as input, can output y, x', y' such that $H(x, y) = H(x', y')$ with non-negligible probability.

(a) Let H be a strong-SPR. Use H to construct a collision resistant hash function H' defined over $(\mathcal{X} \times \mathcal{Y},\ \mathcal{T})$.

(b) Let us show that a function H can be a strong-SPR, but not collision resistant. For example, consider the hash function:

$$H''(0,0) := H''(0,1) := 0 \quad \text{and} \quad H''(x,y) := H(x,y) \text{ for all other inputs.}$$

Prove that if $|\mathcal{X}|$ is super-poly and H is a strong-SPR then so is H''. However, H'' is clearly not collision resistant.

(c) Show that $H_{TCR}(k,(x,y)) := H((k \oplus x), y)$ is a TCR hash function assuming H is a strong-SPR hash function.

8.26 (Enhanced TCR). Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$. We say that H is an **enhanced-TCR** if no efficient adversary can win the following game with non-negligible advantage: the adversary outputs $m \in \mathcal{M}$, is given random $k \in \mathcal{K}$ and outputs (k', m') such that $H(k,m) = H(k', m')$.

(a) Let H be a strong-SPR hash function over $(\mathcal{X} \times \mathcal{Y}, \mathcal{T})$, as defined in Exercise 8.25, where $\mathcal{X} := \{0,1\}^n$. Show that $H'(k,(x,y)) := H((k \oplus x), y)$ is an enhanced-TCR hash function.

(b) Show how to use an enhanced-TCR to extend the domain of a MAC. Let H be a enhanced-TCR defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{X})$ and let (S, V) be a secure MAC defined over $(\mathcal{K}, \mathcal{X}, \mathcal{T})$. Show that the following is a secure MAC:

$$S'(k,m) := \{\ r \xleftarrow{\mathrm{R}} \mathcal{K}_H,\ t \leftarrow S(k, H(r,m)),\ \text{output } (r,t)\}$$
$$V'\big(k, m, (r,t)\big) := \{\ \mathsf{accept} \text{ if } t = V(k, H(r,m))\}$$

8.27 (Weak collision resistance). Let H be a keyed hash function defined over $(\mathcal{K}, \mathcal{M}, \mathcal{T})$. We say that H is a **weak collision resistant (WCR)** if no efficient adversary can win the following game with non-negligible advantage: the challenger chooses a random key $k \in \mathcal{K}$ and lets the adversary query the function $H(k, \cdot)$ at any input of its choice. The adversary wins if it outputs a collision m_0, m_1 for $H(k, \cdot)$.

(a) Show that WCR is a weaker notion than a secure MAC: (1) show that every deterministic secure MAC is WCR, (2) give an example of a secure WCR that is not a secure MAC.

(b) MAC domain extension with a WCR: let (S, V) be a secure MAC and let H be a WCR. Show that the MAC system (S', V') defined by $S'\big((k_0, k_1), m\big) := S\big(k_1, H(k_0, m)\big)$ is secure.

(c) Show that Merkle-Damgård expands a compressing fixed-input length WCR to a variable input length WCR. In particular, let h be a WCR defined over $(\mathcal{K}, \mathcal{X} \times \mathcal{Y}, \mathcal{X})$, where $\mathcal{X} := \{0,1\}^n$ and $\mathcal{Y} := \{0,1\}^\ell$. Define H as a keyed hash function over $(\mathcal{K},\ \{0,1\}^{\le L},\ \mathcal{X})$ as follows:

$$H\big((k_1, k_2), M\big) := \left\{ \begin{array}{l} \text{pad and break } M \text{ into } \ell\text{-bit blocks: } m_1, \ldots, m_s \\ t_0 \leftarrow 0^n \in \mathcal{X} \\ \text{for } i = 1 \text{ to } s \text{ do:} \\ \quad t_i \leftarrow h\big(k_1,\ (t_{i-1}, m_i)\big) \\ \text{encode } s \text{ as a block } b \in \mathcal{Y} \\ t_{s+1} \leftarrow h\big(k_2,\ (t_s, b)\big) \\ \text{output } t_{s+1} \end{array} \right\}$$

Show that H is a WCR if h is.

8.28 (The trouble with random oracles). Let H be a hash function defined over $(\mathcal{K} \times \mathcal{X},\ \mathcal{Y})$. We showed that $H(k,x)$ is a secure PRF when H is modeled as a random oracle. In this exercise we show that this PRF can be tweaked into a new PRF F that uses H as a black-box, and that is a secure PRF when H is modeled as a random model. However, for every concrete instantiation of the hash function H, the PRF F becomes insecure.

For simplicity, assume that $\mathcal{K}$ and $\mathcal{Y}$ consist of bit strings of length n and that $\mathcal{X}$ consists of bit strings of length at most L for some poly-bounded n and L. Assume also that the program for H parses its input as a bit string of the form $k \parallel x$, where $k \in \mathcal{K}$ and $x \in \mathcal{X}$.

Consider a program $Exec(P, v, t)$ that takes as input three bit strings P, v, t. When $Exec(P, v, t)$ runs, it attempts to interpret P as a program written in some programming language (take your pick); it runs P on input v, but stops the execution after $|t|$ steps (if necessary), where $|t|$ is the bit-length of t. The output of $Exec(P, v, t)$ is whatever P outputs on input v, or some special default value if the time bound is exceeded. For simplicity, assume that $Exec(P, v, t)$ always outputs an n-bit string (padding or truncating as necessary). Even though P on input v may run in exponential time (or even fall into an infinite loop), $Exec(P, v, t)$ always runs in time bounded by a polynomial in its input length.

Finally, let T be some arbitrary polynomial, and define

$$F(k, x) := H(k, x) \oplus Exec(x,\ k \parallel x,\ 0^{T(|k|+|x|)}).$$

(a) Show that if H is any hash function that can be implemented by a program P_H whose length is at most L and whose running time on input $k \parallel x$ is at most $T(|k| + |x|)$, then the concrete instantiation of F using this H runs in polynomial time and is not a secure PRF.

Hint: Find a value of x that makes the PRF output 0^n, for all keys $k \in \mathcal{K}$.

(b) Show that F is a secure PRF if H is modeled as a random oracle.

Discussion: Although this is a contrived example, it shakes our confidence in the random oracle model. Nevertheless, the reason why the random oracle model has been so successful in practice is that typically real-world attacks treat the hash function as a black box. The attack on F clearly does not. See also the discussion in [24], which removes the strict time bound restriction on H.

Chapter 9

Authenticated Encryption

Our discussion of encryption in Chapters 2 to 8 leads up to this point. In this chapter we, construct systems that ensure both data secrecy (confidentiality) and data integrity, even against very aggressive attackers that can interact with the sender and receiver quite maliciously and arbitrarily. Such systems are said to provide **authenticated encryption** or are simply said to be AE-secure. This chapter concludes our discussion of symmetric encryption. It is the culmination of our symmetric encryption story.

Recall that in our discussion of CPA security in Chapter 5 we stressed that CPA security does not provide any integrity. An attacker can tamper with the output of a CPA-secure cipher without being detected by the decryptor. We will present many real-world settings where undetected ciphertext tampering comprises both message secrecy and message integrity. Consequently, CPA security by itself is insufficient for almost all applications. Instead, applications should almost always use authenticated encryption to ensure both message secrecy and integrity. We stress that even if secrecy is the only requirement, CPA security is insufficient.

In this chapter we develop the notion of authenticated encryption and construct several AE systems. There are two general paradigms for construction AE systems. The first, called **generic composition**, is to combine a CPA-secure cipher with a secure MAC. There are many ways to combine these two primitives and not all combinations are secure. We briefly consider two examples.

Let (E, D) be a cipher and (S, V) be a MAC. Let k_{enc} be a cipher key and k_{mac} be a MAC key. Two options for combining encryption and integrity immediately come to mind, which are shown in Fig. 9.1 and work as follows:

Encrypt-then-MAC Encrypt the message, $c \xleftarrow{R} E(k_{\text{enc}}, m)$, then MAC the ciphertext, $\text{tag} \xleftarrow{R} S(k_{\text{mac}}, c)$; the result is the ciphertext-tag pair (c, tag). This method is supported in the TLS 1.2 protocol and later versions as well as in the IPsec protocol and in a widely-used NIST standard called GCM (see Section 9.7).

MAC-then-encrypt MAC the message, $\text{tag} \xleftarrow{R} S(k_{\text{mac}}, m)$, then encrypt the message-tag pair, $c \xleftarrow{R} E\big(k_{\text{enc}},\ (m, t)\big)$; the result is the ciphertext c. This method is used in older versions of TLS (e.g., SSL 3.0 and its successor called TLS 1.0) and in the 802.11i WiFi encryption protocol.

As it turns out, only the first method is secure for every combination of CPA-secure cipher and secure MAC. The intuition is that the MAC on the ciphertext prevents any tampering with the ciphertext. We will show that the second method can be insecure — the MAC and cipher can

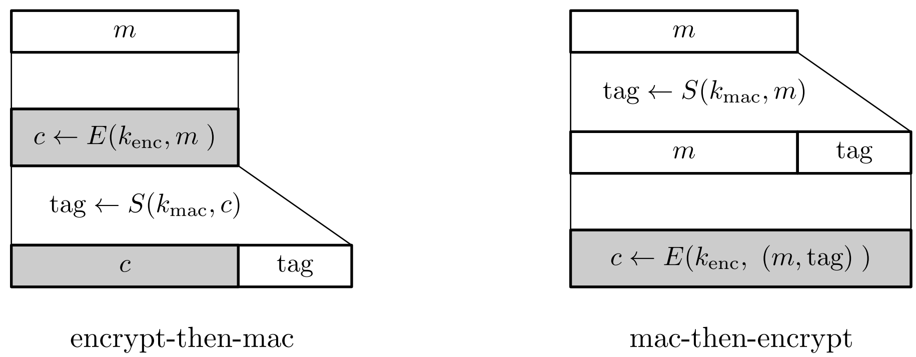


Figure 9.1: Two methods to combine encryption and MAC

interact badly and cause the resulting system to not be AE-secure. This has lead to many attacks on widely deployed systems.

The second paradigm for building authenticated encryption is to build them directly from a block cipher or a PRF without first constructing either a standalone cipher or MAC. These are sometimes called **integrated schemes**. The OCB encryption mode is the primary example in this category (see Exercise 9.17). Other examples include IAPM, XCBC, CCFB, and others.

Authenticated encryption standards. Cryptographic libraries such as OpenSSL often provide an interface for CPA-secure encryption (such as counter mode with a random IV) and a separate interface for computing MACs on messages. In the past, it was up to developers to correctly combine these two primitives to provide authenticated encryption. Every system did it differently and not all incarnations used in practice were secure.

More recently, several standards have emerged for secure authenticated encryption. A popular method called Galois Counter Mode (GCM) uses encrypt-then-MAC to combine random counter mode encryption with a Carter-Wegman MAC (see Section 9.7). We will examine the details of this construction and its security later on in the chapter. Developers are encouraged to use an authenticated encryption mode provided by the underlying cryptographic library and to not implement it themselves.

9.1 Authenticated encryption: definitions

We start by defining what it means for a cipher $\mathcal{E}$ to provide authenticated encryption. It must satisfy two properties. First, $\mathcal{E}$ must be CPA-secure. Second, $\mathcal{E}$ must provide ciphertext integrity, as defined below. Ciphertext integrity is a new property that captures the fact that $\mathcal{E}$ should have properties similar to a MAC. Let $\mathcal{E} = (E, D)$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. We define ciphertext integrity using the following attack game, shown in Fig. 9.2. The game is analogous to the MAC Attack Game 6.1.

Attack Game 9.1 (ciphertext integrity). For a given cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger chooses a random $k \xleftarrow{R} \mathcal{K}$.

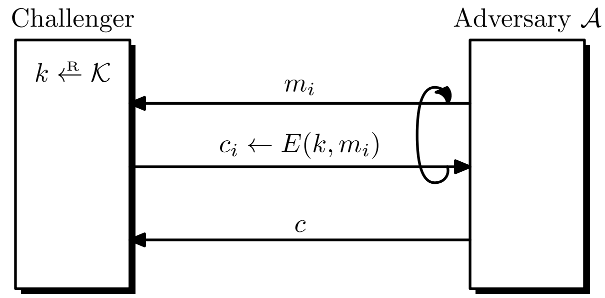


Figure 9.2: Ciphertext integrity game (Attack Game 9.1)

- $\mathcal{A}$ queries the challenger several times. For $i = 1, 2, \ldots$, the ith query consists of a message $m_i \in \mathcal{M}$. The challenger computes $c_i \xleftarrow{R} E(k, m_i)$, and gives c_i to $\mathcal{A}$.
- Eventually $\mathcal{A}$ outputs a candidate ciphertext $c \in \mathcal{C}$ that is not among the ciphertexts it was given, i.e.,
$$c \notin \{c_1, c_2, \ldots\}.$$

We say that $\mathcal{A}$ wins the game if c is a valid ciphertext under k, that is, $D(k, c) \neq \mathsf{reject}$. We define $\mathcal{A}$'s advantage with respect to $\mathcal{E}$, denoted $\mathrm{CI}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$, as the probability that $\mathcal{A}$ wins the game. Finally, we say that $\mathcal{A}$ is a **Q-query adversary** if $\mathcal{A}$ issues at most Q encryption queries. $\square$

Definition 9.1. *We say that a $\mathcal{E} = (E, D)$ provides **ciphertext integrity**, or* CI *for short, if for every efficient adversary $\mathcal{A}$, the value $\mathrm{CI}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.*

CPA security and ciphertext integrity are the properties needed for authenticated encryption. This is captured in the following definition.

Definition 9.2. *We say that a cipher $\mathcal{E} = (E, D)$ provides **authenticated encryption**, or is simply **AE-secure**, if $\mathcal{E}$ is (1) semantically secure under a chosen plaintext attack, and (2) provides ciphertext integrity.*

Why is Definition 9.2 the right definition? In particular, why are we requiring *ciphertext* integrity, rather than some notion of *plaintext* integrity (which might seem more natural)? In Section 9.2, we will describe a very insidious class of attacks called *chosen ciphertext attacks*, and we will see that our definition of AE-security is sufficient (and, indeed, necessary) to prevent such attacks. In Section 9.3, we give a more high-level justification for the definition.

One-time authenticated encryption

In practice, one often uses a symmetric key to encrypt a single message. The key is never used again. For example, when sending encrypted email one often picks an ephemeral key and encrypts the email body under this ephemeral key. The ephemeral key is then encrypted and transmitted in the email header. A new ephemeral key is generated for every email.

In these settings one can use a one-time encryption scheme such as a stream cipher. The cipher must be semantically secure, but need not be CPA-secure. Similarly, it suffices that the cipher provide one-time ciphertext integrity, which is a weaker notion than ciphertext-integrity. In particular, we change Attack Game 9.1 so that the adversary can only obtain the encryption of a single message m.

Definition 9.3. *We say that $\mathcal{E} = (E, D)$ provides* ***one-time ciphertext integrity*** *if for every efficient single-query adversary $\mathcal{A}$, the value* $\mathrm{CI}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

Definition 9.4. *We say that $\mathcal{E} = (E, D)$ provides* ***one-time authenticated encryption****, or is* 1AE*-**secure** for short, if $\mathcal{E}$ is semantically secure and provides one-time ciphertext integrity.*

In applications that only use a symmetric key once, 1AE-security suffices. We will show that the encrypt-then-MAC construction of Fig. 9.1 using a semantically secure cipher and a one-time MAC, provides one-time authenticated encryption. Replacing the MAC by a one-time MAC can lead to efficiency improvements.

9.2 Implications of authenticated encryption

Before constructing AE-secure systems, let us first play with Definition 9.1 a bit to see what it implies. Consider a sender, Alice, and a receiver, Bob, who have a shared secret key k. Alice sends a sequence of messages to Bob over a public network. Each message is encrypted with an AE-secure cipher $\mathcal{E} = (E, D)$ using the key k.

For starters, consider an eavesdropping adversary $\mathcal{A}$. Since $\mathcal{E}$ is CPA-secure this does not help $\mathcal{A}$ learn any new information about messages sent from Alice to Bob.

Now consider a more aggressive adversary $\mathcal{A}$ that attempts to make Bob receive a message that was not sent by Alice. We claim this cannot happen. To see why, consider the following single-message example: Alice encrypts to Bob a message m and the resulting ciphertext c is intercepted by $\mathcal{A}$. The adversary's goal is to create some $\hat{c}$ such that $\hat{m} := D(k, \hat{c}) \neq \mathsf{reject}$ and $\hat{m} \neq m$. This $\hat{c}$ would fool Bob into thinking that Alice sent $\hat{m}$ rather than m. But then $\mathcal{A}$ could also win Attack Game 9.1 with respect to $\mathcal{E}$, contradicting $\mathcal{E}$'s ciphertext integrity. Consequently, $\mathcal{A}$ cannot modify c without being detected. More generally, applying the argument to multiple messages shows that $\mathcal{A}$ cannot cause Bob to receive any messages that were not sent by Alice. The more general conclusion here is that *ciphertext integrity* implies *message integrity*.

9.2.1 Chosen ciphertext attacks: a motivating example

We now consider an even more aggressive type of attack, called a **chosen ciphertext attack** for short. As we will see, an AE-secure cipher provides message secrecy and message integrity even against such a powerful attack.

To motivate chosen ciphertext attacks suppose Alice sends an email message to Bob. For simplicity let us assume that every email starts with the letters `To:` followed by the recipient's email address. So, an email to Bob starts with `To:bob@mail.com` and an email to Mel begins with `To:mel@mail.com`. The mail server decrypts every incoming email and writes it into the recipient's inbox: emails that start with `To:bob@mail.com` are written to Bob's inbox and emails that start with `To:mel@mail.com` are written to Mel's inbox.

Mel, the attacker in this story, wants to read the email that Alice sent to Bob. Unfortunately for Mel, Alice was careful and encrypted the email using a key known only to Alice and to the mail server. When the ciphertext c is received at the mail server it will be decrypted and the resulting message is placed into Bob's inbox. Mel will be unable to read it.

Nevertheless, let us show that if Alice encrypts the email with a CPA-secure cipher such as randomized counter mode or randomized CBC mode then Mel can quite easily obtain the email contents. Here is how: Mel will intercept the ciphertext c en-route to the mail server and modify it to obtain a ciphertext $\hat{c}$ so that the decryption of $\hat{c}$ starts with `To:mel@mail.com`, but is otherwise the same as the original message. Mel then forwards $\hat{c}$ to the mail server. When the mail server receives $\hat{c}$ it will decrypt it and (incorrectly) place the plaintext into Mel's inbox where Mel can easily read it.

To successfully carry out this attack, Mel must first solve the following problem: given an encryption c of some message $(u \parallel m)$ where u is a fixed known prefix (in our case $u :=$ `To:bob@mail.com`), compute a ciphertext $\hat{c}$ that will decrypt to the message $(v \parallel m)$, where v is some other prefix (in our case $v :=$ `To:mel@mail.com`).

Let us show that Mel can easily solve this problem, assuming the encryption scheme is either randomized counter mode or randomized CBC. For simplicity, we also assume that u and v are binary strings whose length is the same as the block size of the underlying block cipher. As usual $c[0]$ and $c[1]$ are the first and second blocks of c where $c[0]$ is the random IV. Mel constructs $\hat{c}$ as follows:

- randomized counter mode: define $\hat{c}$ to be the same as c except that $\hat{c}[1] := c[1] \oplus u \oplus v$.
- randomized CBC mode: define $\hat{c}$ to be the same as c except that $\hat{c}[0] := c[0] \oplus u \oplus v$.

It is not difficult to see that in either case the decryption of $\hat{c}$ starts with the prefix v (see Section 3.3.2). Mel is now able to obtain the decryption of $\hat{c}$ and read the secret message m in the clear.

What just happened? We proved that both encryption modes are CPA secure, and yet we just showed how to break them. This attack is an example of a chosen ciphertext attack — by querying for the decryption of $\hat{c}$, Mel was able to deduce the decryption of c. This attack is also another demonstration of how attackers can exploit the *malleability* of a cipher — we saw another attack based on malleability back in Section 3.3.2.

As we just saw, a CPA-secure system can become completely insecure when an attacker can decrypt certain ciphertexts, even if he cannot directly decrypt a ciphertext that interests him. Put another way, the lack of ciphertext integrity can completely compromise secrecy — even if plaintext integrity is not an explicit security requirement.

We informally argue that if Alice used an AE-secure cipher $\mathcal{E} = (E, D)$ then it would be impossible to mount the attack we just described. Suppose Mel intercepts a ciphertext $c := E(k, m)$. He tries to create another ciphertext $\hat{c}$ such that (1) $\hat{m} := D(k, \hat{c})$ starts with prefix v, and (2) the adversary can recover m from $\hat{m}$, in particular $\hat{m} \neq \mathsf{reject}$. Ciphertext integrity, and therefore AE-security, implies that the attacker cannot create this $\hat{c}$. In fact, the attacker cannot create any new valid ciphertexts and therefore an AE-secure cipher foils the attack.

In the next section, we formally define the notion of a chosen ciphertext attack, and show that if a cipher is AE-secure then it is secure even against this type of attack.

9.2.2 Chosen ciphertext attacks: definition

In this section, we formally define the notion of a *chosen ciphertext attack*. In such an attack, the adversary has all the power of an attacker in a chosen plaintext attack, but in addition, the adversary may obtain decryptions of ciphertexts of its choosing — subject to a restriction. Recall that in a chosen plaintext attack, the adversary obtains a number of ciphertexts from its challenger, in response to encryption queries. The restriction we impose is that the adversary may not ask for the decryptions of any of these ciphertexts. While such a restriction is necessary to make the attack game at all meaningful, it may also seem a bit unintuitive: if the adversary can decrypt ciphertexts of choosing, why would it not decrypt the most important ones? We will explain later (in Section 9.3) more of the intuition behind this definition. We will show below (in Section 9.2.3) that if a cipher is AE-secure then it is secure against chosen ciphertext attack.

Here is the formal attack game:

Attack Game 9.2 (CCA security). For a given cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, we define two experiments. For $b = 0, 1$, we define

Experiment b:

- The challenger selects $k \xleftarrow{R} \mathcal{K}$.
- $\mathcal{A}$ then makes a series of queries to the challenger. Each query can be one of two types:
  - *Encryption query:* for $i = 1, 2, \ldots$, the ith encryption query consists of a pair of messages $(m_{i0}, m_{i1}) \in \mathcal{M}^2$. The challenger computes $c_i \xleftarrow{R} E(k, m_{ib})$ and sends c_i to $\mathcal{A}$.
  - *Decryption query:* for $j = 1, 2, \ldots$, the jth decryption query consists of a ciphertext $\hat{c}_j \in \mathcal{C}$ that is not among the responses to the previous encryption queries, i.e.,

    $$\hat{c}_j \notin \{c_1, c_2, \ldots\}.$$

    The challenger computes $\hat{m}_j \leftarrow D(k, \hat{c}_j)$, and sends $\hat{m}_j$ to $\mathcal{A}$.
- At the end of the game, the adversary outputs a bit $\hat{b} \in \{0, 1\}$.

Let W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b and define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := \big|\Pr[W_0] - \Pr[W_1]\big|. \quad \Box$$

We stress that in the above attack game, the encryption and decryption queries may be arbitrarily interleaved with one another.

Definition 9.5 (CCA security). *A cipher $\mathcal{E}$ is called **semantically secure against chosen ciphertext attack**, or simply **CCA-secure**, if for all efficient adversaries $\mathcal{A}$, the value $\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.*

In some settings, a new key is generated for every message so that a particular key k is only used to encrypt a single message. The system needs to be secure against chosen ciphertext attacks where the attacker fools the user into decrypting multiple ciphertexts using k. For these settings we define security against an adversary that can only issue a *single* encryption query, but many decryption queries.

Definition 9.6 (1CCA security). *In Attack Game 9.2, if the adversary $\mathcal{A}$ is restricted to making a single encryption query, we denote its advantage by $1\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$. A cipher $\mathcal{E}$ is **one-time semantically secure against chosen ciphertext attack**, or simply, **1CCA-secure**, if for all efficient adversaries $\mathcal{A}$, the value $1\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.*

As discussed in Section 2.3.5, Attack Game 9.2 can be recast as a "bit guessing" game, where instead of having two separate experiments, the challenger chooses $b \in \{0, 1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$. In this game, we measure $\mathcal{A}$'s *bit-guessing advantage* $\mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]$ (and $1\mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]$) as $|\Pr[\hat{b} = b] - 1/2|$. The general result of Section 2.3.5 (namely, (2.13)) applies here as well:

$$\mathrm{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]. \tag{9.1}$$

And similarly, for adversaries restricted to a single encryption query, we have:

$$1\mathrm{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 2 \cdot 1\mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]. \tag{9.2}$$

9.2.3 Authenticated encryption implies chosen ciphertext security

We now show that every AE-secure system is also CCA-secure. Similarly, every 1AE-secure system is 1CCA-secure.

Theorem 9.1. *Let $\mathcal{E} = (E, D)$ be a cipher. If $\mathcal{E}$ is AE-secure, then it is CCA-secure. If $\mathcal{E}$ is* 1AE-*secure, then it is 1CCA-secure.*

In particular, suppose $\mathcal{A}$ is a CCA-adversary for $\mathcal{E}$ that makes at most Q_e encryption queries and Q_d decryption queries. Then there exist a CPA-adversary $\mathcal{B}_\mathrm{cpa}$ and a CI-*adversary $\mathcal{B}_\mathrm{ci}$, where $\mathcal{B}_\mathrm{cpa}$ and $\mathcal{B}_\mathrm{ci}$ are elementary wrappers around $\mathcal{A}$, such that*

$$\mathrm{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \le \mathrm{CPA}\mathsf{adv}[\mathcal{B}_\mathrm{cpa}, \mathcal{E}] + 2Q_\mathrm{d} \cdot \mathrm{CI}\mathsf{adv}[\mathcal{B}_\mathrm{ci}, \mathcal{E}]. \tag{9.3}$$

Moreover, $\mathcal{B}_\mathrm{cpa}$ and $\mathcal{B}_\mathrm{ci}$ both make at most Q_e encryption queries.

Before proving this theorem, we point out a converse of sorts: if a cipher is CCA-secure and provides *plaintext* integrity, then it must be AE-secure. You are asked to prove this in Exercise 9.15. These two results together provide strong support for the claim that AE-security is the *right* notion of security for general purpose communication over an insecure network. We also note that it is possible to build a CCA-secure cipher that does not provide ciphertext (or plaintext) integrity — see Exercise 9.12 for an example.

Proof idea. A CCA-adversary $\mathcal{A}$ issues encryption and allowed decryption queries. We first argue that the response to all these decryption queries must be reject. To see why, observe that if the adversary ever issues a valid decryption query c_i whose decryption is not reject, then this c_i can be used to win the ciphertext integrity game. Hence, since all of $\mathcal{A}$'s decryption queries are rejected, the adversary learns nothing by issuing decryption queries and they may as well be discarded. After removing decryption queries we end up with a standard CPA game. The adversary cannot win this game because $\mathcal{E}$ is CPA-secure. We conclude that $\mathcal{A}$ has negligible advantage in winning the CCA game. □

Proof. Let $\mathcal{A}$ be an efficient CCA-adversary attacking $\mathcal{E}$ as in Attack Game 9.2, and which makes at most Q_e encryption queries and Q_d decryption queries. We want to show that $\mathrm{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible, assuming that $\mathcal{E}$ is AE-secure. We will use the bit-guessing versions of the CCA and CPA attack games, and show that

$$\mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}] \le \mathrm{CPA}\mathsf{adv}^*[\mathcal{B}_\mathrm{cpa}, \mathcal{E}] + Q_\mathrm{d} \cdot \mathrm{CI}\mathsf{adv}[\mathcal{B}_\mathrm{ci}, \mathcal{E}]. \tag{9.4}$$

for efficient adversaries $\mathcal{B}_{\mathrm{cpa}}$ and $\mathcal{B}_{\mathrm{ci}}$. Then (9.3) follows from (9.4), along with (9.1) and (5.4). Moreover, as we shall see, the adversary $\mathcal{B}_{\mathrm{cpa}}$ makes at most Q_{e} encryption queries; therefore, if $\mathcal{E}$ is 1AE-secure, it is also 1CCA-secure.

Let us define Game 0 to be the bit-guessing version of Attack Game 9.2. The challenger in this game, called Game 0, works as follows:

$b \xleftarrow{\mathrm{R}} \{0,1\}$ // *$\mathcal{A}$ will try to guess b*
$k \xleftarrow{\mathrm{R}} \mathcal{K}$

upon receiving the ith encryption query (m_{i0}, m_{i1}) from $\mathcal{A}$ do:
    send $c_i \xleftarrow{\mathrm{R}} E(k, m_b)$ to $\mathcal{A}$

upon receiving the jth decryption query $\hat{c}_j$ from $\mathcal{A}$ do:
(1)   send $D(k, \hat{c}_j)$ to $\mathcal{A}$

Eventually the adversary outputs a guess $\hat{b} \in \{0,1\}$. We say that $\mathcal{A}$ wins the game if $b = \hat{b}$ and we denote this event by W_0. By definition, the bit-guessing advantage is

$$\mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}] = |\Pr[W_0] - 1/2|. \tag{9.5}$$

Game 1. We now modify line (1) in the challenger as follows:

(1)   send reject to $\mathcal{A}$

We argue that $\mathcal{A}$ cannot distinguish this challenger from the original. Let Z be the event that in Game 1, $\mathcal{A}$ issues a decryption query $\hat{c}_j$ such that $D(k, \hat{c}_j) \neq \mathsf{reject}$. Clearly, Games 0 and 1 proceed identically as long as Z does not happen. Hence, by the Difference Lemma (i.e., Theorem 4.7) it follows that $|\Pr[W_0] - \Pr[W_1]| \leq \Pr[Z]$.

Using a "guessing strategy" similar to that used in the proof of Theorem 6.1, we can use $\mathcal{A}$ to build a CI-adversary $\mathcal{B}_{\mathrm{ci}}$ that wins the CI attack game with probability at least $\Pr[Z]/Q_{\mathrm{d}}$. Note that in Game 1, the decryption algorithm is not used at all. Adversary $\mathcal{B}_{\mathrm{ci}}$'s strategy is simply to guess a random number $\omega \in \{1, \ldots, Q_{\mathrm{d}}\}$, and then to play the role of challenger to $\mathcal{A}$:

- when $\mathcal{A}$ makes an encryption query, $\mathcal{B}_{\mathrm{ci}}$ forwards this to its own challenger, and returns the response to $\mathcal{A}$;
- when $\mathcal{A}$ makes a decryption query $\hat{c}_j$, $\mathcal{B}_{\mathrm{ci}}$ simply sends reject to $\mathcal{A}$, except that if $j = \omega$, $\mathcal{B}_{\mathrm{ci}}$ outputs $\hat{c}_j$ and halts.

It is not hard to see that $\mathrm{CI}\mathsf{adv}[\mathcal{B}_{\mathrm{ci}}, \mathcal{E}] \geq \Pr[Z]/Q_{\mathrm{d}}$, and so

$$|\Pr[W_0] - \Pr[W_1]| \leq \Pr[Z] \leq Q_{\mathrm{d}} \cdot \mathrm{CI}\mathsf{adv}[\mathcal{B}_{\mathrm{ci}}, \mathcal{E}]. \tag{9.6}$$

Final reduction. Since all decryption queries are rejected in Game 1, this is essentially a CPA attack game. More precisely, we can construct a CPA adversary $\mathcal{B}_{\mathrm{cpa}}$ that plays the role of challenger to $\mathcal{A}$ as follows:

- when $\mathcal{A}$ makes an encryption query, $\mathcal{B}_{\mathrm{cpa}}$ forwards this to its own challenger, and returns the response to $\mathcal{A}$;
- when $\mathcal{A}$ makes a decryption query, $\mathcal{B}_{\mathrm{cpa}}$ simply sends reject to $\mathcal{A}$.

At the end of the game, $\mathcal{B}_{\mathrm{cpa}}$ simply outputs the bit $\hat{b}$ that $\mathcal{A}$ outputs. Clearly,

$$|\Pr[W_1] - 1/2| = \mathrm{CPA}\mathsf{adv}^*[\mathcal{B}_{\mathrm{cpa}}, \mathcal{E}] \tag{9.7}$$

Putting equations (9.5)–(9.7) together gives us (9.4), which proves the theorem. □

9.3 Encryption as an abstract interface

To further motivate the definition of authenticated encryption we show that it precisely captures an intuitive notion of secure encryption as an *abstract interface*. AE-security implies that the real implementation of this interface may be replaced by an idealized implementation in which messages literally jump from sender to receiver, without going over the network at all (even in encrypted form). We now develop this idea more fully.

Suppose a sender S and receiver R are using some arbitrary Internet-based system (e.g, gambling, auctions, banking — whatever). Also, we assume that S and R have already established a shared, random encryption key k. During the protocol, S will send encryptions of messages $m_1, m_2, \ldots$ to R. The messages m_i are determined by the logic of the protocol S is using, whatever that happens to be. We can imagine S placing a message m_i in his "out-box", the precise details of how the out-box works being of no concern to S. Of course, inside S's out-box, we know what happens: an encryption c_i of m_i under k is computed, and this is sent out over the wire to R.

On the receiving end, when a ciphertext $\hat{c}$ is received at R's end of the wire, it is decrypted using k, and if the decryption is a message $\hat{m} \neq \mathsf{reject}$, the message $\hat{m}$ is placed in R's "in-box". Whenever a message appears in his in-box, R can retrieve it and processes it according to the logic of his protocol, without worrying about how the message got there.

An attacker may try to subvert communication between S and R in a number of ways.

- First, the attacker may drop, re-order, or duplicate the ciphertexts sent by S.
- Second, the attacker may modify ciphertexts sent by S, or inject ciphertexts created out of "whole cloth".
- Third, the attacker may have partial knowledge of some of the messages sent by S, or may even be able to influence the choice of some of these messages.
- Fourth, by observing R's behavior, the attacker may be able to glean partial knowledge of some of the messages processed by R. Even the knowledge of whether or not a ciphertext delivered to R was rejected could be useful.

Having described an abstract encryption interface and its implementation, we now describe an *ideal implementation* of this interface that captures in an intuitive way the guarantees ensured by authenticated encryption. When S drops m_i in its out-box, instead of encrypting m_i, the ideal implementation creates a ciphertext c_i by encrypting a dummy message $dummy_i$, that has nothing to do with m_i (except that it should be of the same length). Thus, c_i serves as a "handle" for m_i, but does not contain any information about m_i (other than its length). When c_i arrives at R, the corresponding message m_i is magically copied from S's out-box to R's in-box. If a ciphertext arrives at R that is not among the previously generated c_i's, the ideal implementation simply discards it.

This ideal implementation is just a thought experiment. It obviously cannot be physically realized in any efficient way (without first inventing teleportation). As we shall argue, however, if the underlying cipher $\mathcal{E}$ provides authenticated encryption, the ideal implementation is — for all practical purposes — equivalent to the real implementation. Therefore, a protocol designer need not worry about any of the details of the real implementation or the nuances of cryptographic definitions: he can simply pretend he is using the abstract encryption interface with its ideal implementation, in which ciphertexts are just handles and messages magically jump from S to R.

Hopefully, analyzing the security properties of the higher-level protocol will be much easier in this setting.

Note that even in the ideal implementation, the attacker may still drop, re-order, or duplicate ciphertexts, and these will cause the corresponding messages to be dropped, re-ordered, or duplicated. Using sequence numbers and buffers, it is not hard to deal with these possibilities, but that is left to the higher-level protocol.

We now argue informally that when $\mathcal{E}$ provides authenticated encryption, the real world implementation is indistinguishable from the ideal implementation. The argument proceeds in three steps. We start with the real implementation, and in each step, we make a slight modification.

- First, we modify the real implementation of R's in-box, as follows. When a ciphertext $\hat{c}$ arrives on R's end, the list of ciphertexts $c_1, c_2, \ldots$ previously generated by S is scanned, and if $\hat{c} = c_i$, then the corresponding message m_i is magically copied from S's out-box into R's in-box, without actually running the decryption algorithm.

  The correctness property of $\mathcal{E}$ ensures that this modification behaves exactly the same as the real implementation.

- Second, we modify the implementation on R's in-box again, so that if a ciphertext $\hat{c}$ arrives on R's end that is not among the ciphertexts generated by S, the implementation simply discards $\hat{c}$.

  The only way the adversary could distinguish this modification from the first is if he could create a ciphertext that would not be rejected and was not generated by S. But this is not possible, since $\mathcal{E}$ has ciphertext integrity.

- Third, we modify the implementation of S's out-box, replacing the encryption of m_i with the encryption of dummy_i. The implementation of R's in-box remains as in the second modification. Note that the decryption algorithm is never used in either the second or third modifications. Therefore, an adversary who can distinguish this modification from the second can be used to directly break the CPA-security of $\mathcal{E}$. Hence, since $\mathcal{E}$ is CPA-secure, the two modifications are indistinguishable.

Since the third modification is identical to the ideal implementation, we see that the real and ideal implementations are indistinguishable from the adversary's point of view.

A technical point we have not considered is the possibility that the c_i's generated by S are not unique. Certainly, if we are going to view the c_i's as handles in the ideal implementation, uniqueness would seem to be an essential property. In fact, CPA-security implies that the c_i's generated in the ideal implementation are unique with overwhelming probability — see Exercise 5.11.

9.4 Authenticated encryption ciphers from generic composition

We now turn to constructing authenticated encryption by combining a CPA-secure cipher and a secure MAC. We show that encrypt-then-MAC is always AE-secure, but MAC-then-encrypt is not.

9.4.1 Encrypt-then-MAC

Let $\mathcal{E} = (E, D)$ be a cipher defined over $(\mathcal{K}_\mathrm{e}, \mathcal{M}, \mathcal{C})$ and let $\mathcal{I} = (S, V)$ be a MAC defined over $(\mathcal{K}_\mathrm{m}, \mathcal{C}, \mathcal{T})$. The **encrypt-then-MAC** system $\mathcal{E}_\mathrm{EtM} = (E_\mathrm{EtM}, D_\mathrm{EtM})$, or EtM for short, is defined as follows:

$$
\begin{aligned}
E_\mathrm{EtM}(\ (k_\mathrm{e}, k_\mathrm{m}),\ m) \quad &:= \quad c \xleftarrow{R} E(k_\mathrm{e}, m), \quad t \xleftarrow{R} S(k_\mathrm{m}, c) \\
& \quad \text{Output } (c, t) \\
D_\mathrm{EtM}((k_\mathrm{e}, k_\mathrm{m}),\ (c, t)\ ) \quad &:= \quad \text{if } V(k_\mathrm{m}, c, t) = \mathsf{reject} \text{ then output } \mathsf{reject} \\
& \quad \text{otherwise, output } D(k_\mathrm{e}, c)
\end{aligned}
$$

The EtM system is defined over $(\mathcal{K}_\mathrm{e} \times \mathcal{K}_\mathrm{m},\ \mathcal{M},\ \mathcal{C} \times \mathcal{T})$. The following theorem shows that $\mathcal{E}_\mathrm{EtM}$ provides authenticated encryption.

Theorem 9.2. *Let $\mathcal{E} = (E, D)$ be a cipher and let $\mathcal{I} = (S, V)$ be a MAC system. Then $\mathcal{E}_\mathrm{EtM}$ is* AE*-secure assuming $\mathcal{E}$ is CPA-secure and $\mathcal{I}$ is a secure MAC system. Also, $\mathcal{E}_\mathrm{EtM}$ is* 1AE*-secure assuming $\mathcal{E}$ is semantically secure and $\mathcal{I}$ is a one-time secure MAC system.*

In particular, for every ciphertext integrity adversary $\mathcal{A}_\mathrm{ci}$ that attacks $\mathcal{E}_\mathrm{EtM}$ as in Attack Game 9.1 there exists a MAC adversary $\mathcal{B}_\mathrm{mac}$ that attacks $\mathcal{I}$ as in Attack Game 6.1, where $\mathcal{B}_\mathrm{mac}$ is an elementary wrapper around $\mathcal{A}_\mathrm{ci}$, and which makes no more signing queries than $\mathcal{A}_\mathrm{ci}$ makes encryption queries, such that

$$\mathrm{CI}\mathsf{adv}[\mathcal{A}_\mathrm{ci}, \mathcal{E}_\mathrm{EtM}] = \mathrm{MAC}\mathsf{adv}[\mathcal{B}_\mathrm{mac}, \mathcal{I}].$$

For every CPA adversary $\mathcal{A}_\mathrm{cpa}$ that attacks $\mathcal{E}_\mathrm{EtM}$ as in Attack Game 5.2 there exists a CPA adversary $\mathcal{B}_\mathrm{cpa}$ that attacks $\mathcal{E}$ as in Attack Game 5.2, where $\mathcal{B}_\mathrm{cpa}$ is an elementary wrapper around $\mathcal{A}_\mathrm{cpa}$, and which makes no more encryption queries than does $\mathcal{A}_\mathrm{cpa}$, such that

$$\mathrm{CPA}\mathsf{adv}[\mathcal{A}_\mathrm{cpa}, \mathcal{E}_\mathrm{EtM}] = \mathrm{CPA}\mathsf{adv}[\mathcal{B}_\mathrm{cpa}, \mathcal{E}].$$

Proof. Let us first show that $\mathcal{E}_\mathrm{EtM}$ provides ciphertext integrity. The proof is by a straight forward reduction. Suppose $\mathcal{A}_\mathrm{ci}$ is a ciphertext integrity adversary attacking $\mathcal{E}_\mathrm{EtM}$. We construct a MAC adversary $\mathcal{B}_\mathrm{mac}$ attacking $\mathcal{I}$.

Adversary $\mathcal{B}_\mathrm{mac}$ plays the role of adversary in a MAC attack game for $\mathcal{I}$. It interacts with a MAC challenger $\mathbf{C}_\mathrm{mac}$ that starts by picking a random $k_\mathrm{m} \xleftarrow{R} \mathcal{K}_\mathrm{m}$. Adversary $\mathcal{B}_\mathrm{mac}$ works by emulating a $\mathcal{E}_\mathrm{EtM}$ ciphertext integrity challenger for $\mathcal{A}_\mathrm{ci}$, as follows:

$k_\mathrm{e} \xleftarrow{R} \mathcal{K}_\mathrm{e}$
upon receiving a query $m_i \in \mathcal{M}$ from $\mathcal{A}_\mathrm{ci}$ do:
  $c_i \xleftarrow{R} E(k_\mathrm{e}, m_i)$
  Query $\mathbf{C}_\mathrm{mac}$ on c_i and obtain $t_i \xleftarrow{R} S(k_\mathrm{m}, c_i)$ in response
  Send (c_i, t_i) to $\mathcal{A}_\mathrm{ci}$ // *then* $(c_i, t_i) = E_\mathrm{EtM}(\ (k_\mathrm{e}, k_\mathrm{m}),\ m_i)$
eventually $\mathcal{A}_\mathrm{ci}$ outputs a ciphertext $(c, t) \in \mathcal{C} \times \mathcal{T}$
output the message-tag pair (c, t)

It should be clear that $\mathcal{B}_\mathrm{mac}$ responds to $\mathcal{A}_\mathrm{ci}$'s queries as in a real ciphertext integrity attack game. Therefore, with probability $\mathrm{CI}\mathsf{adv}[\mathcal{A}_\mathrm{ci}, \mathcal{E}_\mathrm{EtM}]$ adversary $\mathcal{A}_\mathrm{ci}$ outputs a ciphertext (c, t) that makes it win Attack Game 9.1 so that $(c, t) \notin \{(c_1, t_1), \ldots\}$ and $V(k_\mathrm{m}, c, t) = \mathsf{accept}$. It follows that (c, t)

is a message-tag pair that lets $\mathcal{B}_{\text{mac}}$ win the MAC attack game and therefore $\mathsf{CIadv}[\mathcal{A}_{\text{ci}}, \mathcal{E}_{\text{EtM}}] = \mathsf{MACadv}[\mathcal{B}_{\text{mac}}, \mathcal{I}]$, as required.

It remains to show that if $\mathcal{E}$ is CPA-secure then so is $\mathcal{E}_{\text{EtM}}$. This simply says that the tag included in the ciphertext, which is computed using the key k_{m} (and does not involve the encryption key k_{e} at all), does not help the attacker break CPA security of $\mathcal{E}_{\text{EtM}}$. This is straightforward and is left as an easy exercise (see Exercise 5.20). $\Box$

Recall that our definition of a secure MAC from Chapter 6 requires that given a message-tag pair (c, t) the attacker cannot come up with a new tag $t' \neq t$ such that (c, t') is a valid message-tag pair. At the time it seemed odd to require this: if the attacker already has a valid tag for c, why do we care if he finds another tag for c? Here we see that if the attacker could come with a new valid tag t' for c then he could break ciphertext integrity for EtM. From an EtM ciphertext (c, t) the attacker could construct a new valid ciphertext (c, t') and win the ciphertext integrity game. Our definition of secure MAC ensures that the attacker cannot modify an EtM ciphertext without being detected.

Common mistakes in implementing encrypt-then-MAC

A common mistake when implementing encrypt-then-MAC is to use the same key for the cipher and the MAC, i.e., setting $k_{\text{e}} = k_{\text{m}}$. The resulting system need not provide authenticated encryption and can be insecure, as shown in Exercise 9.8. In the proof of Theorem 9.2 we relied on the fact that the two keys k_{e} and k_{m} are chosen independently.

Another common mistake is to apply the MAC signing algorithm to only part of the ciphertext. We look at an example. Suppose the underlying CPA-secure cipher $\mathcal{E} = (E, D)$ is randomized CBC mode (Section 5.4.3) so that the encryption of a message m is $(r, c) \xleftarrow{\text{R}} E(k, c)$ where r is a random IV. When implementing encrypt-then-MAC $\mathcal{E}_{\text{EtM}} = (E_{\text{EtM}}, D_{\text{EtM}})$ the encryption algorithm is incorrectly defined as

$$E_{\text{EtM}}\big(\ (k_{\text{e}}, k_{\text{m}}),\ m\big) := \big\{\ (r, c) \xleftarrow{\text{R}} E(k_{\text{e}}, m),\ t \xleftarrow{\text{R}} S(k_{\text{m}}, c),\ \text{output } (r, c, t)\ \big\}.$$

Here, $E(k_{\text{e}}, m)$ outputs the ciphertext (r, c), but the MAC signing algorithm is only applied to c; the IV is not protected by the MAC. This mistake completely destroys ciphertext integrity: given a ciphertext (r, c, t) an attacker can create a new valid ciphertext (r', c, t) for some $r' \neq r$. The decryption algorithm will not detect this modification of the IV and will not output reject. Instead, the decryption algorithm will output $D\big(k_{\text{e}},\ (r', c)\big)$. Since (r', c, t) is a valid ciphertext the adversary wins the ciphertext integrity game. Even worse, if (r, c, t) is the encryption of a message m then changing (r, c, t) to $(r \oplus \Delta, c, t)$ for any Δ causes the CBC decryption algorithm to output a message m' where $m'[0] = m[0] \oplus \Delta$. This means that the attacker can change header information in the first block of m to any value of the attacker's choosing. An early edition of the ISO 19772 standard for authenticated encryption made precisely this mistake [81]. Similarly, in 2013 it was discovered that the `RNCryptor` facility in Apple's iOS, built for data encryption, used a faulty encrypt-then-MAC where the HMAC was not applied to the encryption IV [84].

Another pitfall to watch out for in an implementation is that no plaintext data should be output before the integrity tag over the entire message is verified. See Section 9.9 for an example of this.

9.4.2 MAC-then-encrypt is not generally secure: padding oracle attacks on SSL

Next, we consider the MAC-then-encrypt generic composition of a CPA secure cipher and a secure MAC. We show that this construction need not be AE-secure and can lead to many real-world problems.

To define MAC-then-encrypt precisely, let $\mathcal{I} = (S, V)$ be a MAC defined over $(\mathcal{K}_\mathrm{m}, \mathcal{M}, \mathcal{T})$ and let $\mathcal{E} = (E, D)$ be a cipher defined over $(\mathcal{K}_\mathrm{e},\ \mathcal{M} \times \mathcal{T},\ \mathcal{C})$. The **MAC-then-encrypt** system $\mathcal{E}_\mathrm{MtE} = (E_\mathrm{MtE}, D_\mathrm{MtE})$, or MtE for short, is defined as follows:

$$
\begin{array}{rcl}
E_\mathrm{MtE}(\ (k_\mathrm{e}, k_\mathrm{m}),\ m) & := & t \xleftarrow{\mathrm{R}} S(k_\mathrm{m}, m), \quad c \xleftarrow{\mathrm{R}} E(k_\mathrm{c},\ (m,t)\ ) \\
 & & \text{Output } c \\[1em]
D_\mathrm{EtM}((k_\mathrm{e}, k_\mathrm{m}),\ c\ ) & := & (m,t) \leftarrow D(k_\mathrm{e}, c) \\
 & & \text{if } V(k_\mathrm{m}, m, t) = \mathsf{reject} \text{ then output } \mathsf{reject} \\
 & & \text{otherwise, output } m
\end{array}
$$

The MtE system is defined over $(\mathcal{K}_\mathrm{e} \times \mathcal{K}_\mathrm{m},\ \mathcal{M},\ \mathcal{C})$.

A badly broken MtE cipher. We show that MtE is not guaranteed to be AE-secure even if $\mathcal{E}$ is a CPA-secure cipher and $\mathcal{I}$ is a secure MAC. In fact, MtE can fail to be secure for widely-used ciphers and MACs and this has lead to many significant attacks on deployed systems.

Consider the SSL 3.0 protocol used to protect WWW traffic for over two decades (the protocol is disabled in modern browsers). SSL 3.0 uses MtE to combine randomized CBC mode encryption and a secure MAC. We showed in Chapter 5 that randomized CBC mode encryption is CPA-secure, yet this combination is badly broken: an attacker can effectively decrypt all traffic using a chosen ciphertext attack. This leads to a devastating attack on SSL 3.0 called **POODLE** [18].

Let us assume that the underlying block cipher used in CBC operates on 16 byte blocks, as in AES. Recall that CBC mode encryption pads its input to a multiple of the block length and SSL 3.0 does so as follows: if a pad of length $p > 0$ bytes is needed, the scheme pads the message with $p-1$ arbitrary bytes and adds one additional byte whose value is set to $(p-1)$. If the message length is already a multiple of the block length (16 bytes) then SSL 3.0 adds a dummy block of 16 bytes where the last byte is set to 15 and the first 15 bytes are arbitrary. During decryption the pad is removed by reading the last byte and removing that many more bytes.

Concretely, the cipher $\mathcal{E}_\mathrm{MtE} = (E_\mathrm{MtE}, D_\mathrm{MtE})$ obtained from applying MtE to randomized CBC mode encryption and a secure MAC works as follows:

- $E_\mathrm{MtE}(\ (k_\mathrm{e}, k_\mathrm{m}),\ m)$: First use the MAC signing algorithm to compute a fixed-length tag $t \xleftarrow{\mathrm{R}} S(k_\mathrm{m}, m)$ for m. Next, encrypt $m \parallel t$ with randomized CBC encryption: pad the message and then encrypt in CBC mode using key k_e and a random IV. Thus, the following data is encrypted to generate the ciphertext c:

$$\boxed{\qquad\qquad \text{message } m \qquad\qquad} \boxed{\ \text{tag } t\ } \boxed{\ \text{pad } p\ } \tag{9.8}$$

  Notice that the tag t does not protect the integrity of the pad. We will exploit this to break CPA security using a chosen ciphertext attack.

- $D_\mathrm{MtE}(\ (k_\mathrm{e}, k_\mathrm{m}),\ c)$: Run CBC decryption to obtain the plaintext data in (9.8). Next, remove the pad p by reading the last byte in (9.8) and removing that many more bytes from the data (i.e., if the last byte is 3 then that byte is removed plus 3 additional bytes). Next, verify the MAC tag and if valid return the remaining bytes as the message. Otherwise, output reject.

Both SSL 3.0 and TLS 1.0 use a defective variant of randomized CBC encryption, discussed in Exercise 5.12, but this is not relevant to our discussion here. Here we will assume that a correct implementation of randomized CBC encryption is used.

The chosen ciphertext attack. We show a chosen ciphertext attack on the system $\mathcal{E}_{\text{MtE}}$ that lets the adversary decrypt any ciphertext of its choice. It follows that $\mathcal{E}_{\text{MtE}}$ need not be AE-secure, even though the underlying cipher is CPA-secure. Throughout this section we let (E, D) denote the block cipher used in CBC mode encryption. It operates on 16-byte blocks.

Suppose the adversary intercepts a valid ciphertext $c := E_{\text{MtE}}(\ (k_{\text{e}}, k_{\text{m}}),\ m)$ for some unknown message m. The length of m is such that after a MAC tag t is appended to m the length of $(m \parallel t)$ is a multiple of 16 bytes. This means that a full padding block of 16 bytes is appended during CBC encryption and the last byte of this pad is 15. Then the ciphertext c looks as follows:

$$c \quad = \quad \underbrace{\boxed{\quad c[0] \quad}}_{\text{IV}}\underbrace{\boxed{\quad c[1] \quad}\ \cdots\ }_{\text{encryption of m}}\underbrace{\boxed{c[\ell-1]}}_{\text{encrypted tag}}\underbrace{\boxed{\quad c[\ell] \quad}}_{\text{encrypted pad}}$$

Lets us first show that the adversary can learn something about $m[0]$ (the first 16-byte block of m). This will break semantic security of $\mathcal{E}_{\text{MtE}}$. The attacker prepares a chosen ciphertext query $\hat{c}$ by replacing the last block of c with $c[1]$. That is,

$$\hat{c} \quad := \quad \boxed{\quad c[0] \quad}\boxed{\quad c[1] \quad}\ \cdots\ \boxed{c[\ell-1]}\underbrace{\boxed{\quad c[1] \quad}}_{\text{encrypted pad?}} \tag{9.9}$$

By definition of CBC decryption, decrypting the last block of $\hat{c}$ yields the 16-byte plaintext block

$$v := D\big(k_{\text{e}}, c[1]\big) \oplus c[\ell-1] = m[0] \oplus c[0] \oplus c[\ell-1].$$

If the last byte of v is 15 then during decryption the entire last block will be treated as a padding block and removed. The remaining string is a valid message-tag pair and will decrypt properly. If the last byte of v is not 15 then most likely the response to the decryption query will be reject.

Put another way, if the response to a decryption query for $\hat{c}$ is not reject then the attacker learns that the last byte of $m[0]$ is equal to the last byte of $u := 15 \oplus c[0] \oplus c[\ell-1]$. Otherwise, the attacker learns that the last byte of $m[0]$ is not equal to the last byte of u. This directly breaks semantic security of the $\mathcal{E}_{\text{MtE}}$: the attacker learned something about the plaintext m.

We leave it as an instructive exercise to recast this attack in terms of an adversary in a chosen ciphertext attack game (as in Attack Game 9.2). With a single plaintext query followed by a single ciphertext query the adversary has advantage 1/256 in winning the game. This already proves that $\mathcal{E}_{\text{MtE}}$ is insecure.

Now, suppose the attacker obtains another encryption of m, call it c', using a different IV. The attacker can use the ciphertexts c and c' to form four useful chosen ciphertext queries: it can replace the last block of either c or c' with either of $c[1]$ or $c'[1]$. By issuing these four ciphertext queries the attacker learns if the last byte of $m[0]$ is equal to the last byte of one of

$$15 \oplus c[0] \oplus c[\ell-1], \qquad 15 \oplus c[0] \oplus c'[\ell-1], \qquad 15 \oplus c'[0] \oplus c[\ell-1], \qquad 15 \oplus c'[0] \oplus c'[\ell-1].$$

If these four values are distinct they give the attacker four chances to learn the last byte of $m[0]$. Repeating this multiple times with more fresh encryptions of the message m will quickly reveal the last byte of $m[0]$. Each chosen ciphertext query reveals that byte with probability $1/256$. Therefore, on average, with 256 chosen ciphertext queries the attacker learns the exact value of the last byte of $m[0]$. So, not only can the attacker break semantic security, the attacker can actually recover one byte of the plaintext. Next, suppose the adversary could request an encryption of m shifted one byte to the right to obtain a ciphertext c_1. Plugging $c_1[1]$ into the last block of the ciphertexts from the previous phase (i.e., encryptions of the unshifted m) and issuing the resulting chosen ciphertext queries reveals the second to last byte of $m[0]$. Repeating this for every byte of m eventually reveals all of m. We show next that this gives a real attack on SSL 3.0.

A complete break of SSL 3.0. Chosen ciphertext attacks may seem theoretical, but they frequently translate to devastating real-world attacks. Consider a Web browser and a victim Web server called `bank.com`. The two exchange information encrypted using SSL 3.0. The browser and server have a shared secret called a cookie and the browser embeds this cookie in every request that it sends to `bank.com`. That is, abstractly, requests from the browser to `bank.com` look like:

$$\boxed{\text{GET } \texttt{path} \quad \text{cookie: } \texttt{cookie}}$$

where `path` identifies the name of a resource being requested from `bank.com`. The browser only inserts the cookie into requests it sends to `bank.com`

The attacker's goal is to recover the secret cookie. First it makes the browser visit `attacker.com` where it sends a Javascript program to the browser. This Javascript program makes the browser issue a request for resource "/AA" at `bank.com`. The reason for this particular path is to ensure that the length of the message and MAC is a multiple of the block size (16 bytes), as needed for the attack. Consequently, the browser sends the following request to `bank.com`

$$\boxed{\text{GET /AA} \quad \text{cookie: } \texttt{cookie}} \tag{9.10}$$

encrypted using SSL 3.0. The attacker can intercept this encrypted request c and mounts the chosen ciphertext attack on MtE to learn one byte of the cookie. That is, the attacker prepares $\hat{c}$ as in (9.9), sends $\hat{c}$ to `bank.com` and looks to see if `bank.com` responds with an SSL error message. If no error message is generated then the attacker learns one byte of the cookie. The Javascript can cause the browser to repeatedly issue the request (9.10) giving the adversary the fresh encryptions needed to eventually learn one byte of the cookie.

Once the adversary learns one byte of the cookie it can shift the cookie one byte to the right by making the Javascript program issue a request to `bank.com` for

$$\boxed{\text{GET /AAA} \quad \text{cookie: } \texttt{cookie}}$$

This gives the attacker a block of ciphertext, call it $c_1[2]$, where the cookie is shifted one byte to the right. Resending the requests from the previous phase to the server, but now with the last block replaced by $c_1[2]$, eventually reveals the second byte of the cookie. Iterating this process for every byte of the cookie eventually reveals the entire cookie.

In effect, Javascript in the browser provides the attacker with the means to mount the desired chosen plaintext attack. Intercepting packets in the network, modifying them and observing the server's response, gives the attacker the means to mount the desired chosen ciphertext attack. The combination of these two completely breaks MtE encryption in SSL 3.0.

One minor detail is that whenever `bank.com` responds with an SSL error message the SSL session shuts down. This does not pose a problem: every request that the Javascript running in the browser makes to `bank.com` initiates a new SSL session. Hence, every chosen ciphertext query is encrypted under a different session key, but that makes no difference to the attack: every query tests if one byte of the cookie is equal to one known random byte. With enough queries the attacker learns the entire cookie.

9.4.3 More padding oracle attacks.

TLS 1.0 is an updated version of SSL 3.0. It defends against the attack of the previous section by adding structure to the pad as explained in Section 5.4.4: when padding with p bytes, all bytes of the pad are set to $p - 1$. Moreover, during decryption, the decryptor is required to check that all padding bytes have the correct value and reject the ciphertext if not. This makes it harder to mount the attack of the previous section. Of course our goal was merely to show that MtE is not generally secure and SSL 3.0 made that abundantly clear.

A padding oracle timing attack. Despite the defenses in TLS 1.0 a naive implementation of MtE decryption may still be vulnerable. Suppose the implementation works as follows: first it applies CBC decryption to the received ciphertext; next it checks that the pad structure is valid and if not it rejects the ciphertext; if the pad is valid it checks the integrity tag and if valid it returns the plaintext. In this implementations the integrity tag is checked only if the pad structure is valid. This means that a ciphertext with an invalid pad structure is rejected faster than a ciphertext with a valid pad structure, but an invalid tag. An attacker can measure the time that the server takes to respond to a chosen ciphertext query and if a TLS error message is generated quickly it learns that the pad structure was invalid. Otherwise, it learns that the pad structure was valid.

This timing channel is called a **padding oracle side-channel**. It is a good exercise to devise a chosen ciphertext attack based on this behavior to completely decrypt a secret cookie, as we did for SSL 3.0. To see how this might work, suppose an attacker intercepts an encrypted TLS 1.0 record c. Let m be the decryption of c. Say the attacker wishes to test if the last byte of $m[2]$ is equal to some fixed byte value b. Let B be an arbitrary 16-byte block whose last byte is b. The attacker creates a new ciphertext block $\hat{c}[1] := c[1] \oplus B$ and sends the 3-block record $\hat{c} = (c[0], \hat{c}[1], c[2])$ to the server. After CBC decryption of $\hat{c}$, the last plaintext block will be

$$\hat{m}[2] := \hat{c}[1] \oplus D(k, c[2]) = m[2] \oplus B.$$

If the last byte of $m[2]$ is equal to b then $\hat{m}[2]$ ends in zero which is a valid pad. The server will attempt to verify the integrity tag resulting in a slow response. If the last byte of $m[2]$ is not equal to b then $\hat{m}[2]$ will not end in 0 and will likely end in an invalid pad, resulting in a fast response. By measuring the response time the attacker learns if the last byte of $m[2]$ is equal to b. Repeating this with many chosen ciphertext queries, as we did for SSL 3.0, reveals the entire secret cookie.

An even more sophisticated padding oracle timing attack on MtE, as used in TLS 1.0, is called Lucky13 [3]. It is quite challenging to implement TLS 1.0 decryption in way that hides the timing information exploited by the Lucky13 attack.

Informative error messages. To make matters worse, the TLS 1.0 specification [31] states that the server should send one type of error message (called `bad_record_mac`) when a received ciphertext is rejected because of a MAC verification error and another type of error message (`decryption_failed`) when the ciphertext is rejected because of an invalid padding block. In principle, this tells the attacker if a ciphertext was rejected because of an invalid padding block or because of a bad integrity tag. This could have enabled the chosen ciphertext attack of the previous paragraph without needing to resort to timing measurements. Fortunately, the error messages are encrypted and the attacker cannot see the error code.

Nevertheless, there is an important lesson to be learned here: when decryption fails, the system should never explain why. A generic '`decryption_failed`' code should be sent without offering any other information. This issue was recognized and addressed in TLS 1.1. Moreover, upon decryption failure, a correct implementation should always take the same amount of time to respond, no matter the failure reason.

9.4.4 Secure instances of MAC-then-encrypt

Although MtE is not generally secure when applied to a CPA-secure cipher, it can be shown to be secure for specific CPA ciphers discussed in Chapter 5. We show in Theorem 9.3 below that if $\mathcal{E}$ happens to implement randomized counter mode, then MtE is secure. In Exercise 9.9 we show that the same holds for randomized CBC, assuming there is no message padding.

Theorem 9.3 shows that MAC-then-encrypt with randomized counter mode is AE-secure even if the MAC is only one-time secure. That is, it suffices to use a weak MAC that is only secure against an adversary that makes a *single* chosen message query. Intuitively, the reason we can prove security using such a weak MAC is that the MAC value is encrypted, and consequently it is harder for the adversary to attack the MAC. Since one-time MACs are a little shorter and faster than many-time MACs, MAC-then-encrypt with randomized counter mode has a small advantage over encrypt-then-MAC. Nevertheless, the attacks on MAC-then-encrypt presented in the previous section suggest that it is difficult to implement correctly, and should not be used.

Our starting point is a randomized counter-mode cipher $\mathcal{E} = (E, D)$, as discussed in Section 5.4.2. We will assume that $\mathcal{E}$ has the general structure as presented in the case study on AES counter mode at the end of Section 5.4.2 (page 189). Namely, we use a counter-mode variant where the cipher $\mathcal{E}$ is built from a secure PRF F defined over $(\mathcal{K}_\mathrm{e},\ \mathcal{X} \times \mathbb{Z}_\ell,\ \mathcal{Y})$, where $\mathcal{Y} := \{0,1\}^n$. More precisely, for a message $m \in \mathcal{Y}^{\leq \ell}$ algorithm E works as follows:

$$E(k_\mathrm{e},\ m) := \left\{ \begin{array}{l} x \xleftarrow{\mathrm{R}} \mathcal{X} \\ \text{for } j = 0 \text{ to } |m| - 1: \\ \qquad u[j] \leftarrow F\big(k_\mathrm{e},\ (x, j)\big) \oplus m[j] \\ \text{output } c := (x, u) \in \mathcal{X} \times \mathcal{Y}^{|m|} \end{array} \right\}$$

Algorithm $D(k_\mathrm{e}, c)$ is defined similarly. Let $\mathcal{I} = (S, V)$ be a secure one-time MAC defined over $(\mathcal{K}_\mathrm{m}, \mathcal{M}, \mathcal{T})$ where $\mathcal{M} := \mathcal{Y}^{\leq \ell_\mathrm{m}}$ and $\mathcal{T} := \mathcal{Y}^{\ell_\mathrm{t}}$, and where $\ell_\mathrm{m} + \ell_\mathrm{t} < \ell$.

The MAC-then-encrypt cipher $\mathcal{E}_\mathrm{MtE} = (E_\mathrm{MtE}, D_\mathrm{MtE})$, built from F and $\mathcal{I}$ and taking messages in $\mathcal{M}$, is defined as follows:

$$\begin{aligned} E_\mathrm{MtE}\big(\ (k_\mathrm{e}, k_\mathrm{m}),\ m\big) &:= \big\{\ t \xleftarrow{\mathrm{R}} S(k_\mathrm{m}, m),\quad c \xleftarrow{\mathrm{R}} E\big(k_\mathrm{e},\ (m \parallel t)\big),\quad \text{output } c\ \big\} \\ D_\mathrm{MtE}\big(\ (k_\mathrm{e}, k_\mathrm{m}),\ c\big) &:= \left\{ \begin{array}{l} (m \parallel t) \leftarrow D(k_\mathrm{e}, c) \\ \text{if } V(k_\mathrm{m}, m, t) = \mathsf{reject} \text{ then output } \mathsf{reject} \\ \text{otherwise, output } m \end{array} \right\} \end{aligned} \tag{9.11}$$

As we discussed at the end of Section 9.4.1, and in Exercise 9.8, the two keys k_e and k_m must be chosen independently. Setting $k_e = k_m$ will invalidate the following security theorem.

Theorem 9.3. *The cipher $\mathcal{E}_{\mathrm{MtE}} = (E_{\mathrm{MtE}}, D_{\mathrm{MtE}})$ in (9.11) built from the PRF F and MAC $\mathcal{I}$ provides authenticated encryption assuming $\mathcal{I}$ is a secure one-time MAC and F is a secure PRF where $1/|\mathcal{X}|$ is negligible.*

In particular, for every Q-query ciphertext integrity adversary $\mathcal{A}_{\mathrm{ci}}$ that attacks $\mathcal{E}_{\mathrm{MtE}}$ as in Attack Game 9.1 there exists two MAC adversaries $\mathcal{B}_{\mathrm{mac}}$ and $\mathcal{B}'_{\mathrm{mac}}$ that attack $\mathcal{I}$ as in Attack Game 6.1, and a PRF adversary $\mathcal{B}_{\mathrm{prf}}$ that attacks F as in Attack Game 4.2, each of which is an elementary wrapper around $\mathcal{A}_{\mathrm{ci}}$, such that

$$\begin{aligned} \mathrm{CI}\mathsf{adv}[\mathcal{A}_{\mathrm{ci}}, \mathcal{E}_{\mathrm{MtE}}] \le \mathrm{PRF}\mathsf{adv}[\mathcal{B}_{\mathrm{prf}}, F] &+ \\ Q \cdot \mathrm{MAC}_1\mathsf{adv}[\mathcal{B}_{\mathrm{mac}}, \mathcal{I}] + \mathrm{MAC}_1\mathsf{adv}[\mathcal{B}'_{\mathrm{mac}}, \mathcal{I}] &+ \frac{Q^2}{2|\mathcal{X}|}. \end{aligned} \tag{9.12}$$

For every CPA adversary $\mathcal{A}_{\mathrm{cpa}}$ that attacks $\mathcal{E}_{\mathrm{EtM}}$ as in Attack Game 5.2 there exists a CPA adversary $\mathcal{B}_{\mathrm{cpa}}$ that attacks $\mathcal{E}$ as in Attack Game 5.2, which is an elementary wrapper around $\mathcal{A}_{\mathrm{cpa}}$, such that

$$\mathrm{CPA}\mathsf{adv}[\mathcal{A}_{\mathrm{cpa}}, \mathcal{E}_{\mathrm{MtE}}] = \mathrm{CPA}\mathsf{adv}[\mathcal{B}_{\mathrm{cpa}}, \mathcal{E}]$$

Proof idea. CPA security of the system follows immediately from CPA security of randomized counter mode. The challenge is to prove ciphertext integrity for $\mathcal{E}_{\mathrm{MtE}}$. So let $\mathcal{A}_{\mathrm{ci}}$ be a ciphertext integrity adversary. This adversary makes a series of queries, $m_1, \ldots, m_Q$. For each m_i, the CI challenger gives to $\mathcal{A}_{\mathrm{ci}}$ a ciphertext $c_i = (x_i, u_i)$, where x_i is a random IV, and u_i is a one-time pad encryption of the pair $m_i \parallel t_i$ using a pseudo-random pad r_i derived from x_i using the PRF F. Here, t_i is a MAC tag computed on m_i. At the end of the attack game, adversary $\mathcal{A}_{\mathrm{ci}}$ outputs a ciphertext $c = (x, u)$, which is not among the c_i's, and wins if c is a valid ciphertext. This means that u decrypts to $m \parallel t$ using a pseudo-random pad r derived from x, and t is a valid tag on m.

Now, using the PRF security property and the fact that the x_i's are unlikely to repeat, we can effectively replace the pseudo-random r_i's (and r) with truly random pads, without affecting $\mathcal{A}_{\mathrm{ci}}$'s advantage significantly. This is where the terms $\mathrm{PRF}\mathsf{adv}[\mathcal{B}_{\mathrm{prf}}, F]$ and $Q^2/2|\mathcal{X}|$ in (9.12) come from. Note that after making this modification, the t_i's are perfectly hidden from the adversary.

We then consider two different ways in which $\mathcal{A}_{\mathrm{ci}}$ can win in this modified attack game.

- In the first way, the value x output by $\mathcal{A}_{\mathrm{ci}}$ is not among the x_i's. But in this case, the only way for $\mathcal{A}_{\mathrm{ci}}$ to win is to hope that a random tag on a random message is valid. This is where the term $\mathrm{MAC}_1\mathsf{adv}[\mathcal{B}'_{\mathrm{mac}}, \mathcal{I}]$ in (9.12) comes from.

- In the second way, the value x is equal to x_j for some $j = 1, \ldots, Q$. In this case, to win, the value u must decrypt under the pad r_j to $m \parallel t$ where t is a valid tag on m. Moreover, since $c \neq c_j$, we have $(m, t) \neq (m_j, t_j)$. To turn $\mathcal{A}_{\mathrm{ci}}$ into a one-time MAC adversary, we have to guess the index j in advance: for all indices i different from the guessed index, we can replace the tag t_i by a dummy tag. This guessing strategy is where the term $Q \cdot \mathrm{MAC}_1\mathsf{adv}[\mathcal{B}_{\mathrm{mac}}, \mathcal{I}]$ in (9.12) comes from. $\square$

Proof. To prove ciphertext integrity, we let $\mathcal{A}_{\mathrm{ci}}$ interact with a number of closely related challengers. For $j = 0, 1, 2, 3, 4$ we define W_j to be the event that the adversary wins in Game j.

Game 0. As usual, we begin by letting $\mathcal{A}_{\mathrm{ci}}$ interact with the standard ciphertext integrity challenger in Attack Game 9.1 as it applies to $\mathcal{E}_{\mathrm{MtE}}$, so that $\Pr[W_0] = \mathrm{CI}\mathsf{adv}[\mathcal{A}_{\mathrm{ci}}, \mathcal{E}_{\mathrm{MtE}}]$.

Game 1. Now, we replace the pseudo-random pads in the counter-mode cipher by truly independent one-time pads. Since F is a secure PRF and $1/|\mathcal{X}|$ is negligible, the adversary will not notice the difference. The resulting CI challenger for $\mathcal{E}_{\mathrm{MtE}}$ works as follows.

$k_{\mathrm{m}} \xleftarrow{\mathrm{R}} \mathcal{K}_{\mathrm{m}}$ // *Choose random MAC key*
$\omega \xleftarrow{\mathrm{R}} \{1, \ldots, Q\}$ // *this ω will be used in Game 3*
upon receiving the ith query $m_i \in \mathcal{Y}^{\leq \ell_{\mathrm{m}}}$ for $i = 1, 2, \ldots$ do:
(1) $t_i \leftarrow S(k_{\mathrm{m}}, m_i) \in \mathcal{T}$ // *compute the tag for m_i*
(2) $x_i \xleftarrow{\mathrm{R}} \mathcal{X}$ // *Choose a random IV*
$r_i \xleftarrow{\mathrm{R}} \mathcal{Y}^{|m_i|+\ell_{\mathrm{t}}}$ // *Choose a sufficiently long truly random one-time pad*
$u_i \leftarrow (m_i \parallel t_i) \oplus r_i, \quad c_i \leftarrow (x_i, u_i)$ // *build ciphertext*
send c_i to the adversary

At the end of the game, $\mathcal{A}_{\mathrm{ci}}$ outputs $c = (x, u)$, which is not among $c_1, \ldots, c_Q$, and the winning condition is evaluated as follows:

// *decrypt ciphertext c*
(3) if $x = x_j$ for some j then $(m \parallel t) \leftarrow u \oplus r_j$
(4) otherwise, $r \xleftarrow{\mathrm{R}} \mathcal{Y}^{|u|}$ and $(m \parallel t) \leftarrow u \oplus r$
$\mathcal{A}_{\mathrm{ci}}$ wins if $V(k_{\mathrm{m}}, m, t) = \mathsf{accept}$ // *check resulting message-tag pair*

Note that for specificity, in line (3) if there is more than one j for which $x = x_j$, we can take the smallest such j.

A standard argument shows that there exists an efficient PRF adversary $\mathcal{B}_{\mathrm{prf}}$ such that:

$$|\Pr[W_1] - \Pr[W_0]| \leq \mathrm{PRF}\mathsf{adv}[\mathcal{B}_{\mathrm{prf}}, F] + \frac{Q^2}{2|\mathcal{X}|}. \tag{9.13}$$

Note that if we wanted to be a bit more careful, we would break this argument up into two steps. In the first step, we would play our "PRF card" to replace $F(k_{\mathrm{e}}, \cdot)$ be a truly random function f. This introduces the term $\mathrm{PRF}\mathsf{adv}[\mathcal{B}_{\mathrm{prf}}, F]$ in (9.13). In the second step, we would use the "forgetful gnome" technique to make all the outputs of f independent. Using the Difference Lemma applied to the event that all of the x_i's are distinct introduces the term $Q^2/2|\mathcal{X}|$ in (9.13).

Game 2. Now we restrict the adversary's winning condition to require that the IV used in the final ciphertext c is the same as one of the IVs given to $\mathcal{A}_{\mathrm{ci}}$ during the game. In particular, we replace line (4) with

(4) otherwise, the adversary loses in Game 2.

Let Z_2 be the event that in Game 2, the final ciphertext $c = (x, u)$ from $\mathcal{A}_{\mathrm{ci}}$ is valid despite using a previously unused $x \in \mathcal{X}$. We know that the two games proceed identically, unless event Z_2 happens. When event Z_2 happens in Game 2 then the resulting pair (m, t) is uniformly random in $\mathcal{Y}^{|u|-\ell_{\mathrm{t}}} \times \mathcal{Y}^{\ell_{\mathrm{t}}}$. Such a pair is unlikely to form a valid message-tag pair. Not only that, the challenger in Game 2 effectively encrypts all of the tags t_i generated in line (1) with a one-time pad, so these tags could be replaced by dummy tags, without affecting the probability that Z_2 occurs. Based on these observations, we can easily construct an efficient MAC adversary $\mathcal{B}'_{\mathrm{mac}}$ such that $\Pr[Z_2] \leq \mathrm{MAC}_1\mathsf{adv}[\mathcal{B}'_{\mathrm{mac}}, \mathcal{I}]$. Adversary $\mathcal{B}'_{\mathrm{mac}}$ runs as follows. It plays the role of challenger to $\mathcal{A}_{\mathrm{ci}}$ as in Game 2, except that in line (1) above, it computes $t_i \leftarrow 0^{\ell_{\mathrm{t}}}$. When $\mathcal{A}_{\mathrm{ci}}$ outputs $c = (x, u)$, adversary $\mathcal{B}'_{\mathrm{mac}}$ generates outputs a random pair in $\mathcal{Y}^{|u|-\ell_t} \times \mathcal{Y}^{\ell_t}$. Hence, by the difference lemma, we have

$$|\Pr[W_2] - \Pr[W_1]| \le \mathrm{MAC_1adv}[\mathcal{B}'_{\mathrm{mac}}, \mathcal{I}]. \tag{9.14}$$

Game 3. We further constrain the adversary's winning condition by requiring that the ciphertext forgery use the IV from ciphertext number ω given to $\mathcal{A}_{\mathrm{ci}}$. Here ω is a random number in $\{1, \dots, Q\}$ chosen by the challenger. The only change to the winning condition of Game 2 is that line (3) now becomes:

(3) if $x = x_\omega$ then $(m \parallel t) \leftarrow u \oplus r_\omega$
(4) otherwise, the adversary loses in Game 2.

Since ω is independent of $\mathcal{A}_{\mathrm{ci}}$'s view, we know that

$$\Pr[W_3] \ge (1/Q) \cdot \Pr[W_2] \tag{9.15}$$

Game 4. Finally, we change the challenger so that it only computes a valid tag for query number ω issued by $\mathcal{A}_{\mathrm{ci}}$. For all other queries the challenger just makes up an arbitrary (invalid) tag. Since the tags are encrypted using one-time pads the adversary cannot tell that he is given encryptions of invalid tags. In particular, the only difference from Game 3 is that we replace line (1) by the following two lines:

(1) $t_i \leftarrow (0^n)^{\ell_t} \in \mathcal{T}$
if $i = \omega$ then $t_i \leftarrow S(k_{\mathrm{m}}, m_i) \in \mathcal{T}$ // *only compute correct tag for* m_ω

Since the adversary's view in this game is identical to its view in Game 3 we have

$$\Pr[W_4] = \Pr[W_3] \tag{9.16}$$

Final reduction. We claim that there is an efficient one-time MAC forger $\mathcal{B}_{\mathrm{mac}}$ so that

$$\Pr[W_4] = \mathrm{MAC_1adv}[\mathcal{B}_{\mathrm{mac}}, \mathcal{I}] \tag{9.17}$$

Adversary $\mathcal{B}_{\mathrm{mac}}$ interacts with a MAC challenger **C** and works as follows:

$\omega \xleftarrow{R} \{1, \dots, Q\}$
upon receiving the ith query $m_i \in \{0,1\}^{\ell_m}$ for $i = 1, 2, \dots$ do:
  $t_i \leftarrow (0^n)^{\ell_t} \in \mathcal{T}$
  if $i = \omega$ then query **C** for the tag on m_i and let $t_i \in \mathcal{T}$ be the response
  $x_i \xleftarrow{R} \mathcal{X}$ // *Choose a random IV*
  $r_i \xleftarrow{R} \mathcal{Y}^{|m|+\ell_t}$ // *Choose a sufficiently long random one-time pad*
  $u_i \leftarrow (m_i \parallel t_i) \oplus r_i$, $c_i \leftarrow (x_i, u_i)$
  send c_i to the adversary
when $\mathcal{A}_{\mathrm{ci}}$ outputs $c = (x, u)$ from $\mathcal{A}_{\mathrm{ci}}$ do:
  if $x = x_\omega$ then
    $(m \parallel t) \leftarrow u \oplus r_\omega$
    output (m, t) as the message-tag forgery

Since $c \ne c_\omega$ we know that $(m, t) \ne (m_\omega, t_\omega)$. Hence, whenever $\mathcal{A}_{\mathrm{ci}}$ wins Game 4 we know that $\mathcal{B}_{\mathrm{mac}}$ does not abort, and outputs a pair (m, t) that lets it win the one-time MAC attack game. It follows that $\Pr[W_4] = \mathrm{MAC_1adv}[\mathcal{B}_{\mathrm{mac}}, \mathcal{I}]$ as required. In summary, putting equations (9.13)–(9.17) together proves the theorem. □

9.4.5 Encrypt-then-MAC or MAC-then-encrypt?

So far we proved the following facts about the MtE and EtM modes:

- EtM provides authenticated encryption whenever the cipher is CPA-secure and the MAC is secure. The MAC on the ciphertext prevents any tampering with the ciphertext.
- MtE is not generally secure — there are examples of CPA-secure ciphers for which the MtE system does is not AE-secure. Moreover, MtE is difficult to implement correctly due to a potential timing side-channel that leads to serious chosen ciphertext attacks. However, for specific ciphers, such as randomized counter mode and randomized CBC, the MtE mode is AE-secure even if the MAC is only one-time secure.
- A third mode, called encrypt-and-MAC (EaM), is discussed in Exercise 9.10. The exercise shows that EaM is secure when using randomized counter-mode cipher as long as the MAC is a secure PRF. EaM is inferior to EtM in every respect and should not be used.

These facts, and the example attacks on MtE, suggest that EtM is the better mode to use. Of course, it is critically important that the underlying cipher be CPA-secure and the underlying MAC be a secure MAC. Otherwise, EtM may provide no security at all.

Given all the past mistakes in implementing these modes it is advisable that developers not implement EtM themselves. Instead, it is best to use an encryption standard, like GCM (see Section 9.7), that uses EtM to provide authenticated encryption out of the box.

9.5 Nonce-based authenticated encryption with associated data

In this section we extend the syntax of authenticated encryption to match the way in which it is commonly used. First, as we did for encryption and for MACs, we define nonce-based authenticated encryption where we make the encryption and decryption algorithms deterministic, but let them take as input a unique nonce. This approach can reduce ciphertext size and also improve security.

Second, we extend the encryption algorithm by giving it an additional input message, called **associated data**, whose integrity is protected by the ciphertext, but its secrecy is not. The need for associated data comes up in a number of settings. For example, when encrypting packets in a networking protocol, authenticated encryption protects the packet body, but the header must be transmitted in the clear so that the network can route the packet to its intended destination. Nevertheless, we want to ensure header integrity. The header is provided as the associated data input to the encryption algorithm.

A cipher that supports associated data is called an **AD cipher**. The syntax for a nonce-based AD cipher $\mathcal{E} = (E, D)$ is as follows:

$$c = E(k, m, d, \mathcal{N}),$$

where $c \in \mathcal{C}$ is the ciphertext, $k \in \mathcal{K}$ is the key, $m \in \mathcal{M}$ is the message, $d \in \mathcal{D}$ is the associated data, and $\mathcal{N} \in \mathcal{N}$ is the nonce. Moreover, the encryption algorithm E is required to be deterministic. Likewise, the decryption syntax becomes

$$D(k, c, d, \mathcal{N})$$

which outputs a message m or reject. We say that the nonce-based AD cipher is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{D}, \mathcal{C}, \mathcal{N})$. As usual, we require that ciphertexts generated by E are correctly decrypted by D, *as long as both are given the same nonce and associated data.* That is, for all keys k, all messages m, all associated data d, and all nonces $\mathcal{N} \in \mathcal{N}$:

$$D\big(k,\ E(k,\ m,\ d,\ \mathcal{N}),\ d,\ \mathcal{N}\big) = m.$$

If the message m given as input to the encryption algorithm is the empty message then cipher (E, D) essentially becomes a MAC system for the associated data d.

CPA security. A nonce-based AD cipher is CPA-secure if it does not leak any useful information to an eavesdropper assuming that *no nonce is used more than once* in the encryption process. CPA security for a nonce-based AD cipher is defined as CPA security for a standard nonce-based cipher (Section 5.5). The only difference is in the encryption queries. Encryption queries in Experiment b, for $b = 0, 1$, are processed as follows:

> The ith encryption query is a pair of messages, $m_{i0}, m_{i1} \in \mathcal{M}$, of the same length, associated data $d_i \in \mathcal{D}$, and a unique nonce $\mathcal{N}_i \in \mathcal{N} \setminus \{\mathcal{N}_1, \ldots, \mathcal{N}_{i-1}\}$.
>
> The challenger computes $c_i \leftarrow E(k, m_{ib}, d_i, \mathcal{N}_i)$, and sends c_i to the adversary.

Nothing else changes from the definition in Section 5.5. Note that the associated data d_i is under the adversary's control, as are the nonces $\mathcal{N}_i$, subject to the nonces being unique. For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\mathrm{nCPA_{ad}}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := |\Pr[W_0] - \Pr[W_1]|. \quad \Box$$

Definition 9.7 (CPA security). *A nonce-based AD cipher is called* ***semantically secure against chosen plaintext attack****, or simply* ***CPA-secure****, if for all efficient adversaries $\mathcal{A}$, the quantity* $\mathrm{nCPA_{ad}}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

Ciphertext integrity. A nonce-based AD cipher provides ciphertext integrity if an attacker who can request encryptions under key k for messages, associated data, and nonces of his choice cannot output a new triple $(c, d, \mathcal{N})$ that is accepted by the decryption algorithm. The adversary, however, must never issue an encryption query using a previously used nonce.

More precisely, we modify the ciphertext integrity game (Attack Game 9.1) as follows:

Attack Game 9.3 (ciphertext integrity). For a given AD cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{D}, \mathcal{C}, \mathcal{N})$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger chooses a random $k \xleftarrow{\text{R}} \mathcal{K}$.
- $\mathcal{A}$ queries the challenger several times. For $i = 1, 2, \ldots$, the ith query consists of a message $m_i \in \mathcal{M}$, associated data $d_i \in \mathcal{D}$, and a previously unused nonce $\mathcal{N}_i \in \mathcal{N} \setminus \{\mathcal{N}_1, \ldots, \mathcal{N}_{i-1}\}$. The challenger computes $c_i \xleftarrow{\text{R}} E(k, m_i, d_i, \mathcal{N}_i)$, and gives c_i to $\mathcal{A}$.
- Eventually $\mathcal{A}$ outputs a candidate triple $(c, d, \mathcal{N})$ where $c \in \mathcal{C}$, $d \in \mathcal{D}$, and $\mathcal{N} \in \mathcal{N}$ that is not among the triples it was given, i.e.,

$$(c, d, \mathcal{N}) \notin \{(c_1, d_1, \mathcal{N}_1),\ (c_2, d_2, \mathcal{N}_2),\ \ldots\}.$$

We say that $\mathcal{A}$ wins the game if $D(k, c, d, \mathcal{N}) \neq \mathsf{reject}$. We define $\mathcal{A}$'s advantage with respect to $\mathcal{E}$, denoted $\mathrm{nCI_{ad}}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$, as the probability that $\mathcal{A}$ wins the game. □

Definition 9.8. *We say that a nonce-based AD cipher $\mathcal{E} = (E, D)$ has* ***ciphertext integrity*** *if for all efficient adversaries $\mathcal{A}$, the value $\mathrm{nCI_{ad}}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ is negligible.*

Authenticated encryption. We can now define nonce-based authenticated encryption for an AD cipher. We refer to this notion as a **nonce-based AEAD cipher** which is shorthand for **authenticated encryption with associated data**.

Definition 9.9. *We say that a nonce-based AD cipher $\mathcal{E} = (E, D)$ provides authenticated encryption, or is simply a* ***nonce-based AEAD cipher****, if $\mathcal{E}$ is CPA-secure and has ciphertext integrity.*

Generic encrypt-then-MAC composition. We construct a nonce-based AEAD cipher $\mathcal{E} = (E_{\mathrm{EtM}}, D_{\mathrm{EtM}})$ by combining a nonce-based CPA-secure cipher (E, D) (as in Section 5.5) with a nonce-based secure MAC (S, V) (as in Section 7.5) as follows:

$$
\begin{aligned}
E_{\mathrm{EtM}}(\ (k_{\mathrm{e}}, k_{\mathrm{m}}),\ m, d, \mathcal{N}) \quad &:= \quad c \leftarrow E(k_{\mathrm{e}}, m, \mathcal{N}), \quad t \leftarrow S(k_{\mathrm{m}}, (c, d), \mathcal{N}) \\
& \qquad \text{Output } (c, t) \\
D_{\mathrm{EtM}}((k_{\mathrm{e}}, k_{\mathrm{m}}),\ (c, t),\ d, \mathcal{N}) \quad &:= \quad \text{if } V(k_{\mathrm{m}},\ (c, d),\ t,\ \mathcal{N}) = \mathsf{reject} \text{ then output } \mathsf{reject} \\
& \qquad \text{otherwise, output } D(k_{\mathrm{e}}, c, d, \mathcal{N})
\end{aligned}
$$

The EtM system is defined over $(\mathcal{K}_{\mathrm{e}} \times \mathcal{K}_{\mathrm{m}}, \mathcal{M}, \mathcal{D}, \mathcal{C} \times \mathcal{T}, \mathcal{N})$. The following theorem shows that $\mathcal{E}_{\mathrm{EtM}}$ is a secure AEAD cipher.

Theorem 9.4. *Let $\mathcal{E} = (E, D)$ be a nonce-based cipher and let $\mathcal{I} = (S, V)$ be a nonce-based MAC system. Then $\mathcal{E}_{\mathrm{EtM}}$ is a nonce-based AEAD cipher assuming $\mathcal{E}$ is CPA-secure and $\mathcal{I}$ is a secure MAC system.*

The proof of Theorem 9.4 is essentially the same as the proof of Theorem 9.2.

9.6 One more variation: CCA-secure ciphers with associated data

In Section 9.5, we introduced two new features to our ciphers: nonces and associated data. There are two variations we could consider: ciphers with nonces but without associated data, and ciphers with associated data but without nonces. We could also consider all of these variations with respect to other security notions, such as CCA security. Considering all of these variations in detail would be quite tedious. However, we consider one variation that will be important later in the text, namely CCA-secure ciphers with associated data (but without nonces).

To define this notion, we begin by defining the syntax for a cipher with associated data, or AD cipher, without nonces. For such a cipher $\mathcal{E} = (E, D)$, the encryption algorithm may be probabilistic and works as follows:

$$c \xleftarrow{\mathrm{R}} E(k, m, d),$$

where $c \in \mathcal{C}$ is the ciphertext, $k \in \mathcal{K}$ is the key, $m \in \mathcal{M}$ is the message, and $d \in \mathcal{D}$ is the associated data. The decryption syntax is

$$D(k, c, d),$$

which outputs a message m or reject. We say that the AD cipher is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{D}, \mathcal{C})$. As usual, we require that ciphertexts generated by E are correctly decrypted by D, as long as both are given the same associated data. That is,

$$\Pr\left[D\big(k,\ E(k,\ m,\ d),\ d\big) = m\right] = 1.$$

Definition 9.10 (CCA and 1CCA security with associated data). *The definition of CCA security for ordinary ciphers carries over naturally to AD ciphers. Attack Game 9.2 is modified as follows. For encryption queries, in addition to a pair of messages (m_{i0}, m_{i1}), the adversary also submits associated data d_i, and the challenger computes $c_i \xleftarrow{\text{R}} E(k, m_{ib}, d_i)$. For decryption queries, in addition to a ciphertext $\hat{c}_j$, the adversary submits associated data $\hat{d}_j$, and the challenger computes $\hat{m}_j \leftarrow D(k, \hat{c}_j, \hat{d}_j)$. The restriction is that the* pair *$(\hat{c}_j, \hat{d}_j)$ may not be among the pairs $(c_1, d_1), (c_2, d_2), \ldots$ corresponding to previous encryption queries. An adversary $\mathcal{A}$'s advantage in this game is denoted $\mathrm{CCA_{ad}}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$, and the cipher is said to be **CCA secure** if this advantage is negligible for all efficient adversaries $\mathcal{A}$. If we restrict the adversary to a single encryption query, as in Definition 9.6, the advantage is denoted $\mathrm{1CCA_{ad}}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$, and the cipher is said to be **1CCA secure** if this advantage is negligible for all efficient adversaries $\mathcal{A}$.*

Generic encrypt-then-MAC composition. In later applications, the notion that we will use is 1CCA security, so for simplicity we focus on that notion for now. We construct a 1CCA-secure AD cipher $\mathcal{E} = (E_{\mathrm{EtM}}, D_{\mathrm{EtM}})$ by combining a semantically secure cipher (E, D) with a one-time MAC (S, V) as follows:

$$\begin{aligned}
E_{\mathrm{EtM}}(\ (k_{\mathrm{e}}, k_{\mathrm{m}}),\ m, d) \quad &:= \quad c \xleftarrow{\text{R}} E(k_{\mathrm{e}}, m), \quad t \xleftarrow{\text{R}} S(k_{\mathrm{m}}, (c, d)) \\
& \quad \text{Output } (c, t) \\
D_{\mathrm{EtM}}((k_{\mathrm{e}}, k_{\mathrm{m}}),\ (c, t),\ d) \quad &:= \quad \text{if } V(k_{\mathrm{m}},\ (c, d),\ t) = \mathsf{reject} \text{ then output } \mathsf{reject} \\
& \quad \text{otherwise, output } D(k_{\mathrm{e}}, c, d)
\end{aligned}$$

The EtM system is defined over $(\mathcal{K}_{\mathrm{e}} \times \mathcal{K}_{\mathrm{m}}, \mathcal{M}, \mathcal{D}, \mathcal{C} \times \mathcal{T})$.

Theorem 9.5. *Let $\mathcal{E} = (E, D)$ be a semantically secure cipher and let $\mathcal{I} = (S, V)$ be a one-time secure MAC system. Then $\mathcal{E}_{\mathrm{EtM}}$ is a 1CCA-secure AD cipher.*

The proof of Theorem 9.5 is straightforward, and we leave it as an exercise to the reader.

We observe that in most common implementations of the semantically secure cipher $\mathcal{E} = (E, D)$, the encryption algorithm E is deterministic. Likewise, in the most common implementations of the one-time secure MAC $\mathcal{I} = (S, V)$, the signing algorithm is deterministic. So for such implementations, the resulting 1CCA-secure AD cipher will have a deterministic encryption algorithm.

9.7 Case study: Galois counter mode (GCM)

Galois counter mode (GCM) is a popular nonce-based AEAD cipher standardized by NIST in 2007. GCM is an encrypt-then-MAC cipher combining a CPA-secure cipher and a secure MAC. The CPA secure cipher is nonce-based counter mode, usually using AES. The secure MAC is a Carter-Wegman MAC built from a keyed hash function called GHASH, a variant of the function H_{xpoly} from Section 7.4. When encrypting the empty message the cipher becomes a MAC system called **GMAC** providing integrity for the associated data.

GCM uses an underlying block cipher $\mathcal{E} = (E, D)$ such as AES defined over $(\mathcal{K}, \mathcal{X})$ where $\mathcal{X} := \{0,1\}^{128}$. The block cipher is used for both counter mode encryption and the Carter-Wegman MAC. The GHASH function is defined over $(\mathcal{X}, \mathcal{X}^{\leq \ell}, \mathcal{X})$ for $\ell := 2^{32} - 1$.

GCM can take variable size nonces, but let us first describe GCM using a 96-bit nonce $\mathcal{N}$ which is the simplest case. The GCM encryption algorithm operates as follows:

input: key $k \in \mathcal{K}$, message m, associated data d, and nonce $\mathcal{N} \in \{0,1\}^{96}$

$k_{\mathrm{m}} \leftarrow E(k,\ 0^{128})$ // *first, generate the key for GHASH (a variant of H_{xpoly})*

Compute the initial value of the counter in counter mode encryption:

$x \leftarrow (\mathcal{N} \parallel 0^{31}1) \in \{0,1\}^{128}$

$x' \leftarrow x + 1$ // *initial value of counter*

$c \leftarrow \{$encryption of m using counter mode starting the counter at $x'\}$

$d' \leftarrow \{$pad d with zeros to closest multiple of 128 bits$\}$

$c' \leftarrow \{$pad c with zeros to closest multiple of 128 bits$\}$

Compute the Carter-Wegman MAC:

$$(*) \qquad h \leftarrow \mathrm{GHASH}\Big(k_{\mathrm{m}},\ \big(d' \parallel c' \parallel \mathrm{length}(d) \parallel \mathrm{length}(c)\big)\Big) \ \in \{0,1\}^{128}$$

$t \leftarrow h \oplus E(k, x) \ \in \{0,1\}^{128}$

output (c, t) // *encrypt-then-MAC ciphertext*

Each of the length fields on line $(*)$ is a 64-bit value indicating the length in bytes of the respective field. If the input nonce $\mathcal{N}$ is not 96-bits long, then $\mathcal{N}$ is padded to the closest multiple of 128 bits, yielding the padded string $\mathcal{N}'$, and the initial counter value x is computed as $x \leftarrow \mathrm{GHASH}\big(k_{\mathrm{m}},\ (\mathcal{N}' \parallel \mathrm{length}(\mathcal{N}))\big)$ which is a value in $\{0,1\}^{128}$.

As usual, the integrity tag t can be truncated to whatever length is desired. The shorter the tag t the more vulnerable the system becomes to ciphertext integrity attacks.

Messages to be encrypted must be less than 2^{32} blocks each (i.e., messages must be in $\mathcal{X}^v$ for some $v < 2^{32}$). Recommendations in the standard suggest that a single key k should not be used to encrypt more than 2^{32} messages.

The GCM decryption algorithm takes as input a key $k \in \mathcal{K}$, a ciphertext (c, t), associate data d and a nonce $\mathcal{N}$. It operates as in encrypt-then-MAC: it first derives $k_{\mathrm{m}} \leftarrow E(k, 0^n)$ and checks the Carter-Wegman integrity tag t. If valid it outputs the counter mode decryption of c. We emphasize that decryption must be atomic: no plaintext data is output before the integrity tag is verified over the entire message.

GHASH. It remains to describe the keyed hash function GHASH defined over $(\mathcal{X}, \mathcal{X}^{\leq \ell}, \mathcal{X})$. This hash function is used in a Carter-Wegman MAC and therefore, for security, must be a DUF. In Section 7.4 we showed that the function H_{xpoly} is a DUF and GHASH is essentially the same thing. Recall that $H_{\mathrm{xpoly}}(k, z)$ works by evaluating a polynomial derived from z at the point k. We described H_{xpoly} using arithmetic modulo a prime p so that both blocks of z and the output are elements in $\mathbb{Z}_p$.

The hash function GHASH is almost the same as H_{xpoly}, except that the input message blocks and the output are elements of $\{0,1\}^{128}$. Also, the DUF property holds with respect to the XOR operator $\oplus$, rather than subtraction modulo some number. As discussed in Remark 7.4, to build an XOR-DUF we use polynomials defined over the finite field $\mathrm{GF}(2^{128})$. This is a field of 2^{128}

elements called a **Galois field**, which is where GCM gets its name. This field is defined by the irreducible polynomial $g(X) := X^{128} + X^7 + X^2 + X + 1$. Elements of $\mathrm{GF}(2^{128})$ are polynomials over $\mathrm{GF}(2)$ of degree less than 128, with arithmetic done modulo $g(X)$. While that sounds fancy, an element of $\mathrm{GF}(2^{128})$ can be conveniently represented as a string of 128 bits (each bit encodes one of the coefficients of the polynomial). Addition in the field is just XOR, while multiplication is a bit more complicated, but still not too difficult (see below — many modern computers provide direct hardware support).

With this notation, for $k \in \mathrm{GF}(2^{128})$ and $z \in \big(\mathrm{GF}(2^{128})\big)^v$ the function $\mathrm{GHASH}(k, z)$ is simply polynomial evaluation in $\mathrm{GF}(2^{128})$:

$$\mathrm{GHASH}(k, z) := z[0]k^v + z[1]k^{v-1} + \ldots + z[v-1]k \ \in \mathrm{GF}(2^{128}) \tag{9.18}$$

That's it. Appending the two length fields to the GHASH input on line $(*)$ ensures that the XOR-DUF property is maintained even for messages of different lengths.

Security. The AEAD security of GCM is similar to the analysis we did for generic composition of encrypt-then-MAC (Theorem 9.4), and follows from the security of the underlying block cipher as a PRF. The main difference between GCM and our generic composition is that GCM "cuts a few corners" when it comes to keys: it uses just a single key k and uses $E(k, 0^n)$ as the GHASH key, and $E(k, x)$ as the pad that is used to mask the output of GHASH, which is similar to, but not exactly the sames as, what is done in Carter-Wegman. Importantly, the counter mode encryption begins with the counter value $x' := x + 1$, so that the inputs to the PRF that are used to encrypt the message are guaranteed to be distinct from the inputs used to derive the GHASH key and pad. The above discussion focused on the case where the nonce is 96 bits. The other case, where GHASH is applied to the nonce to compute x, requires a more involved analysis — see Exercise 9.14.

GCM has no nonce re-use resistance. If a nonce is accidentally re-used on two different messages then all secrecy for those message is lost. Even worse, the GHASH secret key k_{m} is exposed (Exercise 7.13) and this can be used to break ciphertext integrity. Hence, it is vital that nonces not be re-used in GCM.

Optimizations and performance. There are many ways to optimize the implementation of GCM and GHASH. In practice, the polynomial in (9.18) is evaluated using Horner's method so that processing each block of plaintext requires only one addition and one multiplication in $\mathrm{GF}(2^{128})$.

Intel recently added a special instruction (called PCLMULQDQ) to their instruction set to quickly carry out binary polynomial multiplication. This instruction cannot be used directly to implement GHASH because of incompatibility with how the standard represents elements in $\mathrm{GF}(2^{128})$. Fortunately, work of Gueron shows how to overcome these difficulties and use the PCLMULQDQ instruction to speed-up GHASH on Intel platforms.

Since GHASH needs only one addition and one multiplication in $\mathrm{GF}(2^{128})$ per block one would expect that the bulk of the time during GCM encryption and decryption is spent on AES in counter mode. However, due to improvements in hardware implementations of AES, especially pipelining of the AES-NI instructions, this is not always the case. On Intel's Haswell processors (introduced in 2013) GCM is about three times slower than pure counter mode due to the extra overhead of GHASH. However, upcoming improvements in the implementation of PCLMULQDQ will likely make GCM just slightly more expensive than pure counter mode, which is the best one can hope for.

We should point out that it already is possible to implement secure authenticated encryption at a cost that is not much more than the cost of AES counter mode — this can be achieved using an integrated scheme such as OCB (see Exercise 9.17).

9.8 Case study: the TLS 1.3 record protocol

The Transport Layer Security (TLS) protocol is by far the most widely deployed security protocol. Virtually every online purchase is protected by TLS. Although TLS is primarily used to protect Web traffic, it is a general protocol that can protect many types of traffic: email, messaging, and many others.

The original version of TLS was designed at Netscape where it was called the Secure Socket Layer protocol or SSL. SSL 2.0 was designed in 1994 to protect Web e-commerce traffic. SSL 3.0, designed in 1995, corrected several significant security problems in SSLv2. For example, SSL 2.0 uses the same key for both the cipher and the MAC. While this is bad practice — it invalidates the proofs of security for MtE and EtM — it also implies that if one uses a weak cipher key, say do to export restrictions, then the MAC key must also be weak. SSL 2.0 supported only a small number of algorithms and, in particular, only supported MD5-based MACs.

The Internet Engineering Task Force (IETF) created the Transport Layer Security (TLS) working group to standardize an SSL-like protocol. The working group produced a specification for the TLS 1.0 protocol in 1999 [31]. TLS 1.0 is a minor variation of SSL 3.0 and is often referred to as SSL version 3.1. TLS is supported by most major browsers and web servers and TLS 1.3 is the recommended protocol to use. We will mostly focus on TLS 1.3 here.

The TLS 1.3 record protocol. Abstractly, TLS consists of two components. The first, called **TLS session setup**, negotiates the cipher suite that will be used to encrypt the session and then sets up a shared secret between the browser and server. The second, called the **TLS record protocol** uses this shared secret to securely transmit data between the two sides. TLS session setup uses public-key techniques and will be discussed later in Chapter 20. Here we focus on the TLS record protocol.

In TLS terminology, the shared secret generated during session setup is called a **master-secret**. This high entropy master secret is used to derive two keys $k_{b\to s}$ and $k_{s\to b}$. The key $k_{b\to s}$ encrypts messages from the browser to the server while $k_{s\to b}$ encrypts messages in the reverse direction. TLS derives the two keys by using the master secret and other randomness as a seed for a key derivation function called HKDF (Section 8.10.5) to derive enough pseudo-random bits for the two keys. This step is carried out by both the browser and server so that both sides have the keys $k_{b\to s}$ and $k_{s\to b}$.

The TLS record protocol sends data in records whose size is at most 2^{14} bytes. If one side needs to transmit more than 2^{14} bytes, the record protocol fragments the data into multiple records each of size at most 2^{14}. Each party maintains a 64-bit **write sequence number** that is initialized to zero and is incremented by one for every record sent by that party.

TLS 1.3 uses a nonce-based AEAD cipher (E, D) to encrypt a record. Which nonce-based AEAD cipher is used is determined by negotiation during TLS session setup. The AEAD encryption algorithm is given the following arguments:

- secret key: $k_{b\to s}$ or $k_{s\to b}$ depending on whether the browser or server is encrypting.
- plaintext data: up to 2^{14} bytes.

- associated data: a concatenation of three fields: the encrypting party's 64-bit write sequence number, a 1-byte record type (a value of 23 means application data), and a 2-byte protocol version (set to 3.1 in TLS 3.1).

- nonce (8 bytes or longer): the nonce is computed by (1) padding the encrypting party's 64-bit write sequence number on the left with zeroes to the expected nonce length and (2) XORing this padded sequence number with a random string (called `client_write_iv` or `server_write_iv`, depending on who is encrypting) that was derived from the master secret during session setup and is fixed for the life of the session. TLS 1.3 could have used an equivalent and slightly easier to comprehend method: choose the initial nonce value at random and then increment it sequentially for each record. The method used by TLS 1.3 is a little easier to implement.

The AEAD cipher outputs a ciphertext c which is then formatted into an encrypted TLS record as follows:

`type`	`version`	`length`	ciphertext c

where `type` is a 1-byte record type (handshake record or application data record), `version` is a 2-byte protocol version set to 3.1 for TLS 3.1, `length` is a 2-byte field indicating the length of c, and c is the ciphertext. The type, version, and length fields are all sent in the clear. Notice that the nonce is not part of the encrypted TLS record. The recipient computes the nonce by itself.

Why is the initial nonce value chosen at random? Why not simply set it to zero? In networking protocols the first message block sent over TLS is usually a fixed public value. If the nonce were set to zero then the first ciphertext would be computed as $c_0 \leftarrow E(k, m_0, d, 0)$ where the adversary knows m_0 and associate data d. This opens up the system to an exhaustive search attack for the key k using a *time-space tradeoff* discussed in Chapter 18. The attack shows that with a large amount of pre-computation and sufficient storage, an attacker can quickly recover k from c_0 with non-negligible advantage — for 128-bit keys, such attacks may be feasible in the not-too-distant future. Randomizing the initial nonce "future proofs" TLS against such attacks.

When a record is received, the receiving party runs the AEAD decryption algorithm to decrypt c. If decryption results in reject then the party sends a fatal `bad_record_mac` alert to its peer and shuts down the TLS session.

The length field. In TLS 1.3, as in earlier versions of TLS, the record length is sent in the clear. Several attacks based on traffic analysis exploit record lengths to deduce information about the record contents. For example, if an encrypted TLS record contains one of two images of different size then the length will reveal to an eavesdropper which image was encrypted. Chen et al. [25] show that the lengths of encrypted records can reveal considerable information about private data that a user supplies to a cloud application. They use an online tax filing system as their example. Other works show attacks of this type on many other systems. Since there is no complete solution to this problem, it is often ignored.

When encrypting a TLS record the length field is not part of the associated data and consequently has no integrity protection. The reason is that due to variable length padding, the length of c may not be known before the encryption algorithm terminates. Therefore, the length cannot be given as input to the encryption algorithm. This does not compromise security: a secure AEAD cipher will reject a ciphertext that is a result of tampering with the length field.

Replay prevention. An attacker may attempt to replay a previous record to cause the wrong action at the recipient. For example, the attacker could attempt to make the same purchase order be processed twice, by simply replaying the record containing the purchase order. TLS uses the 64-bit sequence number to discard such replicated packets. TLS assumes in-order record delivery so that the recipient already knows what sequence number to expect without any additional information in the record. A replicated record will be discarded because the AEAD decryption algorithm will be given the wrong nonce as input.

9.9 Case study: an attack on non-atomic decryption in SSH

SSH (secure shell) is a popular command line tool for securely exchanging information with a remote host. SSH is designed to replace (insecure) UNIX tools such as telnet, rlogin, rsh, and rcp. Here we describe a fascinating vulnerability in an older cipher suite used in SSH. This vulnerability is an example of what can go wrong when decryption is not atomic, that is, when the decryption algorithm releases fragments of a decrypted record before verifying integrity of the entire record.

First, a bit of history. The first version of SSH, called SSHv1, was made available in 1995. It was quickly pointed out that SSHv1 suffers from serious design flaws.

- Most notably, SSHv1 provides data integrity by computing a Cyclic Redundancy Check (CRC) of the plaintext and appending the resulting checksum to the ciphertext in the clear. CRC is a simple keyless, linear function — so not only does this directly leak information about the plaintext, it is also not too hard to break integrity either.
- Another issue is the incorrect use of CBC mode encryption. SSHv1 always sets the CBC initial value (IV) to 0. Consequently, an attacker can tell when two SSHv1 packets contain the same prefix. Recall that for CPA security one must choose the IV at random.
- Yet another problem, the same encryption key was used for both directions (user to server and server to user).

To correct these issues, a revised and incompatible protocol called SSHv2 was published in 1996. Session setup results in two keys $k_{u\to s}$, used to encrypt data from the user to the server, and $k_{s\to u}$, used to encrypt data in the reverse direction. Here we focus only how these keys are used for message transport in SSHv2.

SSHv2 encryption. Let us examine an older cipher suite used in SSHv2. SSHv2 combines a CPA-secure cipher with a secure MAC using encrypt-and-MAC (Exercise 9.10) in an attempt to construct a secure AEAD cipher. Specifically, SSHv2 encryption works as follows (Fig. 9.3):

1. **Pad.** Pad the plaintext with *random* bytes so that the total length of

   $$\texttt{plaintext} := \texttt{packet-length} \parallel \texttt{pad-length} \parallel \texttt{message} \parallel \texttt{pad}$$

   is a multiple of the cipher block length (16 bytes for AES). The pad length can be anywhere from 4 bytes to 255 bytes. The packet length field measures the length of the packet in bytes, not including the integrity tag or the packet-length field itself.

2. **Encrypt.** Encrypt the gray area in Fig. 9.3 using AES in randomized CBC mode with either $k_{u\to s}$ or $k_{s\to u}$, depending on the encrypting party. SSHv2 uses a defective version of randomized CBC mode encryption described in Exercise 5.12.

Gray area is encrypted; Boxed area is authenticated by integrity tag

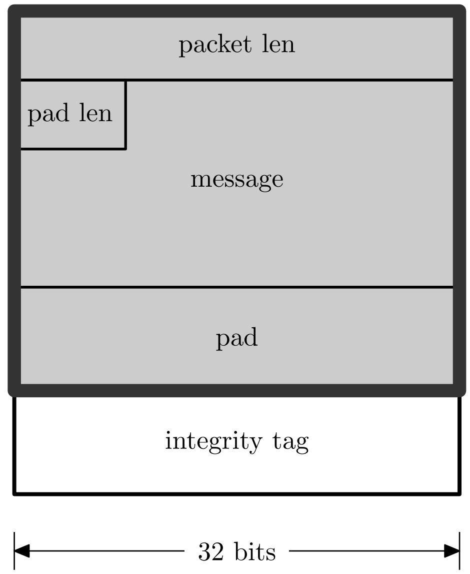


Figure 9.3: An SSHv2 packet

3. **MAC.** A MAC is computed over a `sequence-number` and the plaintext data in the thick box in Fig. 9.3. Here `sequence-number` is a 32-bit sequence number that is initialized to zero for the first packet, and is incremented by one after every packet. SSHv2 can use one of a number of MAC algorithms, but HMAC-SHA1-160 must be supported.

When an encrypted packet is received the decryption algorithm works as follows: first it decrypts the `packet-length` field using either $k_{u \to s}$ or $k_{s \to u}$. Next, it reads that many more packets from the network plus as many additional bytes as needed for the integrity tag. Next it decrypts the rest of the ciphertext and verifies validity of the integrity tag. If valid, it removes the pad and returns the plaintext message.

Although SSH uses encrypt-and-MAC, which is not generally secure, we show in Exercise 9.10 that for certain combinations of cipher and MAC, including the required ones in SSHv2, encrypt-and-MAC provides authenticated encryption.

SSH boundary hiding via length encryption. An interesting aspect of SSHv2 is that the encryption algorithm encrypts the packet length field, as shown in Fig. 9.3. The motivation for this is to ensure that if a sequence of encrypted SSH packets are sent over an insecure network as a stream of bytes, then an eavesdropper should be unable to determine the number of packets sent or their lengths. This is intended to frustrate certain traffic analysis attacks that deduce information about the plaintext from its size.

Hiding message boundaries between consecutive encrypted messages is outside the requirements addressed by authenticated encryption. In fact, many secure AEAD modes do not provide this level of secrecy. TLS 1.0, for example, sends the length of the every record in the clear making it easy to detect boundaries between consecutive encrypted records. Enhancing authenticated encryption to ensure boundary hiding has been formalized by Boldyreva, Degabriele, Paterson, and Stam [20], proposing a number of constructions satisfying the definitions.

An attack on non-atomic decryption. Notice that CBC decryption is done in two steps: first the 32-bit `packet-length` field is decrypted and used to decide how many more bytes to read from the network. Next, the rest of the CBC ciphertext is decrypted.

Generally speaking, AEAD ciphers are not designed to be used this way: plaintext data should not be used until the entire ciphertext decryption process is finished; however, in SSHv2 the decrypted length field is used before its integrity has been verified.

Can this be used to attack SSHv2? A beautiful attack [1] shows how this non-atomic decryption can completely compromise secrecy. Here we only describe the high-level idea, ignoring many details. Suppose an attacker intercepts a 16-byte ciphertext block c and it wants to learn the first four bytes of the decryption of c. It does so by abusing the decryption process as follows: first, it sends the ciphertext block c to the server *as if* it were the first block of a new encrypted packet. The server decrypts c and interprets the first four bytes as a length field ℓ. The server now expects to read ℓ bytes of data from the network before checking the integrity tag. The attacker can slowly send to the server arbitrary bytes, one byte at a time, waiting after each byte to see if the server responds. Once the server reads ℓ bytes it attempts to verify the integrity tag on the bytes it received and this most likely fails causing the server to send back an error message. Thus, once ℓ bytes are read the attacker receives an error message. This tells the attacker the value of ℓ which is what it wanted.

In practice, there are many complications in mounting an attack like this. Nevertheless, it shows the danger of using decrypted data — the length field in this case — before its integrity has been verified. As mentioned above, we refer to [20] for encryption methods that securely hide packet lengths.

A clever traffic analysis attack on SSH. SSHv2 operates by sending one network packet for every user keystroke. This gives rise to an interesting traffic analysis attack reported in [98]. Suppose a network eavesdropper knows that the user is entering a password at his or her keyboard. By measuring timing differences between consecutive packets, the eavesdropper obtains timing information between consecutive keystrokes. This exposes information about the user's password: a large timing gap between consecutive keystrokes reveals information about the keyboard position of the relevant keys. The authors show that this information can significantly speed up an offline password dictionary attack. To make matters worse, password packets are easily identified since applications typically turn off echo during password entry so that password packets do not generate an echo packet from the server.

Some SSH implementations defend against this problem by injecting randomly timed "dummy" messages to make traffic analysis more difficult. Dummy messages are identified by setting the first message byte to `SSH_MSG_IGNORE` and are ignored by the receiver. The eavesdropper cannot distinguish dummy records from real ones thanks to encryption.

9.10 Case study: 802.11b WEP, a badly broken system

The IEEE 802.11b standard ratified in 1999 defines a protocol for short range wireless communication (WiFi). Security is provided by a Wired Equivalent Privacy (WEP) encapsulation of 802.11b

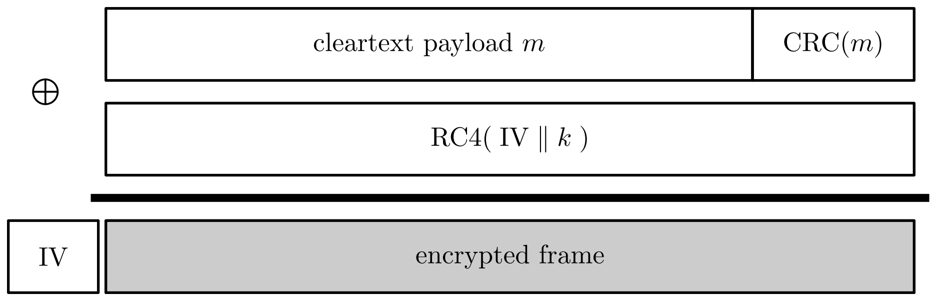


Figure 9.4: WEP Encryption

data frames. The design goal of WEP is to provide data privacy at the level of a wired network. WEP, however, completely fails on this front and gives us an excellent case study illustrating how a weak design can lead to disastrous results.

When WEP is enabled, all members of the wireless network share a long term secret key k. The standard supports either 40-bit keys or 128-bit keys. The 40-bit version complies with US export restrictions that were in effect at the time the standard was drafted. We will use the following notation to describe WEP:

- WEP encryption uses the RC4 stream cipher. We let $\mathrm{RC4}(s)$ denote the pseudo random sequence generated by RC4 given the seed s.
- We let $\mathrm{CRC}(m)$ denote the 32-bit CRC checksum of a message $m \in \{0,1\}^*$. The details of CRC are irrelevant for our discussion and it suffices to view CRC as some fixed function from bit strings to $\{0,1\}^{32}$.

Let m be an 802.11b cleartext frame. The first few bits of m encode the length of m. To encrypt an 802.11b frame m the sender picks a 24-bit IV and computes:

$$
\begin{aligned}
&c \leftarrow \big(m \,\|\, \mathrm{CRC}(m)\big) \;\oplus\; \mathrm{RC4}(\mathrm{IV} \,\|\, k) \\
&c_{\text{full}} \leftarrow (\mathrm{IV},\; c)
\end{aligned}
$$

The WEP encryption process is shown in Fig. 9.4. The receiver decrypts by first computing $c \oplus \mathrm{RC4}(\mathrm{IV} \,\|\, k)$ to obtain a pair (m, s). The receiver accepts the frame if $s = \mathrm{CRC}(m)$ and rejects it otherwise.

Attack 1: IV collisions. The designers of WEP understood that a stream cipher key should never be reused. Consequently, they used the 24-bit IV to derive a per-frame key $k_{\text{f}} := \mathrm{IV} \,\|\, k$. The standard, however, does not specify how to choose the IVs and many implementations do so poorly. We say that an IV collision occurs whenever a wireless station happens to send two frames, say frame number i and frame number j, encrypted using the same IV. Since IVs are sent in the clear, an eavesdropper can easily detect IV collisions. Moreover, once an IV collision occurs the attacker can use the two-time pad attack discussed in Section 3.3.1 to decrypt both frames i and j.

So, how likely is an IV collision? By the birthday paradox, an implementation that chooses a random IV for each frame will cause an IV collision after only an expected $\sqrt{2^{24}} = 2^{12} = 4096$ frames. Since each frame body is at most 1156 bytes, a collision will occur after transmitting about 4MB on average.

Alternatively, an implementation could generate the IV using a counter. The implementation will exhaust the entire IV space after 2^{24} frames are sent, which will take about a day for a wireless access point working at full capacity. Even worse, several wireless cards that use the counter method reset the counter to 0 during power-up. As a result, these cards will frequently reuse low value IVs, making the traffic highly vulnerable to a two-time pad attack.

Attack 2: related keys. A far more devastating attack on WEP encryption results from the use of related RC4 keys. In Chapter 3 we explained that a new and *random* stream cipher key must be chosen for every encrypted message. WEP, however, uses keys $1 \parallel k,\ 2 \parallel k, \ldots$ which are all closely related — they all have the same suffix k. RC4 was never designed for such use, and indeed, is completely insecure in these settings. Fluhrer, Mantin, and Shamir [38] showed that after about a million WEP frames are sent, an eavesdropper can recover the entire long term secret key k. The attack was implemented by Stubblefield, Ioannidis, and Rubin [101] and is now available in a variety of hacking tools such as WepCrack and AirSnort.

Generating per frame keys should have been done using a PRF, for example, setting the key for frame i to $k_i := F(k, \text{IV})$ — the resulting keys would be indistinguishable from random, independent keys. Of course, while this approach would have prevented the related keys problem, it would not solve the IV collision problem discussed above, or the malleability problem discussed next.

Attack 3: malleability. Recall that WEP attempts to provide authenticated encryption by using a CRC checksum for integrity. In a sense, WEP uses the MAC-then-encrypt method, but it uses CRC instead of a MAC. We show that despite the encryption step, this construction utterly fails to provide ciphertext integrity.

The attack uses the linearity of CRC. That is, given $\text{CRC}(m)$ for some message m, it is easy to compute $\text{CRC}(m \oplus \Delta)$ for any Δ. More precisely, there is a public function L such that for any m and $\Delta \in \{0,1\}^\ell$ we have that

$$\text{CRC}(m \oplus \Delta) = \text{CRC}(m) \oplus L(\Delta)$$

This property enables an attacker to make arbitrary modifications to a WEP ciphertext without ever being detected by the receiver. Let c be a WEP ciphertext, namely

$$c = \big(m, \text{CRC}(m)\big) \ \oplus \ \text{RC4}(\text{IV} \parallel k)$$

For any $\Delta \in \{0,1\}^\ell$, an attacker can create a new ciphertext $c' \leftarrow c \oplus \big(\Delta, L(\Delta)\big)$, which satisfies

$$\begin{aligned} c' \ &= \ \text{RC4}(\text{IV} \parallel k) \ \oplus \ \big(m,\ \text{CRC}(m)\big) \ \oplus \ \big(\Delta,\ L(\Delta)\big) = \\ & \quad\ \text{RC4}(\text{IV} \parallel k) \ \oplus \ \big(m \oplus \Delta,\ \ \text{CRC}(m) \oplus L(\Delta)\big) = \\ & \quad\ \text{RC4}(\text{IV} \parallel k) \ \oplus \ \big(m \oplus \Delta,\ \ \text{CRC}(m \oplus \Delta)\big) \end{aligned}$$

Hence, c' decrypts without errors to $m \oplus \Delta$. We see that given the encryption of m, an attacker can create a valid encryption of $m \oplus \Delta$ for any Δ of his choice. We explained in Section 3.3.2 that this can lead to serious attacks.

Attack 4: Chosen ciphertext attack. The protocol is vulnerable to a chosen ciphertext attack called **chop-chop** that lets the attacker decrypt an encrypted frame of its choice. We describe a simple version of this attack in Exercise 9.5.

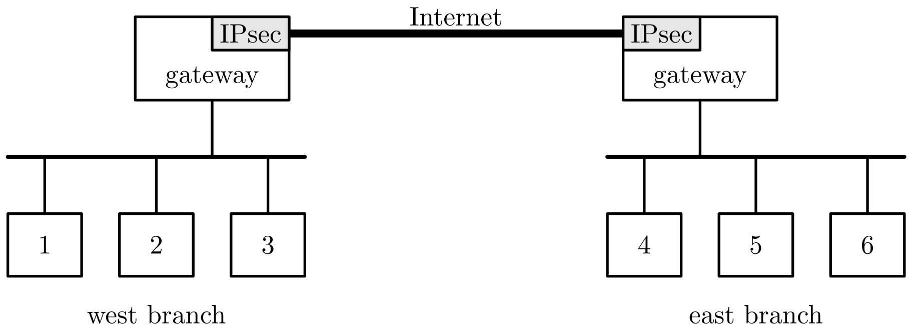


Figure 9.5: A virtual private network (VPN) between east and west office branches

Attack 5: Denial of Service. We briefly mention that 802.11b suffers from a number of serious Denial of Service (DoS) attacks. For example, in 802.11b a wireless client sends a "disassociate" message to the wireless station once the client is done using the network. This allows the station to free memory resources allocates to that client. Unfortunately, the "disassociate" message is unauthenticated, allowing anyone to send a disassociate message on behalf of someone else. Once disassociated, the victim will take a few seconds to re-establish the connection to the base station. As a result, by sending a single "disassociate" message every few seconds, an attacker can prevent a computer of their choice from connecting to the wireless network. These attacks are implemented in 802.11b tools such as `Void11`.

802.11i. Following the failures of the 802.11b WEP protocol, a new standard called 802.11i was ratified in 2004. 802.11i provides authenticated encryption using a MAC-then-encrypt mode called CCM. In particular, CCM uses (raw) CBC-MAC for the MAC and counter mode for encryption. Both are implemented in 802.11i using AES as the underlying PRF. CCM was adopted by NIST as a federal standard [86].

9.11 Case study: IPsec

The IPsec protocol provides confidentiality and integrity for Internet IP packets. The protocol was first published in 1998 and was subsequently updated in 2005. The IPsec protocol consists of many sub-protocols that are not relevant for our discussion here. In this section we will focus on the most commonly used IPsec protocol called **encapsulated security payload** (ESP) in tunnel mode.

Virtual private networks (VPNs) are an important application for IPsec. A VPN enables two office branches to communicate securely over a public Internet channel, as shown in Fig. 9.5. Here, packets from machines 1,2,3 are encrypted at the west gateway using IPsec and transmitted over the public channel. The east gateway decrypts each received packet and forwards it to its destination inside the east branch, namely, one of 4,5,6. We note that all packets sent from west to east are encrypted using the same cryptographic key $k_{w \to e}$. Packets sent from east to west are processed similarly, but encrypted using a different key, $k_{e \to w}$. We will use this VPN example as our motivating example for IPsec.

To understand IPsec one first needs a basic understanding of the IP protocol. Here we focus on IP version 4 (IPv4), which is currently widely deployed. The left side of Fig. 9.6 shows a (cleartext)

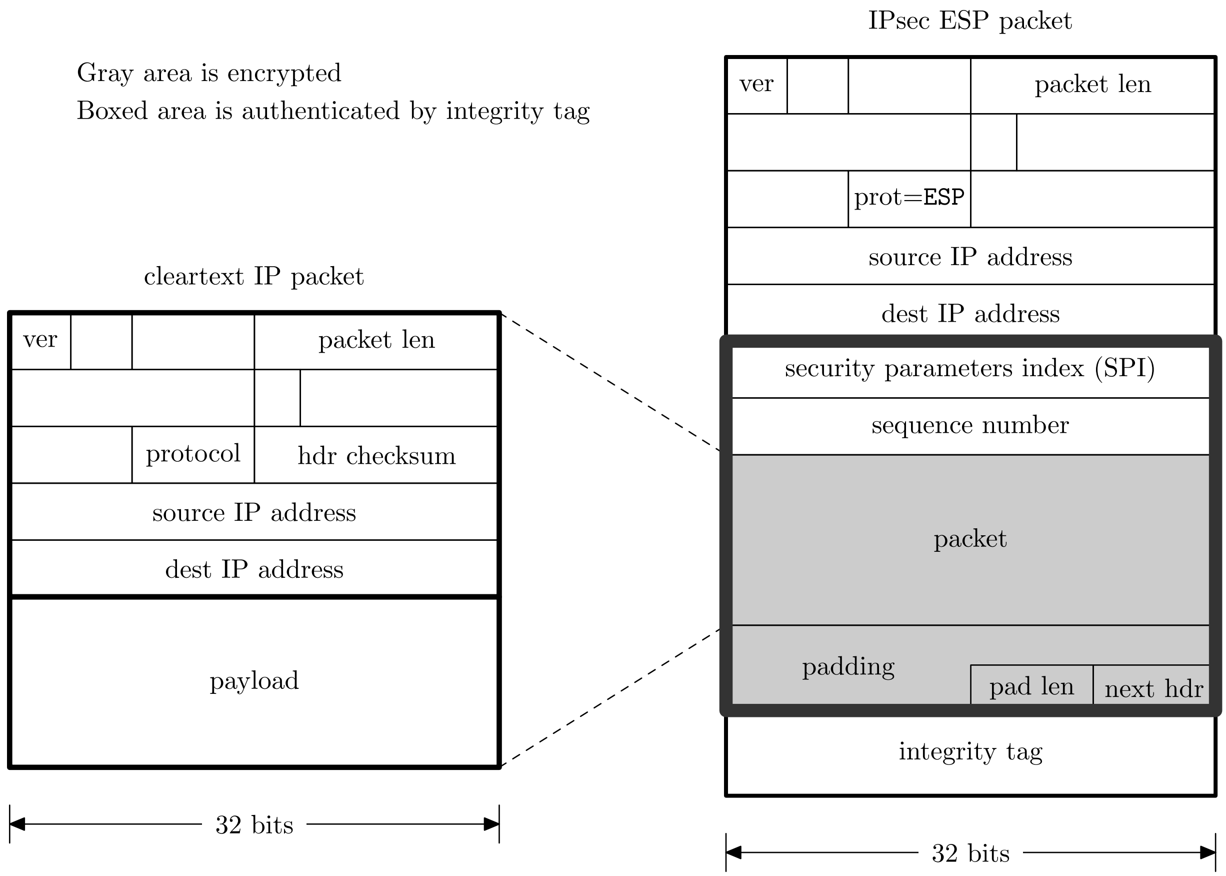


Figure 9.6: Cleartext IPv4 packet and an IPsec ESP packet

IPv4 packet. The packet consists of a packet header and a packet payload. The header contains a bunch of fields, but only a few are relevant to our discussion:

- The first four bits indicate the **version** number which is set to 4 for IPv4.
- The 2-byte **packet length** field contains the length in bytes of the entire packet including the header.
- The 1-byte **protocol** field describes the packet payload For example, protocol = 6 indicates a TCP payload.
- the 2-byte **header checksum** contains a checksum of all header bytes (excluding the checksum field). The checksum is used to detect random transmission errors in the header. Packets with an invalid checksum are dropped at the recipient. The checksum can be computed by anyone and consequently provides no integrity against an attacker. In fact, Internet routers regularly change fields in the packet header as the packet moves from router to router and recompute the checksum.
- The source and destination IP indicate the source and destination addresses for the packet.
- The **payload** contains the packet contents and is variable length.

IPsec encapsulated security payload (ESP). The right side of Fig. 9.6 shows the result of encrypting a packet with ESP in tunnel mode. We first describe the fields in the encrypted packet and then describe the encryption process.

IPsec key management — the SPI field. Every ESP endpoint maintains a **security association database** (SAD). A record in the SAD is called a **security association** (SA) and is identified by a 32 bit identifier called a **security parameters index** (SPI). A SAD record (an SA) contains many connection-specific parameters, such as the ESP encryption algorithm (e.g. 3DES-CBC or AES-CBC), the ESP secret key (e.g. $k_{w\to e}$ or $k_{e\to w}$), the source and destination IP addresses, the SPI, and various key-exchange parameters.

When the east branch gateway sends out a packet, it uses the packet's destination IP address and other parameters to choose a security association (SA) in its security association database (SAD). The gateway embeds the 32-bit SPI of the chosen SA in the packet header and encrypts the packet using the secret key specified in the SA. When the packet arrives at its destination, the recipient locates an appropriate SA in its own SAD using the following algorithm:

1. First, look for an SA matching the received (SPI, dest address, source address);
2. If no match is found, the recipient looks for a match based on the (SPI, dest address) pair;
3. Otherwise, it looks for a match based on the SPI only.

If no SA exists for the received packet, the packet is discarded. Otherwise, the gateway decrypts the packet using the secret key specified in the chosen SA. Most often an SA is used for transmitting packets in one direction, e.g., from east to west. A bi-directional TCP connection between east and west uses two separate SAs — one for packets from east to west and one for packets from west to east. Generally, an ESP endpoint maintains two SAD records for each peer.

The SAD at a particular host is managed semi-manually. Some parameters are managed manually while others are negotiated between the communicating hosts. In particular, an SA secret key can be set manually at both endpoints or it can be negotiated using an IPsec key exchange protocol called IKE [62]. We will not discuss SAD management here.

ESP anti-replay — the sequence number field. The sequence number enables the recipient to detect and discard duplicate packets. Duplication can result from a network error or can be caused by an attacker who is deliberately replaying old packets. Every ESP end point maintains a **sequence number** for each security association. By default the sequence number is 64 bits long (called an extended sequence number), although older versions of ESP use a shorter 32 bit sequence number. The sequence number is initialized to zero when the security association is created and is incremented by one for each packet sent using the SA. The entire 64 bits are included in the MAC calculation. However, only the 32 least significant bits (LSB) are included in the ESP packet header. In other words, ESP endpoints maintain 64-bit counters, of which the 32 MSBs are implicit while the 32 LSBs are explicit in the packet header.

For our discussion of sequence numbers, we assume that there is at most a single host sending packets for each security association (SA). Hence, for a particular SA there is no danger of two hosts sending a packet with the same sequence number. Note that multiple hosts can receive packets for a particular SA, as in the case of multicast. We only disallow multiple hosts from sending packets using a single SA.

For a particular SA, the recipient must discard any packet that contains a 32-bit sequence number that was previously contained in an earlier packet. Since packets can arrive out of order, verifying sequence number unicity at the recipient takes some effort. RFC 4303 recommends that the recipient maintain a window (e.g. bit vector) of size 32. The "right" edge of the window represents the highest, validated sequence number value received on this SA. Packets that contain sequence numbers lower than the "left" edge of the window are discarded. Received packets falling within the window are checked against the list of received packets within the window, and are discarded if their sequence number was already seen. The window shifts whenever a valid packet with a sequence number on the "right" of the current window is received. Consequently, the receiver recovers gracefully from a long sequence of lost packets

If more than 2^{32} consecutive packets are lost, then the 64-bit sequence numbers at the sender and receiver will go out of sync — the 32 MSBs implicitly maintained by the two will differ. As a result, all further packets will be rejected due to MAC validation failure. This explains why the designers of ESP chose to include 32 bits in the packet header — a loss of 2^{32} packets in unlikely. Including fewer bits (e.g. 16 bits) would have greatly increased the chance of communication failure.

Padding and the next header field. ESP first appends a pad to ensure that the length of the data to encrypt is a multiple of the block length of the chosen encryption algorithm (e.g. a multiple of 16 bytes for AES-CBC). It also ensures that the resulting ciphertext length is a multiple of four bytes. The pad length is anywhere from 0 to 255 bytes. An additional pad-length byte is appended to indicate the number of padding bytes preceding it. Finally, a next header (`next-hdr`) byte, is appended to indicate the payload type. Most often the payload type is an IPv4 packet in which case `next-hdr`=4.

ESP supports an optional **traffic flow confidentiality** (TFC) service where the sender attempts to hide the length of the plaintext packet. To do so, the sender appends dummy (unspecified) bytes to the payload before padding takes place. The length of the TFC pad is arbitrary. The packet length field in the plaintext IP header indicates the beginning of the TFC pad. The TFC pad is removed after decryption.

ESP also supports "dummy" packets to defeat traffic analysis. The goal is to prevent an observer from telling when the sender transmits data. For example, one can instruct the sender to transmit a packet every millisecond, whether it has data to send or not. When no data is available, the sender transmits a "dummy" packet which is indicated by setting `next-hdr`=59. Since the `next-hdr` field is encrypted an observer cannot tell dummy packets from real packets. However, at the destination, all dummy packets are discarded immediately after decryption.

The encryption process. ESP implements the encrypt-then-MAC method in four steps. We discuss each step in turn.

1. **Pad.** The pad, including the optional TFC pad and next header field, are appended to the plaintext IP packet.

2. **Encrypt.** The gray area in Fig. 9.6 is encrypted with the algorithm and key specified by the SA. ESP supports a variety of encryption algorithms, but is required to support 3DES-CBC, AES-CBC, and AES counter mode. For CBC modes the IV is prepended to the encrypted payload and is sent in the clear. The encryption algorithm can be set to NULL in which case no encryption takes place. This is used when ESP provides integrity but no confidentiality.

3. **MAC.** An integrity tag is computed using an algorithm and key specified in the SA. The tag is computed over the following data

$$\text{SPI} \parallel \text{64-bit sequence number} \parallel \text{ciphertext}$$

   where ciphertext is the result of Step 2. Note that the tag is computed over the 64 bit sequence number even though only 32 bits are embedded in the packet. The resulting tag is placed in the integrity tag field following the ciphertext. ESP supports a variety of MAC algorithms, but is required to support HMAC-SHA1-96, HMAC-MD5-96, and AES-XCBC-MAC-96 (XCBC-MAC is a variant of CMAC). The integrity tag field is optional and is omitted if the encryption algorithm already provides authenticated encryption, as in the case of GCM.

4. **Encapsulate.** Finally, an IPv4 packet header is prepended to obtain an ESP packet as shown on the right side of Fig. 9.6. The protocol field in the IPv4 header is set to 50 indicating an ESP payload.

Decryption follows a similar process. The recipient first checks the 32-bit sequence number. If the value is repeated or outside the allowed window, the packet is dropped. Next, the recipient checks the tag field, and rejects the packet if MAC verification fails. The packet is then decrypted and the padding removed. If the packet is a dummy packet (i.e. the next header field is equal to 59), the packet is discarded. Finally, the original cleartext packet is reconstructed and sent to the destination. Note that in principle, the sequence number field could have been encrypted. The designers of ESP chose to send the field in the clear so as to reduce the time until a duplicate packet is rejected.

Security. IP packets can arrive at any order, be duplicated, and even modified. By relying on encrypt-then-MAC and on the sequence number, ESP ensures that the recipient sees a data stream identical to the one transmitted by the sender. One issue that haunts ESP is a setting that provides CPA-secure encryption without an integrity check. RFC 4303 states that

ESP allows encryption-only SAs because this may offer considerably better performance and still provide adequate security, e.g., when higher-layer authentication/integrity protection is offered independently.

Relying on a higher application layer for integrity is highly risky. On the sender side the application layer processes data before passing it to the IP layer. Hence, this implements MAC-then-encrypt which from a theoretical point view we know can be insecure. More importantly, in practice it is dangerous to assume that the higher layer will protect the entire IP packet. For example, a higher layer such as SSL may provide integrity without encryption. Combining encryption-only ESP and integrity-only SSL will be insecure since the SSL layer will not provide integrity for the encrypted packet header. As a result, an attacker can tamper with the destination IP field in the encrypted packet. The recipient's IPsec gateway will decrypt the packet and forward the result to an unintended destination, thus causing a serious privacy breach. This and other dangers of the ESP encryption-only mode are discussed in [8, 87].

We note, however, that when the cipher used provides authenticated encryption (such as GCM mode) it is perfectly fine to use encryption without an integrity check, since the cipher already provides authenticated encryption.

9.12 A fun application: private information retrieval

To be written.

9.13 Notes

Citations to the literature to be added.

9.14 Exercises

9.1 (AE-security: simple examples). Let (E, D) be an AE-secure cipher. Consider the following derived ciphers:

(a) $E_1(k,m) := \big(E(k,m),\ E(k,m)\big); \quad D_2\big(k,\ (c_1,c_2)\big) := \begin{cases} D(k,c_1) & \text{if } D(k,c_1) = D(k,c_2) \\ \textsf{reject} & \text{otherwise} \end{cases}$

(b) $E_2(k,m) := \big\{c \leftarrow E(k,m),\ \text{output } (c,c)\big\}; \quad D_2\big(k,\ (c_1,c_2)\big) := \begin{cases} D(k,c_1) & \text{if } c_1 = c_2 \\ \textsf{reject} & \text{otherwise} \end{cases}$

Show that part (b) is AE-secure, but part (a) is not.

9.2 (AE-security: some insecure constructions). Let (E, D) be a CPA-secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ and let $H_1 : \mathcal{M} \to \mathcal{T}$ and $H_2 : \mathcal{C} \to \mathcal{T}$ be collision resistant hash functions. Define the following two ciphers:

$$E_1(k,m) := \big(E(k,m),\ H_1(m)\big); \quad D_1\big(k,\ (c_1,c_2)\big) := \begin{cases} D(k,c_1) & \text{if } H_1(D(k,c_1)) = c_2 \\ \textsf{reject} & \text{otherwise} \end{cases}$$

$$E_2(k,m) := \big(E(k,m),\ H_2(c)\big); \quad D_2\big(k,\ (c_1,c_2)\big) := \begin{cases} D(k,c_1) & \text{if } H_2(c_1) = c_2 \\ \textsf{reject} & \text{otherwise} \end{cases}$$

Show that both ciphers are not AE-secure.

9.3 (An Android Keystore Attack). Let (E, D) be a secure block cipher defined over $(\mathcal{K}, \mathcal{X})$ and let $(E_{\text{cbc}}, D_{\text{cbc}})$ be the cipher derived from (E, D) using randomized CBC mode, as in Section 5.4.3. Let $H : \mathcal{M} \to \mathcal{X}$ be a collision resistant hash function. Consider the following attempt at building an AE-secure cipher:

$$E_1(k,m) := E_{\text{cbc}}\big(k,\ (H(m),m)\big)\ ; \qquad D_1(k,c) := \left\{ \begin{array}{l} (t,m) \leftarrow D_{\text{cbc}}(k,c) \\ \text{if } t = H(m) \text{ output } m, \text{ otherwise } \textsf{reject} \end{array} \right\}$$

Show that (E_1, D_1) is not AE-secure by giving a chosen-ciphertext attack on it. You may assume $m \in \mathcal{X}$ for simplicity. This construction was used to protect secret keys in the Android KeyStore. The chosen-ciphertext attack resulted in a compromise of the key store [93].

9.4 (Redundant message encoding does not give AE***).*** The attack in the previous exercise can be generalized if instead of using CBC encryption as the underlying cipher, we use randomized counter mode, as in Section 5.4.2. Let $(E_{\text{ctr}}, D_{\text{ctr}})$ be such a counter-mode cipher, and assume that its message space is $\{0,1\}^{\ell'}$. Let $f : \{0,1\}^{\ell} \to \{0,1\}^{\ell'}$ be a one-to-one function, and let $g : \{0,1\}^{\ell'} \to \{0,1\}^{\ell} \cup \{\perp\}$ be its inverse, in the sense that $g(m') = m$ whenever $m' = f(m)$ for some m, and $g(m') = \perp$ if m' is not in the image of f. Intuitively, f represents an "error detecting code": a message $m \in \{0,1\}^{\ell}$ is "encoded" as $m' = f(m)$. If m' gets modified into a value $\tilde{m}'$, this modification will be detected if $g(\tilde{m}') = \perp$. Now define a new cipher (E_2, D_2) with message space $\{0,1\}^{\ell}$ as follows:

$$E_2(k,m) := E_{\text{ctr}}\big(k,\ f(m)\big)\ ; \qquad D_1(k,c) := \left\{ \begin{array}{l} m' \leftarrow D_{\text{ctr}}(k,c) \\ \text{if } g(m') \neq \perp \text{ output } g(m'), \text{ otherwise } \textsf{reject} \end{array} \right\}$$

Show that (E_2, D_2) is not AE-secure by giving a chosen-ciphertext attack on it.

9.5 (Chop-chop attack). The parity bit b for a message $m \in \{0,1\}^*$ is just the XOR of all the bits in m. After appending the parity bit, the message $m' = m \parallel b$ has the property that the XOR of all the bits is zero. Parity bits are sometimes used as a very simple form of error detection. They are meant to provide a little protection against low-probability, random errors: if a single bit of m' gets flipped, this can be detected, since the XOR of the bits of the corrupted m' will now be one.

Consider a cipher where encryption is done using randomized counter mode without any padding. Messages are variable length bit strings and ciphertexts are bit strings of the same length as plaintext. No MAC is used, but before the plaintext is encrypted, the sender appends a parity bit to the end of the plaintext. After the receiver decrypts, he checks the parity bit and returns either the plaintext (with the parity bit removed) or reject.

Design a chosen-ciphertext attack that recovers the complete plaintext of every encrypted message.

Hint: Use the fact that the system encrypts variable length messages.

Remark: A variant of this attack, called chopchop, was used successfully against encryption in the 802.11b protocol. The name is a hint for how the attack works. Note that the previous exercise already tells us that this scheme is not CCA-secure, but the attack in this exercise is much more devastating.

9.6 (Nested encryption). Let (E, D) be an AE-secure cipher. Consider the following derived cipher (E', D'):

$$E'\big((k_1, k_2), m\big) := E\big(k_2, E(k_1, m)\big); \qquad D'\big((k_1, k_2),\ c\big) := \begin{cases} D\big(k_1, D(k_2, c)\big) & \text{if } D(k_2, c) \neq \textsf{reject} \\ \textsf{reject} & \text{otherwise} \end{cases}$$

(a) Show that (E', D') is AE-secure even if the adversary knows k_1, but not k_2.

(b) Show that (E', D') is not AE-secure if the adversary knows k_2 but not k_1.

(c) Design a cipher built from (E, D) where keys are pairs $(k_1, k_2) \in \mathcal{K}^2$ and the cipher remains AE-secure even if the adversary knows one of the keys, but not the other.

9.7 (A format oracle attack). Let $\mathcal{E}$ be an arbitrary CPA-secure cipher, and assume that the key space for $\mathcal{E}$ is $\{0,1\}^n$. Show how to "sabotage" $\mathcal{E}$ to obtain another cipher $\mathcal{E}'$ such that $\mathcal{E}'$ is still CPA secure, but $\mathcal{E}'$ is insecure against chosen ciphertext attack, in the following sense. In the attack, the adversary is allowed to make several decryption queries, such that in each query, the adversary only learns whether the result of the decryption was reject or not. Design an adversary that makes a series of decryption queries as above, and then outputs the secret key in its entirety.

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9.8 (Choose independent keys). Let us see an example of a CPA-secure cipher and a secure MAC that are insecure when used in encrypt-then-MAC when the same secret key k is used for both the cipher and the MAC. Let (E, D) be a block cipher defined over $(\mathcal{K}, \mathcal{X})$ where $\mathcal{X} = \{0,1\}^n$ and $|\mathcal{X}|$ is super-poly. Consider randomized CBC mode encryption built from (E, D) as the CPA-secure cipher for single block messages: an encryption of $m \in \mathcal{X}$ is the pair $c := (r,\ E(k,\ r \oplus m))$ where r is the random IV. Use RawCBC built from (E, D) as the secure MAC. This MAC is secure in this context because it is only being applied fixed length messages (messages in $\mathcal{X}^2$): the tag on a ciphertext $c \in \mathcal{X}^2$ is $t := E\big(k,\ E(k, c[0]) \oplus c[1]\big)$. Show that using the same key k for both the cipher and the MAC in encrypt-then-MAC results in a cipher that is not CPA secure.

9.9 (MAC-then-encrypt). Prove that MAC-then-encrypt provides authenticated encryption when the underlying cipher is randomized CBC mode encryption and the MAC is a secure MAC. For concreteness, if the underlying cipher works on blocks of a fixed size, a message m is a sequence of full blocks, and the tag t for the MAC is one full block, so the message that is CBC-encrypted is the block sequence $m \parallel t$.

9.10 (An AEAD from encrypt-and-MAC). Let (E, D) be randomized counter mode encryption defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ where the underlying secure PRF has domain $\mathcal{X}$. We let $E(k, m; r)$ denote the encryption of message m with key k using $r \in \mathcal{X}$ as the IV. Let F be a secure PRF defined over $(\mathcal{K},\ (\mathcal{M} \times \mathcal{D} \times \mathcal{N}),\ \mathcal{X})$. Show that the following cipher (E_1, D_1) is a secure nonce-based

AEAD cipher assuming $|\mathcal{X}|$ is super-poly.

$$E_1\big((k_e, k_m),\ m,\ d,\ \mathcal{N}\ \big) := \big\{ t \leftarrow F\big(k_m,\ (m, d, \mathcal{N})\big),\ c \xleftarrow{R} E(k_c, m; t),\ \text{output } (c, t)\ \big\}$$

$$D_1\big((k_e, k_m),\ (c, t),\ d,\ \mathcal{N})\ \big) := \left\{ \begin{array}{l} m \leftarrow D(k_e, c; t) \\ \text{if } F\big(k_m,\ (m, d, \mathcal{N})\big) \neq t \text{ output } \mathsf{reject}, \text{ otherwise output } m \end{array} \right\}$$

This method is loosely called encrypt-and-MAC because the message m is both encrypted by the cipher and is the input to the MAC signing algorithm, which here is a PRF.

Discussion: This construction is related to the authenticated SIV cipher (Exercise 9.11) and offers similar **nonce re-use resistance**. One down-side of this system is that the tag t cannot be truncated as one often does with a PRF-based MAC.

9.11 (Authenticated SIV). We discuss a modification of the SIV construction, introduced in Exercise 5.8, that provides ciphertext integrity without enlarging the ciphertext any further. We call this the **authenticated SIV** construction. With $\mathcal{E} = (E, D)$, F, and $\mathcal{E}' = (E', D')$ as in Exercise 5.8, we define $\mathcal{E}'' = (E', D'')$, where

$$D''\big((k, k'),\ c\big) := \left\{ \begin{array}{l} m \leftarrow D(k, c) \\ \text{if } E'((k, k'), m) = c \text{ output } m, \text{ otherwise output } \mathsf{reject} \end{array} \right\}$$

Assume that $|\mathcal{R}|$ is super-poly and that for very fixed key $k \in \mathcal{K}$ and $m \in \mathcal{M}$, the function $E(k, m; \cdot) : \mathcal{R} \to \mathcal{C}$ is one to one (which holds for counter and CBC mode encryption). Show that $\mathcal{E}''$ provides ciphertext integrity.

Note: Since the encryption algorithm of $\mathcal{E}''$ is the same as that of $\mathcal{E}'$ we know that $\mathcal{E}''$ is deterministic CPA-secure, assuming that $\mathcal{E}$ is CPA-secure (as was shown in Exercise 5.8).

9.12 (Constructions based on strongly secure block ciphers). Let (E, D) be a block cipher defined over $(\mathcal{K},\ \mathcal{M} \times \mathcal{R})$.

(a) As in Exercise 5.6, let (E', D') be defined as

$$E'(k, m) := \big\{ r \xleftarrow{R} \mathcal{R},\ c \xleftarrow{R} E\big(k,\ (m, r)\big),\ \text{output } c \big\}$$
$$D'(k, c) := \big\{ (m, r') \leftarrow D(k, c),\ \text{output } m \big\}$$

Show that (E', D') is CCA-secure provided (E, D) is a *strongly secure* block cipher and $1/|\mathcal{R}|$ is negligible. This is an example of a CCA-secure cipher that clearly does not provide ciphertext integrity.

(b) Let (E'', D'') be defined as

$$E''(k, m) := \big\{ r \xleftarrow{R} \mathcal{R},\ c \xleftarrow{R} E\big(k,\ (m, r)\big),\ \text{output } (c, r) \big\}$$

$$D''\big(k, (c, r)\big) := \left\{ \begin{array}{l} (m, r') \leftarrow D(k, c) \\ \text{if } r = r' \text{ output } m, \text{ otherwise output } \mathsf{reject} \end{array} \right\}$$

This cipher is defined over $\big(\mathcal{K},\ \mathcal{M},\ (\mathcal{M} \times \mathcal{R}) \times \mathcal{R}\big)$. Show that (E'', D'') is AE-secure provided (E, D) is a strongly secure block cipher and $1/|\mathcal{R}|$ is negligible.

(c) Suppose that $0 \in \mathcal{R}$ and we modify algorithms E'' and D'' to work as follows:

$$\tilde{E}''(k, m) := \{r \leftarrow 0,\ c \xleftarrow{\text{R}} E(k,\ (m, r)),\ \text{output } c\}$$

$$\tilde{D}''(k, c) := \left\{ \begin{array}{l} (m, r') \leftarrow D(k, c) \\ \text{if } r' = 0 \text{ output } m, \text{ otherwise output } \textsf{reject} \end{array} \right\}$$

Show that $(\tilde{E}'', \tilde{D}'')$ is *one-time* AE-secure provided (E, D) is a strongly secure block cipher, and $1/|\mathcal{R}|$ is negligible.

9.13 (MAC from encryption). Let (E, D) be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Define the following MAC system (S, V) also defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$:

$$S(k, m) := E(k, m); \qquad V(k, m, t) := \begin{cases} \textsf{accept} & \text{if } D(k, t) = m \\ \textsf{reject} & \text{otherwise} \end{cases}$$

Show that if (E, D) has ciphertext integrity then (S, V) is a secure MAC system.

9.14 (GCM analysis). Give a complete security analysis of GCM (see Section 9.7). Show that it is nonce-based AEAD secure assuming the security of the underlying block cipher as a PRF and that GHASH is an XOR-DUF. Start out with the easy case when the nonce is 96-bits. Then proceed to the more general case where GHASH may be applied to the nonce to compute x.

9.15 (Plaintext integrity). Consider a weaker notion of integrity called **plaintext integrity**, or simply PI. The PI game is identical to the CI game except that the winning condition is relaxed to:

- $D(k, c) \neq \textsf{reject}$, and
- $D(k, c) \notin \{m_1, m_2, \ldots\}$

Prove that the following holds:

(a) Show that MAC-then-Encrypt is both CPA and PI secure.

Note: The MAC-then-Encrypt counter-example (Section 9.4.2) shows that a system that is CPA and PI secure is not CCA-secure (and, therefore, not AE-secure).

(b) Prove that a system that is CCA- and PI-secure is also AE-secure. The proof only needs a weak version of CCA, namely where the adversary issues a single decryption query and is told whether the ciphertext is accepted or rejected. Also, you may assume a super-poly-sized message space.

9.16 (Encrypted UHF MAC). Let H be a hash function defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{X})$ and (E, D) be a cipher defined over $(\mathcal{K}_E, \mathcal{X}, \mathcal{C})$. Define the **encrypted UHF MAC system** $\mathcal{I} = (S, V)$ as follows: for key (k_1, k_2) and message $m \in \mathcal{M}$ define

$$S\big((k_1, k_2),\ m\big) := E\big(k_1,\ H(k_2, m)\big)$$

$$V\big((k_1, k_2),\ m,\ c\big) := \begin{cases} \textsf{accept} & \text{if } H(k_2, m) = D(k_1, c), \\ \textsf{reject} & \text{otherwise.} \end{cases}$$

Show that $\mathcal{I}$ is a secure MAC system assuming H is a computational UHF and (E, D) provides authenticated encryption. Recall from Section 7.4 that CPA security of (E, D) is insufficient for this MAC system to be secure.

9.17 (Simplified OCB mode). OCB is an elegant and efficient AE cipher built from a tweakable block cipher (as defined in Exercise 4.11). Let (E, D) be a tweakable block cipher defined over $(\mathcal{K}, \mathcal{X}, \mathcal{T})$ where $\mathcal{X} := \{0,1\}^n$ and the tweak set is $\mathcal{T} := \mathcal{N} \times \{-\ell, \dots, \ell\}$. Consider the following nonce-based cipher (E', D') with key space $\mathcal{K}$, message space $\mathcal{X}^{\leq \ell}$, ciphertext space $\mathcal{X}^{\ell+1}$, and nonce space $\mathcal{N}$. For simplicity, the cipher does not support associated data.

$$E'(k, m, \mathcal{N}) := \left\{ \begin{array}{l} \text{create (uninitialized) } c \in \mathcal{X}^{|m|} \\ \text{checksum} \leftarrow 0^n \\ \text{for } i = 0, \dots, |m| - 1 : \\ \quad c[i] \leftarrow E\big(k,\ m[i],\ (\mathcal{N}, i+1)\big) \\ \quad \text{checksum} \leftarrow \text{checksum} \oplus m[i] \\ t \leftarrow E\big(k,\ \text{checksum},\ (\mathcal{N}, -|m|)\big) \\ \text{output } (c, t) \end{array} \right\}$$

$$D'(k, (c,t), \mathcal{N}) := \left\{ \begin{array}{l} \text{create (uninitialized) } m \in \mathcal{X}^{|c|} \\ \text{checksum} \leftarrow 0^n \\ \text{for } i = 0, \dots, |c| - 1 : \\ \quad m[i] \leftarrow D\big(k,\ c[i],\ (\mathcal{N}, i+1)\big) \\ \quad \text{checksum} \leftarrow \text{checksum} \oplus m[i] \\ t' \leftarrow E\big(k,\ \text{checksum},\ (\mathcal{N}, -|c|)\big) \\ \text{if } t = t' \text{ output } m, \text{ else } \mathsf{reject} \end{array} \right\}$$

(a) Prove that (E', D') is a nonce-based AE-secure cipher assuming (E, D) is a strongly secure tweakable block cipher and $|\mathcal{X}|$ is super-poly.

(b) Show that if t were computed as $t \leftarrow E\big(k,\ \text{checksum},\ (\mathcal{N}, 0)\big)$ then the scheme would be insecure: it would have no ciphertext integrity.

9.18 (Non-committing encryption). Let (E, D) be a cipher. We say that the cipher is **non-committing** if an adversary can find a ciphertext c and two keys k_0, k_1 such that c decrypts successfully under both k_0 and k_1 and the resulting plaintexts are different. The non-committing property means that the adversary can transmit c, but if he or she are later required to reveal the decryption key, say for an internal audit, the adversary can "open" the ciphertext in two different ways.

(a) Let (E, D) be an encrypt-then-MAC AE-secure cipher where the underlying encryption is randomized counter mode built using a secure PRF. Show that (E, D) is non-committing.

(b) Show that GCM mode encryption is non-committing.

(c) Describe a simple way in which the ciphers from parts (a) and (b) can be made committing.

9.19 (Middlebox encryption). In this exercise we develop a mode of encryption that lets a middlebox placed between the sender and recipient inspect all traffic in the clear, but prevents the middlebox for modifying traffic en-route. This is often needed in enterprise settings where a middlebox ensures that no sensitive information is accidentally sent out. Towards this goal let us define a middlebox cipher as a tuple of four algorithms (E, D, D', K) where $E(k, m)$ and $D(k, c)$ are the usual encryption and decryption algorithms used by the end-points, K is an algorithm that derives a sub-key k' from the primary key k (i.e., $k' \xleftarrow{R} K(k)$), and $D'(k', c)$ is the decryption algorithm used by the middlebox with the sub-key k'. We require the usual correctness properties: $D(k, c)$ and $D'(k', c)$ output m whenever $c \xleftarrow{R} E(k, m)$ and $k' \xleftarrow{R} K(k)$.

(a) Security for a middlebox cipher (E, D, D', K) captures our desired confidentiality and integrity requirements. In particular, we say that a middlebox cipher is secure if the following three properties hold:

   (i) the cipher is secure against a chosen plaintext attack (CPA security) when the adversary knows nothing about k,

   (ii) the cipher provides ciphertext integrity with respect to the decryption algorithm $D'(k', \cdot)$, and the adversary knows nothing about k, and

   (iii) the cipher provides ciphertext integrity with respect to the decryption algorithm $D(k, \cdot)$, and the adversary is given a sub-key $k' \xleftarrow{R} K(k)$, but again knows nothing about k.

   The second requirement says that the middlebox will only decrypt authentic ciphertexts. The third requirement says that the receiving end-point will only decrypt authentic ciphertexts, even if the middlebox is corrupt.

   Formalize these requirements as attack games.

(b) Give a construction that satisfies your definition from part (a). You can use an AE secure cipher and a secure MAC as building blocks.

Part II

Public key cryptography

In the second part of the book we study how parties who don't share a secret key can communicate over a public network. We start off by introducing the basic tools used in public key cryptography — the RSA and Diffie-Hellman functions. We then show how one party, Alice, can send messages to another party, Bob, given Bob's public key. We then discuss digital signatures and given several constructions. Some constructions are based entirely on tools tools from Part I while other constructions are based on public key tools. The last two chapters in part II explain how to establish a secure session using identification and key exchange.

Chapter 10

Public key tools

We begin our discussion of public-key cryptography by introducing several basic tools that will be used in the remainder of the book. The main applications for these tools will emerge in the next few chapters where we use them for public-key encryption, digital signatures, and key exchange. Since we use some basic algebra and number theory in this chapter, the reader is advised to first briefly scan through Appendix A.

We start with a simple toy problem: generating a shared secret key between two parties so that a passive eavesdropping adversary cannot feasibly guess their shared key. The adversary can listen in on network traffic, but cannot modify messages en-route or inject his own messages. In a later chapter we develop the full machinery needed for key exchange in the presence of an active attacker who may tamper with network traffic.

At the onset we emphasize that security against eavesdropping is typically not sufficient for real world-applications, since an attacker capable of listening to network traffic is often also able to tamper with it; nevertheless, this toy eavesdropping model is a good way to introduce the new public-key tools.

10.1 A toy problem: anonymous key exchange

Two users, Alice and Bob, who never met before talk on the phone. They are worried that an eavesdropper is listening to their conversation and hence they wish to encrypt the session. Since Alice and Bob never met before they have no shared secret key with which to encrypt the session. Thus, their initial goal is to generate a shared secret unknown to the adversary. They may later use this secret as a session-key for secure communication. To do so, Alice and Bob execute a protocol where they take turns in sending messages to each other. The eavesdropping adversary can hear all these messages, but cannot change them or inject his own messages. At the end of the protocol Alice and Bob should have a secret that is unknown to the adversary. The protocol itself provides no assurance to Alice that she is really talking to Bob, and no assurance to Bob that he is talking to Alice — in this sense, the protocol is "anonymous."

More precisely, we model Alice and Bob as communicating machines. A **key exchange protocol** P is a pair of probabilistic machines (A, B) that take turns in sending messages to each other. At the end of the protocol, when both machines terminate, they both obtain the same value k. A **protocol transcript** T_P is the sequence of messages exchanged between the parties in one execution of the protocol. Since A and B are probabilistic machines, we obtain a different transcript every time we run the protocol. Formally, the transcript T_P of protocol P is a random variable, which is a function of the random bits generated by A and B. The eavesdropping adversary $\mathcal{A}$ sees the entire transcript T_P and its goal is to figure out the secret k. We define security of a key exchange protocol using the following game.

Attack Game 10.1 (Anonymous key exchange). For a key exchange protocol $P = (A, B)$ and a given adversary $\mathcal{A}$, the attack game runs as follows.

- The challenger runs the protocol between A and B to generate a shared key k and transcript T_P. It gives T_P to $\mathcal{A}$.
- $\mathcal{A}$ outputs a guess $\hat{k}$ for k.

We define $\mathcal{A}$'s advantage, denoted $\text{AnonKE}\mathsf{adv}[\mathcal{A}, P]$, as the probability that $\hat{k} = k$. $\square$

Definition 10.1. *We say that an anonymous key exchange protocol P is secure against an eavesdropper if for all efficient adversaries $\mathcal{A}$, the quantity* $\text{AnonKE}\mathsf{adv}[\mathcal{A}, P]$ *is negligible.*

This definition of security is extremely weak, for three reasons. First, we assume the adversary is unable to tamper with messages. Second, we only guarantee that the adversary cannot guess k in its entirety. This does not rule out the possibility that the adversary can guess, say, half the bits of k. If we are to use k as a secret session key, the property we would really like is that k is indistinguishable from a truly random key. Third, the protocol provides no assurance of the identities of the participants. We will strengthen Definition 10.1 to meet these stronger requirements in Chapter 20.

Given all the tools we developed in Part 1, it is natural to ask if anonymous key exchange can be done using an arbitrary secure symmetric cipher. The answer is yes, it can be done as we show in Section 10.8, but the resulting protocol is highly inefficient. To develop efficient protocols we must first introduce a few new tools.

10.2 One-way trapdoor functions

In this section, we introduce a tool that will allow us to build an efficient and secure key exchange protocol. In Section 8.11, we introduced the notion of a one-way function. This is a function $F : \mathcal{X} \to \mathcal{Y}$ that is easy to compute, but hard to invert. As we saw in Section 8.11, there are a number of very efficient functions that are plausibly one-way. One-way functions, however, are not sufficient for our purposes. We need one-way functions with a special feature, called a **trapdoor**. A trapdoor is a secret that allows one to efficiently invert the function; however, without knowledge of the trapdoor, the function remains hard to invert.

Let us make this notion more precise.

Definition 10.2 (Trapdoor function scheme). *Let $\mathcal{X}$ and $\mathcal{Y}$ be finite sets. A* ***trapdoor function scheme*** *$\mathcal{T}$, defined over $(\mathcal{X}, \mathcal{Y})$, is a triple of algorithms (G, F, I), where*

- *G is a probabilistic key generation algorithm that is invoked as $(pk, sk) \overset{\text{R}}{\leftarrow} G()$, where pk is called a* ***public key*** *and sk is called a* ***secret key****.*
- *F is a deterministic algorithm that is invoked as $y \leftarrow F(pk, x)$, where pk is a public key (as output by G) and x lies in $\mathcal{X}$. The output y is an element of $\mathcal{Y}$.*

- *I is a deterministic algorithm that is invoked as $x \leftarrow I(sk, y)$, where sk is a secret key (as output by G) and y lies in $\mathcal{Y}$. The output x is an element of $\mathcal{X}$.*

Moreover, the following ***correctness property*** *should be satisfied: for all possible outputs (pk, sk) of $G()$, and for all $x \in \mathcal{X}$, we have $I(sk,\ F(pk, x)\ ) = x$.*

Observe that for every pk, the function $F(pk, \cdot)$ is a function from $\mathcal{X}$ to $\mathcal{Y}$. The correctness property says that sk is the trapdoor for inverting this function; note that this property also implies that the function $F(pk, \cdot)$ is one-to-one. Note that we do not insist that $F(pk, \cdot)$ maps $\mathcal{X}$ onto $\mathcal{Y}$. That is, there may be elements $y \in \mathcal{Y}$ that do not have any preimage under $F(pk, \cdot)$. For such y, we make no requirements on algorithm I — it can return some arbitrary element $x \in \mathcal{X}$ (one might consider returning a special reject symbol in this case, but it simplifies things a bit not to do this).

In the special case where $\mathcal{X} = \mathcal{Y}$, then $F(pk, \cdot)$ is not only one-to-one, but onto. That is, $F(pk, \cdot)$ is a *permutation on the set* $\mathcal{X}$. In this case, we may refer to (G, F, I) as a **trapdoor permutation scheme** defined over $\mathcal{X}$.

The basic security property we want from a trapdoor permutation scheme is a one-wayness property, which basically says that given pk and $F(pk, x)$ for random $x \in \mathcal{X}$, it is hard to compute x without knowledge of the trapdoor sk. This is formalized in the following game.

Attack Game 10.2 (One-way trapdoor function scheme). For a given trapdoor function scheme $\mathcal{T} = (G, F, I)$, defined over $(\mathcal{X}, \mathcal{Y})$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger computes

$$(pk, sk) \overset{R}{\leftarrow} G(), \quad x \overset{R}{\leftarrow} \mathcal{X}, \quad y \leftarrow F(pk, x)$$

  and sends (pk, y) to the adversary.

- The adversary outputs $\hat{x} \in \mathcal{X}$.

We define the adversary's advantage in inverting $\mathcal{T}$, denoted $\mathrm{OW}\mathsf{adv}[\mathcal{A}, \mathcal{T}]$, to be the probability that $\hat{x} = x$. $\square$

Definition 10.3. *We say that a trapdoor function scheme $\mathcal{T}$ is* ***one way*** *if for all efficient adversaries $\mathcal{A}$, the quantity $\mathrm{OW}\mathsf{adv}[\mathcal{A}, \mathcal{T}]$ is negligible.*

Note that in Attack Game 10.2, since the value x is uniformly distributed over $\mathcal{X}$ and $F(pk, \cdot)$ is one-to-one, it follows that the value $y := F(pk, x)$ is uniformly distributed over the image of $F(pk, \cdot)$. In the case of a trapdoor permutation scheme, where $\mathcal{X} = \mathcal{Y}$, the value of y is uniformly distributed over $\mathcal{X}$.

10.2.1 Key exchange using a one-way trapdoor function scheme

We now show how to use a one-way trapdoor function scheme $\mathcal{T} = (G, F, I)$, defined over $(\mathcal{X}, \mathcal{Y})$, to build a secure anonymous key exchange protocol. The protocol runs as follows, as shown in Fig. 10.1:

- Alice computes $(pk, sk) \overset{R}{\leftarrow} G()$, and sends pk to Bob.

- Upon receiving pk from Alice, Bob computes $x \overset{R}{\leftarrow} \mathcal{X}, y \leftarrow F(pk, x)$, and sends y to Alice.

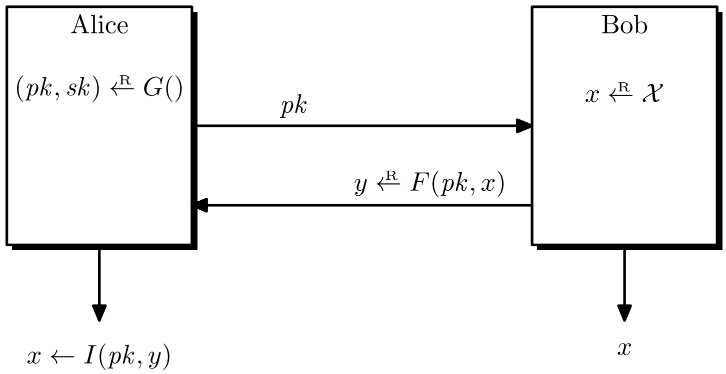


Figure 10.1: Key exchange using a trapdoor function scheme

- Upon receiving y from Bob, Alice computes $x \leftarrow I(sk, y)$.

The correctness property of the trapdoor function scheme guarantees that at the end of the protocol, Alice and Bob have the same value x — this is their shared, secret key. Now consider the security of this protocol, in the sense of Definition 10.1. In Attack Game 10.1, the adversary sees the transcript consisting of the two messages pk and y. If the adversary could compute the secret x from this transcript with some advantage, then this very same adversary could be used directly to break the trapdoor function scheme, as in Attack Game 10.2, with exactly the same advantage.

10.2.2 Mathematical details

We give a more mathematically precise definition of a trapdoor function scheme, using the terminology defined in Section 2.4.

Definition 10.4 (Trapdoor function scheme). *A* ***trapdoor function scheme*** *is a triple of efficient algorithms (G, F, I) along with families of spaces with system parameterization P:*

$$\mathbf{X} = \{\mathcal{X}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \mathbf{Y} = \{\mathcal{Y}_{\lambda,\Lambda}\}_{\lambda,\Lambda}.$$

As usual, $\lambda \in \mathbb{Z}_{\geq 1}$ is a security parameter and $\Lambda \in \text{Supp}(P(\lambda))$ is a domain parameter. We require that

1. $\mathbf{X}$ *is efficiently recognizable and sampleable.*

2. $\mathbf{Y}$ *is efficiently recognizable.*

3. *G is an efficient probabilistic algorithm that on input λ, Λ, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \text{Supp}(P(\lambda))$, outputs a pair (pk, sk), where pk and sk are bit strings whose lengths are always bounded by a polynomial in λ.*

4. *F is an efficient* deterministic *algorithm that on input λ, Λ, pk, x, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \text{Supp}(P(\lambda))$, $(pk, sk) \in \text{Supp}(G(\lambda, \Lambda))$ for some sk, and $x \in \mathcal{X}_{\lambda,\Lambda}$, outputs an element of $\mathcal{Y}_{\lambda,\Lambda}$.*

5. *I is an efficient* deterministic *algorithm that on input λ, Λ, sk, y, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in$ $\mathrm{Supp}(P(\lambda))$, $(pk, sk) \in \mathrm{Supp}(G(\lambda, \Lambda))$ for some pk, and $y \in \mathcal{Y}_{\lambda,\Lambda}$, outputs an element of $\mathcal{X}_{\lambda,\Lambda}$.*

6. *For all $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $(pk, sk) \in \mathrm{Supp}(G(\lambda, \Lambda))$, and $x \in \mathcal{X}_{\lambda,\Lambda}$, we have $I(\lambda, \Lambda; sk, F(\lambda, \Lambda; pk, x)) = x$.*

As usual, in defining the one-wayness security property, we parameterize Attack Game 10.2 by the security parameter λ, and the advantage $\mathrm{OW}\mathsf{adv}[\mathcal{A}, \mathcal{T}]$ is actually a function of λ. Definition 10.3 should be read as saying that $\mathrm{OW}\mathsf{adv}[\mathcal{A}, \mathcal{T}](\lambda)$ is a negligible function.

10.3 A trapdoor permutation scheme based on RSA

We now describe a trapdoor permutation scheme that is plausibly one-way. It is called RSA after its inventors, Rivest, Shamir, and Adleman. Recall that a trapdoor permutation is a special case of a trapdoor function, where the domain and range are the same set. This means that for every public-key, the function is a permutation of its domain, which is why we call it a trapdoor permutation. Despite many years of study, RSA is essentially the only known reasonable candidate trapdoor permutation scheme (there are a few others, but they are all very closely related to the RSA scheme).

Here is how RSA works. First, we describe a probabilistic algorithm RSAGen that takes as input an integer $\ell > 2$, and an odd integer $e > 2$.

RSAGen$(\ell, e) :=$
  generate a random ℓ-bit prime p such that $\gcd(e, p-1) = 1$
  generate a random ℓ-bit prime q such that $\gcd(e, q-1) = 1$ and $q \neq p$
  $n \leftarrow pq$
  $d \leftarrow e^{-1} \bmod (p-1)(q-1)$
  output (n, d).

To efficiently implement the above algorithm, we need an efficient algorithm to generate random ℓ-bit primes. This is discussed in **??**. Also, we use the extended Euclidean algorithm (see **??**) to compute $e^{-1} \bmod (p-1)(q-1)$. Note that since $\gcd(e, p-1) = \gcd(e, q-1) = 1$, it follows that $\gcd(e, (p-1)(q-1)) = 1$, and hence e has a multiplicative inverse modulo $(p-1)(q-1)$.

Now we describe the RSA trapdoor permutation scheme $\mathcal{T}_{\mathrm{RSA}} = (G, F, I)$. It is parameterized by fixed values of ℓ and e.

- Key generation runs as follows:

$$G() := \quad (n, d) \xleftarrow{R} \mathrm{RSAGen}(\ell, e), \quad pk \leftarrow (n, e), \quad sk \leftarrow (n, d)$$
$$\text{output } (pk, sk).$$

- For a given public key $pk = (n, e)$, and $x \in \mathbb{Z}_n$, we define $F(pk, x) := x^e \in \mathbb{Z}_n$.

- For a given secret key $sk = (n, d)$, and $y \in \mathbb{Z}_n$, we define $I(sk, y) := y^d \in \mathbb{Z}_n$.

Note that although the encryption exponent e is considered to be a fixed system parameter, we also include it as part of the public key pk.

A technicality. For each fixed $pk = (n, e)$, the function $F(pk, \cdot)$ maps $\mathbb{Z}_n$ into $\mathbb{Z}_n$; thus, the domain and range of this function actually vary with pk. However, in our definition of a trapdoor permutation scheme, the domain and range of the function are not allowed to vary with the public key. So in fact, this scheme does not quite satisfy the formal syntactic requirements of a trapdoor permutation scheme. One could easily generalize the definition of a trapdoor permutation scheme, to allow for this. However, we shall not do this; rather, we shall state and analyze various schemes based on a trapdoor permutation scheme as we have defined it, and then show how to instantiate these schemes using RSA. Exercise 10.23 explores an idea that builds a proper trapdoor permutation scheme based on RSA.

Ignoring this technical issue for the moment, let us first verify that $\mathcal{T}_{\text{RSA}}$ satisfies the correctness requirement of a trapdoor permutation scheme. This is implied by the following:

Theorem 10.1. *Let $n = pq$ where p and q are distinct primes. Let e and d be integers such that $ed \equiv 1 \pmod{(p-1)(q-1)}$. Then for all $x \in \mathbb{Z}$, we have $x^{ed} \equiv x \pmod{n}$.*

Proof. The hypothesis that $ed \equiv 1 \pmod{(p-1)(q-1)}$ just means that $ed = 1 + k(p-1)(q-1)$ for some integer k. Certainly, if $x \equiv 0 \pmod{p}$, then $x^{ed} \equiv 0 \equiv x \pmod{p}$; otherwise, if $x \not\equiv 0 \pmod{p}$, then by Fermat's little theorem (see **??**), we have

$$x^{p-1} \equiv 1 \pmod{p},$$

and so

$$x^{ed} \equiv x^{1+k(p-1)(q-1)} \equiv x \cdot \left(x^{(p-1)}\right)^{k(q-1)} \equiv x \cdot 1^{k(q-1)} \equiv x \pmod{p}.$$

Therefore,

$$x^{ed} \equiv x \pmod{p}.$$

By a symmetric argument, we have

$$x^{ed} \equiv x \pmod{q}.$$

Thus, $x^{ed} - x$ is divisible by the distinct primes p and q, and must therefore be divisible by their product n, which means

$$x^{ed} \equiv x \pmod{n}. \quad \square$$

So now we know that $\mathcal{T}_{\text{RSA}}$ satisfies the correctness property of a trapdoor permutation scheme. However, it is not clear that it is one-way. For $\mathcal{T}_{\text{RSA}}$, one-wayness means that there is no efficient algorithm that given n and x^e, where $x \in \mathbb{Z}_n$ is chosen at random, can effectively compute x. It is clear that if $\mathcal{T}_{\text{RSA}}$ is one-way, then it must be hard to factor n; indeed, if it were easy to factor n, then one could compute d in exactly the same way as is done in algorithm RSAGen, and then use d to compute $x = y^d$.

It is widely believed that factoring n is hard, provided ℓ is sufficiently large — typically, ℓ is chosen to be between 1000 and 1500. Moreover, the only known efficient algorithm to invert $\mathcal{T}_{\text{RSA}}$ is to first factor n and then compute d as above. However, there is no known *proof* that the assumption that factoring n is hard implies that $\mathcal{T}_{\text{RSA}}$ is one-way. Nevertheless, based on current evidence, it seems reasonable to conjecture that $\mathcal{T}_{\text{RSA}}$ is indeed one-way. We state this conjecture now as an explicit assumption. As usual, this is done using an attack game.

Attack Game 10.3 (RSA). For given integers $\ell > 2$ and odd $e > 2$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger computes

$$(n, d) \xleftarrow{\mathrm{R}} \mathrm{RSAGen}(\ell, e), \quad x \xleftarrow{\mathrm{R}} \mathbb{Z}_n, \quad y \leftarrow x^e \in \mathbb{Z}_n$$

  and gives the input (n, y) to the adversary.

- The adversary outputs $\hat{x} \in \mathbb{Z}_n$.

We define the adversary's advantage in breaking RSA, denoted $\mathrm{RSA}\mathsf{adv}[\mathcal{A}, \ell, e]$, as the probability that $\hat{x} = x$. □

Definition 10.5 (RSA assumption). *We say that the RSA assumption holds for (ℓ, e) if for all efficient adversaries $\mathcal{A}$, the quantity $\mathrm{RSA}\mathsf{adv}[\mathcal{A}, \ell, e]$ is negligible.*

We analyze the RSA assumption and present several known attacks on it later on in Chapter 15.

We next introduce some terminology that will be useful later. Suppose (n, d) is an output of $\mathrm{RSAGen}(\ell, e)$, and suppose that $x \in \mathbb{Z}_n$ and let $y := x^e$. The number n is called an **RSA modulus**, the number e is called an **encryption exponent**, and the number d is called a **decryption exponent**. We call (n, y) an **instance** of the **RSA problem**, and we call x a **solution** to this instance of the RSA problem. The RSA assumption asserts that there is no efficient algorithm that can effectively solve the RSA problem.

10.3.1 Key exchange based on the RSA assumption

Consider now what happens when we instantiate the key exchange protocol in Section 10.2.1 with $\mathcal{T}_{\mathrm{RSA}}$. The protocol runs as follows:

- Alice computes $(n, d) \xleftarrow{\mathrm{R}} \mathrm{RSAGen}(\ell, e)$, and sends (n, e) to Bob.
- Upon receiving (n, e) from Alice, Bob computes $x \xleftarrow{\mathrm{R}} \mathbb{Z}_n$, $y \leftarrow x^e$, and sends y to Alice.
- Upon receiving y from Bob, Alice computes $x \leftarrow y^d$.

The secret shared by Alice and Bob is x. The message flow is the same as in Fig. 10.1. Under the RSA assumption, this is a secure anonymous key exchange protocol.

10.3.2 Mathematical details

We give a more mathematically precise definition of the RSA assumption, using the terminology defined in Section 2.4.

In Attack Game 10.3, the parameters ℓ and e are actually poly-bounded and efficiently computable functions of a security parameter λ. Likewise, $\mathrm{RSA}\mathsf{adv}[\mathcal{A}, \ell, e]$ is a function of λ. As usual, Definition 10.5 should be read as saying that $\mathrm{RSA}\mathsf{adv}[\mathcal{A}, \ell, e](\lambda)$ is a negligible function.

There are a couple of further wrinkles we should point out. First, as already mentioned above, the RSA scheme does not quite fit our definition of a trapdoor permutation scheme, as the definition of the latter does not allow the set $\mathcal{X}$ to vary with the public key. It would not be too difficult to modify our definition of a trapdoor permutation scheme to accommodate this generalization. Second, the specification of RSAGen requires that we generate random prime numbers of a given bit length. In theory, it is possible to do this in (expected) polynomial time; however, the most practical algorithms (see Section **??**) may — with negligible probability — output a number that is not a prime. If that should happen, then it may be the case that the basic correctness requirement — namely, that $I(sk, F(pk, x)) = x$ for all pk, sk, x — is no longer satisfied. It would also not be too difficult to modify our definition of a trapdoor permutation scheme to accommodate this type of generalization as well. For example, we could recast this requirement as an attack game (in which any efficient adversary wins with negligible probability): in this game, the challenger generates $(pk, sk) \xleftarrow{R} G()$ and sends (pk, sk) to the adversary; the adversary wins the game if he can output $x \in \mathcal{X}$ such that $I(sk, F(pk, x)) \neq x$. While this would be a perfectly reasonable definition, using it would require us to modify security definitions for higher-level constructs. For example, if we used this relaxed correctness requirement in the context of key exchange, we would have to allow for the possibility that the two parties end up with different keys with some negligible probability.

10.4 Diffie-Hellman key exchange

In this section, we explore another approach to constructing secure key exchange protocols, which was invented by Diffie and Hellman. Just as with the protocol based on RSA, this protocol will require a bit of algebra and number theory. However, before getting in to the details, we provide a bit of motivation and intuition.

Consider the following "generic" key exchange protocol the makes use of two functions E and F. Alice chooses a random secret α, computes $E(\alpha)$, and sends $E(\alpha)$ to Bob over an insecure channel. Likewise, Bob chooses a random secret β, computes $E(\beta)$, and sends $E(\beta)$ to Alice over an insecure channel. Alice and Bob both somehow compute a shared key $F(\alpha, \beta)$. In this high-level description, E and F are some functions that should satisfy the following properties:

1. E should be easy to compute;
2. given α and $E(\beta)$, it should be easy to compute $F(\alpha, \beta)$;
3. given $E(\alpha)$ and β, it should be easy to compute $F(\alpha, \beta)$;
4. given $E(\alpha)$ and $E(\beta)$, it should be hard to compute $F(\alpha, \beta)$.

Properties 1–3 ensure that Alice and Bob can efficiently implement the protocol: Alice computes the shared key $F(\alpha, \beta)$ using the algorithm from Property 2 and her given data α and $E(\beta)$. Bob computes the same key $F(\alpha, \beta)$ using the algorithm from Property 3 and his given data $E(\alpha)$ and β. Property 4 ensures that the protocol is secure: an eavesdropper who sees $E(\alpha)$ and $E(\beta)$ should not be able to compute the shared key $F(\alpha, \beta)$.

Note that properties 1–4 together imply that E is hard to invert; indeed, if we could compute efficiently α from $E(\alpha)$, then by Property 2, we could efficiently compute $F(\alpha, \beta)$ from $E(\alpha), E(\beta)$, which would contradict Property 4.

To make this generic approach work, we have to come up with appropriate functions E and F. To a first approximation, the basic idea is to implement E in terms of exponentiation to some fixed base g, defining $E(\alpha) := g^\alpha$ and $F(\alpha, \beta) := g^{\alpha\beta}$. Notice then that

$$E(\alpha)^\beta = (g^\alpha)^\beta = F(\alpha, \beta) = (g^\beta)^\alpha = E(\beta)^\alpha.$$

Hence, provided exponentiation is efficient, Properties 1–3 are satisfied. Moreover, if Property 4 is to be satisfied, then at the very least, we require that taking logarithms (i.e., inverting E) is hard.

To turn this into a practical and plausibly secure scheme, we cannot simply perform exponentiation on ordinary integers since the numbers would become too large. Instead, we have to work in an appropriate finite algebraic domain, which we introduce next.

10.4.1 The key exchange protocol

Suppose p is a large prime and that q is a large prime dividing $p-1$ (think of p as being very large random prime, say 2048 bits long, and think of q as being about 256 bits long).

We will be doing arithmetic mod p, that is, working in $\mathbb{Z}_p$. Recall that $\mathbb{Z}_p^*$ is the set of nonzero elements of $\mathbb{Z}_p$. An essential fact is that since q divides $p-1$, $\mathbb{Z}_p^*$ has an element g of order q (see Section **??**). This means that $g^q = 1$ and that all of the powers g^a, for $a = 0, \ldots, q-1$, are distinct. Let $\mathbb{G} := \{g^a : a = 0, \ldots, q-1\}$, so that $\mathbb{G}$ is a subset of $\mathbb{Z}_p^*$ of cardinality q. It is not hard to see that $\mathbb{G}$ is closed under multiplication and inversion; that is, for all $u, v \in \mathbb{G}$, we have $uv \in \mathbb{G}$ and $u^{-1} \in \mathbb{G}$. Indeed, $g^a \cdot g^b = g^{a+b} = g^c$ with $c := (a+b) \bmod q$, and $(g^a)^{-1} = g^d$ with $d := (-a) \bmod q$. In the language of algebra, $\mathbb{G}$ is called a subgroup of the group $\mathbb{Z}_p^*$.

For every $u \in \mathbb{G}$ and integers a and b, it is easy to see that $u^a = u^b$ if $a \equiv b \bmod q$. Thus, the value of u^a depends only on the residue class of a modulo q. Therefore, if $\alpha = [a]_q \in \mathbb{Z}_q$ is the residue class of a modulo q, we can define $u^\alpha := u^a$ and this definition is unambiguous. From here on we will frequently use elements of $\mathbb{Z}_q$ as exponents applied to elements of $\mathbb{G}$.

So now we have everything we need to describe the Diffie-Hellman key exchange protocol. We assume that the description of $\mathbb{G}$, including $g \in \mathbb{G}$ and q, is a system parameter that is generated once and for all at system setup time and shared by all parties involved. The protocol runs as follows, as shown in Fig. 10.2:

1. Alice computes $\alpha \xleftarrow{\text{R}} \mathbb{Z}_q$, $u \leftarrow g^\alpha$, and sends u to Bob.

2. Bob computes $\beta \xleftarrow{\text{R}} \mathbb{Z}_q, v \leftarrow g^\beta$ and sends v to Alice.

3. Upon receiving v from Bob, Alice computes $w \leftarrow v^\alpha$

4. Upon receiving u from Alice, Bob computes $w \leftarrow u^\beta$

The secret shared by Alice and Bob is

$$w = v^\alpha = g^{\alpha\beta} = u^\beta.$$

10.4.2 Security of Diffie-Hellman key exchange

For a fixed element $g \in \mathbb{G}$, different from 1, the function from $\mathbb{Z}_q$ to $\mathbb{G}$ that sends $\alpha \in \mathbb{Z}_q$ to $g^\alpha \in \mathbb{G}$ is called the **discrete exponentiation function**. This function is one-to-one and onto, and its inverse function is called the **discrete logarithm function**, and is usually denoted Dlog_g; thus, for $u \in \mathbb{G}$, $\mathsf{Dlog}_g(u)$ is the unique $\alpha \in \mathbb{Z}_q$ such that $u = g^\alpha$. The value g is called the **base** of the discrete logarithm.

If the Diffie-Hellman protocol has any hope of being secure, it must be hard to compute α from g^α for a random α; in other words, it must be hard to compute the discrete logarithm function. There are a number of candidate group families $\mathbb{G}$ where the discrete logarithm function is believed to be hard to compute. For example, when p and q are sufficiently large, suitably chosen primes,

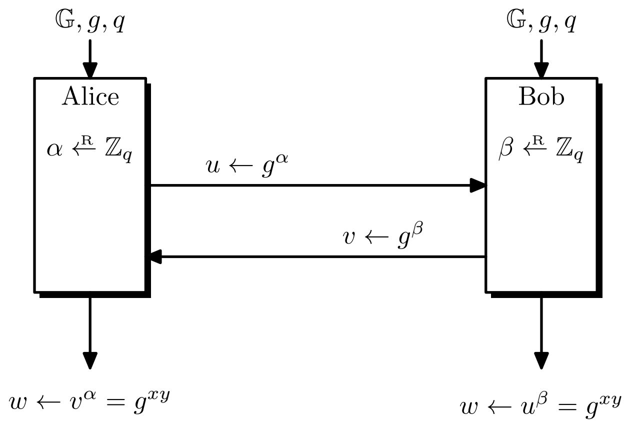


Figure 10.2: Diffie-Hellman key exchange

the discrete logarithm function in the order q subgroup of $\mathbb{Z}_p^*$ is believed to be hard to compute (p should be at least 2048-bits, and q should be at least 256-bits). This assumption is called the **discrete logarithm assumption** and is defined in the next section.

Unfortunately, the discrete logarithm assumption by itself is not enough to ensure that the Diffie-Hellman protocol is secure. Observe that the protocol is secure if and only if the following holds:

given $g^{\alpha}, g^{\beta} \in \mathbb{G}$, where $\alpha \xleftarrow{R} \mathbb{Z}_q$ and $\beta \xleftarrow{R} \mathbb{Z}_q$, it is hard to compute $g^{\alpha\beta} \in \mathbb{G}$.

This security property is called the **computational Diffie-Hellman assumption**. Although the computational Diffie-Hellman assumption is stronger than the discrete logarithm assumption, all evidence still suggests that this is a reasonable assumption in groups where the discrete logarithm assumption holds.

10.5 Discrete logarithm and related assumptions

In this section, we state the discrete logarithm and related assumptions more precisely and in somewhat more generality, and explore in greater detail relationships among them.

The subset $\mathbb{G}$ of $\mathbb{Z}_p^*$ that we defined above in Section 10.4 is a specific instance of a general type of mathematical object known as a *cyclic group*. There are in fact other cyclic groups that are very useful in cryptography, most notably, groups based on *elliptic curves* — we shall study elliptic curve cryptography in Chapter 16. From now on, we shall state assumptions and algorithms in terms of an abstract cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. In general, such groups may be selected by a randomized process, and again, the description of $\mathbb{G}$, including $g \in \mathbb{G}$ and q, is a system parameter that is generated once and for all at system setup time and shared by all parties involved.

We shall use just a bit of terminology from group theory. The reader who is unfamiliar with the concept of a group may wish to refer to **??**; alternatively, for the time being, the reader may simply ignore this abstraction entirely:

- Whenever we refer to a "cyclic group," the reader may safely assume that this means the specific set $\mathbb{G}$ defined above as a subgroup of $\mathbb{Z}_p^*$.

- The "order of $\mathbb{G}$" is just a fancy name for the size of the set $\mathbb{G}$, which is q.
- A "generator of $\mathbb{G}$" is an element $g \in \mathbb{G}$ with the property that every element of $\mathbb{G}$ can be expressed as a power of g.

We begin with a formal statement of the discrete logarithm assumption, stated in our more general language. As usual, we need an attack game.

Attack Game 10.4 (Discrete logarithm). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. For a given adversary $\mathcal{A}$, define the following attack game:

- The challenger computes
$$\alpha \xleftarrow{\text{R}} \mathbb{Z}_q, \quad u \leftarrow g^\alpha,$$
and gives the value u to the adversary.
- The adversary outputs some $\hat{\alpha} \in \mathbb{Z}_q$.

We define $\mathcal{A}$'s **advantage in solving the discrete logarithm problem for** $\mathbb{G}$, denoted $\mathrm{DL}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$, as the probability that $\hat{\alpha} = \alpha$. $\Box$

Definition 10.6 (Discrete logarithm assumption). *We say that the* ***discrete logarithm (DL) assumption*** *holds for* $\mathbb{G}$ *if for all efficient adversaries* $\mathcal{A}$ *the quantity* $\mathrm{DL}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$ *is negligible.*

We say that g^α is an **instance** of the **discrete logarithm (DL) problem (for** $\mathbb{G}$**)**, and that α is a solution to this problem instance. By convention, we assume that the description of $\mathbb{G}$ includes its order q and a generator g. The DL assumption asserts that there is no efficient algorithm that can effectively solve the DL problem.

Note that the DL assumption is defined in terms of a group $\mathbb{G}$ and generator $g \in \mathbb{G}$. As already mentioned, the group $\mathbb{G}$ and generator g are chosen and fixed at system setup time via a process that may be randomized. Also note that all elements of $\mathbb{G} \setminus \{1\}$ are in fact generators for $\mathbb{G}$, but we do not insist that g is chosen uniformly among these (but see Exercise 10.16). Different methods for selecting groups and generators give rise to different DL assumptions (and the same applies to the CDH and DDH assumptions, defined below).

Now we state the computational Diffie-Hellman assumption.

Attack Game 10.5 (Computational Diffie-Hellman). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. For a given adversary $\mathcal{A}$, the attack game runs as follows.

- The challenger computes
$$\alpha, \beta \xleftarrow{\text{R}} \mathbb{Z}_q, \quad u \leftarrow g^\alpha, \quad v \leftarrow g^\beta, \quad w \leftarrow g^{\alpha\beta}$$
and gives the pair (u, v) to the adversary.
- The adversary outputs some $\hat{w} \in \mathbb{G}$.

We define $\mathcal{A}$'s **advantage in solving the computational Diffie-Hellman problem for** $\mathbb{G}$, denoted $\mathrm{CDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$, as the probability that $\hat{w} = w$. $\Box$

Definition 10.7 (Computational Diffie-Hellman assumption). *We say that the* ***computational Diffie-Hellman (CDH)*** *assumption holds for* $\mathbb{G}$ *if for all efficient adversaries* $\mathcal{A}$ *the quantity* $\mathrm{CDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$ *is negligible.*

We say that (g^α, g^β) is an **instance** of the **computational Diffie-Hellman (CDH) problem**, and that $g^{\alpha\beta}$ is a solution to this problem instance. Again, by convention, we assume that the description of $\mathbb{G}$ includes its order q and a generator g. The CDH assumption asserts that there is no efficient algorithm that can effectively solve the CDH problem.

An interesting property of the CDH problem is that there is no general and efficient algorithm to even *recognize* correct solutions to the CDH problem, that is, given an instance (u, v) of the CDH problem, and a group element $\hat{w}$, to determine if $\hat{w}$ is a solution to the given problem instance. This is in contrast to the RSA problem: given an instance (n, e, y) of the RSA problem, and an element $\hat{x}$ of $\mathbb{Z}_n^*$, we can efficiently test if $\hat{x}$ is a solution to the given problem instance simply by testing if $\hat{x}^e = y$. In certain cryptographic applications, this lack of an efficient algorithm to recognize solutions to the CDH problem can lead to technical difficulties. However, this apparent limitation is also an *opportunity*: if we assume not only that solving the CDH problem is hard, but also that recognizing solutions to CDH problem is hard, then we can sometimes prove stronger security properties for certain cryptographic schemes.

We shall now formalize the assumption that recognizing solutions to the CDH problem is hard. In fact, we shall state a stronger assumption, namely, that even distinguishing solutions from random group elements is hard. It turns out that this stronger assumption is equivalent to the weaker one (see Exercise 10.9).

Attack Game 10.6 (Decisional Diffie-Hellman). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. For a given adversary $\mathcal{A}$, we define two experiments.

Experiment b **(**$b = 0, 1$**):**

- The challenger computes

$$\alpha, \beta, \gamma \xleftarrow{\text{R}} \mathbb{Z}_q, \quad u \leftarrow g^\alpha, \quad v \leftarrow g^\beta, \quad w_0 \leftarrow g^{\alpha\beta}, \quad w_1 \leftarrow g^\gamma,$$

  and gives the triple (u, v, w_b) to the adversary.

- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

If W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b, we define $\mathcal{A}$'s **advantage in solving the decisional Diffie-Hellman problem for** $\mathbb{G}$ as

$$\mathrm{DDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \Box$$

Definition 10.8 (Decisional Diffie-Hellman assumption). *We say that the* ***decisional Diffie-Hellman (DDH)*** *assumption holds for* $\mathbb{G}$ *if for all efficient adversaries* $\mathcal{A}$ *the quantity* $\mathrm{DDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$ *is negligible.*

For $\alpha, \beta, \gamma \in \mathbb{Z}_q$, we call $(g^\alpha, g^\beta, g^\gamma)$ a **DH-triple** if $\gamma = \alpha\beta$; otherwise, we call it a **non-DH-triple**. The DDH assumption says that there is no efficient algorithm that can effectively distinguish between random DH-triples and random triples. More precisely, in the language of Section 3.11, the DDH assumptions says that the uniform distribution over DH-triples and the uniform distribution over $\mathbb{G}^3$ are computationally indistinguishable. It is not hard to show the the DDH assumption implies that it is hard to distinguish between random DH-triples and random non-DH-triples (see Exercise 10.6).

Clearly, the DDH assumption implies the CDH assumption: if we could effectively solve the CDH problem, then we could easily determine if a given triple $(u, v, \hat{w})$ is a DH-triple by first computing a correct solution w to the instance (u, v) of the CDH problem, and then testing if $w = \hat{w}$.

In defining the DL, CDH, and DDH assumptions, we have restricted our attention to prime order groups. This is convenient for a number of technical reasons. See, for example, Exercise 10.20, where you are asked to show that the DDH assumption for groups of even order is simply false.

10.5.1 Random self-reducibility

An important property of the discrete-log function in a group $\mathbb{G}$ is that it is either hard almost everywhere in $\mathbb{G}$ or easy everywhere in $\mathbb{G}$. A middle ground where discrete-log is easy for some inputs and hard for others is not possible. We prove this by showing that the discrete-log function has a *random self reduction*.

Consider a specific cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. Suppose $\mathcal{A}$ is an efficient algorithm with the following property: if $u \in \mathbb{G}$ is chosen at random, then $\Pr[\mathcal{A}(u) = \mathsf{Dlog}_g(u)] = \epsilon$. That is, on a random input u, algorithm $\mathcal{A}$ computes the discrete logarithm of u with probability ϵ. Here, the probability is over the random choice of u, as well as any random choices made by $\mathcal{A}$ itself.[1] Suppose $\epsilon = 0.1$. Then the group $\mathbb{G}$ is of little use in cryptography since an eavesdropper can use $\mathcal{A}$ to break 10% of all Diffie-Hellman key exchanges. However, this does not mean that $\mathcal{A}$ is able to compute $\mathsf{Dlog}_g(u)$ with non-zero probability for all $u \in \mathbb{G}$. It could be the case that for 10% of the inputs $u \in \mathbb{G}$, algorithm $\mathcal{A}$ always computes $\mathsf{Dlog}_g(u)$, while for the remaining 90%, it never computes $\mathsf{Dlog}_g(u)$.

We show how to convert $\mathcal{A}$ into an efficient algorithm $\mathcal{B}$ with the following property: for all $u \in \mathbb{G}$, algorithm $\mathcal{B}$ on input u successfully computes $\mathsf{Dlog}_g(u)$ with probability ϵ. Here, the probability is only over the random choices made by $\mathcal{B}$. We so do using a reduction that maps a given discrete-log instance to a random discrete-log instance. Such a reduction is called a **random self reduction**.

Theorem 10.2. *Consider a specific cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. Suppose $\mathcal{A}$ is an efficient algorithm with the following property: if $u \in \mathbb{G}$ is chosen at random, then $\Pr[\mathcal{A}(u) = \mathsf{Dlog}_g(u)] = \epsilon$, with the probability is over the random choice of u and the random choices made by $\mathcal{A}$. Then there is an efficient algorithm $\mathcal{B}$ with the following property: for all $u \in \mathbb{G}$, algorithm $\mathcal{B}$ either outputs* fail *or $\mathsf{Dlog}_g(u)$, and it outputs the latter with probability ϵ, where now the probability is only over the random choices made by $\mathcal{B}$.*

Theorem 10.2 implements the transformation shown in Fig. 10.3. The point is that, unlike $\mathcal{A}$, algorithm $\mathcal{B}$ works for *all* inputs. To compute discrete-log of a particular $u \in \mathbb{G}$ one can iterate $\mathcal{B}$ on the same input u several times, say $n\lceil 1/\epsilon \rceil$ times for some n. Using the handy inequality $1 + x \leq \exp(x)$ (which holds for all x), this iteration will produce the discrete-log with probability $1-(1-\epsilon)^{n\lceil 1/\epsilon \rceil} \geq 1-\exp(-n)$. In particular, if $1/\epsilon$ is poly-bounded, we can efficiently compute the discrete logarithm of any group element with negligible failure probability. In contrast, iterating $\mathcal{A}$ on the same input u many times may never produce a correct answer. Consequently, if discrete-log is easy for a non-negligible fraction of instances, then it will be easy for *all* instances.

[1] Technical note: the probability ϵ is not quite the same as $\mathrm{DL}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$, as the latter is also with respect to the random choice of group/generator made at system setup time; here, we are viewing these as truly fixed.

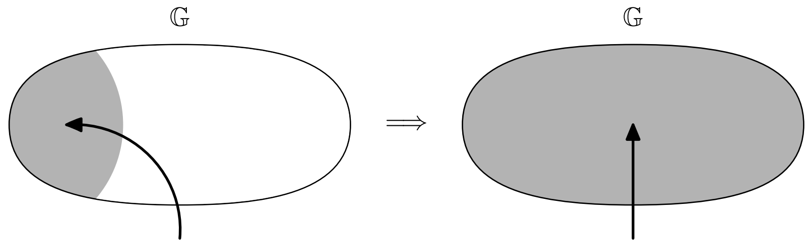


Figure 10.3: The effect of a random self reduction

Proof of Theorem 10.2. Algorithm $\mathcal{B}$ works as follows:

> Input: $u \in \mathbb{G}$
> Output: $\mathsf{Dlog}_g(u)$ or fail
> $\sigma \xleftarrow{\text{R}} \mathbb{Z}_q$
> $u_1 \leftarrow u \cdot g^\sigma \in \mathbb{G}$
> $\alpha_1 \leftarrow \mathcal{A}(u_1)$
> if $g^{\alpha_1} \neq u_1$
>   then output fail
>   else output $\alpha \leftarrow \alpha_1 - \sigma$

Suppose that $u = g^\alpha$. Observe that $u_1 = g^{\alpha+\sigma}$. Since σ is uniformly distributed over $\mathbb{Z}_q$, the group element u_1 is uniformly distributed over $\mathbb{G}$. Therefore, on input u_1, adversary $\mathcal{A}$ will output $\alpha_1 = \alpha + \sigma$ with probability ϵ. When this happens, $\mathcal{B}$ will output $\alpha_1 - \sigma = \alpha$, and otherwise, $\mathcal{B}$ will output fail. $\square$

Why random self reducibility is important. Any hard problem can potentially form the basis of a cryptosystem. For example, an NP-hard problem known as subset sum has attracted attention for many years. Unfortunately, many hard problems, including subset sum, are only hard in the worst case. Generally speaking, such problems are of little use in cryptography, where we need problems that are not just hard in the worst case, but hard *on average* (i.e., for randomly chosen inputs). For a problem with a random self-reduction, if it hard in the worst case, then it must be hard on average. This implication makes such problems attractive for cryptography.

One can also give random self reductions for both the CDH and DDH problems, as well as for the RSA problem (in a more limited sense). These ideas are developed the chapter exercises.

10.5.2 Mathematical details

As in previous sections, we give the mathematical details pertaining to the DL, CDH, and DDH assumptions. We use the terminology introduced in Section 2.4. This section may be safely skipped on first reading with very little loss in understanding.

To state the assumptions asymptotically we introduce a security parameter λ that identifies the group in which the DL, CDH, and DDH games are played. We will require that the adversary's advantage in breaking the assumption is a negligible function of λ. As lambda increases the adversary's advantage in breaking discrete-log in the group defined by λ should quickly go to zero.

To make sense of the security parameter λ we need a family of groups that increase in size as λ increases. As in Section 2.4, this family of groups is parameterized by both λ and an additional system parameter Λ. The idea is that once λ is chosen, a system parameter Λ is generated by a **system parameterization algorithm** P. The pair (λ, Λ) then fully identifies the group $\mathbb{G}_{\lambda,\Lambda}$ where the DL, CDH, and DDH games are played. Occasionally we will refer to Λ as a **group description**. This Λ is a triple

$$\Lambda := (\ \Lambda_1,\ q,\ g\ )$$

where Λ_1 is an arbitrary string, q is prime number that represents the order of the group $\mathbb{G}_{\lambda,\Lambda}$, and g is a generator of $\mathbb{G}_{\lambda,\Lambda}$.

Definition 10.9 (group family). *A **group family** $\mathbb{G}$ consists of an algorithm* Mul *along with a family of spaces:*

$$\mathbf{G} = \{\mathbb{G}_{\lambda,\Lambda}\}_{\lambda,\Lambda}$$

with system parameterization algorithm P, such that

1. $\mathbf{G}$ *is efficiently recognizable.*
2. *Algorithm* Mul *is an efficient deterministic algorithm that on input $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \text{Supp}(P(\lambda))$, $u, v \in \mathbb{G}_{\lambda,\Lambda}$, outputs $w \in \mathbb{G}_{\lambda,\Lambda}$.*
3. *For all $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda = (\Lambda_1, q, g) \in \text{Supp}(P(\lambda))$, algorithm* Mul *is a multiplication operation on $\mathbb{G}_{\lambda,\Lambda}$ that defines a cyclic group of prime order q generated by g.*

The definition implies that all the spaces $\mathbb{G}_{\lambda,\Lambda}$ are efficiently sampleable. Since $\Lambda = (\Lambda_1, q, g)$ we can randomly sample a random element u of $\mathbb{G}_{\lambda,\Lambda}$ by picking a random $\alpha \xleftarrow{R} \mathbb{Z}_q$ and setting $u \leftarrow g^{\alpha}$. Specific group families may allow for a more efficient method that generates a random group element. The group identity element may always be obtained by raising g to the power q, although for specific group families, there are most likely simpler and faster ways to do this.

An example. We define the asymptotic version of a subgroup of prime order q within $\mathbb{Z}_p^*$, where q is a prime dividing $p-1$, and p itself is prime. Here the system parameterization algorithm P takes λ as input and outputs a group description $\Lambda := (p, q, g)$ where p is a random $\ell(\lambda)$-bit prime (for some poly-bounded length function ℓ) and g is an element of $\mathbb{Z}_p^*$ of order q. The group $\mathbb{G}_{\lambda,\Lambda}$ is the subgroup of $\mathbb{Z}_p^*$ generated by g. Elements of $\mathbb{G}_{\lambda,\Lambda}$ may be efficiently recognized as follows: first, one can check that a given bit string properly encodes an element u of $\mathbb{Z}_p^*$; second, one can check that $u^q = 1$.

Armed with the concept of a group family, we now parameterize the DL Attack Game 10.4 by the security parameter λ. In that game, the adversary is given the security parameter λ and a group description $\Lambda = (\Lambda_1, q, g)$, where g is a generator for the group $\mathbb{G}_{\lambda,\Lambda}$. It is also given a random $u \in \mathbb{G}_{\lambda,\Lambda}$, and it wins the game if it computes $\mathsf{Dlog}_g(u)$. Its advantage $\text{DL}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$ is now a function of λ, and for each λ, this advantage is a probability that depends on the random choice of group and generator, as well as the random choices made by the the challenger and adversary. Definition 10.6 should be read as saying that $\text{DL}\mathsf{adv}[\mathcal{A}, \mathbb{G}](\lambda)$ is a negligible function.

We use the same approach to define the asymptotic CDH and DDH assumptions.

10.6 Collision resistant hash functions from number-theoretic primitives

It turns out that the RSA and DL assumptions are extremely versatile, and can be used in many cryptographic applications. As an example, in this section, we show how to build collision-resistant hash functions based on the RSA an DL assumptions.

Recall from Section 8.1 that a hash function H defined over $(\mathcal{M}, \mathcal{T})$ is an efficiently computable function from $\mathcal{M}$ to $\mathcal{T}$. In most applications, we want the message space $\mathcal{M}$ to be much larger than the digest space $\mathcal{T}$. We also defined a notion of collision resistance, which says that for every efficient adversary $\mathcal{A}$, its collision-finding advantage CR$\mathsf{adv}[\mathcal{A}, H]$ is negligible. Here, CR$\mathsf{adv}[\mathcal{A}, H]$ is defined to be probability that $\mathcal{A}$ can produce a collision, i.e., a pair $m_0, m_1 \in \mathcal{M}$ such that $m_0 \neq m_1$ but $H(m_0) = H(m_1)$.

10.6.1 Collision resistance based on DL

Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. We define a hash function H_{dl} defined over $(\mathbb{Z}_q \times \mathbb{Z}_q, \mathbb{G})$. This hash function is parameterized by the group $\mathbb{G}$ and the generator g, along with a randomly chosen $u \in \mathbb{G}$. Thus, the group $\mathbb{G}$, along with the group elements g and u, are chosen once and for all, and together, they define the hash function H_{dl}. For $\alpha, \beta \in \mathbb{Z}_q$, we define

$$H_{\mathrm{dl}}(\alpha, \beta) := g^{\alpha} u^{\beta}.$$

Notice that a collision on H_{dl} consists of $\alpha, \beta, \alpha', \beta' \in \mathbb{Z}_q$ such that

$$(\alpha, \beta) \neq (\alpha', \beta') \quad \text{and} \quad g^{\alpha} u^{\beta} = g^{\alpha'} u^{\beta'}. \tag{10.1}$$

That is, a collision of this type gives us two different ways to represent the same group element as a power of g times a power of u. The problem of finding a collision of this type is sometimes called the **representation problem**.

Theorem 10.3. *The hash function H_{dl} is collision resistant under the DL assumption.*

In particular, for every collision-finding adversary $\mathcal{A}$, there exists a DL adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{CR}\mathsf{adv}[\mathcal{A}, H_{\mathrm{dl}}] = \mathrm{DL}\mathsf{adv}[\mathcal{B}, \mathbb{G}]. \tag{10.2}$$

Proof. Consider a collision as in (10.1). Notice that $g^{\alpha} u^{\beta} = g^{\alpha'} u^{\beta'}$ implies

$$g^{\alpha - \alpha'} u^{\beta - \beta'} = 1, \tag{10.3}$$

where either $\alpha - \alpha' \neq 0$ or $\beta - \beta' \neq 0$. Thus, we have a nontrivial representation of 1 as a power of g times a power of u.

We claim that $\beta - \beta' \neq 0$. To see this, suppose by way of contradiction that $\beta - \beta' = 0$. Then (10.3) implies $g^{\alpha - \alpha'} = 1$, and since g is a generator for $\mathbb{G}$, this would mean $\alpha - \alpha' = 0$. Thus, we would have $\alpha - \alpha' = 0$ and $\beta - \beta' = 0$, which is a contradiction.

Since $\beta - \beta' \neq 0$ and q is prime, it follows that $\beta - \beta'$ has a multiplicative inverse in $\mathbb{Z}_q$, which we can in fact efficiently compute (see Section **??**). So we can rewrite (10.3) as

$$u = g^{(\alpha' - \alpha)/(\beta - \beta')},$$

which means

$$\mathsf{Dlog}_g(u) = (\alpha' - \alpha)/(\beta - \beta'). \tag{10.4}$$

So we use the given collision-finding adversary $\mathcal{A}$ to build a DL adversary $\mathcal{B}$ as follows. When $\mathcal{B}$ receives its challenge $u \in \mathbb{G}$ from its DL-challenger, $\mathcal{B}$ runs $\mathcal{A}$ using H_{dl}, which is defined using $\mathbb{G}$, g, and the given u. If $\mathcal{A}$ produces a collision as in (10.1), adversary $\mathcal{B}$ computes and outputs $\mathsf{Dlog}_g(u)$ as in (10.4). By the above discussion, (10.2) is clear. □

The function $H_{\text{dl}} : \mathbb{Z}_q \times \mathbb{Z}_q \to \mathbb{G}$ maps from a message space of size q^2 to a digest space of size q. The good news is that the message space is larger than the digest space, and so the hash function actually compresses. The bad news is that the set of *encodings* of $\mathbb{G}$ may be much larger than the set $\mathbb{G}$ itself. Indeed, if $\mathbb{G}$ is constructed as recommended in Section 10.4 as a subset of $\mathbb{Z}_p^*$, then elements of $\mathbb{G}$ are encoded as 2048-bit strings, even though the group $\mathbb{G}$ itself has order $\approx 2^{256}$. So if we replace the set $\mathbb{G}$ by the set of encodings, the hash function H_{dl} is not compressing at all. This problem can be avoided by using other types of groups with more compact encodings, such as elliptic curve groups (see Chapter 16). See also Exercise 10.17 and Exercise 10.18.

10.6.2 Collision resistance based on RSA

We shall work with an RSA encryption exponent e that is a prime. For this application, the bigger e is, the more compression we get. Let $I_e := \{0, \dots, e-1\}$. Let n be an RSA modulus, generated as in Section 10.3 using an appropriate length parameter ℓ. We also choose a random $y \in \mathbb{Z}_n^*$. The values e, n, and y are chosen once and for all, and together they determine a hash function H_{rsa} defined over $(\mathbb{Z}_n^* \times I_e,\ \mathbb{Z}_n^*)$ as follows: for $a \in \mathbb{Z}_n^*$ and $b \in I_e$, we define

$$H_{\text{rsa}}(a, b) := a^e y^b.$$

We will show that H_{rsa} is collision resistant under the RSA assumption. Note that H_{rsa} can be used directly as a compression function in the Merkle-Damgård paradigm (see Section 8.4) to build a collision-resistant hash function for arbitrarily large message spaces. In applying Theorem 8.3, we would take $\mathcal{X} = \mathbb{Z}_n^*$ and $\mathcal{Y} = \{0,1\}^{\lfloor \log_2 e \rfloor}$.

To analyze H_{rsa}, we will need a couple of technical results. The first result simply says that in the RSA attack game, it is no easier to compute an eth root of a random element of $\mathbb{Z}_n^*$ than it is to compute an eth root of a random element of $\mathbb{Z}_n$. To make this precise, suppose that we modify Attack Game 10.3 so that the challenger chooses $x \xleftarrow{\text{R}} \mathbb{Z}_n^*$, and keep everything else the same. Note that since x is uniformly distributed over $\mathbb{Z}_n^*$, the value $y := x^e$ is also uniformly distributed over $\mathbb{Z}_n^*$. Denote by $\text{uRSA}\mathsf{adv}[\mathcal{A}, \ell, e]$ the adversary $\mathcal{A}$'s advantage in this modified attack game.

Theorem 10.4. *Let $\ell > 2$ and odd $e > 2$ be integers. For every adversary $\mathcal{A}$, there exists an an adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that* $\text{uRSA}\mathsf{adv}[\mathcal{A}, \ell, e] \leq \text{RSA}\mathsf{adv}[\mathcal{B}, \ell, e]$.

Proof. Let $\mathcal{A}$ be a given adversary. Here is how $\mathcal{B}$ works. Adversary $\mathcal{B}$ receives a random element $y \in \mathbb{Z}_n$. If $y \in \mathbb{Z}_n^*$, then $\mathcal{B}$ gives y to $\mathcal{A}$ and outputs whatever $\mathcal{A}$ outputs. Otherwise, $\mathcal{B}$ computes an eth root x of y as follows. If $y = 0$, $\mathcal{B}$ sets $x := 0$; otherwise, by computing the GCD of y and n, $\mathcal{B}$ can factor n, compute the RSA decryption exponent d, and then compute $x := y^d$.

Let W be the event that $\mathcal{B}$ succeeds. We have

$$\Pr[W] = \Pr[W \mid y \in \mathbb{Z}_n^*] \Pr[y \in \mathbb{Z}_n^*] + \Pr[W \mid y \notin \mathbb{Z}_n^*] \Pr[y \notin \mathbb{Z}_n^*].$$

The result follows from the observations that

$$\Pr[W \mid y \in \mathbb{Z}_n^*] = \text{uRSA}\mathsf{adv}[\mathcal{A}, \ell, e]$$

and

$$\Pr[W \mid y \notin \mathbb{Z}_n^*] = 1 \geq \text{uRSA}\mathsf{adv}[\mathcal{A}, \ell, e]. \quad \Box$$

The above theorem shows that the standard RSA assumption implies a variant RSA assumption, where the preimage is chosen at random from from $\mathbb{Z}_n^*$, rather than $\mathbb{Z}_n$. In Exercise 10.22, you are to show the converse, that is, that this variant RSA assumption implies the standard RSA assumption.

We also need the following technical result, which says that given $y \in \mathbb{Z}_n^*$, along with an integer f that is relatively prime to e, and an eth root of y^f, we can easily compute an eth root of y itself.

Just to get a feeling for the result, suppose $e = 3$ and $f = 2$. We have $w \in Z_n^*$ such that $w^3 = y^2$. We want to compute $x \in \mathbb{Z}_n^*$ such that $x^3 = y$. If we set $x := (y/w)$, then we have

$$x^3 = y^3/w^3 = y^3/y^2 = y.$$

Theorem 10.5 (Shamir's trick). *There is an efficient algorithm that takes as input n, e, f, w, y, where n is a positive integer, e and f are relatively prime integers, and w and y are elements of $\mathbb{Z}_n^*$ that satisfy $w^e = y^f$, and outputs $x \in \mathbb{Z}_n^*$ such that $x^e = y$.*

Proof. Using the extended Euclidean algorithm (see Section **??**), we compute integers s and t such that $es + ft = \gcd(e, f)$, and output $x := y^s w^t$. If $\gcd(e, f) = 1$ and $w^e = y^f$, then

$$x^e = (y^s w^t)^e = y^{es} w^{et} = y^{es} y^{ft} = y^{es+ft} = y^1 = y. \quad \Box$$

Theorem 10.6. *The hash function H_{rsa} is collision resistant under the RSA assumption.*

In particular, for every collision-finding adversary $\mathcal{A}$, there exists an RSA adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that

$$\mathsf{CRadv}[\mathcal{A}, H_{\text{rsa}}] \leq \text{RSA}\mathsf{adv}[\mathcal{B}, \ell, e]. \tag{10.5}$$

Proof. We construct an adversary $\mathcal{B}'$ that plays the alternative RSA attack game considered in Theorem 10.4. We will show that $\mathsf{CRadv}[\mathcal{A}, H_{\text{rsa}}] = \text{uRSA}\mathsf{adv}[\mathcal{B}', \ell, e]$, and the theorem will the follow from Theorem 10.4.

Our RSA adversary $\mathcal{B}'$ runs as follows. It receives (n, y) from its challenger, where n is an RSA modulus and y is a random element of $\mathbb{Z}_n^*$. The values e, n, y define the hash function H_{rsa}, and adversary $\mathcal{B}'$ runs adversary $\mathcal{A}$ with this hash function. Suppose that $\mathcal{A}$ finds a collision. This is a pair of inputs $(a, b) \neq (a', b')$ such that

$$a^e y^b = (a')^e y^{b'},$$

which we may rewrite as

$$(a/a')^e = y^{b'-b}.$$

Using this collision, $\mathcal{B}'$ will compute an eth root of y.

Observe that $b' - b \neq 0$, since otherwise we would have $(a/a') = 1$ and hence $a = a'$. Also observe that since $|b - b'| < e$ and e is prime, we must have $\gcd(e, b - b') = 1$. So now we simply apply Theorem 10.5 with n, e, and y as given, and $w := a/a'$ and $f := b' - b$. $\Box$

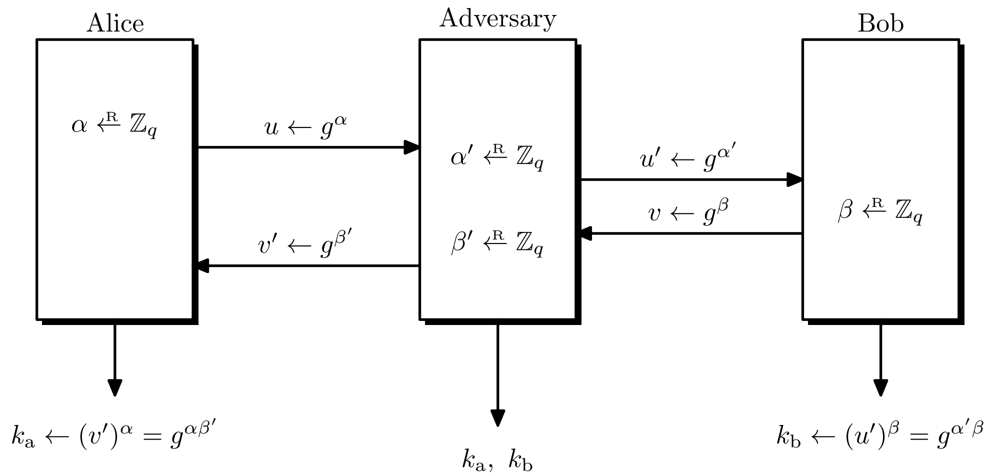


Figure 10.4: Man in the middle attack

10.7 Attacks on the anonymous Diffie-Hellman protocol

The Diffie-Hellman key exchange is secure against a passive eavesdropper. Usually, however, an attacker capable of eavesdropping on traffic is also able to inject its own messages. The protocol completely falls apart in the presence of an active adversary who controls the network. The main reason is the lack of authentication. Alice sets up a shared secret, but she has no idea with whom the secret is shared. The same holds for Bob. An active attacker can abuse this to expose all traffic between Alice and Bob. The attack, called a **man in the middle attack**, works against any key exchange protocol that does not include authentication. It works as follows (see Fig. 10.4):

- Alice sends (g, g^{α}) to Bob. The attacker blocks this message from reaching Bob. He picks a random $\alpha' \xleftarrow{R} \mathbb{Z}_n$ and sends $(g, g^{\alpha'})$ to Bob.
- Bob responds with g^{β}. The attacker blocks this message from reaching Alice. He picks a random $\beta' \xleftarrow{R} \mathbb{Z}_n$ and sends $g^{\beta'}$ to Alice.
- Now Alice computes the key $k_{\text{A}} := g^{\alpha\beta'}$ and Bob computes $k_{\text{B}} := g^{\alpha'\beta}$. The attacker knows both k_{A} and k_{B}.

At this point Alice thinks k_{A} is a secret key shared with Bob and will use k_{A} to encrypt messages to him. Similarly for Bob with his key k_{B}. The attacker can act as a proxy between the two. He intercepts each message $c_i := E(k_{\text{A}}, m_i)$ from Alice, re-encrypts it as $c_i' \leftarrow E(k_{\text{B}}, m_i)$ and forwards c_i' to Bob. He also re-encrypts messages from Bob to Alice. The communication channel works properly for both parties and they have no idea that this proxying is taking place. The attacker, however, sees all plaintexts in the clear.

This generic attack explains why we view key exchange secure against eavesdropping as a toy problem. Protocols secure in this model can completely fall apart once the adversary can tamper with traffic. We will come back to this problem in Chapter 20, where we design protocols secure against active attackers.

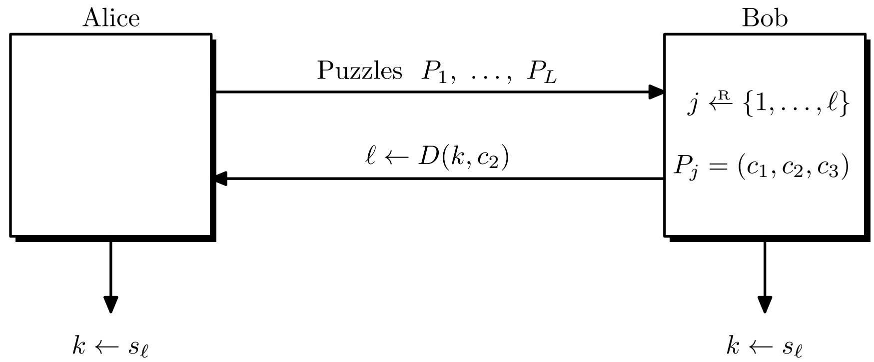


Figure 10.5: Merkle puzzles protocol

10.8 Merkle puzzles: a partial solution to key exchange using block ciphers

Can we build a secure key exchange protocol using symmetric-key primitives? The answer is yes, but the resulting protocol is very inefficient. We show how to do key exchange using a block cipher $\mathcal{E} = (E, D)$ defined over $(\mathcal{K}, \mathcal{M})$. Alice and Bob want to generate a random $s \in \mathcal{M}$ that is unknown to the adversary. They use a protocol called **Merkle puzzles** (due to the same Merkle from the Merkle-Damgård hashing paradigm). The protocol, shown in Fig. 10.5, works as follows:

Protocol 10.1 (Merkle puzzles).

1. Alice picks random triples $(k_i, s_i) \xleftarrow{R} \mathcal{K} \times \mathcal{M}$ for $i = 1, \ldots, L$. We will determine the optimal value for L later. She constructs L puzzles where puzzle P'_i is defined as:

$$P'_i = \big(\ E(k_i, s_i),\ E(k_i, i),\ E(k_i, 0)\ \big)$$

   Next, she sends the L puzzles in a random order to Bob. That is, she picks a random permutation $\pi \xleftarrow{R} \text{Perms}[\{1, \ldots, L\}]$ and sends $(P_1, \ldots, P_L) := (P'_{\pi(1)}, \ldots, P'_{\pi(L)})$ to Bob.

2. Bob picks a random puzzle $P_j = (c_1, c_2, c_3)$ where $j \xleftarrow{R} \{1, \ldots, L\}$. He solves the puzzle by brute force, by trying all keys $k \in \mathcal{K}$ until he finds one such that

$$D(k, c_3) = 0. \tag{10.6}$$

   In the unlikely event that Bob finds two different keys that satisfy (10.6), he indicates to Alice that the protocol failed, and they start over. Otherwise, Bob computes $\ell \leftarrow D(k, c_2)$ and $s \leftarrow D(k, c_1)$, and sends ℓ back to Alice.

3. Alice locates puzzle P'_ℓ and sets $s \leftarrow s_\ell$. Both parties now know the shared secret $s \in \mathcal{M}$.

Clearly, when the protocol terminates successfully, both parties agree on the same secret $s \in \mathcal{M}$. Moreover, when $|\mathcal{M}|$ is much larger than $|\mathcal{K}|$, the protocol is very likely to terminate successfully, because under these conditions (10.6) is likely to have a unique solution.

The work for each party in this protocol is as follows:

$$\text{Alice's work} = O(L), \qquad \text{Bob's work} = O(|\mathcal{K}|).$$

Hence, to make the workload for the two parties about the same we need to set $L \approx |\mathcal{K}|$. Either way, the size of L and $\mathcal{K}$ needs to be within reason so that both parties can perform the computation in a reasonable time. For example, one can set $L \approx |\mathcal{K}| \approx 2^{30}$. When using AES one can force $\mathcal{K}$ to have size 2^{30} by fixing the 98 most significant bits of the key to zero.

Security. The adversary sees the protocol transcript which includes all the puzzles and the quantity ℓ sent by Bob. Since the adversary does not know which puzzle Bob picked, intuitively, he needs to solve all puzzles until he finds puzzle P_ℓ. Thus, to recover $s \in \mathcal{M}$ the adversary must solve L puzzles each one taking $O(|\mathcal{K}|)$ time to solve. Overall, the adversary must spend time $O(L|\mathcal{K}|)$.

One can make this argument precise, by modeling the block cipher $\mathcal{E}$ as an ideal cipher, as we did in Section 4.7. We can assume that $|\mathcal{K}|$ is poly-bounded, and that $|\mathcal{M}|$ is super-poly. Then the analysis shows that if the adversary makes at most Q queries to the ideal cipher, then its probability of learning the secret $s \in \mathcal{M}$ is bounded by approximately $\frac{Q}{L|\mathcal{K}|}$. Working out the complete proof and the exact bound is a good exercise in working with the ideal cipher model.

Performance. Suppose we set $L \approx |\mathcal{K}|$. Then the adversary must spend time $O(L^2)$ to break the protocol, while each participant spends time $O(L)$. Hence, there is a quadratic gap between the work of the participants and the work to break the protocol. Technically speaking, this doesn't satisfy our definitions of security — with constant work the adversary has advantage about $1/L^2$ which is non-negligible. Even worse, in practice one would have to make L extremely large to have a reasonable level of security against a determined attacker. The resulting protocol is then very inefficient.

Nevertheless, the Merkle puzzles protocol is very elegant and shows what can be done using block ciphers alone. As the story goes, Merkle came up with this clever protocol while taking a seminar as an undergraduate student at Berkeley. The professor gave the students the option of submitting a research paper instead of taking the final exam. Merkle submitted his key exchange protocol as the research project. These ideas, however, were too far out and the professor rejected the paper. Merkle still had to take the final exam. Subsequently, for his Ph.D. work, Merkle chose to move to a different school to work with Martin Hellman.

It is natural to ask if a better key exchange protocol, based on block ciphers, can achieve better than quadratic separation between the participants and the adversary. Unfortunately, a result by Impagliazzo and Rudich [57] suggests that one cannot achieve better separation using block ciphers alone.

10.9 Fun application: Pedersen commitments

To be written.

10.10 Notes

Citations to the literature to be added.

10.11 Exercises

10.1 (Computationally unbounded adversaries). Show that an anonymous key exchange protocol P (as in Definition 10.1) cannot be secure against a computationally unbounded adversary. This explains why all protocols in this chapter must rely on computational assumptions.

10.2 (DDH PRG). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Consider the following PRG defined over $(\mathbb{Z}_q^2,\ \mathbb{G}^3)$:

$$G(\alpha, \beta) := (g^\alpha, g^\beta, g^{\alpha\beta}).$$

Show that G is a secure PRG assuming DDH holds in $\mathbb{G}$.

10.3 (The Naor-Reingold PRF). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let us show that the following PRF defined over $\big(\mathbb{Z}_q^{n+1},\ \{0,1\}^n,\ \mathbb{G}\big)$ is secure assuming DDH holds in $\mathbb{G}$:

$$F_{\mathrm{NR}}\Big(\ (\alpha_0, \alpha_1, \dots, \alpha_n),\ (x_1, \dots, x_n)\ \Big) := g^{\left(\alpha_0 \cdot \alpha_1^{x_1} \cdots \alpha_n^{x_n}\right)}$$

This secure PRF is called the Naor-Reingold PRF.

(a) We prove security of F_{NR} using Exercise 4.18. First, show that F_{NR} is an *augmented tree construction* constructed from the PRG: $G_{\mathrm{NR}}(\alpha, g^\beta) := (g^\beta, g^{\alpha\beta})$.

(b) Second, show that G_{NR} satisfies the hypothesis of Exercise 4.18 part (b), assuming DDH holds in $\mathbb{G}$. Use the result of Exercise 10.10.

Security of F_{NR} now follows from Exercise 4.18 part (b).

Discussion: See Exercise 11.1 for a simpler PRF from the DDH assumption, but in the random oracle model.

10.4 (Random self-reduction for CDH (I)). Consider a specific cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. For $u = g^\alpha \in \mathbb{G}$ and $v = g^\beta \in \mathbb{G}$, define $[u, v] = g^{\alpha\beta}$, which is the solution instance (u, v) of the CDH problem. Consider the randomized mapping from $\mathbb{G}^2$ to $\mathbb{G}^2$ that sends (u, v) to $(\tilde{u}, v)$, where

$$\rho \xleftarrow{\mathrm{R}} \mathbb{Z}_q, \quad \tilde{u} \leftarrow g^\rho u.$$

Show that

(a) $\tilde{u}$ is uniformly distributed over $\mathbb{G}$;

(b) $[\tilde{u}, v] = [u, v] \cdot v^\rho$.

10.5 (Random self-reduction for CDH (II)). Continuing with the previous exercise, suppose $\mathcal{A}$ is an efficient algorithm that solves the CDH problem with success probability ϵ on random inputs. That is, if $u, v \in \mathbb{G}$ are chosen at random, then $\Pr[\mathcal{A}(u, v) = [u, v]] = \epsilon$, where the probability is over the random choice of u and v, as well as any random choices made by $\mathcal{A}$. Using $\mathcal{A}$, construct an efficient algorithm $\mathcal{B}$ that solves the CDH problem with success probability ϵ for *all* inputs. More precisely, for all $u, , v \in \mathbb{G}$, we have $\Pr[\mathcal{B}(u, v) = [u, v]] = \epsilon$, where the probability is now only over the random choices made by $\mathcal{B}$.

Remark: If we iterate $\mathcal{B}$ on the same input (u, v) many times, say $n\lceil 1/\epsilon \rceil$ times for some n, at least one of these iterations will output the correct result $[u, v]$ with probability $1 - (1 - \epsilon)^{n\lceil 1/\epsilon \rceil} \geq$

$1 - \exp(-n)$. Unfortunately, assuming the DDH is true, we will have no way of knowing which of these outputs is the correct result.

10.6 (An alternative DDH characterization). Let $\mathbb{G}$ by a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let $\mathcal{P}$ be the uniform distribution over $\mathbb{G}^3$. Let $\mathcal{P}_{\text{dh}}$ be the uniform distribution over the set of all DH-triples $(g^\alpha, g^\beta, g^{\alpha\beta})$. Let $\mathcal{P}_{\text{ndh}}$ be the uniform distribution over the set of all non-DH-triples $(g^\alpha, g^\beta, g^\gamma), \gamma \neq \alpha\beta$.

(a) Show that the statistical distance between $\mathcal{P}$ and $\mathcal{P}_{\text{ndh}}$ is $1/q$.

(b) Using part (a), deduce that under the DDH assumption, the distributions $\mathcal{P}_{\text{dh}}$ and $\mathcal{P}_{\text{ndh}}$ are computationally indistinguishable.

10.7 (Random self-reduction for DDH (I)). Consider a specific cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. Let $\mathbf{DH}$ be the set of all DH-triples, i.e.,

$$\mathbf{DH} := \{(g^\alpha, g^\beta, g^{\alpha\beta}) \in \mathbb{G}^3 : \alpha, \beta \in \mathbb{Z}_q\}.$$

For fixed $u \in \mathbb{G}$, and let $\mathbf{T}_u$ be the subset of $\mathbb{G}^3$ whose first coordinate is u. Consider the randomized mapping from $\mathbb{G}^3$ to $\mathbb{G}^3$ that sends (u, v, w) to (u, v^*, w^*), where

$$\sigma \xleftarrow{\text{R}} \mathbb{Z}_q, \quad \tau \xleftarrow{\text{R}} \mathbb{Z}_q, \quad v^* \leftarrow g^\sigma v^\tau, \quad w^* \leftarrow u^\sigma w^\tau.$$

Prove the following:

(a) if $(u, v, w) \in \mathbf{DH}$, then (u, v^*, w^*) is uniformly distributed over $\mathbf{DH} \cap \mathbf{T}_u$;

(b) if $(u, v, w) \notin \mathbf{DH}$, then (u, v^*, w^*) is uniformly distributed over $\mathbf{T}_u$.

10.8 (Random self-reduction for DDH (II)). Continuing with the previous exercise, consider the randomized mapping from $\mathbb{G}^3$ to $\mathbb{G}^3$ that sends (u, v, w) to $(\tilde{u}, v, \tilde{w})$, where

$$\rho \xleftarrow{\text{R}} \mathbb{Z}_q, \quad \tilde{u} \leftarrow g^\rho u, \quad \tilde{w} \leftarrow v^\rho w.$$

Prove the following:

(a) $\tilde{u}$ is uniformly distributed over $\mathbb{G}$;

(b) $(u, v, w) \in \mathbf{DH} \iff (\tilde{u}, v, \tilde{w}) \in \mathbf{DH}$;

(c) if we apply the randomized mapping from the previous exercise to $(\tilde{u}, v, \tilde{w})$, obtaining the triple $(\tilde{u}, v^*, \tilde{w}^*)$, then we have

- if $(u, v, w) \in \mathbf{DH}$, then $(\tilde{u}, v^*, \tilde{w}^*)$ is uniformly distributed over $\mathbf{DH}$;
- if $(u, v, w) \notin \mathbf{DH}$, then $(\tilde{u}, v^*, \tilde{w}^*)$ is uniformly distributed over $\mathbb{G}^3$.

10.9 (Random self-reduction for DDH (III)). Continuing with the previous exercise, prove the following. Suppose $\mathcal{A}$ is an efficient algorithm that takes as input three group elements and outputs a bit, and which satisfies the following property: if $\alpha, \beta, \gamma \in \mathbb{Z}_q$ are chosen at random, then

$$\left|\Pr[\mathcal{A}(g^\alpha, g^\beta, g^{\alpha\beta}) = 1] - \Pr[\mathcal{A}(g^\alpha, g^\beta, g^\gamma) = 1]\right| = \epsilon,$$

where the probability is over the random choice of α, β, γ, as well as any random choices made by $\mathcal{A}$. Assuming that $1/\epsilon$ is poly-bounded, show how to use $\mathcal{A}$ to build an efficient algorithm $\mathcal{B}$ that for all inputs (u, v, w) correctly decides whether or not $(u, v, w) \in \mathbf{DH}$ with negligible error probability. That is, adversary $\mathcal{B}$ may output an incorrect answer, but for all inputs, the probability that its answer is incorrect should be negligible.

Hint: Use a Chernoff bound.

10.10 (Multi-DDH). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let $n \leq m$ be positive integers. Define the following two distributions over $\mathbb{G}^{n \cdot m+n+m}$:

$$\begin{aligned} \mathcal{D}: \quad & g^{\alpha_i} \quad (i = 1, \ldots, n), \qquad g^{\beta_j} \quad (j = 1, \ldots, m) \\ & g^{\alpha_i \beta_j} \quad (i = 1, \ldots, n, \quad j = 1, \ldots, m), \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}: \quad & g^{\alpha_i} \quad (i = 1, \ldots, n), \qquad g^{\beta_j} \quad (j = 1, \ldots, m) \\ & g^{\gamma_{ij}} \quad (i = 1, \ldots, n, \quad j = 1, \ldots, m). \end{aligned}$$

where the α_i's, β_j's, and γ_{ij}'s are uniformly and independently distributed over $\mathbb{Z}_q$. Show that under the DDH assumption, $\mathcal{D}$ and $\mathcal{R}$ are computationally indistinguishable (as in Definition 3.4). In particular, show that for every adversary $\mathcal{A}$ that distinguishes $\mathcal{D}$ and $\mathcal{R}$, there exists a DDH adversary $\mathcal{B}$ (which is an elementary wrapper around $\mathcal{A}$) such that

$$\mathsf{Distadv}[\mathcal{A}, \mathcal{D}, \mathcal{R}] \leq n \cdot (1/q + \mathrm{DDH}\mathsf{adv}[\mathcal{B}, \mathbb{G}]).$$

Hint: First give a proof for the case $n = 1$ using the results of Exercise 10.6 and Exercise 10.7, and then generalize to arbitrary n using a hybrid argument.

Discussion: This result gives us a DDH-based PRG G defined over $(\mathbb{Z}_q^{n+m},\ \mathbb{G}^{n \cdot m+n+m})$, with a nice expansion rate, given by

$$G\Big(\ \{\alpha_i\}_{i=1}^{n},\ \{\beta_j\}_{j=1}^{m}\ \Big) := \Big(\ \{g^{\alpha_i}\}_{i=1}^{n},\ \{g^{\beta_j}\}_{j=1}^{m},\ \{g^{\alpha_i \beta_j}\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}\ \Big)$$

10.11 (Matrix DDH). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let n and m be positive integers, and assume $n \leq m$. For $A = (\alpha_{ij}) \in \mathbb{Z}_q^{n \times m}$ (i.e., A is an $n \times m$ matrix with entries in $\mathbb{Z}_q$), let g^A be the $n \times m$ matrix whose entry at row i column j is the group element $g^{\alpha_{ij}}$. For $k = 1, \ldots, n$, define the random variable $R(k)$ to be a random matrix uniformly distributed over all $n \times m$ matrices over $\mathbb{Z}_q$ of of rank k. Let $1 \leq k_1 < k_2 \leq n$. Show that $g^{R(k_1)}$ and $g^{R(k_2)}$ are computationally indistinguishable under the DDH. In particular, show that for every adversary $\mathcal{A}$ that distinguishes $g^{R(k_1)}$ and $g^{R(k_2)}$ there exists a DDH adversary $\mathcal{B}$ (which is an elementary wrapper around $\mathcal{A}$) such that

$$\mathsf{Distadv}[\mathcal{A}, g^{R(k_1)}, g^{R(k_2)}] \leq (k_2 - k_1) \cdot (1/q + \mathrm{DDH}\mathsf{adv}[\mathcal{B}, \mathbb{G}]).$$

Hint: Use the fact that if $A \in \mathbb{Z}_q^{n \times m}$ is a fixed matrix of rank k, and if $U \in \mathbb{Z}_q^{n \times n}$ and $V \in \mathbb{Z}_q^{m \times m}$ are a random *invertible* matrices, then the matrix $UAV \in \mathbb{Z}_q^{n \times m}$ is uniformly distributed over all $n \times m$ matrices of rank k. You might also try to prove this fact, which is not too hard.

Discussion: For $k_1 = 1$ and $k_2 = n$, this result implies a closely related, but slightly weaker form of Exercise 10.10. In this sense, this exercise is a generalization of Exercise 10.10.

10.12 (A trapdoor test). Consider a specific cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. Let $u \in \mathbb{G}$ and $f : \mathbb{G} \to \mathbb{G}^3$. Now set

$$\sigma \xleftarrow{R} \mathbb{Z}_q, \quad \tau \xleftarrow{R} \mathbb{Z}_q, \quad \bar{u} \leftarrow g^{\sigma} u^{\tau}, \quad (v, w, \bar{w}) \leftarrow f(\bar{u}).$$

Let S be the the event that (u, v, w) and $(\bar{u}, v, \bar{w})$ are both DH-triples. Let T be the event that $\bar{w} = v^{\sigma} w^{\tau}$. Show that:

(a) $\bar{u}$ is uniformly distributed over $\mathbb{G}$;

(b) $\Pr[S \wedge \neg T] = 0$;

(c) $\Pr[\neg S \wedge T] \le 1/q$.

Remark: This result gives us a kind of trapdoor test. Suppose a group element $u \in \mathbb{G}$ is given (it could be chosen at random or adversarially chosen). Then we can generate a random element $\bar{u}$ and a "trapdoor" (σ, τ). Using this trapdoor, given group elements $v, w, \bar{w} \in \mathbb{G}$ (possibly adversarially chosen in a way that depends on $\bar{u}$), we can reliably test if (u, v, w) and $(\bar{u}, v, \bar{w})$ are both DH-triples, even though we do not know either $\mathsf{Dlog}_g(u)$ or $\mathsf{Dlog}_g(\bar{u})$, and even though we cannot tell whether (u, v, w) and $(\bar{u}, v, \bar{w})$ are individually DH-triples. This rather technical result has several nice applications, one of which is developed in the following exercise.

10.13 (A CDH self-corrector). Consider a specific cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. Let $\mathcal{A}$ be an efficient algorithm with the following property: if $\alpha, \beta \in \mathbb{Z}_q$ are chosen at random, then $\Pr[\mathcal{A}(g^{\alpha}, g^{\beta}) = g^{\alpha\beta}] = \epsilon$. Here, the probability is over the random choice of α and β, as well as any random choices made by $\mathcal{A}$. Assuming $1/\epsilon$ is poly-bounded and $|\mathbb{G}|$ is super-poly, show how to use $\mathcal{A}$ to build an efficient algorithm $\mathcal{B}$ that solves the CDH problem on all inputs with negligible error probability; that is, on every input (g^{α}, g^{β}), algorithm $\mathcal{B}$ outputs a single group element w, and $w \neq g^{\alpha\beta}$ with negligible probability (and this probability is just over the random choices made by $\mathcal{B}$).

Here is a high-level sketch of how $\mathcal{B}$ might work on input (u, v).

> somehow choose $\bar{u} \in \mathbb{G}$
> somehow use $\mathcal{A}$ to generate lists $L, \bar{L}$ of group elements
> for each w in L and each $\bar{w}$ in $\bar{L}$ do
>   if (u, v, w) and $(\bar{u}, v, \bar{w})$ are both DH-triples then
>   output w and halt
> output an arbitrary group element

As stated, this algorithm is not fully specified. Nevertheless, you can use this rough outline, combined with the CDH random self reduction in Exercise 10.4 and the trapdoor test in Exercise 10.12, to prove the desired result.

For the next problem, we need the following notions from complexity theory:

- *We say problem A is* deterministic poly-time reducible *to problem B if there exists a deterministic algorithm R for solving problem A on all inputs that makes calls to a subroutine that solves problem B on all inputs, where the running time of R (not including the running time for the subroutine for B) is polynomial in the input length.*

- *We say that A and B are* deterministic poly-time equivalent *if A is deterministic poly-time reducible to B and B is deterministic poly-time reducible to A.*

10.14 (Problems equivalent to DH). Consider a specific cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. Show that the following problems are deterministic poly-time equivalent:

(a) Given g^α and g^β, compute $g^{\alpha\beta}$ (this is just the Diffie-Hellman problem).

(b) Given g^α, compute $g^{(\alpha^2)}$.

(c) Given g^α with $\alpha \neq 0$, compute $g^{1/\alpha}$.

(d) Given g^α and g^β with $\beta \neq 0$, compute $g^{\alpha/\beta}$.

Note that all problem instances are defined with respect to the same group $\mathbb{G}$ and generator $g \in \mathbb{G}$.

10.15 (System parameters). In formulating the discrete-log Attack Game 10.4, we assume that the description of $\mathbb{G}$, including $g \in \mathbb{G}$ and q, is a system parameter that is generated once and for all at system setup time and shared by all parties involved. This parameter may be generated via some randomized process, in which case the advantage $\epsilon = \mathrm{DL}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$ is a probability over the choice of system parameter, as well as the random choice of $\alpha \in \mathbb{Z}_q$ made by the challenger and any random choices made by adversary. So we can think of the system parameter as a random variable Λ, and for any specific system parameter Λ_0, we can consider the corresponding conditional advantage $\epsilon(\Lambda_0)$ given that $\Lambda = \Lambda_0$, which is a probability just over the random choice of $\alpha \in \mathbb{Z}_q$ made by the challenger and any random choices made by adversary. Let us call Λ_0 a "vulnerable" parameter if $\epsilon(\Lambda_0) \geq \epsilon/2$.

(a) Prove that the probability that Λ is vulnerable is at least $\epsilon/2$.

Note that even if an adversary breaks the DL with respect to a randomly generated system parameter, there could be many particular system parameters for which the adversary cannot or will not break the DL (it is helpful to imagine an adversary that is all powerful yet capricious, who simply refuses to break the DL for certain groups and generators which he finds distasteful). This result says, however, that there is still a non-negligible fraction of vulnerable system parameters for which the adversary breaks the DL.

(b) State and prove an analogous result for the CDH problem.

(c) State and prove an analogous result for the DDH problem.

10.16 (Choice of generators). In formulating the DL, CDH, and DDH assumptions, we work with a cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$. We do not specify how the generator g is chosen. Indeed, it may be desirable to choose a specific g that allows for more efficient implementations. Conceivably, such a g could be a "weak" generator that makes it easier for an adversary to break the DL, CDH, or DDH assumptions. So to be on the safe side, we might insist that the generator g is uniformly distributed over $\mathbb{G}\setminus\{1\}$. If we do this, we obtain new assumptions, which we call the rDL, rCDH, and rDDH assumptions. Show that:

(a) the rDL and DL assumptions are equivalent;

(b) the rCDH and CDH assumptions are equivalent;

(c) the DDH assumption implies the rDDH assumption.

Hint: To start with, you might first consider the setting where we are working with a specific group, then generalize your result to incorporate all the aspects of the asymptotic attack game (see Section 10.5.2), including the security parameter and the system parameter (where the group is selected at system setup time).

Remark: The rDDH assumption is not known to imply the DDH assumption, so for applications that use the DDH assumption, it seems safest to work with a random generator.

10.17 (Collision resistance from discrete-log). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let n be a poly-bounded parameter. We define a hash function H defined over $(\mathbb{Z}_q^n, \mathbb{G})$. The hash function is parameterized by the group $\mathbb{G}$ and n randomly chosen group elements $g_1, \ldots, g_n \in \mathbb{G}$. For $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_q^n$, we define

$$H(\alpha_1, \ldots, \alpha_n) := g_1^{\alpha_1} \cdots g_n^{\alpha_n}.$$

Prove that H is collision resistant under the DL assumption for $\mathbb{G}$. In particular, show that for every collision-finding adversary $\mathcal{A}$, there exists a DL adversary $\mathcal{B}$, which is an elementary wrapper around $\mathcal{A}$, such that $\text{CR}\mathsf{adv}[\mathcal{A}, H] \leq \text{DL}\mathsf{adv}[\mathcal{B}, \mathbb{G}] + 1/q$.

10.18 (Collision resistance in $\mathbb{Z}_p^*$). This exercise asks you to prove that the hash function presented in Section 8.5.1 is collision resistant under an appropriate DL assumption. Let us define things a bit more precisely. Let p be a large prime such that $q := (p-1)/2$ is also prime. The prime q is called a *Sophie Germain* prime, and p is sometimes called a "strong" prime. Such primes are often very convenient to use in cryptography. Suppose x is a randomly chosen integer in the range $[2, q]$ and y is a randomly chosen integer in the range $[1, q]$. These parameters define a hash function H that takes as input two integers in $[1, q]$ and outputs an integer in $[1, q]$, as specified in (8.3). Let $\mathbb{G}$ be the subgroup of order q in $\mathbb{Z}_p^*$, and consider the DL assumption for $\mathbb{G}$ with respect to a randomly chosen generator. Show that H is collision resistant under this DL assumption.

Hint: Use the fact that and that the map that sends $\alpha \in \mathbb{Z}_p^*$ to $\alpha^2 \in \mathbb{Z}_p^*$ is a group homomorphism with image $\mathbb{G}$ and kernel ± 1; also use the fact that there is an efficient algorithm for taking square roots in $\mathbb{Z}_p^*$.

10.19 (A broken CRHF). Consider the following variation of the hash construction in the previous exercise. Let p be a large prime such that $q := (p-1)/2$ is also prime. Let x and y be randomly chosen integers in the range $[2, p-2]$ (so neither can be $\pm 1 \pmod{p}$). These parameters define a hash function H that takes as input two integers in $[1, p-1]$ and outputs an integer in $[1, p-1]$, as follows:

$$H(a, b) := x^a y^b \bmod p.$$

Give an efficient, deterministic algorithm that takes as input p, x, y as above, and computes a collision on the corresponding H. Your algorithm should work for all inputs p, x, y.

10.20 (DDH is easy in groups of even order). We have restricted the DL, CDH, and DDH assumptions to prime order groups $\mathbb{G}$. Consider the DDH assumption for a cyclic group $\mathbb{G}$ of *even* order q with generator $g \in \mathbb{G}$. Except for dropping the restriction that q is prime, the attack game is identical to Attack Game 10.6. Give an efficient adversary that has advantage $1/2$ in solving the DDH for $\mathbb{G}$.

Remark: For a prime $p > 2$, the group $\mathbb{Z}_p^*$ is a cyclic group of even order $p-1$. This exercise shows that the DDH assumption is false in this group. Exercise 10.19 gives another reason to restrict ourselves to groups of prime order.

10.21 (RSA variant (I)). Let n be an RSA modulus generated by RSAGen(ℓ, e). Let X and X^* be random variables, where X is uniformly distributed over $\mathbb{Z}_n$ and X^* is uniformly distributed over $\mathbb{Z}_n^*$. Show that the statistical distance $\Delta[X, X^*]$ is less than $2^{-(\ell-2)}$.

10.22 (RSA variant (II)). In Theorem 10.4, we considered a variant of the RSA assumption where the challenger chooses the preimage x at random from $\mathbb{Z}_n^*$, rather than $\mathbb{Z}_n$. That theorem showed that the standard RSA assumption implies this variant RSA assumption. In this exercise, you are to show the converse. In particular, show that $\text{RSA}\mathsf{adv}[\mathcal{A}, \ell, e] \leq \text{uRSA}\mathsf{adv}[\mathcal{B}, \ell, e] + 2^{-(\ell-2)}$ for every adversary $\mathcal{A}$.

Hint: Use the result of the previous exercise.

10.23 (A proper trapdoor permutation scheme based on RSA). As discussed in Section 10.3, our RSA-based trapdoor permutation scheme does not quite satisfy our definitions, simply because the domain on which it acts varies with the public key. This exercise shows one way to patch things up. Let ℓ and e be parameters used for RSA key generation, and let G be the key generation algorithm, which outputs a pair (pk, sk). Recall that $pk = (n, e)$, where n is an RSA modulus, which is the product of two ℓ-bit primes, and e is the encryption exponent. The secret key is $sk = (n, d)$, where d is the decryption exponent corresponding to the encryption exponent e. Choose a parameter L that is a substantially larger than 2ℓ, so that $n/2^L$ is negligible. Let $\mathcal{X}$ be the set of integers in the range $[0, 2^L)$. We shall present a trapdoor permutation scheme (G, F^*, I^*), defined over $\mathcal{X}$. The function F^* takes two inputs: a public key pk as above and an integer $x \in \mathcal{X}$, and outputs an integer $y \in \mathcal{X}$, computed as follows. Divide x by n to obtain the integer quotient Q and remainder R, so that $x = nQ + R$ and $0 \leq R < n$. If $Q > 2^L/n - 1$, then set $S := R$; otherwise, set $S := R^e \bmod n$. Finally, set $y := nQ + S$.

(a) Show that $F^*(pk, \cdot)$ is a permutation on $\mathcal{X}$, and give an efficient inversion function I^* that satisfies $I^*(sk, F^*(pk, x)) = x$ for all $x \in \mathcal{X}$.

(b) Show under the RSA assumption, (G, F^*, I^*) is one-way.

10.24 (Random self-reduction for RSA). Suppose we run $(n, d) \xleftarrow{R} \text{RSAGen}(\ell, e)$. There could be "weak" RSA moduli n for which an adversary can break the the RSA assumption with some probability ϵ. More precisely, suppose that there is an efficient algorithm $\mathcal{A}$ such that for any such "weak" modulus n, if $x \in \mathbb{Z}_n^*$ is chosen at random, then $\Pr[\mathcal{A}(x^e) = x] \geq \epsilon$, where the probability is over the random choice of x, as well as any random choices made by $\mathcal{A}$. Using $\mathcal{A}$, construct an efficient algorithm $\mathcal{B}$ such that for every "weak" modulus n, and every $x \in \mathbb{Z}_n$, we have $\Pr[\mathcal{A}(x^e) = x] \geq \epsilon$, where the probability is now only over the random choices made by $\mathcal{B}$.

Hint: Use the randomized mapping from $\mathbb{Z}_n^*$ to $\mathbb{Z}_n^*$ that sends y to $\tilde{y}$, where $r \xleftarrow{R} \mathbb{Z}_n^*$, $\tilde{y} \leftarrow r^e y$. Show that for every $y \in \mathbb{Z}_n^*$, the value $\tilde{y}$ is uniformly distributed over $\mathbb{Z}_n^*$.

10.25 (n-product CDH). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. The following attack game defines the **n-product CDH problem** (here, n is a poly-bounded parameter, not necessarily constant). The challenger begins by choosing $\alpha_i \xleftarrow{R} \mathbb{Z}_q$ for $i = 1, \ldots, n$. The adversary then makes a sequence of queries. In each query, the adversary submits proper a subset of indices $S \subsetneq \{1, \ldots, n\}$, and the challenger responds with

$$g^{\prod_{i \in S} \alpha_i}.$$

The adversary wins the game if he outputs

$$g^{\alpha_1 \cdots \alpha_n}.$$

We relate the hardness of solving the n-product CDH problem to another problem, called the **n-power CDH problem**. In the attack game for this problem, the challenger begins by choosing $\alpha \xleftarrow{R} \mathbb{Z}_q^*$, and gives

$$g, g^{\alpha}, \ldots, g^{\alpha^{n-1}}$$

to the adversary. The adversary wins the game if he outputs g^{α^n}.

Show that if there is an efficient adversary $\mathcal{A}$ that breaks n-product CDH with non-negligible probability, then there is an efficient adversary $\mathcal{B}$ that breaks n-power CDH with non-negligible probability.

10.26 (Trapdoor collison resistance). Let us show that the collision resistant hash functions H_{dl} and H_{rsa}, presented in Section 10.6, are *trapdoor* collision resistant.

(a) Recall that H_{dl} is defined as $H_{\text{dl}}(\alpha, \beta) := g^{\alpha} u^{\beta} \in \mathbb{G}$, where g and u are parameters chosen at setup. Show that anyone who knows the discrete-log of u base g (the trapdoor), can break the 2nd-preimage resistance of H_{dl}. That is, given (α, β) as input, along with the trapdoor, one can efficiently compute $(\alpha', \beta') \neq (\alpha, \beta)$ such that $H_{\text{dl}}(\alpha', \beta') = H_{\text{dl}}(\alpha, \beta)$.

(b) Recall that H_{rsa} is defined as $H_{\text{rsa}}(a, b) := a^e y^b \in \mathbb{Z}_n$, where n, e and y are parameters chosen at setup. Show that anyone who knows the eth root of y in $\mathbb{Z}_n$ (the trapdoor), can break the 2nd-preimage resistance of H_{rsa}.

(c) Continuing with part (b), show that anyone who knows the factorization of n (the trapdoor), can invert H_{rsa}. That is, given $z \in Z_n$ as input, one can find (a, b) such that $H_{\text{rsa}}(a, b) = z$.

Discussion: Part (c) shows that the factorization of n is a "stronger" trapdoor for H_{rsa} than the eth root of y. The latter only breaks 2nd-preimage resistance of H_{rsa}, where as the former enables complete inversion. Both trapdoors break collision resistance.

Chapter 11

Public key encryption

In this chapter, we consider again the basic problem of encryption. As a motivating example, suppose Alice wants to send Bob an encrypted email message, even though the two of them do not share a secret key (nor do they share a secret key with some common third party). Surprisingly, this can be done using a technology called **public-key encryption**.

The basic idea of public-key encryption is that the receiver, Bob in this case, runs a key generation algorithm G, obtaining a pair of keys:

$$(pk, sk) \xleftarrow{\mathrm{R}} G().$$

The key pk is Bob's *public key*, and sk is Bob's *secret key*. As their names imply, Bob should keep sk secret, but may publicize pk.

To send Bob an encrypted email message, Alice needs two things: Bob's email address, and Bob's public key pk. How Alice reliably obtains this information is a topic we shall explore later in Section 13.8. For the moment, one might imagine that this information is placed by Bob in some kind of public directory to which Alice has read-access.

So let us assume now that Alice has Bob's email address and public key pk. To send Bob an encryption of her email message m, she computes the ciphertext

$$c \xleftarrow{\mathrm{R}} E(pk, m).$$

She then sends c to Bob, using his email address. At some point later, Bob receives the ciphertext c, and decrypts it, using his *secret key*:

$$m \leftarrow D(sk, c).$$

Public-key encryption is sometimes called **asymmetric encryption** to denote the fact that the encryptor uses one key, pk, and the decryptor uses a different key, sk. This is in contrast with *symmetric encryption*, discussed in Part 1, where both the encryptor and decryptor use the same key.

A few points deserve further discussion:

- Once Alice obtains Bob's public key, the only interaction between Alice and Bob is the actual transmission of the ciphertext from Alice to Bob: no further interaction is required. In fact, we chose encrypted email as our example problem precisely to highlight this feature, as email delivery protocols do not allow any interaction beyond delivery of the message.

- As we will discuss later, the same public key may be used many times. Thus, once Alice obtains Bob's public key, she may send him encrypted messages as often as she likes. Moreover, other users besides Alice may send Bob encrypted messages using the same public key pk.
- As already mentioned, Bob may publicize his public key pk. Obviously, for any secure public-key encryption scheme, it must be hard to compute sk from pk, since anyone can decrypt using sk.

11.1 Two further example applications

Public-key encryption is used in many real-world settings. We give two more examples.

11.1.1 Sharing encrypted files

In many modern file systems, a user can store encrypted files to which other users have read access: the owner of the file can selectively allow others to read the unencrypted contents of the file. This is done using a combination of public-key encryption and an ordinary, symmetric cipher.

Here is how it works. Alice encrypts a file f under a key k, using an ordinary, symmetric cipher. The resulting ciphertext c is stored on the file system. If Alice wants to grant Bob access to the contents of the file, she encrypts k under Bob's public key; that is, she computes $c_{\mathrm{B}} \xleftarrow{\mathrm{R}} E(pk_{\mathrm{B}}, k)$, where pk_{B} is Bob's public key. The ciphertext c_{B} is then stored on the file system near the ciphertext c, say, as part of the file header, which also includes file metadata (such as the file name, modification time, and so on). Now when Bob wants to read the file f, he can decrypt c_{B} using his secret key sk_{B}, obtaining k, using which he can decrypt c using the symmetric cipher. Also, so that Alice can read the file herself, she grants access to herself just as she does to Bob, by encrypting k under her own public key pk_{A}.

This scheme scales very nicely if Alice wants to grant access to f to a number of users. Only one copy of the encrypted file is stored on the file system, which is good if the file is quite large (such as a video file). For each user that is granted access to the file, only an encryption of the key k is stored in the file header. Each of these ciphertexts is fairly small (on the order of a few hundred bytes), even if the file itself is very big.

11.1.2 Key escrow

Consider a company that deploys an encrypted file system such as the one described above. One day Alice is traveling, but her manager needs to read one of her files to prepare for a meeting with an important client. Unfortunately, the manager is unable to decrypt the file because it is encrypted and Alice is unreachable.

Large companies solve this problem using a mechanism called **key escrow**. The company runs a key escrow server that works as follows: at setup time the key escrow server generates a secret key sk_{ES} and a corresponding public key pk_{ES}. It keeps the secret key to itself and makes the public key available to all employees.

When Alice stores the encryption c of a file f under a symmetric key k, she also encrypts k under pk_{ES}, and then stores the resulting ciphertext c_{ES} in the file header. Every file created by company employees is encrypted this way. Now, if Alice's manager later needs access to f and Alice is unreachable, the manager sends c_{ES} to the escrow service. The server decrypts c_{ES}, obtaining k, and sends k to the manager, who can then use this to decrypt c and obtain f.

Public-key encryption makes it possible for the escrow server to remain offline, until someone needs to decrypt an inaccessible file. Also, notice that although the escrow service allows Alice's manager to read her files, the escrow service itself cannot read Alice's files, since the escrow service never sees the encryption of the file.

11.2 Basic definitions

We begin by defining the basic syntax and correctness properties of a public-key encryption scheme.

Definition 11.1. *A* ***public-key encryption scheme*** $\mathcal{E} = (G, E, D)$ *is a triple of efficient algorithms: a* ***key generation algorithm*** G*, an* ***encryption algorithm*** E*, a* ***decryption algorithm*** D*.*

- G *is a probabilistic algorithm that is invoked as* $(pk, sk) \xleftarrow{R} G()$*, where* pk *is called a* ***public key*** *and* sk *is called a* ***secret key****.*

- E *is a probabilistic algorithm that is invoked as* $c \xleftarrow{R} E(pk, m)$*, where* pk *is a public key (as output by* G*),* m *is a message, and* c *is a ciphertext.*

- D *is a deterministic algorithm that is invoked as* $m \leftarrow D(sk, c)$*, where* sk *is a secret key (as output by* G*),* c *is a ciphertext, and* m *is either a message, or a special* reject *value (distinct from all messages).*

- *As usual, we require that decryption undoes encryption; specifically, for all possible outputs* (pk, sk) *of* G*, and all messages* m*, we have*

$$\Pr[D(sk,\ E(pk,\ m)\ ) = m] = 1.$$

- *Messages are assumed to lie in some finite* ***message space*** $\mathcal{M}$*, and ciphertexts in some finite* ***ciphertext space*** $\mathcal{C}$*. We say that* $\mathcal{E} = (G, E, D)$ *is defined over* $(\mathcal{M}, \mathcal{C})$*.*

We next define the notion of semantic security for a public-key encryption scheme. We stress that this notion of security only models an eavesdropping adversary. We will discuss stronger security properties in the next chapter.

Attack Game 11.1 (semantic security). For a given public-key encryption scheme $\mathcal{E} = (G, E, D)$, defined over $(\mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, we define two experiments.

Experiment b $(b = 0, 1)$:

- The challenger computes $(pk, sk) \xleftarrow{R} G()$, and sends pk to the adversary.

- The adversary computes $m_0, m_1 \in \mathcal{M}$, of the same length, and sends them to the challenger.

- The challenger computes $c \xleftarrow{R} E(pk, m_b)$, and sends c to the adversary.

- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

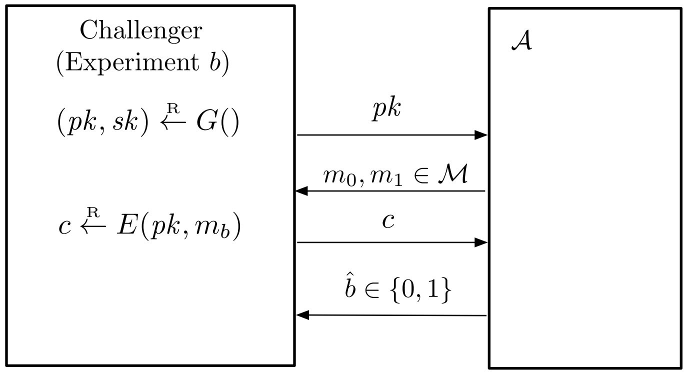


Figure 11.1: Experiment b of Attack Game 11.1

If W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b, we define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] := \Big| \Pr[W_0] - \Pr[W_1] \Big|. \quad \Box$$

Note that in the above game, the events W_0 and W_1 are defined with respect to the probability space determined by the random choices made by the key generation and encryption algorithms, and the random choices made by the adversary. See Fig. 11.1 for a schematic diagram of Attack Game 11.1.

Definition 11.2 (semantic security). *A public-key encryption scheme $\mathcal{E}$ is* ***semantically secure*** *if for all efficient adversaries $\mathcal{A}$, the value $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}]$ is negligible.*

As discussed in Section 2.3.5, Attack Game 11.1 can be recast as a "bit guessing" game, where instead of having two separate experiments, the challenger chooses $b \in \{0, 1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$. In this game, we measure $\mathcal{A}$'s *bit-guessing advantage* $\mathsf{SSadv}^*[\mathcal{A}, \mathcal{E}]$ as $|\Pr[\hat{b} = b] - 1/2|$. The general result of Section 2.3.5 (namely, (2.13)) applies here as well:

$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \mathsf{SSadv}^*[\mathcal{A}, \mathcal{E}]. \tag{11.1}$$

11.2.1 Mathematical details

We give a more mathematically precise definition of a public-key encryption scheme, using the terminology defined in Section 2.4.

Definition 11.3 (public-key encryption scheme). *A* ***public-key encryption scheme*** *consists of a three algorithms, G, E, and D, along with two families of spaces with system parameterization P:*

$$\mathbf{M} = \{\mathcal{M}_{\lambda,\Lambda}\}_{\lambda,\Lambda} \quad \textit{and} \quad \mathbf{C} = \{\mathcal{C}_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$

such that

1. $\mathbf{M}$ *and* $\mathbf{C}$ *are efficiently recognizable.*

2. **M** *has an effective length function.*

3. *Algorithm G is an efficient probabilistic algorithm that on input λ, Λ, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, outputs a pair (pk, sk), where pk and sk are bit strings whose lengths are always bounded by a polynomial in λ.*

4. *Algorithm E is an efficient probabilistic algorithm that on input λ, Λ, pk, m, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $(pk, sk) \in \mathrm{Supp}(G(\lambda, \Lambda))$ for some sk, and $m \in \mathcal{M}_{\lambda,\Lambda}$, always outputs an element of $\mathcal{C}_{\lambda,\Lambda}$.*

5. *Algorithm D is an efficient* deterministic *algorithm that on input λ, Λ, sk, c, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $(pk, sk) \in \mathrm{Supp}(G(\lambda, \Lambda))$ for some pk, and $c \in \mathcal{C}_{\lambda,\Lambda}$, outputs either an element of $\mathcal{M}_{\lambda,\Lambda}$, or a special symbol* $\mathsf{reject} \notin \mathcal{M}_{\lambda,\Lambda}$.

6. *For all $\lambda, \Lambda, pk, sk, m, c$, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $(pk, sk) \in \mathrm{Supp}(G(\lambda, \Lambda))$, $k \in \mathcal{K}_{\lambda,\Lambda}$, $m \in \mathcal{M}_{\lambda,\Lambda}$, and $c \in \mathrm{Supp}(E(\lambda, \Lambda; pk, m))$, we have $D(\lambda, \Lambda; sk, c) = m$.*

As usual, the proper interpretation of Attack Game 11.1 is that both challenger and adversary receive λ as a common input, and that the challenger generates Λ and sends this to the adversary before the game proper begins. The advantage is actually a function of λ, and security means that this is a negligible function of λ.

11.3 Implications of semantic security

Before constructing semantically secure public-key encryption schemes, we first explore a few consequences of semantic security. We first show that any semantically secure public-key scheme must use a *randomized* encryption algorithm. We also show that in the public-key setting, semantic security implies CPA security. This was not true for symmetric encryption schemes: the one-time pad is semantically secure, but not CPA secure.

11.3.1 The need for randomized encryption

Let $\mathcal{E} = (G, E, D)$ be a semantically secure public-key encryption scheme defined over $(\mathcal{M}, \mathcal{C})$ where $|\mathcal{M}| \geq 2$. We show that the encryption algorithm E must be a randomized, otherwise the scheme cannot be semantically secure.

To see why, suppose E is deterministic. Then the following adversary $\mathcal{A}$ breaks semantic security of $\mathcal{E} = (G, E, D)$:

- $\mathcal{A}$ receives a public key pk from its challenger.
- $\mathcal{A}$ chooses two distinct messages m_0 and m_1 in $\mathcal{M}$ and sends them to its challenger. The challenger responds with $c := E(pk, m_b)$ for some $b \in \{0, 1\}$.
- $\mathcal{A}$ computes $c_0 := E(pk, m_0)$ and outputs 0 if $c = c_0$. Otherwise, it outputs 1.

Because E is deterministic, we know that $c = c_0$ whenever $b = 0$. Therefore, when $b = 0$ the adversary always outputs 0. Similarly, when $b = 1$ it always outputs 1. Therefore

$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] = 1$$

showing that $\mathcal{E}$ is insecure.

This generic attack explains why semantically secure public-key encryption schemes must be randomized. All the schemes we construct in this chapter and the next use randomized encryption. This is quite different from the symmetric key settings where a deterministic encryption scheme can be semantically secure; for example, the one-time pad.

11.3.2 Semantic security against chosen plaintext attack

Recall that when discussing symmetric ciphers, we introduced two distinct notions of security: semantic security, and semantic security against chosen plaintext attack (or CPA security, for short). We showed that for symmetric ciphers, semantic security *does not* imply CPA security. However, for public-key encryption schemes, semantic security *does* imply CPA security. Intuitively, this is because in the public-key setting, the adversary can encrypt any message he likes, without knowledge of any secret key material. The adversary does so using the given public key and never needs to issue encryption queries to the challenger. In contrast, in the symmetric key setting, the adversary cannot encrypt messages on his own.

The attack game defining CPA security in the public-key setting is the natural analog of the corresponding game in the symmetric setting (see Attack Game 5.2 in Section 5.3):

Attack Game 11.2 (CPA security). For a given public-key encryption scheme $\mathcal{E} = (G, E, D)$, defined over $(\mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, we define two experiments.

Experiment b $\quad(b = 0, 1)$:

- The challenger computes $(pk, sk) \xleftarrow{\text{R}} G()$, and sends pk to the adversary.
- The adversary submits a sequence of queries to the challenger.

  For $i = 1, 2, \ldots$, the ith query is a pair of messages, $m_{i0}, m_{i1} \in \mathcal{M}$, of the same length.

  The challenger computes $c_i \xleftarrow{\text{R}} E(pk, m_{ib})$, and sends c_i to the adversary.
- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

If W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b, then we define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \Box$$

Definition 11.4 (CPA security). *A public-key encryption scheme $\mathcal{E}$ is called* ***semantically secure against chosen plaintext attack****, or simply* ***CPA secure****, if for all efficient adversaries $\mathcal{A}$, the value* $\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

Theorem 11.1. *If a public-key encryption scheme $\mathcal{E}$ is semantically secure, then it is also CPA secure.*

In particular, for every CPA adversary $\mathcal{A}$ that plays Attack Game 11.2 with respect to $\mathcal{E}$, and which makes at most Q queries to its challenger, there exists an SS adversary $\mathcal{B}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = Q \cdot \text{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}].$$

Proof. The proof is a straightforward hybrid argument, and is very similar to the proof of Theorem 5.1. Suppose $\mathcal{E} = (G, E, D)$ is defined over $(\mathcal{M}, \mathcal{C})$. Let $\mathcal{A}$ be a CPA adversary that plays Attack Game 11.2 with respect to $\mathcal{E}$, and which makes at most Q queries to its challenger.

We describe the relevant hybrid games. For $j = 0, \ldots, Q$, Hybrid j is played between $\mathcal{A}$ and a challenger who works as follows:

> $(pk, sk) \xleftarrow{R} G()$
> Send pk to $\mathcal{A}$
> Upon receiving the ith query $(m_{i0}, m_{i1}) \in \mathcal{M}^2$ from $\mathcal{A}$ do:
>   if $i > j$
>   then $c_i \xleftarrow{R} E(pk, m_{i0})$
>   else $c_i \xleftarrow{R} E(pk, m_{i1})$
>   send c_i to $\mathcal{A}$.

Put another way, the challenger in Hybrid j encrypts

$$m_{11}, \ldots, m_{j1}, \quad m_{(j+1)0}, \ldots, m_{Q0},$$

As usual, we define p_j to be the probability that $\mathcal{A}$ outputs 1 in Hybrid j. Clearly,

$$\text{CPA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = |p_Q - p_0|.$$

Next, we define an appropriate adversary $\mathcal{B}$ that plays Attack Game 11.1 with respect to $\mathcal{E}$:

> First, $\mathcal{B}$ chooses $\omega \in \{1, \ldots, Q\}$ at random.
>
> Then, $\mathcal{B}$ plays the role of challenger to $\mathcal{A}$: it obtains a public key pk from its own challenger, and forwards this to $\mathcal{A}$; when $\mathcal{A}$ makes a query (m_{i0}, m_{i1}), $\mathcal{B}$ computes its response c_i as follows:
>
>   if $i > \omega$ then
> $c \xleftarrow{R} E(pk, m_{i0})$
>   else if $i = \omega$ then
> $\mathcal{B}$ submits (m_{i0}, m_{i1}) to its own challenger
> c_i is set to the challenger's response
>   else // $i < \omega$
> $c_i \xleftarrow{R} E(pk, m_{i1})$.
>
> Finally, $\mathcal{B}$ outputs whatever $\mathcal{A}$ outputs.

The crucial difference between the proof of this theorem and that of Theorem 5.1 is that for $i \neq \omega$, adversary $\mathcal{B}$ can encrypt the relevant message using the public key.

For $b = 0, 1$, let W_b be the event that $\mathcal{B}$ outputs 1 in Experiment 0 of its attack game. It is clear that for $j = 1, \ldots, Q$,

$$\Pr[W_0 \mid \omega = j] = p_{j-1} \quad \text{and} \quad \Pr[W_1 \mid \omega = j] = p_j,$$

and the theorem follows by the usual telescoping sum calculation. □

One can also consider multi-key CPA security, where the adversary sees many encryptions under many public keys. In the public-key setting, semantic security implies not only CPA security, but multi-key CPA security — see Exercise 11.9.

11.4 Encryption based on a trapdoor function scheme

In this section, we show how to use a trapdoor function scheme (see Section 10.2) to build a semantically secure public-key encryption scheme. In fact, this scheme makes use of a hash function, and our proof of security works only when we model the hash function as a random oracle (see Section 8.10.2). We then present a concrete instantiation of this scheme, based on RSA (see Section 10.3).

Our encryption scheme is called $\mathcal{E}_{\mathrm{TDF}}$, and is built out of several components:

- a trapdoor function scheme $\mathcal{T} = (G, F, I)$, defined over $(\mathcal{X}, \mathcal{Y})$,
- a symmetric cipher $\mathcal{E}_{\mathrm{s}} = (E_{\mathrm{s}}, D_{\mathrm{s}})$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$,
- a hash function $H : \mathcal{X} \to \mathcal{K}$.

The message space for $\mathcal{E}_{\mathrm{TDF}}$ is $\mathcal{M}$, and the ciphertext space is $\mathcal{Y} \times \mathcal{C}$. We now describe the key generation, encryption, and decryption algorithms for $\mathcal{E}_{\mathrm{TDF}}$.

- The key generation algorithm for $\mathcal{E}_{\mathrm{TDF}}$ is the key generation algorithm for $\mathcal{T}$.
- For a given public key pk, and a given message $m \in \mathcal{M}$, the encryption algorithm runs as follows:

$$\begin{aligned} E(pk, m) := \quad & x \xleftarrow{\text{R}} \mathcal{X}, \quad y \leftarrow F(pk, x), \quad k \leftarrow H(x), \quad c \xleftarrow{\text{R}} E_{\mathrm{s}}(k, m) \\ & \text{output } (y, c). \end{aligned}$$

- For a given secret key sk, and a given ciphertext $(y, c) \in \mathcal{Y} \times \mathcal{C}$, the decryption algorithm runs as follows:

$$\begin{aligned} D(sk,\ (y, c)\ ) := \quad & x \leftarrow I(sk, y), \quad k \leftarrow H(x), \quad m \leftarrow D_{\mathrm{s}}(k, c) \\ & \text{output } m. \end{aligned}$$

Thus, $\mathcal{E}_{\mathrm{TDF}} = (G, E, D)$, and is defined over $(\mathcal{M}, \mathcal{Y} \times \mathcal{C})$.

The correctness property for $\mathcal{T}$ immediately implies the correctness property for $\mathcal{E}_{\mathrm{TDF}}$. If H is modeled as a random oracle (see Section 8.10), one can prove that $\mathcal{E}_{\mathrm{TDF}}$ is semantically secure, assuming that $\mathcal{T}$ is one-way, and that $\mathcal{E}_{\mathrm{s}}$ is semantically secure.

Recall that in the random oracle model, the function H is modeled as a random function $\mathcal{O}$ chosen at random from the set of all functions $\mathrm{Funs}[\mathcal{X}, \mathcal{K}]$. More precisely, in the random oracle version of Attack Game 11.1, the challenger chooses $\mathcal{O}$ at random. In any computation where the challenger would normally evaluate H, it evaluates $\mathcal{O}$ instead. In addition, the adversary is allowed to ask the challenger for the value of the function $\mathcal{O}$ at any point of its choosing. The adversary may make any number of such "random oracle queries" at any time of its choosing. We use $\mathrm{SS^{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\mathrm{TDF}}]$ to denote $\mathcal{A}$'s advantage against $\mathcal{E}_{\mathrm{TDF}}$ in the random oracle version of Attack Game 11.1.

Theorem 11.2. *Assume $H : \mathcal{X} \to \mathcal{K}$ is modeled as a random oracle. If $\mathcal{T}$ is one-way and $\mathcal{E}_{\mathrm{s}}$ is semantically secure, then $\mathcal{E}_{\mathrm{TDF}}$ is semantically secure.*

In particular, for every SS adversary $\mathcal{A}$ that attacks $\mathcal{E}_{\mathrm{TDF}}$ as in the random oracle version of Attack Game 11.1, there exist an inverting adversary $\mathcal{B}_{\mathrm{ow}}$ that attacks $\mathcal{T}$ as in Attack Game 10.2, *and an SS adversary $\mathcal{B}_{\rm s}$ that attacks $\mathcal{E}_{\rm s}$ as in Attack Game 2.1, where $\mathcal{B}_{\rm ow}$ and $\mathcal{B}_{\rm s}$ are elementary wrappers around $\mathcal{A}$, such that*

$$\mathrm{SS^{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\rm TDF}] \leq 2 \cdot \mathrm{OW}\mathsf{adv}[\mathcal{B}_{\rm ow}, \mathcal{T}] + \mathrm{SS}\mathsf{adv}[\mathcal{B}_{\rm s}, \mathcal{E}_{\rm s}]. \tag{11.2}$$

Proof idea. Suppose the adversary sees the ciphertext (y, c), where $y = F(pk, x)$. If H is modeled as a random oracle, then intuitively, the only way the adversary can learn anything at all about the symmetric key k used to generate c is to explicitly evaluate the random oracle representing H at the point x; however, if he could so this, we could easily convert the adversary into an adversary that inverts the function $F(pk, \cdot)$, contradicting the one-wayness assumption. Therefore, from the adversary's point of view, k is completely random, and semantic security for $\mathcal{E}_{\rm TDF}$ follows directly from the semantic security of $\mathcal{E}_{\rm s}$. In the detailed proof, we implement the random oracle using the same "faithful gnome" technique as was used to efficiently implement random functions (see Section 4.4.2); that is, we represent the random oracle as a table of input/output pairs corresponding to points at which the adversary actually queried the random oracle (as well as the point at which the challenger queries the random oracle when it runs the encryption algorithm). We also use many of the same proof techniques introduced in Chapter 4, specifically, the "forgetful gnome" technique (introduced in the proof of Theorem 4.6) and the Difference Lemma (Theorem 4.7). □

Proof. It is convenient to prove the theorem using the bit-guessing versions of the semantic security game. We prove:

$$\mathrm{SS^{ro}}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}_{\rm TDF}] \leq \mathrm{OW}\mathsf{adv}[\mathcal{B}_{\rm ow}, \mathcal{T}] + \mathrm{SS}\mathsf{adv}^*[\mathcal{B}_{\rm s}, \mathcal{E}_{\rm s}]. \tag{11.3}$$

Then (11.2) follows by (11.1) and (2.12).

Define Game 0 to be the game played between $\mathcal{A}$ and the challenger in the bit-guessing version of Attack Game 11.1 with respect to $\mathcal{E}_{\rm TDF}$. We then modify the challenger to obtain Game 1. In each game, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. Also, for $j = 0, 1$, we define W_j to be the event that $\hat{b} = b$ in Game j. We will show that $|\Pr[W_1] - \Pr[W_0]|$ is negligible, and that $\Pr[W_1]$ is negligibly close to 1/2. From this, it follows that

$$\mathrm{SS^{ro}}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}_{\rm TDF}] = |\Pr[W_0] - 1/2| \tag{11.4}$$

is also negligible.

Game 0. Note that the challenger in Game 0 also has to respond to the adversary's random oracle queries. The adversary can make any number of random oracle queries, but at most one encryption query. Recall that in addition to direct access the random oracle via explicit random oracle queries, the adversary also has indirect access to the random oracle via the encryption query, where the challenger also makes use of the random oracle. In describing this game, we directly implement the random oracle as a "faithful gnome." This is done using an associative array $Map : \mathcal{X} \to \mathcal{K}$. The details are in Fig. 11.2. In the initialization step, the challenger prepares some quantities that will be used later in processing the encryption query. In particular, in addition to computing $(pk, sk) \xleftarrow{\rm R} G()$, the challenger precomputes $x \xleftarrow{\rm R} \mathcal{X}$, $y \leftarrow F(pk, x)$, $k \xleftarrow{\rm R} \mathcal{K}$. It also sets $Map[x] \leftarrow k$, which means that the value of the random oracle at x is equal to k.

Game 1. This game is precisely the same as Game 0, except that we make our gnome "forgetful" by deleting line (3) in Fig. 11.2.

Let Z be the event that the adversary queries the random oracle at the point x in Game 1. Clearly, Games 0 and 1 proceed identically unless Z occurs, and so by the Difference Lemma, we

$$
\begin{array}{ll}
 & \text{initialization:} \\
(1) & \quad (pk, sk) \xleftarrow{R} G(),\ x \xleftarrow{R} \mathcal{X},\ y \leftarrow F(pk, x) \\
 & \quad \text{initialize an empty associative array } Map : \mathcal{X} \to \mathcal{K} \\
(2) & \quad k \xleftarrow{R} \mathcal{K},\ b \xleftarrow{R} \{0,1\} \\
(3) & \quad Map[x] \leftarrow k \\
 & \quad \text{send the public key } pk \text{ to } \mathcal{A}; \\
 & \text{upon receiving an encryption query } (m_0, m_1) \in \mathcal{M}^2\text{:} \\
(4) & \quad c \leftarrow E_{\mathrm{s}}(k, m_b) \\
 & \quad \text{send } (y, c) \text{ to } \mathcal{A}; \\
 & \text{upon receiving a random oracle query } \hat{x} \in \mathcal{X}\text{:} \\
 & \quad \text{if } \hat{x} \notin \text{Domain}(Map) \text{ then } Map[\hat{x}] \xleftarrow{R} \mathcal{K} \\
 & \quad \text{send } Map[\hat{x}] \text{ to } \mathcal{A}
\end{array}
$$

Figure 11.2: Game 0 challenger

have

$$|\Pr[W_1] - \Pr[W_0]| \le \Pr[Z]. \tag{11.5}$$

If event Z happens, then one of the adversary's random oracle queries is the inverse of y under $F(pk, \cdot)$. Moreover, in Game 1, the value x is used only to define $y = F(pk, x)$, and nowhere else. Thus, we can use adversary $\mathcal{A}$ to build an efficient adversary $\mathcal{B}_{\mathrm{ow}}$ that breaks the one-wayness assumption for $\mathcal{T}$ with an advantage equal to $\Pr[Z]$.

Here is how adversary $\mathcal{B}_{\mathrm{ow}}$ works in detail. This adversary plays Attack Game 10.2 against a challenger $\mathbf{C}_{\mathrm{ow}}$, and plays the role of challenger to $\mathcal{A}$ as in Fig. 11.2, except with the following lines modified as indicated:

$$
\begin{array}{ll}
(1) & \quad \text{obtain } (pk, y) \text{ from } \mathbf{C}_{\mathrm{ow}} \\
(3) & \quad \textit{(deleted)}
\end{array}
$$

Additionally,

$$
\begin{array}{l}
\text{when } \mathcal{A} \text{ terminates:} \\
\quad \text{if } F(pk, \hat{x}) = y \text{ for some } \hat{x} \in \text{Domain}(Map) \\
\quad\quad \text{then output } \hat{x} \\
\quad\quad \text{else output "failure".}
\end{array}
$$

To analyze $\mathcal{B}_{\mathrm{ow}}$, we may naturally view Game 1 and the game played between $\mathcal{B}_{\mathrm{ow}}$ and $\mathbf{C}_{\mathrm{ow}}$ as operating on the same underlying probability space. By definition, Z occurs if and only if $x \in \text{Domain}(Map)$ when $\mathcal{B}_{\mathrm{ow}}$ finishes its game. Therefore,

$$\Pr[Z] = \text{OW}\mathsf{adv}[\mathcal{B}_{\mathrm{ow}}, \mathcal{T}]. \tag{11.6}$$

Observe that in Game 1, the key k is only used to encrypt the challenge plaintext. As such, the adversary is essentially attacking $\mathcal{E}_{\mathrm{s}}$ as in the bit-guessing version of Attack Game 2.1 at this point. More precisely, we derive an efficient SS adversary $\mathcal{B}_s$ based on Game 1 that uses $\mathcal{A}$ as a subroutine, such that

$$|\Pr[W_1] - 1/2| = \mathsf{SSadv}^*[\mathcal{B}_s, \mathcal{E}_s]. \tag{11.7}$$

Adversary $\mathcal{B}_s$ plays the bit-guessing version of Attack Game 2.1 against a challenger $\mathbf{C}_s$, and plays the role of challenger to $\mathcal{A}$ as in Fig. 11.2, except with the following lines modified as indicated:

(2) *(deleted)*
(3) *(deleted)*
(4) forward (m_0, m_1) to $\mathbf{C}_s$, obtaining c

Additionally,

when $\mathcal{A}$ outputs $\hat{b}$:
  output $\hat{b}$

To analyze $\mathcal{B}_s$, we may naturally view Game 1 and the game played between $\mathcal{B}_s$ and $\mathbf{C}_s$ as operating on the same underlying probability space. By construction, $\mathcal{B}_s$ and $\mathcal{A}$ output the same thing, and so (11.7) holds.

Combining (11.4), (11.5), (11.6), and (11.7), yields (11.3). $\square$

11.4.1 Instantiating $\mathcal{E}_{\mathrm{TDF}}$ with RSA

Suppose we now use RSA (see Section 10.3) to instantiate $\mathcal{T}$ in the above encryption scheme $\mathcal{E}_{\mathrm{TDF}}$. This scheme is parameterized by two quantities: the length ℓ of the prime factors of the RSA modulus, and the encryption exponent e, which is an odd, positive integer. Recall that the RSA scheme does not quite fit the definition of a trapdoor permutation scheme, because the domain of the trapdoor permutation is not a fixed set, but varies with the public key. Let us assume that $\mathcal{X}$ is a fixed set into which we may embed $\mathbb{Z}_n$, for every RSA modulus n generated by $\mathrm{RSAGen}(\ell, e)$ (for example, we could take $\mathcal{X} = \{0,1\}^{2\ell}$). The scheme also makes use of a symmetric cipher $\mathcal{E}_s = (E_s, D_s)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, as well as a hash function $H : \mathcal{X} \to \mathcal{K}$.

The basic RSA encryption scheme is $\mathcal{E}_{\mathrm{RSA}} = (G, E, D)$, with message space $\mathcal{M}$ and ciphertext space $\mathcal{X} \times \mathcal{C}$, where

- the key generation algorithm runs as follows:

$$\begin{aligned} G() := \quad & (n,d) \xleftarrow{\text{R}} \mathrm{RSAGen}(\ell, e), \quad pk \leftarrow (n,e), \quad sk \leftarrow (n,d) \\ & \text{output } (pk, sk); \end{aligned}$$

- for a given public key $pk = (n, e)$, and message $m \in \mathcal{M}$, the encryption algorithm runs as follows:

$$\begin{aligned} E(pk, m) := \quad & x \xleftarrow{\text{R}} \mathbb{Z}_n, \quad y \leftarrow x^e, \quad k \leftarrow H(x), \quad c \xleftarrow{\text{R}} E_s(k, m) \\ & \text{output } (y, c) \in \mathcal{X} \times \mathcal{C}; \end{aligned}$$

- for a given secret key $sk = (n, d)$, and a given ciphertext $(y, c) \in \mathcal{X} \times \mathcal{C}$, where y represents an element of $\mathbb{Z}_n$, the decryption algorithm runs as follows:

$$\begin{aligned} D(sk,\ (y,c)\ ) := \quad & x \leftarrow y^d, \quad k \leftarrow H(x), \quad m \leftarrow D_s(k, c) \\ & \text{output } m. \end{aligned}$$

Theorem 11.3. *Assume $H : \mathcal{X} \to \mathcal{K}$ is modeled as a random oracle. If the RSA assumption holds for parameters (ℓ, e), and $\mathcal{E}_s$ is semantically secure, then $\mathcal{E}_{RSA}$ is semantically secure.*

> *In particular, for any SS adversary $\mathcal{A}$ that attacks $\mathcal{E}_{RSA}$ as in the random oracle version of Attack Game 11.1, there exist an RSA adversary $\mathcal{B}_{rsa}$ that breaks the RSA assumption for (ℓ, e) as in Attack Game 10.3, and an SS adversary $\mathcal{B}_s$ that attacks $\mathcal{E}_s$ as in Attack Game 2.1, where $\mathcal{B}_{rsa}$ and $\mathcal{B}_s$ are elementary wrappers around $\mathcal{A}$, such that*
>
> $$\mathrm{SS}^{\mathrm{ro}}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}_{\mathrm{RSA}}] \le \mathrm{RSA}\mathsf{adv}[\mathcal{B}_{\mathrm{rsa}}, \ell, e] + \mathrm{SS}\mathsf{adv}^*[\mathcal{B}_s, \mathcal{E}_s].$$

Proof. The proof of Theorem 11.2 carries over, essentially unchanged. □

11.5 ElGamal encryption

In this section we show how to build a public-key encryption scheme from Diffie-Hellman. Security will be based on either the CDH or DDH assumptions from Section 10.5.

The encryption scheme is a variant of a scheme first proposed by ElGamal, and we call it $\mathcal{E}_{EG}$. It is built out of several components:

- a cyclic group $\mathbb{G}$ of prime order q with generator $g \in \mathbb{G}$,
- a symmetric cipher $\mathcal{E}_s = (E_s, D_s)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$,
- a hash function $H : \mathbb{G} \to \mathcal{K}$.

The message space for $\mathcal{E}_{EG}$ is $\mathcal{M}$, and the ciphertext space is $\mathbb{G} \times \mathcal{C}$. We now describe the key generation, encryption, and decryption algorithms for $\mathcal{E}_{EG}$.

- the key generation algorithm runs as follows:

$$\begin{aligned} G() := \quad & \alpha \xleftarrow{R} \mathbb{Z}_q, \quad u \leftarrow g^\alpha \\ & pk \leftarrow u, \quad sk \leftarrow \alpha \\ & \text{output } (pk, sk); \end{aligned}$$

- for a given public key $pk = u \in \mathbb{G}$ and message $m \in \mathcal{M}$, the encryption algorithm runs as follows:

$$\begin{aligned} E(pk, m) := \quad & \beta \xleftarrow{R} \mathbb{Z}_q, \quad v \leftarrow g^\beta, \quad w \leftarrow u^\beta, \quad k \leftarrow H(w), \quad c \leftarrow E_s(k, m) \\ & \text{output } (v, c); \end{aligned}$$

- for a given secret key $sk = \alpha \in \mathbb{Z}_q$ and a ciphertext $(v, c) \in \mathbb{G} \times \mathcal{C}$, the decryption algorithm runs as follows:

$$\begin{aligned} D(sk,\ (v, c)\ ) := \quad & w \leftarrow v^\alpha, \quad k \leftarrow H(w), \quad m \leftarrow D_s(k, c) \\ & \text{output } m. \end{aligned}$$

Thus, $\mathcal{E}_{EG} = (G, E, D)$, and is defined over $(\mathcal{M}, \mathbb{G} \times \mathcal{C})$.

Note that the description of the group $\mathbb{G}$ and generator $g \in \mathbb{G}$ is considered to be a system parameter, rather than part of the public key.

11.5.1 Semantic security of ElGamal in the random oracle model

We shall analyze the security of $\mathcal{E}_{\text{EG}}$ under two different sets of assumptions. In this section we do the analysis modeling $H : \mathbb{G} \to \mathcal{K}$ as a random oracle, under the CDH assumption for $\mathbb{G}$, and the assumption that $\mathcal{E}_{\text{s}}$ is semantically secure. In the next section we analyze $\mathcal{E}_{\text{EG}}$ without the random oracle model, but using the stronger DDH assumption for $\mathbb{G}$.

Theorem 11.4. *Assume $H : \mathbb{G} \to \mathcal{K}$ is modeled as a random oracle. If the CDH assumption holds for $\mathbb{G}$, and $\mathcal{E}_{\text{s}}$ is semantically secure, then $\mathcal{E}_{\text{EG}}$ is semantically secure.*

In particular, for every SS adversary $\mathcal{A}$ that plays the random oracle version of Attack Game 11.1 with respect to $\mathcal{E}_{\text{EG}}$, and makes at most Q queries to the random oracle, there exist a CDH adversary $\mathcal{B}_{\text{cdh}}$ that plays Attack Game 10.5 with respect to $\mathbb{G}$, and an SS adversary $\mathcal{B}_{\text{s}}$ that plays Attack Game 2.1 with respect to $\mathcal{E}_{\text{s}}$, where $\mathcal{B}_{\text{cdh}}$ and $\mathcal{B}_{\text{s}}$ are elementary wrappers around $\mathcal{A}$, such that

$$\text{SS}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\text{EG}}] \le 2Q \cdot \text{CDH}\mathsf{adv}[\mathcal{B}_{\text{cdh}}, \mathbb{G}] + \text{SS}\mathsf{adv}[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}]. \tag{11.8}$$

Proof idea. Suppose the adversary sees the ciphertext (v, c), where $v = g^{\beta}$. If H is modeled as a random oracle, then intuitively, the only way the adversary can learn anything at all about the symmetric key k used to generate c is to explicitly evaluate the random oracle representing H at the point $w = v^{\alpha}$; however, if he could so this, we could convert the adversary into an adversary that breaks the CDH assumption for $\mathbb{G}$. One wrinkle is that we cannot recognize the correct solution to the CDH problem when we see it (if the DDH assumption is true), so we simply guess by choosing at random from among all of the adversary's random oracle queries. This is where the factor of Q in (11.8) comes from. So unless the adversary can break the CDH assumption, from the adversary's point of view, k is completely random, and semantic security for $\mathcal{E}_{\text{EG}}$ follows directly from the semantic security of $\mathcal{E}_{\text{s}}$. □

Proof. It is convenient to prove the theorem using the bit-guessing version of the semantic security game. We prove:

$$\text{SS}^{\text{ro}}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}_{\text{EG}}] \le Q \cdot \text{CDH}\mathsf{adv}[\mathcal{B}_{\text{cdh}}, \mathbb{G}] + \text{SS}\mathsf{adv}^*[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}]. \tag{11.9}$$

Then (11.8) follows from (11.1) and (2.12).

We define Game 0 to be the game played between $\mathcal{A}$ and the challenger in the bit-guessing version of Attack Game 11.1 with respect to $\mathcal{E}_{\text{EG}}$. We then modify the challenger to obtain Game 1. In each game, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. Also, for $j = 0, 1$, we define W_j to be the event that $\hat{b} = b$ in Game j. We will show that $|\Pr[W_1] - \Pr[W_0]|$ is negligible, and that $\Pr[W_1]$ is negligibly close to 1/2. From this, it follows that

$$\text{SS}^{\text{ro}}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}_{\text{EG}}] = |\Pr[W_0] - 1/2| \tag{11.10}$$

is negligible.

Game 0. The adversary can make any number of random oracle queries, but at most one encryption query. Again, recall that in addition to direct access the random oracle via explicit random oracle queries, the adversary also has indirect access to the random oracle via the encryption query, where the challenger also makes use of the random oracle. The random oracle is implemented using an associative array $Map : \mathbb{G} \to \mathcal{K}$. The details are in Fig. 11.3. At line (3), we effectively set the random oracle at the point w to k.

	initialization:
(1)	$\alpha, \beta \xleftarrow{R} \mathbb{Z}_q$, $u \leftarrow g^\alpha$, $v \leftarrow g^\beta$, $w \leftarrow g^{\alpha\beta}$
	initialize an empty associative array $Map : \mathbb{G} \to \mathcal{K}$
(2)	$k \xleftarrow{R} \mathcal{K}$, $b \xleftarrow{R} \{0,1\}$
(3)	$Map[w] \leftarrow k$
	send the public key u to $\mathcal{A}$;
	upon receiving an encryption query $(m_0, m_1) \in \mathcal{M}^2$:
(4)	$c \leftarrow E_s(k, m_b)$
	send (v, c) to $\mathcal{A}$;
	upon receiving a random oracle query $\hat{w} \in \mathbb{G}$:
	if $\hat{w} \notin \text{Domain}(Map)$ then $Map[\hat{w}] \xleftarrow{R} \mathcal{K}$
	send $Map[\hat{w}]$ to $\mathcal{A}$

Figure 11.3: Game 0 challenger

Game 1. This is the same as Game 0, except we delete line (3) in Fig. 11.3.

Let Z be the event that the adversary queries the random oracle at w in Game 1. Clearly, Games 0 and 1 proceed identically unless Z occurs, and so by the Difference Lemma, we have

$$|\Pr[W_1] - \Pr[W_0]| \leq \Pr[Z]. \tag{11.11}$$

If event Z happens, then one of the adversary's random oracle queries is the solution w to the instance (u, v) of the CDH problem. Moreover, in Game 1, the values α and β are only needed to compute u and v, and nowhere else. Thus, we can use adversary $\mathcal{A}$ to build an adversary $\mathcal{B}_{\text{cdh}}$ to break the CDH assumption: we simply choose one of the adversary's random oracle queries at random, and output it — with probability at least $\Pr[Z]/Q$, this will be the solution to the given instance of the CDH problem.

In more detail, adversary $\mathcal{B}_{\text{cdh}}$ plays Attack Game 10.5 against a challenger $\mathbf{C}_{\text{cdh}}$, and plays the role of challenger to $\mathcal{A}$ as in Fig. 11.3, except with the following lines modified as indicated:

(1)	obtain (u, v) from $\mathbf{C}_{\text{cdh}}$
(3)	*(deleted)*

Additionally,

when $\mathcal{A}$ terminates:
  if $\text{Domain}(Map) \neq \emptyset$
    then $\hat{w} \xleftarrow{R} \text{Domain}(Map)$, output $\hat{w}$
    else output "failure"

To analyze $\mathcal{B}_{\text{cdh}}$, we may naturally view Game 1 and the game played between $\mathcal{B}_{\text{cdh}}$ and $\mathbf{C}_{\text{cdh}}$ as operating on the same underlying probability space. By definition, Z occurs if and only if $w \in \text{Domain}(Map)$ when $\mathcal{B}_{\text{cdh}}$ finishes its game. Moreover, since $|\text{Domain}(Map)| \leq Q$, it follows that

$$\text{CDH}\mathsf{adv}[\mathcal{B}_{\text{cdh}}, \mathbb{G}] \geq \Pr[Z]/Q. \tag{11.12}$$

Observe that in Game 1, the key k is only used to encrypt the challenge plaintext. We leave it to the reader to describe an efficient SS adversary $\mathcal{B}_s$ that uses $\mathcal{A}$ as a subroutine, such that

$$|\Pr[W_1] - 1/2| = \mathrm{SS}\mathsf{adv}^*[\mathcal{B}_s, \mathcal{E}_s]. \tag{11.13}$$

Combining (11.10), (11.11), (11.12), and (11.13), yields (11.9), which completes the proof of the theorem. □

11.5.2 Semantic security of ElGamal without random oracles

As we commented in Section 8.10.2, security results in the random oracle model do not necessarily imply security in the real world. When it does not hurt efficiency, it is better to avoid the random oracle model. By replacing the CDH assumption by the stronger, but still reasonable, DDH assumption, and by making an appropriate, but reasonable, assumption about H, we can prove that the same system $\mathcal{E}_{\mathrm{EG}}$ is semantically secure without resorting to the random oracle model.

We thus obtain two security analyses of $\mathcal{E}_{\mathrm{EG}}$: one in the random oracle model, but using the CDH assumption. The other, without the random oracle model, but using the stronger DDH assumption. We are thus using the random oracle model as a hedge: in case the DDH assumption turns out to be false in the group $\mathbb{G}$, the scheme remains secure assuming CDH holds in $\mathbb{G}$, but in a weaker random oracle semantic security model. In Exercise 11.13 we develop yet another analysis of ElGamal without random oracles, but using a weaker assumption than DDH called **hash Diffie-Hellman** (HDH) which more accurately captures the exact requirement to needed to prove security.

To carry out the analysis using the DDH assumption in $\mathbb{G}$ we make a specific assumption about the hash function $H : \mathbb{G} \to \mathcal{K}$, namely that H is a **secure key derivation function**, or **KDF** for short. We already introduced a very general notion of a key derivation function in Section 8.10. What we describe here is more focused and tailored precisely to our current situation.

Intuitively, $H : \mathbb{G} \to \mathcal{K}$ is a secure KDF if no efficient adversary can effectively distinguish between $H(w)$ and k, where w is randomly chosen from $\mathbb{G}$, and k is randomly chosen from $\mathcal{K}$. To be somewhat more general, we consider an arbitrary, efficiently computable hash function $F : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary, finite sets.

Attack Game 11.3 (secure key derivation). For a given hash function $F : \mathcal{X} \to \mathcal{Y}$, and for a given adversary $\mathcal{A}$, we define two experiments.

Experiment b $\quad$ ($b = 0, 1$):

- The challenger computes

$$x \xleftarrow{\mathrm{R}} \mathcal{X}, \quad y_0 \leftarrow F(x), \quad y_1 \xleftarrow{\mathrm{R}} \mathcal{Y},$$

  and sends y_b to the adversary.

- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

If W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b, then we define $\mathcal{A}$'s **advantage** with respect to F as

$$\mathrm{KDF}\mathsf{adv}[\mathcal{A}, F] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \square$$

Definition 11.5 (secure key derivation). *A hash function $F : \mathcal{X} \to \mathcal{Y}$ is a **secure KDF** if for every efficient adversary $\mathcal{A}$, the value* $\mathrm{KDF}\mathsf{adv}[\mathcal{A}, F]$ *is negligible.*

It is plausible to conjecture that an "off the shelf" hash function, like SHA256 or HKDF (see Section 8.10.5), is a secure KDF. In fact, one may justify this assumption modeling the hash function as a random oracle; however, using this explicit computational assumption, rather than the random oracle model, yields more meaningful results.

One may even build a secure KDF without making any assumptions at all: the construction in Section 8.10.4 based on a universal hash function and the leftover hash lemma yields an unconditionally secure KDF. Even though this construction is theoretically attractive and quite efficient, it may not be a wise choice from a security point of view: as already discussed above, if the DDH turns out to be false, we can still rely on the CDH in the random oracle model, but for that, it is better to use something based on SHA256 or HKDF, which can more plausibly be modeled as a random oracle.

Theorem 11.5. *If the DDH assumption holds for $\mathbb{G}$, $H : \mathbb{G} \to \mathcal{K}$ is a secure KDF, and $\mathcal{E}_{\mathrm{s}}$ is semantically secure, then $\mathcal{E}_{\mathrm{EG}}$ is semantically secure.*

In particular, for every SS adversary $\mathcal{A}$ that plays Attack Game 11.1 with respect to $\mathcal{E}_{\mathrm{EG}}$, there exist a DDH adversary $\mathcal{B}_{\mathrm{ddh}}$ that plays Attack Game 10.6 with respect to $\mathbb{G}$, a KDF adversary $\mathcal{B}_{\mathrm{kdf}}$ that plays Attack Game 11.3 with respect to H, and an SS adversary $\mathcal{B}_{\mathrm{s}}$ that plays Attack Game 2.1 with respect to $\mathcal{E}_{\mathrm{s}}$, where $\mathcal{B}_{\mathrm{ddh}}$, $\mathcal{B}_{\mathrm{kdf}}$, and $\mathcal{B}_{\mathrm{s}}$ are elementary wrappers around $\mathcal{A}$, such that

$$\mathrm{SS}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\mathrm{EG}}] \le 2 \cdot \mathrm{DDH}\mathsf{adv}[\mathcal{B}_{\mathrm{ddh}}, \mathbb{G}] + 2 \cdot \mathrm{KDF}\mathsf{adv}[\mathcal{B}_{\mathrm{kdf}}, H] + \mathrm{SS}\mathsf{adv}[\mathcal{B}_{\mathrm{s}}, \mathcal{E}_{\mathrm{s}}]. \tag{11.14}$$

Proof idea. Suppose the adversary sees the ciphertext (v, c), where $v = g^{\beta}$ and c is a symmetric encryption created using the key $k := H(u^{\beta})$. Suppose the challenger replaces $w = u^{\beta}$ by a random independent group element $\tilde{w} \in \mathbb{G}$ and constructs k as $k := H(\tilde{w})$. By the DDH assumption the adversary cannot tell the difference between u^{β} and $\tilde{w}$ and hence its advantage is only negligibly changed. Under the KDF assumption, $k := H(\tilde{w})$ looks like a random key in $\mathcal{K}$, independent of the adversary's view, and therefore security follows by semantic security of $\mathcal{E}_{\mathrm{s}}$. $\Box$

Proof. More precisely, it is convenient to prove the theorem using the bit-guessing version of the semantic security game. We prove:

$$\mathrm{SS}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}_{\mathrm{EG}}] \le \mathrm{DDH}\mathsf{adv}[\mathcal{B}_{\mathrm{ddh}}, \mathbb{G}] + \mathrm{KDF}\mathsf{adv}[\mathcal{B}_{\mathrm{kdf}}, H] + \mathrm{SS}\mathsf{adv}^*[\mathcal{B}_{\mathrm{s}}, \mathcal{E}_{\mathrm{s}}]. \tag{11.15}$$

Then (11.14) follows by (11.1) and (2.12).

Define Game 0 to be the game played between $\mathcal{A}$ and the challenger in the bit-guessing version of Attack Game 11.1 with respect to $\mathcal{E}_{\mathrm{EG}}$. We then modify the challenger to obtain Games 1 and 2. In each game, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. Also, for $j = 0, 1, 2$, we define W_j to be the event that $\hat{b} = b$ in Game j. We will show that $|\Pr[W_2] - \Pr[W_0]|$ is negligible, and that $\Pr[W_2]$ is negligibly close to $1/2$. From this, it follows that

$$\mathrm{SS}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}_{\mathrm{EG}}] = |\Pr[W_0] - 1/2| \tag{11.16}$$

is negligible.

Game 0. The logic of the challenger in this game is presented in Fig. 11.4.

initialization:
(1) $\quad\alpha, \beta \xleftarrow{\text{R}} \mathbb{Z}_q,\ \gamma \leftarrow \alpha\beta,\ u \leftarrow g^{\alpha},\ v \leftarrow g^{\beta},\ w \leftarrow g^{\gamma}$
(2) $\quad k \leftarrow H(w)$
$\quad b \xleftarrow{\text{R}} \{0,1\}$
$\quad$ send the public key u to $\mathcal{A}$;

upon receiving $(m_0, m_1) \in \mathcal{M}^2$:
$\quad c \leftarrow E_{\text{s}}(k, m_b)$, send (v, c) to $\mathcal{A}$

Figure 11.4: Game 0 challenger

Game 1. We first play our "DDH card." The challenger in this game is as in Fig. 11.4, except that line (1) is modified as follows:

(1) $\quad\alpha, \beta \xleftarrow{\text{R}} \mathbb{Z}_q,\ \gamma \xleftarrow{\text{R}} \mathbb{Z}_q,\ u \leftarrow g^{\alpha},\ v \leftarrow g^{\beta},\ w \leftarrow g^{\gamma}$

We describe an efficient DDH adversary $\mathcal{B}_{\text{ddh}}$ that uses $\mathcal{A}$ as a subroutine, such that

$$|\Pr[W_0] - \Pr[W_1]| = \text{DDH}\mathsf{adv}[\mathcal{B}_{\text{ddh}}, \mathbb{G}]. \tag{11.17}$$

Adversary $\mathcal{B}_{\text{ddh}}$ plays Attack Game 10.6 against a challenger $\mathbf{C}_{\text{ddh}}$, and plays the role of challenger to $\mathcal{A}$ as in Fig. 11.4, except with line (1) modified as follows:

(1) $\quad$ obtain (u, v, w) from $\mathbf{C}_{\text{ddh}}$

Additionally,

when $\mathcal{A}$ outputs $\hat{b}$:
$\quad$ if $b = \hat{b}$ then output 1 else output 0

Let p_0 be the probability that $\mathcal{B}_{\text{ddh}}$ outputs 1 when $\mathbf{C}_{\text{ddh}}$ is running Experiment 0 of the DDH Attack Game 10.6, and let p_1 be the probability that $\mathcal{B}_{\text{ddh}}$ outputs 1 when $\mathbf{C}_{\text{ddh}}$ is running Experiment 1. By definition, $\text{DDH}\mathsf{adv}[\mathcal{B}_{\text{ddh}}, \mathbb{G}] = |p_1 - p_0|$. Moreover, if $\mathbf{C}_{\text{ddh}}$ is running Experiment 0, then adversary $\mathcal{A}$ is playing our Game 0, and so $p_0 = \Pr[W_0]$, and if $\mathbf{C}_{\text{ddh}}$ is running Experiment 1, then $\mathcal{A}$ is playing our Game 1, and so $p_1 = \Pr[W_1]$. Equation (11.17) now follows immediately.

Game 2. Observe that in Game 1, w is completely random, and is used only as an input to H. This allows us to play our "KDF card." The challenger in this game is as in Fig. 11.4, except with the following lines modified as indicated:

(1) $\quad\alpha, \beta \xleftarrow{\text{R}} \mathbb{Z}_q,\ \gamma \xleftarrow{\text{R}} \mathbb{Z}_q,\ u \leftarrow g^{\alpha},\ v \leftarrow g^{\beta},\ w \leftarrow g^{\gamma}$
(2) $\quad k \xleftarrow{\text{R}} \mathcal{K}$

We may easily derive an efficient KDF adversary $\mathcal{B}_{\text{kdf}}$ that uses $\mathcal{A}$ as a subroutine, such that

$$|\Pr[W_1] - \Pr[W_2]| = \text{KDF}\mathsf{adv}[\mathcal{B}_{\text{kdf}}, H]. \tag{11.18}$$

Adversary $\mathcal{B}_{\text{kdf}}$ plays Attack Game 11.3 against a challenger $\mathbf{C}_{\text{kdf}}$, and plays the role of challenger to $\mathcal{A}$ as in Fig. 11.4, except with the following lines modified as indicated:

(1) $\alpha, \beta \xleftarrow{R} \mathbb{Z}_q$, $u \leftarrow g^\alpha$, $v \leftarrow g^\beta$, $\gamma \xleftarrow{R} \mathbb{Z}_q$, $w \leftarrow g^\gamma$
(2) obtain k from $\mathbf{C}_{\text{kdf}}$

Additionally,

when $\mathcal{A}$ outputs $\hat{b}$:
if $b = \hat{b}$ then output 1 else output 0

We leave it to the reader to verify (11.18).

Observe that in Game 2, the key k is only used to encrypt the challenge plaintext. As such, the adversary is essentially just playing the SS game with respect to $\mathcal{E}_\text{s}$ at this point. We leave it to the reader to describe an efficient SS adversary $\mathcal{B}_\text{s}$ that uses $\mathcal{A}$ as a subroutine, such that

$$|\Pr[W_2] - 1/2| = \text{SS}\mathsf{adv}^*[\mathcal{B}_\text{s}, \mathcal{E}_\text{s}]. \tag{11.19}$$

Combining (11.16), (11.17), (11.18), and (11.19), yields (11.15), which completes the proof of the theorem. □

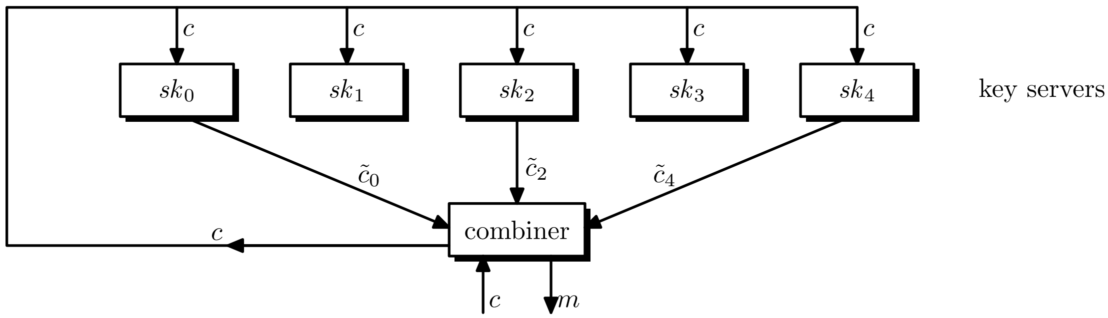
11.6 Threshold decryption

We next discuss an important technique used to protect the secret key *sk* in a public key encryption scheme. Suppose *sk* is stored on a server, and that server is used to decrypt incoming ciphertexts. If the server is compromised, and the key is stolen, then all ciphertexts ever encrypted under the corresponding public-key can be decrypted by the attacker. For this reason, important secret keys are sometimes stored in a special hardware component, called a **hardware security module** (HSM) that responds to decryption requests, but never exports the secret key in the clear. An attacker who compromises the server can temporarily use the key, but cannot steal the key and use it offline.

Another approach to protecting a secret key is to split it into a number of pieces, called **shares**, and require that all the shares must be present in order to decrypt a ciphertext. Each share can be stored on a different machine so that all the machines must cooperate in order to decrypt a ciphertext. Decryption fails if even one machine does not participate. Consequently, to steal the secret key, an attacker must break the security of all the machines, and this can be harder than compromising a single machine. In what follows, we use s to denote the total number of shares.

While splitting the key makes it harder to steal, it also hurts availability. If even a single share is lost, decryption becomes impossible. For this reason we often require that decryption can proceed even if only t of the s shares are available, for some $0 < t \leq s$. For security, $t-1$ shares should reveal nothing about the key *sk*, and should not help the adversary decrypt ciphertexts. Typical values for t and s are 3-out-of-5 or 5-out-of-8; however some applications require larger values for t and s. In a 3-out-of-5 sharing, stealing only two shares should reveal nothing helpful to the adversary.

Threshold decryption. Ideally, during decryption, the secret key *sk* is never reconstituted in a single location. This ensures that there is no single point of failure that an adversary can attack to steal the key. In such a system, there are s **key servers**, and an additional entity called a **combiner** that orchestrates the decryption process. The combiner takes as input a ciphertext c to decrypt, and forwards c to all the key servers. Every online server applies its key share to c, and



The combiner sends the given ciphertext c to all five key servers. Three servers respond, enabling the combiner to construct and output the plaintext message m.

Figure 11.5: Threshold decryption using three responses from five key servers.

sends back a "partial decryption." Once t responses are received from the key servers, the combiner can construct the complete decryption of c. The entire process is shown in Fig. 11.5. Overall, the system should decrypt c without reconstituting the key sk in a single location. Such a system is said to support threshold decryption.

Definition 11.6. *A* ***public-key threshold decryption scheme*** $\mathcal{E} = (G, E, D, C)$ *is a tuple of four efficient algorithms:*

- *G is a probabilistic algorithm that is invoked as $(pk, sk_1, \ldots, sk_s) \xleftarrow{R} G(s,t)$ to generate a t-out-of-s shared key. It outputs a public key pk and s shares $SK := \{sk_1, \ldots, sk_s\}$ of the decryption key.*

- *E is an encryption algorithm as in a public key encryption scheme, invoked as $c \xleftarrow{R} E(pk, m)$.*

- *D is a deterministic algorithm that is invoked as $c' \leftarrow D(sk_i, c)$, where sk_i is one of the key shares output by G, c is a ciphertext, and c' is a partial decryption of c using sk_i.*

- *C is a deterministic algorithm that is invoked as $m \leftarrow C(c, c'_1, \ldots, c'_t)$, where c is a ciphertext, and $c'_1, \ldots, c'_t$ are some t partial decryptions of c, computed using t distinct key shares.*

- *As usual, decryption should correctly decrypt well-formed ciphertexts; specifically, for all possible outputs $(pk, sk_1, \ldots, sk_s)$ of $G(s,t)$, all messages m, and all t-size subsets $\{sk'_1, \ldots, sk'_t\}$ of sk, for all outputs c of $E(pk, m)$, we have $C(\ c,\ D(sk'_1, c),\ \ldots,\ D(sk'_t, c)\ ) = m$.*

A public-key threshold decryption scheme is secure if an adversary that completely compromises $t-1$ of the key servers, and can eavesdrop on the output of the remaining key servers, cannot break semantic security. We will define security more precisely after we look at some constructions.

Note that Definition 11.6 requires that t and s be specified at key generation time. However, all the schemes in this section can be extended so that both t and s can be changed after the secret key shares are generated, without changing the public key pk.

Combinatorial threshold decryption. Recall that in Exercise 2.21 we saw how a symmetric decryption key k can be split into three shares, so that any two shares can be used to decrypt a given ciphertext, but a single share cannot. The scheme can be generalized so that k can be split into s shares and any $t \leq s$ can be used to decrypt, but $t-1$ shares cannot. The communication pattern during decryption is a little different than the one shown in Fig. 11.5, but nevertheless, the system satisfies our goal of decrypting without ever reconstituting the key k in a single location.

The difficulty with the scheme in Exercise 2.21 is that its performance degrades rapidly as t and s grow. Even supporting a small number of shares, say a 5-out-of-8 sharing, requires a ciphertext that is over fourteen times as long as a non-threshold ciphertext.

ElGamal threshold decryption. As we will shortly see, the ElGamal encryption scheme (Section 11.5) supports a very efficient threshold decryption mechanism, even for large t and s. In Exercise 11.16 we look at RSA threshold decryption.

11.6.1 Shamir's secret sharing scheme

Our threshold version of ElGamal encryption is based on a technique, which has numerous other application, called **secret sharing**.

Suppose Alice has a secret $\alpha \in Z$, where Z is some finite set. She wishes to generate s shares of α, each belonging to some finite set Z', and denoted $\alpha_1, \ldots, \alpha_s \in Z'$, so that the following property is satisfied: any t of the s shares are sufficient to reconstruct α, but every set of $t-1$ shares reveals nothing about α. This sharing lets Alice give one share to each of her s friends, so that any t friends can help her recover α, but $t-1$ friends learn nothing. Such a scheme is called a **secret sharing scheme**.

Definition 11.7. *A **secret sharing scheme over** Z is a pair of efficient algorithms (G, C):*

- *G is a probabilistic algorithm that is invoked as $(\alpha_1, \ldots, \alpha_s) \xleftarrow{R} G(s, t, \alpha)$, where $0 < t \leq s$ and $\alpha \in Z$, to generate a t-out-of-s sharing of α. It outputs s shares $SK := \{\alpha_1, \ldots, \alpha_s\}$.*
- *C is a deterministic algorithm that is invoked as $\alpha \leftarrow C(\alpha'_1, \ldots, \alpha'_t)$, to recover α.*
- *Correctness: we require that for every $\alpha \in Z$, every set of s shares SK output by $G(s, t, \alpha)$, and every t-size subset $\{\alpha'_1, \ldots, \alpha'_t\}$ of SK, we have that $C(\alpha'_1, \ldots, \alpha'_t) = \alpha$.*

Intuitively, a secret sharing scheme is secure if every set of $t-1$ shares output by $G(s, t, \alpha)$ reveals nothing about α. To define this notion formally, it will be convenient to use the following notation: for a set $S \subseteq \{1, \ldots, s\}$, we denote by $G(s, t, \alpha)[S]$ the set of shares output by G at positions indicated by S. For example, $G(s, t, \alpha)[\{1, 3, 4\}]$ is the set $\{\alpha_1, \alpha_3, \alpha_4\}$.

Definition 11.8. *A secret sharing scheme (G, C) over Z is **secure** if for every $\alpha, \alpha' \in Z$, and every subset S of $\{1, \ldots, s\}$ of size $t-1$, the distribution $G(s, t, \alpha)[S]$ is identical to the distribution $G(s, t, \alpha')[S]$.*

The definition implies that by looking at $t-1$ shares, one cannot tell if the secret is α or α', for all α and α' in Z. Hence, looking at only $t-1$ shares reveals nothing about the secret.

Shamir secret sharing. An elegant secret sharing scheme over $\mathbb{Z}_q$, where q is prime, is due to Shamir. This scheme makes use of the following general fact about **polynomial interpolation**: a polynomial of degree at most $t-1$ is completely determined by t points on the polynomial. For example, two points determine a line, and three points determine a parabola. This general fact not only holds for the real numbers and complex numbers, but over any algebraic domain in which all non-zero elements have a multiplicative inverse. Such a domain is called a *field*. When q is prime, $\mathbb{Z}_q$ is a field, and so this general fact holds here as well.

Shamir's scheme $(G_{\text{sh}}, C_{\text{sh}})$ is a t-out-of-s secret sharing scheme over $\mathbb{Z}_q$ that requires that $q > s$, and works as follows:

- $G_{\text{sh}}(s, t, \alpha)$: choose random $a_1, \ldots, a_{t-1} \xleftarrow{\text{R}} \mathbb{Z}_q$ and define the polynomial

$$f(x) := a_{t-1}x^{t-1} + a_{t-2}x^{t-2} + \ldots + a_1 x + \alpha \quad \in \mathbb{Z}_q[x].$$

  Notice that f has degree at most $t-1$ and that $f(0) = \alpha$.

  Next, choose arbitrary s non-zero points $x_1, \ldots, x_s$ in $\mathbb{Z}_q$ (for example, we could just use the points $1, \ldots, s$ in $\mathbb{Z}_q$).

  For $i = 1, \ldots, s$ compute $y_i \leftarrow f(x_i) \in \mathbb{Z}_q$, and define $\alpha_i := (x_i, y_i)$.
  Output the s shares $\alpha_1, \ldots, \alpha_s \in \mathbb{Z}_q^2$.

- $C_{\text{sh}}(\alpha'_1, \ldots, \alpha'_t)$: an input of t valid shares corresponds to t points on the polynomial f, and these t points completely determine f. Algorithm C_{sh} interpolates the polynomial f and outputs $\alpha := f(0)$.

The description of algorithm C_{sh} needs a bit more explanation. A simple method for interpolating the polynomial of degree at most $t-1$ from t points is called **Lagrange interpolation**. Let us see how it works.

Given t shares $\alpha'_i = (x'_i, y'_i)$ for $i = 1, \ldots, t$, define t polynomials:

$$L_i(x) := \prod_{\substack{j=1 \\ j \neq i}}^{t} \frac{x - x'_j}{x'_i - x'_j} \quad \in \mathbb{Z}_q[x] \qquad \text{for } i = 1, \ldots, t.$$

It is not difficult to verify that: $L_i(x'_i) = 1$ and $L_i(x'_j) = 0$ for all $j \neq i$ in $\{1, \ldots, t\}$. Next, consider the polynomial

$$g(x) := L_1(x) \cdot y'_1 + \ldots + L_t(x) \cdot y'_t \quad \in \mathbb{Z}_q[x]$$

Again, it is not difficult to see that $g(x'_i) = y'_i = f(x'_i)$ for all $i = 1, \ldots, t$. Since both f and g are polynomials of degree $t-1$, and they match at t points, they must be the same polynomial (here is we use our general fact about polynomial interpolation). Therefore, $\alpha = f(0) = g(0)$, and in particular

$$\alpha = g(0) = \sum_{i=1}^{t} \lambda_i \cdot y'_i \qquad \text{where} \qquad \lambda_i := L_i(0) = \prod_{\substack{j=1 \\ j \neq i}}^{t} \frac{-x'_j}{x'_i - x'_j} \in \mathbb{Z}_q. \tag{11.20}$$

The scalars $\lambda_1, \ldots, \lambda_t \in \mathbb{Z}_q$ are called **Lagrange coefficients**.

Using (11.20) we can now describe algorithm C_{sh} in more detail. Given a set of $t-1$ shares, the algorithm first computes the Lagrange coefficients $\lambda_1, \ldots, \lambda_t \in \mathbb{Z}_q$. Computing these quantities requires division, but since q is prime, this is always well defined. It then computes α using the linear combination on the left side of (11.20).

Note that the Lagrange coefficients $\lambda_1, \ldots, \lambda_t$ do not depend on the secret α, and can be precomputed if one knows ahead of time which shares will be used to reconstruct α.

Security. It remains to show that this secret sharing scheme is secure, as in Definition 11.8.

Theorem 11.6. *Shamir's secret sharing scheme* $(G_{\rm sh}, C_{\rm sh})$ *is secure.*

Proof. To prove the theorem, we shall show that for every $\alpha \in \mathbb{Z}_q$, any set of $t-1$ shares $(x'_1, y'_1), \ldots, (x'_{t-1}, y'_{t-1})$ has the property that the y-coordinates $y'_1, \ldots, y'_{t-1}$ are uniformly and independently distributed over $\mathbb{Z}_q$. So let α and $x'_1, \ldots, x'_{t-1}$ be fixed.

Claim. Consider the map that sends $(a_1, \ldots, a_{t-1}) \in \mathbb{Z}_q^{t-1}$ (as chosen by $G_{\rm sh}(s, t, \alpha)$) to $(y'_1, \ldots, y'_{t-1}) \in \mathbb{Z}_q^{t-1}$, which are the y-coordinates of the shares whose x-coordinates are $x'_1, \ldots, x'_{t-1}$. Then this map is one-to-one.

The theorem follows from the claim, since if $(a_1, \ldots, a_{t-1})$ is chosen uniformly over $\mathbb{Z}_q^{t-1}$, then $(y'_1, \ldots, y'_{t-1})$ must also be uniformly distributed over $\mathbb{Z}_q^{t-1}$.

Finally, to prove the claim, suppose by way of contradiction that this map is not one-to-one. This would imply the existence of two distinct polynomials $g(x), h(x) \in \mathbb{Z}_q[x]$ of degree at most $t-2$, such that the polynomials $\alpha + xg(x)$ and $\alpha + xh(x)$ agree at the $t-1$ *non-zero* points $x'_1, \ldots, x'_{t-1}$. But then this implies that $g(x)$ and $h(x)$ themselves agree at these same $t-1$ points, which contradicts our basic fact about polynomial interpolation. □

11.6.2 ElGamal threshold decryption

For any public-key encryption scheme, one can use Shamir secret sharing to share the secret decryption key *sk*, in a t-out-of-s fashion, among s servers. Then any t servers can help the combiner reconstruct the secret key and decrypt a given ciphertext. However, this creates a single point of failure: an adversary who compromises the combiner during decryption will learn *sk* in the clear.

In this section we show how to enhance ElGamal decryption, so that decryption can be done with the help of t servers, as in Fig. 11.5, but without reconstituting the key at a single location. We first describe the scheme, and then define and prove security.

ElGamal threshold decryption. Recall that the ElGamal encryption scheme (Section 11.5) uses a group $\mathbb{G}$ of prime order q with generator $g \in \mathbb{G}$, a symmetric cipher $\mathcal{E}_{\rm s} = (E_{\rm s}, D_{\rm s})$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and a hash function $H : \mathbb{G} \to \mathcal{K}$. The secret key *sk* is an element $\alpha \in \mathbb{Z}_q$, and a ciphertext $(v, c) \in \mathbb{G} \times \mathcal{C}$ is decrypted by first computing $w \leftarrow v^{\alpha}$.

To support t-out-of-s threshold decryption, the key generation algorithm first generates a t-out-of-s Shamir secret sharing of the ElGamal decryption key $\alpha \in \mathbb{Z}_q$. The resulting shares, (x_i, y_i) for $i = 1, \ldots, s$, are the shares of the decryption key α, and each key server is given one share.

Now, to decrypt an ElGamal ciphertext (v, c), it suffices for some t key servers to send the partial decryption $(x_i,\ v^{y_i}) \in \mathbb{Z}_q \times \mathbb{G}$ to the combiner. Once the combiner receives t partial decryptions $c'_i = (x_i,\ v^{y_i})$ for $i = 1, \ldots, t$, it decrypts the ciphertext as follows: First, the combiner uses $x_1, \ldots, x_t$ to compute the Lagrange coefficients $\lambda_1, \ldots, \lambda_t \in \mathbb{Z}_q$ as in Eq. (11.20). Next, it computes

$$w \leftarrow (v^{y_1})^{\lambda_1} \cdot (v^{y_2})^{\lambda_2} \cdots (v^{y_t})^{\lambda_t} \ \in \mathbb{G}.$$

By (11.20) we know that

$$w = v^{(y_1 \cdot \lambda_1 + \cdots + y_t \cdot \lambda_t)} = v^{\alpha}. \tag{11.21}$$

This $w = v^{\alpha}$ is sufficient to decrypt the ciphertext (v, c), as in normal ElGamal decryption. Observe that during decryption, the ElGamal decryption key α was never assembled in a single location.

The complete ElGamal threshold decryption system $\mathcal{E}_{\rm thEG} = (G, E, D, C)$ works as follows:

- Key generation runs as follows, using Shamir's secret sharing scheme $(G_{\text{sh}}, C_{\text{sh}})$:

$$\begin{aligned} G(s,t) := \quad & \alpha \xleftarrow{\text{R}} \mathbb{Z}_q, \qquad pk := u \leftarrow g^{\alpha} \\ & (x_1, y_1), \ldots, (x_s, y_s) \xleftarrow{\text{R}} G_{\text{sh}}(s, t, \alpha) \\ & \text{for } i = 1, \ldots, s \text{ set } sk_i := (x_i, y_i) \\ & \text{output } (pk, sk_1, \ldots, sk_s) \end{aligned}$$

- The encryption algorithm $E(pk, m)$ is the same as in ElGamal encryption in Section 11.5. It outputs a pair $(v, c) \in \mathbb{G} \times \mathcal{C}$.

- for a given secret key share $sk_i = (x, y) \in \mathbb{Z}_q \times \mathbb{G}$ and a ciphertext $(v, c) \in \mathbb{G} \times \mathcal{C}$, the partial decryption algorithm runs as follows:

$$\begin{aligned} D(sk_i,\ (v,c)\ ) := \quad & w \leftarrow v^y, \\ & \text{output } c' := (x, w) \in \mathbb{Z}_q \times \mathbb{G}. \end{aligned}$$

- given a ciphertext $(v, c) \in \mathbb{G} \times \mathcal{C}$, and t partial decryptions $c'_i = (x_i, w_i)$ for $i = 1, \ldots, t$, the combine algorithm runs as follows:

$$\begin{aligned} & C\big((v,c),\ c'_1, \ldots, c'_t\big) := \\ & \qquad \text{use } x_1, \ldots, x_t \text{ to compute } \lambda_1, \ldots, \lambda_t \in \mathbb{Z}_q \text{ as in (11.20)} \\ (*) \quad & \qquad \text{set } w \leftarrow w_1^{\lambda_1} \cdot w_2^{\lambda_2} \cdots w_t^{\lambda_t} \in \mathbb{G}, \quad k \leftarrow H(w), \quad m \leftarrow D_{\text{s}}(k, c) \\ & \qquad \text{output } m \end{aligned}$$

The combine algorithm works correctly because, as explained in (11.21), the quantity w computed on line $(*)$ satisfies $w = v^{\alpha}$, which is then used to derive the symmetric encryption key k needed to decrypt c.

ElGamal threshold decryption is secure. First, let us define more precisely what it means for a threshold decryption scheme to be secure. As usual, this is done by defining an attack game. Just as in Attack Game 11.1, our adversary will be allowed to make a single encryption query, in which he submits a pair of messages to the challenger, and obtains an encryption of one of them. However, to capture the notion of security we are looking for in a threshold decryption scheme, in addition to the public key, the adversary also gets to see $t - 1$ shares of the secret key of its choice. Additionally, we want to capture the notion that the combiner cannot become a single point of failure. To this end, we allow the adversary to make any number of *combiner queries*: in such a query, the adversary sumbits a single message to the challenger, and gets to see not only its encryption, but also all s of the corresponding partial decryptions of the ciphertext.

Our security definition, given below, allows the adversary to eavesdrop on all traffic sent to the combiner. A more powerful adversary might completely compromise the combiner, and tamper with what it sends to the key servers. We do not consider such adversaries here, but will come back to this question in Chapter 16.

Attack Game 11.4 (threshold decryption semantic security). For a public-key threshold decryption scheme $\mathcal{E} = (G, E, D, C)$ defined over $(\mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, we define two experiments, parameterized by integers $0 < t \leq s$.

Experiment b $\;(b = 0, 1)$**:**

- *Setup:* the adversary chooses a set $S \subseteq \{1, \ldots, s\}$ of size $t-1$ and gives it to the challenger. The challenger runs $(pk, sk_1, \ldots, sk_s) \xleftarrow{R} G(s,t)$ and sends pk and $\{sk_i\}_{i \in S}$ to the adversary.

- The adversary queries the challenger several times. Each query can be one of two types:

  - *Combiner query*: for $j = 1, 2, \ldots,$ the jth such query is a message $m_j \in \mathcal{M}$. The challenger computes $c_j \xleftarrow{R} E(pk, m_j)$ and the s partial decryptions $c'_{j,i} \leftarrow D(sk_i, c_j)$, for $i = 1, \ldots, s$. The challenger sends c_j and $c'_{j,1}, \ldots, c'_{j,s}$ to the adversary.
  - *Single encryption query:* The adversary sends $m_0, m_1 \in \mathcal{M}$, of the same length, to the challenger. The challenger computes $c \xleftarrow{R} E(pk, m_b)$, and sends c to the adversary. The adversary may only issue a single encryption query (which may be preceded or followed by any number of combiner queries).

- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

If W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b, define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\text{thSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := \Big|\Pr[W_0] - \Pr[W_1]\Big|. \quad \Box$$

Definition 11.9 (threshold decryption semantic security). *A public-key threshold decryption scheme $\mathcal{E}$ is* ***semantically secure*** *if for all efficient adversaries $\mathcal{A}$, the value* $\text{thSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

Next, we argue that the ElGamal threshold decryption scheme $\mathcal{E}_{\text{thEG}}$ is semantically secure. The proof is very similar to the proof of Theorem 11.5.

Theorem 11.7. *If $\mathcal{E}_{\text{EG}}$ is semantically secure, then $\mathcal{E}_{\text{thEG}}$ is threshold decryption semantically secure.*

In particular, for every adversary $\mathcal{A}$ that attacks $\mathcal{E}_{\text{thEG}}$ as in Attack Game 11.4, there exists an adversary $\mathcal{B}$ that attacks $\mathcal{E}_{\text{EG}}$ as in Attack Game 11.1, such that

$$\text{thSS}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\text{thEG}}] = \text{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}_{\text{EG}}].$$

Proof. We design $\mathcal{B}$ to play the role of challenger to $\mathcal{A}$. When $\mathcal{A}$ receives $pk = g^{\alpha}$ from its own challenger, we need to have $\mathcal{A}$ provide to $\mathcal{B}$ not only pk, but also $t-1$ key shares. By Theorem 11.6, we know that $(G_{\text{sh}}, C_{\text{sh}})$ satisfies Definition 11.7, which means that we can generate the required $t-1$ key shares by just running $G_{\text{sh}}(\hat{\alpha}, r, s)$ for an *arbitrary* $\hat{\alpha} \in \mathbb{Z}_q$. In fact, by the proof of of Theorem 11.6, we know that we can just generate the y-coordinates of the required shares by choosing elements of $\mathbb{Z}_q$ uniformly and independently.

When $\mathcal{A}$ makes its single encryption query, $\mathcal{B}$ forwards this query to its own challenger, and forwards the response from the challenger back to $\mathcal{A}$.

Whenever $\mathcal{A}$ outputs a bit $\hat{b} \in \{0, 1\}$, our adversary $\mathcal{B}$ outputs the same bit $\hat{b}$.

To finish the proof, we have to show how our $\mathcal{B}$ can faithfully respond to all of $\mathcal{A}$'s combiner queries. Once we do this, the proof will be finished: $\mathcal{B}$ will have the same advantage in its attack game that $\mathcal{A}$ has in its attack game.

Let (x'_i, y'_i) for $i = 1, \ldots, t-1$ be the key shares that were given to $\mathcal{A}$. Let $m \in \mathcal{M}$ be a combiner query. Our $\mathcal{B}$ first encrypts m by choosing a random $\beta \xleftarrow{R} \mathbb{Z}_q$ and computing $v \leftarrow g^{\beta}$, $w \leftarrow u^{\beta}$, $c \leftarrow E_{\text{s}}(H(w), m)$. Now, let (x, y) be some key share. Our $\mathcal{B}$ needs to compute the partial decryption $c' := (x,\ v^{y})$. There are two cases:

- If $x \in \{x'_1, \ldots, x'_{t-1}\}$ then $\mathcal{B}$ knows y and can easily compute $c' := (x, v^y)$.
- Otherwise, our $\mathcal{B}$ can compute v^y without knowing y, as follows. It uses (11.20) to compute the t Lagrange coefficients $\lambda, \lambda_1, \ldots, \lambda_{t-1} \in \mathbb{Z}_q$ corresponding to the t points $x, x'_1, \ldots, x'_{t-1} \in \mathbb{Z}_q$. Although $\mathcal{B}$ does not know α or y, it knows that

  $$\alpha = \lambda \cdot y + \lambda_1 \cdot y'_1 + \ldots + \lambda_{t-1} \cdot y'_{t-1}.$$

  By multiplying both sides by β and exponentiating, it follows that

  $$u^\beta = g^{\beta \cdot \alpha} = g^{\beta \cdot \lambda \cdot y} \cdot g^{\beta(\lambda_1 \cdot y'_1 + \cdots + \lambda_{t-1} \cdot y'_{t-1})} = (v^y)^\lambda \cdot g^{\beta(\lambda_1 \cdot y'_1 + \cdots + \lambda_{t-1} \cdot y'_{t-1})}.$$

  Since v^y is the only unknown in this equation, $\mathcal{B}$ can easily solve for v^y, and obtain the required value.

In conclusion, we see that $\mathcal{B}$ can compute all the required partial decryptions $c' := (x, v^y)$, and send them to the adversary, along with the ciphertext (v, c). □

Further enhancements. The threshold decryption scheme $\mathcal{E}_{\text{thEG}}$ can be strengthened in several ways. First, the system $\mathcal{E}_{\text{thEG}}$ easily generalizes to more flexible access structures than strict threshold. For example, it is easy to extend the scheme to support the following access structure: decryption is possible if key server number 1 participates, and at least t of the remaining $s-1$ key servers participate. We explore more general access structures in Exercise 11.15.

Another enhancement, called **proactive security**, further strengthens the system by forcing the adversary to break into all s servers *within a short period of time*, say ten minutes [53]. Otherwise, the adversary gets nothing. This is done by having the key servers proactively refresh the sharing of their secret key every ten minutes, without changing the public key.

Finally, key generation can be strengthened so that the secret key α is not generated in a central location. Instead, the s key servers engage in a distributed computation to generate the key shares [45]. This way the secret key α is always stored in shared form, from inception to final retirement.

11.7 Fun application: oblivious transfer from DDH

To be written.

11.8 Notes

Citations to the literature to be added.

11.9 Exercises

11.1 (Simple PRF from DDH). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let $H : \mathcal{M} \to \mathbb{G}$ be a hash function, which we shall model as a random oracle (see Section 8.10.2). Let F be the PRF defined over $(\mathbb{Z}_q, \mathcal{M}, \mathbb{G})$ as follows:

$$F(k, m) := H(m)^k \quad \text{for } k \in \mathbb{Z}_q,\ m \in \mathcal{M}.$$

Show that F is a secure PRF in the random oracle model for H under the DDH assumption for $\mathbb{G}$.

Hint: Use the results of Exercises 10.6 and 10.7.

11.2 (Simple PRF from CDH). Continuing with Exercise 11.1, let $\hat{H} : \mathbb{G} \times \mathbb{G} \to \mathcal{Y}$ be a hash function, which we again model as a random oracle. Let $\hat{F}$ be the PRF defined over $(\mathbb{Z}_q, \mathcal{M}, \mathcal{Y})$ as follows:

$$\hat{F}(k, m) := \hat{H}\left(H(m),\ H(m)^k\right) \quad \text{for } k \in \mathbb{Z}_q,\ m \in \mathcal{M}.$$

Show that $\hat{F}$ is a secure PRF in the random oracle model for H and $\hat{H}$ under the CDH assumption for $\mathbb{G}$.

Hint: Use the result of Exercise 10.4.

11.3 (Oblivious PRF from DDH). Your proof that the PRF F F presented in Exercise 11.1 should still go through even if the value g^k is publicly known. Using this fact, we can design a protocol that allows F to be evaluated *obliviously*. This means that if Bob has a key k and Alice has an input m, there is a simple protocol that lets Alice compute $F(k, m)$ in such a way that Bob does not learn anything about m and Alice learns nothing about k besides $F(k, m)$ and g^k.

Hint: Alice starts by sending Bob $H(m) \cdot g^\rho$ for random $\rho \in \mathbb{Z}_q$ — see also Exercise 10.4.

11.4 (Broken variant of RSA). Consider the following broken version of the RSA public-key encryption scheme: key generation is as in $\mathcal{E}_{\text{RSA}}$, but to encrypt a message $m \in \mathbb{Z}_n$ with public key $pk = (n, e)$ do $E(pk, m) := m^e$. Decryption is done using the RSA trapdoor.

Clearly this scheme is not semantically secure. Even worse, suppose one encrypts a *random* message $m \in \{0, 1, \ldots, 2^{64}\}$ to obtain $c := m^e \bmod n$. Show that for 35% of plaintexts in $[0, 2^{64}]$, an adversary can recover the complete plaintext m from c using only 2^{35} eth powers in $\mathbb{Z}_n$.

Hint: Use the fact that about 35% of the integers m in $[0, 2^{64}]$ can be written as $m = m_1 \cdot m_2$ where $m_1, m_2 \in [0, 2^{34}]$.

11.5 (Multiplicative ElGamal). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Consider a simple variant of the ElGamal encryption system $\mathcal{E}_{\text{MEG}} = (G, E, D)$ that is defined over $(\mathbb{G}, \mathbb{G}^2)$. The key generation algorithm G is the same as in $\mathcal{E}_{\text{EG}}$, but encryption and decryption work as follows:

- for a given public key $pk = u \in \mathbb{G}$ and message $m \in \mathbb{G}$:

$$E(pk, m) := \quad \beta \xleftarrow{\text{R}} \mathbb{Z}_q,\ \ v \leftarrow g^\beta,\ \ c \leftarrow m \cdot u^\beta,\ \ \text{output } (v, c)$$

- for a given secret key $sk = \alpha \in \mathbb{Z}_q$ and a ciphertext $(v, c) \in \mathbb{G}^2$:

$$D(sk,\ (v, c)\ ) := \quad \text{output } c/v^\alpha$$

(a) Show that $\mathcal{E}_{\text{MEG}}$ is semantically secure assuming the DDH assumption holds in $\mathbb{G}$. In particular, you should show that the advantage of any adversary $\mathcal{A}$ in breaking the semantic security of $\mathcal{E}_{\text{MEG}}$ is bounded by 2ϵ, where ϵ is the advantage of an adversary $\mathcal{B}$ (which is an elementary wrapper around $\mathcal{A}$) in the DDH attack game.

(b) Show that $\mathcal{E}_{\text{MEG}}$ is not semantically secure if the DDH assumption does not hold in $\mathbb{G}$.

(c) Show that $\mathcal{E}_{\text{MEG}}$ has the following property: given a public key pk, and two ciphertexts $c_1 \xleftarrow{\text{R}} E(pk, m_1)$ and $c_2 \xleftarrow{\text{R}} E(pk, m_2)$, it is possible to create a new ciphertext c which is an encryption of $m_1 \cdot m_2$. This property is called a **multiplicative homomorphism**.

11.6 (An attack on multiplicative ElGamal). Let p and q be large primes such that q divides $p-1$. Let $\mathbb{G}$ be the order q subgroup of $\mathbb{Z}_p^*$ generated by $g \in \mathbb{G}$ and assume that the DDH assumption holds in $\mathbb{G}$. Suppose we instantiate the ElGamal system from Exercise 11.5 with the group $\mathbb{G}$. However, plaintext messages are chosen from the entire group $\mathbb{Z}_p^*$ so that the system is defined over $(\mathbb{Z}_p^*, \mathbb{G} \times \mathbb{Z}_p^*)$. Show that the resulting system is not semantically secure.

11.7 (Extending the message space). Suppose that we have a public-key encryption scheme $\mathcal{E} = (G, E, D)$ with message space $\mathcal{M}$. From this, we would like to build an encryption scheme with message space $\mathcal{M}^2$. To this end, consider the following encryption scheme $\mathcal{E}^2 = (G^2, E^2, D^2)$, where

$$\begin{aligned}
G^2() \quad &:= \quad (pk_0, sk_0) \xleftarrow{\text{R}} G(), \quad (pk_1, sk_1) \xleftarrow{\text{R}} G(), \\
& \text{output } pk := (pk_0, pk_1) \text{ and } sk := (sk_0, sk_1) \\
E^2\big(pk, (m_0, m_1)\big) \quad &:= \quad \big(E(pk_0, m_0),\ E(pk_1, m_1)\big) \\
D^2\big(sk, (c_0, c_1)\big) \quad &:= \quad \big(D(sk_0, c_0),\ D(sk_1, c_1)\big)
\end{aligned}$$

Show that $\mathcal{E}^2$ is semantically secure, assuming $\mathcal{E}$ itself is semantically secure.

11.8 (Modular hybrid construction). Both of the encryption schemes presented in this chapter, $\mathcal{E}_{\text{TDF}}$ in Section 11.4 and $\mathcal{E}_{\text{EG}}$ in Section 11.5, as well as many other schemes used in practice, have a "hybrid" structure that combines an asymmetric component and a symmetric component in a fairly natural and modular way. The symmetric part is, of course, the symmetric cipher $\mathcal{E}_{\text{s}} = (E_{\text{s}}, D_{\text{s}})$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. The asymmetric part can be understood in abstract terms as what is called a **key encapsulation mechanism**, or **KEM**.

A KEM $\mathcal{E}_{\text{kem}}$ consists of a tuple of algorithms $(G, E_{\text{kem}}, D_{\text{kem}})$. Algorithm G is invoked as $(pk, sk) \xleftarrow{\text{R}} G()$. Algorithm E_{kem} is invoked as $(k, c_{\text{kem}}) \xleftarrow{\text{R}} E_{\text{kem}}(pk)$, where $k \in \mathcal{K}$ and $c_{\text{kem}} \in \mathcal{C}_{\text{kem}}$. Algorithm D_{kem} is invoked as $k \leftarrow D_{\text{kem}}(sk, c_{\text{kem}})$, where $k \in \mathcal{K} \cup \{\mathsf{reject}\}$ and $c_{\text{kem}} \in \mathcal{C}_{\text{kem}}$. We say that $\mathcal{E}_{\text{kem}}$ is **defined over** $(\mathcal{K}, \mathcal{C}_{\text{kem}})$. We require that $\mathcal{E}_{\text{kem}}$ satisfies the following **correctness requirement**: for all possible outputs (pk, sk) of $G()$, and all possible outputs (k, c_{kem}) of $E_{\text{kem}}(pk)$, we have $D_{\text{kem}}(sk, c_{\text{kem}}) = k$.

We can define a notion of semantic security in terms of an attack game between a challenger and an adversary $\mathcal{A}$, as follows. In Experiment b, for $b = 0, 1$, the challenger computes

$$(pk, sk) \xleftarrow{\text{R}} G(), \ \ (k_0, c_{\text{kem}}) \xleftarrow{\text{R}} E_{\text{kem}}(pk), \ \ k_1 \xleftarrow{\text{R}} \mathcal{K},$$

and sends (k_b, c_{kem}) to $\mathcal{A}$. Finally, $\mathcal{A}$ outputs $\hat{b} \in \{0,1\}$. As usual, if W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b, we define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}_{\text{kem}}$ as $\mathsf{SSadv}[\mathcal{A}, \mathcal{E}_{\text{kem}}] := |\Pr[W_0] - \Pr[W_1]|$, and if this advantage is negligible for all efficient adversaries, we say that $\mathcal{E}_{\text{kem}}$ is **semantically secure**.

Now consider the **hybrid public-key encryption scheme** $\mathcal{E} = (G, E, D)$, constructed out of $\mathcal{E}_{\text{kem}}$ and $\mathcal{E}_{\text{s}}$, and defined over $(\mathcal{M}, \mathcal{C}_{\text{kem}} \times \mathcal{C})$. The key generation algorithm for $\mathcal{E}$ is the same as that of $\mathcal{E}_{\text{kem}}$. The encryption algorithm E works as follows:

$$E(pk, m) := \big\{ \ \ (k, c_{\text{kem}}) \xleftarrow{\text{R}} E_{\text{kem}}(pk), \ \ c \xleftarrow{\text{R}} E_{\text{s}}(k, m), \ \ \text{output } (c_{\text{kem}}, c) \ \ \big\}.$$

The decryption algorithm D works as follows:

$$D(sk, (c_{\text{kem}}, c)) := \{ \ m \leftarrow \textsf{reject}, \ k \leftarrow D_{\text{kem}}(sk, c_{\text{kem}}), \ \text{if } k \neq \textsf{reject} \text{ then } m \leftarrow D_{\text{s}}(k, c), \\ \text{output } m \ \}.$$

(a) Prove that $\mathcal{E}$ satisfies the correctness requirement for a public key encryption scheme, assuming $\mathcal{E}_{\text{kem}}$ and $\mathcal{E}_{\text{s}}$ satisfy their corresponding correctness requirements.

(b) Prove that $\mathcal{E}$ is semantically secure, assuming that $\mathcal{E}_{\text{kem}}$ and $\mathcal{E}_{\text{s}}$ are semantically secure. You should prove a concrete security bound that says that for every adversary $\mathcal{A}$ attacking $\mathcal{E}$, there are adversaries $\mathcal{B}_{\text{kem}}$ and $\mathcal{B}_{\text{s}}$ (which are elementary wrappers around $\mathcal{A}$) such that

$$\mathsf{SSadv}[\mathcal{A}, \mathcal{E}] \leq 2 \cdot \mathsf{SSadv}[\mathcal{B}_{\text{kem}}, \mathcal{E}_{\text{kem}}] + \mathsf{SSadv}[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}].$$

(c) Describe the KEM corresponding to $\mathcal{E}_{\text{TDF}}$ and prove that it is semantically secure (in the random oracle model, assuming $\mathcal{T}$ is one way).

(d) Describe the KEM corresponding to $\mathcal{E}_{\text{EG}}$ and prove that it is semantically secure (in the random oracle model, under the CDH assumption for $\mathbb{G}$).

(e) Let $\mathcal{E}_{\text{a}} = (G, E_{\text{a}}, D_{\text{a}})$ be a public-key encryption scheme defined over $(\mathcal{K}, \mathcal{C}_{\text{a}})$. Define the KEM $\mathcal{E}_{\text{kem}} = (G, E_{\text{kem}}, D_{\text{a}})$, where

$$E_{\text{kem}}(pk) := \{ \ k \xleftarrow{\text{R}} \mathcal{K}, \ c_{\text{kem}} \xleftarrow{\text{R}} E_{\text{a}}(pk, k), \ \text{output } (k, c_{\text{kem}}) \ \}.$$

Show that $\mathcal{E}_{\text{kem}}$ is semantically secure, assuming that $\mathcal{E}_{\text{a}}$ is semantically secure.

Discussion: Part (e) shows that one can always build a KEM from a public-key encryption scheme by just using the encryption scheme to encrypt a symmetric key; however, parts (c) and (d) show that there are more direct and efficient ways to do this.

11.9 (Multi-key CPA security). Generalize the definition of CPA security for a public-key encryption scheme to the multi-key setting. In this attack game, the adversary gets to obtain encryptions of many messages under many public keys. Show that semantic security implies multi-key CPA security. You should show that security degrades linearly in $Q_{\text{k}}Q_{\text{e}}$, where Q_{k} is a bound on the number of keys, and Q_{e} is a bound on the number of encryption queries per key. That is, the advantage of any adversary $\mathcal{A}$ in breaking the multi-key CPA security of a scheme is at most $Q_{\text{k}}Q_{\text{e}} \cdot \epsilon$, where ϵ is the advantage of an adversary $\mathcal{B}$ (which is an elementary wrapper around $\mathcal{A}$) that breaks the scheme's semantic security.

11.10 (A tight reduction for multiplicative ElGamal). We proved in Exercise 11.9 that semantic security for a public-key encryption scheme implies multi-key CPA security; however, the security degrades significantly as the number of keys and encryptions increases. Consider the multiplicative ElGamal encryption scheme $\mathcal{E}_{\text{MEG}}$ from Exercise 11.5. You are to show show a *tight* reduction from multi-key CPA security for $\mathcal{E}_{\text{MEG}}$ to the DDH assumption, which does not degrade at all as the number of keys and encryptions increases. In particular, you should show that the advantage of any adversary $\mathcal{A}$ in breaking the multi-key CPA security of $\mathcal{E}_{\text{MEG}}$ is bounded by $2(\epsilon + 1/q)$, where ϵ is the advantage of an adversary $\mathcal{B}$ (which is an elementary wrapper around $\mathcal{A}$) in the DDH attack game.

Note: You should assume that in the multi-key CPA game, the same group $\mathbb{G}$ and generator $g \in \mathbb{G}$ is used throughout.

Hint: Use the results of Exercises 10.6, 10.7, and 10.8.

11.11 (An easy discrete-log group). Let n be a large integer and consider the following subset of $\mathbb{Z}_{n^2}^*$:

$$\mathbb{G}_n := \big\{ \ [an+1]_{n^2} \in \mathbb{Z}_{n^2}^* : \quad a \in \{0, \ldots, n-1\}\big\}$$

(a) Show that $\mathbb{G}_n$ is a multiplicative subgroup of $\mathbb{Z}_{n^2}^*$ of order n.

(b) Which elements of $\mathbb{G}_n$ are generators?

(c) Choose an arbitrary generator $g \in \mathbb{G}_n$ and show that the discrete-log problem in $\mathbb{G}_n$ is easy.

11.12 (Pallier encryption). Let us construct another public-key encryption scheme (G, E, D) that makes use of RSA composites:

- The key generation algorithm is parameterized by a fixed value ℓ and runs as follows:

$$\begin{aligned} G(\ell) := \quad & \text{generate two distinct random } \ell\text{-bit primes } p \text{ and } q, \\ & n \leftarrow pq, \quad d \leftarrow (p-1)(q-1)/2 \\ & pk \leftarrow n, \quad sk \leftarrow d \\ & \text{output } (pk, sk) \end{aligned}$$

- for a given public key $pk = n$ and message $m \in \{0, \ldots, n-1\}$, set $g := [n+1]_{n^2} \in \mathbb{Z}_{n^2}^*$. The encryption algorithm runs as follows:

$$E(pk, m) := \quad h \xleftarrow{R} \mathbb{Z}_{n^2}^*, \quad c \xleftarrow{R} g^m h^n \in \mathbb{Z}_{n^2}^*, \quad \text{output } c.$$

(a) Explain how the decryption algorithm $D(sk, c)$ works.

Hint: Using the notation of Exercise 11.11, observe that c^d falls in the subgroup $\mathbb{G}_n$ which has an easy discrete-log.

(b) Show that this public-key encryption scheme is semantically secure under the following assumption:

let n be a product of two random ℓ-bit primes,
let u be uniform in $\mathbb{Z}_{n^2}^*$,
let v be uniform in the subgroup $(\mathbb{Z}_{n^2})^n := \{h^n \ : \ h \in \mathbb{Z}_{n^2}^*\}$,

then the distribution (n, u) is computationally indistinguishable from the distribution (n, v).

Discussion: This encryption system, called **Pallier encryption**, has a useful property called an additive homomorphism: for ciphertexts $c_0 \xleftarrow{R} E(pk, m_0)$ and $c_1 \xleftarrow{R} E(pk, m_1)$, the product $c \leftarrow c_0 \cdot c_1$ is an encryption of $m_0 + m_1 \bmod n$.

11.13 (Hash Diffie-Hellman). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let $H : \mathbb{G} \to \mathcal{K}$ be a hash function. We say that the **Hash Diffie-Hellman** (HDH) assumption holds for $(\mathbb{G}, H)$ if the distribution $\big(g^\alpha, g^\beta, H(g^{\alpha\beta})\big)$ is computationally indistinguishable from the distribution (g^α, g^β, k) where $\alpha, \beta \xleftarrow{R} \mathbb{Z}_q$ and $k \xleftarrow{R} \mathcal{K}$.

(a) Show that if H is modeled as a random oracle and the CDH assumption holds for $\mathbb{G}$, then the HDH assumption holds for $(\mathbb{G}, H)$.

(b) Show that if H is a secure KDF and the DDH assumption holds for $\mathbb{G}$, then the HDH assumption holds for $(\mathbb{G}, H)$.

(c) Prove that the ElGamal public-key encryption scheme $\mathcal{E}_{\text{EG}}$ is semantically secure if the HDH assumption holds for $(\mathbb{G}, H)$.

11.14 (Anonymous public-key encryption). Suppose t people publish their public-keys $pk_1, \ldots, pk_t$. Alice sends an encrypted message to one of them, say pk_5, but she wants to ensure that no one (other than user 5) can tell which of the t users is the intended recipient. You may assume that every user, other than user 5, who tries to decrypt Alice's message with their secret key, obtains fail.

(a) Define a security model that captures this requirements. The adversary should be given t public keys $pk_1, \ldots, pk_t$ and it then selects the message m that Alice sends. Upon receiving a challenge ciphertext, the adversary should learn nothing about which of the t public keys is the intended recipient. A system that has this property is said to be **an anonymous public-key encryption scheme**.

(b) Show that the ElGamal public-key encryption system $\mathcal{E}_{\text{EG}}$ is anonymous.

(c) Show that the RSA public-key encryption system $\mathcal{E}_{\text{RSA}}$ is not anonymous. Assume that all t public keys are generated using the same RSA parameters ℓ and e.

11.15 (Access structures). Generalize the ElGamal threshold decryption scheme of Section 11.6.2 to the following settings: The s key servers are split into two disjoint groups S_1 and S_2, and decryption should be possible only if the combiner receives at least t_1 responses from the set S_1, and at least t_2 responses from the set S_2, where $t_1 \leq |S_1|$ and $t_2 \leq |S_2|$. Adapt the security definition to these settings, and prove that your scheme is secure.

Discussion: An access structure is the set of subsets of $\{0, \ldots, s-1\}$ that should be able to decrypt. In Section 11.6.2 we looked at a threshold access structure, and this exercise looks at a slightly more general threshold access structure. Other access structures can be achieved using more general secret sharing schemes, as long as the secret is reconstructed using a *linear* function of the given shares. Such schemes, called linear secret sharing schemes (LSSS), are surveyed in [5].

11.16 (RSA threshold decryption). Let us show how to enable simple threshold decryption for the RSA public key encryption scheme of Section 11.4.1.

(a) Recall that the key generation algorithm generates numbers n, e, d, where n is the RSA modulus, e is the encryption exponent, and d is the decryption exponent. We extend the key generation algorithm with two more steps: choose a random integer d_1 in $[1, n^2]$ and set $d_2 = d_1 - d \in \mathbb{Z}$. Then output the two key shares $sk_1 := (n, d_1)$ and $sk_2 := (n, d_2)$, and the public key $pk := (n, e)$. Explain how to use this setup for 2-out-of-2 threshold decryption, to match the framework of Definition 11.6.

Hint: Show that the distribution of the key share d_2 is statistically close to the uniform distribution on $\{1, \ldots, n^2\}$.

(b) Prove that your scheme from part (a) satisfies the security definition for 2-out-of-2 threshold decryption (Definition 11.9).

(c) Generalize the scheme to provide 2-out-of-3 threshold decryption, using the mechanism of Exercise 2.20. Prove that the scheme is secure.

11.17 (Proxy re-encryption). Bob works for the Acme corporation and publishes a public-key pk_{bob} so that all incoming emails to Bob are encrypted under pk_{bob}. When Bob goes on vacation he instructs the company's mail server to forward all his incoming encrypted email to Alice. Alice's public key is pk_{alice}. The mail server needs a way to translate an email encrypted under public-key pk_{bob} into an email encrypted under public-key pk_{alice}. This would be easy if the mail server had sk_{bob}, but then the mail server can read all of Bob's incoming email.

Suppose that pk_{bob} and pk_{alice} are public keys for the ElGamal encryption scheme $\mathcal{E}_{\text{EG}}$ discussed in Section 11.5, both based on the same group $\mathbb{G}$ with generator $g \in \mathbb{G}$. Then the mail server can do the translation from pk_{bob} to pk_{alice} while learning nothing about the email contents.

(a) Suppose $pk_{\text{alice}} = g^{\alpha}$ and $pk_{\text{bob}} = g^{\alpha'}$. Show that giving $\tau := \alpha/\alpha'$ to the mail server lets it translate an email encrypted under pk_{bob} into an email encrypted under pk_{alice}, and vice-versa.

(b) Assume that $\mathcal{E}_{\text{EG}}$ is semantically secure. Show that the adversary cannot break semantic security for Alice, even if it is given Bob's public key $g^{\alpha'}$ along with the translation key τ.

11.18 (A voting system). Consider an election system where voters vote for one of two parties and their vote is either 0 and 1. The election service publishes an ElGamal public-key pk and every voter sends to the election service its vote $b_i \in \{0, 1\}$, encoded as the group element g^{b_i}, encrypted under pk using the multiplicative ElGamal system from Exercise 11.5. The election service needs to determine how many people voted 0 and how many voted 1. This is equivalent to computing $S := \sum_{i=1}^{n} b_i$ where n is the total number of voters who sent in their encrypted votes. You may assume that n is at most 10^9.

(a) Suppose the election service is partitioned into two components, a tabulation service and a decryption authority. Incoming votes are received by the tabulation service and the decryption authority is an offline box that holds sk and only communicates with the tabulation service. Show that the tabulation service can send a *single* ElGamal ciphertext c^* to the decryption authority who then decrypts c^* and outputs S in the clear. If both parties are honestly following your protocol then neither one learns anything other than S about the individual votes. Explain how the tabulation service constructs c^*.

Hint: Use Exercise 11.5 part (c).

(b) Show that a single malicious voter can make S come out to be whatever value that voter wants.

Discussion: While part (b) shows that this voting system is insecure as is, this idea can form the basis of a secure election system. See [28] for details.

Chapter 12

Chosen ciphertext secure public key encryption

In Chapter 11, we introduced the notion of public-key encryption. We also defined a basic form of security called *semantic security*, which is completely analogous to the corresponding notion of semantic security in the symmetric-key setting. We observed that in the public-key setting, semantic security implies security against a chosen plaintext attack, i.e., CPA security.

In this chapter, we study the stronger notion of security against chosen ciphertext attack, or **CCA security**. In the CPA attack game, the decryption key is never used, and so CPA security provides no guarantees in any real-world setting in which the decryption key is actually used to decrypt messages. The notion of CCA security is designed to model a wide spectrum of real-world attacks, and it is considered the "gold standard" for security in the public-key setting.

We briefly introduced the notion of CCA security in the symmetric-key setting in Section 9.2, and the definition in the public-key setting is a straightforward translation of the definition in the symmetric-key setting. However, it turns out CCA security plays a more fundamental role in the public-key setting than in the symmetric-key setting.

12.1 Basic definitions

As usual, we formulate this notion of security using an attack game, which is a straightforward adaptation of the CCA attack game in the symmetric settings (Attack Game 9.2) to the public-key setting.

Attack Game 12.1 (CCA security). For a given public-key encryption scheme $\mathcal{E} = (G, E, D)$, defined over $(\mathcal{M}, \mathcal{C})$, and for a given adversary $\mathcal{A}$, we define two experiments.

Experiment b **(**$b = 0, 1$**):**

- The challenger computes $(pk, sk) \xleftarrow{R} G()$ and sends pk to the adversary.
- $\mathcal{A}$ then makes a series of queries to the challenger. Each query can be one of two types:
  - *Encryption query:* for $i = 1, 2, \ldots$, the ith encryption query consists of a pair of messages $(m_{i0}, m_{i1}) \in \mathcal{M}^2$, of the same length. The challenger computes $c_i \xleftarrow{R} E(pk, m_{ib})$ and sends c_i to $\mathcal{A}$.

– *Decryption query:* for $j = 1, 2, \ldots$, the jth decryption query consists of a ciphertext $\hat{c}_j \in \mathcal{C}$ that is not among the responses to the previous encryption queries, i.e.,

$$\hat{c}_j \notin \{c_1, c_2, \ldots\}.$$

The challenger computes $\hat{m}_j \leftarrow D(sk, \hat{c}_j)$, and sends $\hat{m}_j$ to $\mathcal{A}$.

- At the end of the game, the adversary outputs a bit $\hat{b} \in \{0, 1\}$.

Let W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b and define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}$ as

$$\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] := \big|\Pr[W_0] - \Pr[W_1]\big|. \quad \Box$$

Definition 12.1 (CCA Security). *A public-key encryption scheme $\mathcal{E}$ is called* ***semantically secure against a chosen ciphertext attack****, or simply* ***CCA secure****, if for all efficient adversaries $\mathcal{A}$, the value* $\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

Just as we did in the symmetric-key setting, we can consider a restricted attack game in which the adversary makes only a single encryption query:

Definition 12.2 (1CCA security). *In Attack Game 12.1, if the adversary $\mathcal{A}$ is restricted to making a single encryption query, we denote its advantage by* $1\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$*. A public-key encryption scheme $\mathcal{E}$ is* ***one-time semantically secure against chosen ciphertext attack****, or simply,* ***1CCA secure****, if for all efficient adversaries $\mathcal{A}$, the value* $1\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$ *is negligible.*

Notice that if we strip away the decryption queries, 1CCA security corresponds to semantic security, and CCA security corresponds to CPA security. We showed in Theorem 11.1 that semantic security for a public-key encryption scheme implies CPA security. A similar result holds with respect to chosen ciphertext security, namely, that 1CCA security implies CCA security.

Theorem 12.1. *If a public-key encryption scheme $\mathcal{E}$ is 1CCA secure, then it is also CCA secure.*

In particular, for every CCA adversary $\mathcal{A}$ that plays Attack Game 12.1 with respect to $\mathcal{E}$, and which makes at most Q_e encryption queries to its challenger, there exists a 1CCA adversary $\mathcal{B}$ as in Definition 12.2, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = Q_e \cdot 1\text{CCA}\mathsf{adv}[\mathcal{B}, \mathcal{E}].$$

The proof is a simple hybrid argument that is almost identical to that of Theorem 11.1, and we leave the details as an easy exercise to the reader. Using another level of hybrid argument, one can also extend this to the multi-key setting as well — see Exercise 12.5.

Since 1CCA security implies CCA security, if we want to prove that a particular public-key encryption scheme is CCA secure, we will typically simply prove 1CCA security. So it will be helpful to study the 1CCA attack game in a bit more detail. We can view the 1CCA attack game as proceeding in a series of phases:

Initialization phase: the challenger generates $(pk, sk) \xleftarrow{R} G()$ and sends pk to the adversary.

Phase 1: the adversary submits a series of decryption queries to the challenger; each such query is a ciphertext $\hat{c} \in \mathcal{C}$, to which the challenger responds with $\hat{m} \leftarrow D(sk, \hat{c})$.

Encryption query: the adversary submits a single encryption query (m_0, m_1) to the challenger; in Experiment b (where $b = 0, 1$), the challenger responds with $c \xleftarrow{R} E(pk, m_b)$.

Phase 2: the adversary again submits a series of decryption queries to the challenger; each such query is a ciphertext $\hat{c} \in \mathcal{C}$, subject to the restriction that $\hat{c} \neq c$, to which the challenger responds with $\hat{m} \leftarrow D(sk, \hat{c})$.

Finish: at the end of the game, the adversary outputs a bit $\hat{b} \in \{0, 1\}$.

As usual, as discussed in Section 2.3.5, Attack Game 12.1 can be recast as a "bit guessing" game, where instead of having two separate experiments, the challenger chooses $b \in \{0, 1\}$ at random, and then runs Experiment b against the adversary $\mathcal{A}$. In this game, we measure $\mathcal{A}$'s *bit-guessing advantage* $\mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]$ (and $1\mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]$) as $|\Pr[\hat{b} = b] - 1/2|$. The general result of Section 2.3.5 applies here as well:

$$\mathrm{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 2 \cdot \mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]. \tag{12.1}$$

And similarly, for adversaries restricted to a single encryption query, we have:

$$1\mathrm{CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] = 2 \cdot 1\mathrm{CCA}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}]. \tag{12.2}$$

12.2 Understanding CCA security

The definition of CCA security may seem rather unintuitive at first. Indeed, one might ask: in the attack game, why can the adversary get any message decrypted except the ones he really wants to decrypt? One answer is that without this restriction, it would be impossible to satisfy the definition. However, this is not a very satisfying answer, and it begs the question as to whether the entire definitional framework makes sense.

In this section, we explore the definition of CCA security from several angles. Hopefully, by the end, the reader will understand why this definition makes sense, and what it is good for.

12.2.1 CCA security and ciphertext malleability

Our first example illustrates an important property of CCA secure systems: they are **non-malleable**. That is, given an encryption c of some message m, the attacker cannot create a different ciphertext c' that decrypts to a message m' that is somehow related to m. The importance of this will become clear in the example below.

Consider a professor, Bob, who collects homework by email. Moreover, assume that Bob generates a public key/secret key pair (pk, sk) for a public-key encryption scheme, and gives pk to all of his students. When a student Alice submits an email, she encrypts it under pk.

To make things concrete, suppose that the public-key encryption scheme is the semantically secure scheme $\mathcal{E}_{\mathrm{TDF}}$ presented in Section 11.4, which is based on a trapdoor function along with some symmetric cipher $\mathcal{E}_{\mathrm{s}}$. The only requirement on $\mathcal{E}_{\mathrm{s}}$ is that it is semantically secure, so let us assume that $\mathcal{E}_{\mathrm{s}}$ is a stream cipher (such as AES in counter mode).

When Alice encrypts the email message m containing her homework using $\mathcal{E}_{\mathrm{TDF}}$ and pk, the resulting ciphertext is of the form (y, c), where $y = F(pk, x)$ and $c = G(H(x)) \oplus m$. Here, H is a hash function and G is a PRG.

As we saw in Section 3.3.2, any stream cipher is extremely *malleable*, and the public-key scheme $\mathcal{E}_{\text{TDF}}$ inherits this weakness. In particular, an attacker Molly can do essentially the same thing here as she did in Section 3.3.2. Namely, assuming that Alice's email message m starts with the header `From:Alice`, by flipping a few bits of the symmetric-key ciphertext c, Molly obtains another ciphertext c' that decrypts (under the same symmetric key) to a message m' that is identical to m, except that the header now reads `From:Molly`.

Using the above technique, Molly can "steal" Alice's homework as follows. She intercepts Alice's ciphertext (y, c). She then modifies the symmetric-key ciphertext c to obtain c' as above, and sends the public-key ciphertext (y, c') to Bob. Now, when Professor Bob decrypts (y, c'), he will essentially see Alice's homework, but Bob will mistakenly think that the homework was submitted by Molly, and give Molly credit for it.

The attack described so far is a good example of a chosen ciphertext attack, which could not succeed if the public-key encryption scheme were actually CCA secure. Indeed, if given (y, c) it is possible for Molly to create a new ciphertext (y, c') where the header `From:Alice` is changed to `From:Molly`, then the system cannot be CCA secure. For such a system, we can design a simple CCA adversary $\mathcal{A}$ that has advantage 1 in the CCA security game. Here is how.

- Create a pair of messages, each with the same header, but different bodies. Our adversary $\mathcal{A}$ submits this pair as an encryption query, obtaining (y, c).
- $\mathcal{A}$ then uses Molly's algorithm to create a ciphertext (y, c'), which should encrypt a message with a different header but the same body.
- $\mathcal{A}$ then submits (y, c') as a decryption query, and outputs 0 or 1, depending on which body it sees.

As we have shown, if Alice encrypts her homework using a CCA-secure system, she is assured that no one can steal her homework by modifying the ciphertext she submitted. CCA security, however, does not prevent all attacks on this homework submission system. An attacker can maliciously submit a homework on behalf of Alice, and possibly hurt her grade in the class. Indeed, anyone can send an encrypted homework to the professor, and in particular, a homework that begins with `From:Alice`. Preventing this type of attack requires tools that we will develop later. In Section 13.7, where we develop the notion of signcryption, which is one way to prevent this attack.

12.2.2 CCA security vs authentication

When we first encountered the notion of CCA security in the symmetric-key setting, back in Section 9.2, we saw that CCA security was implied by AE security, i.e., ciphertext integrity plus CPA security. Moreover, we saw that ciphertext integrity could be easily added to any CPA-secure encryption scheme using the encrypt-then-MAC method. We show here that this does not work in the public-key setting: simply adding an authentication wrapper does not make the system CCA secure.

Consider again the homework submission system example in the previous section. If we start with a scheme, like $\mathcal{E}_{\text{TDF}}$, which is not itself CCA secure, we might hope to make it CCA secure using encrypt-then-MAC: Alice wraps the ciphertext (y, c) with some authentication data computed from (y, c). Say, Alice computes a MAC tag t over (y, c) using a secret key that she shares with Bob and sends (y, c, t) to Bob (or, instead of a MAC, she computes a digital signature on (y, c), a concept discussed in Chapter 13). Bob can check the authentication data to make sure the ciphertext was generated by Alice. However, regardless of the authentication wrapper used, Molly can still carry out the attack described in the previous section. Here is how. Molly intercepts Alice's ciphertext (y, c, t), and computes (y, c') exactly as before. Now, since Molly is a registered student in Bob's course, she presumably is using the same authentication mechanism as all other students, so she simply computes her own authentication tag t' on ciphertext (y, c') and sends (y, c', t') to Bob. Bob receives (y, c', t'), and believes the authenticity of the ciphertext. When Bob decrypts (y, c'), the header `From:Molly` will look perfectly consistent with the authentication results.

What went wrong? Why did the strategy of authenticating ciphertexts provide us with CCA security in the symmetric-key setting, but not in the public-key setting? The reason is simply that in the public-key setting, anyone is allowed to send an encrypted message to Bob using Bob's public key. The added flexibility that public-key encryption provides makes it more challenging to achieve CCA security, yet CCA security is vital for security in real-world systems. (We will discuss in detail how to securely combine CCA-secure public-key encryption and digital signatures when we discuss signcryption in Section 13.7.)

12.2.3 CCA security and key escrow

Consider again the key escrow example discussed in Section 11.1.2. Recall that in that example, Alice encrypts a file f using a symmetric key k. Among other things, Alice stores along with the encrypted file an *escrow* of the file's encryption key. Here, the escrow is an encryption c_{ES} of k under the public key of some *escrow service*. If Alice works for some company, then if need be, Alice's manager or other authorized entity can retrieve the file's encryption key by presenting c_{ES} to the escrow service for decryption.

If the escrow service uses a CCA-secure encryption scheme, then it is possible to implement an *access control policy* which can mitigate against potential abuse. This can be done as follows. Suppose that in forming the escrow-ciphertext c_{ES}, Alice encrypts the pair (k, h) under the escrow service's public key, where h is a collision-resistant hash of the metadata md associated with the file f: this might include the name of the file, the time that it was created and/or modified, and perhaps the identity of the owner of the file (Alice, in this case). Let us also assume that all of this metadata md is stored on the file system in the clear along with the encrypted file.

Now suppose a requesting entity presents the escrow-ciphertext c_{ES} to the escrow service, along with the corresponding metadata md. The escrow service may impose some type of access control policy, based on the given metadata, along with the identity or credentials of the requesting entity. Such a policy could be very specific to a particular company or organization. For example, the requesting entity may be Alice's manager, and it is company policy that Alice's manager should have access to all files owned by Alice. Or the requesting entity may be an external auditor that is to have access to all files created by certain employees on a certain date.

To actually enforce this access control policy, not only must the escrow service verify that the requesting identity's credentials and the supplied metadata conform to the access control policy, the escrow service must also perform the following check: after decrypting the escrow-ciphertext c_{ES} to obtain the pair (k, h), it must check that h matches the hash of the metadata supplied by the requesting entity. Only if these match does the escrow service release the key k to the requesting entity.

This type of access control can prevent certain abuses. For example, consider the external auditor who has the right to access all files created by certain employees on a certain date. Suppose the auditor himself is a bit too nosy, and during the audit, wants to find out some information in a personal file of Alice that is not one of the files targeted by the audit. The above implementation of the escrow service, along with CCA security, ensures that the nosy auditor cannot obtain this unauthorized information. Indeed, suppose c_{ES} is the escrow-ciphertext associated with Alice's personal file, which is not subject to the audit, and that this file has metadata md. Suppose the auditor submits a pair (c'_{ES}, md') to the escrow service. There are several cases to consider:

- if $md' = md$, then the escrow service will reject the request, as the metadata md of Alice's personal file does not fit the profile of the audit;
- if $md' \neq md$ and $c'_{\text{ES}} = c_{\text{ES}}$, then the collision resistance of the hash ensures that the escrow service will reject the request, as the hash embedded in the decryption of c'_{ES} will not not match the hash of the supplied metadata md';
- if $md' \neq md$ and $c'_{\text{ES}} \neq c_{\text{ES}}$, then the escrow service may or may not accept the request, but even if it does, CCA security and the fact that $c'_{\text{ES}} \neq c_{\text{ES}}$ ensures that no information about the encryption key for Alice's personal file is revealed.

This implementation of an escrow service is pretty good, but it is far from perfect:

- It assumes that Alice follows the protocol of actually encrypting the file encryption key along with the correct metadata. Actually, this may not be such an unreasonable assumption, as these tasks will be performed automatically by the file system on Alice's behalf, and so it may not be so easy for a misbehaving Alice to circumvent this protocol.
- It assumes that the requesting entity and the escrow service do not collude.

Treating the metadata as associated data. In Section 12.7 we define public-key encryption with associated data, which is the public-key analogue of symmetric encryption with associated data from Section 9.5. Here the public-key encryption and decryption algorithms take a third input called associated data. The point is that decryption reveals no useful information if the given associated data used in decryption is different from the one used in encryption.

The metadata information md in the escrow system above can be treated as associated data, instead of appending it to the plaintext. This will result in a smaller ciphertext while achieving the same security goals. In fact, associating metadata to a ciphertext for the purpose described above is a very typical application of associated data in a public-key encryption scheme.

12.2.4 Encryption as an abstract interface

To conclude our motivational discussion of CCA security we show that it abstractly captures a "correct" and very natural notion of security. We do this by describing encryption as an abstract interface, as discussed in Section 9.3 in the symmetric case.

The setting is as follows. We have a sender S and receiver R, who are participating in some protocol, during which S drops messages $m_1, m_2, \ldots$ into his out-box, and R retrieves messages from his in-box. While S and R do not share a secret key, we assume that R has generated public key/secret key pair (pk, sk), and that S knows R's public key pk.

That is the abstract interface. In a real implementation, when m_i is placed in S's out-box, it is encrypted under pk, yielding a corresponding ciphertext c_i, which is sent over the wire to R. On the receiving end, when a ciphertext $\hat{c}$ is received at R's end of the wire, it is decrypted using sk, and if the decryption is a message $\hat{m} \neq \mathsf{reject}$, the message $\hat{m}$ is placed in R's in-box.

Note that while we are syntactically restricting ourselves to a single sender S, this restriction is superficial: in system with many users, all of them have access to R's public key, and so we can model such a system by allowing all users to place messages in S's out-box.

Just as in Section 9.3, an attacker may attempt to subvert communication in several ways:

- The attacker may drop, re-order, or duplicate the ciphertexts sent by S.
- The attacker may modify ciphertexts sent by S, or inject ciphertexts computed in some arbitrary fashion.
- The attacker may have partial knowledge — or even influence the choice — of the messages sent by S.
- The attacker can obtain partial knowledge of some of the messages retrieved by R, and determine if a given ciphertext delivered to R was rejected.

We now describe an *ideal implementation* of this interface. It is slightly different from the ideal implementation in Section 9.3 — in that section, we were working with the notion of AE security, while here we are working with the notion of CCA security. When S drops m_i in its out-box, instead of encrypting m_i, the ideal implementation creates a ciphertext c_i by encrypting a dummy message $dummy_i$, that has nothing to do with m_i (except that it should be of the same length). Thus, c_i serves as a "handle" for m_i, but does not contain any information about m_i (other than its length). When c_i arrives at R, the corresponding message m_i is magically copied from S's out-box to R's in-box. If a ciphertext $\hat{c}$ arrives at R that is not among the previously generated c_i's, the ideal implementation *decrypts $\hat{c}$ using sk as usual*.

CCA security implies that this ideal implementation of the service is for all practical purposes equivalent to the real implementation. In the ideal implementation, we see that messages magically jump from S to R, in spite of any information the adversary may glean by getting R to decrypt other ciphertexts — the ciphertexts generated by S in the ideal implementation serve simply as handles for the corresponding messages, but do not carry any other useful information. Hopefully, analyzing the security properties of a higher-level protocol will be much easier using this ideal implementation.

Note that even in the ideal implementation, the attacker may still drop, re-order, or duplicate ciphertexts, and these will cause the corresponding messages to be dropped, re-ordered, or duplicated. A higher-level protocol can easily take measures to deal with these issues.

We now argue informally that when $\mathcal{E}$ is CCA secure, the real world implementation is indistinguishable from the ideal implementation. The argument is similar to that in Section 9.3. It proceeds in two steps, starting with the real implementation, and in each step, we make a slight modification.

- First, we modify the real implementation of R's in-box, as follows. When a ciphertext $\hat{c}$ arrives on R's end, the list of ciphertexts $c_1, c_2, \ldots$ previously generated by S is scanned, and if $\hat{c} = c_i$, then the corresponding message m_i is magically copied from S's out-box into R's in-box, without actually running the decryption algorithm.

  The correctness property of $\mathcal{E}$ ensures that this modification behaves exactly the same as the real implementation. Note that in this modification, any ciphertext that arrives at R's end that is not among the ciphertexts previously generated by S will be decrypted as usual using sk.

- Second, we modify the implementation of S's out-box, replacing the encryption of m_i with the encryption of $dummy_i$. The implementation of R's in-box remains as in the first modification.

  Here is where we use the CCA security property: if the attacker could distinguish the second modification from the first, we could use the attacker to break the CCA security of $\mathcal{E}$.

Since the second modification is identical to the ideal implementation, we see that the real and ideal implementations are indistinguishable from the adversary's point of view.

Just as in Section 9.3, we have ignored the possibility that the c_i's generated by S are not unique. Certainly, if we are going to view the c_i's as handles in the ideal implementation, uniqueness would seem to be an essential property. Just as in the symmetric case, CPA security (which is implied by CCA security) guarantees that the c_i's are unique with overwhelming probability (the reader can verify that the result of Exercise 5.11 holds in the public-key setting as well).

12.3 CCA-secure encryption from trapdoor function schemes

We now turn to constructing CCA-secure public-key encryption schemes. We begin with a construction from a general trapdoor function scheme satisfying certain properties. We use this to obtain a CCA-secure system from RSA. Later, in Section 12.6, we will show how to construct suitable trapdoor functions (in the random oracle model) from arbitrary, CPA-secure public-key encryption schemes. Using the result in this section, all these trapdoor functions give us CCA-secure encryption schemes.

Consider again the public-key encryption scheme $\mathcal{E}_{\mathrm{TDF}} = (G, E, D)$ discussed in Section 11.4, which is based on an arbitrary trapdoor function scheme $\mathcal{T} = (G, F, I)$, defined over $(\mathcal{X}, \mathcal{Y})$. Let us briefly recall this scheme: it makes use of a symmetric cipher $\mathcal{E}_{\mathrm{s}} = (E_{\mathrm{s}}, D_{\mathrm{s}})$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and a hash function $H : \mathcal{X} \to \mathcal{K}$, which we model as a random oracle. The message space for $\mathcal{E}_{\mathrm{TDF}}$ is $\mathcal{M}$ and the ciphertext space is $\mathcal{Y} \times \mathcal{C}$. The key generation algorithm for $\mathcal{E}_{\mathrm{TDF}}$ is the same as the key generation algorithm for $\mathcal{T}$, and encryption and decryption work as follows:

$$
\begin{aligned}
E(pk, m) \quad &:= \quad x \xleftarrow{\mathrm{R}} \mathcal{X},\ y \leftarrow F(pk, x),\ k \leftarrow H(x),\ c \xleftarrow{\mathrm{R}} E_{\mathrm{s}}(k, m) \\
&\qquad\ \text{output } (y, c); \\
D(sk,\ (y, c)\ ) \quad &:= \quad x \leftarrow I(sk, y),\ k \leftarrow H(x),\ m \leftarrow D_{\mathrm{s}}(k, c) \\
&\qquad\ \text{output } m.
\end{aligned}
$$

If $\mathcal{X} \neq \mathcal{Y}$, that is, if $\mathcal{T}$ is not a trapdoor permutation scheme, we have to modify the scheme slightly to get a scheme that is CCA secure. Basically, we modify the decryption algorithm to explicitly check that the given value $y \in \mathcal{Y}$ is actually in the image of $F(pk, \cdot)$. So the scheme we will analyze is $\mathcal{E}'_{\mathrm{TDF}} = (G, E, D')$, where

$$
\begin{aligned}
D'(sk,\ (y, c)\ ) \quad := \quad & x \leftarrow I(sk, y) \\
& \text{if } F(pk, x) = y \\
& \qquad \text{then } k \leftarrow H(x),\ m \leftarrow D_{\mathrm{s}}(k, c) \\
& \qquad \text{else } \ m \leftarrow \textsf{reject} \\
& \text{output } m.
\end{aligned}
$$

We will prove that $\mathcal{E}'_{\mathrm{TDF}}$ is CCA secure if we model H as a random oracle, under appropriate assumptions. The first assumption we will make is that $\mathcal{E}_{\mathrm{s}}$ is 1CCA secure (see Section 9.6). We also have to assume that $\mathcal{T}$ is one-way. However, when $\mathcal{X} \neq \mathcal{Y}$, we need a somewhat stronger assumption: that $\mathcal{T}$ is one-way even given access to an "image oracle". Essentially, this means that given pk and $y = F(pk, x)$ for randomly chosen $x \in \mathcal{X}$, it is hard to compute x, even given access to an oracle that will answer arbitrary questions of the form "does a given $\hat{y} \in \mathcal{Y}$ lie in the image of $F(pk, \cdot)$?". We formalize this notion by giving an attack game that is similar to Attack Game 10.2, but where the adversary has access to an image oracle.

Attack Game 12.2 (One-way trapdoor function scheme even with image oracle). For a given trapdoor function scheme $\mathcal{T} = (G, F, I)$, defined over $(\mathcal{X}, \mathcal{Y})$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger computes

$$(pk, sk) \xleftarrow{\mathrm{R}} G(), \quad x \xleftarrow{\mathrm{R}} \mathcal{X}, \quad y \leftarrow F(pk, x)$$

  and sends (pk, y) to the adversary.

- The adversary makes a series of *image oracle queries* to the challenger. Each such query is of the form $\hat{y} \in \mathcal{Y}$, to which the challenger replies "yes" if $F(pk, I(sk, \hat{y})) = \hat{y}$, and "no" otherwise.

- The adversary outputs $\hat{x} \in \mathcal{X}$.

We define the adversary's advantage in inverting $\mathcal{T}$ given access to an image oracle, denoted $\mathrm{IOW}\mathsf{adv}[\mathcal{A}, \mathcal{T}]$, to be the probability that $\hat{x} = x$. $\square$

Definition 12.3. *We say that a trapdoor function scheme $\mathcal{T}$ is* ***one way given an image oracle*** *if for all efficient adversaries $\mathcal{A}$, the quantity $\mathrm{IOW}\mathsf{adv}[\mathcal{A}, \mathcal{T}]$ is negligible.*

In Exercise 12.13 we show that (in the random oracle model) every one way trapdoor function scheme can be easily converted into one that is one way given an image oracle.

The next theorem proves the CCA security of $\mathcal{E}'_{\mathrm{TDF}}$, assuming $\mathcal{T}$ is one-way given an image oracle, $\mathcal{E}_{\mathrm{s}}$ is 1CCA secure (see Definition 9.6), and H is modeled as a random oracle. In Exercise 12.12 we explore an alternative analysis of this scheme under different assumptions.

In proving this theorem, we just prove that $\mathcal{E}'_{\mathrm{TDF}}$ is 1CCA secure (see Definition 12.2). By virtue of Theorem 12.1, this is sufficient. Recall that in the random oracle model (see Section 8.10), the function H is modeled as a random function $\mathcal{O}$ chosen at random from the set of all functions $\mathrm{Funs}[\mathcal{X}, \mathcal{K}]$. This means that in the random oracle version of the 1CCA attack game, the challenger chooses $\mathcal{O}$ at random. In any computation where the challenger would normally evaluate H, it evaluates $\mathcal{O}$ instead. In addition, the adversary is allowed to ask the challenger for the value of the function $\mathcal{O}$ at any point of its choosing. The adversary may make any number of such "random oracle queries" at any time of its choosing, arbitrarily interleaved with its usual encryption and decryption queries. We use $1\mathrm{CCA}^{\mathrm{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}'_{\mathrm{TDF}}]$ to denote $\mathcal{A}$'s advantage against $\mathcal{E}'_{\mathrm{TDF}}$ in the random oracle version of the 1CCA attack game.

Theorem 12.2. *Assume $H : \mathcal{X} \to \mathcal{K}$ is modeled as a random oracle. If $\mathcal{T}$ is one-way given an image oracle, and $\mathcal{E}_{\mathrm{s}}$ is 1CCA secure, then $\mathcal{E}'_{\mathrm{TDF}}$ is CCA secure.*

In particular, for every 1CCA adversary $\mathcal{A}$ that attacks $\mathcal{E}'_{\text{TDF}}$ as in the random oracle version of Definition 12.2, there exist an inverting adversary $\mathcal{B}_{\text{iow}}$ that breaks the one-wayness assumption for $\mathcal{T}$ as in Attack Game 12.2, and a 1CCA adversary $\mathcal{B}_{\text{s}}$ that attacks $\mathcal{E}_{\text{s}}$ as in Definition 9.6, where $\mathcal{B}_{\text{iow}}$ and $\mathcal{B}_{\text{s}}$ are elementary wrappers around $\mathcal{A}$, such that

$$\text{1CCA}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}'_{\text{TDF}}] \leq 2 \cdot \text{IOW}\mathsf{adv}[\mathcal{B}_{\text{iow}}, \mathcal{T}] + \text{1CCA}\mathsf{adv}[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}]. \tag{12.3}$$

For applications of this theorem in the sequel, we record here some further technical properties that the adversary $\mathcal{B}_{\text{iow}}$ satisfies.

If $\mathcal{A}$ makes at most Q_{d} decryption queries, then $\mathcal{B}_{\text{iow}}$ makes at most Q_{d} image-oracle queries. Also, the only dependence of $\mathcal{B}_{\text{iow}}$ on the function F is that it invokes $F(pk, \cdot)$ as a subroutine, at most Q_{ro} times, where Q_{ro} is a bound on the number of random-oracle queries made by $\mathcal{A}$; moreover, if $\mathcal{B}_{\text{iow}}$ produces an output $\hat{x}$, it always evaluates $F(pk, \cdot)$ at $\hat{x}$.

Proof idea. The crux of the proof is to show that the adversary's decryption queries do not help him in any significant way. What this means technically is that we have to modify the challenger so that it can compute responses to the decryption queries *without using the secret key sk*. The trick to achieve this is to exploit the fact that our challenger is in charge of implementing the random oracle, maintaining a table of all input/output pairs. Assume the target ciphertext (i.e., the one resulting from the encryption query) is (y, c), where $y = F(pk, x)$, and suppose the challenger is given a decryption query $(\hat{y}, \hat{c})$, where $y \neq \hat{y} = F(pk, \hat{x})$.

- If the adversary has previously queried the random oracle at $\hat{x}$, and if $\hat{k}$ was the output of the random oracle at $\hat{x}$, then the challenger simply decrypts $\hat{c}$ using $\hat{k}$.

- Otherwise, if the adversary has not made such a random oracle query, then the challenger does not know the correct value of the symmetric key — but neither does the adversary. The challenger is then free to choose a key $\hat{k}$ at random, and decrypt $\hat{c}$ using this key; however, the challenger must do some extra book-keeping to ensure consistency, so that if the adversary ever queries the random oracle in the future at the point $\hat{x}$, then the challenger "back-patches" the random oracle, so that its output at $\hat{x}$ is set to $\hat{k}$.

We also have to deal with decryption queries of the form $(y, \hat{c})$, where $\hat{c} \neq c$. Intuitively, under the one-wayness assumption for $\mathcal{T}$, the adversary will never query the random oracle at x, and so from the adversary's point of view, the symmetric key k used in the encryption query, and used in decryption queries of the form $(y, \hat{c})$, is as good as random, and so CCA security for $\mathcal{E}'_{\text{TDF}}$ follows immediately from 1CCA security for $\mathcal{E}_{\text{s}}$.

In the above, we have ignored ciphertext queries of the form $(\hat{y}, \hat{c})$ where $\hat{y}$ has no preimage under $F(pk, \cdot)$. The real decryption algorithm rejects such queries. This is why we need to assume $\mathcal{T}$ is one-way given an image oracle — in the reduction, we need this image oracle to reject ciphertexts of this form. □

Proof. It is convenient to prove the theorem using the bit-guessing versions of the 1CCA attack games. We prove:

$$\text{1CCA}^{\text{ro}}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}'_{\text{TDF}}] \leq \text{IOW}\mathsf{adv}[\mathcal{B}_{\text{iow}}, \mathcal{T}] + \text{1CCA}\mathsf{adv}^*[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}]. \tag{12.4}$$

Then (12.3) follows by (12.2) and (9.2).

As usual, we define Game 0 to be the game played between $\mathcal{A}$ and the challenger in the bit-guessing version of the 1CCA attack game with respect to $\mathcal{E}'_{\mathrm{TDF}}$. We then modify the challenger to obtain Game 1. In each game, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. Also, for $j = 0, 1$, we define W_j to be the event that $\hat{b} = b$ in Game j.

Game 0. The logic of the challenger is shown in Fig. 12.1. The challenger has to respond to random oracle queries, in addition to encryption and decryption queries. The adversary can make any number of random oracle queries, and any number of decryption queries, but at most one encryption query. Recall that in addition to direct access to the random oracle via explicit random oracle queries, the adversary also has indirect access to the random oracle via the encryption and decryption queries, where the challenger also makes use of the random oracle. In the initialization step, the challenger computes $(pk, sk) \xleftarrow{R} G()$; we also have our challenger make those computations associated with the encryption query that can be done without yet knowing the challenge plaintext. To facilitate the proof, we want our challenger to use the secret key sk as little as possible in processing decryption queries. This will motivate a somewhat nontrivial strategy for implementing the decryption and random oracle queries.

As usual, we will make use of an associative array to implement the random oracle. In the proof of Theorem 11.2, which analyzed the semantic security of $\mathcal{E}_{\mathrm{TDF}}$, we did this quite naturally by using an associative array $Map : \mathcal{X} \to \mathcal{K}$. We could do the same thing here, but because we want our challenger to use the secret key as little as possible, we adopt a different strategy. Namely, we will represent the random oracle using associative array $Map' : \mathcal{Y} \to \mathcal{K}$, with the convention that if the value of the oracle at $\hat{x} \in \mathcal{X}$ is equal to $\hat{k} \in \mathcal{K}$, then $Map'[\hat{y}] = \hat{k}$, where $\hat{y} = F(pk, \hat{x})$. We will also make use of an associative array $Pre : \mathcal{Y} \to \mathcal{X}$ that is used to track explicit random oracle queries made by the adversary: if $Pre[\hat{y}] = \hat{x}$, this means that the adversary queried the oracle at the point $\hat{x}$, and $\hat{y} = F(pk, \hat{x})$. Note that Map' will in general be defined at points other than those at which Pre is defined, since the challenger also makes random oracle queries.

In preparation for the encryption query, in the initialization step, the challenger precomputes $x \xleftarrow{R} \mathcal{X}$, $y \leftarrow F(pk, x)$, $k \xleftarrow{R} \mathcal{K}$. It also sets $Map'[y] \leftarrow k$, which means that the value of the random oracle at x is equal to k. Also note that in the initialization step, the challenger sets $c \leftarrow \perp$, and in processing the encryption query, overwrites c with a ciphertext in $\mathcal{C}$. Thus, decryption queries processed while $c = \perp$ are phase 1 queries, while those processed while $c \neq \perp$ are phase 2 queries.

To process a decryption query $(\hat{y}, \hat{c})$, making minimal use of the secret key, the challenger uses the following strategy.

- If $\hat{y} = y$, the challenger just uses the prepared key k directly to decrypt $\hat{c}$.

- Otherwise, the challenger checks if Map' is defined at the point $\hat{y}$, and if not, it assigns to $Map'[\hat{y}]$ a random value $\hat{k}$. If $\hat{y}$ has a preimage $\hat{x}$ and Map' was not defined at $\hat{y}$, this means that neither the adversary nor the challenger previously queried the random oracle at $\hat{x}$, and so this new random value $\hat{k}$ represents the value or the random oracle at $\hat{x}$; in particular, if the adversary later queries the random oracle at the point $\hat{x}$, this same value of $\hat{k}$ will be used. If $\hat{y}$ has no preimage, then assigning $Map'[\hat{y}]$ a random value $\hat{k}$ has no real effect — it just streamlines the logic a bit.

- Next, the challenger tests if $\hat{y}$ is in the image of $F(pk, \cdot)$. If $\hat{y}$ is not in the image, the challenger just rejects the ciphertext. In Fig. 12.1, we implement this by invoking the function

initialization:

$(pk, sk) \xleftarrow{R} G()$, $x \xleftarrow{R} \mathcal{X}$, $y \leftarrow F(pk, x)$
$c \leftarrow \perp$
initialize empty associative arrays $Pre : \mathcal{Y} \to \mathcal{X}$ and $Map' : \mathcal{Y} \to \mathcal{K}$
$k \xleftarrow{R} \mathcal{K}$, $b \xleftarrow{R} \{0, 1\}$
(1) $Map'[y] \leftarrow k$
send the public key pk to $\mathcal{A}$;

upon receiving an encryption query $(m_0, m_1) \in \mathcal{M}^2$:

$b \xleftarrow{R} \{0, 1\}$, $c \xleftarrow{R} E_s(k, m_b)$, send (y, c) to $\mathcal{A}$;

upon receiving a decryption query $(\hat{y}, \hat{c}) \in \mathcal{X} \times \mathcal{C}$, where $(\hat{y}, \hat{c}) \neq (y, c)$:

if $\hat{y} = y$ then
  $\hat{m} \leftarrow D_s(k, \hat{c})$
else
  if $\hat{y} \notin \text{Domain}(Map')$ then $Map'[\hat{y}] \xleftarrow{R} \mathcal{K}$
(2) if $Image(pk, sk, \hat{y}) =$ "no" // *i.e., $\hat{y}$ is not in the image of $F(pk, \cdot)$*
    then $\hat{m} \leftarrow$ reject
    else $\hat{k} \leftarrow Map'[\hat{y}]$, $\hat{m} \leftarrow D_s(\hat{k}, \hat{c})$
send $\hat{m}$ to $\mathcal{A}$;

upon receiving a random oracle query $\hat{x} \in \mathcal{X}$:

$\hat{y} \leftarrow F(pk, \hat{x})$, $Pre[\hat{y}] \leftarrow \hat{x}$
if $\hat{y} \notin \text{Domain}(Map')$ then $Map'[\hat{y}] \xleftarrow{R} \mathcal{K}$
send $Map'[\hat{y}]$ to $\mathcal{A}$

Figure 12.1: Game 0 challenger

$Image(pk, sk, \hat{y})$. For now, we can think of $Image$ as being implemented as follows:

$$Image(pk, sk, \hat{y}) \quad := \quad \Big\{ \text{ return "yes" if } F(pk, I(sk, \hat{y})) = \hat{y} \text{ and "no" otherwise } \Big\}.$$

This is the only place where our challenger makes use of the secret key.

- Finally, if $\hat{y}$ is in the range of $F(pk, \cdot)$, the challenger simply decrypts $\hat{c}$ directly using the symmetric key $\hat{k} = Map'[\hat{y}]$, which at this point is guaranteed to be defined, and represents the value of the random oracle at the preimage $\hat{x}$ of $\hat{y}$. Note that our challenger can do this, *without actually knowing $\hat{x}$.* This is the crux of the proof.

Despite this somewhat involved bookkeeping, it should be clear that our challenger behaves *exactly* as in the usual attack game.

Game 1. This game is precisely the same as Game 0, except that we delete the line marked (1) in Fig. 12.1.

Let Z be the event that the adversary queries the random oracle at x in Game 1. Clearly, Games 0 and 1 proceed identically unless Z occurs, and so by the Difference Lemma, we have

$$|\Pr[W_1] - \Pr[W_0]| \leq \Pr[Z]. \tag{12.5}$$

If event Z happens, then at the end of Game 1, we have $Pre[y] = x$. What we want to do, therefore, is use $\mathcal{A}$ to build an efficient adversary $\mathcal{B}_{\text{iow}}$ that breaks the one-wayness assumption for $\mathcal{T}$ with an advantage equal to $\Pr[Z]$, with the help of an image oracle. The logic of $\mathcal{B}_{\text{iow}}$ is very straightforward. Basically, after obtaining the public key pk and $y \in \mathcal{Y}$ from its challenger in Attack Game 12.2, $\mathcal{B}_{\text{iow}}$ plays the role of challenger to $\mathcal{A}$ as in Game 1. The value of x is never explicitly used in that game (other than to compute y), and the value of the secret key sk is not used, except in the evaluation of the *Image* function, and for this, $\mathcal{B}_{\text{iow}}$ can use the image oracle provided to it in Attack Game 12.2. At the end of the game, if $y \in \text{Domain}(Pre)$, then $\mathcal{B}_{\text{iow}}$ outputs $x = Pre[y]$. It should be clear, by construction, that

$$\Pr[Z] = \text{OW}\mathsf{adv}[\mathcal{B}_{\text{iow}}, \mathcal{T}]. \tag{12.6}$$

Finally, note that in Game 1, the key k is only used to encrypt the challenge plaintext, and to process decryption queries of the form $(y, \hat{c})$, where $\hat{c} \neq c$. As such, the adversary is essentially just playing the 1CCA attack game against $\mathcal{E}_{\text{s}}$ at this point. More precisely, we can easily derive an efficient 1CCA adversary $\mathcal{B}_{\text{s}}$ based on Game 1 that uses $\mathcal{A}$ as a subroutine, such that

$$|\Pr[W_1] - 1/2| = \text{1CCA}\mathsf{adv}^*[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}]. \tag{12.7}$$

This adversary $\mathcal{B}_{\text{s}}$ generates (pk, sk) itself and uses sk to answer queries from $\mathcal{A}$.

Combining (12.5), (12.6) and (12.7), we obtain (12.4). That completes the proof of the theorem. □

12.3.1 Instantiating $\mathcal{E}'_{\text{TDF}}$ with RSA

Suppose we instantiate $\mathcal{E}'_{\text{TDF}}$ using RSA just as we did in Section 11.4.1. The underlying trapdoor function is actually a permutation on $\mathbb{Z}_n$. This implies two things. First, we can omit the check in the decryption algorithm that y is in the image of the trapdoor function, and so we end up with exactly the same scheme $\mathcal{E}_{\text{RSA}}$ as was presented in Section 11.4.1. Second, the implementation of the image oracle in Attack Game 12.2 is trivial to implement, and so we end up back with Attack Game 10.2. Theorem 12.2 specializes as follows:

Theorem 12.3. *Assume $H : \mathcal{X} \to \mathcal{K}$ is modeled as a random oracle. If the RSA assumption holds for parameters (ℓ, e), and $\mathcal{E}_{\text{s}}$ is 1CCA secure, then $\mathcal{E}_{\text{RSA}}$ is CCA secure.*

In particular, for every 1CCA adversary $\mathcal{A}$ that attacks $\mathcal{E}_{\text{RSA}}$ as in the random oracle version of Definition 12.2, there exist an RSA adversary $\mathcal{B}_{\text{rsa}}$ that breaks the RSA assumption for (ℓ, e) as in Attack Game 10.3, and a 1CCA adversary $\mathcal{B}_{\text{s}}$ that attacks $\mathcal{E}_{\text{s}}$ as in Definition 9.6, where $\mathcal{B}_{\text{rsa}}$ and $\mathcal{B}_{\text{s}}$ are elementary wrappers around $\mathcal{A}$, such that

$$\text{1CCA}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\text{RSA}}] \leq 2 \cdot \text{RSA}\mathsf{adv}[\mathcal{B}_{\text{rsa}}, \ell, e] + \text{1CCA}\mathsf{adv}[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}].$$

12.4 CCA-secure ElGamal encryption

We saw that the basic RSA encryption scheme $\mathcal{E}_{\text{RSA}}$ could be shown to be CCA secure in the random oracle model under the RSA assumption (and assuming the underlying symmetric cipher was 1CCA secure). It is natural to ask whether the basic ElGamal encryption scheme $\mathcal{E}_{\text{EG}}$, discussed in Section 11.5, is CCA secure in the random oracle model, under the CDH assumption. Unfortunately, this is not the case: it turns out that a slightly stronger assumption than the CDH assumption is both necessary and sufficient to prove the security of $\mathcal{E}_{\text{EG}}$.

12.4.1 CCA security for basic ElGamal encryption

Recall that the basic ElGamal encryption scheme, $\mathcal{E}_{\text{EG}} = (G, E, D)$, introduced in Section 11.5. It is defined in terms of a cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$, a symmetric cipher $\mathcal{E}_{\text{s}} = (E_{\text{s}}, D_{\text{s}})$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and a hash function $H : \mathbb{G} \to \mathcal{K}$. The message space of $\mathcal{E}_{\text{EG}}$ is $\mathcal{M}$ and the ciphertext space is $\mathbb{G} \times \mathcal{C}$. Public keys are of the form $u \in \mathbb{G}$ and secret keys are of the form $\alpha \in \mathbb{Z}_q$. The algorithms G, E, and D are defined as follows:

$$
\begin{aligned}
G() \quad &:= \quad \alpha \xleftarrow{\text{R}} \mathbb{Z}_q, \;\; u \leftarrow g^{\alpha}, \;\; pk \leftarrow u, \;\; sk \leftarrow \alpha \\
&\qquad \text{output } (pk, sk); \\
E(u, m) \quad &:= \quad \beta \xleftarrow{\text{R}} \mathbb{Z}_q, \;\; v \leftarrow g^{\beta}, \;\; w \leftarrow u^{\beta}, \;\; k \leftarrow H(w), \;\; c \xleftarrow{\text{R}} E_{\text{s}}(k, m) \\
&\qquad \text{output } (v, c); \\
D(\alpha, \; (v, c)\;) \quad &:= \quad w \leftarrow v^{\alpha}, \;\; k \leftarrow H(w), \;\; m \leftarrow D_{\text{s}}(k, c) \\
&\qquad \text{output } m.
\end{aligned}
$$

To see why the CDH assumption by itself is not sufficient to establish the security of $\mathcal{E}_{\text{EG}}$ against chosen ciphertext attack, suppose the public key is $u = g^{\alpha}$. Now, suppose an adversary selects group elements $\hat{v}$ and $\hat{w}$ in some arbitrary way, and computes $\hat{k} \leftarrow H(\hat{w})$ and $\hat{c} \xleftarrow{\text{R}} E_{\text{s}}(\hat{k}, \hat{m})$ for some arbitrary message $\hat{m}$. Further, suppose the adversary can obtain the decryption m^* of the ciphertext $(\hat{v}, \hat{c})$. Now, it is very likely that $\hat{m} = m^*$ if and only if $\hat{w} = \hat{v}^{\alpha}$, or in other words, if and only if $(u, \hat{v}, \hat{w})$ is a DH-triple. Thus, in the chosen ciphertext attack game, decryption queries can be effectively used by the adversary to answer questions of the form "is $(u, \hat{v}, \hat{w})$ a DH-triple?" for group elements $\hat{v}$ and $\hat{w}$ of the adversary's choosing. In general, the adversary would not be able to efficiently answer such questions on his own (this is the DDH assumption), and so these decryption queries may potentially leak some information about the secret key α. Based on the current state of our knowledge, this leakage does not seem to compromise the security of the scheme; however, we do need to state this as an explicit assumption.

Intuitively, the **interactive CDH assumption** states that given a random instance (g^{α}, g^{β}) of the DH problem, it is hard to compute $g^{\alpha\beta}$, even when given access to a "DH-decision oracle" that recognizes DH-triples of the form $(g^{\alpha}, \cdot, \cdot)$. More formally, this assumption is defined in terms of the following attack game.

Attack Game 12.3 (Interactive Computational Diffie-Hellman). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. For a given adversary $\mathcal{A}$, the attack game runs as follows.

- The challenger computes

  $$\alpha, \beta \xleftarrow{\text{R}} \mathbb{Z}_q, \;\; u \leftarrow g^{\alpha}, \;\; v \leftarrow g^{\beta}, \;\; w \leftarrow g^{\alpha\beta}$$

  and gives (u, v) to the adversary.

- The adversary makes a sequence of *DH-decision oracle queries* to the challenger. Each query is of the form $(\tilde{v}, \tilde{w}) \in \mathbb{G}^2$. Upon receiving such a query, the challenger tests if $\tilde{v}^\alpha = \tilde{w}$; if so, he sends "yes" to the adversary, and otherwise, sends "no" to the adversary.

- Finally, the adversary outputs some $\hat{w} \in \mathbb{G}$.

We define $\mathcal{A}$'s **advantage in solving the interactive computational Diffie-Hellman problem**, denoted $\mathrm{ICDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$, as the probability that $\hat{w} = w$. □

We stress that in the above attack game, the adversary can ask the challenger for help in determining whether certain triples are DH-triples, but only triples of the form $(u, \cdot, \cdot)$, where u is generated by the challenger.

Definition 12.4 (Interactive Computational Diffie-Hellman assumption). *We say that the **interactive computational Diffie-Hellman (ICDH)** assumption holds for* $\mathbb{G}$ *if for all efficient adversaries* $\mathcal{A}$ *the quantity* $\mathrm{ICDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$ *is negligible.*

By the above discussion, we see (at least heuristically) that the ICDH assumption is necessary to establish the CCA security of $\mathcal{E}_{\mathrm{EG}}$. Conversely, one can prove that $\mathcal{E}_{\mathrm{EG}}$ is CCA secure in the random oracle model under the ICDH assumption (and assuming also that $\mathcal{E}_{\mathrm{s}}$ is 1CCA secure); however, we shall instead analyze a slight variation of $\mathcal{E}_{\mathrm{EG}}$, for which the reduction is simpler and more efficient. This encryption scheme, which we denote $\mathcal{E}'_{\mathrm{EG}}$, is exactly the same as $\mathcal{E}_{\mathrm{EG}}$, except that the symmetric key k is derived by hashing both v and w, instead of just w; that is, the hash function H is now of the form $H : \mathbb{G}^2 \to \mathcal{K}$, and the symmetric key k is computed as $k = H(v, w)$.

For completeness, we describe the scheme $\mathcal{E}'_{\mathrm{EG}} = (G, E, D)$ in its entirety. It is defined in terms of a cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$, a symmetric cipher $\mathcal{E}_{\mathrm{s}} = (E_{\mathrm{s}}, D_{\mathrm{s}})$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and a hash function $H : \mathbb{G}^2 \to \mathcal{K}$. Public keys are of the form $u \in \mathbb{G}$ and secret keys are of the form $\alpha \in \mathbb{Z}_q$. The algorithms G, E, and D are defined as follows:

$$
\begin{aligned}
G() \quad &:= \quad \alpha \xleftarrow{\mathrm{R}} \mathbb{Z}_q, \ \ u \leftarrow g^\alpha, \ \ pk \leftarrow u, \ \ sk \leftarrow \alpha \\
&\qquad \text{output } (pk, sk); \\
E(u, m) \quad &:= \quad \beta \xleftarrow{\mathrm{R}} \mathbb{Z}_q, \ \ v \leftarrow g^\beta, \ \ w \leftarrow u^\beta, \ \ \boxed{k \leftarrow H(v, w),} \ \ c \xleftarrow{\mathrm{R}} E_{\mathrm{s}}(k, m) \\
&\qquad \text{output } (v, c); \\
D(\alpha, \ (v, c)\ ) \quad &:= \quad w \leftarrow v^\alpha, \ \ \boxed{k \leftarrow H(v, w),} \ \ m \leftarrow D_{\mathrm{s}}(k, c) \\
&\qquad \text{output } m.
\end{aligned}
$$

The message space is $\mathcal{M}$ and the ciphertext space is $\mathbb{G} \times \mathcal{C}$. We have highlighted the differences between $\mathcal{E}'_{\mathrm{EG}}$ and $\mathcal{E}_{\mathrm{EG}}$.

Theorem 12.4. *Assume* $H : \mathbb{G}^2 \to \mathcal{K}$ *is modeled as a random oracle. If the ICDH assumption holds for* $\mathbb{G}$*, and* $\mathcal{E}_{\mathrm{s}}$ *is 1CCA secure, then* $\mathcal{E}'_{\mathrm{EG}}$ *is CCA secure.*

In particular, for every 1CCA adversary $\mathcal{A}$ *that attacks* $\mathcal{E}'_{\mathrm{EG}}$ *as in the random oracle version of Definition 12.2, there exist an ICDH adversary* $\mathcal{B}_{\mathrm{icdh}}$ *for* $\mathbb{G}$ *as in Attack Game 12.3, and a 1CCA adversary* $\mathcal{B}_{\mathrm{s}}$ *that attacks* $\mathcal{E}_{\mathrm{s}}$ *as in Definition 9.6, where* $\mathcal{B}_{\mathrm{icdh}}$ *and* $\mathcal{B}_{\mathrm{s}}$ *are elementary wrappers around* $\mathcal{A}$*, such that*

$$
1\mathrm{CCA}^{\mathrm{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}'_{\mathrm{EG}}] \leq 2 \cdot \mathrm{ICDH}\mathsf{adv}[\mathcal{B}_{\mathrm{icdh}}, \mathbb{G}] + 1\mathrm{CCA}\mathsf{adv}[\mathcal{B}_{\mathrm{s}}, \mathcal{E}_{\mathrm{s}}]. \tag{12.8}
$$

In addition, the number of DH-decision oracle queries made by $\mathcal{B}_{\mathrm{icdh}}$ *is bounded by the number of random oracle queries made by* $\mathcal{A}$*.*

Proof. The basic structure of the proof is very similar to that of Theorem 12.2. As in that proof, it is convenient to use the bit-guessing versions of the 1CCA attack games. We prove

$$1\text{CCA}^{\text{ro}}\mathsf{adv}^*[\mathcal{A}, \mathcal{E}'_{\text{EG}}] \leq \text{ICDH}\mathsf{adv}[\mathcal{B}_{\text{icdh}}, \mathbb{G}] + 1\text{CCA}\mathsf{adv}^*[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}]. \tag{12.9}$$

Then (12.8) follows by (12.2) and (9.2).

We define Games 0 and 1. Game 0 is the bit-guessing version of Attack Game 12.1 played by $\mathcal{A}$ with respect to $\mathcal{E}'_{\text{EG}}$. In each game, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. For $j = 0, 1$, we define W_j to be the event that $\hat{b} = b$ in Game j.

Game 0. The logic of the challenger is shown in Fig. 12.2. The adversary can make any number of random oracle queries, and any number of decryption queries, but at most one encryption query. As usual, in addition to direct access the random oracle via explicit random oracle queries, the adversary also has indirect access to the random oracle via the encryption and decryption queries, where the challenger also makes use of the random oracle.

In the initialization step, the challenger computes the secret key $\alpha \in \mathbb{Z}_q$ and the public key $u = g^\alpha$; it also makes those computations associated with the encryption query that can be done without yet knowing the challenge plaintext. As in the proof of Theorem 12.2, we want our challenger to use the secret key α as little as possible in processing decryption queries, and again, we use a somewhat nontrivial strategy for implementing the decryption and random oracle queries. Nevertheless, despite the significant superficial differences, this implementation will be logically equivalent to the actual attack game.

As usual, we will implement the random oracle using an associative array $Map : \mathbb{G}^2 \to \mathcal{K}$. However, we will also make use of an auxiliary associative array $Map' : \mathbb{G} \to \mathcal{K}$. The convention is that if $(u, \hat{v}, \hat{w})$ is a DH-triple, and the value of the random oracle at the point $(\hat{v}, \hat{w})$ is $\hat{k}$, then $Map[\hat{v}, \hat{w}] = Map'[\hat{v}] = \hat{k}$. However, in processing a decryption query $(\hat{v}, \hat{c})$, we may speculatively assign a random value $\hat{k}$ to $Map'[\hat{v}]$, and then later, if the adversary queries the random oracle at the point $(\hat{v}, \hat{w})$, where $(u, \hat{v}, \hat{w})$ is a DH-triple, we assign the value $\hat{k}$ to $Map[\hat{v}, \hat{w}]$, in order to maintain consistency.

Now for more details. In preparation for the encryption query, in the initialization step, the challenger precomputes $\beta \xleftarrow{R} \mathbb{Z}_q$, $v \leftarrow g^\beta$, $w \leftarrow g^{\alpha\beta}$, $k \xleftarrow{R} \mathcal{K}$. It also sets $Map[v, w]$ and $Map'[v]$ to k, which means that the value of the random oracle at (v, w) is equal to k. Also note that in the initialization step, the challenger sets $c \leftarrow \perp$, and in processing the encryption query, overwrites c with a ciphertext in $\mathcal{C}$. Thus, decryption queries processed while $c = \perp$ are phase 1 queries, while those processed while $c \neq \perp$ are phase 2 queries.

Processing random oracle queries. When processing a random oracle query $(\hat{v}, \hat{w})$, if $Map[\hat{v}, \hat{w}]$ has not yet been defined, the challenger proceeds as follows.

- First, it tests if $(u, \hat{v}, \hat{w})$ is a DH-triple. In Fig. 12.2, we implement this by invoking the function $DHP(\alpha, \hat{v}, \hat{w})$. For now, we can think of DHP as being implemented as follows:

  $$DHP(\alpha, \hat{v}, \hat{w}) \quad := \quad \hat{v}^\alpha = \hat{w}.$$

  This is the only place where our challenger makes use of the secret key.

- If $(u, \hat{v}, \hat{w})$ is a DH-triple, the challenger sets $Map'[\hat{v}]$ to a random value, if it is not already defined, and then sets $Map[\hat{v}, \hat{w}] \leftarrow Map'[\hat{v}]$. It also sets $Sol[\hat{v}] \leftarrow \hat{w}$, where $Sol : \mathbb{G} \to \mathbb{G}$ is another associative array. The idea is that Sol records solutions to Diffie-Hellman instances $(u, \hat{v})$ that are discovered while processing random oracle queries.

- If $(u, \hat{v}, \hat{w})$ is not a DH-triple, then the challenger just sets $Map[\hat{v}, \hat{w}]$ to a random value.

The result of the random oracle query is always $Map[\hat{v}, \hat{w}]$.

Processing decryption queries. In processing a decryption query $(\hat{v}, \hat{c})$, the challenger proceeds as follows.

- If $\hat{v} = v$, the challenger just uses the prepared key k directly to decrypt $\hat{c}$.
- Otherwise, the challenger checks if Map' is defined at the point $\hat{v}$, and if not, it assigns to $Map'[\hat{v}]$ a random value. It then uses the value $\hat{k} = Map'[\hat{v}]$ directly to decrypt $\hat{c}$. Observe that our challenger performs the decryption without using the solution $\hat{w}$ to the instance $(u, \hat{v})$ of the CDH problem. However, if the adversary queries the random oracle at the point $(\hat{v}, \hat{w})$, the adversary will see the same value $\hat{k}$, and so consistency is maintained.

Hopefully, it is clear that our challenger behaves *exactly* as in the usual attack game, despite the more elaborate bookkeeping.

Game 1. This game is the same as Game 0, except that we delete line (1) in Fig. 12.2.

Let Z be the event that $\mathcal{A}$ queries the random oracle at (v, w) in Game 1. It is not hard to see that Games 0 and 1 proceed identically, unless Z occurs. By the Difference Lemma, we have

$$|\Pr[W_1] - \Pr[W_0]| \leq \Pr[Z]. \tag{12.10}$$

If event Z happens, then at the end of Game 1, we have $Sol[v] = w$. What we want to do, therefore, is use $\mathcal{A}$ to build an efficient adversary $\mathcal{B}_{\text{icdh}}$ that breaks the CDH assumption for $\mathbb{G}$, with the help of a DH-decision oracle, with an advantage equal to $\Pr[Z]$. The logic of $\mathcal{B}_{\text{icdh}}$ is very straightforward. Basically, after obtaining u and v from its challenger in Attack Game 12.3, $\mathcal{B}_{\text{icdh}}$ plays the role of challenger to $\mathcal{A}$ as in Game 1. Besides the computation of u, the value of α is never explicitly used in that game, other than in the evaluation of the *DHP* function, and for this, $\mathcal{B}_{\text{icdh}}$ can use the DH-decision oracle provided to it in Attack Game 12.3. At the end of the game, if $v \in \text{Domain}(Sol)$, then $\mathcal{B}_{\text{icdh}}$ outputs $w = Sol[v]$.

By construction, it is clear that

$$\Pr[Z] = \text{ICDH}\mathsf{adv}[\mathcal{B}_{\text{icdh}}, \mathbb{G}]. \tag{12.11}$$

Finally, note that in Game 1, the key k is only used to encrypt the challenge plaintext, and to process decryption queries of the form $(v, \hat{c})$, where $\hat{c} \neq c$. As such, the adversary is essentially just playing the 1CCA attack game against $\mathcal{E}_{\text{s}}$ at this point. More precisely, we can easily derive an efficient 1CCA adversary $\mathcal{B}_{\text{s}}$ based on Game 1 that uses $\mathcal{A}$ as a subroutine, such that

$$|\Pr[W_1] - 1/2| = \text{1CCA}\mathsf{adv}^*[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}]. \tag{12.12}$$

We leave the details of $\mathcal{B}_{\text{s}}$ to the reader.

Combining (12.10), (12.11), and (12.12), we obtain (12.9). That completes the proof of the theorem. □

initialization:

$\alpha, \beta \xleftarrow{R} \mathbb{Z}_q$, $u \leftarrow g^{\alpha}$, $v \leftarrow g^{\beta}$, $w \leftarrow g^{\alpha\beta}$

$k \xleftarrow{R} \mathcal{K}$, $b \xleftarrow{R} \{0,1\}$

$c \leftarrow \perp$

initialize three empty associative arrays

$Map : \mathbb{G}^2 \to \mathcal{K}$, $Map' : \mathbb{G} \to \mathcal{K}$, and $Sol : \mathbb{G} \to \mathbb{G}$

(1) $Map[v, w] \leftarrow k$, $Map'[v] \leftarrow k$

send the public key u to $\mathcal{A}$;

upon receiving an encryption query $(m_0, m_1) \in \mathcal{M}^2$:

$c \xleftarrow{R} E_s(k, m_b)$, send (v, c) to $\mathcal{A}$;

upon receiving a decryption query $(\hat{v}, \hat{c}) \in \mathbb{G} \times \mathcal{C}$, where $(\hat{v}, \hat{c}) \neq (v, c)$:

if $\hat{v} = v$ then

$\hat{m} \leftarrow D_s(k, \hat{c})$

else

if $\hat{v} \notin \text{Domain}(Map')$ then $Map'[\hat{v}] \xleftarrow{R} \mathcal{K}$

$\hat{k} \leftarrow Map'[\hat{v}]$, $\hat{m} \leftarrow D_s(\hat{k}, \hat{c})$

send $\hat{m}$ to $\mathcal{A}$;

upon receiving a random oracle query $(\hat{v}, \hat{w}) \in \mathbb{G}^2$:

if $(\hat{v}, \hat{w}) \notin \text{Domain}(Map)$ then

if $\text{DHP}(\alpha, \hat{v}, \hat{w})$ then

if $\hat{v} \notin \text{Domain}(Map')$ then $Map'[\hat{v}] \xleftarrow{R} \mathcal{K}$

$Map[\hat{v}, \hat{w}] \leftarrow Map'[\hat{v}]$, $Sol[\hat{v}] \leftarrow \hat{w}$

else

$Map[\hat{v}, \hat{w}] \xleftarrow{R} \mathcal{K}$

send $Map[\hat{v}, \hat{w}]$ to $\mathcal{A}$

Figure 12.2: Game 0 challenger

Discussion. We proved that $\mathcal{E}'_{\text{EG}}$ is CCA-secure, in the random oracle model, under the ICDH assumption. Is the ICDH assumption reasonable? On the one hand, in Chapter 16 we will see groups $\mathbb{G}$ where the ICDH assumption is equivalent to the CDH assumption. In such groups there is no harm in assuming ICDH. On the other hand, the ElGamal system is most commonly implemented in groups where ICDH is not known to be equivalent to CDH. Is it reasonable to assume ICDH in such groups? Currently, we do not know of any group where CDH holds, but ICDH does not hold. As such, it appears to be a reasonable assumption to use when constructing cryptographic schemes. Later, in Section 12.6.2, we will see a variant of ElGamal encryption that is CCA-secure, in the random oracle model, under the normal CDH assumption.

12.5 CCA security from DDH without random oracles

In Section 11.5.2, we proved that $\mathcal{E}_{\mathrm{EG}}$ was semantically secure without relying on the random oracle model. Rather, we used the DDH assumption (among other assumptions). Unfortunately, it seems unlikely that we can ever hope to prove that $\mathcal{E}_{\mathrm{EG}}$ (or $\mathcal{E}'_{\mathrm{EG}}$, for that matter) is CCA secure without relying on random oracles.

In this section, we present a public key encryption scheme that can be proved CCA secure without relying on the random oracle heuristic. The scheme is based on the DDH assumption (as well as a few other standard assumptions). The scheme is a variant of one designed by Cramer and Shoup, and we call it $\mathcal{E}_{\mathrm{CS}}$. It is built out of several components:

- a cyclic group $\mathbb{G}$ of prime order q with generator $g \in \mathbb{G}$,
- a symmetric cipher $\mathcal{E}_{\mathrm{s}} = (E_{\mathrm{s}}, D_{\mathrm{s}})$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$,
- a hash function $H : \mathbb{G} \to \mathcal{K}$,
- a hash function $H' : \mathbb{G} \times \mathbb{G} \to \mathbb{Z}_q$.

The message space for $\mathcal{E}_{\mathrm{CS}}$ is $\mathcal{M}$, and the ciphertext space is $\mathbb{G}^3 \times \mathcal{C}$. We now describe the key generation, encryption, and decryption algorithms for $\mathcal{E}_{\mathrm{CS}}$.

- the key generation algorithm runs as follows:

$$
\begin{aligned}
G() :=\quad & \alpha \xleftarrow{\mathrm{R}} \mathbb{Z}_q, \quad u \leftarrow g^{\alpha} \\
& \text{for } i = 1, \ldots, 3\text{: } \sigma_i, \tau_i \xleftarrow{\mathrm{R}} \mathbb{Z}_q,\ u_i \leftarrow g^{\sigma_i} u^{\tau_i} \\
& pk \leftarrow (u, u_1, u_2, u_3), \quad sk \leftarrow (\sigma_1, \tau_1, \sigma_2, \tau_2, \sigma_3, \tau_3) \\
& \text{output } (pk, sk);
\end{aligned}
$$

- for a given public key $pk = (u, u_1, u_2, u_3) \in \mathbb{G}^4$ and message $m \in \mathcal{M}$, the encryption algorithm runs as follows:

$$
\begin{aligned}
E(pk, m) :=\quad & \beta \xleftarrow{\mathrm{R}} \mathbb{Z}_q, \quad v \leftarrow g^{\beta}, \quad w \leftarrow u^{\beta}, \quad \rho \leftarrow H'(v, w) \\
& w_1 \leftarrow u_1^{\beta}, \quad w_2 \leftarrow (u_2 u_3^{\rho})^{\beta} \\
& k \leftarrow H(w_1), \quad c \xleftarrow{\mathrm{R}} E_{\mathrm{s}}(k, m) \\
& \text{output } (v, w, w_2, c);
\end{aligned}
$$

- for a given secret key $sk = (\sigma_1, \tau_1, \sigma_2, \tau_2, \sigma_3, \tau_3) \in \mathbb{Z}_q^6$ and a ciphertext $(v, w, w_2, c) \in \mathbb{G}^3 \times \mathcal{C}$, the decryption algorithm runs as follows:

$$
\begin{aligned}
D(sk,\ (v, w, w_2, c)\ ) :=\quad & \rho \leftarrow H'(v, w) \\
& \text{if } v^{\sigma_2 + \rho\sigma_3} w^{\tau_2 + \rho\tau_3} = w_2 \\
& \qquad \text{then } w_1 \leftarrow v^{\sigma_1} w^{\tau_1}, \quad k \leftarrow H(w_1), \quad m \leftarrow D_{\mathrm{s}}(k, c) \\
& \qquad \text{else} \quad m \leftarrow \mathsf{reject} \\
& \text{output } m.
\end{aligned}
$$

We first argue that $\mathcal{E}_{\mathrm{CS}}$ satisfies the basic correctness property, i.e., that decryption undoes encryption. Consider an arbitrary encryption of a message m, which has the form (v, w, w_2, c), where

$$
v = g^{\beta},\ \ w = u^{\beta},\ \ \rho = H'(v, w),\ \ w_1 = u_1^{\beta},\ \ w_2 = (u_2 u_3^{\rho})^{\beta},\ \ k = H(w_1),\ \ c = E_{\mathrm{s}}(k, m).
$$

First, observe that

$$v^{\sigma_2+\rho\sigma_3} w^{\tau_2+\rho\tau_3} = g^{\beta(\sigma_2+\rho\sigma_3)} u^{\beta(\tau_2+\rho\tau_3)} = (u_2 u_3^{\rho})^{\beta} = w_2.$$

This implies that the test in the decryption algorithm succeeds. Second, observe that

$$v^{\sigma_1} w^{\tau_1} = g^{\beta\sigma_1} u^{\beta\tau_1} = u_1^{\beta} = w_1.$$

This implies that the decryption algorithm derives the same symmetric key k as was used in encryption, and correctness for $\mathcal{E}_{\rm CS}$ follows from correctness for $\mathcal{E}_{\rm s}$.

We shall prove that $\mathcal{E}_{\rm CS}$ is CCA secure under the following assumptions:

- the DDH assumption holds in $\mathbb{G}$;
- $\mathcal{E}_{\rm s}$ is 1CCA secure;
- H is a secure KDF (see Definition 11.5);
- H' is collision resistant (see Definition 8.1).

One can in fact prove security of $\mathcal{E}_{\rm CS}$ under a weaker assumption on H' (namely, target collision resistance — see Definition 8.5). Moreover, a variation of $\mathcal{E}_{\rm CS}$ can be proved secure under an assumption that is somewhat weaker than the DDH assumption (namely, the Hash Diffie-Hellman assumption, discussed in Exercise 11.13). These results are developed below in the exercises.

Theorem 12.5. *If the DDH assumption holds in $\mathbb{G}$, $\mathcal{E}_{\rm s}$ is 1CCA secure, H is a secure KDF, and H' is collision resistant, then $\mathcal{E}_{\rm CS}$ is CCA secure.*

In particular, for every 1CCA adversary $\mathcal{A}$ that attacks $\mathcal{E}_{\rm CS}$ as in Definition 12.2, and makes at most $Q_{\rm d}$ decryption queries, there exist a DDH adversary $\mathcal{B}_{\rm ddh}$ for $\mathbb{G}$ as in Attack Game 10.6, a 1CCA adversary $\mathcal{B}_{\rm s}$ that attacks $\mathcal{E}_{\rm s}$ as in Definition 9.6, a KDF adversary $\mathcal{B}_{\rm kdf}$ that attacks H as in Attack Game 11.3, and a collision-finding adversary $\mathcal{B}_{\rm cr}$ that attacks H' as in Attack Game 8.1, where $\mathcal{B}_{\rm ddh}$, $\mathcal{B}_{\rm s}$, $\mathcal{B}_{\rm kdf}$, $\mathcal{B}_{\rm cr}$ are elementary wrappers around $\mathcal{A}$, such that

$$\begin{aligned}1\mathrm{CCA}\mathsf{adv}[\mathcal{A},\mathcal{E}_{\rm CS}] \le 2\Big(&\mathrm{DDH}\mathsf{adv}[\mathcal{B}_{\rm ddh},\mathbb{G}] + \mathrm{KDF}\mathsf{adv}[\mathcal{B}_{\rm kdf},H] \\ &+ \mathrm{CR}\mathsf{adv}[\mathcal{B}_{\rm cr},H'] + \frac{Q_{\rm d}+1}{q}\Big) + 1\mathrm{CCA}\mathsf{adv}[\mathcal{B}_{\rm s},\mathcal{E}_{\rm s}].\end{aligned} \tag{12.13}$$

Proof. As usual, it is convenient to use the bit-guessing versions of the 1CCA attack games. We prove

$$\begin{aligned}1\mathrm{CCA}\mathsf{adv}^*[\mathcal{A},\mathcal{E}_{\rm CS}] \le\ &\mathrm{DDH}\mathsf{adv}[\mathcal{B}_{\rm ddh},\mathbb{G}] + \mathrm{KDF}\mathsf{adv}[\mathcal{B}_{\rm kdf},H] \\ &+ \mathrm{CR}\mathsf{adv}[\mathcal{B}_{\rm cr},H'] + \frac{Q_{\rm d}+1}{q} + 1\mathrm{CCA}\mathsf{adv}^*[\mathcal{B}_{\rm s},\mathcal{E}_{\rm s}].\end{aligned} \tag{12.14}$$

Then (12.13) follows by (12.2) and (9.2).

We define a series of games, Game j for $j = 0, \ldots, 6$. Game 0 is the bit-guessing version of Attack Game 12.1 played by $\mathcal{A}$ with respect to $\mathcal{E}_{\rm CS}$. In each game, b denotes the random bit chosen by the challenger, while $\hat{b}$ denotes the bit output by $\mathcal{A}$. For $j = 0, \ldots, 6$, we define W_j to be the event that $\hat{b} = b$ in Game j.

initialization:

$\alpha, \beta \xleftarrow{R} \mathbb{Z}_q$

(1) $\gamma \leftarrow \alpha\beta$

$u \leftarrow g^{\alpha}$, $v \leftarrow g^{\beta}$, $w \leftarrow g^{\gamma}$

$\rho \leftarrow H'(v, w)$

for $i = 1, \ldots, 3$: $\sigma_i, \tau_i \xleftarrow{R} \mathbb{Z}_q$, $u_i \leftarrow g^{\sigma_i} u^{\tau_i}$

(2) $w_1 \leftarrow u_1^{\beta}$

(3) $w_2 \leftarrow (u_2 u_3^{\rho})^{\beta}$

(4) $k \leftarrow H(w_1)$

$b \xleftarrow{R} \{0, 1\}$, $c \leftarrow \perp$

send the public key (u, u_1, u_2, u_3) to $\mathcal{A}$;

upon receiving an encryption query $(m_0, m_1) \in \mathcal{M}^2$:

$c \xleftarrow{R} E_{\mathrm{s}}(k, m_b)$, send (v, w, w_2, c) to $\mathcal{A}$;

upon receiving a decryption query $(\hat{v}, \hat{w}, \hat{w}_2, \hat{c}) \in \mathbb{G}^3 \times \mathcal{C}$, where $(\hat{v}, \hat{w}, \hat{w}_2, \hat{c}) \neq (v, w, w_2, c)$:

if $(\hat{v}, \hat{w}, \hat{w}_2) = (v, w, w_2)$ then

$\hat{m} \leftarrow D_{\mathrm{s}}(k, \hat{c})$

else

$\hat{\rho} \leftarrow H'(\hat{v}, \hat{w})$

(5) if $\hat{v}^{\sigma_2 + \hat{\rho}\sigma_3} \hat{w}^{\tau_2 + \hat{\rho}\tau_3} = \hat{w}_2$ then

(6) $\hat{w}_1 \leftarrow \hat{v}^{\sigma_1} \hat{w}^{\tau_1}$

$\hat{k} \leftarrow H(\hat{w}_1)$, $\hat{m} \leftarrow D_{\mathrm{s}}(\hat{k}, \hat{c})$

else

$\hat{m} \leftarrow \mathsf{reject}$

send $\hat{m}$ to $\mathcal{A}$.

Figure 12.3: Game 0 challenger

Game 0. The logic of the challenger is shown in Fig. 12.3. The adversary can make any number of decryption queries, but at most one encryption query. Note that in the initialization step, the challenger performs those computations associated with the encryption query that it can, without yet knowing the challenge plaintext. Also note that in the initialization step, the challenger sets $c \leftarrow \perp$, and in processing the encryption query, overwrites c with a ciphertext in $\mathcal{C}$. Thus, decryption queries processed while $c = \perp$ are phase 1 queries, while those processed while $c \neq \perp$ are phase 2 queries.

Game 1. We replace the lines marked (2) and (3) in Fig. 12.3 as follows:

(2) $w_1 \leftarrow v^{\sigma_1} w^{\tau_1}$

(3) $w_2 \leftarrow v^{\sigma_2 + \rho\sigma_3} w^{\tau_2 + \rho\tau_3}$

Basically, we have simply replaced the formulas used to generate w_1 and w_2 in the encryption procedure with those used in the decryption procedure. As we already argued above in analyzing the correctness property for $\mathcal{E}_{\text{CS}}$, these formulas are equivalent. In particular:

$$\Pr[W_1] = \Pr[W_0]. \tag{12.15}$$

The motivation for making this change is that now, the only place where we use the exponents α, β, and γ is in the definition of the group elements u, v, and w, which allows us to then play the "DDH card" in the next step of the proof.

Game 2. We replace the line marked (1) in Fig. 12.3 with

(1) $\quad \gamma \xleftarrow{\text{R}} \mathbb{Z}_q$

After this change, the lines marked (1), (2), and (3) in Fig. 12.3 now read as follows:

(1) $\quad \gamma \xleftarrow{\text{R}} \mathbb{Z}_q$
(2) $\quad w_1 \leftarrow v^{\sigma_1} w^{\tau_1}$
(3) $\quad w_2 \leftarrow v^{\sigma_2+\rho\sigma_3} w^{\tau_2+\rho\tau_3}$

It is easy to see that

$$\bigl|\Pr[W_1] - \Pr[W_2]\bigr| \leq \mathsf{DDHadv}[\mathcal{B}_{\text{ddh}}, \mathbb{G}] \tag{12.16}$$

for an efficient DDH adversary $\mathcal{B}_{\text{ddh}}$, which works as follows. After it obtains its DDH problem instance (u, v, w) from its own challenger, adversary $\mathcal{B}_{\text{ddh}}$ plays the role of challenger to $\mathcal{A}$ in Game 0, but using the given values u, v, w. If (u, v, w) is a random DH-triple, then this is equivalent to Game 0, and if (u, v, w) is a random triple, this is equivalent to Game 1. At the end of the game, $\mathcal{B}_{\text{ddh}}$ outputs 1 if $\hat{b} = b$ and 0 otherwise.

Game 3. We replace the line marked (1) in Fig. 12.3 with

(1) $\quad \gamma \xleftarrow{\text{R}} \mathbb{Z}_q \setminus \{\alpha\beta\}$

After this change, the lines marked (1), (2), and (3) in Fig. 12.3 now read as follows:

(1) $\quad \gamma \xleftarrow{\text{R}} \mathbb{Z}_q \setminus \{\alpha\beta\}$
(2) $\quad w_1 \leftarrow v^{\sigma_1} w^{\tau_1}$
(3) $\quad w_2 \leftarrow v^{\sigma_2+\rho\sigma_3} w^{\tau_2+\rho\tau_3}$

Since the statistical distance between the uniform distribution on all triples and the uniform distribution on all non-DH-triples is $1/q$ (see Exercise 10.6), it follows that:

$$\bigl|\Pr[W_2] - \Pr[W_3]\bigr| \leq \frac{1}{q}. \tag{12.17}$$

Interlude. Before continuing with the proof, let us see what the changes so far have accomplished. Consider any fixed values of α, β, and $\gamma \neq \alpha\beta$. Moreover, consider the group elements u_1, w_1 generated by the challenger. These satisfy the equations

$$u_1 = g^{\sigma_1} u^{\tau_1} = g^{\sigma_1+\alpha\tau_1} \quad \text{and} \quad w_1 = v^{\sigma_1} w^{\tau_1} = g^{\beta\sigma_1+\gamma\tau_1}.$$

Taking discrete logarithms, we can write this as a matrix equation

$$\begin{pmatrix} \mathsf{Dlog}_g u_1 \\ \mathsf{Dlog}_g w_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}}_{=:M} \begin{pmatrix} \sigma_1 \\ \tau_1 \end{pmatrix}. \tag{12.18}$$

Now, the matrix M is non-singular. One way to see this is to calculate its determinant $\det(M) = \gamma - \alpha\beta \neq 0$. Another way to see this is to observe that the second row of M cannot be a scalar multiple of the first: if it were, then by looking at the first column of M, the second row of M would have to be equal to β times the first, and by looking at the second column of M, this would imply $\gamma = \alpha\beta$, which is not the case.

Next, observe that σ_1 and τ_1 are uniformly and independently distributed over $\mathbb{Z}_q$. Since M is non-singular, it follows from (12.18) that $\mathsf{Dlog}_g u_1$ and $\mathsf{Dlog}_g w_1$ are also uniformly and independently distributed over $\mathbb{Z}_q$. Equivalently, u_1 and w_1 are uniformly and independently distributed over $\mathbb{G}$.

If the adversary does not submit any decryption oracle queries, he learns nothing more about u_1 and w_1, and since w_1 is only used to derive the key k and then encrypt m_b, security follows easily from the assumptions that H is a secure KDF and $\mathcal{E}_\mathrm{s}$ is semantically secure.

Unfortunately, if the adversary does make decryption queries, these could potentially leak information about w_1. Specifically, suppose the adversary submits a ciphertext $(\hat{v}, \hat{w}, \hat{w}_2, \hat{c})$ such that $(u, \hat{v}, \hat{w})$ is not a DH-triple, yet passes the test at line (5). Then the value of $\hat{w}_1 = \hat{v}^{\sigma_1}\hat{w}^{\tau_1}$ computed on line (6), together with the value u_1 in the public key, completely determine the values of σ_1 and τ_1, and hence the value of w_1. This can be seen by again considering a matrix equation as above. Indeed, if $\hat{\beta} := \mathsf{Dlog}_g \hat{v}$ and $\hat{\gamma} = \mathsf{Dlog}_g \hat{w}$, with $\hat{\gamma} \neq \alpha\hat{\beta}$, then

$$\begin{pmatrix} \mathsf{Dlog}_g u_1 \\ \mathsf{Dlog}_g \hat{w}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \alpha \\ \hat{\beta} & \hat{\gamma} \end{pmatrix}}_{=:\widehat{M}} \cdot \begin{pmatrix} \sigma_1 \\ \tau_1 \end{pmatrix}.$$

Again, the matrix $\widehat{M}$ is non-singular, and so the values $\mathsf{Dlog}_g u_1$ and $\mathsf{Dlog}_g \hat{w}_1$ completely determine σ_1 and τ_1.

So to complete the proof, we shall argue that with overwhelming probability, the scenario described in the previous paragraph does not occur. That is, we shall argue that whenever the adversary submits a ciphertext $(\hat{v}, \hat{w}, \hat{w}_2, \hat{c})$, where $(u, \hat{v}, \hat{w})$ is not a DH-triple, the test at line (5) will pass with only negligible probability. That is the point of including the extra group elements u_2 and u_3 in the public key and the extra group element w_2 in the ciphertext.

Game 4. This is the same as Game 3, except we replace lines (5) and (6) by

(5) if $\hat{v}^{\alpha} = \hat{w}$ and $\hat{v}^{\alpha_2 + \hat{\rho}\alpha_3} = \hat{w}_2$ then
(6) $\quad\hat{w}_1 \leftarrow \hat{v}^{\alpha_1}$

where we define

$$\alpha_i := \sigma_i + \alpha\tau_i \qquad (i = 1, \ldots, 3). \tag{12.19}$$

Observe that if $(u, \hat{v}, \hat{w})$ is not a DH-triple, then the modified test in line (5) will not pass; otherwise, if it is a DH-triple (i.e., $\hat{v}^{\alpha} = \hat{w}$), one can verify that this test passes if and only if the original test in Game 3 passes, and the computation of $\hat{w}_1$ on line (6) is equivalent to that in Game 3. In particular, this new test is strictly stronger than the test in Game 3. Also notice that the computations in lines (5) and (6) in Game 4 do not depend directly on the individual values of σ_1, τ_1, σ_2, τ_2, σ_3, and τ_3, but rather, only indirectly, via the values α_1, α_2, and α_3, defined in (12.19). The importance of this will become evident later in the proof.

After this change, the lines marked (1), (2), (3), (5), and (6) in Fig. 12.3 now read as follows:

(1) $\gamma \xleftarrow{\text{R}} \mathbb{Z}_q \setminus \{\alpha\beta\}$
(2) $w_1 \leftarrow v^{\sigma_1} w^{\tau_1}$
(3) $w_2 \leftarrow v^{\sigma_2+\rho\sigma_3} w^{\tau_2+\rho\tau_3}$
(5) if $\hat{v}^{\alpha} = \hat{w}$ and $\hat{v}^{\alpha_2+\hat{\rho}\alpha_3} = \hat{w}_2$ then
(6) $\hat{w}_1 \leftarrow \hat{v}^{\alpha_1}$

Define Z to be the event that in Game 4, for some decryption query, the test in line (5) is performed, and we have $\hat{w} \neq \hat{v}^{\alpha}$ but $\hat{w}_2 = w_2^*$, where

$$w_2^* := \hat{v}^{\sigma_2+\hat{\rho}\sigma_3} \hat{w}^{\tau_2+\hat{\rho}\tau_3}.$$

Such a ciphertext is rejected in Game 4, but not in Game 3. However, the two games proceed identically unless Z occurs, and so by the Difference Lemma, we have

$$\big|\Pr[W_3] - \Pr[W_4]\big| \leq \Pr[Z]. \tag{12.20}$$

To bound $\Pr[Z]$, it will also be convenient to consider the event Z' that for the relevant decryption query, we have $(v, w) \neq (\hat{v}, \hat{w})$ but $H'(v, w) = H'(\hat{v}, \hat{w})$, that is, (v, w) and $(\hat{v}, \hat{w})$ form a collision under H'. Clearly, we have

$$\Pr[Z] \leq \Pr[Z'] + \Pr[\neg Z' \wedge Z]. \tag{12.21}$$

It should be clear that

$$\Pr[Z'] \leq \mathrm{CR}\mathsf{adv}[\mathcal{B}_{\mathrm{cr}}, H'] \tag{12.22}$$

for an efficient collision-finding adversary $\mathcal{B}_{\mathrm{cr}}$. Indeed, adversary $\mathcal{B}_{\mathrm{cr}}$ just plays Game 4 and waits for the event Z' to happen.

So now we are left to bound $\Pr[\neg Z' \wedge Z]$. We claim that

$$\Pr[\neg Z' \wedge Z] \leq \frac{Q_{\mathrm{d}}}{q}, \tag{12.23}$$

where Q_{d} is an upper bound on the number of decryption queries. To prove (12.23), it will suffice to consider the event $\neg Z' \wedge Z$ for just a single decryption query and apply the union bound.

So consider a fixed decryption query $(\hat{v}, \hat{w}, \hat{w}_2, \hat{c})$, and suppose that $\neg Z' \wedge Z$ occurs at this query. We must have $(\hat{v}, \hat{w}, \hat{w}_2) \neq (v, w, w_2)$, as otherwise, we would not even reach the test at line (5). We must also have $(\hat{v}, \hat{w}) \neq (v, w)$, as otherwise $w_2^* = w_2 \neq \hat{w}_2$, and so event Z could not have occurred at this query. Moreover, since Z' does not occur at this query, we must have $\hat{\rho} \neq \rho$. Let $\hat{\beta} := \mathsf{Dlog}_g \hat{v}$ and $\hat{\gamma} = \mathsf{Dlog}_g \hat{w}$. Since Z occurs at this query, we must have $\hat{\gamma} \neq \alpha\hat{\beta}$.

Summarizing, if $\neg Z' \wedge Z$ occurs at this query, we must have

$$\hat{\rho} \neq \rho, \quad \hat{\gamma} \neq \alpha\hat{\beta}, \quad \text{and} \quad \hat{w}_2 = w_2^*.$$

We can express the relationship between the values σ_2, τ_2, σ_3, τ_3 and the values $\mathsf{Dlog}_g u_2$, $\mathsf{Dlog}_g u_3$, $\mathsf{Dlog}_g w_2$, $\mathsf{Dlog}_g w_2^*$ as a matrix equation:

$$\begin{pmatrix} \mathsf{Dlog}_g u_2 \\ \mathsf{Dlog}_g u_3 \\ \mathsf{Dlog}_g w_2 \\ \mathsf{Dlog}_g w_2^* \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ \beta & \gamma & \rho\beta & \rho\gamma \\ \hat{\beta} & \hat{\gamma} & \hat{\rho}\hat{\beta} & \hat{\rho}\hat{\gamma} \end{pmatrix}}_{=:\overline{M}} \begin{pmatrix} \sigma_2 \\ \tau_2 \\ \sigma_3 \\ \tau_3 \end{pmatrix}. \tag{12.24}$$

An essential fact is that the matrix $\overline{M}$ is non-singular. Indeed, one can again just compute the determinant
$$\det(\overline{M}) = (\rho - \hat{\rho})(\gamma - \alpha\beta)(\hat{\gamma} - \alpha\hat{\beta}),$$
which is nonzero under our assumptions.

Since σ_2, τ_2, σ_3, and τ_3 are uniformly and independently distributed over $\mathbb{Z}_q$, and $\overline{M}$ is non-singular, the values $\mathsf{Dlog}_g u_2$, $\mathsf{Dlog}_g u_3$, $\mathsf{Dlog}_g w_2$, and $\mathsf{Dlog}_g w_2^*$ are also uniformly and independently distributed over $\mathbb{Z}_q$. Moreover, in Game 4, the only information the adversary obtains about σ_2, τ_2, σ_3, and τ_3 is that implied by the values $\mathsf{Dlog}_g u_2$, $\mathsf{Dlog}_g u_3$, and $\mathsf{Dlog}_g w_2$. This is where we use the fact that the test at line (5) is now implemented in terms of the values $\alpha_2 = \mathsf{Dlog}_g u_2$ and $\alpha_3 = \mathsf{Dlog}_g u_3$, defined in (12.19). That is, the test itself only uses information that is already present in the public key. It follows that the value $\hat{w}_2$ computed by the adversary is independent of the correct value w_2^*; therefore, $\hat{w}_2 = w_2^*$ with probability $1/q$. The bound (12.23) then follows from the union bound.

Game 5. We replace the line marked (2) with

$$\begin{array}{ll}
(2) & w_1 \xleftarrow{\text{R}} \mathbb{G}
\end{array}$$

After this change, the lines marked (1), (2), (3), (5), and (6) in Fig. 12.3 now read as follows:

$$\begin{array}{ll}
(1) & \gamma \xleftarrow{\text{R}} \mathbb{Z}_q \setminus \{\alpha\beta\} \\
(2) & w_1 \xleftarrow{\text{R}} \mathbb{G} \\
(3) & w_2 \leftarrow v^{\sigma_2+\rho\sigma_3} w^{\tau_2+\rho\tau_3} \\
(5) & \quad\quad \text{if } \hat{v}^{\alpha} = \hat{w} \text{ and } \hat{v}^{\alpha_2+\hat{\rho}\alpha_3} = \hat{w}_2 \text{ then} \\
(6) & \quad\quad\quad\quad \hat{w}_1 \leftarrow \hat{v}^{\alpha_1}
\end{array}$$

We claim that
$$\Pr[W_5] = \Pr[W_4]. \tag{12.25}$$
This is because, as already argued in the analysis of Game 2, the values $\mathsf{Dlog}_g u_1$ and $\mathsf{Dlog}_g w_1$ are related to the random values σ_1 and τ_1 by the matrix equation (12.18), where the matrix M is non-singular. Moreover, in Game 4, the only information the adversary obtains about σ_1 and τ_1 is that implied by $\mathsf{Dlog}_g u_1$ and $\mathsf{Dlog}_g w_1$. This is where we use the fact that the computation at line (6) is implemented in terms of $\alpha_1 = \mathsf{Dlog}_g u_1$. That is, the computation of $\hat{w}_1$ at line (6) only uses information that is already present in the public key. Thus, replacing w_1 by a truly random group element does not really change the game at all.

Game 6. Finally, the stage is set to play our "KDF card" and "1CCA card". We replace the line marked (4) by

$$\begin{array}{ll}
(4) & k \xleftarrow{\text{R}} \mathcal{K}
\end{array}$$

After this change, the lines marked (1)–(6) in Fig. 12.3 now read as follows:

$$\begin{array}{ll}
(1) & \gamma \xleftarrow{\text{R}} \mathbb{Z}_q \setminus \{\alpha\beta\} \\
(2) & w_1 \xleftarrow{\text{R}} \mathbb{G} \\
(3) & w_2 \leftarrow v^{\sigma_2+\rho\sigma_3} w^{\tau_2+\rho\tau_3} \\
(4) & k \xleftarrow{\text{R}} \mathcal{K} \\
(5) & \quad\quad \text{if } \hat{v}^{\alpha} = \hat{w} \text{ and } \hat{v}^{\alpha_2+\hat{\rho}\alpha_3} = \hat{w}_2 \text{ then} \\
(6) & \quad\quad\quad\quad \hat{w}_1 \leftarrow \hat{v}^{\alpha_1}
\end{array}$$

It should be clear that

$$\left|\Pr[W_5] - \Pr[W_6]\right| \le \text{KDF}\mathsf{adv}[\mathcal{B}_{\text{kdf}}, H] \tag{12.26}$$

and

$$\left|\Pr[W_6] - 1/2\right| = 1\text{CCA}\mathsf{adv}^*[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}], \tag{12.27}$$

where $\mathcal{B}_{\text{kdf}}$ is an efficient adversary attacking H as a KDF, and $\mathcal{B}_{\text{s}}$ is a 1CCA adversary attacking $\mathcal{E}_{\text{s}}$.

The bound (12.14) now follows directly from (12.15), (12.16), (12.17), (12.20), (12.21), (12.22), (12.23), (12.25), (12.26), and (12.27). That completes the proof of the theorem. □

12.6 CCA security via a generic transformation

We have presented several constructions of CCA-secure public key encryption schemes. In Section 12.3, we saw how to achieve CCA security in the random oracle model using a trapdoor function scheme, and in particular (in Section 12.3.1) with RSA. In Section 12.4, we saw how to achieve CCA security in the random oracle model under the interactive CDH assumption, and with a bit more effort, we were able to achieve CCA security in Section 12.5 without resorting to the random oracle model, but under the DDH assumption.

It is natural to ask if there is a generic transformation that converts any CPA-secure public key encryption scheme into one that is CCA-secure, as we did for symmetric encryption in Chapter 9. The answer is yes. In the random oracle model it is possible to give a simple and efficient transformation from CPA-security to CCA-security. This transformation, called the **Fujisaki-Okamoto transformation**, allows one to efficiently convert any public-key encryption scheme that satisfies a very weak security property (weaker than CPA security) into a public-key encryption scheme that is CCA-secure in the random oracle model. It is possible, in principle, to give a similar transformation without relying on random oracles, however, the known constructions are too inefficient to be used in practice [33].

Applications. We show in Section 12.6.2 that applying the Fujisaki-Okamoto transformation to a variant of ElGamal encryption, gives a public key encryption scheme that is CCA-secure in the random oracle model under the ordinary CDH assumption, rather than the stronger, interactive CDH assumption. (Exercise 12.23 develops another approach to achieving the same result, with a tighter security reduction to the CDH assumption).

Beyond ElGamal, the Fujisaki-Okamoto transformation can be applied to other public key encryption schemes, such as Regev's lattice-based encryption scheme discussed in Chapter 17, the McEliece coding-based scheme [73], and the NTRU scheme [54]. All these systems can be made CCA secure, in the random oracle model, using the technique in this section.

The Fujisaki-Okamoto transformation. It is best to understand the Fujisaki-Okamoto transformation as a technique that allows us to build a trapdoor function scheme $\mathcal{T}_{\text{FO}}$ that is one way, even given an image oracle (as in Definition 12.3), starting from any one-way, probabilistic public-key encryption scheme $\mathcal{E}_{\text{a}} = (G_{\text{a}}, E_{\text{a}}, D_{\text{a}})$. We can then plug $\mathcal{T}_{\text{FO}}$ into the construction $\mathcal{E}'_{\text{TDF}}$ presented in Section 12.3, along with a 1CCA symmetric cipher, to obtain a public-key encryption scheme $\mathcal{E}_{\text{FO}}$ that is CCA secure in the random oracle model.

Let $\mathcal{E}_a = (G_a, E_a, D_a)$ be an arbitrary public-key encryption scheme with message space $\mathcal{X}$ and ciphertext space $\mathcal{Y}$.

- The encryption algorithm E_a may be probabilistic, and in this case, it will be convenient to make its random coin tosses explicit. To this end, let us view E_a as a *deterministic* algorithm that takes three inputs: a public key pk, a message $x \in \mathcal{X}$, and a randomizer $r \in \mathcal{R}$, where $\mathcal{R}$ is some finite **randomizer space**. To encrypt a message $x \in \mathcal{X}$ under a public key pk, one chooses $r \in \mathcal{R}$ at random, and then computes the ciphertext $E_a(pk, x; r)$.

- In general, the decryption algorithm D_a may return the special symbol reject; however, we will assume that this is not the case. That is, we will assume that D_a always returns an element in the message space $\mathcal{X}$. This is not a serious restriction, as we can always modify the decryption algorithm so as to return some default message instead of reject. This assumption will simplify the presentation somewhat.

The Fujisaki-Okamoto transformation applied to $\mathcal{E}_a = (G_a, E_a, D_a)$ works as follows. We will also need a hash function $U : \mathcal{X} \to \mathcal{R}$, mapping messages to randomizers, which will be modeled as a random oracle in the security analysis. The trapdoor function scheme is $\mathcal{T}_{\text{FO}} = (G_a, F, D_a)$, defined over $(\mathcal{X}, \mathcal{Y})$, where

$$F(pk, x) := E_a(pk, x; U(x)). \tag{12.28}$$

To prove that $\mathcal{T}_{\text{FO}}$ is one way given an image oracle, in addition to modeling U as a random oracle, we will need to make the following assumptions, which will be made more precise below:

1. $\mathcal{E}_a$ is **one way**, which basically means that given an encryption of a random message $x \in \mathcal{X}$, it is hard to compute x;

2. $\mathcal{E}_a$ is **unpredictable**, which basically means that a random re-encryption of any ciphertext $y \in \mathcal{Y}$ is unlikely to be equal to y.

We now make the above assumptions more precise. As usual, the one-wayness property is defined in terms of an attack game.

Attack Game 12.4 (One-way encryption). For a given public-key encryption scheme $\mathcal{E}_a = (G_a, E_a, D_a)$ with message space $\mathcal{X}$, ciphertext space $\mathcal{Y}$, and randomizer space $\mathcal{R}$, and a given adversary $\mathcal{A}$, the attack game proceeds as follows:

- The challenger computes

  $$(pk, sk) \xleftarrow{\text{R}} G_a(), \ \ x \xleftarrow{\text{R}} \mathcal{X}, \ \ r \xleftarrow{\text{R}} \mathcal{R}, \ \ y \leftarrow E_a(pk, r; s),$$

  and sends (pk, y) to the adversary.

- The adversary outputs $\hat{x} \in \mathcal{R}$.

We say $\mathcal{A}$ wins the above game if $\hat{x} = x$, and we define $\mathcal{A}$'s advantage $\text{OW}\mathsf{adv}[\mathcal{A}, \mathcal{E}_a]$ to be the probability that $\mathcal{A}$ wins the game. $\square$

Definition 12.5 (One-way encryption). *A public-key encryption scheme $\mathcal{E}_a$ is **one way** if for every efficient adversary $\mathcal{A}$, the value $\text{OW}\mathsf{adv}[\mathcal{A}, \mathcal{E}_a]$ is negligible.*

Note that because $\mathcal{E}_a$ may be probabilistic, an adversary that wins Attack Game 12.4 may not even know that they have won the game.

We define unpredictable encryption as follows.

Definition 12.6 (Unpredictable encryption). *Let $\mathcal{E}_a = (G_a, E_a, D_a)$ be a given public-key encryption scheme with message space $\mathcal{X}$, ciphertext space $\mathcal{Y}$, and randomizer space $\mathcal{R}$. We say $\mathcal{E}_a$ is **ϵ-unpredictable** if for every possible output (pk, sk) of G_a and every $y \in \mathcal{Y}$, if we choose $r \in \mathcal{R}$ at random, then we have*

$$\Pr[E_a(pk, D_a(sk, y); r) = y] \leq \epsilon.$$

*We say $\mathcal{E}_a$ is **unpredictable** if it is ϵ-unpredictable for negligible ϵ.*

We note that the one-wayness assumption is implied by semantic security (see Exercise 12.9). We also note that, any public-key encryption scheme that is semantically secure typically is also unpredictable, even though this is not implied by the definition. Moreover, any public-key encryption scheme can be easily transformed into one that satisfies this assumption, without affecting the one-wayness assumption (see Exercise 12.10).

Theorem 12.6. *If U is modeled as a random oracle, and if $\mathcal{E}_a$ is one way and unpredictable, then the trapdoor function scheme $\mathcal{T}_{FO}$, resulting from the Fujisaki-Okamoto transformation (12.28), is one way given an image oracle.*

In particular, assume that $\mathcal{E}_a$ is ϵ-unpredictable. Also assume that adversary $\mathcal{A}$ attacks $\mathcal{T}_{FO}$ as in the random oracle version of Attack Game 12.2, and makes at most Q_{io} image oracle queries and Q_{ro} random oracle queries. Moreover, assume that $\mathcal{A}$ always includes its output among its random oracle queries. Then there exists an adversary $\mathcal{B}_{ow}$ that breaks the one-wayness assumption for $\mathcal{E}_a$ as in Attack Game 12.4, where $\mathcal{B}_{ow}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathrm{OW^{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{T}_{FO}] \leq Q_{io} \cdot \epsilon + Q_{ro} \cdot \mathrm{OW}\mathsf{adv}[\mathcal{B}, \mathcal{E}_a]. \tag{12.29}$$

Proof. We define Game 0 to be the game played between $\mathcal{A}$ and the challenger in the random oracle version of Attack Game 12.2 with respect to $\mathcal{T}_{FO} = (G_a, F, D_a)$. We then modify the challenger several times to obtain Games 1, 2, and so on. In each game, x denotes the random element of $\mathcal{X}$ chosen by the challenger. For $j = 0, 1, \ldots,$ we define W_j to be the event that x is among the random oracle queries made by $\mathcal{A}$ in Game j. As stated above, we assume that $\mathcal{A}$ always queries the random oracle at its output value: this is a reasonable assumption, and we can always trivially modify an any adversary to ensure that it behaves this way, increasing its random-oracle queries by at most 1. Clearly, we have

$$\mathrm{OW^{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{T}_{FO}] \leq \Pr[W_0]. \tag{12.30}$$

Game 0. The challenger in Game 0 has to respond to random oracle queries, in addition to image oracle queries. We make use of an associative array $Map : \mathcal{X} \to \mathcal{R}$ to implement the random oracle representing the hash function U. The logic of the challenger is shown in Fig. 12.4. The adversary can make any number of random oracle queries and any number of image queries. The associative array $Pre : \mathcal{Y} \to \mathcal{X}$ is used to track the adversary's random oracle queries. Basically, $Pre[\hat{y}] = \hat{x}$ means that $\hat{y}$ is the image of $\hat{x}$ under $F(pk, \cdot)$.

Game 1. In this game, we make the following modification to the challenger. The line marked (2) in the logic for processing decryption queries is modified as follows:

initialization:
    $(pk, sk) \stackrel{R}{\leftarrow} G_a()$, $x \stackrel{R}{\leftarrow} \mathcal{X}$, $r \stackrel{R}{\leftarrow} \mathcal{R}$, $y \leftarrow E_a(pk, x; r)$
    initialize empty associative arrays $Map : \mathcal{X} \to \mathcal{R}$ and $Pre : \mathcal{Y} \to \mathcal{X}$
(1) $Map[x] \leftarrow r$
    send the public key pk to $\mathcal{A}$;

upon receiving an image oracle query $\hat{y} \in \mathcal{Y}$:
    if $\hat{y} = y$ then
        $result \leftarrow$ "yes"
    else
        $\hat{x} \leftarrow D_a(sk, \hat{y})$
        if $\hat{x} \notin \text{Domain}(Map)$ then $Map[\hat{x}] \stackrel{R}{\leftarrow} \mathcal{X}$
        $\hat{r} \leftarrow Map[\hat{x}]$
(2)   if $E_a(pk, \hat{x}; \hat{r}) = \hat{y}$
            then $result \leftarrow$ "yes"
            else $result \leftarrow$ "no"
    send $result$ to $\mathcal{A}$;

upon receiving a random oracle query $\hat{x} \in \mathcal{X}$:
    if $\hat{x} \notin \text{Domain}(Map)$ then $Map[\hat{x}] \stackrel{R}{\leftarrow} \mathcal{R}$
    $\hat{r} \leftarrow Map[\hat{x}]$, $\hat{y} \leftarrow E_a(pk, \hat{x}; \hat{r})$, $Pre[\hat{y}] \leftarrow \hat{x}$
    send $\hat{r}$ to $\mathcal{A}$

Figure 12.4: Game 0 challenger

(2)   if $\hat{y} \in \text{Domain}(Pre)$

Let Z_1 be the event that in Game 1, the adversary submits an image oracle query $\hat{y}$ such that

$$\hat{y} \neq y, \quad \hat{y} \notin \text{Domain}(Pre), \quad \text{and} \quad E_a(pk, \hat{x}; \hat{r}) = \hat{y},$$

where $\hat{x}$ and $\hat{r}$ are computed as in the challenger. It is clear that Games 0 and 1 proceed identically unless Z_1 occurs, and so by the Difference Lemma, we have

$$|\Pr[W_1] - \Pr[W_0]| \leq \Pr[Z_1]. \tag{12.31}$$

We argue that

$$\Pr[Z_1] \leq Q_{io} \cdot \epsilon, \tag{12.32}$$

where we are assuming that $\mathcal{E}_a$ is ϵ-unpredictable. Indeed, observe that in Game 1, if $\mathcal{A}$ makes an image query $\hat{y}$ with

$$\hat{y} \neq y \quad \text{and} \quad \hat{y} \notin \text{Domain}(Pre),$$

then either

- $\hat{x} = x$, and so $E_a(pk, \hat{x}; \hat{r}) = y \neq \hat{y}$ with certainty, or

upon receiving an image oracle query $\hat{y} \in \mathcal{Y}$:
    if $\hat{y} \in \{y\} \cup \text{Domain}(Pre)$ then
        then $result \leftarrow$ "yes"
        else $result \leftarrow$ "no"
    send $result$ to $\mathcal{A}$

Figure 12.5: Modified logic for image oracle queries

- $\hat{x} \neq x$, and so $\hat{r}$ is independent of $\mathcal{A}$'s view, from which it follows that $E_{\text{a}}(pk, \hat{x}; \hat{r}) = \hat{y}$ with probability at most ϵ.

The inequality (12.32) then follows by the union bound.

Game 2. This game is the same Game 1, except that we implement the image oracle queries using the logic described in Fig. 12.5. The idea is that in Game 1, we do not really need to use the secret key to implement the image oracle queries.

It should be clear that

$$\Pr[W_2] = \Pr[W_1]. \tag{12.33}$$

Since we do not use the secret key at all in Game 2, this makes it easy to play our "one-wayness card."

Game 3. In this game, we delete the line marked (1) in Fig. 12.4.

We claim that

$$\Pr[W_3] = \Pr[W_2]. \tag{12.34}$$

Indeed, Games 2 and 3 proceed identically until $\mathcal{A}$ queries the random oracle at x. So if W_2 does not occur, neither does W_3, and if W_3 does not occur, neither does W_2. That is, W_2 and W_3 are identical events.

We sketch the design an efficient adversary $\mathcal{B}$ such that

$$\Pr[W_3] \leq Q_{\text{ro}} \cdot \text{OW}\mathsf{adv}[\mathcal{B}, \mathcal{E}_{\text{a}}]. \tag{12.35}$$

The basic idea, as usual, is that $\mathcal{B}$ plays the role of challenger to $\mathcal{A}$, as in Game 3, except that the values pk, sk, x, r, and y are generated by $\mathcal{B}$'s OW challenger, from which $\mathcal{B}$ obtains the values pk and y. Adversary $\mathcal{B}$ interacts with $\mathcal{A}$ just as the challenger in Game 3. The key observation is that $\mathcal{B}$ does not need to know the values sk, x, and r in order to carry out its duties. At the end of the game, if $\mathcal{A}$ made a random oracle query at the point x, then the value x will be contained in the set Domain(Map). In general, it may not be easy to determine which of the values in this set is the correct decryption of y, and so we use our usual guessing strategy; namely, $\mathcal{B}$ simply chooses an element at random from Domain(Map) as its guess at the decryption of y. It is clear that the inequality (12.35) holds.

The inequality (12.29) now follows from (12.30)–(12.35). That proves the theorem. □

12.6.1 A generic instantiation

Putting all the pieces together, we get the following public-key encryption scheme $\mathcal{E}_{\rm FO}$. The components consist of:

- a public-key encryption scheme $\mathcal{E}_{\rm a} = (G_{\rm a}, E_{\rm a}, D_{\rm a})$, with message space $\mathcal{X}$, ciphertext space $\mathcal{Y}$, and randomizer space $\mathcal{R}$;
- a symmetric cipher $\mathcal{E}_{\rm s} = (E_{\rm s}, D_{\rm s})$, with key space $\mathcal{K}$ and message space $\mathcal{M}$;
- hash functions $H : \mathcal{X} \to \mathcal{K}$ and $U : \mathcal{X} \to \mathcal{R}$.

The scheme $\mathcal{E}_{\rm FO} = (G_{\rm a}, E, D)$ has message space $\mathcal{M}$ and ciphertext space $\mathcal{Y} \times \mathcal{C}$. Encryption and decryption work as follows:

$$
\begin{array}{rcl}
E(pk, m) & := & x \xleftarrow{\text{R}} \mathcal{X},\ \ r \leftarrow U(X),\ \ y \leftarrow E_{\rm a}(pk, x; r) \\
 & & k \leftarrow H(x),\ \ c \xleftarrow{\text{R}} E_{\rm s}(k, m) \\
 & & \text{output } (y, c); \\
D(sk,\ (y,c)\ ) & := & x \leftarrow I(sk, y),\ \ r \leftarrow U(x) \\
 & & \text{if } E_{\rm a}(pk, x; r) \neq y \\
 & & \quad\quad \text{then } m \leftarrow \mathsf{reject} \\
 & & \quad\quad \text{else}\ \ k \leftarrow H(x),\ \ m \leftarrow D_{\rm s}(k, c) \\
 & & \text{output } m.
\end{array}
$$

Combining Theorem 12.2 and Theorem 12.6, we immediately get the following:

Theorem 12.7. *If H and U are modeled as a random oracles, $\mathcal{E}_{\rm a}$ is one way and unpredictable, and $\mathcal{E}_{\rm s}$ is 1CCA secure, then the above public-key encryption scheme $\mathcal{E}_{\rm FO}$ is CCA secure.*

In particular, assume that that $\mathcal{E}_{\rm a}$ is ϵ-unpredictable. Then for every 1CCA adversary $\mathcal{A}$ that attacks $\mathcal{E}_{\rm FO}$ as in the random oracle version of Definition 12.2, and which makes at most $Q_{\rm d}$ decryption queries, Q_H queries to the random oracle for H, and Q_U queries to the random oracle for U, there exist an adversary $\mathcal{B}_{\rm ow}$ that breaks the one-wayness assumption for $\mathcal{E}_{\rm a}$ as in Attack Game 12.4, and a 1CCA adversary $\mathcal{B}_{\rm s}$ that attacks $\mathcal{E}_{\rm s}$ as in Definition 9.6, where $\mathcal{B}_{\rm ow}$ and $\mathcal{B}_{\rm s}$ are elementary wrappers around $\mathcal{A}$, such that

$$
\text{1CCA}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\rm FO}] \leq 2(Q_H + Q_U) \cdot \text{OW}\mathsf{adv}[\mathcal{B}_{\rm ow}, \mathcal{E}_{\rm a}] + 2Q_{\rm d} \cdot \epsilon + \text{1CCA}\mathsf{adv}[\mathcal{B}_{\rm s}, \mathcal{E}_{\rm s}]. \tag{12.36}
$$

12.6.2 A concrete instantiation with ElGamal

In the Fujisaki-Okamoto transformation, we can easily use a variant of ElGamal encryption in the role of $\mathcal{E}_{\rm a}$. Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. We define a public-key encryption scheme $\mathcal{E}_{\rm a} = (G_{\rm a}, E_{\rm a}, D_{\rm a})$, with message space $\mathbb{G}$, ciphertext space $\mathbb{G}^2$, and randomizer space $\mathbb{Z}_q$. Public keys are of the form $u \in \mathbb{G}$ and secret keys of the form $\alpha \in \mathbb{Z}_q$. Key generation, encryption, and decryption work as follows:

$$
\begin{array}{rcl}
G_{\rm a}() & := & \alpha \xleftarrow{\text{R}} \mathbb{Z}_q,\ \ u \leftarrow g^\alpha,\ \ pk \leftarrow u,\ \ sk \leftarrow \alpha \\
 & & \text{output } (pk, sk); \\
E_{\rm a}(u, x; \beta) & := & v \leftarrow g^\beta,\ \ w \leftarrow u^\beta,\ \ y \leftarrow wx \\
 & & \text{output } (v, y); \\
D_{\rm a}(\alpha,\ (v, y)) & := & w \leftarrow v^\alpha,\ \ x \leftarrow y/w \\
 & & \text{output } x.
\end{array}
$$

We called this scheme multiplicative ElGamal in Exercise 11.5, where we showed that it is semantically secure under the DDH assumption. It easily verified that $\mathcal{E}_a$ has the following properties:

- $\mathcal{E}_a$ is one-way under the CDH assumption. Indeed, an adversary $\mathcal{A}$ that breaks the one-wayness assumption for $\mathcal{E}_a$ is easily converted to an adversary $\mathcal{B}$ that breaks the CDH with same advantage. Given an instance $(u, v) \in \mathbb{G}^2$ of the CDH problem, adversary $\mathcal{B}$ plays the role of challenger against $\mathcal{A}$ in Attack Game 12.4 as follows:

  - $\mathcal{B}$ sets $y \xleftarrow{R} \mathbb{G}$, and gives $\mathcal{A}$ the public key u and the ciphertext (v, y);
  - when $\mathcal{A}$ outputs $x \in \mathbb{G}$, adversary $\mathcal{B}$ outputs $w \leftarrow y/x$.

  Clearly, if x is the decryption of (v, y), then $w = y/x$ is the solution to the given instance (u, v) of the CDH problem.

- $\mathcal{E}_a$ is $1/q$-unpredictable. Moreover, under the CDH assumption, it must be the case that $1/q$ is negligible.

Putting all the pieces together, we get the following public-key encryption scheme $\mathcal{E}_{\rm FO}^{\rm EG} = (G, E, D)$. The components consist of:

- a cyclic group $\mathbb{G}$ of prime order q generated by $g \in \mathbb{G}$;
- a symmetric cipher $\mathcal{E}_s = (E_s, D_s)$, with key space $\mathcal{K}$ and message space $\mathcal{M}$;
- hash functions $H : \mathbb{G} \to \mathcal{K}$ and $U : \mathbb{G} \to \mathbb{Z}_q$.

The message space of $\mathcal{E}_{\rm FO}^{\rm EG}$ is $\mathcal{M}$ and its ciphertext space is $\mathbb{G}^2 \times \mathcal{C}$. Public keys are of the form $u \in \mathbb{G}$ and secret keys of the form $\alpha \in \mathbb{Z}_q$. The key generation, encryption, and decryption algorithms work as follows:

$$
\begin{array}{rcl}
G() := & := & \alpha \xleftarrow{R} \mathbb{Z}_q,\;\; u \leftarrow g^{\alpha},\;\; pk \leftarrow u,\;\; sk \leftarrow \alpha \\
 & & \text{output } (pk, sk); \\
E(u, m) & := & x \xleftarrow{R} \mathbb{G},\;\; \beta \leftarrow U(x),\;\; v \leftarrow g^{\beta},\;\; w \leftarrow u^{\beta},\;\; y \leftarrow w \cdot x \\
 & & k \leftarrow H(x),\;\; c \xleftarrow{R} E_s(k, m) \\
 & & \text{output } (v, y, c); \\
D(\alpha,\; (v, y, c)) & := & w \leftarrow v^{\alpha},\;\; x \leftarrow y/w,\;\; \beta \leftarrow U(x) \\
 & & \text{if } g^{\beta} = v \\
 & & \quad\quad \text{then } k \leftarrow H(x),\;\; m \leftarrow D_s(k, c) \\
 & & \quad\quad \text{else } \;\; m \leftarrow \mathsf{reject} \\
 & & \text{output } m.
\end{array}
$$

Here, we have optimized the decryption algorithm a bit: if $v = g^{\beta}$, then it follows that $E_a(pk, x; \beta) = (g^{\beta}, u^{\beta} x) = (v, y)$, and so it is unnecessary to execute all of algorithm E_a.

As a special case of Theorem 12.7, we get the following:

Theorem 12.8. *If H and U are modeled as a random oracles, the CDH assumption holds for $\mathbb{G}$, and $\mathcal{E}_s$ is 1CCA secure, then the above public-key encryption scheme $\mathcal{E}_{\rm FO}^{\rm EG}$ is CCA secure.*

In particular, for every 1CCA adversary $\mathcal{A}$ that attacks $\mathcal{E}_{\mathrm{FO}}^{\mathrm{EG}}$ as in the random oracle version of Definition 12.2, and which makes at most Q_{d} decryption queries, Q_H queries to the random oracle for H, and Q_U queries to the random oracle for U, there exist an adversary $\mathcal{B}_{\mathrm{cdh}}$ that breaks the CDH assumption for $\mathbb{G}$ as in Attack Game 10.5, and a 1CCA adversary $\mathcal{B}_{\mathrm{s}}$ that attacks $\mathcal{E}_{\mathrm{s}}$ as in Definition 9.6, where $\mathcal{B}_{\mathrm{cdh}}$ and $\mathcal{B}_{\mathrm{s}}$ are elementary wrappers around $\mathcal{A}$, such that

$$1\mathrm{CCA}^{\mathrm{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\mathrm{FO}}^{\mathrm{EG}}] \leq 2(Q_H + Q_U) \cdot \mathrm{CDH}\mathsf{adv}[\mathcal{B}_{\mathrm{cdh}}, \mathbb{G}] + 2Q_{\mathrm{d}}/q + 1\mathrm{CCA}\mathsf{adv}[\mathcal{B}_{\mathrm{s}}, \mathcal{E}_{\mathrm{s}}]. \tag{12.37}$$

Contrast this result to the construction in Section 12.4.1: to achieve CCA security, instead of the ordinary CDH assumption, that scheme requires the stronger, interactive CDH assumption.

12.7 CCA-secure public-key encryption with associated data

In Section 9.6, we introduced the notion of CCA security for symmetric-key ciphers with associated data. In this section, we briefly sketch how this notion can be adapted to public-key encryption.

First, we have to deal with the syntactic changes. A public-key encryption scheme $\mathcal{E} = (G, E, D)$ with associated data, or **AD public-key encryption scheme**, has the same basic structure as an ordinary public-key encryption scheme, except that the encryption algorithm E and decryption algorithm D each take an additional input d, called the **associated data**. Thus, E gets invoked as $c \xleftarrow{\mathrm{R}} E(pk, m, d)$, and D gets invoked as $m \leftarrow D(sk, c, d)$. As usual, we require that ciphertexts generated by E are correctly decrypted by D, as long as both are given the same associated data. That is, for all possible outputs (pk, sk) of G, and all messages m and associated data d, we have

$$\Pr[D(sk,\ E(pk,\ m,\ d),\ d\ ) = m] = 1.$$

Messages lie in some finite message space $\mathcal{M}$, ciphertexts in some finite ciphertext space $\mathcal{C}$, and associated data in some finite space $\mathcal{D}$. We say that $\mathcal{E}$ is defined over $(\mathcal{M}, \mathcal{D}, \mathcal{C})$.

Definition 12.7 (CCA and 1CCA security with associated data). *The definition of CCA security for ordinary public-key encryption schemes carries over naturally to AD public-key encryption schemes. Attack Game 12.1 is modified as follows. For encryption queries, in addition to a pair of messages (m_{i0}, m_{i1}), the adversary also submits associated data d_i, and the challenger computes $c_i \xleftarrow{\mathrm{R}} E(pk, m_{ib}, d_i)$. For decryption queries, in addition to a ciphertext $\hat{c}_j$, the adversary submits associated data $\hat{d}_j$, and the challenger computes $\hat{m}_j \leftarrow D(sk, \hat{c}_j, \hat{d}_j)$. The restriction is that the* pair *$(\hat{c}_j, \hat{d}_j)$ may not be among the pairs $(c_1, d_1), (c_2, d_2), \ldots$ corresponding to previous encryption queries. An adversary $\mathcal{A}$'s advantage in this game is denoted* $\mathrm{CCA_{ad}}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$*, and the scheme is said to be **CCA secure** if this advantage is negligible for all efficient adversaries $\mathcal{A}$. If we restrict the adversary to a single encryption query, as in Definition 12.2, the advantage is denoted* $1\mathrm{CCA_{ad}}\mathsf{adv}[\mathcal{A}, \mathcal{E}]$*, and the scheme is said to be **1CCA secure** if this advantage is negligible for all efficient adversaries $\mathcal{A}$.*

Observations. We make a couple of simple observations.

- Theorem 12.1 carries over to AD schemes. That is, if an AD public-key encryption scheme is 1CCA secure, then it is also CCA secure. The proof and concrete security bounds go through with no real changes.

- All of the CCA-secure public-key encryption schemes presented in this chapter can be trivially converted to CCA-secure AD public-key encryption schemes, simply by replacing the symmetric cipher $\mathcal{E}_{\mathrm{s}}$ used in each construction with a 1CCA-secure AD cipher. The associated data for the AD public-key scheme is simply passed through to the AD symmetric-key cipher, in both the encryption and decryption algorithms.

Applications. CCA-secure AD public-key encryption has a number of natural applications. One such application is the key-escrow application, which we discussed in Section 12.2.3. In this application, we escrowed a file-encryption key k by encrypting the pair (k, h) under the public-key of a key escrow service. Here, h was the collision-resistant hash of some metadata md associated with the file, and the public-key encryption scheme used by the escrow service was assumed CCA secure. By encrypting the pair (k, h), the escrow service could enforce various access control policies, based on the metadata and the identity or credentials of an entity requesting the key k. However, the metadata itself was considered public information, and it did not really need to be encrypted, except that we wanted it to be bundled in some non-malleable way with the key k. This same effect can be achieved more naturally and efficiently by using a CCA-secure AD public-key encryption scheme, as follows. When the key k is escrowed, the escrow-ciphertext is generated by encrypting k using the metadata md as associated data. When a requesting entity presents a pair (c, md) to the escrow service, the service checks that that the requesting identity's credentials and the supplied metadata conform to the access control policy, and if so, decrypts c using the supplied metadata md as associated data. The access control policy is enforced by the CCA-security property: attempting to decrypt the escrow-ciphertext using non-matching metadata as associated data will not leak any information about the corresponding file-encryption key.

We will also make use use of CCA-secure AD public-key encryption in building signcryption schemes (see Section 13.7.3).

12.8 Case study: PKCS1, OAEP, OAEP+, and SAEP

The most widely used public-key encryption scheme using RSA is described in a standard from RSA Labs called PKCS1. This scheme is quite different from the scheme $\mathcal{E}_{\mathrm{RSA}}$ we presented in Section 12.3.1.

Why does the PKCS1 standard not use $\mathcal{E}_{\mathrm{RSA}}$? The reason is that when encrypting a short message — much shorter than the RSA modulus n — a PKCS1 ciphertext is more compact than an $\mathcal{E}_{\mathrm{RSA}}$ ciphertext. The $\mathcal{E}_{\mathrm{RSA}}$ scheme outputs a ciphertext (y, c) where y is in $\mathbb{Z}_n$ and c is a symmetric ciphertext, while a PKCS1 ciphertext is only a single element of $\mathbb{Z}_n$.

Public-key encryption for short messages is used in a variety of settings. For example, in some key exchange protocols, public-key encryption is only applied to short messages: a symmetric key and some metadata. Similarly, in some access control systems, one encrypts a short access token and nothing else. In these settings, schemes like PKCS1 are more space efficient than $\mathcal{E}_{\mathrm{RSA}}$. It is worth noting, however, that the ElGamal scheme $\mathcal{E}'_{\mathrm{EG}}$ can produce even shorter ciphertexts (although encryption time with ElGamal is typically higher than with RSA).

Our goal in this section is to study PKCS1, and more generally, public-key encryption schemes based on a trapdoor function $\mathcal{T} = (G, F, I)$ defined over $(\mathcal{X}, \mathcal{Y})$, where the ciphertext is just a single element of $\mathcal{Y}$.

12.8.1 Padding schemes

Let $\mathcal{T} = (G, F, I)$ be a trapdoor function defined over $(\mathcal{X}, \mathcal{Y})$, and let $\mathcal{M}$ be some message space, where $|\mathcal{M}| \ll |\mathcal{X}|$. Our goal is to design a public-key encryption scheme where a ciphertext is just a single element in $\mathcal{Y}$. To do so, we use the following general paradigm: to encrypt a message $m \in \mathcal{M}$, the encryptor "encodes" the given message as an element of $\mathcal{X}$, and then applies the trapdoor function to the encoded element to obtain a ciphertext $c \in \mathcal{Y}$. The decryptor inverts the trapdoor function at c, and decodes the resulting value to obtain the message m.

As a first naive attempt, suppose $\mathcal{X} := \{0,1\}^t$ and $\mathcal{M} := \{0,1\}^s$, where, say, $t = 2048$ and $s = 256$. To encrypt a message $m \in \mathcal{M}$ using the public key pk do

$$E(pk, m) := F\big(pk,\ 0^{t-s} \parallel m\big).$$

Here we pad the message m in $\mathcal{M}$ with zeros so that it is in $\mathcal{X}$. To decrypt a ciphertext c, invert the trapdoor function by computing $I(sk, c)$ and strip off the $(t - s)$ zeros on the left.

This naive scheme uses deterministic encryption and is therefore not even CPA secure. It should never be used. Instead, to build a secure public-key scheme we need a better way to encode the message $m \in \mathcal{M}$ into the domain $\mathcal{X}$ of the trapdoor function. The encoding should be invertible to enable decryption, and should be randomized to have some hope of providing CPA security, let alone CCA security. Towards this goal, let us define the notion of a padding scheme.

Definition 12.8. *A* ***padding scheme*** $\mathcal{PS} = (P, U)$*, defined over* $(\mathcal{M}, \mathcal{R}, \mathcal{X})$*, is a pair of efficient algorithms,* P *and* U*, where* $P : \mathcal{M} \times \mathcal{R} \to \mathcal{X}$ *and* $U : \mathcal{X} \to \mathcal{M} \cup \{\ \mathsf{reject}\ \}$ *is its inverse in the following sense:* $U(x) = m$ *whenever* $x = P(m, r)$ *for some* $(m, r) \in \mathcal{M} \times \mathcal{R}$*, and* $U(x) = \mathsf{reject}$ *if* x *is not in the image of* P*.*

For a given padding scheme (P, U) defined over $(\mathcal{M}, \mathcal{R}, \mathcal{X})$, let us define the following pubic-key encryption scheme $\mathcal{E}_{\text{pad}} = (G, E, D)$ derived from the trapdoor function $\mathcal{T} = (G, F, I)$:

$$\begin{array}{ll} E(pk,m) \ := & D(sk,c) \ := \\ \quad r \xleftarrow{R} \mathcal{R},\ \ x \leftarrow P(m,r), & \quad x \leftarrow I(sk,c), \\ \quad c \leftarrow F(pk,x), & \quad m \leftarrow U(x), \\ \quad \text{output } c; & \quad \text{output } m. \end{array} \tag{12.38}$$

When the trapdoor function $\mathcal{T}$ is RSA it will be convenient to call this scheme RSA-$\mathcal{PS}$ encryption. For example, when RSA is coupled with PKCS1 padding we obtain RSA-PKCS1 encryption.

The challenge now is to design a padding scheme $\mathcal{PS}$ for which $\mathcal{E}_{\text{pad}}$ can be proven CCA secure, in the random oracle, under the assumption that $\mathcal{T}$ is one way. Many such padding schemes have been developed with varying properties. In the next subsections we describe several such schemes, their security properties, and limitations.

12.8.2 PKCS1 padding

The oldest padding scheme, which is still in use today, is called PKCS1 padding.

To describe this padding scheme let assume from now on that the domain $\mathcal{X}$ of the trapdoor function is $0^8 \times \{0,1\}^{t-8}$, where t is a multiple of 8. That is, $\mathcal{X}$ consists of all t-bit strings whose left-most 8 bits are zero. These zero bits are meant to accommodate a t-bit RSA modulus, so that all such strings are binary encodings of numbers that are less than the RSA modulus. The message

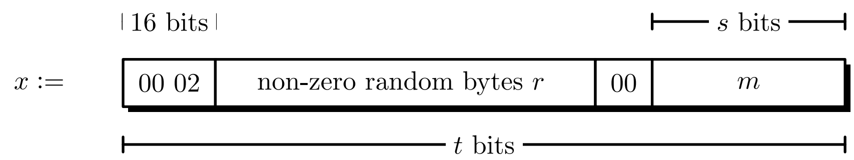


Figure 12.6: PKCS1 padding (mode 2)

space $\mathcal{M}$ consists of all bit strings whose length is a multiple of 8, but at most $t - 88$. The PKCS1 standard is very much byte oriented, which is why all bit strings are multiples of 8. The number 88 is specified in the standard: the message to be encrypted must be at least 11 bytes (88 bits) shorter than the RSA modulus. For an RSA modulus of size 2048 bits, the message can be at most 245 bytes (1960 bits). In practice, messages are often only 32 bytes (256 bits).

The PKCS1 padding algorithm is shown in Fig. 12.6. A double-digit number, like 00 or 02, in the figure denotes a one-byte (8-bit) value in hexadecimal notation. Here, s is the length of the message m. The randomizer r shown in the figure is a sequence of $(t - s)/8 - 3$ random *non-zero* bytes.

The PKCS1 padding scheme (P, U) works as follows. We can take the randomizer space $\mathcal{R}$ to be the set of of all strings r' of non-zero bytes of length $t/8 - 3$; to pad a particular message m, we use a prefix r of r' of appropriate length so that the resulting string x is exactly t-bits long. Here are the details of algorithms P and U.

Algorithm $P(m, r')$:
output $x := \big(00 \parallel 02 \parallel r \parallel 00 \parallel m\big) \in \{0,1\}^t$,
where r is the appropriate prefix of r'

Algorithm $U(x)$:
(1) parse x as $\big(00 \parallel 02 \parallel \text{non-zero bytes } r \parallel 00 \parallel m\big)$
if x cannot be parsed this way, output reject
else, output m

Because the string r contains only non-zero bytes, parsing x in line (1) can be done unambiguously by scanning the string x from left to right. The 16 bits representing 00 02 at the left of the string is the reason why this padding is called PKCS1 mode 2 (mode 1 is discussed in the next chapter).

By coupling PKCS1 padding with RSA, as in (12.38), we obtain the **RSA-PKCS1** encryption scheme. What can we say about the security of RSA-PKCS1? As it turns out, not much. In fact, there is a devastating chosen ciphertext attack on it, which we discuss next.

12.8.3 Bleichenbacher's attack on the RSA-PKCS1 encryption scheme

The RSA-PKCS1 standard, although widely deployed, is not secure against chosen ciphertext attacks. We describe an attack, due to Bleichenbacher, as it applies to the TLS protocol between a client and a server. More recent versions of TLS defend against this attack, as discussed below. The only details of TLS relevant to this discussion is the following:

- During session setup, the client chooses a random 48-byte (192-bit) string, called the `pre_master_secret`, and encrypts it with RSA-PKCS1 under the server's public-key. It sends the resulting ciphertext c to the server in a message called `client_key_exchange`.

- When the server receives a `client_key_exchange` message it extracts the ciphertext c and attempts to decrypt it. If PKCS1 decoding returns reject, the server sends an abort message to the client. Otherwise, it continues normally with session setup.

Let us show a significant vulnerability in this system that is a result of a chosen ciphertext attack on RSA-PKCS1. Suppose the attacker has a ciphertext c that it intercepted from an earlier TLS session with the server. This c is an encryption generated using the server's RSA public key (n, e), with RSA modulus n and encryption exponent e. The attacker's goal is to decrypt c. Let x be the eth root of c in $\mathbb{Z}_n$, so that $x^e = c$ in $\mathbb{Z}_n$. We show how the attacker can learn x, which is sufficient to decrypt c.

The attacker's strategy is based on the following observation: let r be some element in $\mathbb{Z}_n$ and define $c' \leftarrow c \cdot r^e$ in $\mathbb{Z}_n$; then

$$c' = c \cdot r^e = (x \cdot r)^e \ \in \mathbb{Z}_n.$$

The attacker plays the role of a client and attempts to establish a TLS connection with the server. The attacker creates a `client_key_exchange` message that contains c' as the encrypted `pre_master_secret` and sends the message to the server. The server, following the protocol, computes the eth root of c' to obtain $x' = x \cdot r$ in $\mathbb{Z}_n$. Next, the server checks if x' is a proper PKCS1 encoding: does x' begin with the two bytes 00 02, and if so, is it followed by non-zero bytes, then a zero byte, and then 48 additional (message) bytes? If not, the server sends an abort message to the attacker. Otherwise, decryption succeeds and it sends the next TLS message to the attacker. Consequently, the server's response to the attacker's `client_key_exchange` message reveals some information about $x' = x \cdot r$. It tells the attacker if x' is a valid PKCS1 encoding.

The attacker can repeat this process over and over with different values of $r \in \mathbb{Z}_n$ of its choosing. Every time the attacker learns if $x \cdot r$ is a valid PKCS1 encoding or not. In effect, the server becomes an oracle that implements the following predicate for the attacker:

$$P_x(r) := \begin{cases} 1 & \text{if } x \cdot r \text{ in } \mathbb{Z}_n \text{ is a valid PKCS1 encoding;} \\ 0 & \text{otherwise.} \end{cases}$$

The attacker can query this predicate for any $r \in \mathbb{Z}_n$ of its choice and as many times as it wants.

Bleichenbacher showed that for a 2048-bit RSA modulus, this oracle is sufficient to recover all of x with several million queries to the server. Exercise 12.16 gives a simple example of this phenomenon.

This attack is a classic example of a real-world chosen ciphertext attack. The adversary has a challenge ciphertext c that it wants to decrypt. It does so by creating a number of related ciphertexts and asks the server to "partially decrypt" those ciphertexts (i.e., evaluate the predicate P_x). After enough queries, the adversary is able to obtain the decryption of c. Clearly, this attack would not be possible if RSA-PKCS1 were CCA-secure: CCA security implies that such attacks are not possible even given a *full* decryption oracle, let alone a partial decryption oracle like P_x.

This devastating attack lets the attacker eavesdrop on any TLS session of its choice. Given the wide deployment of RSA-PKCS1 in TLS, the question then is how to best defend against it.

The TLS defense. When Bleichenbacher's attack was discovered in 1998, there was a clear need to fix TLS. Moving away from PKCS1 to a completely different padding scheme would have been difficult since it would have required updating both clients and servers, and this can take decades for everyone to update. The challenge was to find a solution that requires only server-side changes, so that deployment can be done server-side only. This will protect all clients, old and new, connecting to an updated server.

The solution, implemented in TLS 1.0 and later, changes the RSA-PKCS1 server-side decryption process to the following procedure:

1. generate a string r of 48 random bytes,
2. decrypt the RSA-PKCS1 ciphertext to recover the plaintext m,
3. if the PKCS1 padding is invalid, or the length of m is not exactly 48 bytes:
4.   set $m \leftarrow r$
5. return m

In other words, when PKCS1 parsing fails, simply choose a random plaintext r and use this r as the decrypted value. Clearly, the TLS session setup will fail further down the line and setup will abort, but presumably doing so at that point reveals no useful information about the decryption of c. Some justification for this process is provided by Jonsson and Kaliski [59]. The TLS 1.2 standard goes further and includes the following warning about this decryption process:

> In any case, a TLS server MUST NOT generate an alert if processing an RSA-encrypted pre-master secret message fails [...] Instead, it MUST continue the handshake with a randomly generated pre-master secret. It may be useful to log the real cause of failure for troubleshooting purposes; however, care must be taken to avoid leaking the information to an attacker (through, e.g., timing, log files, or other channels.)

Note the point about side channels, such as timing attacks, in the last sentence. Suppose the server takes a certain amount of time to respond to a `client_key_exchange` message when the PKCS1 padding is valid, and a different amount of time when it is invalid. Then by measuring the server's response time, the Bleichenbacher attack is easily made possible again.

The DROWN attack. To illustrate the cost of cryptographic mistakes, we mention an interesting attack called DROWN [4]. While implementations of TLS 1.0 and above are immune to Bleichenbacher's attack, an old version of the TLS protocol, called SSL 2.0, is still vulnerable. Although SSL 2.0 is quite old and vulnerable, many Internet servers still support SSL 2.0 so that old clients can connect to them. The trouble is that, in a common TLS deployment, the server has only one TLS public-key pair. The *same* public key is used to establish a session when the latest version of TLS is used, as when the old SSL 2.0 is used. As a result, an attacker can record the ciphertext c used in a modern TLS session, encrypted under the server's public key, and then use Bleichenbacher's attack on the SSL 2.0 implementation to decrypt this c. This lets the attacker decrypt the TLS session, despite the fact that TLS is immune to Bleichenbacher's attack. Effectively, the old SSL 2.0 implementation compromises the modern TLS.

This attack shows that once a cryptographically flawed protocol is deployed, it is very difficult to get rid of it. Even more troubling is that flaws in a protocol can be used to attack later versions of the protocol that have supposedly corrected those flaws. The lesson is: make sure to get the cryptography right the first time. The best way to do that is to only use schemes that have been properly analyzed.

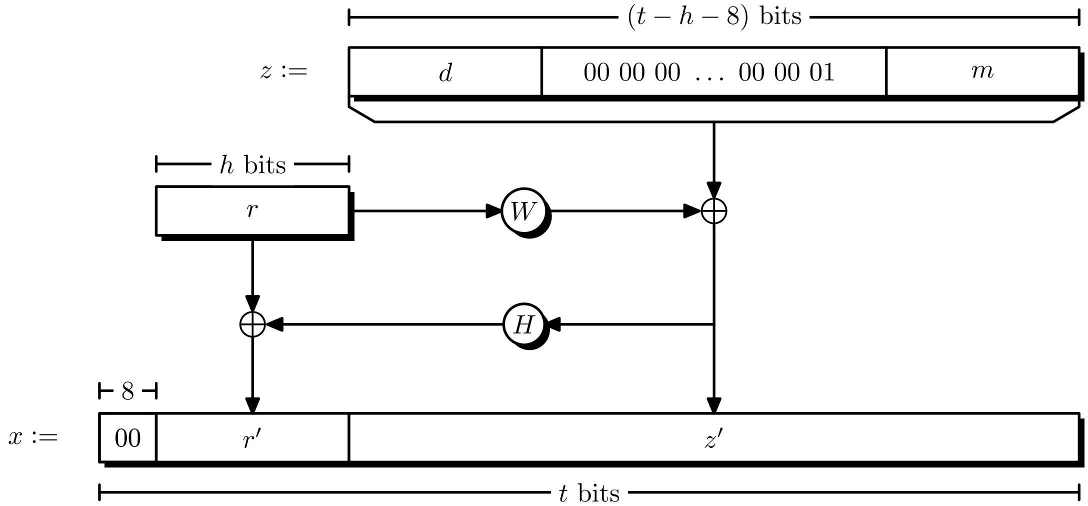


Figure 12.7: OAEP padding using hash functions H and W, and optional associated data d

12.8.4 Optimal Asymmetric Encryption Padding (OAEP)

The failure of RSA-PKCS1 leaves us with the original question: is there a padding scheme (P, U) so that the resulting encryption scheme $\mathcal{E}_{\text{pad}}$ from (12.38) can be shown to be CCA-secure, in the random oracle model, based on the one-wayness of the trapdoor function?

The answer is yes, and the first attempt at such a padding scheme was proposed by Bellare and Rogaway in 1994. This padding, is called Optimal Asymmetric Encryption Padding (OAEP), and the derived public-key encryption scheme was standardized in the PKCS1 version 2.0 standard. It is called "optimal" because the ciphertext is a single element of $\mathcal{Y}$, and nothing else.

The OAEP padding scheme (P, U) is defined over $(\mathcal{M}, \mathcal{R}, \mathcal{X})$, where $\mathcal{R} := \{0,1\}^h$ and $\mathcal{X} := 0^8 \times \{0,1\}^{t-8}$. As usual, we assume that h and t are multiples of eight so that lengths can be measured in bytes. As before, in order to accommodate a t-bit RSA modulus, we insist that the left-most 8 bits of any element in $\mathcal{X}$ are zero. The message space $\mathcal{M}$ consists of all bit strings whose length is a multiple of 8, but at most $t - 2h - 16$.

The scheme also uses two hash functions H and W, where

$$H : \{0,1\}^{t-h-8} \to \mathcal{R}\ , \qquad W : \mathcal{R} \to \{0,1\}^{t-h-8}. \tag{12.39}$$

The set $\mathcal{R}$ should be sufficiently large to be the range of a collision resistant hash. Typically, SHA256 is used as the function H and we set $h := 256$. The function W is derived from SHA256 (see Section 8.10.3 for recommended derivation techniques).

OAEP padding is used to build a public-key encryption scheme with associated data (as discussed in Section 12.7). As such, the padding algorithm P takes an optional third argument $d \in \mathcal{R} = \{0,1\}^h$, representing the associated data. To support associated data that is more than h bits long one can first hash the associated data using a collision resistant hash to obtain an element of $\mathcal{R}$. If no associated data is provided as input to P, then d is set to a constant that identifies the hash function H, as specified in the standard. For example, for SHA256, one sets d to the following

256-bit hex value:

$$d := \texttt{E3B0C442 98FC1C14 9AFBF4C8 996FB924 27AE41E4 649B934C A495991B 7852B855}.$$

Algorithm $P(m, r, d)$ is shown in Fig. 12.7. Every pair of digits in the figure represents one byte (8 bits). The variable length string of zeros in z is chosen so that the total length of z is exactly $(t - h - 8)$ bits. The algorithm outputs an $x \in \mathcal{X}$.

The inverse algorithm U, on input $x \in \mathcal{X}$ and $d \in \mathcal{R}$, is defined as follows:

$$
\begin{array}{ll}
 & \text{parse } x \text{ as } (00 \parallel r' \parallel z') \text{ where } r' \in \mathcal{R} \text{ and } z' \in \{0,1\}^{t-h-8} \\
(1) & \text{if } x \text{ cannot be parsed this way, set } m \leftarrow \mathsf{reject} \\
 & \text{else} \\
 & \quad r \leftarrow H(z') \oplus r', \quad z \leftarrow W(r) \oplus z' \\
 & \quad \text{parse } z \text{ as } (d \parallel 00 \ \ldots \ 00\ 01 \parallel m) \text{ where } d \in \mathcal{R} \text{ and } m \in \mathcal{M} \\
 & \quad \text{if } z \text{ cannot be parsed this way, set } m \leftarrow \mathsf{reject} \\
 & \text{output } m
\end{array}
$$

Finally, the public-key encryption scheme RSA-OAEP is obtained by combining the RSA trapdoor function with the OAEP padding scheme, as in (12.38). When referring to OAEP coupled with a general trapdoor function $\mathcal{T} = (G, F, I)$, we denote the resulting encryption scheme by $\mathcal{E}_{\text{OAEP}} = (G, E, D)$.

The security of $\mathcal{E}_{\text{OAEP}}$. One might hope to prove CCA security of $\mathcal{E}_{\text{OAEP}}$ in the random oracle model using only the assumption that $\mathcal{T}$ is one-way. Unfortunately, that is unlikely because of a counter-example: there is a plausible trapdoor function $\mathcal{T}$ for which the resulting $\mathcal{E}_{\text{OAEP}}$ is vulnerable to a CCA attack. See Exercise 12.18.

Nevertheless, it is possible to prove security of $\mathcal{E}_{\text{OAEP}}$ by making a stronger one-wayness assumption about $\mathcal{T}$, called partial one-wayness. Recall that in the game defining a one-way function, the adversary is given pk and $y \leftarrow F(pk, x)$, for some pk and random $x \in \mathcal{X}$, and is asked to produce x. In the game defining a partial one-way function, the adversary is given pk and y, but is only asked to produce, say, certain bits of x. If no efficient adversary can accomplish even this simpler task, then we say that $\mathcal{T}$ is partial one-way. More generally, instead of producing some bits of x, the adversary is asked to produce a particular function f of x. This is captured in the following game.

Attack Game 12.5 (Partial one-way trapdoor function scheme). For a given trapdoor function scheme $\mathcal{T} = (G, F, I)$, defined over $(\mathcal{X}, \ \mathcal{Y})$, a given efficiently computable function $f : \mathcal{X} \to \mathcal{Z}$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger computes

  $$(pk, sk) \xleftarrow{\text{R}} G(), \quad x \leftarrow \mathcal{X}, \quad y \leftarrow F(pk, x)$$

  and sends (pk, y) to the adversary.

- The adversary outputs $\hat{z} \in \mathcal{Z}$.

We define the adversary's advantage, denoted $\text{POW}\mathsf{adv}[\mathcal{A}, \mathcal{T}, f]$, to be the probability that $\hat{z} = f(x)$.
□

Definition 12.9. *We say that a trapdoor function scheme $\mathcal{T}$ defined over $(\mathcal{X},\ \mathcal{Y})$ is* ***partial one way with respect to*** *$f : \mathcal{X} \to \mathcal{Z}$ if, for all efficient adversaries $\mathcal{A}$, the quantity* $\mathrm{POW}\mathsf{adv}[\mathcal{A}, \mathcal{T}, f]$ *is negligible.*

Clearly, a partial one-way trapdoor function is also a one-way trapdoor function: if an adversary can recover x it can also recover $f(x)$. Therefore, the assumption that a trapdoor function is partial one way is at least as strong as assuming that the trapdoor function is one way.

The following theorem, due to Fujisaki, Okamoto, Pointcheval, and Stern, shows that $\mathcal{E}_{\mathrm{OAEP}}$ is CCA-secure in the random oracle model, assuming $\mathcal{T}$ is partial one-way. The proof can be found in their paper [41].

Theorem 12.9. *Let t, h, $\mathcal{X}$, H, and W be as in the OAEP construction. Assume H and W are modeled as a random oracles. Let $\mathcal{T} = (G, F, I)$ be a trapdoor function defined over $(\mathcal{X}, \mathcal{Y})$. Let $f : \mathcal{X} \to \{0,1\}^{t-h-8}$ be the function that returns the right-most $(t-h-8)$ bits of its input. If $\mathcal{T}$ is partial one way with respect to f, and 2^h is super-poly, then $\mathcal{E}_{\mathrm{OAEP}}$ is CCA secure.*

Given Theorem 12.9 the question is then: is RSA a partial one-way function? We typically assume RSA is one-way, but is it partial one-way when the adversary is asked to compute only $(t-h-8)$ bits of the pre-image? As it turns out, if RSA is one-way then it is also partial one-way. More precisely, suppose there is an efficient adversary $\mathcal{A}$ that given an RSA modulus n and encryption exponent e, along with $y \leftarrow x^e \in \mathbb{Z}_n$ as input, outputs more than half the least significant bits of x. Then there is an efficient adversary $\mathcal{B}$ that uses $\mathcal{A}$ and recovers all the bits of x. See Exercise 12.19.

As a result of this wonderful fact, we obtain as a corollary of Theorem 12.9 that RSA-OAEP is CCA-secure in the random oracle model assuming only that RSA is a one-way function. However, the concrete security bounds obtained when proving CCA security of RSA-OAEP based on the one-wayness of RSA are quite poor.

Manger's timing attack. RSA-OAEP is tricky to implement securely. Suppose the OAEP algorithm $U(x, d)$ were implemented so that it takes a certain amount of time when the input is rejected because of the test on line (1), and a different amount of time when the test succeeds. Notice that rejection on line (1) occurs when the eight most significant bits of x are not all zero. Now, consider again the setting of Bleichenbacher's attack on PKCS1. The adversary has a ciphertext c, generated using under the server's RSA public key, with RSA modulus n and encryption exponent e. The adversary wants to decrypt c. It can repeatedly interact with the server, sending it $c' \leftarrow c \cdot r^e$ in $\mathbb{Z}_n$, for various values of r of the adversary's choice. By measuring the time that the server takes to respond, the attacker can tell if rejection happened because of line (1). Therefore, the attacker learns if the eight most significant bits of $(c')^{1/e}$ in $\mathbb{Z}_n$ are all zero. As in Bleichenbacher's attack, this partial decryption oracle is sufficient to decrypt all of c. See Exercise 12.16, or Manger [69], for the full details.

12.8.5 OAEP+ and SAEP+

In the previous section we saw that RSA-OAEP is CCA-secure assuming RSA is a one-way function. However, for other one-way trapdoor functions, the derived scheme $\mathcal{E}_{\mathrm{OAEP}}$ may not be CCA-secure.

The next question is then: is there a padding scheme (P, U) that, when coupled with a general trapdoor function, gives a CCA-secure scheme in the random oracle model? The answer is yes, and a padding scheme that does so, called OAEP+, is a variation of OAEP [97]. The difference, essentially, is that the block of zero bytes in Fig. 12.7 is replaced with the value $H'(m,r)$ for some hash function H'. This block is verified during decryption by recomputing $H'(m,r)$ from the recovered values for m and r. The ciphertext is rejected if the wrong value is found in this block.

For RSA specifically, it is possible to use a simpler CCA-secure padding scheme. This simpler padding scheme, called SAEP+, eliminates the hash function H and the corresponding xor on the left of H in Fig. 12.7. The randomizer r needs to be longer than in OAEP. Specifically, r must be slightly longer than half the size of the modulus, that is, slightly more than $t/2$ bits. RSA-SAEP+ is CCA-secure, in the random oracle model, assuming the RSA function is one-way [21]. It provides a simple alternative padding scheme for RSA.

12.9 Fun application: sealed bid auctions

To be written.

12.10 Notes

Citations to the literature to be added.

12.11 Exercises

12.1 (Insecurity of multiplicative ElGamal). Show that multiplicative ElGamal from Exercise 11.5 is not CCA secure. Your adversary should have an advantage of 1 in the 1CCA attack game.

12.2 (Sloppy CCA). Let $\mathcal{E} = (G, E, D)$ be a CCA-secure public-key encryption scheme defined over $(\mathcal{M}, \mathcal{C})$ where $\mathcal{C} := \{0,1\}^{\ell}$. Consider the encryption scheme $\mathcal{E}' = (G, E', D')$ defined over $(\mathcal{M}, \mathcal{C}')$ where $\mathcal{C} := \{0,1\}^{\ell+1}$ as follows:

$$E'(pk, m) := E(k, m) \parallel 0 \quad \text{and} \quad D'(sk, c) := D(sk, c[0 \mathinner{..} \ell-1]).$$

That is, the last ciphertext bit can be 0 or 1, but the decryption algorithm ignores this bit. Show that $\mathcal{E}'$ is not CCA secure. Your adversary should have an advantage of 1 in the 1CCA attack game.

Discussion: Clearly, adding a bit to the ciphertext does not harm security in practice, yet it breaks CCA security of the scheme. This issue suggests that the definition of CCA security may be too strong. A different notion, called **generalized CCA** (gCCA), weakens the definition of CCA security so that simple transformations of the ciphertext, like the one in $\mathcal{E}'$, do not break gCCA security. More formally, we assume that for each key pair (pk, sk), there is an equivalence relation $\equiv_{pk}$ on ciphertexts such that

$$c \equiv_{pk} c' \implies D(sk, c) = D(sk, c').$$

Moreover, we assume that given pk, c, c', it is easy to tell if $c \equiv_{pk} c'$. Note that the relation $\equiv_{pk}$ is specific to the particular encryption scheme. Then, in Attack Game 12.1, we insist each decryption query is not equivalent to (as opposed to not equal to) any ciphertext arising from a previous encryption query.

12.3 (Extending the message space). Continuing with Exercise 11.7. Show that even if $\mathcal{E}$ is CCA secure, $\mathcal{E}^2$ is not CCA secure. For this, you should assume $\mathcal{M}$ is non-trivial (i.e., contains at least two messages of the same length).

Note: The next exercise presents a correct way to extend the message space of a CCA-secure encryption scheme.

12.4 (Modular hybrid construction). All of the public-key encryption schemes presented in this chapter can be viewed as special cases of the general hybrid construction introduced in Exercise 11.8.

Consider a KEM $\mathcal{E}_{\text{kem}} = (G, E_{\text{kem}}, D_{\text{kem}})$, defined over $(\mathcal{K}, \mathcal{C}_{\text{kem}})$. We define 1CCA security for $\mathcal{E}_{\text{kem}}$ in terms of an attack game, played between a challenger and an adversary $\mathcal{A}$, as follows. In Experiment b, for $b = 0, 1$, the challenger first computes

$$(pk, sk) \xleftarrow{\text{R}} G(), \;\; (k_0, c_{\text{kem}}) \xleftarrow{\text{R}} E_{\text{kem}}(pk), \;\; k_1 \xleftarrow{\text{R}} \mathcal{K},$$

and sends (k_b, c_{kem}) to $\mathcal{A}$. Next, the adversary submits a sequence of decryption queries to the challenger. Each such query is of the form $\hat{c}_{\text{kem}} \in \mathcal{C}_{\text{kem}}$, to which the challenger responds with $D_{\text{kem}}(sk, \hat{c}_{\text{kem}})$. Finally, $\mathcal{A}$ outputs $\hat{b} \in \{0, 1\}$. As usual, if W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b, we define $\mathcal{A}$'s **advantage** with respect to $\mathcal{E}_{\text{kem}}$ as $\text{1CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\text{kem}}] := |\Pr[W_0] - \Pr[W_1]|$, and if this advantage is negligible for all efficient adversaries, we say that $\mathcal{E}_{\text{kem}}$ is **1CCA secure**.

If $\mathcal{E}_{\text{s}}$ is a symmetric cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, then as in Exercise 11.8, we also consider the hybrid public-key encryption scheme $\mathcal{E} = (G, E, D)$, defined over $(\mathcal{M}, \mathcal{C}_{\text{kem}} \times \mathcal{C})$, constructed out of $\mathcal{E}_{\text{kem}}$ and $\mathcal{E}_{\text{s}}$.

(a) Prove that $\mathcal{E}$ is CCA secure, assuming that $\mathcal{E}_{\text{kem}}$ and $\mathcal{E}_{\text{s}}$ are 1CCA secure. You should prove a concrete security bound that says that for every adversary $\mathcal{A}$ attacking $\mathcal{E}$, there are adversaries $\mathcal{B}_{\text{kem}}$ and $\mathcal{B}_{\text{s}}$ (which are elementary wrappers around $\mathcal{A}$) such that

$$\text{1CCA}\mathsf{adv}[\mathcal{A}, \mathcal{E}] \leq 2 \cdot \text{1CCA}\mathsf{adv}[\mathcal{B}_{\text{kem}}, \mathcal{E}_{\text{kem}}] + \text{1CCA}\mathsf{adv}[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}].$$

(b) Describe the KEM corresponding to $\mathcal{E}'_{\text{TDF}}$ and prove that it is 1CCA secure (in the random oracle model, assuming $\mathcal{T}$ is one way given an image oracle).

(c) Describe the KEM corresponding to $\mathcal{E}'_{\text{EG}}$ and prove that it is 1CCA secure (in the random oracle model, under the ICDH assumption for $\mathbb{G}$).

(d) Give examples that show that if one of $\mathcal{E}_{\text{kem}}$ and $\mathcal{E}_{\text{s}}$ is 1CCA secure, while the other is only semantically secure, then $\mathcal{E}$ need not be CCA secure.

(e) Consider the KEM built $\mathcal{E}_{\text{kem}}$ constructed out of the encryption scheme $\mathcal{E}_{\text{a}}$, as in part (e) of Exercise 11.8. Show that $\mathcal{E}_{\text{kem}}$ is 1CCA secure, assuming that $\mathcal{E}_{\text{a}}$ is 1CCA secure.

Discussion: Using this result, one can arbitrarily extend the message space of any CCA-secure encryption scheme whose message space is already large enough to contain the key space for a 1CCA-secure symmetric cipher. For example, in practice, a 128-bit message space suffices. Interestingly, one can arbitrarily extend the message space even when starting from a CCA-secure scheme for 1-bit messages [80, 55].

12.5 (Multi-key CCA security). Generalize the definition of CCA security for a public-key encryption scheme to the multi-key setting. In this attack game, the adversary gets to obtain encryptions of many messages under many public keys, and can make as decryption queries with respect to any of these keys. Show that 1CCA security implies multi-key CCA security. You should show that security degrades linearly in $Q_k Q_e$, where Q_k is a bound on the number of keys, and Q_e is a bound on the number of encryption queries per key. That is, the advantage of any adversary $\mathcal{A}$ in breaking the multi-key CCA security of a scheme is at most $Q_k Q_e \cdot \epsilon$, where ϵ is the advantage of an adversary $\mathcal{B}$ (which is an elementary wrapper around $\mathcal{A}$) that breaks the scheme's 1CCA security.

12.6 (Multi-key CCA security of ElGamal). Consider a slight modification of the public-key encryption scheme $\mathcal{E}'_{\mathrm{EG}}$, which was presented an analyzed in Section 12.4. This new scheme, which we call $x\mathcal{E}'_{\mathrm{EG}}$, is exactly the same as $\mathcal{E}'_{\mathrm{EG}}$, except that instead of deriving the symmetric key as $k = H(v, w)$, we derive it as $k = H(u, v, w)$. Consider the security of $x\mathcal{E}'_{\mathrm{EG}}$ in the multi-key CCA attack game, discussed above in Exercise 12.5. In that attack game, suppose Q_{te} is a bound on the total number of encryptions — clearly, Q_{te} is at most $Q_k Q_e$, but it could be smaller. Let $\mathcal{A}$ be an adversary that attacks the multi-key CCA security of $x\mathcal{E}'_{\mathrm{EG}}$. Show that $\mathcal{A}$'s advantage is at most

$$2\epsilon_{\mathrm{icdh}} + Q_{\mathrm{te}} \cdot \epsilon_{\mathrm{s}},$$

where ϵ_{icdh} is that advantage of an ICDH adversary $\mathcal{B}_{\mathrm{icdh}}$ attacking $\mathbb{G}$ and ϵ_{s} is the advantage of a 1CCA adversary $\mathcal{B}_{\mathrm{s}}$ attacking $\mathcal{E}_{\mathrm{s}}$ (where both $\mathcal{B}_{\mathrm{icdh}}$ and $\mathcal{B}_{\mathrm{s}}$ are elementary wrappers around $\mathcal{A}$).

Hint: Use the random self reduction for CDH (see Exercise 10.4).

12.7 (Multi-key CCA security of Fujisaki-Okamoto with ElGamal). Consider a slight modification of Fujisaki-Okamoto transformation, in which we include the public key in the hash function, and suppose we instantiate this scheme with ElGamal encryption as in Section 12.6.2. Call this new scheme $x\mathcal{E}_{\mathrm{FO}}^{\mathrm{EG}}$. The only difference is that we include the public key u in the hash functions, so we compute $\beta \leftarrow U(u, x)$ and $k \leftarrow H(u, x)$. Consider the security of $x\mathcal{E}_{\mathrm{FO}}^{\mathrm{EG}}$ in the multi-key CCA attack game, discussed above in Exercise 12.5. In that attack game, suppose Q_{te} is a bound on the total number of encryptions — clearly, Q_{te} is at most $Q_k Q_e$, but it could be smaller. Also, let Q_{ro} be a bound on the total number of random oracle queries, and Q_{td} be a bound on the total number of decryptions. Let $\mathcal{A}$ be an adversary that attacks the multi-key CCA security of $x\mathcal{E}_{\mathrm{FO}}^{\mathrm{EG}}$. Show that $\mathcal{A}$'s advantage is at most

$$2Q_{\mathrm{ro}} \cdot \epsilon_{\mathrm{cdh}} + 2Q_{\mathrm{td}}/q + Q_{\mathrm{te}} \cdot \epsilon_{\mathrm{s}},$$

where ϵ_{cdh} is that advantage of a CDH adversary $\mathcal{B}_{\mathrm{cdh}}$ attacking $\mathbb{G}$ and ϵ_{s} is the advantage of a 1CCA adversary $\mathcal{B}_{\mathrm{s}}$ attacking $\mathcal{E}_{\mathrm{s}}$ (where both $\mathcal{B}_{\mathrm{cdh}}$ and $\mathcal{B}_{\mathrm{s}}$ are elementary wrappers around $\mathcal{A}$).

Hint: Use the random self reduction for CDH (see Exercise 10.4).

12.8 (Fujisaki-Okamoto with verifiable ciphertexts). Consider the Fujisaki-Okamoto transformation presented in Section 12.6. Suppose that the asymmetric cipher $\mathcal{E}_{\mathrm{a}}$ has *verifiable ciphertexts*, which means that there is an efficient algorithm that given a public key pk, along with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, determines whether or not y is an encryption of x under pk. Under this assumption, improve the security bound (12.29) to

$$\mathrm{OW}^{\mathrm{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{T}_{\mathrm{FO}}] \leq Q_{\mathrm{io}} \cdot \epsilon + \mathrm{OW}\mathsf{adv}[\mathcal{B}, \mathcal{E}_{\mathrm{a}}].$$

Notice that this bound does not degrade as Q_{ro} grows.

12.9. Show that any semantically secure public-key encryption scheme with a super-poly-sized message space is one way (as in Definition 12.5).

12.10 (Any cipher can be made unpredictable). Let (G_a, E_a, D_a) be a public key encryption scheme with message space $\mathcal{X}$, ciphertext space $\mathcal{Y}$, and randomizer space $\mathcal{R}$. Let $\mathcal{S}$ be some super-poly-sized finite set. Consider the encryption scheme (G_a, E'_a, D'_a), with message space $\mathcal{X}$, ciphertext space $\mathcal{Y} \times \mathcal{S}$, and randomizer space $\mathcal{R} \times \mathcal{S}$, where $E'_a(pk, x; (r, s)) := (E_a(pk, x; r), s)$ and $D'_a(sk, (y, s)) := D_a(sk, y)$. Show that (G_a, E'_a, D'_a) is unpredictable (as in Definition 12.6). Also show that if (G_a, E_a, D_a) is one way (as in Definition 12.5), then so is (G_a, E'_a, D'_a).

12.11 (Fujisaki-Okamoto with semantically secure encryption). Consider the Fujisaki-Okamoto transformation presented in Section 12.6. Suppose that the asymmetric cipher $\mathcal{E}_a$ is semantically secure. Under this assumption, improve the security bound (12.29) to

$$\mathrm{OW}^{\mathrm{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{T}_{\mathrm{FO}}] \le Q_{\mathrm{io}} \cdot \epsilon + \mathrm{SS}\mathsf{adv}[\mathcal{B}, \mathcal{E}_a] + Q_{\mathrm{ro}}/|\mathcal{X}|.$$

12.12 (An analysis of $\mathcal{E}'_{\mathrm{TDF}}$ without image oracles). Theorem 12.2 shows that $\mathcal{E}'_{\mathrm{TDF}}$ is CCA-secure assuming the trapdoor function scheme $\mathcal{T}$ is one-way given access to an image oracle, and $\mathcal{E}_s$ is 1CCA secure. It is possible to prove security of $\mathcal{E}'_{\mathrm{TDF}}$ assuming only that $\mathcal{T}$ is one-way (i.e., without assuming it is one-way given access to an image oracle), provided that $\mathcal{E}_s$ is 1AE secure. Note that we are making a slightly stronger assumption about $\mathcal{E}_s$ (1AE instead of 1CCA), but prove security under a weaker assumption on $\mathcal{T}$. Prove the following statement: if $H : \mathcal{X} \to \mathcal{K}$ is modeled as a random oracle, $\mathcal{T}$ is one-way, and $\mathcal{E}_s$ is 1AE secure, then $\mathcal{E}'_{\mathrm{TDF}}$ is CCA secure.

Hint: The proof is similar to the proof of Theorem 12.2. Let $(\hat{y}, \hat{c})$ be a decryption query from the adversary where $\hat{y} \neq y$. If $\mathcal{E}_s$ provides ciphertext integrity, then in testing whether $\hat{y}$ is in the image of $F(pk, \cdot)$, we can instead test if the adversary queried the random oracle at a preimage $\hat{x}$ of $\hat{y}$. If not, we can safely reject the ciphertext — ciphertext integrity implies that the original decryption algorithm would have anyway rejected the ciphertext with overwhelming probability.

Discussion: The analysis in this exercise requires that when a ciphertext (y, c) fails to decrypt, the adversary does not learn why. In particular, the adversary must not learn if decryption failed because the inversion of y failed, or because the symmetric decryption of c failed. This means, for example, if the time to decrypt is not the same in both cases, and this discrepancy is detectable by the adversary, then the analysis in this exercise no longer applies. By contrast, the analysis in Theorem 12.2 is unaffected by this side-channel leak: the adversary is given an image oracle and can determine, by himself, the reason for a decryption failure. In this respect, the analysis of Theorem 12.2 is more robust to side-channel attacks and is the preferable way to think of this system.

12.13 (Immunizing against image queries). Let (G, F, I) be a trapdoor function scheme defined over $(\mathcal{X}, \mathcal{Y})$. Let $U : \mathcal{X} \to \mathcal{R}$ be a hash function. Consider the trapdoor function scheme (G, F', I') defined over $(\mathcal{X}, \mathcal{Y} \times \mathcal{R})$, where $F'(pk, x) := (F(pk, x), U(x))$ and $I'(sk, (y, r)) := I(sk, y)$. Show that if U is modeled as a random oracle, (G, F, I) is one way, and $|\mathcal{R}|$ is super-poly, then (G, F', I') is one way given an image oracle.

12.14 (A broken CPA to CCA transformation). Consider the following attempt at transforming a CPA-secure scheme to a CCA-secure one. Let (G, E, D) be a CPA-secure encryption scheme defined over $(\mathcal{K} \times \mathcal{M},\ \mathcal{C})$, and let (S, V) be a secure MAC with key space $\mathcal{K}$. We construct a new encryption scheme (G, E', D'), with message space $\mathcal{M}$, as follows:

$$E'(pk, m) := \left\{ \begin{array}{l} k \xleftarrow{R} \mathcal{K}, \\ c \xleftarrow{R} E\big(pk,\ (k, m)\big), \\ t \xleftarrow{R} S(k, c), \\ \text{output } (c, t) \end{array} \right\} \qquad D'\big(sk, (c, t)\big) := \left\{ \begin{array}{l} (k, m) \leftarrow D(sk, c), \\ \text{if } V(k, c, t) = \textsf{accept} \text{ output } m, \\ \text{otherwise output } \textsf{reject} \end{array} \right\}$$

One might expect this scheme to be CCA-secure because a change to a ciphertext (c, t) will invalidate the MAC tag t. Show that this is incorrect. That is, show a CPA-secure encryption scheme (G, E, D) for which (G, E', D') is not CCA-secure (for any choice of MAC).

12.15 (Public-key encryption with associated data). In Section 12.7 we defined public-key encryption with associated data. We mentioned that the CCA-secure schemes in this chapter can be made into public-key encryption schemes with associated data by replacing the symmetric cipher used with an AD symmetric cipher. Here we develop another approach. Consider the scheme $\mathcal{E}'_{\text{TDF}}$ from Section 12.3. Show that defining the encryption algorithm E as:

$$\begin{array}{rcl} E(pk, m, d) & := & x \xleftarrow{R} \mathcal{X},\ \ y \leftarrow F(pk, x),\ \ k \leftarrow H(x, d),\ \ c \xleftarrow{R} E_{\text{s}}(k, m) \\ & & \text{output } (y, c); \end{array}$$

and making the corresponding change to the $\mathcal{E}'_{\text{TDF}}$ decryption algorithm D', gives a secure public-key encryption with associated data, under the same assumptions used in the analysis of $\mathcal{E}'_{\text{TDF}}$.

12.16 (Baby Bleichenbacher attack). Consider an RSA public key (n, e), where n is an RSA modulus, and e is an encryption exponent. For $x \in \mathbb{Z}_n$, consider the predicate $P_x : \mathbb{Z}_n \to \{0, 1\}$ defined as:

$$P_x(r) := \left\{ \begin{array}{l} y \leftarrow x \cdot r \in \mathbb{Z}_n \\ \text{treat } y \text{ as an integer in the interval } [0, n) \\ \text{if } y > n/2, \text{ output } 1 \\ \text{else, output } 0 \end{array} \right\}$$

(a) Show that by querying the predicate P_x at about $\log_2 n$ points, it is possible to learn the value of x.

(b) Suppose an attacker obtains an RSA public key and an element $c \in \mathbb{Z}_n$. It wants to compute the eth root of c in $\mathbb{Z}_n$. To do so, the attacker can query an oracle that takes $z \in \mathbb{Z}$ as input, and outputs 1 when $[z^{1/e} \bmod n] > n/2$, and outputs 0 otherwise. Here $[z^{1/e} \bmod n]$ is an integer w in the interval $[0, n)$ such that $w^e \equiv z \bmod n$. Use part (a) to show how the adversary can recover the eth root of c.

12.17 (OAEP is CPA-secure for any trapdoor function). Let $\mathcal{T} = (G, F, I)$ be a trapdoor function defined over $(\mathcal{X}, \mathcal{Y})$ where $\mathcal{X} = 0^8 \times \{0, 1\}^{t-8}$. Consider the OAEP padding scheme from Fig. 12.7, omitting the associated data input d, and let $\mathcal{E}_{\text{OAEP}}$ be the public key encryption scheme that results from coupling $\mathcal{T}$ with OAEP, as in (12.38). Show that $\mathcal{E}_{\text{OAEP}}$ is CPA secure in the random oracle model.

12.18 (A counter-example to the CCA-security of OAEP). Let $\mathcal{T}_0 = (G, F_0, I_0)$ be a one-way trapdoor permutation defined over $\mathcal{R} := \{0, 1\}^h$. Suppose, $\mathcal{T}_0$ is xor-homomorphic in the following sense: there is an efficient algorithm C that for all pk output by G and all $r, \Delta \in \mathcal{R}$, we have $C(F_0(pk, r)) = F_0(pk, r \oplus \Delta)$. Next, if $t > 2h + 16$, let $\mathcal{T} = (G, F, I)$ be the trapdoor permutation defined over $0^8 \times \{0,1\}^{t-8}$ as follows:

$$F\big(pk,\ (00 \parallel r \parallel z)\big) = 00 \parallel F_0(pk, r) \parallel z.$$

Notice that from $F\big(pk,\ (00 \parallel r \parallel z)\big)$ it is easy to recover z, but not the entire preimage. Consider the public-key encryption $\mathcal{E}_{\text{OAEP}}$ obtained by coupling this $\mathcal{T}$ with OAEP as in (12.38). Show a CCA attack on this scheme that has advantage 1 in winning the CCA game. Your attack shows that for some one-way trapdoor functions, the scheme $\mathcal{E}_{\text{OAEP}}$ may not be CCA-secure.

12.19 (RSA is partial one-way). Consider an RSA public key (n, e), where n is an RSA modulus, and e is an encryption exponent. Suppose n is a t-bit integer where t is even, and let T be an integer that is a little bit smaller than $2^{(t/2)}$. Let x be a random integer in the interval $[0, n)$ and $y := (x^e \bmod n) \in \mathbb{Z}_n$. Suppose $\mathcal{A}$ is an algorithm so that

$$\Pr\Big[\mathcal{A}(n, e, y) = z \text{ and } 0 \le x - zT < T\Big] > \epsilon.$$

The fact that the integer zT is so close to x means that z reveals half of the most significant bits of x. Hence, $\mathcal{A}$ is an RSA partial one-way adversary for the most significant bits.

(a) Construct an algorithm $\mathcal{B}$ that takes (n, e, y) as input, and outputs x with probability ϵ^2. For this, you should determine a more precise value for the parameter T.

Hint: Algorithm $\mathcal{B}$ works by choosing a random $r \in \mathbb{Z}_n$ and running $z_0 \leftarrow \mathcal{A}(n, e, y)$ and $z_1 \leftarrow \mathcal{A}(n, e,\ y \cdot r^e)$. If $\mathcal{A}$ outputs valid z_0 and z_1 both times — an event that happens with probability ϵ^2 (explain why) — then

$$\begin{aligned} x &\equiv z_0 T + \Delta_0 \pmod{n} \\ x \cdot r &\equiv z_1 T + \Delta_1 \pmod{n} \end{aligned}$$

where $0 \le \Delta_0, \Delta_1 < T$. Show an efficient algorithm that given such r, z_0, z_1, outputs x, Δ_0, Δ_1, with high probability. Your algorithm $\mathcal{B}$ should make use of an algorithm for finding shortest vectors in 2-dimensional lattices (see, for example, [102]). If you get stuck, see [41].

Discussion: This result shows that if RSA is one-way, then an adversary cannot even compute the most significant bits of a preimage.

(b) Show that a similar result holds if an algorithm $\mathcal{A}'$ outputs more than half the *least* significant bits of x.

12.20 (Multiplicative Cramer-Shoup encryption). Consider the following multiplicative version of the Cramer-Shoup encryption scheme (presented in Section 12.5). Let $\mathbb{G}$ be a cyclic group of prime order q with generator $g \in \mathbb{G}$. Let $H' : \mathbb{G}^3 \to \mathbb{Z}_q$ be a hash function. The encryption scheme $\mathcal{E}_{\text{MCS}} = (G, E, D)$ is defined over $(\mathbb{G}, \mathbb{G}^4)$ as follows. Key generation is exactly as in $\mathcal{E}_{\text{CS}}$. For a given public key $pk = (u, u_1, u_2, u_3) \in \mathbb{G}^4$ and message $m \in \mathbb{G}$, the encryption algorithm runs as follows:

$$\begin{aligned} E(pk, m) := \quad & \beta \xleftarrow{\text{R}} \mathbb{Z}_q,\ \ v \leftarrow g^\beta,\ \ w \leftarrow u^\beta,\ \ c \leftarrow u_1^\beta \cdot m \\ & \rho \leftarrow H'(v, w, c),\ \ w_2 \leftarrow (u_2 u_3^\rho)^\beta,\ \ \text{output } (v, w, w_2, c). \end{aligned}$$

For a given secret key $sk = (\sigma_1, \tau_1, \sigma_2, \tau_2, \sigma_3, \tau_3) \in \mathbb{Z}_q^6$ and a ciphertext $(v, w, w_2, c) \in \mathbb{G}^4$, the decryption algorithm runs as follows:

$$
\begin{aligned}
D(sk,\ (v, w, w_2, c)\ ) := \quad & \rho \leftarrow H'(v, w, c) \\
& \text{if } v^{\sigma_2 + \rho\sigma_3} w^{\tau_2 + \rho\tau_3} = w_2 \\
& \qquad \text{then output } c/(v^{\sigma_1} w^{\tau_1}) \\
& \qquad \text{else} \ \ \text{output } \mathsf{reject}.
\end{aligned}
$$

Show that $\mathcal{E}_{\text{MCS}}$ is CCA secure, provided H' is collision resistant and the DDH assumption holds for $\mathbb{G}$.

12.21 (Non-adaptive CCA security and Cramer-Shoup lite). One can define a weaker notion of CCA security, corresponding to a variant of the CCA attack game in which the adversary must make all of his decryption queries before making any of his decryption queries. Moreover, just as we did for ordinary CCA security, it suffices to assume that the adversary makes just a single encryption query. Let us call the corresponding security notion **non-adaptive 1CCA security**.

Now consider the following simplified version of the encryption scheme in the previous exercise. Again, $\mathbb{G}$ is a cyclic group of prime order q with generator $g \in \mathbb{G}$. The encryption scheme $\mathcal{E}_{\text{MCSL}} = (G, E, D)$ is defined over $(\mathbb{G}, \mathbb{G}^4)$ as follows. The key generation algorithm runs as follows:

$$
\begin{aligned}
G() := \quad & \alpha \xleftarrow{\text{R}} \mathbb{Z}_q, \quad u \leftarrow g^{\alpha} \\
& \text{for } i = 1, \ldots, 2\text{: } \sigma_i, \tau_i \xleftarrow{\text{R}} \mathbb{Z}_q,\ u_i \leftarrow g^{\sigma_i} u^{\tau_i} \\
& pk \leftarrow (u, u_1, u_2), \quad sk \leftarrow (\sigma_1, \tau_1, \sigma_2, \tau_2) \\
& \text{output } (pk, sk).
\end{aligned}
$$

For a given public key $pk = (u, u_1, u_2) \in \mathbb{G}^3$ and message $m \in \mathbb{G}$, the encryption algorithm runs as follows:

$$
\begin{aligned}
E(pk, m) := \quad & \beta \xleftarrow{\text{R}} \mathbb{Z}_q, \quad v \leftarrow g^{\beta}, \quad w \leftarrow u^{\beta}, \quad c \leftarrow u_1^{\beta} \cdot m \\
& w_2 \leftarrow u_2^{\beta}, \quad \text{output } (v, w, w_2, c).
\end{aligned}
$$

for a given secret key $sk = (\sigma_1, \tau_1, \sigma_2, \tau_2) \in \mathbb{Z}_q^4$ and a ciphertext $(v, w, w_2, c) \in \mathbb{G}^4$, the decryption algorithm runs as follows:

$$
\begin{aligned}
D(sk,\ (v, w, w_2, c)\ ) := \quad & \text{if } v^{\sigma_2} w^{\tau_2} = w_2 \\
& \qquad \text{then output } c/(v^{\sigma_1} w^{\tau_1}) \\
& \qquad \text{else} \ \ \text{output } \mathsf{reject}.
\end{aligned}
$$

(a) Show that $\mathcal{E}_{\text{MCSL}}$ is non-adaptive 1CCA secure, provided the DDH assumption holds for $\mathbb{G}$.

(b) Show that $\mathcal{E}_{\text{MCSL}}$ is not CCA secure.

12.22 (The twin CDH problem). In Section 12.4.1, we saw that the basic ElGamal encryption scheme could not be proved secure under the ordinary CDH assumption, even in the random oracle model. To analyze the scheme, we had to introduce a new, stronger assumption, called the interactive CDH (ICDH) assumption (see Definition 13.9). In this exercise and the next, we show how to avoid this stronger assumption with just a slightly more involved encryption scheme.

Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. The **Twin CDH (2CDH)** problem is this: given

$$g^{\alpha_1}, g^{\alpha_2}, g^{\beta}$$

compute the *pair*

$$(g^{\alpha_1\beta}, g^{\alpha_2\beta}).$$

A tuple of the form

$$(g^{\alpha_1}, g^{\alpha_2}, g^{\beta}, g^{\alpha_1\beta}, g^{\alpha_2\beta})$$

is called **Twin DH (2DH) tuple**. The **interactive Twin CDH (I2CDH) assumption** is this: it is hard to solve a random instance $(g^{\alpha_1}, g^{\alpha_2}, g^{\beta})$ of the 2DH problem, given access to an oracle that recognizes 2DH-tuples of the form $(g^{\alpha_1}, g^{\alpha_2}, \cdot, \cdot, \cdot)$.

(a) Flesh out the details of the I2CDH assumption by giving an attack game analogous to Attack Game 12.3. In particular, you should define an analogous advantage I2CDH$\mathsf{adv}[\mathcal{A}, \mathbb{G}]$ for an adversary $\mathcal{A}$ in this attack game.

(b) Using the trapdoor test in Exercise 10.12, show that the CDH assumption implies the I2CDH assumption. In particular, show that for every I2CDH adversary $\mathcal{A}$, there exists a CDH adversary $\mathcal{B}$ (where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$), such that

$$\text{I2CDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}] \leq \text{CDH}\mathsf{adv}[\mathcal{B}, \mathbb{G}] + \frac{Q_{\text{ro}}}{q},$$

where Q_{ro} is an upper bound on the number of oracle queries made by $\mathcal{A}$.

12.23 (Twin CDH encryption). The **Twin CDH encryption scheme**, $\mathcal{E}_{\text{2cdh}} = (G, E, D)$, is a public-key encryption scheme whose CCA security (in the random oracle model) is based on the I2CDH assumption (see previous exercise). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. a symmetric cipher We also need a symmetric cipher $\mathcal{E}_{\text{s}} = (E_{\text{s}}, D_{\text{s}})$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and a hash function $H : \mathbb{G}^3 \to \mathcal{K}$. The algorithms G, E, and D are defined as follows:

$$\begin{array}{rcl}
G() & := & \alpha_1 \xleftarrow{\text{R}} \mathbb{Z}_q, \;\; \alpha_2 \xleftarrow{\text{R}} \mathbb{Z}_q, \;\; u_1 \leftarrow g^{\alpha_1}, \;\; u_2 \leftarrow g^{\alpha_2} \\
 & & pk \leftarrow (u_1, u_2), \;\; sk \leftarrow (\alpha_1, \alpha_2) \\
 & & \text{output } (pk, sk); \\
E(pk, m) & := & \beta \xleftarrow{\text{R}} \mathbb{Z}_q, \;\; v \leftarrow g^{\beta}, \;\; w_1 \leftarrow u_1^{\beta}, \;\; w_2 \leftarrow u_2^{\beta} \\
 & & k \leftarrow H(v, w_1, w_2), \;\; c \xleftarrow{\text{R}} E_{\text{s}}(k, m) \\
 & & \text{output } (v, c); \\
D(sk, \; (v, c) \;) & := & w_1 \leftarrow v^{\alpha_1}, \;\; w_2 \leftarrow v^{\alpha_2}, \;\; k \leftarrow H(v, w_1, w_2), \;\; m \leftarrow D_{\text{s}}(k, c) \\
 & & \text{output } m.
\end{array}$$

The message space is $\mathcal{M}$ and the ciphertext space is $\mathbb{G} \times \mathcal{C}$.

(a) Suppose that we model the hash function H as a random oracle. Show that $\mathcal{E}_{\text{2cdh}}$ is CCA secure under the I2CDH assumption, also assuming that $\mathcal{E}_{\text{s}}$ is 1CCA secure. In particular, show that for every 1CCA adversary $\mathcal{A}$ attacking $\mathcal{E}_{\text{2cdh}}$, there exist an I2CDH adversary $\mathcal{B}_{\text{i2cdh}}$ and a 1CCA adversary $\mathcal{B}_{\text{s}}$, where $\mathcal{B}_{\text{i2cdh}}$ and $\mathcal{B}_{\text{s}}$ are elementary wrappers around $\mathcal{A}$, such that

$$\text{1CCA}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{\text{2cdh}}] \leq 2 \cdot \text{I2CDH}\mathsf{adv}[\mathcal{B}_{\text{i2cdh}}, \mathbb{G}] + \text{1CCA}\mathsf{adv}[\mathcal{B}_{\text{s}}, \mathcal{E}_{\text{s}}].$$

(b) Now use the result of part (b) of the previous exercise to show that $\mathcal{E}_{2\text{cdh}}$ is secure in the random oracle model under the ordinary CDH assumption for $\mathbb{G}$ (along with the assumption that $\mathcal{E}_\text{s}$ is 1CCA secure). In particular, show that for every 1CCA adversary $\mathcal{A}$ attacking $\mathcal{E}_{2\text{cdh}}$, there exist a CDH adversary $\mathcal{B}_\text{cdh}$ and a 1CCA adversary $\mathcal{B}_\text{s}$, where $\mathcal{B}_\text{cdh}$ and $\mathcal{B}_\text{s}$ are elementary wrappers around $\mathcal{A}$, such that

$$1\text{CCA}^\text{ro}\mathsf{adv}[\mathcal{A}, \mathcal{E}_{2\text{cdh}}] \leq 2 \cdot \text{CDH}\mathsf{adv}[\mathcal{B}_\text{cdh}, \mathbb{G}] + \frac{2Q_\text{ro}}{q} + 1\text{CCA}\mathsf{adv}[\mathcal{B}_\text{s}, \mathcal{E}_\text{s}],$$

where Q_ro is a bound on the number of random oracle queries made by $\mathcal{A}$.

Discussion: Compared to the ElGamal encryption scheme, $\mathcal{E}'_\text{EG}$, which we analyzed in Section 12.4, this scheme achieves CCA security under the CDH assumption, rather than the stronger ICDH assumption. Also, compared to the instantiation of the Fujisaki-Okamoto transformation with ElGamal, $\mathcal{E}_\text{FO}^\text{EG}$, which we analyzed in Section 12.6.2, the reduction to CDH here is much tighter, as we do not need to multiply $\text{CDH}\mathsf{adv}[\mathcal{B}_\text{cdh}, \mathbb{G}]$ by a factor of Q_ro as in (12.37). This tight reduction even extends to the more general multi-key CCA setting, as explored in the next exercise.

12.24 (Multi-key CCA security of Twin CDH). Consider a slight modification of the public-key encryption scheme $\mathcal{E}_{2\text{cdh}}$ from the previous exercise. This new scheme, which we call $x\mathcal{E}_{2\text{cdh}}$, is exactly the same as $\mathcal{E}_{2\text{cdh}}$, except that instead of deriving the symmetric key as $k = H(v, w_1, w_2)$, we derive it as $k = H(u_1, u_2, v, w_1, w_2)$. Consider the security of $x\mathcal{E}_{2\text{cdh}}$ in the multi-key CCA attack game, discussed above in Exercise 12.5. In that attack game, suppose Q_te is a bound on the total number of encryptions. Also, let Q_ro be a bound on the total number of random oracle queries. Let $\mathcal{A}$ be an adversary that attacks the multi-key CCA security of $x\mathcal{E}_{2\text{cdh}}$. Show that $\mathcal{A}$'s advantage is at most

$$2 \cdot \epsilon_\text{cdh} + \frac{2Q_\text{ro}}{q} + Q_\text{te} \cdot \epsilon_\text{s},$$

where ϵ_cdh is that advantage of a CDH adversary $\mathcal{B}_\text{cdh}$ attacking $\mathbb{G}$ and ϵ_s is the advantage of a 1CCA adversary $\mathcal{B}_\text{s}$ attacking $\mathcal{E}_\text{s}$ (where both $\mathcal{B}_\text{cdh}$ and $\mathcal{B}_\text{s}$ are elementary wrappers around $\mathcal{A}$).

Hint: Use the random self reduction for CDH (see Exercise 10.4).

12.25 (The twin HDH problem). This exercise and the next develop a public-key encryption scheme that is CCA secure without random oracles and under the weaker Hash Diffie-Hellman (HDH) assumption (see Exercise 11.13).

First, we explore a decisional variant of the 2CDH problem discussed above in Exercise 12.22. Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. Let $H : \mathbb{G} \to \mathcal{K}$ be a hash function. The **interactive twin HDH (I2HDH)** assumption for $(\mathbb{G}, H)$ asserts that it is hard to distinguish random tuples of the form

$$(g^{\alpha_1}, g^{\alpha_2}, g^\beta, H(g^{\alpha_1\beta}))$$

from random tuples of the form

$$(g^{\alpha_1}, g^{\alpha_2}, g^\beta, k),$$

where and $\alpha_1, \alpha_2, \beta \in \mathbb{Z}_q$ and $k \in \mathcal{K}$, even when given access to an oracle that recognizes 2DH-tuples of the form $(g^{\alpha_1}, g^{\alpha_2}, \cdot, \cdot, \cdot)$.

(a) Flesh out the details of the I2HDH assumption by giving an appropriate attack game and defining an appropriate notion of advantage.

(b) Using the trapdoor test in Exercise 10.12, show that the HDH assumption for $(\mathbb{G}, H)$ implies the I2HDH assumption for $(\mathbb{G}, H)$, giving a concrete security bound.

12.26 (Twin HDH encryption). The **Twin HDH encryption scheme**, $\mathcal{E}_{2\text{hdh}} = (G, E, D)$, is a public-key encryption scheme whose CCA security (without random oracles) is based on the I2HDH assumption (see previous exercise). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. We also need a symmetric cipher $\mathcal{E}_s = (E_s, D_s)$, defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, and hash functions $H : \mathbb{G} \to \mathcal{K}$ and $H' : \mathbb{G} \to \mathbb{Z}_q$. The algorithms G, E, and D are defined as follows:

$$
\begin{array}{rcl}
G() & := & \alpha_1, \alpha_2, \tilde{\alpha}_1, \tilde{\alpha}_2 \xleftarrow{\text{R}} \mathbb{Z}_q \\
& & u_1 \leftarrow g^{\alpha_1},\ u_2 \leftarrow g^{\alpha_2},\ \tilde{u}_1 \leftarrow g^{\tilde{\alpha}_1},\ \tilde{u}_2 \leftarrow g^{\tilde{\alpha}_2} \\
& & pk \leftarrow (u_1, u_2, \tilde{u}_1, \tilde{u}_2), \quad sk \leftarrow (\alpha_1, \alpha_2, \tilde{\alpha}_1, \tilde{\alpha}_2) \\
& & \text{output } (pk, sk); \\
E(pk, m) & := & \beta \xleftarrow{\text{R}} \mathbb{Z}_q, \quad v \leftarrow g^{\beta}, \quad \rho \leftarrow H'(v) \\
& & x_1 \leftarrow (u_1^{\rho} \tilde{u}_1)^{\beta}, \quad x_2 \leftarrow (u_2^{\rho} \tilde{u}_2)^{\beta}, \quad w_1 \leftarrow u_1^{\beta}, \quad k \leftarrow H(w_1) \\
& & c \xleftarrow{\text{R}} E_s(k, m) \\
& & \text{output } (v, x_1, x_2, c); \\
D(sk,\ (v, x_1, x_2, c)\ ) & := & \rho \leftarrow H'(v), \\
& & \text{if } v^{\alpha_1 \rho + \tilde{\alpha}_1} = x_1 \text{ and } v^{\alpha_2 \rho + \tilde{\alpha}_2} = x_2 \\
& & \qquad \text{then } w_1 \leftarrow v^{\alpha_1}, \quad k \leftarrow H(w_1), \quad m \leftarrow D_s(k, c) \\
& & \qquad \text{else} \quad m \leftarrow \mathsf{reject} \\
& & \text{output } m.
\end{array}
$$

(a) Show that $\mathcal{E}_{2\text{hdh}}$ is CCA secure under the I2HDH assumption for $(\mathbb{G}, H)$, also assuming that $\mathcal{E}_s$ is 1CCA secure and that H' is collision resistant, giving a concrete security bound.

You may wish to structure your proof as follows. Working with the bit-guessing version of the 1CCA attack game, define a sequence of games:

Game 0: This is the original attack game, where the challenger starts the game by generating not only the public key and secret key, but also the components (including v, x_1, and x_2) of the target ciphertext other than c.

Game 1: Modify the challenger so that it rejects all decryption queries $(\hat{v}, \hat{x}_1, \hat{x}_2, \hat{c})$ such that $\hat{v} \neq v$ but $H'(\hat{v}) = H'(v)$. Argue that the advantage in Game 1 is negligibly close to that in Game 0, under the collision resistance assumption for H'.

Game 2: Modify the challenger so that the elements $\tilde{u}_1, \tilde{u}_2$ in public key and the elements x_1, x_2 in the target ciphertext are generated as follows:

$$\sigma_1, \sigma_2 \xleftarrow{\text{R}} \mathbb{Z}_q, \quad \tilde{u}_1 \leftarrow g^{\sigma_1} u_1^{-\rho}, \quad \tilde{u}_2 \leftarrow g^{\sigma_2} u_2^{-\rho}, \quad x_1 \leftarrow v^{\sigma_1}, \quad x_2 \leftarrow v^{\sigma_2},$$

where $\rho := H'(v)$. With this change, the exponents $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are implicitly defined as

$$\tilde{\alpha}_1 = \sigma_1 - \alpha_1 \rho \quad \text{and} \quad \tilde{\alpha}_2 = \sigma_2 - \alpha_2 \rho.$$

Show how to further modify the challenger, so that all decryption queries $(\hat{v}, \hat{x}_1, \hat{x}_2, \hat{c})$ with $\hat{\rho} := H'(\hat{v}) \neq \rho$ are answered without using the values α_1, α_2 at all, but instead, using an oracle to test if

$$(u_1, u_2, \hat{v}, \hat{w}_1, \hat{w}_2)$$

is a 2DH-tuple, where

$$\hat{w}_1 := (\hat{x}_1/\hat{v}^{\sigma_1})^{1/(\hat{\rho}-\rho)} \quad \text{and} \quad \hat{w}_2 := (\hat{x}_2/\hat{v}^{\sigma_2})^{1/(\hat{\rho}-\rho)}.$$

With these modifications, the values $\alpha_1, \alpha_2, \beta$ should not be used in Game 2, except to define the group elements u_1, u_2, v, w_1, and implicitly in the implementation of the 2DH-oracle. The advantage in Game 2 should be identical to that in Game 1.

Note: This is an example of what is known as an *all-but-one simulation strategy*, where we set up a public parameter in such a way that a simulator can answer all but one query of a certain type.

Game 3: Replace the key k used to generate the target ciphertext by a random key. Argue that the advantage in Game 3 is negligibly close to that in Game 2, under the I2HDH assumption for $(\mathbb{G}, H)$. Argue that the advantage in Game 3 is negligible, assuming that $\mathcal{E}_{\mathrm{s}}$ is 1CCA secure.

(b) Now use part (b) of the previous exercise to show that $\mathcal{E}_{\mathrm{2hdh}}$ is CCA secure under the HDH assumption for $(\mathbb{G}, H)$, also assuming that $\mathcal{E}_{\mathrm{s}}$ is 1CCA secure and that H' is collision resistant, giving a concrete security bound.

Chapter 13

Digital signatures

In this chapter and the next we develop the concept of a digital signature. Although there are some parallels between physical world signatures and digital signatures, the two are quite different. We motivate digital signatures with three examples.

Example 1: Software distribution. Suppose a software company, SoftAreUs, releases a software update for its product. Customers download the software update file U by some means, say from a public distribution site or from a peer-to-peer network. Before installing U on their machine, customers want to verify that U really is from SoftAreUs. To facilitate this, SoftAreUs appends a short tag to U, called a signature. Only SoftAreUs can generate a signature on U, but anyone in the world can verify it. Note that there are no secrecy issues here — the update file U is available in the clear to everyone. A MAC system is of no use in this setting because SoftAreUs does not maintain a shared secret key with each of its customers. Some software distribution systems use collision resistant hashing, but that requires an online read-only server that every customer uses to check that the hash of the received file U matches the hash value on the read-only server.

To provide a clean solution, with no additional security infrastructure, we need a new cryptographic mechanism called a digital signature. The signing process works as follows:

- First, SoftAreUs generates a secret signing key sk along with some corresponding public key denoted pk. SoftAreUs keeps the secret key sk to itself. The public key pk is hard-coded into all copies of the software sold by SoftAreUs and is used to verify signatures issued using sk.
- To sign a software update file U, SoftAreUs runs a signing algorithm S that takes (sk, U) as input. The algorithm outputs a short signature σ. SoftAreUs then ships the pair (U, σ) to all its customers.
- A customer Bob, given the update (U, σ) and the public key pk, checks validity of this message-signature pair using a signature verification algorithm V that takes (pk, U, σ) as input. The algorithm outputs either accept or reject depending on whether the signature is valid or not. Recall that Bob obtains pk from the pre-installed software system from SoftAreUs.

This mechanism is widely used in practice in a variety of software update systems. For security we must require that an adversary, who has pk, cannot generate a valid signature on a fake update file. We will make this precise in the next section.

We emphasize that a digital signature σ is a function of the data U being signed. This is very different from signatures in the physical world where the signature is always the same no matter what document is being signed.

Example 2: Authenticated email. As a second motivating example, suppose Bob receives an email claiming to be from his friend Alice. Bob wants to verify that the email really is from Alice. A MAC system would do the job, but requires that Alice and Bob have a shared secret key. What if they never met before and do not share a secret key? Digital signatures provide a simple solution. First, Alice generates a public/secret key pair (pk, sk). For now, we assume Alice places pk in a public read-only directory. We will discuss how to get rid of this directory in just a minute.

When sending an email m to Bob, Alice generates a signature σ on m derived using her secret key. She then sends (m, σ) to Bob. Bob receives (m, σ) and verifies that m is from Alice in two steps. First, Bob retrieves Alice's public key pk. Second, Bob runs the signature verification algorithm on the triple (pk, m, σ). If the algorithm outputs accept then Bob is assured that the message came from Alice. More precisely, Bob is assured that the message was sent by someone who knows Alice's secret key. Normally this would only be Alice, but if Alice's key is stolen then the message could have come from the thief.

As a more concrete example of this, the *domain keys identified mail (DKIM)* system is an email-signing system that is widely used on the Internet. An organization that uses DKIM generates a public/secret key pair (pk, sk) and uses sk to sign every outgoing email from the organization. The organization places the public key pk in the DNS records associated with the organization, so that anyone can read pk. An email recipient verifies the signature on every incoming DKIM email to ensure that the email source is the claimed organization. If the signature is valid the email is delivered, otherwise it is dropped. DKIM is widely used as a mechanism to make it harder for spammers to send spam email that pretends to be from a reputable source.

Example 3: Certificates. As a third motivating example for digital signatures, we consider their most widely used application. In Chapter 11 and in the authenticated email system above, we assumed public keys are obtained from a read-only public directory. In practice, however, there is no public directory. Instead, Alice's public key pk is certified by some third party called a **certificate authority** or CA for short. We will see how this process works in more detail in Section 13.8. For now, we briefly explain how signatures are used in the certification process.

To generate a certified public key, Alice first generates a public/private key pair (pk, sk) for some public-key cryptosystem, such as a public-key encryption scheme or a signature scheme. Next, Alice presents her public key pk to the CA. The CA then verifies that Alice is who she claims to be, and once the CA is convinced that it is speaking with Alice, the CA constructs a statement m saying "public key pk belongs to Alice." Finally, the CA signs the message m using its own secret key sk_{CA} and sends the pair $Cert := (m, \sigma_{\mathrm{CA}})$ back to Alice. This pair $Cert$ is called a **certificate** for pk. When Bob needs Alice's public key, he first obtains Alice's certificate from Alice and verifies the CA's signature in the certificate. If the signature is valid, Bob has some confidence that pk is Alice's public key. The main purpose of the CA's digital signature is to prove to Bob that the statement m was issued by the CA. Of course, to verify the CA's signature, Bob needs the CA's public key pk_{CA}. Typically, CA public keys come pre-installed with an operating system or a Web browser. In other words, we simply assume that the CA's public key is already available on Bob's machine.

Of course, the above can be generalized so that the CA's certificate for Alice associates several public keys with her identity, such as public keys for both encryption and signatures.

Non-repudiation. An interesting property of the authenticated email system above is that Bob now has evidence that the message m is from Alice. He could show the pair (m, σ) to a judge who could also verify Alice's signature. Thus, for example, if m says that Alice agrees to sell her car to Bob, then Alice is (in some sense) committed to this transaction. Bob can use Alice's signature as proof that Alice agreed to sell her car to Bob — the signature binds Alice to the message m. This property provided by digital signatures is called **non-repudiation**.

Unfortunately, things are not quite that simple. Alice can repudiate the signature by claiming that the public key pk is not hers and therefore the signature was not issued by her. Or she can claim that her secret key sk was stolen and the signature was issued by the thief. After all, computers are compromised and keys are stolen all the time. Even worse, Alice could deliberately leak her secret key right after generating it thereby invalidating all her signatures. The judge at this point has no idea who to believe.

These issues are partially the reason why digital signatures are not often used for legal purposes. Digital signatures are primarily a cryptographic tool used for authenticating data in computer systems. They are a useful building block for higher level mechanisms such as key-exchange protocols, but have little to do with the legal system. Several legislative efforts in the U.S. and Europe attempt to clarify the process of digitally signing a document. In the U.S., for example, electronically signing a document does not require a cryptographic digital signature. We discuss the legal aspects of digital signatures in Section 13.9.

Non-repudiation does not come up in the context of MACs because MACs are non-binding. To see why, suppose Alice and Bob share a secret key and Alice sends a message to Bob with an attached MAC tag. Bob cannot use the tag to convince a judge that the message is from Alice since Bob could have just as easily generated the tag himself using the MAC key. Hence Alice can easily deny ever sending the message. The asymmetry of a signature system — the signer has sk while the verifier has pk — makes it harder (though not impossible) for Alice to deny sending a signed message.

13.1 Definition of a digital signature

Now that we have an intuitive feel for how digital signature schemes work, we can define them more precisely. Functionally, a digital signature is similar to a MAC. The main difference is that in a MAC, both the signing and verification algorithms use the same secret key, while in a signature scheme, the signing algorithm uses one key, sk, while the verification algorithm uses another, pk.

Definition 13.1. *A* ***signature scheme*** $\mathcal{S} = (G, S, V)$ *is a triple of efficient algorithms,* G, S *and* V*, where* G *is called a* ***key generation algorithm****,* S *is called a* ***signing algorithm****, and* V *is called a* ***verification algorithm****. Algorithm* S *is used to generate signatures and algorithm* V *is used to verify signatures.*

- *G is a probabilistic algorithm that takes no input. It outputs a pair (pk, sk), where sk is called a secret* ***signing key*** *and pk is called a public* ***verification key****.*

- *S is a probabilistic algorithm that is invoked as $\sigma \xleftarrow{R} E(sk, m)$, where sk is a secret key (as output by G) and m is a message. The algorithm outputs a* ***signature*** *σ.*

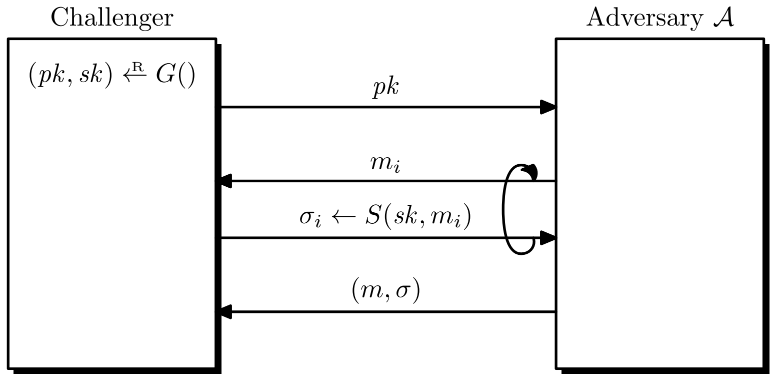


Figure 13.1: Signature attack game (Attack Game 13.1)

- *V is a deterministic algorithm invoked as $V(pk, m, \sigma)$. It outputs either* accept *or* reject*.*
- *We require that a signature generated by S is always accepted by V. That is, for all (pk, sk) output by G and all messages m, we have*

$$\Pr[V(pk,\ m,\ S(sk,\ m)\ ) = \mathsf{accept}] = 1.$$

As usual, we say that messages lie in a finite ***message space*** *$\mathcal{M}$, and signatures lie in some finite* ***signature space*** *Σ. We say that $\mathcal{S} = (G, S, V)$ is defined over $(\mathcal{M}, \Sigma)$.*

13.1.1 Secure signatures

The definition of a secure signature scheme is similar to the definition of secure MAC. We give the adversary the power to mount a **chosen message attack**, namely the attacker can request the signature on any message of his choice. Even with such power, the adversary should not be able to create an **existential forgery**, namely the attacker cannot output a valid message-signature pair (m, σ) for some new message m. Here "new" means a message that the adversary did not previously request a signature for.

More precisely, we define secure signatures using an attack game between a challenger and an adversary $\mathcal{A}$. The game is described below and in Fig. 13.1.

Attack Game 13.1 (Signature security). For a given signature scheme $\mathcal{S} = (G, S, V)$, defined over $(\mathcal{M}, \Sigma)$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The challenger runs $(pk, sk) \xleftarrow{R} G()$ and sends pk to $\mathcal{A}$.
- $\mathcal{A}$ queries the challenger several times. For $i = 1, 2, \ldots,$ the ith *signing query* is a message $m_i \in \mathcal{M}$. Given m_i, the challenger computes $\sigma_i \xleftarrow{R} S(sk, m_i)$, and then gives σ_i to $\mathcal{A}$.
- Eventually $\mathcal{A}$ outputs a candidate forgery pair $(m, \sigma) \in \mathcal{M} \times \Sigma$.

We say that the adversary wins the game if the following two conditions hold:

- $V(pk, m, \sigma) = \mathsf{accept}$, and

- m is new, namely $m \notin \{m_1, m_2, \ldots\}$.

We define $\mathcal{A}$'s advantage with respect to $\mathcal{S}$, denoted $\mathrm{SIG}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$, as the probability that $\mathcal{A}$ wins the game. Finally, we say that $\mathcal{A}$ is a **q-query adversary** if $\mathcal{A}$ issues at most q signing queries. □

Definition 13.2. *We say that a signature scheme $\mathcal{S}$ is secure if for all efficient adversaries $\mathcal{A}$, the quantity* $\mathrm{SIG}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$ *is negligible.*

In case the adversary wins Attack Game 13.1, the pair (m, σ) it outputs is called an **existential forgery**. Systems that satisfy Definition 13.2 are said to be **existentially unforgeable under a chosen message attack**.

Verification queries. In our discussion of MACs we proved Theorem 6.1, which showed that tag verification queries do not help the adversary forge MACs. In the case of digital signatures, verification queries are a non-issue — the adversary can always verify message-signature pairs for himself. Hence, there is no need for an analogue to Theorem 6.1 for digital signatures.

Security against multi-key attacks. In real systems there are many users, and each one of them can have a signature key pair (pk_i, sk_i) for $i = 1, \ldots, n$. Can a chosen message attack on pk_1 help the adversary forge signatures for pk_2? If that were possible then our definition of secure signature would be inadequate since it would not model real-world attacks. Just as we did for other security primitives, one can generalize the notion of a secure signatures to the multi-key setting, and prove that a secure signature is also secure in the multi-key settings. See Exercise 13.1. We proved a similar fact for a secure MAC system in Exercise 6.3.

Strongly unforgeable signatures Our definition of existential forgery is a little different than the definition of secure MACs. Here we only require that the adversary cannot forge a signature on a new message m. We do not preclude the adversary from producing a new signature on m from some other signature on m. That is, a signature scheme is secure even if the adversary can transform a valid pair (m, σ) into a new valid pair (m, σ').

In contrast, for MAC security we insisted that given a message-tag pair (m, t) the adversary cannot create a new valid tag $t' \neq t$ for m. This was necessary for proving security of the encrypt-then-MAC construction in Section 9.4.1. It was also needed for proving that MAC verification queries do not help the adversary (see Theorem 6.1 and Exercise 6.7).

One can similarly strengthen Definition 13.2 to require this more stringent notion of existential unforgeability. We capture this in the following modified attack game.

Attack Game 13.2. For a given signature scheme $\mathcal{S} = (G, S, V)$, and a given adversary $\mathcal{A}$, the game is identical to Attack Game 13.1, except that the second bullet in the winning condition is changed to:

- (m, σ) is new, namely $(m, \sigma) \notin \big\{(m_1, \sigma_1),\ (m_2, \sigma_2), \ldots\big\}$

We define $\mathcal{A}$'s advantage with respect to $\mathcal{S}$, denoted $\mathrm{stSIG}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$, as the probability that $\mathcal{A}$ wins the game. □

Definition 13.3. *We say that a signature scheme $\mathcal{S}$ is* ***strongly secure*** *if for all efficient adversaries $\mathcal{A}$, the quantity* $\mathrm{stSIG}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$ *is negligible.*

Strong security ensures that for a secure signature scheme, the adversary cannot create a new signature on a previously signed message, as we required for MACs. There are a few specific situations that require signatures satisfying this stronger security notion, such as [33, 22] and a signcryption construction described in Section 13.7. However, most often Definition 13.2 is sufficient. At any rate, any secure signature scheme $\mathcal{S} = (G, S, V)$ can be converted into a strongly secure signature scheme $\mathcal{S}' = (G', S', V')$. See Exercise 14.6.

Limitations of the security definition. Definition 13.2 ensures that generating valid message-signature pairs is difficult without the secret key. The definition, however, does not capture several additional desirable properties for a signature scheme:

- **Binding signatures**. Definition 13.2 does not require that the signer be bound to messages she signs. That is, suppose the signer generates a signature σ on some message m. The definition does not preclude the signer from producing another message $m' \neq m$ for which σ is a valid signature. The message m might say "Alice owes Bob ten dollars" while m' says "Alice owes Bob one dollar." Since σ is a valid signature on both messages, a judge cannot tell what message Alice actually signed. See Exercise 13.2.

  For many applications of digital signatures we do not need the signer to be bound to signed messages. Consequently, we do not require signature schemes to enforce this property. Nevertheless, many of the constructions in this chapter and the next do bind the signer to the message. That is, the signer cannot produce two distinct messages with the same signature.

- **Duplicate Signature Key Selection (DSKS)**. Let $\mathcal{S} = (G, S, V)$ be a signature scheme and let (m, σ) be a valid message-signature pair with respect to some public key pk. The signature scheme $\mathcal{S}$ is said to be vulnerable to DSKS if an attacker, who sees (m, σ), can generate a key pair pk', sk' such that (m, σ) is also valid with respect to the public key pk'. We require that the attacker can produce both pk' and sk'. Exercise 13.3 gives examples of signature schemes that are vulnerable to DSKS.

  A DSKS vulnerability can lead to a number of undesirable consequences. For example, suppose (m, σ) is a signed homework solution set submitted by a student Alice. After the submission deadline, an attacker Molly, who did not submit a solution set, can use a DSKS attack to claim that the homework submission (m, σ) is hers. To do so, Molly uses the DSKS attack to generate a key pair pk', sk' such that (m, σ) is a valid message-signature pair for the key pk'. Because the assignment is properly signed under both public keys pk and pk', the Professor cannot tell who submitted the assignment (assuming the homework m does not identify Alice). In practice, DSKS attacks have been used to attack certain key exchange protocols, as discussed in Chapter 20.

  Definition 13.2 does not preclude DSKS attacks. However, it is quite easy to immunize a signature scheme against DSKS attacks: the signer simply attaches his or her public key to the message before signing the message. The verifier does the same before verifying the signature. This way, the signing public key is authenticated along with the message (see Exercise 13.4). Attaching the public key to the message prior to signing is good practice and is recommended in many real-world applications.

13.1.2 Mathematical details

As usual, we give a more mathematically precise definition of a signature, using the terminology defined in Section 2.4. This section may be safely skipped on first reading.

Definition 13.4 (Signature). *A **signature** scheme is a triple of efficient algorithms (G, S, V), along with two families of spaces with system parameterization P:*

$$\mathbf{M} = \{\mathcal{M}_{\lambda,\Lambda}\}_{\lambda,\Lambda}, \quad \textit{and} \quad \mathbf{\Sigma} = \{\Sigma_{\lambda,\Lambda}\}_{\lambda,\Lambda},$$

As usual, $\lambda \in \mathbb{Z}_{\geq 1}$ is a security parameter and $\Lambda \in \mathrm{Supp}(P(\lambda))$ is a system parameter. We require that

1. *$\mathbf{M}$ and $\mathbf{\Sigma}$ are efficiently recognizable.*

2. *Algorithm G is an efficient probabilistic algorithm that on input λ, Λ, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, outputs a pair (pk, sk), where pk and sk are bit strings whose lengths are always bounded by a polynomial in λ.*

3. *Algorithm S is an efficient probabilistic algorithm that on input λ, Λ, sk, m, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $(pk, sk) \in \mathrm{Supp}(G(\lambda, \Lambda))$ for some pk, and $m \in \mathcal{M}_{\lambda,\Lambda}$, always outputs an element of $\Sigma_{\lambda,\Lambda}$.*

4. *Algorithm V is an efficient deterministic algorithm that on input $\lambda, \Lambda, pk, m, \sigma$, where $\lambda \in \mathbb{Z}_{\geq 1}$, $\Lambda \in \mathrm{Supp}(P(\lambda))$, $(pk, sk) \in \mathrm{Supp}(G(\lambda, \Lambda))$ for some sk, $m \in \mathcal{M}_{\lambda,\Lambda}$, and $\sigma \in \Sigma_{\lambda,\Lambda}$, and outputs either* accept *or* reject.

In defining security, we parameterize Attack Game 13.1 by the security parameter λ which is given to both the adversary and the challenger. The advantage $\mathrm{SIG}\mathsf{adv}[\mathcal{A}, \mathcal{S}]$ is then a function of λ. Definition 13.2 should be read as saying that $\mathrm{SIG}\mathsf{adv}[\mathcal{A}, \mathcal{S}](\lambda)$ is a negligible function. Similarly for Definition 13.3.

13.2 Extending the message space with collision resistant hashing

Suppose we are given a secure digital signature scheme with a small message space, say $\mathcal{M} = \{0,1\}^{256}$. We show how to extend the message space to much larger messages using a collision resistant hash function. We presented a similar construction for MACs in Fig. 8.1. Let $\mathcal{S} = (G, S, V)$ be a signature scheme defined over $(\mathcal{M}, \Sigma)$ and let $H : \mathcal{M}' \to \mathcal{M}$ be a hash function, where the set $\mathcal{M}'$ is much larger than $\mathcal{M}$. Define a new signature scheme $\mathcal{S}' = (G, S', V')$ over $(\mathcal{M}', \Sigma)$ as

$$S'(sk,\ m) := S(sk,\ H(m)) \quad \text{and} \quad V'(pk,\ m,\ \sigma) := V(pk,\ H(m),\ \sigma) \tag{13.1}$$

The new scheme signs much larger message than the original scheme. This approach is often called the **hash-and-sign paradigm**. As a concrete example, suppose we take H to be SHA256. Then any signature scheme capable of signing 256-bit messages can be securely extended to a signature scheme capable of signing arbitrary long messages. Hence, from now on it suffices to focus on building signature schemes for short 256-bit messages.

The following simple theorem shows that this construction is secure. Its proof is essentially identical to the proof of Theorem 8.1.

Theorem 13.1. *Suppose the signature scheme $\mathcal{S}$ is secure and the hash function H is collision resistant. Then the derived signature scheme $\mathcal{S}' = (G, S', V')$ defined in (13.1) is a secure signature.*

In particular, suppose $\mathcal{A}$ is a signature adversary attacking $\mathcal{S}'$ (as in Attack Game 13.1). Then there exist an efficient signature adversary $\mathcal{B}_{\mathcal{S}}$ and an efficient collision finder $\mathcal{B}_H$, which are elementary wrappers around $\mathcal{A}$, such that

$$\mathrm{SIG}\mathsf{adv}[\mathcal{A}, \mathcal{S}'] \le \mathrm{SIG}\mathsf{adv}[\mathcal{B}_{\mathcal{S}}, \mathcal{S}] + \mathrm{CR}\mathsf{adv}[\mathcal{B}_H, H]$$

13.2.1 Extending the message space using TCR functions

We briefly show that collision resistance is not necessary for extending the message space of a signature scheme. A second pre-image resistant (SPR) hash function is sufficient. Recall that in Section 8.11.2 we used SPR hash functions to build target collision resistant (TCR) hash functions. We then used a TCR hash function to extend the message space of a MAC. We can do the same here to extend the message space of a signature scheme.

Let H be a TCR hash function defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{T})$. Let $\mathcal{S} = (G, S, V)$ be a signature scheme for short messages in $\mathcal{K}_H \times \mathcal{T}$. We build a new signature scheme $\mathcal{S}' = (G, S', V')$ for signing messages in $\mathcal{M}$ as follows:

$$\begin{array}{ll}
S'(sk,\ m) := & V'(pk,\ m,\ (\sigma, r)\ ) := \\
\qquad r \xleftarrow{\text{R}} \mathcal{K}_H & \qquad h \leftarrow H(r, m) \\
\qquad h \leftarrow H(r, m) & \qquad \text{Output } V(pk,\ (r, h),\ \sigma) \\
\qquad \sigma \leftarrow S\big(sk,\ (r, h)\big) & \\
\qquad \text{Output } (\sigma, r) &
\end{array} \tag{13.2}$$

The signing procedure chooses a random TCR key r, includes r as part of the message being signed, and outputs r as part of the final signature. As a result, signatures produced by this scheme are longer than signatures produced by extending the domain using a collision resistant hash, as above. Using the TCR construction from Fig. 8.13, the length of r is logarithmic in the size of the message being signed. This extra logarithmic size key must be included in every signature. Exercise 13.6 proposes a way to get shorter signatures.

The benefit of the TCR construction is that security only relies on H being TCR, which is a much weaker property than collision resistance and hence more likely to hold for H. For example, the function SHA256 may eventually be broken as a collision-resistant hash, but the function $H(r, m) := \mathrm{SHA256}(r \parallel m)$ may still be secure as a TCR.

The following theorem proves security of the construction in (13.2) above. The theorem and its proof are almost identical to the same theorem and proof applied to MAC systems (Theorem 8.12). Note that the concrete bound in the theorem below has an extra factor of Q that does not appear in Theorem 13.1 above. The reason for this extra Q factor is the same as in the proof for MAC systems (Theorem 8.12).

Theorem 13.2. *Suppose $\mathcal{S} = (G, S, V)$ is a secure signature scheme and the hash function H is TCR. Then the derived signature scheme $\mathcal{S}' = (G, S', V')$ defined in (13.2) is secure.*

In particular, for every signature adversary $\mathcal{A}$ attacking $\mathcal{S}'$ (as in Attack Game 13.1) that issues at most Q signing queries, there exist an efficient signature adversary $\mathcal{B}_{\mathcal{S}}$ and an efficient TCR adversary $\mathcal{B}_H$, which are elementary wrappers around $\mathcal{A}$, such that

$$\mathrm{SIG}\mathsf{adv}[\mathcal{A}, \mathcal{S}'] \le \mathrm{SIG}\mathsf{adv}[\mathcal{B}_{\mathcal{S}}, \mathcal{S}] + Q \cdot \mathrm{TCR}\mathsf{adv}[\mathcal{B}_H, H].$$

13.3 Signatures from trapdoor permutations: the full domain hash

We now turn to constructing signature schemes. All the constructions in this chapter are proven secure in the random oracle model. We present practical non-random-oracle constructions in Chapter 16 and in the next chapter. We will see more random oracle signature schemes in Chapter 19.

We begin with a simple construction based on trapdoor permutations. We then present a concrete signature scheme from the only trapdoor permutation we have, namely RSA. Recall that a trapdoor permutation scheme defined over $\mathcal{X}$ is a triple of algorithms $\mathcal{T} = (G, F, I)$, where G generates a public key/secret key pair (pk, sk), $F(pk, \cdot)$ evaluates a permutation on $\mathcal{X}$ in the forward direction, and $I(sk, \cdot)$ evaluates the permutation in the reverse direction. See Section 10.2 for details.

We show that a trapdoor permutation $\mathcal{T}$ gives a simple signature scheme. The only other ingredient we need is a hash function H that maps messages in $\mathcal{M}$ to elements in $\mathcal{X}$. This function will be modeled as a random oracle in the security analysis. The signature scheme, called **full domain hash** (FDH), denoted $\mathcal{S}_{\text{FDH}}$, works as follows:

- The key generation algorithm for $\mathcal{S}_{\text{FDH}}$ is the key generation algorithm G of the trapdoor permutation scheme $\mathcal{T}$. It outputs a pair (pk, sk).
- The signature on m is simply the inverse of $H(m)$ with respect to the function $F(pk, \cdot)$. That is, to sign a message $m \in \mathcal{M}$ using sk, the signing algorithm S runs as follows:

$$\begin{aligned} S(sk, m) := \quad & y \leftarrow H(m), \quad \sigma \leftarrow I(sk, y) \\ & \text{output } \sigma. \end{aligned}$$

- To verify a signature σ on a message m the verification algorithm V checks that $F(pk, \sigma)$ is equal to $H(m)$. More precisely, V works as follows:

$$\begin{aligned} V(pk, m, \sigma) := \quad & y \leftarrow F(pk, \sigma) \\ & \text{if } y = H(m) \text{ output } \mathsf{accept}; \text{ otherwise, output } \mathsf{reject}. \end{aligned}$$

We will analyze $\mathcal{S}_{\text{FDH}}$ by modeling the hash function H as a random oracle. Recall that in the random oracle model (see Section 8.10), the function H is modeled as a random function $\mathcal{O}$ chosen at random from the set of all functions $\text{Funs}[\mathcal{M}, \mathcal{X}]$. More precisely, in the random oracle version of Attack Game 13.1, the challenger chooses $\mathcal{O}$ at random. In any computation where the challenger would normally evaluate H, it evaluates $\mathcal{O}$ instead. In addition, the adversary is allowed to ask the challenger for the value of the function $\mathcal{O}$ at any point of its choosing. The adversary may make any number of such "random oracle queries" at any time of its choosing. We use $\text{SIG}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{S}_{\text{FDH}}]$ to denote $\mathcal{A}$'s advantage against $\mathcal{S}_{\text{FDH}}$ in the random oracle version of Attack Game 13.1.

Theorem 13.3. *Let $\mathcal{T} = (G, F, I)$ be a one-way trapdoor permutation defined over $\mathcal{X}$. Let $H : \mathcal{M} \to \mathcal{X}$ be a hash function. Then the derived FDH signature scheme $\mathcal{S}_{\text{FDH}}$ is a secure signature scheme when H is modeled as a random oracle.*

In particular, let $\mathcal{A}$ be an efficient adversary attacking $\mathcal{S}_{\text{FDH}}$ in the random oracle version of Attack Game 13.1. Moreover, assume that $\mathcal{A}$ issues at most Q_{ro} random oracle queries and Q_{s} signing queries. Then there exists an efficient inverting adversary $\mathcal{B}$ that attacks $\mathcal{T}$ as in Attack Game 10.2, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\text{SIG}^{\text{ro}}\mathsf{adv}[\mathcal{A}, \mathcal{S}_{\text{FDH}}] \leq (Q_{\text{ro}} + Q_{\text{s}} + 1) \cdot \text{OW}\mathsf{adv}[\mathcal{B}, \mathcal{T}] \tag{13.3}$$

An overview of the proof of security for $\mathcal{S}_{\mathrm{FDH}}$. We defer the full proof of Theorem 13.3 to Section 13.4.2. For now, we sketch the main ideas. To forge a signature on a message m, an adversary has to compute $\sigma = I(sk, y)$, where $y = H(m)$. With H modeled as a random oracle, the value y is essentially just a random point in $\mathcal{X}$, and so this should be hard to do, assuming $\mathcal{T}$ is one way. Unfortunately, this argument does not deal with the fact that in a chosen message attack, the adversary can get arbitrary messages signed before producing its forgery. Again, since H is modeled as a random oracle, this effectively means that to break the signature scheme, the adversary must win the following game: after seeing several random points $y_1, y_2, \ldots$ in $\mathcal{X}$ (corresponding to the hash outputs on various messages), the adversary can ask to see preimages of some of the y_i's (corresponding to the signing queries), and then turn around and produce the preimage of one of the remaining y_i's. It turns out that winning this game is not too much easier than breaking the one-wayness of $\mathcal{T}$ in the usual sense. This will be proved below in Lemma 13.5 using a a kind of "guessing argument": in the reduction, we will have to guess in advance at which of the random points the adversary will invert $F(pk, \cdot)$. This is where the factor $Q_{\mathrm{ro}} + Q_{\mathrm{s}} + 1$ in (13.3) comes from.

Unique signatures. The $\mathcal{S}_{\mathrm{FDH}}$ scheme is a **unique signature scheme**: for a given public key, every message m has a *unique* signature σ that will be accepted as valid for m by the verification algorithm. This means that if $\mathcal{S}_{\mathrm{FDH}}$ is secure, it must also be strongly secure in the sense of Definition 13.3.

The importance of hashing. The hash function H is crucial to the security of $\mathcal{S}_{\mathrm{FDH}}$. Without first hashing the message, the system is trivially insecure. To see why, suppose we incorrectly define the signature on $m \in \mathcal{X}$ as $\sigma := I(sk, m)$. That is, we apply I without first hashing m. Then to forge a signature, the adversary simply chooses a random $\sigma \in \mathcal{X}$ and computes $m \leftarrow F(pk, \sigma)$. The pair (m, σ) is an existential forgery. Note that this forgery is created without using the chosen message attack. Of course this m is likely to be gibberish, but is a valid existential forgery.

This attack shows that the hash function H plays a central role in ensuring that $\mathcal{S}_{\mathrm{FDH}}$ is secure. Unfortunately, we can only prove security when H is modeled as a random oracle. We cannot prove security of $\mathcal{S}_{\mathrm{FDH}}$, when H is a concrete hash function, using standard assumptions about $\mathcal{T}$ and H.

13.3.1 Signatures based on the RSA trapdoor permutation

We instantiate the $\mathcal{S}_{\mathrm{FDH}}$ construction with the only trapdoor permutation at our disposal, namely RSA. We obtain the **RSA full domain hash** signature scheme, denoted $\mathcal{S}_{\text{RSA-FDH}}$. Recall that parameters for RSA are generated using algorithm $\mathrm{RSAGen}(\ell, e)$ which outputs a pair (pk, sk) where $pk = (n, e)$. Here n is a product of two ℓ-bit primes. The RSA trapdoor permutation $F(pk, \cdot) : \mathbb{Z}_n \to \mathbb{Z}_n$ is defined as $F(pk, x) := x^e$.

For each public key $pk = (n, e)$, the $\mathcal{S}_{\text{RSA-FDH}}$ system needs a hash function H that maps messages in $\mathcal{M}$ to $\mathbb{Z}_n$. This is a problem — the output space of H depends on n which is different for every public key. Since hash functions generally have a fixed output space, it is preferable that the range of H be fixed and independent of n. To do so, we define the range of H to be $\mathcal{Y} := \{1, \ldots, 2^{2\ell-2}\}$ which, when embedded in $\mathbb{Z}_n$, covers a large fraction of $\mathbb{Z}_n$, for all the RSA moduli n output by $\mathrm{RSAGen}(\ell, e)$.

We describe the signature scheme $\mathcal{S}_{\text{RSA-FDH}}$ using a hash function H defined over $(\mathcal{M}, \mathcal{Y})$. We chose $\mathcal{Y}$ as above so that $|\mathcal{Y}| \geq n/4$ for all n output by $\mathrm{RSAGen}(\ell, e)$. This is necessary for the proof of security. Because an RSA modulus n is large, at least 2048 bits, the hash function H must produce a large output, approximately 2048 bits long. One cannot simply use SHA256. We described appropriate long-output hash functions in Section 8.10.2.

For a given hash function $H : \mathcal{M} \to \mathcal{Y}$, the $\mathcal{S}_{\text{RSA-FDH}}$ signature scheme works as follows:

- the key generation algorithm G uses parameters ℓ and e and runs as follows:

$$\begin{aligned} G() := \quad & (n, d) \xleftarrow{\text{R}} \text{RSAGen}(\ell, e), \quad pk \leftarrow (n, e), \quad sk \leftarrow (n, d) \\ & \text{output } (pk, sk); \end{aligned}$$

- for a given secret key $sk = (n, d)$, and message $m \in \mathcal{M}$, algorithm S runs as follows:

$$\begin{aligned} S(sk, m) := \quad & y \leftarrow H(m) \in \mathcal{Y}, \qquad \sigma \leftarrow y^d \in \mathbb{Z}_n \\ & \text{output } \sigma; \end{aligned}$$

- for a given public key $pk = (n, e)$ the verification algorithm runs as follows:

$$\begin{aligned} V(pk, m, \sigma) := \quad & y \leftarrow \sigma^e \in \mathbb{Z}_n \\ & \text{if } y = H(m) \text{ output } \mathsf{accept}; \text{ otherwise, output } \mathsf{reject}. \end{aligned}$$

Signing and verification speed. Recall that typically the public key exponent e is small, often $e = 3$ or $e = 65537$, while the secret key exponent d is as large as n. Consequently, signature generation, which uses a d exponentiation, is much slower than signature verification. In fact, RSA has the fastest signature verification algorithm among all the standardized signature schemes. This makes RSA very attractive for applications where a signature is generated offline, but needs to be quickly verified online. Certificates used in a public key infrastructure are a good example where fast verification is attractive. We discuss ways to speed-up the RSA signing procedure in Chapter 15.

Signature size. One downside of RSA is that the signatures are much longer than in other signature schemes, such as the ones presented in Chapter 19. To ensure that factoring the RSA modulus n is sufficiently difficult, the size of n must be at least 2048 bits (256 bytes). As a result, RSA signatures are 256 bytes, which is considerably longer than in other schemes. This causes difficulties in heavily congested or low bandwidth networks as well as in applications where space is at a premium. For example, at one point the post office looked into printing digital signatures on postage stamps. The signatures were intended to authenticate the recipient's address and were to be encoded as a two dimensional bar code on the stamp. RSA signatures were quickly ruled out because there is not enough space on a postage stamp. We will discuss short signatures in Section 16.3.

The importance of hashing. We showed above that $\mathcal{S}_{\text{FDH}}$ is insecure without first hashing the message. In particular, consider the **unhashed RSA system** where a signature on $m \in \mathbb{Z}_n$ is defined as $\sigma := m^d$. We showed that this system is insecure since anyone can create an existential forgery (m, σ). Recall, however, that this attack typically forges a signature on a message m that is likely to be gibberish.

We can greatly strengthen the attack on this unhashed RSA using the random self-reducibility property of RSA (see Exercise 10.24). In particular, we show that an attacker can obtain the signature on any message m of his choice by issuing a single signing query for a random $\hat{m} \in \mathbb{Z}_n^*$. Let (n, e) be an RSA public key and let $m \in \mathbb{Z}_n$ be some message. As the reader should verify, we may assume that $m \in \mathbb{Z}_n^*$. To obtain the signature on m the attacker does the following:

$r \xleftarrow{\text{R}} \mathbb{Z}_n^*, \quad \hat{m} \leftarrow m \cdot r^e$
Request the signature on $\hat{m}$ and obtain $\hat{\sigma}$
Output $\sigma \leftarrow \hat{\sigma}/r$

Indeed, if $\hat{\sigma}^e = \hat{m}$ then $\sigma \in \mathbb{Z}_n$ is a valid signature on m since

$$\sigma^e = (\hat{\sigma}/r)^e = \hat{\sigma}^e/r^e = \hat{m}/r^e = m. \tag{13.4}$$

The attack shows that by fooling the user into signing a random message $\hat{m}$ the adversary can obtain the signature on a message m of his choice. We say that unhashed RSA signatures are **universally forgeable** and thus should never be used.

Surprisingly, the fact that an attacker can convert a signature on a random message into a signature on a chosen message turns out to play a central role in the construction of so called blind signatures. Blind signatures are used in protocols for anonymous electronic cash and anonymous electronic voting. In both applications blind signatures are the main ingredient for ensuring privacy (see Exercise 13.12).

Security of RSA full domain hash. Recall that the security proof for the general full domain hash $\mathcal{S}_{\text{FDH}}$ (Theorem 13.3) was very loose: an adversary $\mathcal{A}$ with advantage ϵ in attacking $\mathcal{S}_{\text{FDH}}$ gives an adversary $\mathcal{B}$ with advantage $\epsilon/(Q_{\text{ro}} + Q_{\text{s}} + 1)$ in attacking the underlying trapdoor permutation.

Can we do better? Indeed, we can: using the random self-reducibility property of RSA, we can prove security with a much tighter bound, as shown in Theorem 13.4 below. In particular, the factor $Q_{\text{ro}}+Q_{\text{s}}+1$ is replaced by (approximately) Q_{s}. This is significant, because in a typical attack, the number of signing queries Q_{s} is likely to be much smaller than the number of random oracle queries Q_{ro}. Indeed, on the one hand, Q_{ro} represents the number of times an attacker evaluates the hash function H. These computations can be done by the attacker "off line," and the attacker is only bounded by how own computing resources. On the other hand, each signing query requires that an honest user sign a message. Concretely, a conservative bound on Q_{ro} could perhaps be as large as 2^{128}, while Q_{s} could perhaps be reasonably bounded by 2^{40}. We thus obtain a much tighter reduction for $\mathcal{S}_{\text{RSA-FDH}}$ than for $\mathcal{S}_{\text{FDH}}$ with a general trapdoor permutation. However, even for $\mathcal{S}_{\text{RSA-FDH}}$ the reduction is not tight due to the Q_{s} factor. We will address that later in Section 13.5.

As in the proof of $\mathcal{S}_{\text{FDH}}$, our security proof for $\mathcal{S}_{\text{RSA-FDH}}$ models the hash function $H : \mathcal{M} \to \mathcal{Y}$ as a random oracle. The proof requires that $\mathcal{Y}$ is a large subset of $\mathbb{Z}_n$ (we specifically assume that $|\mathcal{Y}| \geq n/4$, but any constant fraction would do). In what follows, we use 2.72 as an upper bound on the base of the natural logarithm $e \approx 2.718$ (not to be confused with the RSA public exponent e).

Theorem 13.4. *Let $H : \mathcal{M} \to \mathcal{Y}$ be a hash function, where $\mathcal{Y} = \{1, \ldots, 2^{2\ell-2}\}$. If the RSA assumption holds for (ℓ, e), then $\mathcal{S}_{\text{RSA-FDH}}$ with parameters (ℓ, e) is a secure signature scheme, when H is modeled as a random oracle.*

In particular, let $\mathcal{A}$ be an efficient adversary attacking $\mathcal{S}_{\text{RSA-FDH}}$ in the random oracle version of Attack Game 13.1. Moreover, assume that $\mathcal{A}$ issues at most Q_{s} signing queries. Then there *exists an efficient RSA adversary $\mathcal{B}$ as in Attack Game 10.3, where and $\mathcal{B}$ are elementary wrappers around $\mathcal{A}$, such that*

$$\mathrm{SIG^{ro}adv}[\mathcal{A}, \mathcal{S}_{\text{RSA-FDH}}] \leq 2.72 \cdot (Q_{\mathrm{s}} + 1) \cdot \mathrm{RSAadv}[\mathcal{B}, \ell, e] \tag{13.5}$$

We defer the proof of Theorem 13.4 to Section 13.4.2.

13.4 Security analysis of full domain hash

The goal of this section is to analyze the security of the the full domain hash signature scheme; specifically, we prove Theorems 13.3 and 13.4. We begin with a tool that will be helpful, and is interesting and useful in its own right.

13.4.1 Repeated one-way functions: a useful lemma

Let f be a one-way function over $(\mathcal{X}, \mathcal{Y})$. Briefly, this means that given $y \leftarrow f(x)$ for a random $x \in \mathcal{X}$, it is difficult to find a pre-image of y. This notion was presented in Definition 8.4.

Consider the following, seemingly easier, problem: we give the adversary $\big(f(x_1), \ldots, f(x_t)\big)$ and allow the adversary to request some, but not all, of the x_i's. To win, the adversary must produce one of the remaining x_i's. We refer to this as the **t-repeated one-way problem**. More precisely, the problem is defined using the following game.

Attack Game 13.3 (t-repeated one-way problem). For a given positive integer t and a given adversary $\mathcal{A}$, the game runs as follows:

- The challenger computes

$$x_1, \ldots, x_t \xleftarrow{\text{R}} \mathcal{X}, \quad y_1 \leftarrow f(x_1), \ldots, y_t \leftarrow f(x_t)$$

  and sends $(y_1, \ldots, y_t)$ to the adversary.

- $\mathcal{A}$ makes a sequence of *reveal queries*. Each reveal query consists of an index $j \in \{1, \ldots, t\}$. Given j, the challenger sends x_j to $\mathcal{A}$.

- Eventually, $\mathcal{A}$ the adversary outputs (ν, x), where $\nu \in \{1, \ldots, t\}$ and $x \in \mathcal{X}$.

We say that $\mathcal{A}$ wins the game if index ν is not among $\mathcal{A}$'s reveal queries, and $f(x) = y_\nu$. We define $\mathcal{A}$'s advantage, denoted $\mathrm{rOWadv}[\mathcal{A}, f, t]$, as the probability that $\mathcal{A}$ wins the game. $\square$

The following lemma shows that the repeated one-way problem is equivalent to the standard one-way problem given in Definition 8.4. That is, winning in Attack Game 13.3 is not much easier than inverting f.

Lemma 13.5. *For every t-repeated one-way adversary $\mathcal{A}$ there exists a standard one-way adversary $\mathcal{B}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that*

$$\mathrm{rOWadv}[\mathcal{A}, f, t] \leq t \cdot \mathrm{OWadv}[\mathcal{B}, f]. \tag{13.6}$$

Proof idea. The proof is a kind of "guessing argument", somewhat similar to what we did, for example, in the proof of Theorem 6.1. We want to use $\mathcal{A}$ to build an adversary $\mathcal{B}$ that breaks the one-wayness of f. So $\mathcal{B}$ starts with $y_* \in \mathcal{Y}$ and wants to find a preimage of y_* under f, using $\mathcal{A}$ as a subroutine. The first thing that $\mathcal{B}$ does is make a guess ω at the value of the index ν that $\mathcal{A}$ will ultimately choose. Our adversary $\mathcal{B}$ then prepares values $y_1, \ldots, y_t \in \mathcal{Y}$ as follows: for $i \neq \omega$, it sets $y_i \leftarrow f(x_i)$ for random $x_i \in \mathcal{X}$; it also sets $y_\omega \leftarrow y_*$. It then sends $(y_1, \ldots, y_t)$ to $\mathcal{A}$, as in Attack Game 13.3. If $\mathcal{B}$'s guess was correct (which happens with probability $1/t$), it will be able to respond to all of $\mathcal{A}$'s queries, and $\mathcal{A}$'s final output will provide the preimage of y that $\mathcal{B}$ was looking for. □

Proof. In more detail, our adversary $\mathcal{B}$ is given $y_* := f(x_*)$ for a random $x_* \in \mathcal{X}$, and then plays the role of challenger to $\mathcal{A}$ as in Attack Game 13.3 as follows:

Initialize:
- $x_1, \ldots, x_t \xleftarrow{\text{R}} \mathcal{X}$
- $y_1 \leftarrow f(x_1), \ldots, y_t \leftarrow f(x_t)$
- $\omega \xleftarrow{\text{R}} \{1, \ldots, t\}, \quad y_\omega \leftarrow y_* \quad$ // *Plug y_* at position ω*
- Send $(y_1, \ldots, y_t)$ to $\mathcal{A}$

// *$\mathcal{B}$ now knows pre-images for all y_i's other than y_ω*

Upon receiving a query $j \in \{1, \ldots, t\}$ from $\mathcal{A}$:
- if $j \neq \omega$
  - then send x_j to $\mathcal{A}$
  - else output fail and stop

When $\mathcal{A}$ outputs a pair (ν, x):
- if $\nu = \omega$
  - then output x and stop
  - else output fail and stop

Now we argue that the inequality (13.6) holds.

Define Game 0 to be the game played between $\mathcal{A}$ and the challenger in Attack Game 13.3, and let W_0 be the event that $\mathcal{A}$ wins the game.

Now define a new Game 1, which is the same as Game 0, except that the challenger chooses $\omega \in \{1, \ldots, t\}$ at random. Also, we say that $\mathcal{A}$ wins Game 1 if it wins as in Game 0 with output (ν, x) such that $\nu = \omega$. Define W_1 to be the event that $\mathcal{A}$ wins Game 1.

We can think of Games 0 and 1 as operating on the same underlying probability space. Really, the two games are exactly the same: all that changes is the winning condition. Moreover, as ω is independent of everything else, we have

$$\Pr[W_1] = \Pr[W_0 \wedge \nu = \omega] = \Pr[W_0] \cdot \Pr[\nu = \omega \mid W_0] = (1/t) \cdot \Pr[W_0].$$

Moreover, it is clear that $\mathrm{OW}\mathsf{adv}[\mathcal{B}, f] = \Pr[W_1]$; indeed, adversary $\mathcal{B}$ is really just playing Game 1 — it only aborts when it is clear that it will not win Game 1 anyway — and it wins Game 1 if and only if is succeeds in find a preimage of y_*. □

Application to trapdoor functions. Lemma 13.5 applies equally well to trapdoor functions. If $\mathcal{T} = (G, F, I)$ is a trapdoor function scheme defined over $(\mathcal{X}, \mathcal{Y})$, then $\mathcal{T}$ is one way in the sense of Definition 10.3 if and only if $f := F(pk, \cdot)$ is one way in the sense of Definition 8.4. Indeed, for any adversary, the respective advantages in the corresponding attack games are equal. Technically, with $f := F(pk, \cdot)$, the public key pk is viewed as a "system parameter" defining f.

A tighter reduction for RSA. For a general one-way function f, the concrete bound in Lemma 13.5 is quite poor: if adversary $\mathcal{A}$ has advantage ϵ in winning the t-repeated one-way game, then the lemma constructs a one-way attacker with advantage only ϵ/t.

When f is derived from the RSA function we can obtain a tighter reduction using the random self-reducibility property of RSA. We replace the factor t by a factor of (about) Q, where Q is the number of reveal queries from $\mathcal{A}$. This Q is usually much smaller than t.

We first restate Attack Game 13.3 as it applies to the RSA function. We slightly tweak the game and require that the images $y_1, \ldots, y_t$ given to $\mathcal{A}$ lie in a certain large subset of $\mathbb{Z}_n$ denoted $\mathcal{Y}$. For RSA parameters ℓ and e, we set $\mathcal{Y} := \{1, 2, \ldots, 2^{2\ell-2}\}$ so that for all n generated by RSAGen(ℓ, e), we have $|\mathcal{Y}| \geq n/4$.

Attack Game 13.4 (t-repeated RSA). For given RSA parameters ℓ and e, a given positive integer t, and a given adversary $\mathcal{A}$, the game runs as follows:

- The challenger computes

  $$(n, d) \xleftarrow{\text{R}} \text{RSAGen}(\ell, e)$$
  $$y_1, \ldots, y_t \xleftarrow{\text{R}} \mathcal{Y} \quad // \quad \textit{Recall that } \mathcal{Y} := \{1, 2, \ldots, 2^{2\ell-2}\}$$

  and sends (n, e) and $(y_1, \ldots, y_t)$ to $\mathcal{A}$.

- $\mathcal{A}$ makes a sequence of *reveal queries*. Each reveal query consists of an index $j \in \{1, \ldots, t\}$. Given j, the challenger sends $x_j := y_j^d \in \mathbb{Z}_n$ to $\mathcal{A}$.

- Eventually the adversary outputs (ν, x), where $\nu \in \{1, \ldots, t\}$ and $x \in \mathbb{Z}_n$.

We say that $\mathcal{A}$ wins the game if index ν is not among $\mathcal{A}$'s reveal queries, and $x^e = y_\nu$. We define $\mathcal{A}$'s advantage, denoted rRSA$\mathsf{adv}[\mathcal{A}, \ell, e, t]$, as the probability that $\mathcal{A}$ wins the game. □

We show that the t-repeated RSA problem is equivalent to the basic RSA problem, but with a tighter concrete bound than in Lemma 13.5. In particular, the factor of t is replaced by $2.72 \cdot (Q+1)$. The constant 2.72 is an upper on the base of the natural logarithm $e \approx 2.718$.

Lemma 13.6. *Let ℓ and e be RSA parameters. For every t-repeated RSA adversary $\mathcal{A}$ that makes at most Q reveal queries, there exists a standard RSA adversary $\mathcal{B}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that*

$$\text{rRSA}\mathsf{adv}[\mathcal{A}, \ell, e, t] \leq 2.72 \cdot (Q+1) \cdot \text{RSA}\mathsf{adv}[\mathcal{B}, \ell, e]. \tag{13.7}$$

Proof idea. The proof is similar to that of Lemma 13.5. In that proof, we plugged the challenge instance y_* of the one-way attack game at a random position among the y_i's, and using $\mathcal{A}$, we succeed if $\mathcal{A}$ does not issue a reveal query at the plugged position, and its output inverts at the plugged position. Now, using the random self-reducibility property for RSA, we take the challenge y_*, and "spread it around," plugging related, randomized versions of y_* at *many* randomly chosen positions. We succeed if $\mathcal{A}$'s reveal queries avoid the plugged positions, but its output inverts at one of them. By increasing the number of plugged positions, the chance of hitting one at the output

Initialize: // *Generate random* $y_1, \ldots, y_t \in \mathcal{Y}$
    $\Omega \leftarrow \emptyset$
    for $i = 1, \ldots, t$:
        flip a biased coin $c_i \in \{0, 1\}$ such that $\Pr[c_i = 1] = 1/(Q+1)$
        if $c_i = 1$ then $\Omega \leftarrow \Omega \cup \{i\}$
(1)   repeat
            $x_i \xleftarrow{R} \mathbb{Z}_n, \quad y_i \leftarrow x_i^e \cdot y_*^{c_i}$ // *So* $y_i = x_i^e$ *or* $y_i = x_i^e \cdot y_*$
        until $y_i \in \mathcal{Y}$
    Send (n, e) and $(y_1, \ldots, y_t)$ to $\mathcal{A}$

// $\mathcal{B}$ *now knows pre-images for all* y_i *where* $i \notin \Omega$
Upon receiving a reveal query $j \in \{1, \ldots, t\}$ from $\mathcal{A}$:
    if $j \notin \Omega$
        then send x_j to $\mathcal{A}$
        else output fail and stop

When $\mathcal{A}$ outputs a pair (ν, x):
    if $\nu \in \Omega$
(2)   then $\tilde{x} \leftarrow x/x_\nu$, output $\tilde{x}$
        else output fail and stop

Figure 13.2: Algorithm $\mathcal{B}$ in the proof of Lemma 13.6

stage increases (which is good), but the chance of avoiding them during a reveal query decreases (which is bad). Using a clever strategy for sampling the set of plugged positions, we can optimize the success probability to get the desired result. □

Proof. We describe an adversary $\mathcal{B}$ that is given (n, e) and a random $y_* \in \mathbb{Z}_n$, and then attempts to compute an eth root of y_*.

We first deal with an annoying corner case. It may happen (albeit with very small probability) that $y_* \notin \mathbb{Z}_n^*$. However, in this case, it is easy to compute the eth root of y_*: if $y_* = 0$, the eth root is 0; otherwise, $\gcd(y_*, n)$ gives us the prime factorization of n, which allows us to compute the decryption exponent d, and hence the eth root of y_*.

So from now on, we assume $y_* \in \mathbb{Z}_n^*$. Adversary $\mathcal{B}$ uses $\mathcal{A}$ to compute an eth root of y_* as shown in Fig. 13.2. First, $\mathcal{B}$ generates t random values $y_1, \ldots, y_t \in \mathcal{Y}$ and sends them to $\mathcal{A}$. For each $i = 1, \ldots, t$, either $y_i = x_i^e$, in which case $\mathcal{B}$ knows an eth root of y_i and can respond to a reveal query for i, or $y_i = x_i^e \cdot y_*$ in which case $\mathcal{B}$ does not know an eth root of y_i. Here, Ω is the set of indices i for which $\mathcal{B}$ does not know an eth root of y_i.

If $\mathcal{B}$ reaches the line marked (2) and x is an eth root of y_ν, we have

$$\tilde{x}^e = (x/x_\nu)^e = x^e/x_\nu^e = y_\nu/x_\nu^e = (x_\nu^e \cdot y_*)/x_\nu^e = y_*,$$

and so $\mathcal{B}$'s output $\tilde{x}$ is an eth root of y_*.

Actually, we have ignored another corner case. Namely, it may happen (again, with very small probability) that the value x_ν computed above does not lie in $\mathbb{Z}_n^*$. However, if that happens, it must be the case that $x_\nu \neq 0$ (since $0 \notin \mathcal{Y}$), and as in the other corner case, we can use x_ν to factor n and compute the decryption exponent.

Let us analyze the repeat/until loop at the line marked (1) for a fixed $i = 1, \ldots, t$. Since $y_* \in \mathbb{Z}_n^*$, each candidate value for y_i generated in the loop body is uniformly distributed over $\mathbb{Z}_n$. Since $|\mathcal{Y}| \geq n/4$, the probability that each candidate y_i lies in $\mathcal{Y}$ at at least $1/4$. Therefore, the expected number of loop iterations is at most 4. Moreover, when the loop terminates, the final value of y_i is uniformly distributed over $\mathcal{Y}$.

We now argue that (13.7) holds. The basic structure of the argument is the same as in Lemma 13.5. Define Game 0 to be the game played between $\mathcal{A}$ and the challenger in Attack Game 13.4, and let W_0 be the event that $\mathcal{A}$ wins the game.

Now define a new Game 1, which is the same as Game 0, except that the challenger generates a set of indices $\Omega \subseteq \{1, \ldots, t\}$, as follows: each $i = 1, \ldots, t$ is independently added to Ω with probability $1/(Q+1)$. Let $\mathcal{R}$ be the set of reveal queries made by $\mathcal{A}$. We say that $\mathcal{A}$ wins Game 1 if it wins as in Game 0 with output (ν, x), and in addition, $\mathcal{R} \cap \Omega = \emptyset$ and $\nu \in \Omega$. Define W_1 to be the event that $\mathcal{A}$ wins Game 1. We have

$$\Pr[W_1] = \Pr[W_0 \text{ and } \mathcal{R} \cap \Omega = \emptyset \text{ and } \nu \in \Omega] = \Pr[W_0] \cdot \Pr[\mathcal{R} \cap \Omega = \emptyset \wedge \nu \in \Omega \mid W_0].$$

Moreover, it is not hard to see that

$$\mathrm{RSA}\mathsf{adv}[\mathcal{B}, \ell, e] \geq \Pr[W_1].$$

Indeed, when $\mathcal{B}$'s input y_* lies in $\mathbb{Z}_n^*$, adversary $\mathcal{B}$ is essentially just playing Game 1: the distributions of $(y_1, \ldots, y_t, \Omega)$ are identical in both games. The condition $\mathcal{R} \cap \Omega = \emptyset$ corresponds to the condition that $\mathcal{B}$ does not abort in processing one of $\mathcal{A}$'s reveal queries. The condition $\nu \in \Omega$ corresponds to the condition that $\mathcal{B}$ does not abort at $\mathcal{A}$'s output stage. When $\mathcal{B}$'s input y_* lies outside of $\mathbb{Z}_n^*$, adversary $\mathcal{B}$ always wins.

Since Ω is independent of the $\mathcal{A}$'s view, it suffices to prove the following:

> *Claim.* Let Ω be a randomly generated subset of $\{1, \ldots, t\}$, as above. Let $\mathcal{R} \subseteq \{1, \ldots, t\}$ be a fixed set of at most Q indices, and let $\nu \in \{1, \ldots, t\}$ be a fixed index not in $\mathcal{R}$. Let X be the event that $\mathcal{R} \cap \Omega = \emptyset$ and $\nu \in \Omega$. Then we have
>
> $$\Pr[X] \geq \frac{1}{2.72 \cdot (Q+1)}.$$

The claim is trivially true if $Q = 0$; otherwise, we have:

$$\Pr[X] = \Pr[\mathcal{R} \cap \Omega = \emptyset] \cdot \Pr[\nu \in \Omega] \geq \left(1 - \frac{1}{Q+1}\right)^Q \cdot \frac{1}{Q+1} \geq \frac{1}{2.72 \cdot (Q+1)}.$$

Here, we have made use of the handy inequality $1 + x \leq \exp(x)$, which holds for all real numbers x. That proves the claim and the theorem. $\square$

13.4.2 Proofs of Theorems 13.3 and 13.4

Armed with Lemma 13.5, the proof of Theorem 13.3 is quite straightforward. Let $\mathcal{A}$ be an adversary attacking $\mathcal{S}_{\mathrm{FDH}}$ as in the theorem statement. Using $\mathcal{A}$, we wish to construct an adversary $\mathcal{B}$ that breaks the one-wayness of $\mathcal{T}$ with advantage as in (13.3).

We would like to make a few of simplifying assumptions about $\mathcal{A}$. First, whenever $\mathcal{A}$ makes a signing query on a message, it has previously queried the random oracle at that message. Second, when $\mathcal{A}$ outputs its forgery on a particular message, it has previously queried the random oracle on that message. Third, $\mathcal{A}$ never makes the same random oracle query twice, that is, all of its random oracle queries are distinct. If $\mathcal{A}$ does not already satisfy these properties, we can always convert it to and adversary $\mathcal{A}'$ that does, increasing the number of random oracle queries by at most $Q_{\mathrm{s}} + 1$.

So from now on, let us work with the more convenient adversary $\mathcal{A}'$, which makes at most $t := Q_{\mathrm{ro}} + Q_{\mathrm{s}} + 1$ random oracle queries, and whose advantage in breaking the signature scheme $\mathcal{S}_{\mathrm{FDH}}$ is the same as that of $\mathcal{A}$. From $\mathcal{A}'$, we construct an adversary $\mathcal{B}'$ that wins the t-repeated one-way attack game against $f := F(pk, \cdot)$, where $t := Q_{\mathrm{ro}} + Q_{\mathrm{s}} + 1$, with the same advantage that $\mathcal{A}'$ wins the signature game. After we have $\mathcal{B}'$, the theorem follows immediately from Lemma 13.5.

Adversary $\mathcal{B}'$ works as follows. It obtains $(y_1, \ldots, y_t)$ from its own t-repeated one-way challenger. It responds to the ith random oracle query from $\mathcal{A}'$ with y_i. Whenever $\mathcal{A}'$ asks to sign a particular message, by assumption, the random oracle has already been queried at that message; if this was the jth random oracle query, $\mathcal{B}'$ makes a reveal query at position j to obtain x_j, and forwards x_j to $\mathcal{A}'$. Finally, when $\mathcal{A}'$ outputs its candidate forgery (m, σ), then by assumption, the random oracle query was already queried at m; if this was query number ν, then $\mathcal{B}'$ outputs (ν, σ).

Clearly, $\mathcal{B}'$ simulates the signature attack game perfectly for $\mathcal{A}'$, and wins its attack game precisely when $\mathcal{A}'$ wins its game.

Proof of Theorem 13.4. This is almost identical to the proof of Theorem 13.4. The only difference is that we use Lemma 13.6 instead of Lemma 13.5. In the application of Lemma 13.6, the the of reveal queries Q in Attack Game 13.4 is bounded by Q_{s}.

13.5 An RSA-based signature scheme with tighter security proof

Theorem 13.4 shows that $\mathcal{S}_{\text{RSA-FDH}}$ is a secure signature scheme in the random oracle model, but with a relatively loose security reduction. In particular, let $\mathcal{A}$ be an adversary attacking $\mathcal{S}_{\text{RSA-FDH}}$ that issues at most Q_{s} signing queries and succeeds in breaking $\mathcal{S}_{\text{RSA-FDH}}$ with probability ϵ. Then $\mathcal{A}$ can be used to break the RSA assumption with probability about ϵ/Q_{s}. It is unlikely that $\mathcal{S}_{\text{RSA-FDH}}$ has a tighter security reduction to the RSA assumption.

Surprisingly, a small modification to $\mathcal{S}_{\text{RSA-FDH}}$ gives a signature scheme that has a tight reduction to the RSA assumption in the random oracle model. The only difference is that instead of computing an eth root of $H(m)$, the signing algorithm computes an eth root of $H(b, m)$ for some random bit $b \in \{0, 1\}$. The signature includes the eth root along with the bit b. We call this modified signature scheme $\mathcal{S}'_{\text{RSA-FDH}}$.

We describe $\mathcal{S}'_{\text{RSA-FDH}}$ using the notation of Section 13.3.1. Let $\mathcal{M}' := \{0, 1\} \times \mathcal{M}$. We will need a hash function $H : \mathcal{M}' \to \mathcal{Y}$. Furthermore, we will need a PRF F defined over $(\mathcal{K}, \mathcal{M}, \{0, 1\})$. The $\mathcal{S}'_{\text{RSA-FDH}}$ signature scheme is defined as follows:

- The key generation algorithm G uses fixed RSA parameters ℓ and e, and runs as follows:

$$\begin{aligned} G() := \quad & k \xleftarrow{\mathrm{R}} \mathcal{K}, \quad (n, d) \xleftarrow{\mathrm{R}} \mathrm{RSAGen}(\ell, e) \\ & pk \leftarrow (n, e), \quad sk \leftarrow (k, n, d) \\ & \text{output } (pk, sk). \end{aligned}$$

- For a given secret key $sk = (k, n, d)$ and $m \in \mathcal{M}$, the signing algorithm S runs as follows:

$$S(sk, m) := \quad \begin{array}{l} b \leftarrow F(k, m) \in \{0,1\} \\ y \leftarrow H(b, m) \in \mathcal{Y}, \quad \sigma \leftarrow y^d \in \mathbb{Z}_n \\ \text{output } (b, \sigma). \end{array}$$

- For a given public key $pk = (n, e)$ and signature (b, σ), the verification algorithm does:

$$V\big(pk,\ m,\ (b,\sigma)\big) := \quad \begin{array}{l} y \leftarrow H(b, m) \\ \text{if } y = \sigma^e \text{ output } \textsf{accept}\text{; otherwise, output } \textsf{reject}. \end{array}$$

Security. The $\mathcal{S}'_{\text{RSA-FDH}}$ system can be shown to be secure under the RSA assumption, when H is modeled as a random oracle. The security proof uses the random self reduction of RSA to obtain a tight reduction to the RSA problem. The point is that the factor $2.72(Q_s + 1)$ in Theorem 13.4 is replaced by a factor of 2 in the theorem below.

Theorem 13.7. *Let $H : \mathcal{M}' \to \mathcal{Y}$ be a hash function. Assume that the RSA assumption holds for (ℓ, e), and F is a secure PRF. Then $\mathcal{S}'_{\text{RSA-FDH}}$ is a secure signature scheme when H is modeled as a random oracle.*

In particular, let $\mathcal{A}$ be an efficient adversary attacking $\mathcal{S}'_{\text{RSA-FDH}}$. Then there exist an efficient RSA adversary $\mathcal{B}$ and a PRF adversary $\mathcal{B}_F$, where $\mathcal{B}$ and $\mathcal{B}_F$ are elementary wrappers around $\mathcal{A}$, such that

$$\text{SIG}^{\text{ro}}\textsf{adv}[\mathcal{A}, \mathcal{S}'_{\text{RSA-FDH}}] \le 2 \cdot \text{RSA}\textsf{adv}[\mathcal{B}, \ell, e] + \text{PRF}\textsf{adv}[F, \mathcal{B}_F]$$

Proof idea. Suppose the PRF F is a random function $f : \mathcal{M} \to \{0,1\}$. We build an algorithm $\mathcal{B}$ that uses an existential forger $\mathcal{A}$ to break the RSA assumption. Let $(n, d) \xleftarrow{\text{R}} \text{RSAGen}(\ell, e), x_* \xleftarrow{\text{R}} \mathbb{Z}_n$, and $y_* \leftarrow x_*^e \in \mathbb{Z}_n$. Algorithm $\mathcal{B}$ is given n, y_* and its goal is to output x_*. First $\mathcal{B}$ sends the public key $pk = (n, e)$ to $\mathcal{A}$. Now $\mathcal{A}$ issues random oracle queries and signing queries. To obtain a tight reduction, $\mathcal{B}$ must properly answer all signing queries from $\mathcal{A}$. In other words, $\mathcal{B}$ must be able to sign every message in $\mathcal{M}$. But this seems impossible — if $\mathcal{B}$ already knows the signature on all messages, how can an existential forgery from $\mathcal{A}$ possibly help $\mathcal{B}$ solve the challenge (n, y_*)? The signature produced by $\mathcal{A}$ seems to give $\mathcal{B}$ no new information.

The solution comes from the extra bit in the signature. Recall that in $\mathcal{S}'_{\text{RSA-FDH}}$ every message $m \in \mathcal{M}$ has two valid signatures, namely $\sigma_0 = (0,\ H(m,0)^d)$ and $\sigma_1 = (1,\ H(m,1)^d)$. Algorithm $\mathcal{B}$ sets things up so that it knows exactly one of these signatures for every message. In particular, $\mathcal{B}$ will know the signature $(b, H(b, m))$ where $b \leftarrow f(m)$. The forger $\mathcal{A}$ will output an existential forgery $(m, (b, \sigma))$ where, with probability 1/2, (b, σ) is the signature on m that $\mathcal{B}$ does not know. We will use the random self reduction of RSA to ensure that any such signature enables $\mathcal{B}$ to solve the original challenge. For this to work, $\mathcal{A}$ must not know which of the two signatures $\mathcal{B}$ knows. Otherwise, a malicious $\mathcal{A}$ could always output a signature forgery that is of no use to $\mathcal{B}$. This is the purpose of the PRF.

To implement this idea, $\mathcal{B}$ responds to random oracle queries and signing queries as follows. We let $\mathcal{O}$ denote the random oracle implementing H.

- upon receiving a random oracle query $(b, m) \in \mathcal{M}'$ from $\mathcal{A}$ do:

$$\begin{array}{l} \text{if } b = f(m) \text{ then } c \leftarrow 0 \text{ else } c \leftarrow 1 \\ \text{repeat until } y \in \mathcal{Y} \\ \qquad x \xleftarrow{\text{R}} \mathbb{Z}_n, \quad y \leftarrow x^e \cdot y_*^c \in \mathbb{Z}_n \quad // \quad \textit{So } y = x^e \textit{ or } y = x^e \cdot y_* \\ \text{send } y \text{ to } \mathcal{A} \quad // \quad \textit{This defines } \mathcal{O}(b, m) := y \end{array}$$

Observe that in either case $\mathcal{O}(b,m)$ is a uniform value in $\mathcal{Y}$ as required. In particular, $\mathcal{A}$ learns nothing about the value of $f(m)$.

When $b = f(m)$ the random oracle value $\mathcal{O}(b,m)$ is a random value y for which $\mathcal{B}$ knows an eth root, namely x. When $b \neq f(m)$ then $\mathcal{O}(b,m)$ is a random value y for which $\mathcal{B}$ does not know an eth root. In fact, an eth root of $y = x^e \cdot y_*$ will solve the original challenge — if σ is an eth root of y then $x_* = \sigma/x \in \mathbb{Z}_n$ is an eth root of y_*, since:

$$x_*^e = \sigma^e/x^e = y/x^e = (x^e \cdot y_*)/x^e = y_*. \tag{13.8}$$

In effect, $\mathcal{B}$ uses the random self reduction of RSA to map the original challenge y_* to a random challenge y. It then maps $\mathcal{O}(b,m)$ to this random y.

- Upon receiving a signing query $m \in \mathcal{M}$ from $\mathcal{A}$, respond as follows. First, compute $b \leftarrow f(m)$ and let $y \leftarrow \mathcal{O}(b,m) \in \mathcal{Y}$. By construction, $\mathcal{B}$ defined $\mathcal{O}(b,m) = x^e$ for some random $x \in \mathbb{Z}_n$ chosen by $\mathcal{B}$. Hence, $\mathcal{B}$ has an eth root x for this y. It sends $\mathcal{A}$ the signature (b,x).

So far, $\mathcal{B}$ simulates the challenger perfectly. Its responses to $\mathcal{A}$'s oracle queries are uniform and random in $\mathcal{Y}$ and all its responses to signing queries are valid. Therefore, $\mathcal{A}$ produces an existential forgery (b,σ) on some message m. Then $\sigma^e = \mathcal{O}(b,m)$. Now, if $b \neq f(m)$ then $\mathcal{O}(b,m) = x^e \cdot y_*$ and hence $x_* = \sigma/x$ as in (13.8).

In summary, assuming $b \neq f(m)$, algorithm $\mathcal{B}$ obtains a solution to the challenge y_*. But, by construction of $\mathcal{O}$, the adversary learns no information about the function f. In particular, $f(m)$ is a random bit, and is independent of the adversary's view. Therefore, $b \neq f(m)$ happens with probability $1/2$. This is the source of the factor of 2 in Theorem 13.7. □

So what does this mean? The $\mathcal{S}'_{\text{RSA-FDH}}$ system is a minor modification of $\mathcal{S}_{\text{RSA-FDH}}$. Signatures include an additional bit which leads to a tighter reduction to the RSA assumption. Despite this tighter reduction, $\mathcal{S}'_{\text{RSA-FDH}}$ has not gained much acceptance in practice. Most practitioners do not view the extra complexity as a worthwhile tradeoff against the tighter reduction, especially since this reduction is ultimately heuristic, as it models H as a random oracle. It is not clear that $\mathcal{S}'_{\text{RSA-FDH}}$ is any more secure than $\mathcal{S}_{\text{RSA-FDH}}$ for any particular instantiation of H. This is an open question. Conversely, Exercise 13.7 shows that for every instantiation of H, the signature scheme $\mathcal{S}'_{\text{RSA-FDH}}$ is *no less* secure than $\mathcal{S}_{\text{RSA-FDH}}$.

13.6 Case study: PKCS1 signatures

The most widely deployed standard for RSA signatures is known as PKCS1 version 1.5 mode 1. This RSA signing method is commonly used for signing X.509 certificates. Let n be an t-bit RSA modulus. The standard requires that t is a multiple of 8. Let e be the encryption exponent (or signature verification exponent). To sign a message m, the standard specifies the following steps:

- Hash m to an h-bit hash value using a collision resistant hash function H, where h is also required to be a multiple of 8. The standard requires that $h < t - 88$.

- Let $D \in \{0,1\}^t$ be the binary string shown in Fig. 13.3. The string starts with the two bytes 00 01. It then contains a padding sequence of FF-bytes that ends with a single 00 byte. Next a short DigestInfo (DI) field is appended that encodes the name of the hash function H used

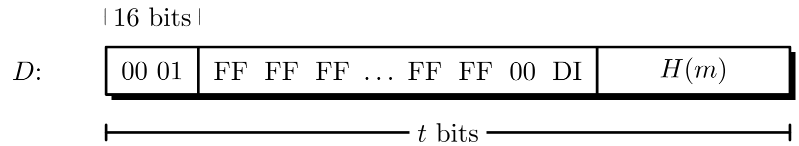


Figure 13.3: PKCS1 signatures: the quantity D signed by RSA

to hash m. For example, when SHA256 is used the DigestInfo field is a *fixed* 19-byte string. Finally, $H(m)$ is appended. The length of the padding sequence of FF-bytes is such that D is exactly t bits.

- View D as an t-bit integer, which we further interpret as an element of $\mathbb{Z}_n$, and output the eth root of D as the signature σ.

To verify the signature, first compute $\sigma^e \in \mathbb{Z}_n$, and then interpret this as an t-bit string D. Finally, verify that D contains all the fields shown in Fig. 13.3, and no other fields.

The reason for prepending the fixed PKCS1 pad to the hash value prior to signing is to avoid a chosen message attack due to Desmedt and Odlyzko [30]. The attack is based on the following idea. Suppose PKCS1 directly signed a 256-bit message digest with RSA, without first expanding it to a long string as in Fig. 13.3. Further, suppose the attacker finds three messages m_1, m_2, m_3 such that

$$H(m_1) = p_1, \quad H(m_2) = p_2, \quad H(m_3) = p_1 \cdot p_2, \tag{13.9}$$

where $H(m_1), H(m_2), H(m_3)$ are viewed as integers in the interval $[0, 2^{256})$. The attacker can request the signatures on m_1 and m_2 and from them deduce the signature on m_3 by multiplying the two given signatures. Hence, the attacker obtains an existential forgery by issuing two chosen message queries. The attack of Desmedt and Odlyzko extends this basic idea so that the attack succeeds with high probability using many chosen message queries. The reason for the padding in Fig. 13.3 is so that the numbers for which an eth root is computed are much longer than 256 bits. As a result, it is much less likely that an attacker can find messages satisfying a condition such as (13.9).

Security. PKCS1 is an example of a **partial domain hash** signature. The message m is hashed into an h-bit string that is mapped into a fixed interval I inside of $\mathbb{Z}_n$. The interval has size $|I| = 2^h$. Typically, the hash size h is 160 or 256 bits, and the modulus size t is at least 2048 bits. Hence, I is a tiny subset of $\mathbb{Z}_n$.

Unfortunately, the proof of Theorem 13.4 requires that the output of the hash function H be uniformly distributed over a large subset $\mathcal{Y}$ of $\mathbb{Z}_n$. This was necessary for the proof of Lemma 13.6. The set $\mathcal{Y}$ had to be large so that we could pick a random $y \in \mathcal{Y}$ for which we knew an eth root.

When hashing into a tiny subset I of $\mathbb{Z}_n$ the proof of Lemma 13.6 breaks down. The problem is that we cannot pick a random $y \in I$ so that an eth root of y is known. More precisely, the obstruction to the proof is the following problem:

($*$) given an RSA modulus n, output a pair (y, x) where y is uniformly distributed in a subset $I \subseteq \mathbb{Z}_n$ and x is an eth root of y.

A solution to this problem will enable us to prove security of PKCS1 under the assumption that computing eth roots is hard in the interval I. Problem $(*)$ is currently open. The best known algorithm [27] solves the problem for $e = 2$ whenever $|I| \geq n^{2/3}$. However, typically in PKCS1, $|I|$ is far smaller than $n^{2/3}$ (and for RSA we use $e > 2$).

In summary, although PKCS1 v1.5 is a widely used standard for signing using RSA, we cannot prove it secure under the standard RSA assumption. An updated version of PKCS1 known as PKCS1 v2.1 includes an additional RSA-based signature method called PSS, discussed in the chapter notes.

13.6.1 Bleichenbacher's attack on PKCS1 signatures

Implementing cryptography is not easy. In this section, we give a clever attack on a once-popular implementation of PKCS1 that illustrates its fragility. Let $pk = (n, 3)$ be an RSA public key for the PKCS1 signature scheme: n is an t-bit RSA modulus and the signature verification exponent is 3. We assume $t \geq 2048$.

When signing a message m using PKCS1 the signer forms the block D shown in Fig. 13.3, and then, treating D as an integer, computes the cube root of D modulo n as the signature σ.

Consider the following erroneous implementation of the verification algorithm. To verify a message-signature pair (m, σ), with SHA256 as the hash function, the verifier does:

1. compute $\sigma^e \in \mathbb{Z}_n$, and then interpret this as a t-bit string D
2. parse D from left to right as follows:
   (a) reject if the top most 2 bytes are not 00 01
   (b) skip over all FF-bytes until reaching a 00 byte and skip over it too
   (c) reject if the next bytes are not the DigestInfo field for the SHA256 function
   (d) read the following 32 bytes (256 bits), compare them to the hash value SHA256(m), and reject if not equal
3. if all the checks above pass successfully, accept the signature

While this procedure appears to correctly verify the signature, it ignores one very crucial step: it does not check that D contains nothing to the right of the hash value. In particular, this verification procedure accepts an t-bit block D^* that looks as follows:

$$D^* := \boxed{00\ 01}\boxed{\text{FF}\ldots\text{FF 00 DI}}\boxed{\text{hash}}\boxed{\text{more bits } J}$$

Here J is some sequence of bits chosen by the attacker. The attacker shortened the variable length padding block of FF's to make room for the quantity J, so that the total length of D^* is still t bits.

This minor-looking oversight leads to a complete break of the signature scheme. An attacker can generate a valid signature on any message m of its choice, as we now proceed to demonstrate.

Let $w \in \mathbb{Z}$ be the largest multiple of eight smaller than $t/3 - 3$. To forge the signature on m, the attacker first computes $H(m) = \text{SHA256}(m)$ and constructs the block D, as in Fig. 13.3, but where D is only w bits long (note that $w \approx t/3$). To make D this short, simply make the variable length padding block sufficiently short. Next, viewing D as an integer, the attacker computes:

$$s \leftarrow \sqrt[3]{D \cdot 2^{t-w}} \in \mathbb{R}, \qquad x \leftarrow \lceil s \rceil \in \mathbb{Z}, \qquad \text{output } x.$$

Here, the cube root s of $D \cdot 2^{t-w}$ is computed over the real numbers and rounded up to the next integer x.

We show that x, when viewed as an element of $\mathbb{Z}_n$, will be accepted as a valid signature on m. Since $0 \leq x - s < 1$, we obtain

$$0 \leq x^3 - (D \cdot 2^{t-w}) = x^3 - s^3 = (x-s)(x^2 + xs + s^2) < 3(s+1)^2.$$

Observe that $s^3 = D \cdot 2^{t-w} < 2^t$, because the leading bits of D are zero. Moreover, for $s \geq 3$, we have that $(s+1)^2 \leq 2s^2 < 2 \cdot 2^{(2/3)t}$, and therefore

$$0 \leq x^3 - (D \cdot 2^{t-w}) < 3(s+1)^2 < 6 \cdot 2^{(2/3)t} < 2^{t-[(t/3)-3]} < 2^{t-w}.$$

In other words, $x^3 = (D \cdot 2^{t-w}) + J$ where $0 \leq J < 2^{t-w}$.

It follows that if we treat x as an element of $\mathbb{Z}_n$, it will be accepted as a signature on m. Indeed, x^3 will be strictly less than n, so the computation of $x^3 \bmod n$ will not wrap around at all. Moreover, when the verifier interprets x^3 as an t-bit string D^*, the w most significant bits of D^* are equal to D, ensuring that x will be accepted as a signature on m with respect to the public key $(n, 3)$.

This attack applies to RSA public keys that use a small public exponent, such as $e = 3$. When it was originally discovered, it was shown to work well against several popular PKCS1 implementations. The attack exploits a bug in the implementation of PKCS1 that is easily mitigated: the verifier must reject the signature if D is not the correct length, or there are bits in D to the right of the hash value. Nevertheless, it is a good illustration of the difficulty of correctly implementing cryptographic primitives. A simple misunderstanding in reading the PKCS1 specification resulted in a devastating attack on its implementation.

13.7 Signcryption: combining signatures and encryption

A signcryption scheme lets a sender, Alice, send an encrypted message to a recipient, Bob, so that (1) only Bob can read the message, and (2) Bob is convinced that the message came from Alice. Signcryption schemes are needed in messaging systems that provide end-to-end security, but where Bob may be offline at the time that Alice sends the message. Because Bob is offline, Alice cannot interact with Bob to establish a shared session key. Instead, she encrypts the message intended for Bob, and Bob receives and decrypts it at a later time. The ciphertext she sends to Bob must convince Bob that the message is from Alice.

Since anyone can generate public-private key pairs, signcryption only makes sense in an environment where every identity is publicly bound to one or more public keys. More precisely, Bob can tell what public keys are bound to Alice's identity, and an attacker cannot cause Bob to associate an incorrect public key to Alice. If this were not the case, that is, if an attacker can generate a public-private key pair and convince Bob that this public key belongs to Alice, then the goals of signcryption cannot be achieved: the attacker could send a message on behalf of Alice, and Bob could not tell the difference; similarly, the attacker could decrypt messages that Bob thinks he is sending to Alice.

To capture this requirement on public keys and identities, we assign to every user X of the system a unique identity id_{X}. Moreover, we assume that any other user can fetch the public key pk_{X} that is bound to the identity id_{X}. So, Alice can obtain a public key bound to Bob, and she can be reasonably confident that only Bob knows the corresponding private key. Abstractly, one can think of a public directory that maintains a mapping from identities to public keys. Anyone can read the directory, but only the user with identity id_{X} can update the record associated with id_{X} (in today's technology, Facebook user profiles serve as such a global directory). In Section 13.8 we will see that certificates are another way to reliably bind public keys to identities.

We will denote the sender's identity by id_{S} and the recipient's identity by id_{R}. We denote the sender's public-private key pair by pk_{S} and sk_{S} and the recipients key pair by pk_{R} and sk_{R}. To encrypt a message m intended for a specific recipient, the sender needs its own identity id_{S} and secret key sk_{S} as well as the recipients identity id_{R} and public key pk_{R}. To decrypt an incoming ciphertext, the recipient needs the sender's identity id_{S} and public key pk_{S} as well as its own identity id_{R} and secret key sk_{R}. With this in place we can define the syntax for signcryption.

Definition 13.5. *A **signcryption** scheme $\mathcal{SC} = (G, E, D)$ is a triple of efficient algorithms, G, E and D, where G is called a **key generation algorithm**, E is called an **encryption algorithm**, and D is called a **decryption algorithm**.*

- *G is a probabilistic algorithm that takes no input. It outputs a pair (pk, sk), where sk is called a **secret key** and pk is called a **public key**.*
- *E is a probabilistic algorithm that is invoked as $c \xleftarrow{\mathrm{R}} E\big(sk_{\mathrm{S}}, id_{\mathrm{S}}, pk_{\mathrm{R}}, id_{\mathrm{R}},\ m\big)$, where sk_{S} and id_{S} are the secret key and identity of the sender, pk_{R} and id_{R} are the public key and identity of the recipient, and m is a message. The algorithm outputs a **ciphertext** c.*
- *D is a deterministic algorithm invoked as $D\big(pk_{\mathrm{S}}, id_{\mathrm{S}}, sk_{\mathrm{R}}, id_{\mathrm{R}},\ c\big)$. It outputs either a message m or a special symbol* reject.
- *We require that a ciphertext generated by E is always accepted by D. That is, for all possible outputs $(pk_{\mathrm{S}}, sk_{\mathrm{S}})$ and $(pk_{\mathrm{R}}, sk_{\mathrm{R}})$ of G, all identities $id_{\mathrm{S}}, id_{\mathrm{R}}$, and all messages m*

$$\Pr\Big[D\big(pk_{\mathrm{S}}, id_{\mathrm{S}}, sk_{\mathrm{R}}, id_{\mathrm{R}},\ E(sk_{\mathrm{S}}, id_{\mathrm{S}}, pk_{\mathrm{R}}, id_{\mathrm{R}},\ m)\big) = m\Big] = 1.$$

*As usual, we say that messages lie in a finite **message space** $\mathcal{M}$, ciphertexts lie in some finite **ciphertext space** $\mathcal{C}$, and identities lie in some finite **identity space** $\mathcal{I}$. We say that $\mathcal{SC} = (G, E, D)$ is defined over $(\mathcal{M}, \mathcal{C}, \mathcal{I})$.*

We can think of signcryption as the public-key analogue of authenticated encryption for symmetric ciphers. Authenticated encryption is designed to achieve the same confidentiality and authenticity goals as signcryption, but assuming the sender and recipient have already established a shared secret key. Signcryption is intended for a non-interactive setting where no shared secret key is available. With this analogy in mind we can consider two signcryption constructions, similar to the ones in Chapter 9:

- The signcryption analogue of encrypt-then-MAC is encrypt-then-sign: first encrypt the message with the recipient's public encryption key and then sign the resulting ciphertext with the sender's secret signing key.
- The signcryption analogue of MAC-then-encrypt is sign-then-encrypt: first sign the message with the sender's secret signing key and then encrypt the message-signature pair with the recipient's public encryption key.

Which of these is secure? Is one method better than the other? To answer these questions we must first formally define what it means for a signcryption scheme to be secure, and then analyze these and other signcryption schemes.

We begin in Section 13.7.1 with a formal definition of security for signcryption. Admittedly, our definition of secure signcryption is a bit lengthy, and it may not be immediately clear that it captures the "right" properties. In Section 13.7.2, we discuss how this definition can be used to derive more intuitive security properties of signcryption in a multi-user setting. It is precisely these implications that give us confidence that the basic definition in Section 13.7.1 is sufficiently strong. In Sections 13.7.3 and 13.7.4 we turn to the problem of constructing secure signcryption schemes. Finally, in Section 13.7.5, we investigate some additional desirable security properties for signcryption, called forward-secrecy and non-repudiation, and show how to achieve them.

13.7.1 Secure signcryption

We begin with the basic security requirements for a signcryption scheme. As we did for authenticated encryption, we define secure signcryption using two games. One game captures data confidentiality: an adversary who does not have Alice's or Bob's secret key cannot break semantic security for a set of challenge ciphertexts from Alice to Bob. The other game captures data authenticity: an adversary who does not have Alice's or Bob's secret key cannot make Bob accept a ciphertext that was not generated by Alice with the intent of sending it to Bob.

In both games the adversary is active. In addition to asking Alice to encrypt messages intended for Bob, and asking Bob to decrypt messages supposedly coming from Alice, the adversary is free to ask Alice to encrypt messages intended for any other user of the adversary's choosing, and to ask Bob to decrypt messages supposedly coming from any other user of the adversary's choosing. Moreover, the attack game reflects the fact that while Alice may be sending messages to Bob, she may also be receiving messages from other users. Therefore, the adversary is free to ask Alice to decrypt messages supposedly coming from any other user of the adversary's choosing. Similarly, modeling the fact that Bob may also be playing the role of sender, the adversary is free to ask Bob to encrypt messages intended for any other user of the adversary's choosing.

Ciphertext integrity. We start with the data authenticity game, which is an adaptation of the ciphertext integrity game used in the definition of authenticated encryption (Attack Game 9.1).

Attack Game 13.5 (ciphertext integrity). For a given signcryption scheme $\mathcal{SC} = (G, E, D)$ defined over $(\mathcal{M}, \mathcal{C}, \mathcal{I})$, and a given adversary $\mathcal{A}$, the attack game runs as follows:

- The adversary chooses two distinct identities $id_{\rm S}$ (the sender identity) and $id_{\rm R}$ (the receiver identity), and gives these to the challenger. The challenger runs G twice to obtain $(pk_{\rm S}, sk_{\rm S})$ and $(pk_{\rm R}, sk_{\rm R})$ and gives $pk_{\rm S}$ and $pk_{\rm R}$ to $\mathcal{A}$.

- $\mathcal{A}$ issues a sequence of queries to the challenger. Each query is one of the following types:

  S $\rightarrow$ R *encryption query:* a message m.
  The challenger computes $c \xleftarrow{\rm R} E(sk_{\rm S}, id_{\rm S}, pk_{\rm R}, id_{\rm R}, m)$, and gives c to $\mathcal{A}$.

  X $\rightarrow$ Y *encryption query:* a tuple $(id_{\rm X}, id_{\rm Y}, pk_{\rm Y}, m)$, where $id_{\rm X} \in \{id_{\rm S}, id_{\rm R}\}$ and $(id_{\rm X}, id_{\rm Y}) \neq (id_{\rm S}, id_{\rm R})$. The challenger responds to $\mathcal{A}$ with c, computed as follows:

$$\text{if } id_X = id_S \text{ then } c \xleftarrow{R} E(sk_S, id_S, pk_Y, id_Y, m),$$
$$\text{if } id_X = id_R \text{ then } c \xleftarrow{R} E(sk_R, id_R, pk_Y, id_Y, m).$$

X $\to$ Y *decryption query:* a tuple $(id_X, id_Y, pk_X, \hat{c})$, where $id_Y \in \{id_S, id_R\}$ and $(id_X, id_Y) \neq (id_S, id_R)$. The challenger responds to $\mathcal{A}$ with $\hat{m}$, computed as follows:

$$\text{if } id_Y = id_S \text{ then } \hat{m} \leftarrow D(pk_X, id_X, sk_S, id_S, \hat{c}),$$
$$\text{if } id_Y = id_R \text{ then } \hat{m} \leftarrow D(pk_X, id_X, sk_R, id_R, \hat{c}).$$

- Finally, $\mathcal{A}$ outputs a candidate ciphertext forgery $c' \in \mathcal{C}$, where c' is not among the responses to an S $\to$ R encryption query.

We say that $\mathcal{A}$ wins the game if its candidate ciphertext forgery c' is a valid ciphertext from id_S to id_R, that is, $D(pk_S, id_S, sk_R, id_R, c') \neq \mathsf{reject}$. We define $\mathcal{A}$'s advantage, denoted $\mathrm{SCI}\mathsf{adv}[\mathcal{A}, \mathcal{SC}]$, as the probability that $\mathcal{A}$ wins the game. □

Definition 13.6. *We say that $\mathcal{SC} = (G, E, D)$ provides **signcryption ciphertext integrity**, or* SCI *for short, if for every efficient adversary $\mathcal{A}$, the value* $\mathrm{SCI}\mathsf{adv}[\mathcal{A}, \mathcal{SC}]$ *is negligible.*

Security against a chosen ciphertext attack. Next, we define the data confidentiality game, which is an adaptation of the game used to define chosen ciphertext security (Attack Game 12.1). Note that in this game, the syntax of the X $\to$ Y encryption and decryption queries are exactly the same as in Attack Game 13.5.

Attack Game 13.6 (CCA security). For a given signcryption scheme $\mathcal{SC} = (G, E, D)$, defined over $(\mathcal{M}, \mathcal{C}, \mathcal{I})$, and for a given adversary $\mathcal{A}$, we define two experiments.

Experiment b $(b = 0, 1)$:

- The adversary chooses two distinct identities id_S (the sender identity) and id_R (the receiver identity), and gives these to the challenger. The challenger runs G twice to obtain (pk_S, sk_S) and (pk_R, sk_R) and gives pk_S and pk_R to $\mathcal{A}$.

- $\mathcal{A}$ issues a sequence of queries to the challenger. Each query is one of the following types:

S $\to$ R *encryption query:* a pair of equal-length messages (m_0, m_1).
The challenger computes $c \xleftarrow{R} E(sk_S, id_S, pk_R, id_R, m_b)$, and gives c to $\mathcal{A}$.

S $\to$ R *decryption query:* a ciphertext $\hat{c}$, where $\hat{c}$ is not among the outputs of any previous S $\to$ R encryption query.
The challenger computes $\hat{m} \xleftarrow{R} D(pk_S, id_S, sk_R, id_R, \hat{c})$, and gives $\hat{c}$ to $\mathcal{A}$.

X $\to$ Y *encryption query:* a tuple (id_X, id_Y, pk_Y, m), where $id_X \in \{id_S, id_R\}$ and $(id_X, id_Y) \neq (id_S, id_R)$. The challenger responds to $\mathcal{A}$ with c, computed as follows:

$$\text{if } id_X = id_S \text{ then } c \xleftarrow{R} E(sk_S, id_S, pk_Y, id_Y, m),$$
$$\text{if } id_X = id_R \text{ then } c \xleftarrow{R} E(sk_R, id_R, pk_Y, id_Y, m).$$

X $\to$ Y *decryption query:* a tuple $(id_X, id_Y, pk_X, \hat{c})$, where $id_Y \in \{id_S, id_R\}$ and $(id_X, id_Y) \neq (id_S, id_R)$. The challenger responds to $\mathcal{A}$ with $\hat{m}$, computed as follows:

$$\text{if } id_Y = id_S \text{ then } \hat{m} \leftarrow D(pk_X, id_X, sk_S, id_S, \hat{c}),$$
$$\text{if } id_Y = id_R \text{ then } \hat{m} \leftarrow D(pk_X, id_X, sk_R, id_R, \hat{c}).$$

- At the end of the game, the adversary outputs a bit $\hat{b} \in \{0, 1\}$.

Let W_b is the event that $\mathcal{A}$ outputs 1 in Experiment b and define $\mathcal{A}$'s advantage as

$$\mathrm{SCCA}\mathsf{adv}[\mathcal{A}, \mathcal{SC}] := \big|\Pr[W_0] - \Pr[W_1]\big|. \quad \Box$$

Definition 13.7 (CCA Security). *A signcryption scheme $\mathcal{SC}$ is called* ***semantically secure against a chosen ciphertext attack****, or simply* ***CCA secure****, if for all efficient adversaries $\mathcal{A}$, the value* $\mathrm{SCCA}\mathsf{adv}[\mathcal{A}, \mathcal{SC}]$ *is negligible.*

Finally, we define a secure signcryption scheme as one that is both CCA secure and has ciphertext integrity.

Definition 13.8. *We say that a signcryption scheme $\mathcal{SC} = (G, E, D)$ is secure if $\mathcal{SC}$ is (1) CCA secure, and (2) provides signcryption ciphertext integrity.*

From two users to multiple users. While this security definition focuses on just two honest users, it actually implies a strong security property in a multi-user setting. We will flesh this out below in Section 13.7.2.

Replay attacks. One thing the definition does not prevent is a "replay" attack: an attacker can record a valid ciphertext c from Alice to Bob and at a later time, say a week later, resend the same c to Bob. Bob receives the replayed ciphertext c and, because it is a valid ciphertext, he might mistakenly believe that Alice sent him the same message again. For example, if the message from Alice is "please transfer \$10 to Charlie," then Bob might incorrectly transfer another \$10 to Charlie.

Signcryption is not designed to prevent replay attacks. Higher level protocols that use signcryption must themselves take measures to counter-act them. We will discuss replay attacks and how to prevent them when we discuss authenticated key exchange in Chapter 20.

Statically vs adaptively chosen user IDs. Our definition of secure signcryption is subject to a rather subtle criticism, related to the manner in which user IDs are chosen. While we leave it to the adversary to choose the user IDs of the sender and receiver (that is, id_{S} and id_{R}), this choice is "static" in the the sense that it is made at the very beginning of the game. A more robust definition would allow a more "adaptive" strategy, in which the adversary gets to choose these IDs after seeing one or both of the public keys, or even after seeing the response to one or more X $\rightarrow$ Y queries. For most realistic schemes (including all of those discussed here), this distinction makes no difference, but it is possible to dream up contrived schemes where it does (see Exercise 13.16). We have presented the definition with statically chosen IDs mainly for the sake of simplicity (and because, arguably, honest users choose their IDs in a manner than is not so much under an adversary's control).

13.7.2 Signcryption as an abstract interface

Our definition of secure signcryption may seem a bit technical, and it is perhaps useful to discuss how this definition can applied. Much as we did in Sections 9.3 and 12.2.4, we do so by describing signcryption as an abstract interface. However, unlike in those two sections, it makes more sense here to explicitly model a system consisting of many users who are trying to send messages to one another over an insecure network.

The setting is as follows. We have a system of many users: some are "honest" and some are "corrupt." The honest users are assumed to follow the specified communication protocol correctly, while the corrupt users may do anything they like to try and subvert the protocol. The corrupt users may collude with each other, and may also attempt to subvert communications by eavesdropping on and tampering with network communication. In fact, we can just assume there is a single attacker who orchestrates the behavior of all the corrupt users and completely controls the network. Moreover, this attacker may have some knowledge of or influence over messages sent by honest users, and may have some knowledge of messages received by honest users.

To start with, we assume that each honest user somehow registers with the system by providing a user ID and a public key. We do not worry about the details of this registration process, except that we require each honest user to have a unique ID and to generate its public key using the key generation algorithm of the signcryption scheme (and, of course, keep the corresponding secret key to itself).

We require that the corrupt users also register with the system. While we insist that all users (honest and corrupt) have unique IDs, we do not make any requirements on how the corrupt users generate their public keys: they may use the prescribed key generation algorithm, or they may do something else entirely, including computing their public key as some function of one or more honest users' public keys. In fact, we may even allow the corrupt users to register with the system after it has been running for a while, choosing their public keys (and even their user IDs) in some way that depends in some malicious way on everything that has happened so far (including all network traffic).

We model the communication interface as a collection of in-boxes and out-boxes.

For each *honest user* $id_{\rm S}$ and each *registered user* (honest or corrupt) $id_{\rm R} \neq id_{\rm S}$, we have an *out-box* denoted $Out(id_{\rm S}, id_{\rm R})$. If $id_{\rm R}$ belongs to an honest user, we say that the out-box is *safe*; otherwise, we say that it is *unsafe*. From time to time, user $id_{\rm S}$ may want to send a message to user $id_{\rm R}$, and he does so by dropping the message in the out-box $Out(id_{\rm S}, id_{\rm R})$.

For each *registered user* (honest or corrupt) $id_{\rm S}$ and each *honest user* $id_{\rm R} \neq id_{\rm S}$, we have an *in-box* denoted $In(id_{\rm S}, id_{\rm R})$. If $id_{\rm S}$ belongs to an honest user, we say that the in-box is *safe*; otherwise, we say that it is *unsafe*. From time to time, a message may appear in the in-box $In(id_{\rm S}, id_{\rm R})$, which user $id_{\rm R}$ may then retrieve.

That is the abstract interface. We now describe the real implementation.

First, consider an out-box $Out(id_{\rm S}, id_{\rm R})$ associated with an honest user $id_{\rm S}$. The user $id_{\rm R}$ may or may not be honest. When user $id_{\rm S}$ user drops a message in the out-box, the message is encrypted using the secret key associated with user $id_{\rm S}$ and the public key associated with user $id_{\rm R}$ (along with the given user IDs). The resulting ciphertext is sent out of the network.

In a properly functioning network, if user $id_{\rm R}$ is an honest user, this ciphertext will eventually be presented to the matching in-box $In(id_{\rm S}, id_{\rm R})$.

Now consider an in-box $In(id_{\rm S}, id_{\rm R})$ associated with an honest user $id_{\rm R}$. The user $id_{\rm S}$ may or may not be honest. Whenever the network presents a ciphertext to this in-box, it is decrypted using the public key of $id_{\rm S}$ and the secret key $id_{\rm R}$ (along with the given user IDs). If the ciphertext is not rejected, the resulting message is placed in the in-box for later consumption by user $id_{\rm R}$.

We now describe an *ideal implementation* of this interface.

Here is what happens when an honest user drops a message in an out-box $Out(id_{\rm S}, id_{\rm R})$. If the out-box is safe (i.e., user id_{R} is an honest user), instead of encrypting the given message, a dummy message is encrypted. This dummy message has nothing to do with the real message (except that it should be of the same length), and the resulting ciphertext just serves as a "handle". Otherwise, if the out-box is unsafe, the real message is encrypted as in the real implementation.

Here is what happens when the network presents a ciphertext to an in-box $In(id_{\mathrm{S}}, id_{\mathrm{R}})$. If the in-box is safe (i.e., user id_{S} is an honest user), the ideal implementation checks if this ciphertext was previously generated as a handle by the matching out-box $Out(id_{\mathrm{S}}, id_{\mathrm{R}})$, and if so, copies the corresponding message directly from the out-box to the in-box; otherwise, the ciphertext is discarded. If the in-box is unsafe, the ciphertext is decrypted as in the real implementation.

We hope that it is intuitively clear that this ideal implementation provides all the security one could possibly hope for. In this ideal implementation, messages magically "jump" from honest senders to honest receivers: the attacker cannot tamper with or glean any information about these messages, even if honest users interact with corrupt users. At worst, an attacker reorders or duplicates messages by reordering or duplicating the corresponding handles (indeed, as already mentioned, our definition of secure signcryption does not rule out "replay" attacks). Typically, this is an issue that a higher level protocol can easily deal with.

We now argue informally that if the signcryption scheme is secure, as in Definition 13.8, then the real world implementation is indistinguishable from the ideal implementation. The argument proceeds in three steps. We start with the real implementation, and in each step, we make a slight modification.

- First, we modify the behavior of the safe in-boxes. Whenever the network presents a ciphertext to the in-box that came from the matching out-box, the corresponding message is copied directly from the out-box to the in-box.

  The correctness property of the signcryption scheme ensures that this modification behaves exactly the same as the real implementation.

- Second, we modify the behavior of the safe in-boxes again. Whenever the network presents a ciphertext to the in-box that did not came from the matching out-box, the ciphertext is discarded.

  The ciphertext integrity property ensures that this modification is indistinguishable from the first. To reduce from the multi-user setting to the two-user setting, one must employ a "guessing argument".

- Third, we modify the behavior of the safe out-boxes, so that dummy messages are encrypted in place of the real messages.

  The CCA security property ensures that this modification is indistinguishable from the second. To reduce from the multi-user setting to the two-user setting, one must employ a "hybrid argument".

Just as in Sections 9.3 and 12.2.4, we have ignored the possibility that the ciphertexts generated in a safe out-box are not unique. If we are going to view these ciphertexts as handles in the ideal implementation, uniqueness is an essential property. However, just as in those cases, the CCA security property implies that these ciphertexts are unique with overwhelming probability.

13.7.3 Constructions: encrypt-then-sign and sign-then-encrypt

We begin by analyzing the two most natural constructions. Both are a combination of a CCA-secure public-key encryption scheme and a secure signature scheme. Getting these combinations right is a little tricky and small variations can be insecure. We explore some insecure variations in Exercise 13.15.

Let $\mathcal{E} = (G_{\text{ENC}}, E, D)$ be a public-key encryption scheme with associated data (see Section 12.7). Recall that this means that E is invoked as $c \xleftarrow{R} E(pk, m, d)$, and D is invoked as $m \xleftarrow{R} D(sk, c, d)$, where d is the "associated data". Also, let $\mathcal{S} = (G_{\text{SIG}}, S, V)$ be a signature scheme. Define algorithm G as:

$$\begin{aligned} G() \quad := \quad & (pk_{\text{ENC}},\ sk_{\text{ENC}}) \xleftarrow{R} G_{\text{ENC}}(), \quad (pk_{\text{SIG}},\ sk_{\text{SIG}}) \xleftarrow{R} G_{\text{SIG}}() \\ & \text{output } pk := (pk_{\text{ENC}},\ pk_{\text{SIG}}) \text{ and } sk := (sk_{\text{ENC}},\ sk_{\text{SIG}}) \end{aligned}$$

In what follows we use the shorthand $E(pk, m, d)$ to mean $E(pk_{\text{ENC}}, m, d)$ and $S(sk, m)$ to mean $S(sk_{\text{SIG}}, m)$, for some message m. We use a similar shorthand for $V(pk, m, \sigma)$ and $D(sk, c, d)$. We next define two natural signcryption schemes, each of which has a message space $\mathcal{M}$ and an identity space $\mathcal{I}$.

Encrypt-then-sign. The scheme $\mathcal{SC}_{\text{EtS}} = (G,\ E_{\text{EtS}},\ D_{\text{EtS}})$ is defined as

$$\begin{aligned} E_{\text{EtS}}(sk_{\text{S}}, id_{\text{S}}, pk_{\text{R}}, id_{\text{R}},\ m) \quad &:= \quad c \xleftarrow{R} E\big(pk_{\text{R}},\ m,\ id_{\text{S}}\big), \quad \sigma \xleftarrow{R} S\big(sk_{\text{S}},\ (c, id_{\text{R}})\big) \\ & \qquad \text{output } (c, \sigma); \\ D_{\text{EtS}}\big(pk_{\text{S}}, id_{\text{S}}, sk_{\text{R}}, id_{\text{R}},\ (c, \sigma)\big) \quad &:= \quad \text{if } V(pk_{\text{S}},\ (c, id_{\text{R}}),\ \sigma) = \mathsf{reject}, \text{ output } \mathsf{reject} \\ & \qquad \text{otherwise, output } D(sk_{\text{R}},\ c,\ id_{\text{S}}). \end{aligned}$$

Here the encryption scheme $\mathcal{E}$ is assumed to be defined over $(\mathcal{M},\ \mathcal{I},\ \mathcal{C})$, so that $\mathcal{I}$ is the associated data space for $\mathcal{E}$. The signature scheme $\mathcal{S}$ is assumed to be defined over $(\mathcal{C} \times \mathcal{I},\ \Sigma)$.

Sign-then-encrypt. The scheme $\mathcal{SC}_{\text{StE}} = (G,\ E_{\text{StE}},\ D_{\text{StE}})$ is defined as

$$\begin{aligned} E_{\text{StE}}(sk_{\text{S}}, id_{\text{S}}, pk_{\text{R}}, id_{\text{R}},\ m) \quad &:= \quad \sigma \xleftarrow{R} S\big(sk_{\text{S}},\ (m, id_{\text{R}})\big), \quad c \xleftarrow{R} E\big(pk_{\text{R}},\ (m, \sigma),\ id_{\text{S}}\big) \\ & \qquad \text{output } c; \\ D_{\text{EtS}}\big(pk_{\text{S}}, id_{\text{S}}, sk_{\text{R}}, id_{\text{R}},\ c\big) \quad &:= \quad \text{if } D(sk_{\text{R}}, c, id_{\text{S}}) = \mathsf{reject}, \text{ output } \mathsf{reject}, \text{ otherwise:} \\ & \qquad\quad (m, \sigma) \leftarrow D(sk_{\text{R}},\ c,\ id_{\text{S}}) \\ & \qquad\quad \text{if } V(pk_{\text{S}},\ (m, id_{\text{R}}),\ \sigma) = \mathsf{reject}, \text{ output } \mathsf{reject} \\ & \qquad\quad \text{otherwise, output } m. \end{aligned}$$

Here the encryption scheme $\mathcal{E}$ is assumed to be defined over $(\mathcal{M} \times \Sigma,\ \mathcal{I},\ \mathcal{C})$, where $\mathcal{I}$ is the associated data space. The signature scheme $\mathcal{S}$ is assumed to be defined over $(\mathcal{M} \times \mathcal{I},\ \Sigma)$. Moreover, we shall assume that the signatures are bit strings whose length only depends on the message being signed (this technical requirement will be required in the security analysis).

The following two theorems show that both schemes are secure signcryption schemes. Notice that the corresponding symmetric constructions analyzed in Section 9.4 were not both secure. Encrypt-then-MAC provides authenticated encryption while MAC-then-encrypt might not. In the signcryption setting, both constructions are secure. The reason sign-then-encrypt is secure is that we are starting from a CCA-secure public-key system $\mathcal{E}$, where as MAC-then-encrypt was built from a CPA-secure cipher. In fact, we know by Exercise 9.15 that MAC-then-encrypt, where the encryption scheme is CCA secure, provides authenticated encryption. Therefore, it should not be too surprising that sign-then-encrypt is secure.

Unlike the encrypt-then-MAC construction, the encrypt-then-sign method requires a CCA-secure encryption scheme for security, rather than just a CPA-secure encryption scheme. We already touched on this issue back in Section 12.2.2 as one of the motivations for studying CCA-secure public-key encryption.

The encrypt-then-sign method requires a *strongly secure* signature scheme for security, as defined in Definition 13.3. Without this, the scheme can be vulnerable to a CCA attack: if an adversary, given a challenge ciphertext (c, σ), can produce a new valid signature σ' on the same data, then the adversary can win the CCA attack game by asking for a decryption of (c, σ'). To prevent this, we require that the signature scheme is strongly secure. This is perhaps to be expected, as in the symmetric setting, the encrypt-then-MAC construction requires a secure MAC, and our definition of a secure MAC is the direct analogue of our definition of a strongly secure signature scheme. In contrast, sign-then-encrypt requires just a secure signature scheme — the scheme need not be strongly secure.

We now present the security theorems for both schemes.

Theorem 13.8. *$\mathcal{SC}_{\mathrm{EtS}}$ is a secure signcryption scheme assuming $\mathcal{E}$ is a CCA-secure public-key encryption scheme with associated data and $\mathcal{S}$ is a strongly secure signature scheme.*

In particular, for every ciphertext integrity adversary $\mathcal{A}_{\mathrm{ci}}$ that attacks $\mathcal{SC}_{\mathrm{EtS}}$ as in Attack Game 13.5 there exists a strong signature adversary $\mathcal{B}_{\mathrm{sig}}$ that attacks $\mathcal{S}$ as in Attack Game 13.2, where $\mathcal{B}_{\mathrm{sig}}$ is an elementary wrapper around $\mathcal{A}_{\mathrm{ci}}$, such that

$$\mathrm{SCI}\mathsf{adv}[\mathcal{A}_{\mathrm{ci}}, \mathcal{SC}_{\mathrm{EtS}}] = \mathrm{stSIG}\mathsf{adv}[\mathcal{B}_{\mathrm{sig}}, \mathcal{S}].$$

In addition, for every CCA adversary $\mathcal{A}_{\mathrm{cca}}$ that attacks $\mathcal{SC}_{\mathrm{EtS}}$ as in Attack Game 13.6 there exists a CCA adversary $\mathcal{B}_{\mathrm{cca}}$ that attacks $\mathcal{E}$ as in Definition 12.7, and a strong signature adversary $\mathcal{B}_{\mathrm{sig}}$ that attacks $\mathcal{S}$ as in Attack Game 13.2, where $\mathcal{B}_{\mathrm{cca}}$ and $\mathcal{B}'_{\mathrm{sig}}$ are elementary wrappers around $\mathcal{A}_{\mathrm{cca}}$, such that

$$\mathrm{SCCA}\mathsf{adv}[\mathcal{A}_{\mathrm{cca}}, \mathcal{SC}_{\mathrm{EtS}}] \le \mathrm{CCA}_{\mathrm{ad}}\mathsf{adv}[\mathcal{B}_{\mathrm{cca}}, \mathcal{E}] + \mathrm{stSIG}\mathsf{adv}[\mathcal{B}'_{\mathrm{sig}}, \mathcal{S}].$$

Proof sketch. We have to prove both ciphertext integrity and security against chosen ciphertext attack. Both proofs make essential use of the placement of the identifiers id_{S} and id_{R} as defined in the encryption and decryption algorithms. We start with ciphertext integrity.

Proving ciphertext integrity. We begin by constructing adversary $\mathcal{B}_{\mathrm{sig}}$ that interacts with a signature challenger for $\mathcal{S}$, while playing the role of challenger to $\mathcal{A}_{\mathrm{ci}}$ in Attack Game 13.5. $\mathcal{B}_{\mathrm{sig}}$ first obtains a signature public key pk^*_{SIG} from its own challenger.

Next, $\mathcal{A}_{\mathrm{ci}}$ supplies two identities id_{S} and id_{R}. $\mathcal{B}_{\mathrm{sig}}$ then uses G_{ENC} and G_{SIG} to generate two public-key encryption key-pairs $(pk_{\mathrm{ENC,S}}, sk_{\mathrm{ENC,S}})$ and $(pk_{\mathrm{ENC,R}}, sk_{\mathrm{ENC,R}})$, and one signature key-pair $(pk_{\mathrm{SIG,R}}, sk_{\mathrm{SIG,R}})$. It sends to $\mathcal{A}_{\mathrm{ci}}$ the two public keys

$$pk_{\mathrm{S}} := (pk_{\mathrm{ENC,S}},\ pk^*_{\mathrm{SIG}}) \qquad \text{and} \qquad pk_{\mathrm{R}} := (pk_{\mathrm{ENC,R}},\ pk_{\mathrm{SIG,R}}).$$

Note that $\mathcal{B}_{\mathrm{sig}}$ knows all the corresponding secret keys, except for the secret key corresponding to pk^*_{SIG}, which is the challenge signature public key that $\mathcal{B}_{\mathrm{sig}}$ is trying to attack.

$\mathcal{A}_{\text{ci}}$ then issues several encryption and decryption queries.

To process an encryption query, $\mathcal{B}_{\text{sig}}$ begins by encrypting the given message m using the encryption algorithm E with the appropriate public key. This generates a ciphertext c. Next, $\mathcal{B}_{\text{sig}}$ must generate an appropriate signature σ. For an S $\rightarrow$ R encryption query, $\mathcal{B}_{\text{sig}}$ obtains a signature σ under pk^*_{SIG} on the message (c, id_{R}) by using its own signature challenger. For an X $\rightarrow$ Y encryption query with $id_{\text{X}} = id_{\text{S}}$, $\mathcal{B}_{\text{sig}}$ obtains a signature σ under pk^*_{SIG} on the message (c, id_{Y}), again, by using its own signature challenger. For an X $\rightarrow$ Y encryption query with $id_{\text{X}} = id_{\text{R}}$, $\mathcal{B}_{\text{sig}}$ generates σ by signing the the message (c, id_{Y}) directly, using the secret key $sk_{\text{SIG,R}}$. In any case, $\mathcal{B}_{\text{sig}}$ responds to the encryption query with the ciphertext/signature pair (c, σ).

$\mathcal{B}_{\text{sig}}$ answers decryption queries from $\mathcal{A}_{\text{ci}}$ by simply running algorithm D_{EtS} on the given data in the query. Indeed, $\mathcal{B}_{\text{sig}}$ has all the required keys to do so.

Eventually, $\mathcal{A}_{\text{ci}}$ outputs a valid ciphertext forgery (c', σ'), where σ' is a valid signature on the message (c', id_{R}). We argue that the message-signature pair $\big((c', id_{\text{R}}),\ \sigma'\big)$ is a strong existential forgery for the signature scheme $\mathcal{S}$. The only way this can fail is if $\mathcal{B}_{\text{sig}}$ had previously asked its challenger for a signature on (c', id_{R}) and the challenger responded with σ'. Observe that the only reason $\mathcal{B}_{\text{sig}}$ would ask for a signature on (c', id_{R}) is as part of responding to an S $\rightarrow$ R encryption query from $\mathcal{A}_{\text{ci}}$. This is where we make essential use the fact that the identity id_{R} is included in the data being signed. We conclude that the signature from the challenger cannot be σ' because the ciphertext forgery (c', σ') must be different from all the S $\rightarrow$ R ciphertexts generated by $\mathcal{B}_{\text{sig}}$. It follows that $\big((c', id_{\text{R}}),\ \sigma'\big)$ is a valid strong existential forgery on $\mathcal{S}$, as required.

Proving chosen ciphertext security. Next, we sketch the proof of CCA security. It is convenient to modify the attack game slightly. Let Game 0 be the original signcryption CCA game between a $\mathcal{SC}_{\text{EtS}}$ challenger and an adversary $\mathcal{A}_{\text{cca}}$. We then define Game 1, which is the same as Game 0, except that we add a “special rejection rule” in the challenger's logic for processing S $\rightarrow$ R decryption queries. Namely, given an S $\rightarrow$ R decryption query $(\hat{c}, \hat{\sigma})$, where $\hat{\sigma}$ is a valid signature on $(\hat{c}, id_{\text{R}})$, and $\hat{c}$ is the first component of a response to a previous S $\rightarrow$ R encryption query, the challenger returns reject without further processing.

It is not difficult to see that Games 0 and 1 proceed identically, unless the challenger rejects a ciphertext $(\hat{c}, \hat{\sigma})$ in Game 1 that would not be rejected in Game 0. However, if $(\hat{c}, \hat{\sigma})$ is such a ciphertext, then $\big((\hat{c}, id_{\text{R}}), \hat{\sigma}\big)$ is a strong existential forgery for $\mathcal{S}$. Therefore, we can construct an adversary $\mathcal{B}'_{\text{sig}}$ whose advantage in strong existential forgery game against $\mathcal{S}$ is equal to the probability that such a ciphertext gets rejected in Game 1.

We now construct an adversary $\mathcal{B}_{\text{cca}}$ whose CCA advantage is the same as $\mathcal{A}_{\text{cca}}$'s advantage in Game 1. As usual, $\mathcal{B}_{\text{cca}}$ interacts with its own CCA challenger, while playing the role of challenger to $\mathcal{A}_{\text{cca}}$ in Game 1.

Adversary $\mathcal{B}_{\text{cca}}$ first obtains an encryption public key pk^*_{ENC} from its own challenger.

Next, $\mathcal{A}_{\text{cca}}$ supplies two identities id_{S} and id_{R}. $\mathcal{B}_{\text{cca}}$ then runs the key-generation algorithm for the signature scheme twice and the key-generation algorithm for the encryption scheme once, and sends to $\mathcal{A}_{\text{cca}}$ the two public keys

$$pk_{\text{S}} := (pk_{\text{ENC,S}},\ pk_{\text{SIG,S}}) \qquad \text{and} \qquad pk_{\text{R}} := (pk^*_{\text{ENC}},\ pk_{\text{SIG,R}}),$$

where it knows all the corresponding secret keys, except for the secret key corresponding to pk^*_{ENC}.

$\mathcal{A}_{\text{cca}}$ then issues several encryption and decryption queries.

Processing encryption queries. Adversary $\mathcal{B}_{\text{cca}}$ answers an S $\rightarrow$ R encryption query for message pair (m_0, m_1) by issuing an encryption query for (m_0, m_1) to its challenger, relative to the associated data id_S. It gets back a ciphertext c, signs (c, id_R) to get σ, and sends (c, σ) to $\mathcal{A}_{cca}$ as a response to the query.

To answer an X $\to$ Y encryption query, $\mathcal{B}_{cca}$ runs algorithm E_{EtS} on the given data in the query. Indeed, $\mathcal{B}_{cca}$ has all the required keys to do so.

Processing decryption queries. Consider first an S $\to$ R decryption query $(\hat{c}, \hat{\sigma})$. Our adversary $\mathcal{B}_{cca}$ uses the following steps:

1. return reject if $\hat{\sigma}$ is an invalid signature on $(\hat{c}, id_R)$ under $pk_{SIG,S}$;

2. return reject if $\hat{c}$ is the first component of any response to an S $\to$ R encryption query (this is the special rejection rule we introduced in Game 1);

3. ask the CCA challenger to decrypt $\hat{c}$ using the associated data id_S, and return the result (note that because of the logic of Steps 1 and 2, $\mathcal{B}_{cca}$ has not issued an encryption query to its own challenger corresponding to $(\hat{c}, id_S)$).

The logic for processing an X $\to$ Y decryption query $(id_X, id_Y, pk_X, (\hat{c}, \hat{\sigma}))$ with with $id_Y = id_R$ is similar:

1. return reject if $\hat{\sigma}$ is an invalid signature on $(\hat{c}, id_R)$ under pk_X;

2. ask the CCA challenger to decrypt $\hat{c}$ using the associated data id_X, and return the result (note that because $id_X \neq id_S$, $\mathcal{B}_{cca}$ has not issued an encryption query to its own challenger corresponding to $(\hat{c}, id_X)$).

For other decryption queries, we have all the keys necessary to perform the decryption directly.

Finishing up. Eventually, $\mathcal{A}_{cca}$ outputs a guess $\hat{b} \in \{0, 1\}$. This guess gives $\mathcal{B}_{cca}$ the same advantage against its CCA challenger that $\mathcal{A}_{cca}$ has in Game 1. $\Box$

Theorem 13.9. *$\mathcal{SC}_{StE}$ is a secure signcryption scheme assuming $\mathcal{E}$ is a CCA-secure public-key encryption scheme with associated data and $\mathcal{S}$ is a secure signature scheme.*

> *In particular, for every ciphertext integrity adversary $\mathcal{A}_{ci}$ that attacks $\mathcal{SC}_{EtS}$ as in Attack Game 13.5 there exists a signature adversary $\mathcal{B}_{sig}$ that attacks $\mathcal{S}$ as in Attack Game 13.1, and a CCA adversary $\mathcal{B}_{cca}$ that attacks $\mathcal{E}$ as in Definition 12.7, where $\mathcal{B}_{sig}$ and $\mathcal{B}'_{cca}$ are elementary wrappers around $\mathcal{A}_{ci}$, such that*
>
> $$\mathrm{SCI}\mathsf{adv}[\mathcal{A}_{ci}, \mathcal{SC}_{EtS}] \leq \mathrm{SIG}\mathsf{adv}[\mathcal{B}_{sig}, \mathcal{S}] + \mathrm{CCA_{ad}}\mathsf{adv}[\mathcal{B}'_{cca}, \mathcal{E}]$$
>
> *In addition, for every CCA adversary $\mathcal{A}_{cca}$ that attacks $\mathcal{SC}_{EtS}$ as in Attack Game 13.6 there exists a CCA adversary $\mathcal{B}_{cca}$ that attacks $\mathcal{E}$ as in Definition 12.7, where $\mathcal{B}_{cca}$ is an elementary wrapper around $\mathcal{A}_{cca}$, such that*
>
> $$\mathrm{SCCA}\mathsf{adv}[\mathcal{A}_{cca}, \mathcal{SC}_{EtS}] = \mathrm{CCA_{ad}}\mathsf{adv}[\mathcal{B}_{cca}, \mathcal{E}]$$

Proof idea. CCA security for the signcryption scheme follows almost immediately from the CCA security of $\mathcal{E}$. The reader can easily fill in the details.

Proving CI for the signcryption scheme is slightly trickier. Let Game 0 be the original CI attack game. We modify Game 0 so that for each S $\to$ R encryption query, instead of computing

$$c \xleftarrow{R} E(pk_R, (m, \sigma), id_S)$$

where

$$\sigma \xleftarrow{R} S(sk_S, (m, id_R)),$$

the challenger instead computes

$$c \xleftarrow{R} E(pk_R, (m, dummy), id_S).$$

Call this Game 1. Under CCA security for $\mathcal{E}$, the adversary's advantage in breaking CI in Game 0 must be negligibly close to the corresponding advantage in Game 1. However, in Game 1, since the challenger never signs any message of the form $(\cdot, id_R)$, breaking CI in Game 1 is tantamount to forging a signature on just such a message.

In proving both security properties, we need to make use of the technical requirement that signatures are bit strings whose length only depends on the message being signed. $\Box$

13.7.4 A construction based on Diffie-Hellman key exchange

Our next signcryption construction does not use signatures at all. Instead, we use a non-interactive variant of the Diffie-Hellman key exchange protocol from Section 10.4.1. The protocol uses a group $\mathbb{G}$ of prime order q with generator $g \in \mathbb{G}$. This variant is said to be non-interactive because once every party publishes its contribution to the protocol — g^α for some random $\alpha \in \mathbb{Z}_q$ — no more interaction is needed to establish a shared key between any pair of parties. For example, once Alice publishes g^α and Bob publishes g^β, their shared secret is derived from $g^{\alpha\beta}$. The signcryption scheme we describe can be built from any non-interactive key exchange, but here we present it concretely using Diffie-Hellman key exchange.

The signcryption scheme $\mathcal{SC}_{\text{DH}}$ is built from three ingredients:

- a symmetric cipher $\mathcal{E} = (E_s, D_s)$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$,
- a group $\mathbb{G}$ of prime order q with generator $g \in \mathbb{G}$, and
- a hash function $H : \mathbb{G}^3 \times \mathcal{I}^2 \to \mathcal{K}$.

Given these ingredients, the system $\mathcal{SC}_{\text{DH}}$ is defined over $(\mathcal{M}, \mathcal{C}, \mathcal{I})$ and works as follows:

- The key generation algorithm G runs as follows:

  $$\alpha \xleftarrow{R} \mathbb{Z}_q, \quad h \leftarrow g^\alpha.$$

  The public key is $pk := h$, and the secret key is $sk := \alpha$. We use h_X to denote the public key associated with identity id_X and use α_X to denote the associated secret key.

- $E\big(\alpha_S, id_S, h_R, id_R,\ m\big)$ works by first deriving the Diffie-Hellman secret between users S and R, namely $h_{SR} := g^{\alpha_S \cdot \alpha_R}$, and then encrypting the message m using the symmetric cipher with a key derived from h_{SR}. More precisely, encryption works as follows, where $h_S := g^{\alpha_S}$:

  $$h_{SR} \leftarrow (h_R)^{\alpha_S} = g^{\alpha_S \cdot \alpha_R}, \qquad k \leftarrow H\big(h_S, h_R, h_{SR},\ id_S, id_R\big), \qquad \text{output } c \xleftarrow{R} E_s(k, m).$$

- $D\big(h_S, id_S, \alpha_R, id_R,\ c\big)$ works as follows, where $h_R := g^{\alpha_R}$:

  $$h_{SR} \leftarrow (h_S)^{\alpha_R} = g^{\alpha_S \cdot \alpha_R}, \qquad k \leftarrow H\big(h_S, h_R, h_{SR},\ id_S, id_R\big), \qquad \text{output } D_s(k, c).$$

It is easy to verify that $\mathcal{SC}_{\rm DH}$ is correct. To state the security theorem we must first introduce a new assumption, called the *double-interactive CDH assumption*. The assumption is related to, but a little stronger than, the interactive CDH assumption introduced in Section 12.4.1.

Intuitively, the double-interactive CDH assumption states that given a random instance (g^α, g^β) of the DH problem, it is hard to compute $g^{\alpha\beta}$, even when given access to a DH-decision oracle that recognizes DH-triples of the form $(g^\alpha, \cdot, \cdot)$ or of the form $(\cdot, g^\beta, \cdot)$. More formally, this assumption is defined in terms of the following attack game.

Attack Game 13.7 (Double-Interactive Computational Diffie-Hellman). Let $\mathbb{G}$ be a cyclic group of prime order q generated by $g \in \mathbb{G}$. For a given adversary $\mathcal{A}$, the attack game runs as follows.

- The challenger computes

$$\alpha, \beta \xleftarrow{\textsc{r}} \mathbb{Z}_q, \qquad u \leftarrow g^\alpha, \qquad v \leftarrow g^\beta, \qquad w \leftarrow g^{\alpha\beta}$$

  and gives (u, v) to the adversary.

- The adversary makes a sequence of queries to the challenger. Each query is one of the following types:

  α-query: given $(\tilde{v}, \tilde{w}) \in \mathbb{G}^2$, the challenger tests if $\tilde{v}^\alpha = \tilde{w}$;

  β-query: given $(\tilde{u}, \tilde{w}) \in \mathbb{G}^2$, the challenger tests if $\tilde{u}^\beta = \tilde{w}$.

  In either case, if equality holds the challenger sends "yes" to the adversary, and otherwise, sends "no" to the adversary.

- Finally, the adversary outputs some $\hat{w} \in \mathbb{G}$.

We define $\mathcal{A}$'s **advantage in solving the double-interactive computational Diffie-Hellman problem**, denoted $\mathrm{I^2CDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$, as the probability that $\hat{w} = w$. $\Box$

Definition 13.9 (Double-Interactive computational Diffie-Hellman assumption). *We say that the **double-interactive computational Diffie-Hellman** (*$\mathrm{I^2CDH}$*) assumption holds for* $\mathbb{G}$ *if for all efficient adversaries* $\mathcal{A}$ *the quantity* $\mathrm{I^2CDH}\mathsf{adv}[\mathcal{A}, \mathbb{G}]$ *is negligible.*

The following theorem shows $\mathcal{SC}_{\rm DH}$ is a secure signcryption scheme where security is defined as in the previous section (Definition 13.8).

Theorem 13.10. $\mathcal{SC}_{\rm DH}$ *is a secure signcryption scheme assuming* $\mathcal{E}$ *is an* AE*-secure cipher, the* $\mathrm{I^2CDH}$ *assumption holds for* $\mathbb{G}$*, and the hash function* H *is modeled as a random oracle.*

> *In particular, for every ciphertext integrity adversary* $\mathcal{A}_{\rm ci}$ *that attacks* $\mathcal{SC}_{\rm DH}$ *as in the random oracle variant of Attack Game 13.5, there exists a ciphertext integrity adversary* $\mathcal{B}_{\rm ci}$ *that attacks* $\mathcal{E}$ *as in Attack Game 9.1, and an* $\mathrm{I^2CDH}$ *adversary* $\mathcal{B}_{\rm dh}$ *for* $\mathbb{G}$*, where* $\mathcal{B}_{\rm ci}$ *and* $\mathcal{B}_{\rm dh}$ *are elementary wrappers around* $\mathcal{A}_{\rm ci}$*, such that*
>
> $$\mathrm{SCI}\mathsf{adv}[\mathcal{A}_{\rm ci}, \mathcal{SC}_{\rm DH}] \le \mathrm{CI}\mathsf{adv}[\mathcal{B}_{\rm ci}, \mathcal{E}] + \mathrm{I^2CDH}\mathsf{adv}[\mathcal{B}_{\rm dh}, \mathbb{G}]$$
>
> *In addition, for every CCA adversary* $\mathcal{A}_{\rm cca}$ *that attacks* $\mathcal{SC}_{\rm DH}$ *as in the random oracle variant of Attack Game 13.6, there exists a CCA adversary* $\mathcal{B}_{\rm cca}$ *that attacks* $\mathcal{E}$ *as in Attack Game 9.2, and an* $\mathrm{I^2CDH}$ *adversary* $\mathcal{B}'_{\rm dh}$ *for* $\mathbb{G}$*, where* $\mathcal{B}_{\rm cca}$ *and* $\mathcal{B}'_{\rm iidh}$ *are elementary wrappers around* $\mathcal{A}_{\rm ci}$*, such that*
>
> $$\mathrm{SCCA}\mathsf{adv}[\mathcal{A}_{\rm cca}, \mathcal{SC}_{\rm DH}] \le \mathrm{CCA}\mathsf{adv}[\mathcal{B}_{\rm cca}, \mathcal{E}] + 2 \cdot \mathrm{I^2CDH}\mathsf{adv}[\mathcal{B}'_{\rm dh}, \mathbb{G}]$$

The proof of Theorem 13.10 follows easily from the analysis of Diffie-Hellman as a non-interactive key exchange scheme. This analysis is given in Section 20.10 and we defer proving the theorem to that section.

13.7.5 Additional desirable properties

So far we looked at three signcryption schemes: $\mathcal{SC}_{\rm DH}$ presented in the previous section and the two schemes presented in Section 13.7.3. All three schemes satisfy the signcryption security definition (Definition 13.8). However, there are significant differences between $\mathcal{SC}_{\rm DH}$ and the two schemes in Section 13.7.3. One difference between $\mathcal{SC}_{\rm DH}$ and the others is a simple inter-operability issue: it requires all users of the system to use the same group $\mathbb{G}$ for generating their keys. This may be acceptable in some settings but not in others, and is inherent to how $\mathcal{SC}_{\rm DH}$ operates.

There are two other, more fundamental, differences that are worth examining further. We explore these differences by defining two new signcryption properties: (1) forward secrecy, and (2) non-repudiation.

Property I: forward secrecy. Suppose Alice encrypts a message to Bob and sends the resulting ciphertext c to Bob. A week later the adversary corrupts Bob and steals his secret key. Because Bob can decrypt c, so can the adversary. There is no hope of maintaining security under such an attack. However, suppose that instead of corrupting Bob, the adversary corrupts Alice and steals her secret key a week after she sent c. Bob's key remains intact and only known to Bob. One might reasonably expect that the adversary not be able to decrypt c using Alice's secret key. We refer to this property as *forward secrecy.*

Let us define more precisely what it means for a signcryption scheme to provide forward secrecy. The goal is to ensure that CCA security is maintained even if the adversary obtains the sender's secret key. To do so we make a small tweak to the CCA security game (Attack Game 13.6).

Attack Game 13.8 (CCA security with forward secrecy). The game is identical to Attack Game 13.6 except that we change the setup step as follows: in addition to giving the adversary the public keys $pk_{\rm S}$ and $pk_{\rm R}$, the challenger gives the adversary the sender's secret key $sk_{\rm S}$. The corresponding advantage is denoted $\mathrm{SCCA'}\mathsf{adv}[\mathcal{A}, \mathcal{SC}]$. $\square$

Definition 13.10. *A signcryption scheme $\mathcal{SC}$ is said to provide* ***forward secrecy*** *if for all efficient adversaries $\mathcal{A}$, the value $\mathrm{SCCA'}\mathsf{adv}[\mathcal{A}, \mathcal{SC}]$ is negligible.*

Forward secrecy for sign-then-encrypt. The sign-then-encrypt construction provides forward secrecy: the secret key $sk_{\rm S}$ is only used for signing messages and does not help to decrypt anything. Indeed, from the concrete security bound given in Theorem 13.9, one can see that the bound on the SCCA advantage does not depend at all on the security of the signature scheme.

Forward secrecy for encrypt-then-sign. One might be tempted to say the same thing for encrypt-then-sign; however, this is not quite true in general. Observe that in the concrete security bound in Theorem 13.8, the bound on the SCCA advantage depends on the security of both the signature scheme and the encryption scheme. Indeed, as we already discussed in relation to the need for a strongly secure signature scheme, if the adversary obtains a ciphertext (c, σ) in response to an $\mathrm{S} \to \mathrm{R}$ encryption query, and could compute a valid signature $\sigma' \neq \sigma$ on $(c, id_{\rm R})$, then by the rules of the CCA attack game, the adversary would be free to submit (c, σ') as an $\mathrm{S} \to \mathrm{R}$ decryption query, completely breaking CCA security.

Now, without the sender's signing key, this attack would be infeasible. But with the signing key, it is easy if the signature algorithm is probabilistic (we will see such signature schemes later): the adversary can use the sender's signing key to generate a different signature on an inner $\mathrm{S} \to \mathrm{R}$ ciphertext and obtain a "new" encrypt-then-sign ciphertext that it can submit to the decryption oracle.

However, all is not lost. There are a couple of ways to salvage the forward secrecy property of encrypt-then-sign. One way is to salvage the the situation is to employ a signature scheme that has *unique signatures* (i.e., for every public key and message, there is at most one valid signature — full domain hash is such a scheme). Then the above attack becomes impossible, even with the signing key. See also Exercise 13.17, which discusses a modification of encrypt-then-sign which achieves forward secrecy more generically.

Another way is to salvage the situation is to weaken the security definition slightly, by simply not allowing the adversary to submit a decryption query for the ciphertext (c, σ') in the attack game. Is this reasonable? Arguably, it is, as anyone can easily tell that the (c, σ) and (c, σ') decrypt to the same thing if σ and σ' are both valid signatures on c. Indeed, such a restriction on the adversary corresponds to the notion of gCCA security discussed in Exercise 12.2, and is actually quite acceptable for most applications.

Forward secrecy for $\mathcal{SC}_{\mathrm{DH}}$. The $\mathcal{SC}_{\mathrm{DH}}$ signcryption system is does not provide forward secrecy at all: given the secret key of the sender, the adversary can decrypt any ciphertext generated by the sender that it wants. Fortunately, we can enhance $\mathcal{SC}_{\mathrm{DH}}$ to provide forward secrecy.

Enhanced $\mathcal{SC}_{\mathrm{DH}}$. Using the notation of Section 13.7.4, the enhanced $\mathcal{SC}_{\mathrm{DH}}$ signcryption system, denoted $\mathcal{SC}'_{\mathrm{DH}}$, is defined over $(\mathcal{M},\ \mathbb{G} \times \mathcal{C},\ \mathcal{I})$ and works as follows:

- The key generation algorithm G is as in $\mathcal{SC}_{\mathrm{DH}}$. We use h_{X} to denote the public key associated with identity id_{X} and use α_{X} to denote the associated secret key.
- $E\big(\alpha_{\mathrm{S}}, id_{\mathrm{S}}, h_{\mathrm{R}}, id_{\mathrm{R}},\ m\big)$ works as follows, where $h_{\mathrm{S}} := g^{\alpha_{\mathrm{S}}}$:

$$\begin{aligned}
&\beta \xleftarrow{\mathrm{R}} \mathbb{Z}_q, \quad v \leftarrow g^{\beta}, \\
&h_{\mathrm{SR}} \leftarrow (h_{\mathrm{R}})^{\alpha_{\mathrm{S}}}, \quad w \leftarrow (h_{\mathrm{R}})^{\beta}, \\
&k \leftarrow H\big(v, w, h_{\mathrm{S}}, h_{\mathrm{R}}, h_{\mathrm{SR}},\ id_{\mathrm{S}},\ id_{\mathrm{R}}\big), \quad c \leftarrow E_{\mathrm{s}}(k, m) \\
&\text{output } (v, c).
\end{aligned}$$

- $D\big(h_{\mathrm{S}}, id_{\mathrm{S}}, \alpha_{\mathrm{R}}, id_{\mathrm{R}},\ (v, c)\big)$ works as follows, where $h_{\mathrm{R}} := g^{\alpha_{\mathrm{R}}}$:

$$h_{\mathrm{SR}} \leftarrow (h_{\mathrm{S}})^{\alpha_{\mathrm{R}}}, \quad w \leftarrow v^{\alpha_{\mathrm{R}}}, \quad k \leftarrow H\big(v, w, h_{\mathrm{S}}, h_{\mathrm{R}}, h_{\mathrm{SR}},\ id_{\mathrm{S}},\ id_{\mathrm{R}}\big), \quad \text{output } D_{\mathrm{s}}(k, c).$$

In this scheme, the symmetric encryption key is derived from the long term secret key $h_{\mathrm{SR}} = g^{\alpha_{\mathrm{S}} \cdot \alpha_{\mathrm{R}}}$ along with an ephemeral secret key $w = g^{\beta \cdot \alpha_{\mathrm{R}}}$. The ephemeral secret key ensures CCA security even when the attacker knows the sender's secret key α_{S}. The long term secret key ensures ciphertext integrity, as before.

The following theorem proves security of $\mathcal{SC}'_{\mathrm{DH}}$ in this stronger signcryption security model. Interestingly, the proof of CCA security for $\mathcal{SC}'_{\mathrm{DH}}$ only relies on the simpler interactive Diffie-Hellman assumption from Section 12.4.1, not the double-interactive assumption $\mathrm{I}^2\mathrm{CDH}$ that we used in proving CCA-security for $\mathcal{SC}_{\mathrm{DH}}$.

Theorem 13.11. *$\mathcal{SC}'_{\mathrm{DH}}$ is a secure signcryption scheme that provides forward secrecy assuming $\mathcal{E}$ is an* AE*-secure cipher, the* $\mathrm{I}^2\mathrm{CDH}$ *assumption holds in $\mathbb{G}$, and the hash function H is modeled as a random oracle.*

In particular, for every ciphertext integrity adversary $\mathcal{A}_{\mathrm{ci}}$ that attacks $\mathcal{SC}'_{\mathrm{DH}}$ as in the random oracle variant of Attack Game 13.5, there exists a ciphertext integrity adversary $\mathcal{B}_{\mathrm{ci}}$ that attacks $\mathcal{E}$ as in Attack Game 9.1, and an $\mathrm{I}^2\mathrm{CDH}$ *adversary $\mathcal{B}_{\mathrm{dh}}$ for $\mathbb{G}$, where $\mathcal{B}_{\mathrm{s}}$ and $\mathcal{B}_{\mathrm{dh}}$ are elementary wrappers around $\mathcal{A}_{\mathrm{ci}}$, such that*

$$\mathrm{SCI}\mathsf{adv}[\mathcal{A}_{\mathrm{ci}}, \mathcal{SC}_{\mathrm{DH}}] \leq \mathrm{CI}\mathsf{adv}[\mathcal{B}_{\mathrm{ci}}, \mathcal{E}] + \mathrm{I}^2\mathrm{CDH}\mathsf{adv}[\mathcal{B}_{\mathrm{dh}}, \mathbb{G}].$$

In addition, for every CCA adversary $\mathcal{A}_{\mathrm{cca}}$ that attacks $\mathcal{SC}_{\mathrm{DH}}$ as in the random oracle variant of Attack Game 13.6, there exists a 1CCA adversary $\mathcal{B}_{\mathrm{1cca}}$ that attacks $\mathcal{E}$ as in Definition 9.6, and an ICDH adversary $\mathcal{B}'_{\mathrm{dh}}$ for $\mathbb{G}$, where $\mathcal{B}_{\mathrm{s}}$ and $\mathcal{B}'_{\mathrm{dh}}$ are elementary wrappers around $\mathcal{A}_{\mathrm{ci}}$, such that

$$\mathrm{SCCA}'\mathsf{adv}[\mathcal{A}_{\mathrm{cca}}, \mathcal{SC}_{\mathrm{DH}}] \leq 1\mathrm{CCA}\mathsf{adv}[\mathcal{B}_{\mathrm{1cca}}, \mathcal{E}] + 2 \cdot \mathrm{ICDH}\mathsf{adv}[\mathcal{B}'_{\mathrm{dh}}, \mathbb{G}].$$

Proof idea. The proof of ciphertext integrity is very similar to the proof in Theorem 13.10. The proof of CCA security with forward secrecy, where the adversary is given the sender's secret key, is almost identical to the proof of ElGamal CCA security (Theorem 12.4), together with the random self reduction for CDH (see Exercise 10.4); as such, the ICDH assumption is sufficient for the proof. $\Box$

Property II: non-repudiation. Suppose Alice encrypts a message m to Bob and obtains the ciphertext c. The question is, does c, together with Bob's secret key, provide Bob with enough evidence to convince a third party that Alice actually sent the message m to Bob? We call this property non-repudiation. We explained at the beginning of the chapter that such evidence is inherently limited in its persuasive powers: Alice can simply claim that her secret key was stolen from her and that someone else produced c, or she can deliberately leak her secret key in order to repudiate c. Nevertheless, since non-repudiation may be required in some situations, we define it and show how to construct signcryption schemes that provide it.

We define the non-repudiation property by slightly tweaking the ciphertext integrity game (Attack Game 13.5). The goal is to ensure that ciphertext integrity is maintained even if the adversary obtains the recipient's secret key. The modified game is as follows:

Attack Game 13.9 (Ciphertext integrity with non-repudiation). The game is identical to Attack Game 13.5 except that we change the setup step as follows: in addition to giving the adversary the public keys pk_{S} and pk_{R}, the challenger gives the adversary the receiver's secret key sk_{R}. The corresponding advantage is denoted $\mathrm{SCI}'\mathsf{adv}[\mathcal{A}, \mathcal{SC}]$. $\Box$

Definition 13.11. *A signcryption scheme $\mathcal{SC}$ is said to provide* ***non-repudiation****, if for all efficient adversaries $\mathcal{A}$, the value $\mathrm{SCI}'\mathsf{adv}[\mathcal{A}, \mathcal{SC}]$ is negligible.*

Non-repudiation for encrypt-then-sign. The encrypt-then-sign construction provides non-repudiation: the secret key sk_{R} is only used to decrypt ciphertexts and does not help in signing anything. Indeed, in the concrete security bound given in Theorem 13.8, one can see that bound on SCI advantage does not depend at all on the security of the signature scheme.

Non-repudiation for sign-then-encrypt. The same argument cannot be made for the sign-then-encrypt construction. Observe that in the concrete security bound given in Theorem 13.9, the bound on the SCCI advantage depends on both the security of the encryption scheme and the signature scheme. In fact, it is easy to see that this scheme cannot provide non-repudiation as we have defined it. Indeed, given the decryption key, one can always decrypt a ciphertext encrypting (m, σ) and then simply re-encrypt it, obtaining a different, but still valid, ciphertext.

Although sign-then-encrypt does not satisfy our definition of non-repudiation, it does satisfy a weaker notion that corresponds to plaintext integrity, rather than ciphertext integrity. Roughly speaking, this property corresponds to a modification of Attack Game 13.9 in which the winning condition is changed: to win the game, it candidate forgery $\hat{c}$ must decrypt to a message that was never submitted as an $\mathrm{S} \to \mathrm{R}$ encryption query. We leave it the reader to flesh out the details of this definition, and to show that sign-then-encrypt satisfies this weaker notion of non-repudiation. See also Exercise 9.15.

Non-repudiation for $\mathcal{SC}_{\mathrm{DH}}$. The $\mathcal{SC}_{\mathrm{DH}}$ scheme does not provide non-repudiation, in a very strong sense: the recipient can encrypt any message just as well as the sender. The same is true for $\mathcal{SC}'_{\mathrm{DH}}$. Because of this property, both these schemes provide *complete deniability* — the sender can always claim (correctly) that any ciphertext it generated could have been generated by the receiver. In real-world settings this deniability property may be considered a feature rather than a bug.

Summary. Forward secrecy is clearly a desirable property in real-world systems. Non-repudiation, in the context of signcryption, is not usually needed, except for some niche applications. In situations where forward secrecy is desirable, but non-repudiation is not, the $\mathcal{SC}'_{\mathrm{DH}}$ scheme is a very efficient solution. In situations where both properties are needed, encrypt-then-sign is a safer option than sign-then-encrypt, despite only providing a slightly weaker notion of CCA security, as discussed above. Exercise 13.17 is a variation of encrypt-then-sign that is also an attractive option to ensure both forward secrecy and non-repudiation.

13.8 Certificates and the public-key infrastructure

We next turn to one of the central applications of digital signatures, namely, their use in certificates and public-key infrastructure. In its simplest form, a certificate is a blob of data that binds a public-key to an identity. This binding is asserted by a third party called a **certificate authority**, or simply a CA. We first discuss the mechanics of how certificates are issued and then discuss some real-world complications in managing certificates — specifically, how to cope with misbehaving CAs and how to revoke certificates.

Obtaining a certificate. Say Alice wishes to obtain a certificate for her domain `alice.com`. She sends a **certificate signing request** (CSR) to the CA, that contains Alice's identity, her email

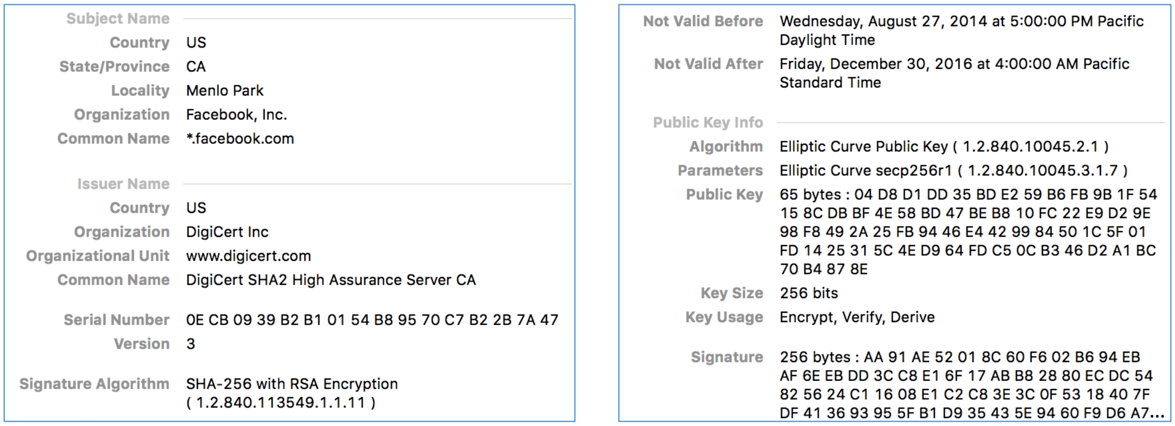


Figure 13.4: An example X.509 certificate

address, and the public key that she wishes to bind to her domain.

Once the CA receives the CSR, it checks that Alice is who she claims to be. In some cases this check is as naive as sending a challenge email to Alice's address and verifying that she can read the email. In other cases this is done by requiring notarized documents proving Alice's identity. We emphasize that certifying Alice's real-world identity is the primary service that the CA provides. If all the checks succeed, the CA assembles the relevant data into a certificate structure, and signs it using the CA's secret signing key. The resulting signed blob is a certificate that binds the public key in the CSR to Alice's identity. Some CAs issue certificates for free, while others require payment from Alice to issue a certificate.

The resulting signed certificate can be sent to anyone that needs to communicate securely with Alice. Anyone who has the CA's verification key can verify the certificate and gain some confidence that the certified public key belongs to Alice.

X.509 certificates. Certificates are formatted according to a standard called X.509. Fig. 13.4 gives an example X.509 certificate that binds a public key to an entity identified in the subject field. Here the entity happens to be Facebook Inc., and its public key is an (elliptic-curve) ElGamal public key, shown on the right side of the figure. The certificate was issued by a CA called DigiCert Inc., who used its RSA signing key to sign the certificate using the PKCS1 standard with SHA256 as the hash function. A portion of the CA's signature is shown on the bottom right of the figure. To verify this certificate one would need the public key for DigiCert Inc.

Every X.509 certificate has a serial number that plays a role in certificate revocation, as explained in Section 13.8.2 below. Certificates also have a validity window: a time when the certificate becomes active, and a time when the certificate expires. A certificate is considered invalid outside of its validity window, and should be rejected by the verifier. The validity window is typically one or two years, but can be longer or shorter. For example, the certificate in Fig. 13.4 has a validity window of about seventeen months. The reason for limiting certificate lifetime is to ensure that if the private key is stolen by an attacker, that attacker can only abuse the key for a limited period of time. The longer the validity window, the longer an attacker can abuse a stolen secret key. We discuss this further is Section 13.8.2 where we discuss certificate revocation.

A certificate issued by a CA can be verified by anyone who has that CA's public key. If there were only one CA in the world then everyone could store a copy of that CA's public key and use it to verify all certificates. However, a single global CA would not work well. First, every country wants to run a CA for local businesses in its region. Second, to keep the price of certificates low, it is best to enable multiple CAs to compete for the business of issuing certificates. Currently there are thousands of active CAs issuing certificates.

Certificate chains. Since there are multiple CAs issuing certificates, and new ones can appear at any time, the challenge is to distribute CA public keys to end-users who need to verify certificates. The solution, called a **certificate chain**, is to allow one CA to certify the public key of another CA. This process can repeat recursively, resulting in a chain of certificates where every certificate in the chain certifies the public key of the next CA in the chain.

The public key of top level CAs, called **root CAs**, are pre-installed on all clients that need to verify certificates. There are several hundred such root CAs that ship with every standard operating system. A root CA can issue a certificate to an **intermediate CA**, and an intermediate CA can issue a certificate to another intermediate CA. Continuing this way we obtain a chain of certificates starting from the root and containing one or more intermediate CAs. Finally, the CA at the bottom of the chain issues a client certificate for the end identity, such as Facebook in Fig. 13.4.

The certificate chain for the Facebook certificate is shown in Fig. 13.5. The root CA is DigiCert Inc., but its secret key is kept offline to reduce the risk of theft. The root secret key is only used for one thing: to issue a certificate for an intermediate CA, that is also owned by DigiCert Inc. That intermediate CA then uses its secret key to issue client certificates to customers like Facebook. If the intermediate CA's secret key is lost or stolen, the corresponding certificate can be revoked, and the root CA can issue a new certificate for the intermediate CA.

To verify this certificate chain of length three, the verifier needs a local trusted copy of the public key of the root CA. That public key lets the verifier check validity of the certificate issued to the intermediate CA. If valid, it has some assurance that the intermediate CA can be trusted. The verifier then checks validity of the certificate issued to Facebook by the intermediate CA. If valid, the verifier has some assurance that it has the correct public key for Facebook.

Certificate chains and basic constraints. X.509 certificates contain many fields and we only scratched the surface in our discussion above. In the context of certificate chains we mention two fields that play an important security role. In Fig. 13.5 we saw that the certificate chain issued to Facebook has length three. What is to prevent Facebook from behaving like a CA and generating a certificate chain of length four for another identity, say `alice.com`? This certificate chain, unbeknownst to Alice, would enable Facebook to impersonate `alice.com` and even eavesdrop on traffic to `alice.com` by acting as a "man in the middle," similar to what we saw in Section 10.7.

The reason Facebook cannot issue certificates is because of a **basic constraint** field that every

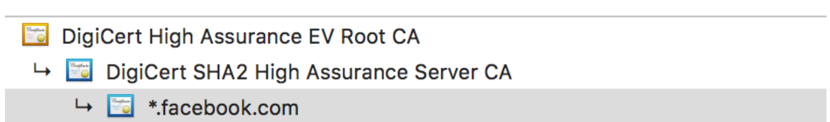


Figure 13.5: An example certificate chain

CA must embed in the certificates that it issues. This field, called the "CA" field, is set to true if the entity being certified is allowed to act as a CA, and is set to false otherwise. For a certificate chain of length ℓ to be valid, it must be the case that the top $\ell - 1$ certificates in the chain have their CA basic constraint set to true. If not, the chain must be rejected by the verifier. Facebook's certificate has its CA field set to "false," preventing Facebook from acting as an intermediate CA.

Certificate validation includes many other such subtle checks, and is generally quite tricky to implement correctly. Many systems that implement custom certificate validation were found to be insecure [46], making them vulnerable to impersonation and man-in-the-middle attacks.

13.8.1 Coping with malicious or negligent certificate authorities

By now it should be clear that CAs have a lot of power. Any CA can issue a rogue certificate and bind the wrong public key to Facebook. If left unchecked, a rogue certificate would enable an adversary to mount a man-in-the-middle attack on traffic to Facebook and eavesdrop on all traffic between Facebook and unsuspecting users. We will discuss these attacks in detail in Chapter 20 after we discuss the TLS session setup mechanism. Several commercial tools make this quite easy to do in practice.

There are currently thousands of intermediate CAs operating on the Internet and all are trusted to issue certificates. Due to the large number of CAs, it is not surprising that wrong certificates are routinely discovered. Here is a small sample of incidents:

- Diginotar was a Dutch certificate authority that was hacked in 2011. The attacker obtained a Diginotar signed certificate for `*.google.com`, and for many other domains, letting the attacker mount a man-in-the-middle attack on all these domains. In response, major Web browser vendors revoked trust in all certificates issued by the Diginotar CA, causing Diginotar to declare bankruptcy in Sep. 2011.

- India NIC in 2013 erroneously issued certificates for several Google and Yahoo domains [65]. This intermediate CA was certified by India CCA, a root CA trusted by Microsoft Windows. As a result, the Chrome browser no longer trusts certificates issued by India NIC. Furthermore, following this incident, the India CCA root CA is only trusted to issue certificates for domains ending in `.in`, such as `google.co.in`.

- Verisign in 2001 erroneously issued a Microsoft code-signing certificate to an individual masquerading as a Microsoft employee [75]. This certificate enabled that individual to distribute code that legitimately looked like it was written by Microsoft. In response, Microsoft issued a Windows software patch that revoked trust in this certificate.

As we can see, many of these events are due to an erroneous process at the CA. Any time a certificate is issued that binds a wrong public key to a domain, that certificate enables a man-in-the-middle attack on the target domain. The end result is that the attacker can inspect and modify traffic to and from the victim domain.

The question then is how to identify and contain misbehaving CAs. We discuss two ideas below.

Certificate pinning. The reader must be wondering how the incidents mentioned above were discovered in the first place. The answer is a mechanism called **certificate pinning**, which is now widely supported by Web browsers. The basic idea is that browsers are pre-configured to know that the only CA authorized to issue certificates for the domain `facebook.com` is "DigiCert SHA2 High Assurance Server CA," as shown in Fig. 13.5. If a browser ever sees a certificate for `facebook.com` that is issued by a different CA, it does two things: first, it treats the certificate as invalid and closes the connection, and second, it optionally alerts an administrator at Facebook that a rogue certificate was discovered. The incident discussed above, involving `India NIC`, was discovered thanks to a certificate pin for `gmail.com`. Browsers in India alerted Google to the existence of a rogue certificate chain for `gmail.com`. Google then took action to revoke the chain and launch an investigation. The signatures in the rogue chain provide irrefutable evidence that something went wrong at the issuing CA.

In more detail, certificate pinning works as follows. Every browser maintains a pinning database, where, roughly speaking, every row in the database is a tuple of the form

$$(\text{domain},\ \text{hash}_0,\ \text{hash}_1,\ \ldots).$$

Each hash_i is the output of a hash function (so for SHA256, a 32-byte string). The data for each record is provided by the domain owner. Facebook, for example, provides the hashes for the `facebook.com` domain.

When the browser connects to a domain using HTTPS, that domain sends its certificate chain to the browser. If the domain is in the pinning database, the browser computes the hash of each certificate in the chain. Let S be the resulting set of hash values. Let T be the set of hash values in the pinning record for this domain. If the intersection of S and T is empty, the certificate chain is rejected, and the browser optionally sends an alert to the domain administrator indicating that a rogue certificate chain was encountered.

To see how this works, consider again the example chain in Fig. 13.5. The pinning record for the domain `facebook.com` is just a single hash, namely the hash of the certificate for "DigiCert SHA2 High Assurance Server CA." In other words, the set T contains a single hash value. If the browser encounters a certificate chain for `facebook.com` where none of the certificates in the chain hash to the pinned value, the certificate chain is rejected. More generally, domains that purchase certificates from multiple CAs include the hash of all those CA certificates in their pinning record.

Why does Facebook write the hash of its CA certificate in the Facebook pinning record? Why not write the hash of the Facebook certificate from Fig. 13.4 in the pinning record? In fact, writing the CA certificate in the pinning record seems insecure; it makes it possible for DigiCert to issue a rogue certificate for `facebook.com` that will be accepted by browsers, despite the pinning record. If instead, Facebook wrote the Facebook certificate in Fig. 13.4 as the only hash value in the pinning record, then DigiCert would be unable to issue a rogue certificate for `facebook.com`. The only certificate for `facebook.com` that browsers would accept would be the certificate in Fig. 13.4. However, there is enormous risk in doing so. If Facebook somehow lost its own secret key, then no browser in the world will be able to connect to `facebook.com`. Pinning the CA certificate lets Facebook recover from key loss by simply asking DigiCert to issue a new certificate for `facebook.com`. Thus, the risk of bringing down the site outweighs the security risk of DigiCert issuing a rogue certificate. While losing the secret key may not be a concern for a large site like Facebook, it is a significant concern for smaller sites who use certificate pinning.

Finally we mention that there are two mechanisms for creating a pinning record: static and dynamic. Static pins are maintained by the browser vendor and shipped with the browser. Dynamic pins allow a domain to declare its own pins via an HTTP header, sent from the server to the browser, as follows:

```
Public-Key-Pins:  pin-sha256="hash"; max-age=expireTime
                  [; report-uri="reportURI"] [; includeSubDomains]
```

Here `pin-sha256` is the hash value to pin to, `max-age` indicates when the browser will forget the pin, and `report-uri` is an optional address where to report pin validation failures. The HTTP header is accepted by the browser only if it is sent over an encrypted HTTPS session. The header is ignored when sent over unencrypted HTTP. This prevents a network attacker from injecting invalid pins.

Certificate transparency. A completely different approach to coping with misbehaving CAs is based on public certificate logs. Suppose there existed a public **certificate log** that contained a list of all the certificates ever issued. Then a company, like Facebook, could monitor the log and learn when someone issues a rogue certificate for `facebook.com`. This idea, called certificate transparency, is compelling, but is not easy to implement. How do we ensure that every certificate ever issued is on the log? How do we ensure that the log is append-only so that a rogue certificate cannot be removed from the log? How do we ensure that everyone in the world sees the same version of the log?

Certificate transparency provides answers to all these questions. Here, we just sketch the architecture. When a CA decides to support certificate transparency, it chooses one of the public certificate logs and augments its certificate issuance procedure as follows: (1) before signing a new certificate, the CA sends the certificate data to the log, (2) the log signs the certificate data and sends back the signature, called a **signed certificate timestamp** (SCT), (3) the CA adds the SCT as an extension to the certificate data and signs the resulting structure, to obtain the final issued certificate. The SCT is embedded as an extension in the newly issued certificate.

The SCT is a promise by the certificate log to post the certificate to its log within a certain time period, say one day. At noon every day, the certificate log appends all the new certificates it received during that day to the log. It then computes a hash of the entire log and signs the hash along with the current timestamp. The log data and the signature are made publicly available for download by anyone.

The next piece of the architecture is a set of auditors that run all over the world and ensure that the certificate logs are behaving honestly — they are posting to the log as required, and they never remove data from the log. Every day the auditors download all the latest logs and their signatures, and check that no certificates were removed from the logs. If they find that a certificate on some day t is missing from the log on day $t+1$, then the log signatures from days t and $t+1$ are evidence that the certificate log is misbehaving.

Moreover, every auditor crawls the Internet looking for certificates. For each certificate that contains an SCT extension, the auditor does an **inclusion check**: it verifies that the certificate appears on the latest version of the log that the SCT points to. If not, then the signed SCT along with the signed log, are evidence that the certificate log is misbehaving. This process ensures that all deployed certificates with an SCT extension must appear on one of the logs; otherwise one of the certificate logs is caught misbehaving. Anyone can run the auditor protocol. In particular, every Web browser can optionally function as an auditor and run the inclusion check before choosing to trust a certificate. If the inclusion check fails, the browser notifies the browser vendor who can launch an investigation into the practices of the certificate log in question. We note that by using a data structure, called a Merkle hash tree, the inclusion check can be done very efficiently, without having to download the entire log. We discuss Merkle hash trees and their applications in

Section 8.9.

Unfortunately, auditing is not enough. A devious certificate log can misbehave in a way that will not be caught by the auditing process above. Suppose that a CA issues a rogue certificate for `facebook.com` and writes it to a certificate log, as required. Now, the certificate log creates two signed versions of the log: one with the rogue certificate and one without. Whenever an auditor downloads the log, it is given the version of the log with the rogue certificate. To the auditor, all seems well. However, when Facebook reads the log to look for rogue `facebook.com` certificates, it is given the version without the rogue certificate. This prevents Facebook from discovering the rogue certificate, even though all the auditors believe that the certificate log is behaving honestly. The architecture mitigates this attack in two ways. First, every certificate must be written to at least two logs, so that both certificate logs must be corrupt for the attack to succeed. Second, there is a broadcast mechanism in which the daily hash of all the logs is broadcast to all entities in the system. A log that does not match the broadcast hash is simply ignored.

The final piece of the architecture is mandating certificate transparency on all CAs. At some point in the future, browser vendors could decide to reject all certificates that do not have a valid SCT from a trusted certificate log. This will effectively force universal adoption of certificate transparency by all CAs. At that point, if a rogue certificate is issued, it will be discovered on one of the certificate logs and revoked. We note that many of the large CAs already support certificate transparency.

13.8.2 Certificate revocation

We next look at the question of revoking certificates. The goal of certificate revocation is to ensure that, after a certificate is revoked, all clients treat that certificate as invalid.

There are many reasons why a certificate may need to be revoked. The certificate could have been issued in error, as discussed in the previous subsection. The private key corresponding to the certificate may have been stolen, in which case the certificate owner will want to revoke the certificate so it cannot be abused. This happens all the time; sites get hacked and their secrets are stolen. One well-publicized example is the **heartbleed** event. Heartbleed is a bug in the OpenSSL library that was introduced in 2012. The bug was publicly discovered and fixed in 2014, but during those two years, from 2012 to 2014, a remote attacker could have easily extracted the secret key from every server that used OpenSSL, by simply sending a particular malformed request to the server. When the vulnerability was discovered in 2014, thousands of certificates had to be revoked because of concern that the corresponding secret keys were compromised.

Given the need to revoke certificates, we next describe a few techniques to do so.

Short-lived certificates. Recall that every certificate has a validity period and the certificate is no longer valid after its expiration date. Usually, when an entity like Facebook buys a one-year certificate, the CA issues a certificate that expires a year after it was issued. Imagine that instead, the CA generated 365 certificates, where each one is valid for exactly one day during that year. All 365 certificates are for the same public key; the only difference is the validity window. These certificates are called **short-lived certificates** because each is valid for only one day.

The CA keeps all these certificates to itself, and releases each one at most a week before it becomes valid. So, the certificate to be used on January 28 is made available on January 21, but no sooner. Every day Facebook connects to a public site provided by the CA and fetches the certificate to be used a week later. This is a simple process to automate, and if anything goes wrong, there is an entire week to fix the problem.

Now, when Facebook needs to revoke its certificate, it simply instructs the CA to stop releasing short-lived certificates for its domain. This effectively makes the stolen private key useless after at most one week. If faster revocation is needed, the CA can be told to release each short-lived certificate only an hour before it becomes valid, in which case the secret key becomes useless at most 25 hours after it is revoked.

The use of short-lived certificates is the simplest and most practical technique for certificate revocation available, yet it is not widely used. The next two techniques are more cumbersome, but are the ones most often used by CAs.

Certificate revocation lists (CRLs). A very different approach is to have the CA collect all certificate revocation requests from all its customers, and on a weekly basis issue a signed list of all certificates that were revoked during that week. This list, called a **certificate revocation list** (CRL), contains the serial numbers of all the certificates that were revoked during that week. The list is signed by the CA.

Every certificate includes a special extension field called **CRL Distribution Points**, as shown in Fig. 13.6. This field instructs the verifier where to obtain the CRL from the issuing CA. The CA must run a public server that serves this list to anyone who asks for it.

When a client needs to validate a certificate, it is expected to download the CRL from the CRL distribution point, and reject the certificate if its serial number appears in the CRL. For performance reasons, the CRL has a validity period of, say one week, and the client can cache the CRL for that period. As a result, it may take a week from the time a revocation request is issued until all clients learn that the certificate has been revoked.

There are two significant difficulties with this approach. First, what should the client do if the CRL server does not respond to a CRL download request? If the client were to accept the certificate, then this opens up a very serious attack. An attacker can cause the client to accept a revoked certificate by simply blocking its connection to the CRL server. Clearly the safe thing to do is to reject the certificate; however, this is also problematic. It means that if the CRL server run by Facebook's CA were to accidentally crash, then no one could connect to Facebook until the CA fixes the CRL server. As you can imagine, this does not go over well with Facebook.

A second difficulty with CRLs is that they force the client to download a large list of revoked certificates that the client does not need. The client is only interested in learning the validity status of a single certificate: the one it is trying to validate. The client does not need, and is not interested in, the status of other certificates. This inefficiency is addressed by a better mechanism called OCSP, which we discuss next.

The online certificate status protocol (OCSP). A client that needs to validate a certificate can use the OCSP protocol to query the CA about the status of that specific certificate. To make this work, the CA includes an OCSP extension field in the certificate, as shown in Fig. 13.6. This field tells the client where to send its OCSP query. In addition, the CA must setup a server, called an **OCSP responder**, that responds to OCSP queries from clients.

When the client needs to validate a certificate, it sends the certificate's serial number to the OCSP responder. Roughly speaking, the responder sends back a signed message saying "valid" or

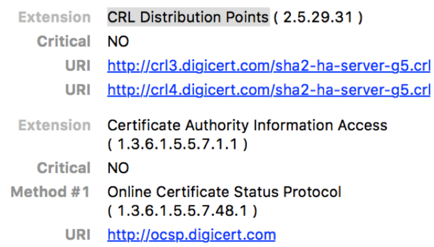


Figure 13.6: The CRL and OCSP fields in the certificate from Fig. 13.4.

"invalid". If "invalid" the client rejects the certificate. OCSP responses can be cached for, say a week, and consequently revocation only takes effect a week after a request is issued.

As with CRLs, it is not clear what the client should do when the OCSP responder simply does not respond. Moreover, OCSP introduces yet another problem. Because a client, such as a Web browser, sends to the CA the serial number of every certificate it encounters, the CA can effectively learn what web sites the user is visiting. This is a breach of user privacy. The problem can be partially mitigated by an extension to OCSP, called **OCSP stapling**, but this extension is rarely used.

13.9 Case study: legal aspects of digital signatures

While cryptographers say that a signature scheme is secure if it existentially unforgeable under a chosen message attack, the legal standard for what constitutes a valid digital signature on an electronic document is quite different. The legal definition tries to capture the notion of intent: a signature is valid if the signer "intended" to sign the document. Here we briefly review a few legislative efforts that try to articulate this notion. This discussion shows that a cryptographic digital signature is very different from a legally binding electronic signature.

Electronic signatures in the United States. On June 30, 2000, the U.S. Congress enacted the Electronic Signatures in Global and National Commerce Act, known as E-SIGN. The goal of E-SIGN is to facilitate the use of electronic signatures in interstate and foreign commerce.

The U.S. statute of frauds requires that contracts for the sale of goods in excess of $500 be signed. To be enforceable under U.S. law, E-SIGN requires that an electronic signature possess three elements: (1) a symbol or sound, (2) attached to or logically associated with an electronic record, and (3) made with the intent to sign the electronic record. Here we only discuss the first element. The U.S. definition of electronic signatures recognizes that there are many different methods by which one can sign an electronic record. Examples of electronic signatures that qualify under E-SIGN include:

1. a name typed at the end of an e-mail message by the sender,
2. a digitized image of a handwritten signature that is attached to an electronic document,
3. a secret password or PIN to identify the sender to the recipient,
4. a mouse click, such as on an "I accept" button,

5. a sound, such as the sound created by pressing '9' on a phone,
6. a cryptographic digital signature.

Clearly, the first five examples are easily forgeable and thus provide little means of identifying the signatory. However, recall that under U.S. law, signing a paper contract with an 'X' constitutes a binding signature, as long as one can establish intent of the signatory to sign the contract. Hence, the first five examples should be treated as the legal equivalent of signing with an 'X'.

United nations treaty on electronic signatures. In November 2005 the United Nations adopted its convention on the use of electronic communications in international contracts. The signature requirements of the 2005 U.N. convention go beyond those required under E-SIGN. In particular, the convention focuses on the issue of security, by requiring the use of a method that (1) identifies the signer, and (2) is reliable. In particular, the convention observes that there is a big difference between an electronic signature that merely satisfies the basic requirements of applicable U.S. law (e.g., a mouse click) and a trustworthy electronic signature. Thus, under the U.N. convention a mouse click qualifies as a digital signature only if it allows the proponent to ultimately prove "who" clicked, and to establish the intention behind the click.

European Community framework for electronic signatures. in December 1999 the European Parliament adopted the Electronic Signatures Directive. The directive addresses three forms of electronic signatures. The first can be as simple as signing an e-mail message with a person's name or using a PIN-code. The second is called the "advanced electronic signature" (AES). The directive is technology neutral but, in practice, AES refers mainly to a cryptographic digital signature based on a public key infrastructure (PKI). An AES is considered to be more secure, and thus enjoys greater legal acceptability. An electronic signature qualifies as an AES if it is: (1) uniquely linked to the signatory, (2) capable of identifying the signatory, (3) created using means that the signatory can maintain under his sole control, and (4) is linked to the data to which it relates in such a manner that any subsequent change of the data is detectable.

13.10 A fun application: private information retrieval

To be written.

13.11 Notes

Citations to the literature to be added.

13.12 Exercises

13.1 (Multi-key signature security). Just as we did for secure MACs in Exercise 6.3, show that security in the single-key signature setting implies security in the multi-key signature setting.

(a) Show how to extend Attack Game 13.1 so that an attacker can submit signing queries with respect to several signing keys. This is analogous to the multi-key generalization described in Exercise 6.3.

(b) Show that every efficient adversary $\mathcal{A}$ that wins your multi-key attack game with probability ϵ can be transformed into an efficient adversary $\mathcal{B}$ that wins Attack Game 13.1 with probability ϵ/Q. The proof uses the same "plug-and-pray" technique as in Exercise 6.3.

13.2 (Non-binding signatures). In Section 13.1.1 we mentioned that secure signatures can be non-binding: for a given (pk, sk), the signer can find two distinct messages m_0 and m_1 where the same signature σ is valid for both messages under pk. We explained that this can cause problems. Give an example of a secure signature that is non-binding.

Hint: Consider using the hash-and-sign paradigm of Section 13.2, but with the collision resistant hash functions discussed in Exercise 10.26.

13.3 (DSKS attack on RSA). Let us show show that $\mathcal{S}_{\text{FDH}}$ is vulnerable to the DSKS attack discussed in Section 13.1.1. Let (n, e) be Alice's public key and $\sigma \in Z_n$ be a signature on some message m. Then $\sigma^e = H(m)$ in $\mathbb{Z}_n$. Show that an adversary can efficiently come up with a new public key $pk' = (n', e')$ and the corresponding secret key, such that (m, σ) is valid message-signature pair with respect to pk'.

Hint: We show in Section 15.2.2 that for some primes p, the discrete-log problem in $\mathbb{Z}_p^*$ can be solved efficiently. For example, when $p = 2^\ell + 1$ is prime, and ℓ is poly-bounded, the discrete-log problem in $\mathbb{Z}_p^*$ is easy. Show that by forming n' as a product of two such primes, the adversary can come up with an e' such that $\sigma^{(e')} = H(m)$ in $\mathbb{Z}_{n'}$.

13.4 (Preventing DSKS attacks). In this exercise we explore a general defense against DSKS vulnerabilities discussed in Section 13.1.1.

(a) Define a security game capturing the fact that a signature scheme is secure against DSKS attacks: the attacker mounts a chosen message attack on some pk and wins if it outputs a (pk', sk'), where $pk' \neq pk$, such that at least one of the given message-signature pairs verifies under pk'. Moreover, sk' is a valid signing key for pk' (assume that you have an algorithm $T(pk', sk')$ that returns accept only when sk' is a valid signing key for pk').

(b) In Section 13.1.1 we describe a general approach to immunizing existentially unforgeable signature schemes against DSKS attacks. Prove that this approach satisfies the security definition from part (a).

13.5 (Derandomizing signatures). Let $\mathcal{S} = (G, S, V)$ be a secure signature scheme defined over $(\mathcal{M}, \Sigma)$, where the signing algorithm S is probabilistic. In particular, algorithm S uses randomness chosen from a space $\mathcal{R}$. We let $S(sk, m; r)$ denote the execution of algorithm S with randomness r. Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{M}, \mathcal{R})$. Show that the following signature scheme $\mathcal{S}' = (G', S', V)$ is secure:

$$G'() := \big\{(pk, sk) \xleftarrow{\text{R}} G(), \quad k \xleftarrow{\text{R}} \mathcal{K}, \quad sk' := (sk, k), \quad \text{output } (pk, sk')\big\}\,;$$
$$S'(sk', m) := \{r \leftarrow F(k, m), \quad \sigma \leftarrow S(sk, m; r), \quad \text{output } \sigma\}\,.$$

Now the signing algorithm for $\mathcal{S}'$ is deterministic.

13.6 (Extending the domain using enhanced TCR). In Exercise 8.26 we defined the notion of an enhanced-TCR. Show how to use an enhanced-TCR to efficiently extend the domain of a signature. In particular, let H be an enhanced-TCR defined over $(\mathcal{K}_H, \mathcal{M}, \mathcal{X})$ and let $\mathcal{S} = (G, S, V)$

be a secure signature scheme with message space $\mathcal{X}$. Show that $\mathcal{S}' = (G, S', V')$ is a secure signature scheme:

$$S'(pk, m) := \{\ r \xleftarrow{\text{R}} \mathcal{K}_H,\ \sigma \leftarrow S\big(sk, H(r, m)\big),\ \text{output } (\sigma, r)\}\ ;$$
$$V'\big(pk, m, (\sigma, r)\big) := \{\ \textsf{accept} \text{ if } \sigma = V(pk, H(r, m))\}\ .$$

The benefit over the construction in Section 13.2.1 is that r is not part of the message given to the signing procedure.

13.7 (FDH variant). Show that the signature scheme $\mathcal{S}'_{\text{RSA-FDH}}$ (defined in Section 13.5) is *no less secure* than the signature scheme $\mathcal{S}_{\text{RSA-FDH}}$ (defined in Section 13.3.1). You should show that if $\mathcal{A}$ is an adversary that succeeds with probability ϵ in breaking $\mathcal{S}'_{\text{RSA-FDH}}$ (which has message space $\mathcal{M}$), then there exists an adversary $\mathcal{B}$ (whose running time is roughly the same as that of $\mathcal{A}$) that succeeds with probability ϵ in breaking $\mathcal{S}_{\text{RSA-FDH}}$ (with message space $\mathcal{M}' = \{0, 1\} \times \mathcal{M}$). This should hold for any hash function H.

13.8 (Probabilistic full domain hash). Consider the following signature scheme $\mathcal{S} = (G, S, V)$ with message space $\mathcal{M}$, and using a hash function $H : \mathcal{M} \times \mathcal{R} \to \mathbb{Z}_n$:

$$G() := \{(n, d) \xleftarrow{\text{R}} \text{RSAGen}(\ell, e),\quad pk := (n, e),\quad sk := (n, d),\quad \text{output } (pk, sk)\}\ ;$$
$$S(sk, m) := \Big\{r \xleftarrow{\text{R}} \mathcal{R},\quad y \leftarrow H(m, r),\quad \sigma \leftarrow y^d \in \mathbb{Z}_n,\quad \text{output } (\sigma, r)\Big\}\ ;$$
$$V\big(pk, m, (\sigma, r)\big) := \{y \leftarrow H(m, r),\quad \textsf{accept} \text{ if } y = \sigma^e \text{ and } \textsf{reject} \text{ otherwise}\}\ .$$

Show that this signature is secure if the RSA assumption holds for (ℓ, e), the quantity $1/|\mathcal{R}|$ is negligible, and H is modeled as a random oracle. Moreover, the reduction to inverting RSA is tight.

Discussion: While $\mathcal{S}'_{\text{RSA-FDH}}$, from Section 13.5, also has a tight reduction, the construction here does not use a PRF. The cost is that signatures are longer because r is included in the signature.

13.9 (Batch RSA). Let us show how to speed up signature generation in $\mathcal{S}_{\text{RSA-FDH}}$.

(a) Let $n = pq$ such that neither 3 nor 5 divide $(p-1)(q-1)$. We are given p, q and $y_1, y_2 \in \mathbb{Z}_n$. Show how to compute both $x_1 := y_1^{1/3} \in \mathbb{Z}_n$ and $x_2 := y_2^{1/5} \in \mathbb{Z}_n$ by just computing the 15th root of $t := (y_1)^5 (y_2)^3 \in \mathbb{Z}_n$ and doing a bit of extra arithmetic. In other words, show that given $t^{1/15} \in \mathbb{Z}_n$, it is possible to compute both x_1 and x_2 using a constant number of arithmetic operations in $\mathbb{Z}_n$.

(b) Describe an algorithm for computing a 15th root in $\mathbb{Z}_n$ using a single exponentiation, for n as in part (a).

(c) Explain how to use parts (a) and (b) to speed up the $\mathcal{S}_{\text{RSA-FDH}}$ signature algorithm. Specifically, show that the signer can sign two messages at once using about the same work as signing a single message. The first message will be signed under the public key $(n, 3)$ and the other under the public key $(n, 5)$. This method generalizes to fast RSA signature generation in larger batches.

13.10 (Signature with message recovery). Let $\mathcal{T} = (G, F, I)$ be a one-way trapdoor permutation defined over $\mathcal{X} := \{0, 1\}^n$. Let $\mathcal{R} := \{0, 1\}^\ell$ and $\mathcal{U} := \{0, 1\}^{n-\ell}$, for some $0 < \ell < n$. Let H

be a hash function defined over $(\mathcal{M} \times \mathcal{U},\ \mathcal{R})$, and let W be a hash function defined over $(\mathcal{R}, \mathcal{U})$. Consider the following signature scheme $\mathcal{S} = (G, S, V)$ defined over $(\mathcal{M} \times \mathcal{U},\ \mathcal{X})$ where

$$S\big(sk,\ (m_0, m_1)\big) := \Big\{ h \leftarrow H(m_0, m_1), \quad \sigma \leftarrow I\big(sk,\ h \,\|\, (W(h) \oplus m_1)\big), \quad \text{output } \sigma \Big\}$$

(a) Explain how the verification algorithm works.

(b) Show that the scheme is secure assuming $\mathcal{T}$ is one-way, $1/|\mathcal{R}|$ is negligible, and H and W are modeled as random oracles.

(c) Show that just given (m_0, σ), where σ is a valid signature on the message (m_0, m_1), it is possible to recover m_1. A signature scheme that has this property is called a **signature with message recovery**. It lets the signer send shorter transmissions: the signer need only transmit (m_0, σ) and the recipient can recover m_1 by itself. This can somewhat mitigate the cost of long signatures with RSA.

(d) Can the technique of Section 13.5 be used to provide a tight security reduction for this construction?

13.11 (An insecure signature with message recovery). Let $\mathcal{T} = (G, F, I)$ be a one-way trapdoor permutation defined over $\mathcal{X} := \{0,1\}^n$. Let H be a hash function defined over $(\mathcal{M}_0, \mathcal{X})$. Consider the following signature scheme $\mathcal{S} = (G, S, V)$ defined over $(\mathcal{M}_0 \times \mathcal{X},\ \mathcal{X})$ where

$$\begin{aligned} S\big(sk,\ (m_0, m_1)\big) &:= \big\{ \sigma \leftarrow I(sk,\ H(m_0) \oplus m_1), \quad \text{output } \sigma \big\} \\ V\big(pk,\ (m_0, m_1),\ \sigma\big) &:= \big\{ y \leftarrow F(pk, \sigma), \quad \textsf{accept} \text{ if } y = H(m_0) \oplus m_1 \text{ and } \textsf{reject} \text{ otherwise} \big\} \end{aligned}$$

(a) Show that just given (m_0, σ), where σ is a valid signature on the message (m_0, m_1), it is possible to recover m_1.

(b) Show that this signature scheme is insecure, even when $\mathcal{T}$ is one-way and H is modeled as a random oracle. You may assume that algorithm I has the following property: for all (sk, pk) output by G, and all $x \in \mathcal{X}$, given only $I(sk, x)$ as input, one can easily compute $I(sk,\ x \oplus 1^n)$.

13.12 (Blind signatures). At the end of Section 13.3.1 we mentioned the RSA signatures can be adapted to give blind signatures. A **blind signature scheme** lets one party, Alice, obtain a signature on a message m from Bob, so that Bob learns nothing about m. Blind signatures are used in e-cash systems and anonymous voting systems.

Let $(n, d) \xleftarrow{R} \text{RSAGen}(\ell, e)$ and set (n, e) as Bob's RSA public key and (n, d) as his corresponding private key. As usual, let $H : \mathcal{M} \to \mathbb{Z}_n$ be a hash function. Alice wants Bob to sign a message $m \in \mathcal{M}$. They engage in the following three-message protocol:

(1) Alice chooses $r \xleftarrow{R} \mathbb{Z}_n$, sets $m' \leftarrow H(m) \cdot r^e \in \mathbb{Z}_n$, and sends m' to Bob,
(2) Bob computes $\sigma' \leftarrow (m')^d \in \mathbb{Z}_n$ and sends σ' to Alice,
(3) Alice computes the signature σ on m as $\sigma \leftarrow \sigma'/r \in \mathbb{Z}_n$.

Equation (13.4) shows that σ is a valid signature on m. Observe that in this process Bob sees a random message m' in $\mathbb{Z}_n$ that is independent of m. As such, he learns nothing about m.

(a) We say that a blind signature protocol is secure if the adversary, given a public key and the ability to request Q blind signatures on messages of his choice, cannot produce $Q+1$ valid message-signature pairs. Write out the precise definition of security.

(b) Show that the RSA blind signature is secure assuming the RSA assumption holds for (ℓ, e), and H is modeled as a random oracle.

13.13 (Threshold RSA signatures). In Exercise 11.16 we showed how a secret RSA decryption key can be split into three shares, so that two shares are needed to decrypt a given ciphertext, but a single share reveals nothing. In this exercise we show that the same can be done for RSA signatures, namely two shares are needed to generate a signature, but one share reveals nothing.

(a) Define what is a threshold signature scheme by adapting Definition 11.6 to the context of signature schemes. Then adapt Attack Game 11.4, used to define security for threshold decryption, to define secure threshold signatures.

(b) Use Exercise 11.16 to construct a 2-out-of-3 threshold RSA signature scheme.

(c) Prove that your scheme from part (b) satisfies the security definition from part (a).

13.14 (Insecure signcryption). Let $\mathcal{E} = (G_{\mathrm{E}}, E, D)$ be a CCA-secure public-key encryption scheme with associated data and let $\mathcal{S} = (G_{\mathrm{S}}, S, V)$ be a strongly secure signature scheme. Define algorithm G as in Section 13.7.3. Show that the following encrypt-then-sign signcryption scheme (G, E', D') is insecure:

$$
\begin{aligned}
E'(sk_{\mathrm{S}}, id_{\mathrm{S}}, pk_{\mathrm{R}}, id_{\mathrm{R}},\ m) \quad &:= \quad c \xleftarrow{\mathrm{R}} E\big(pk_{\mathrm{R}},\ m,\ id_{\mathrm{R}}\big), \quad \sigma \xleftarrow{\mathrm{R}} S\big(sk_{\mathrm{S}},\ (c, id_{\mathrm{S}})\big) \\
&\qquad \text{output } (c, \sigma) \\
D'\big(pk_{\mathrm{S}}, id_{\mathrm{S}}, sk_{\mathrm{R}}, id_{\mathrm{R}},\ (c, \sigma)\big) \quad &:= \quad \text{if } V(pk_{\mathrm{S}},\ (c, id_{\mathrm{S}}),\ \sigma) = \mathsf{reject}, \text{ output } \mathsf{reject} \\
&\qquad \text{otherwise, output } D(sk_{\mathrm{R}},\ c,\ id_{\mathrm{R}})
\end{aligned}
$$

13.15 (The iMessage attack). Let $\mathcal{E} = (G_{\mathrm{E}}, E, D)$ be a CCA-secure public-key encryption scheme and let $\mathcal{S} = (G_{\mathrm{S}}, S, V)$ be a strongly secure signature scheme. Let $(E_{\mathrm{sym}}, D_{\mathrm{sym}})$ be a symmetric cipher with key space $\mathcal{K}$ that implements deterministic counter mode. Define algorithm G as in Section 13.7.3. Consider the following encrypt-then-sign signcryption scheme (G, E', D'):

$$
\begin{aligned}
E'(sk_{\mathrm{S}}, id_{\mathrm{S}}, pk_{\mathrm{R}}, id_{\mathrm{R}},\ m) \quad &:= \quad k \xleftarrow{\mathrm{R}} \mathcal{K}, \quad c_1 \leftarrow E_{\mathrm{sym}}\big(k,\ (id_{\mathrm{S}}, m)\big), \quad c_0 \xleftarrow{\mathrm{R}} E(pk_{\mathrm{R}},\ k) \\
&\qquad \sigma \xleftarrow{\mathrm{R}} S\big(sk_{\mathrm{S}},\ (c_0, c_1, id_{\mathrm{R}})\big) \\
&\qquad \text{output } (c_0, c_1, \sigma) \\
D'\big(pk_{\mathrm{S}}, id_{\mathrm{S}}, sk_{\mathrm{R}}, id_{\mathrm{R}},\ (c_0, c_1, \sigma)\big) \quad &:= \quad \text{if } V(pk_{\mathrm{S}},\ (c_0, c_1, id_{\mathrm{R}}),\ \sigma) = \mathsf{reject}, \text{ output } \mathsf{reject} \\
&\qquad k \leftarrow D(sk_{\mathrm{R}},\ c_0), \quad (id, m) \leftarrow D_{\mathrm{sym}}(k, c_1) \\
&\qquad \text{if } id \neq id_{\mathrm{S}} \text{ output } \mathsf{reject} \\
&\qquad \text{otherwise, output } m
\end{aligned}
$$

Because the symmetric ciphertext c_1 is part of the data being signed by the sender, the designers assumed that there is no need to use an AE cipher and that deterministic counter mode is sufficient. Show that this system is an insecure signcryption scheme by giving a CCA attack. At one point, a variant of this scheme was used by Apple's iMessage system and this lead to a significant breach of iMessage [43]. Because every plaintext message m included a checksum (CRC), an adversary could decrypt arbitrary encrypted messages using a chopchop-like attack (Exercise 9.5).

13.16 (Signcryption: statically vs adaptively chosen user IDs). In the discussion following Definition 13.8, we briefly discussed the possibility of a more robust security definition in which the adversary is allowed to choose the sender and receiver user IDs adaptively, after seeing one or both of the public keys, or even after seeing the response to one or more X $\rightarrow$ Y queries.

(a) Work out the details of this more robust definition, defining corresponding SCI and SCCA attack games.

(b) Give an example of a signcryption scheme that satisfies Definition 13.8 but does not satisfy your more robust definition. To this end, you should start with a scheme that satisfies Definition 13.8, and then "sabotage" the scheme somehow so that it still satisfies Definition 13.8, but no longer satisfies your more robust definition. You may make use of any other standard cryptographic primitives, as convenient.

13.17 (Signcryption: encrypt-and-sign-then-sign). In this exercise, we develop a variation on encrypt-then-sign called encrypt-and-sign-then-sign. As does the scheme $\mathcal{SC}_{\mathrm{EtS}}$, this new scheme, denoted $\mathcal{SC}_{\mathrm{EaStS}}$, makes use of a public-key encryption scheme with associated data $\mathcal{E} = (G_{\mathrm{ENC}}, E, D)$, and a signature scheme $\mathcal{S} = (G_{\mathrm{SIG}}, S, V)$. Key generation for $\mathcal{SC}_{\mathrm{EaStS}}$ is identical to that in $\mathcal{SC}_{\mathrm{EtS}}$. However, $\mathcal{SC}_{\mathrm{EaStS}}$ makes use of another signature scheme $\mathcal{S}' = (G'_{\mathrm{SIG}}, S', V')$. The encryption algorithm $E_{\mathrm{EaStS}}(sk_{\mathrm{S}}, id_{\mathrm{S}}, pk_{\mathrm{R}}, id_{\mathrm{R}}, m)$ runs as follows:

$$\begin{aligned}
&(pk', sk') \xleftarrow{\mathrm{R}} G'_{\mathrm{SIG}}, \quad c \xleftarrow{\mathrm{R}} E(pk_{\mathrm{R}}, m, pk'), \quad \sigma \xleftarrow{\mathrm{R}} S(sk_{\mathrm{S}}, pk'), \\
&\sigma' \xleftarrow{\mathrm{R}} S'(sk', (c, \sigma, id_{\mathrm{S}}, id_{\mathrm{R}})), \quad \text{output } (pk', c, \sigma, \sigma')
\end{aligned}$$

The decryption algorithm $D_{\mathrm{EaStS}}(pk_{\mathrm{S}}, id_{\mathrm{S}}, sk_{\mathrm{R}}, id_{\mathrm{R}}, (pk', c, \sigma, \sigma'))$ runs as follows:

$$\begin{aligned}
&\text{if } V(pk_{\mathrm{S}}, pk', \sigma) = \mathsf{reject} \text{ or } V'(pk', (c, \sigma, id_{\mathrm{S}}, id_{\mathrm{R}}), \sigma') = \mathsf{reject} \\
&\qquad \text{then output } \mathsf{reject} \\
&\qquad \text{else output } D(sk_{\mathrm{R}}, c, pk')
\end{aligned}$$

Here, the value ephemeral public verification key pk' is used as associated data for the encryption scheme $\mathcal{E}$.

Your task is to show that $\mathcal{SC}_{\mathrm{EaStS}}$ is a secure signcryption scheme that provides both forward secrecy and non-repudiation, under the following assumptions: (i) $\mathcal{E}$ is CCA secure; (ii) $\mathcal{S}$ is secure (not necessarily strongly secure); (iii) $\mathcal{S}'$ is strongly secure (just against 1-query adversaries).

Discussion: Note that we have to run the key generation algorithm $\mathcal{S}'$ every time we encrypt, thereby generating an ephemeral signing key that is only used to sign a single message. The fact that we only need security against 1-query adversaries means that it is possible to very efficiently implement $\mathcal{S}'$ under reasonable assumptions. This is the topic of the next chapter.

Another feature is that in algorithm E_{EaStS}, we can run algorithms E and S in parallel; moreover, we can even run algorithms G'_{SIG} and S before algorithm E_{EaStS} is invoked (as discussed in Section 14.4.1). Similarly, in algorithm D_{EaStS}, we can run algorithms V, V', and D in parallel.

Part IV

Appendices

Appendix A

Basic number theory

A.1 Cyclic groups

Notation: for a finite cyclic group $\mathbb{G}$ we let $\mathbb{G}^*$ denote the set of generators of $\mathbb{G}$.

A.2 Arithmetic modulo primes

A.2.1 Basic concepts

We use the letters p and q to denote prime numbers. We will be using large primes, e.g. on the order of 300 digits (1024 bits).

1. For a prime p let $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$.
   Elements of $\mathbb{Z}_p$ can be added modulo p and multiplied modulo p. For $x, y \in \mathbb{Z}_p$ we write $x + y$ and $x \cdot y$ to denote the sum and product of x and y modulo p.

2. Fermat's theorem: $g^{p-1} = 1$ for all $0 \neq g \in \mathbb{Z}_p$
   Example: $3^4 \bmod 5 = 81 \bmod 5 = 1$

3. The *inverse* of $x \in \mathbb{Z}_p$ is an element a satisfying $a \cdot x = 1$.
   The inverse of x in $\mathbb{Z}_p$ is denoted by x^{-1}.
   Example: 1. 3^{-1} in $\mathbb{Z}_5$ is 2 since $2 \cdot 3 = 1 \bmod 5$.
   2. 2^{-1} in $\mathbb{Z}_p$ is $\frac{p+1}{2}$.

4. All elements $x \in \mathbb{Z}_p$ except for $x = 0$ are invertible.
   Simple (but inefficient) inversion algorithm: $x^{-1} = x^{p-2} \bmod p$.
   Indeed, $x^{p-2} \cdot x = x^{p-1} = 1 \bmod p$.

5. We denote by $\mathbb{Z}_p^*$ the set of invertible elements in $\mathbb{Z}_p$. Then $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$.

6. We now have algorithm for solving linear equations in $\mathbb{Z}_p$: $a \cdot x = b$.
   Solution: $x = b \cdot a^{-1} = b \cdot a^{p-2}$.
   What about an algorithm for solving quadratic equations?

A.2.2 Structure of $\mathbb{Z}_p^*$

1. $\mathbb{Z}_p^*$ is a *cyclic group*.
   In other words, there exists $g \in \mathbb{Z}_p^*$ such that $\mathbb{Z}_p^* = \{1, g, g^2, g^3, \ldots, g^{p-2}\}$.
   Such a g is called a *generator* of $\mathbb{Z}_p^*$.
   Example: in $\mathbb{Z}_7^*$: $\langle 3 \rangle = \{1, 3, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{1, 3, 2, 6, 4, 5\} \pmod 7 = \mathbb{Z}_7^*$.

2. Not every element of $\mathbb{Z}_p^*$ is a generator.
   Example: in $\mathbb{Z}_7^*$ we have $\langle 2 \rangle = \{1, 2, 4\} \neq \mathbb{Z}_7^*$.

3. The *order* of $g \in \mathbb{Z}_p^*$ is the smallest positive integer a such that $g^a = 1$.
   The order of $g \in \mathbb{Z}_p^*$ is denoted $\text{order}_p(g)$.
   Example: $\text{order}_7(3) = 6$ and $\text{order}_7(2) = 3$.

4. Lagrange's theorem: for all $g \in \mathbb{Z}_p^*$ we have that $\text{order}_p(g)$ divides $p-1$. Observe that Fermat's theorem is a simple corollary:
   $$\text{for } g \in \mathbb{Z}_p^* \text{ we have } g^{p-1} = (g^{\text{order}(g)})^{(p-1)/\text{order}(g)} = (1)^{(p-1)/\text{order}(g)} = 1.$$

5. If the factorization of $p-1$ is known then there is a simple and efficient algorithm to determine $\text{order}_p(g)$ for any $g \in \mathbb{Z}_p^*$.

A.2.3 Quadratic residues

1. The *square root* of $x \in \mathbb{Z}_p$ is a number $y \in \mathbb{Z}_p$ such that $y^2 = x \bmod p$.
   Example: 1. $\sqrt{2}$ in $\mathbb{Z}_7$ is 3 since $3^2 = 2 \bmod 7$.
   2. $\sqrt{3}$ in $\mathbb{Z}_7$ does not exist.

2. An element $x \in \mathbb{Z}_p^*$ is called a *Quadratic Residue* (QR for short) if it has a square root in $\mathbb{Z}_p$.

3. How many square roots does $x \in \mathbb{Z}_p$ have?
   If $x^2 = y^2$ in $\mathbb{Z}_p$ then $0 = x^2 - y^2 = (x-y)(x+y)$.
   $\mathbb{Z}_p$ is an "integral domain" which implies that $x - y = 0$ or $x + y = 0$, namely $x = \pm y$.
   Hence, elements in $\mathbb{Z}_p$ have either zero square roots or two square roots.
   If a is the square root of x then $-a$ is also a square root of x in $\mathbb{Z}_p$.

4. Euler's theorem: $x \in \mathbb{Z}_p$ is a QR if and only if $x^{(p-1)/2} = 1$.
   Example: $2^{(7-1)/2} = 1$ in $\mathbb{Z}_7$ but $3^{(7-1)/2} = -1$ in $\mathbb{Z}_7$.

5. Let $g \in \mathbb{Z}_p^*$. Then $a = g^{(p-1)/2}$ is a square root of 1. Indeed, $a^2 = g^{p-1} = 1$ in $\mathbb{Z}_p$.
   Square roots of 1 in $\mathbb{Z}_p$ are 1 and -1.
   Hence, for $g \in \mathbb{Z}_p^*$ we know that $g^{(p-1)/2}$ is 1 or -1.

6. Legendre symbol: for $x \in Z_p$ define $\left(\frac{x}{p}\right) := \begin{cases} 1 & \text{if } x \text{ is a QR in } \mathbb{Z}_p \\ -1 & \text{if } x \text{ is not a QR in } \mathbb{Z}_p \\ 0 & \text{if } x = 0 \bmod p \end{cases}$.

7. By Euler's theorem we know that $\left(\frac{x}{p}\right) = x^{(p-1)/2}$ in $\mathbb{Z}_p$.
   $\Longrightarrow$ the Legendre symbol can be efficiently computed.

8. Easy fact: let g be a generator of $\mathbb{Z}_p^*$. Let $x = g^r$ for some integer r.
Then x is a QR in $\mathbb{Z}_p$ if and only if r is even.
$\Longrightarrow$ **the Legendre symbol reveals the parity of** r.

9. Since $x = g^r$ is a QR if and only if r is even it follows that exactly half the elements of $\mathbb{Z}_p$ are QR's.

10. When $p = 3 \bmod 4$ computing square roots of $x \in \mathbb{Z}_p$ is easy.
Simply compute $a = x^{(p+1)/4}$ in $\mathbb{Z}_p$.
$a = \sqrt{x}$ since $a^2 = x^{(p+1)/2} = x \cdot x^{(p-1)/2} = x \cdot 1 = x$ in $\mathbb{Z}_p$.

11. When $p = 1 \bmod 4$ computing square roots in $\mathbb{Z}_p$ is possible but somewhat more complicated; it requires a randomized algorithm.

12. We now have an algorithm for solving quadratic equations in $\mathbb{Z}_p$.
We know that if a solution to $ax^2 + bx + c = 0 \bmod p$ exists then it is given by:
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
in $\mathbb{Z}_p$. Hence, the equation has a solution in $\mathbb{Z}_p$ if and only if $\Delta = b^2 - 4ac$ is a QR in $\mathbb{Z}_p$. Using our algorithm for taking square roots in $\mathbb{Z}_p$ we can find $\sqrt{\Delta} \bmod p$ and recover x_1 and x_2.

13. What about cubic equations in $\mathbb{Z}_p$? There exists an efficient randomized algorithm that solves any equation of degree d in time polynomial in d.

A.2.4 Computing in $\mathbb{Z}_p$

1. Since p is a huge prime (e.g. 1024 bits long) it cannot be stored in a single register.

2. Elements of $\mathbb{Z}_p$ are stored in buckets where each bucket is 32 or 64 bits long depending on the processor's chip size.

3. Adding two elements $x, y \in \mathbb{Z}_p$ can be done in linear time in the *length* of p.

4. Multiplying two elements $x, y \in \mathbb{Z}_p$ can be done in quadratic time in the *length* of p. If p is n bits long, better algorithms work in time $O(n^{1.7})$ (rather than $O(n^2)$).

5. Inverting an element $x \in \mathbb{Z}_p$ can be done in quadratic time in the length of p.

6. Using the repeated squaring algorithm, $x^r \bmod p$ can be computed in time $(\log_2 r)O(n^2)$ where p is n bits long. Note, the algorithm takes linear time in the length of r.

A.2.5 Summary: arithmetic modulo primes

Let p be a 1024 bit prime. Easy problems in $\mathbb{Z}_p$:

1. Generating a random element. Adding and multiplying elements.

2. Computing $g^r \bmod p$ is easy even if r is very large.

3. Inverting an element. Solving linear systems.

4. Testing if an element is a QR and computing its square root if it is a QR.

5. Solving polynomial equations of degree d can be done in polynomial time in d.

Problems that are believed to be hard in $\mathbb{Z}_p$:

1. Let g be a generator of $\mathbb{Z}_p^*$. Given $x \in \mathbb{Z}_p^*$ find an r such that $x = g^r \bmod p$. This is known as the *discrete log problem*.

2. Let g be a generator of $\mathbb{Z}_p^*$. Given $x, y \in \mathbb{Z}_p^*$ where $x = g^{r_1}$ and $y = g^{r_2}$. Find $z = g^{r_1 r_2}$. This is known as the *Diffie-Hellman problem*.

A.3 Arithmetic modulo composites

We are dealing with integers n on the order of 300 digits long, (1024 bits). Unless otherwise stated, we assume that n is the product of two equal size primes, e.g. on the order of 150 digits each (512 bits).

1. For a composite n let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$.
   Elements of $\mathbb{Z}_n$ can be added and multiplied modulo n.

2. The inverse of $x \in \mathbb{Z}_n$ is an element $y \in \mathbb{Z}_n$ such that $x \cdot y = 1 \bmod n$.
   An element $x \in \mathbb{Z}_n$ has an inverse if and only if x and n are relatively prime. In other words, $\gcd(x, n) = 1$.

3. Elements of $\mathbb{Z}_n$ can be efficiently inverted using Euclid's algorithm. If $\gcd(x, n) = 1$ then using Euclid's algorithm it is possible to efficiently construct two integers $a, b \in \mathbb{Z}$ such that $ax + bn = 1$. Reducing this relation modulo n leads to $ax = 1 \bmod n$. Hence $a = x^{-1} \bmod n$.
   note: this inversion algorithm also works in $\mathbb{Z}_p$ for a prime p and is more efficient than inverting x by computing $x^{p-2} \bmod p$.

4. We let $\mathbb{Z}_n^*$ denote the set of invertible elements in $\mathbb{Z}_n$.

5. We now have an algorithm for solving linear equations: $\quad a \cdot x = b \bmod n$.
   Solution: $\quad x = b \cdot a^{-1}$ where a^{-1} is computed using Euclid's algorithm.

6. How many elements are in $\mathbb{Z}_n^*$? We denote by $\varphi(n)$ the number of elements in $\mathbb{Z}_n^*$. We already know that $\varphi(p) = p - 1$ for a prime p.

7. One can show that if $n = p_1^{e_1} \cdots p_m^{e_m}$ then $\varphi(n) = n \cdot \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)$.
   In particular, when $n = pq$ we have that $\varphi(n) = (p-1)(q-1) = n - p - q + 1$.
   Example: $\varphi(15) = \big|\{1, 2, 4, 7, 8, 11, 13, 14\}\big| = 8 = 2 * 4$.

8. Euler's theorem: all $a \in \mathbb{Z}_n^*$ satisfy $a^{\varphi(n)} = 1$ in $\mathbb{Z}_n$.
   note: For primes p Euler's theorem implies that $a^{\varphi(p)} = a^{p-1} = 1$ for all $a \in \mathbb{Z}_p^*$. Hence, Euler's theorem is a generalization of Fermat's theorem.

Structure of $\mathbb{Z}_n$

Theorem A.1 (Chinese Remainder Theorem (CRT)). *state theorem*

Summary

Let n be a 1024 bit integer which is a product of two 512 bit primes. Easy problems in $\mathbb{Z}_n$:

1. Generating a random element. Adding and multiplying elements.
2. Computing $g^r \bmod n$ is easy even if r is very large.
3. Inverting an element. Solving linear systems.

Problems that are believed to be hard if the factorization of n is unknown, but become easy if the factorization of n is known:

1. Finding the prime factors of n.
2. Testing if an element is a QR in $\mathbb{Z}_n$.
3. Computing the square root of a QR in $\mathbb{Z}_n$. This is provably as hard as factoring n. When the factorization of $n = pq$ is known one computes the square root of $x \in \mathbb{Z}_n^*$ by first computing the square root in $\mathbb{Z}_p$ of $x \bmod p$ and the square root in $\mathbb{Z}_q$ of $x \bmod q$ and then using the CRT to obtain the square root of x in $\mathbb{Z}_n$.
4. Computing e'th roots modulo n when $\gcd(e, \varphi(n)) = 1$.
5. More generally, solving polynomial equations of degree $d > 1$. This problem is easy if the factorization of n is known: one first finds the roots of the polynomial equation modulo the prime factors of n and then uses the CRT to obtain the roots in $\mathbb{Z}_n$.

Problems that are believed to be hard in $\mathbb{Z}_n$:

1. Let g be a generator of $\mathbb{Z}_n^*$. Given $x \in \mathbb{Z}_n^*$ find an r such that $x = g^r \bmod n$. This is known as the *discrete log problem*.
2. Let g be a generator of $\mathbb{Z}_n^*$. Given $x, y \in \mathbb{Z}_n^*$ where $x = g^{r_1}$ and $y = g^{r_2}$. Find $z = g^{r_1 r_2}$. This is known as the *Diffie-Hellman problem*.

Appendix B

Basic probability theory

Includes a description of statistical distance.

B.1 Birthday Paradox

Theorem B.1. *Let $\mathcal{M}$ be a set of size n and let $X_1, \ldots, X_k$ be k independent random variables uniform in $\mathcal{M}$. Let C be the event that for some distinct $i, j \in \{1, \ldots, k\}$ we have that $X_i = X_j$. Then*

(i) $\quad \Pr[C] \geq 1 - e^{-k(k-1)/2n} \geq \min\left\{\frac{k(k-1)}{4n}, 0.63\right\}$, *and*

(ii) $\quad \Pr[C] \leq 1 - e^{-k(k-1)/n}$ *when* $k < n/2$.

Proof. These all follow easily from the inequality

$$1 - x \leq e^{-x} \leq 1 - x/2,$$

which holds for all $x \in [0, 1]$. □

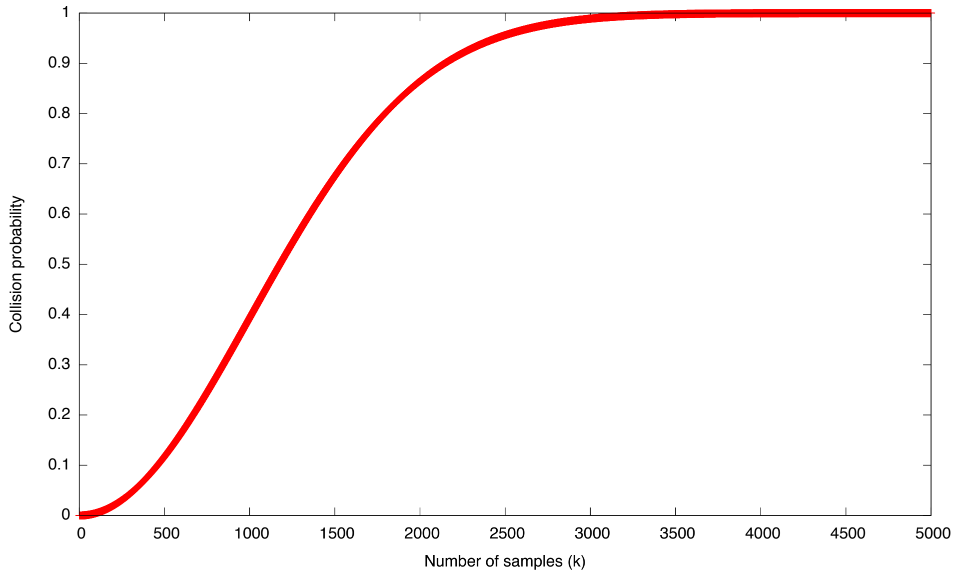
Most frequently we will use the lower bound to say that a collision happens with *at least* a certain probability. But occasionally we will use the upper bound to argue that collisions do not happen.

It is well documented that birthdays are not really uniform throughout the year. For example, in the U.S. the percentage of births in September is higher than in any other month. We show next that this non-uniformity only increases the probability of collision.

We present a stronger version of the birthday paradox that applies to independent random variables that are not necessarily uniform in $\mathcal{M}$. We do, however, require that all random variables are identically distributed. Such random variables are called i.i.d (independent and identically distributed). This version of the birthday paradox is due to Blom [Blom, D. (1973), "A birthday problem", American Mathematical Monthly, vol. 80, pp. 1141-1142].

Corollary B.2. *Let $\mathcal{M}$ be a set of size n and let $X_1, \ldots, X_k$ be k i.i.d random variables over $\mathcal{M}$ where $k \geq 2$. Let C be the event that for some distinct $i, j \in \{1, \ldots, k\}$ we have that $X_i = X_j$. Then*

$$\Pr[C] \geq 1 - e^{-k(k-1)/2n} \geq \min\left\{\frac{k(k-1)}{4n}, 0.63\right\}.$$



The graph shows that collision probability for $n = 10^6$ elements and k ranging from one sample to 5000 samples. It illustrates the threshold phenomenon around the square root. At the square root, $\sqrt{n} = 1000$, the collision probability is about 0.4. Already at $4\sqrt{n} = 4000$ the collision probability is almost 1. At $0.5\sqrt{n} = 500$ the collision probability is small.

Figure B.1: Birthday Paradox

Proof. Let X be a random variable distributed as X_1. Let $\mathcal{M} = \{a_1, \ldots, a_n\}$ and let $p_i = \Pr[X = a_i]$. Let I be the set of all k-tuples over $\mathcal{M}$ containing distinct elements. Then I contains $\binom{n}{k}k!$ tuples. Since the variables are independent we have that:

$$\Pr[\neg C] = \sum_{(b_1,\ldots,b_k)\in I} \Pr[X_1 = b_1 \wedge \ldots \wedge X_k = b_k] = \sum_{(b_1,\ldots,b_k)\in I} \prod_{j=1}^{k} p_{b_j} \tag{B.1}$$

We show that this sum is maximized when $p_1 = p_2 = \ldots = p_n = 1/n$. This will mean that the probability of collision is minimized when all the variables are uniform. The Corollary will then follow from Theorem B.1.

Suppose some p_i is not $1/n$, say $p_i < 1/n$. Since $\sum_{j=1}^{n} p_i = 1$ there must be another p_j such that $p_j > 1/n$. Let $\epsilon = \min((1/n) - p_i,\ p_j - 1/n)$ and note that $p_j - p_i > \epsilon$. We show that replacing p_i by $p_i + \epsilon$ and p_j by $p_j - \epsilon$ increases the value of the sum in (B.1). Clearly, the resulting $p_1, \ldots, p_n$ still sum to 1. Hence, the resulting $p_1, \ldots, p_n$ form a distribution over $\mathcal{M}$ in which there is one less value that is not $1/n$. Furthermore, the probability of no collision in this distribution is greater than in the unmodified distribution. Repeating this replacement process at most n times will show that the sum is maximized when all the p_i's are equal to $1/n$. Again, this means that the probability of not getting a collision is maximized when the variables are uniform.

Now, consider the sum in (B.1). There are four types of terms. First, there are terms that do not contain either p_i or p_j. These terms are unaffected by the change to p_i and p_j. Second, there are terms that contain exactly one of p_i or p_j. These terms pair up. For every k-tuple that contains i but not j there is a corresponding tuple that contains j but not i. Then the sum of the corresponding two terms in (B.1) looks like $A(p_i + \epsilon) + A(p_j - \epsilon)$ for some $A \in [0, 1]$. Since this equals $Ap_i + Ap_j$ the sum of these two terms is not affected by the change to p_i and p_j. Finally, there are terms in (B.1) that contain both p_i and p_j. These terms change by

$$B(p_i + \epsilon)(p_j - \epsilon) - Bp_ip_j = B[\epsilon(p_j - p_i) - \epsilon^2]$$

for some $B \in [0, 1]$. By definition of ϵ we know that $p_j - p_i > \epsilon$ and therefore $\epsilon(p_j - p_i) - \epsilon^2 > 0$. Hence, the sum with modified p_i and p_j is larger than the sum with the unmodified values.

Overall, we proved that the modification to p_i and p_j increases the value of the sum in (B.1), as required. This completes the proof of the Corollary. □

B.1.1 More collision bounds

Consider the sequence $x_i \leftarrow f(x_{i-1})$ for a random function $f : \mathcal{X} \to \mathcal{X}$. Analyze the cycle time of this walk (needed for Pollard). Now, consider the same sequence for a permutation $\pi : \mathcal{X} \to \mathcal{X}$. Analyze the cycle time (needed for analysis of SecurID identification).

B.1.2 A simple distinguisher

We describe a simple algorithm that distinguishes two distributions on strings in $\{0, 1\}^n$. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be independent random variables taking values in $\{0, 1\}$. Then

$$X := (X_1, \ldots, X_n) \qquad \text{and} \qquad Y := (Y_1, \ldots, Y_n)$$

are elements of $\{0, 1\}^n$. Suppose, that for $i = 1, \ldots, n$ we have

$$\Pr[X_i = 1] = p \qquad \text{and} \qquad \Pr[Y_i = 1] = (1 + 2\epsilon) \cdot p$$

for some $p \in [0, 1]$ and some small $\epsilon > 0$ so that $(1 + 2\epsilon) \cdot p \leq 1$. Then X and Y induce two distinct distributions on $\{0, 1\}^n$.

We are given an n-bit string T and are told that it is either sampled according to the distribution X or the distribution Y, so that both p and ϵ are known to us. Our goal is to decide which distribution T was sampled from. Consider the following simple algorithm $\mathcal{A}$:

input: $T = (t_1, \ldots, t_n) \in \{0, 1\}^n$
output: 1 if T is sampled from X and 0 otherwise
$s \leftarrow (1/n) \cdot \sum_{i=1}^n t_i$
if $s > p \cdot (1 + \epsilon)$ output 0 else output 1

We are primarily interested in the quantity

$$\Delta := \big| \Pr[\mathcal{A}(T_x) = 1] - \Pr[\mathcal{A}(T_y) = 1] \big| \quad \in [0, 1]$$

where $T_x \xleftarrow{R} X$ and $T_y \xleftarrow{R} Y$. This quantity captures how well $\mathcal{A}$ distinguishes the distributions X and Y. For a good distinguisher Δ will be close to 1. For a weak distinguisher Δ will be close to 0. The following theorem shows that when n is about $1/(p\epsilon^2)$ then Δ is about $1/2$.

Theorem B.3. *For all $p \in [0, 1]$ and $\epsilon < 0.3$, if $n = 4\lceil 1/(p\epsilon^2) \rceil$ then $\Delta > 0.5$*

Proof. The proof follows directly from the Chernoff bound. When T is sampled from X the Chernoff bound implies that

$$\Pr[\mathcal{A}(T_x) = 1] = \Pr[s > p(1+\epsilon)] \leq e^{-n\cdot(p\epsilon^2/2)} \leq e^{-2} \leq 0.135$$

When T is sampled from Y then the Chernoff bound implies that

$$\Pr[\mathcal{A}(T_y) = 0] = \Pr[s < p(1+\epsilon)] \leq e^{-n\cdot(p\epsilon^2/4)} \leq e^{-1} \leq 0.368$$

Hence, $\Delta > |(1 - 0.368) - 0.135| = 0.503$ and the bound follows. $\square$

Appendix C

Basic complexity theory

To be written.

Appendix D

Probabilistic algorithms

To be written.

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