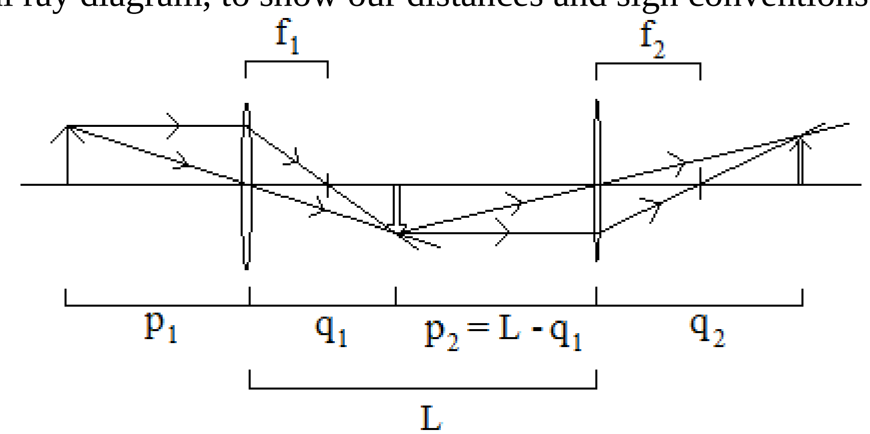
Lecture 9 Notes: 07 / 13

Multiple-lens systems

If we have one lens behind another, we can simply treat the image formed by the first lens as an object for the second lens. For example, suppose we have two convergent lenses, with focal lengths f_1 and f_2, separated by a distance L. The object is located at a distance p_1 in front of the first lens. We want to locate the image and find the magnification.

This is a typical ray diagram, to show our distances and sign conventions:



For this particular diagram, all the quantities are positive. If one of the lenses was divergent, its focal length would be negative; if one or both of the images was on the same side as its corresponding object, the value of q for that image would be negative.

Let us calculate q_2 for given p_1, L, f_1 and f_2. The equations for the two lenses are

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{f_1} \qquad \frac{1}{L - q_1} + \frac{1}{q_2} = \frac{1}{f_2}$$

We can simply use the first equation to calculate q_1 and plug into the second. For example, suppose that the object is 50*cm* away, the lenses are 120*cm* apart, the first lens has a focal length of 30*cm* and the second lens has a focal length of 100*cm*. Then, our equation will give us

$$q_1 = \frac{1}{1/f_1 - 1/p_1} = \frac{1}{1/30cm - 1/50cm} = 75cm$$
$$q_2 = \frac{1}{1/f_2 - 1/(L - q_1)} = \frac{1}{1/100cm - 1/(120 - 75)cm} = -82cm$$

This means that the final image will be virtual, and will be 82 centimeters in front of the second lens (in the direction of the object). This is unlike the ray diagram above, where the final image is real and behind the second lens.

We can calculate the magnification of this two-lens contraption. Since the magnification of the first lens is $M1 = -h_1 / h$, where h is the height of the object and h_1 is the height of the first image, and $M2 = -h_2 / h_1$, since the first image is now the object and h_2 is the height of the second image, $M_1 M_2 = h_2 / h = M$ is the total magnification from the object to the final image (the sign cancels because a sequence of two inverted real images give an upright real image, so the magnification for a real image is positive.) Since $M_1 = -q_1 / p_1$ and $M_2 = -q_2 / p_2 = q_2 / (L - q_1)$, the magnification is

$$M = M_1 M_2 = \frac{q_1 q_2}{p_1 (L - q_1)}$$

For our example, this gives

$$M = \frac{75cm(-82cm)}{50cm(120cm - 75cm)} = -2.7$$

The final image is therefore inverted, and 2.7 times larger than the original.

Example: We might ask how the magnification depends on the distance L; perhaps our apparatus is adjustable, and we want find a length that gives a certain magnification.

$M_1 = -q_1 / p_1$ does not depend on the distance L, but M_2 does. We have

$$M_2 = -\frac{q_2}{L - q_1} = -\frac{1}{(L - q_1)/f_2 - 1} = -\frac{f_2}{L - q_1 - f_2}$$

The overall magnification is thus equal to

$$M = M_1 M_2 = \frac{q_1}{p_1} \frac{f_2}{L - q_1 - f_2}$$

What should we make the length of our apparatus if we want a magnification of, say, -5? Solve this equation for L:

$$L = \left(\frac{1}{M} \frac{q_1}{p_1} + 1 \right) f_2 + q_1 = \left(-\frac{1}{5} \frac{75cm}{50cm} + 1 \right) \times 100cm + 75cm = 145cm$$

So if we separated the lenses by 145*cm* and left the object 50*cm* from the front lens, the magnification would now be -5.

Suppose we have an object at infinity, and we look at it using lenses of focal lengths f_1 and f_2 separated by a distance L. Where does the image form?

The first lens will form an image at the focal point, since parallel rays coming from the far-away object will cross at the focal distance. Thus, $q_1 = f_1$. For the second image,

$$\frac{1}{q_2} = \frac{1}{f_2} - \frac{1}{p_2} = \frac{1}{f_2} - \frac{1}{L - f_1}$$

The location of the second image for a far-away source can be treated as an effective focal point of the entire two-lens image. Thus,

$$\frac{1}{f} = \frac{1}{f_2} - \frac{1}{L - f_1}$$

If the separation between the lenses is very small compared to the focal length of the first lens (for example, if we put two lenses together, front to back), we can neglect L, and obtain the following expression:

$$\frac{1}{f} = \frac{1}{f_2} + \frac{1}{f_1}$$

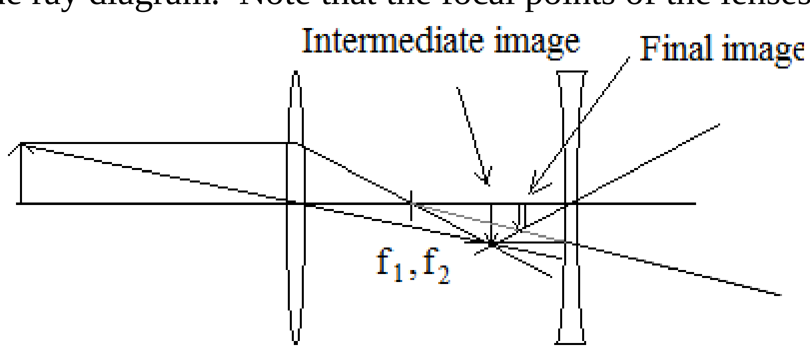
The two lenses thus function together as a single lens, with a combined focal length. Note that this equation is only valid if the lenses are held very close together; otherwise the more general equation above should be used.

We define the ***power of a lens*** as the reciprocal of its focal length: $P = 1 / f$. The units of lens power are just m^{-1}, but these units are traditionally referred to as ***diopters*** when used in this context, just like Hertz are seconds^{-1} in the context of frequency. As we can see, lens power is additive when two lenses are put close together:

$$P = P_1 + P_2$$

Example: Consider a system with a convergent lens, $f_1 = 40$ *cm*, followed by a divergent lens, $f_2 = -60$ *cm*, at a distance of 100 *cm*. An object is placed 100 *cm* in front of the convergent lens. What kind of image is formed, and where?

First we'll draw the ray diagram. Note that the focal points of the lenses coincide.



Thus we expect a small, inverted virtual image located fairly close to the second lens. Our lens equations give us

$$q_1 = \frac{1}{1/f_1 - 1/p_1} = \frac{1}{1/40cm - 1/100cm} = 66.7cm$$

$$q_2 = \frac{1}{1/f_2 - 1/(L - q_1)} = \frac{1}{-1/60cm - 1/(100 - 66.7)cm} = -21.4cm$$

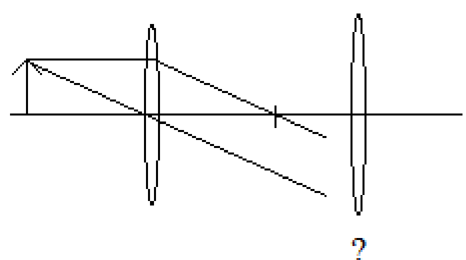
$$M = \frac{q_1}{p_1}\frac{q_2}{L - q_1} = \frac{66.7cm}{100cm}\frac{-21.4cm}{(100 - 66.7)cm} = -0.43$$

The image is thus a bit to the left of the second lens (by about 1/3 of its focal length), is upside-down and smaller than the object.

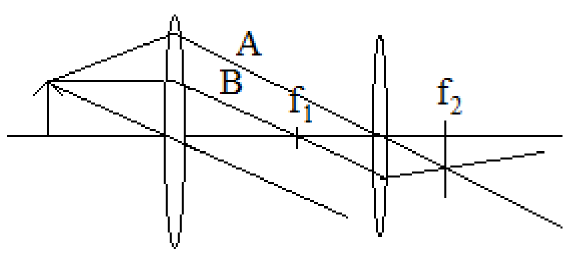
Potential problems with ray diagrams for multiple-lens systems

When trying to handle multiple-lens systems, some troublesome special cases can occur. The first case we'll consider is when the object happens to be at the focal point of the first lens. In this case, the first lens doesn't form an image; the rays coming out of it are parallel. However, the second lens can focus these parallel rays, thus forming a final image. However, we don't have an intermediate image to use as an object for the second lens.

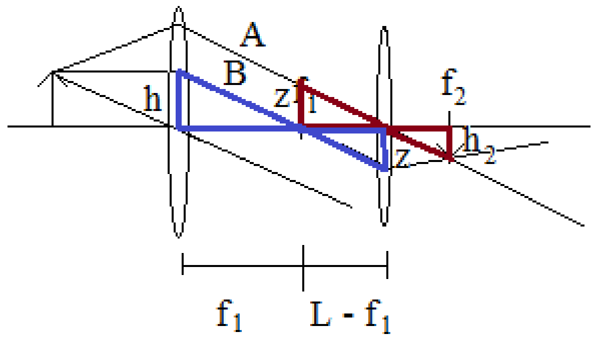
This case can be resolved with a ray diagram. This is a ray diagram for an object at the focal point of the first lens:



Since the rays coming out of the first lens are parallel, there is no intermediate image for the second lens to use as an object. What we can do, however, is add another parallel ray *A* that will go through the center of the second lens:



We know that A must be parallel to B (and any other rays starting from the tip of the object arrow) because the object is at the focal point. Also, since A and B, together with all other rays from the tip of the arrow, are parallel, after passing through the second lens, they will all cross at that lens's focal point. To determine the magnification, mark off the following two triangles and use the fact that they are similar:

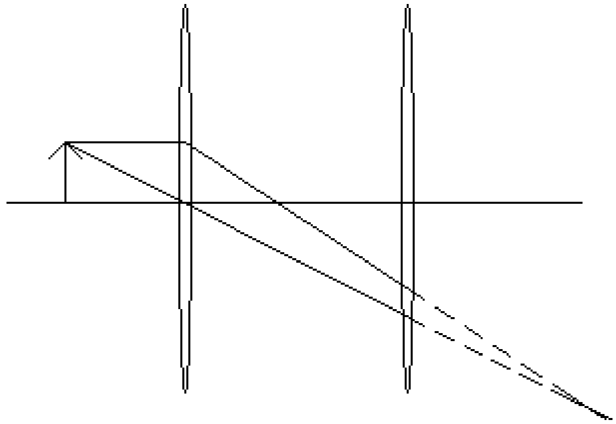


The blue triangles give us $h / f_1 = z / L - f_1$, while the red triangles give us $h_2 / f_2 = z / (L - f_1)$. Dividing the second equation by the first gives

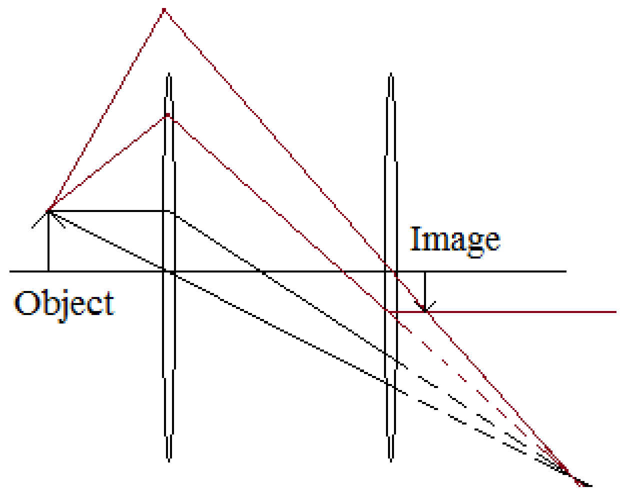
$$\frac{h_2}{h}\frac{f_1}{f_2} = 1 \qquad M = -\frac{h_2}{h} = -\frac{f_2}{f_1}$$

Note the minus sign, since the final image turns out to be upside-down.

Another problem to consider is, what happens when the first lens forms its image behind the second lens? It is not clear that we can use this image as the second lens's object, since it is no longer in front of the lens, where the lens can receive light from it. For example, consider an object 50*cm* in front of a lens with a focal length of 40*cm*. The image will form at a distance of 1 / (1/40-1/50) = 200*cm* behind this lens. What if the second lens, with a focal length of 30*cm*, is placed just 100*cm* behind the lens, a full meter in front of where the image will form? Again, a ray diagram will clarify the situation. First, draw it as if the second lens didn't exist:



The problem is, our rays don't go into the second lens in any way that would make it easy to predict their behavior. However, we can always add more rays. We know that any rays emanating from the tip of the object will go towards the image point in the lower-right corner, so we add a ray that does that but also goes through the center of the second lens, and another ray that goes through the focal point of the second lens. We know how to properly continue these rays:



We have found our image. We could do geometry at this point and determine the location and magnification of the image, but let us try to just mindlessly use the lens equations for the first and second lens, without worrying whether the second lens has an image to use as an object:

The first equation gives us q_1 = 200 *cm*. The second equation is

$$q_2 = \frac{1}{1/f_2 - 1/(L - q_2)} = \frac{1}{1/30cm - 1/(100 - 200)cm} =$$

$$= \frac{1}{1/30cm + 1/100cm} = 23cm$$

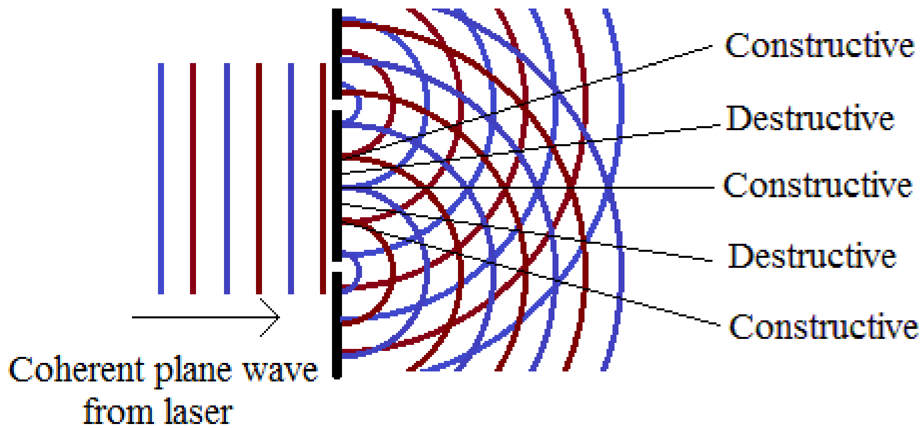
$$M = \frac{q_1}{p_1}\frac{q_2}{L - q_1} = \frac{200cm}{50cm}\frac{23cm}{(100 - 200)cm} = -0.92$$

Turns out this is the right result. The image is slightly smaller than the object, inverted and slightly within the second lens' focal distance. So, this case, unlike the case with the object at the focal point, is not really a problem in terms of applying the lens equations. It is only a problem when one has to draw the ray diagram, and this problem is solved through introduction of additional rays.

Intro to wave optics: the double-slit experiment

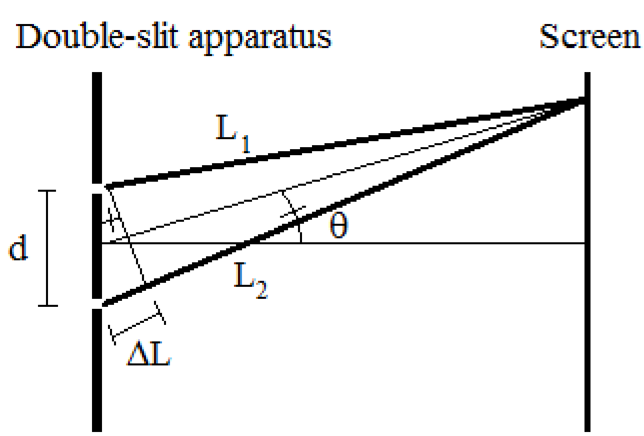
When we are dealing with light interacting with objects of size comparable to the wavelength, we must take into account the wave nature of light. One of the simplest examples of wave optics is the interference of light from two small slits in a screen.

Suppose we have two slits, illuminated by a coherent light source such as a laser (so that the waves emerge from the two slits with the same phase). If the slits are smaller than or comparable to the wavelength, each will act as a point source. In the diagram below, red lines indicate wave crests (maxima) and red lines indicate troughs (minima).



In some directions (labeled "constructive"), the crests overlap with the crests and the troughs overlap with the troughs. The wave amplitudes from the two point sources add in these directions, and if we place a screen where the labels are, there will be a bright spot on the screen (called a ***fringe***). In other directions (labeled "destructive"), the crests overlap with the troughs, and the waves tend to cancel out. On our screen, we would see dark spots along these directions. This is called a double-slit interference pattern.

Consider the geometry of the double-slit apparatus. As the light travels from each slit to a point on the screen, the path lengths from the two slits are different by a distance ΔL :



Let θ be the angle from between the line from the center of the slit apparatus to the center of the screen and the line to the point on the screen. Then, it is easy to show that the angle subtended by ΔL is also θ. Therefore, $\Delta L = d \sin \theta$. We get constructive interference if the path lengths are different by a multiple of the wavelength, and destructive interference if the path lengths are different by half a wavelength:

$$d \sin \theta = n\lambda \qquad \text{Constructive}$$

$$d \sin \theta = \left(n + \frac{1}{2}\right)\lambda \qquad \text{Destructive}$$

Thus we see the alternating pattern of spots of constructive and destructive interference on the screen, as expected. The central point on the screen has a spot where the path lengths are equal, so it is a bright spot; the flanking spots are where the path length difference is $\lambda / 2$, which are dark, followed by bright spots again, etc.

Example: Red light with a wavelength of 650 *nm* goes through double slits with a separation of 1 *mm*. The interference pattern is projected on a screen 10 meters away. What is the distance between the first bright spot and the second one? What if blue light, with a wavelength of 450 *nm*, is used instead?

The first spot occurs at $\theta = 0$. The second spot is at $d \sin \theta = \lambda$, or $\sin \theta = \lambda / d$, $\theta = 6.5 \times 10^{-4}$ radians. The separation on the screen is $R\theta$, where R is the distance to the screen, so the separation is $y = 6.5 \times 10^{-3} m = 6.5 mm$.

For blue light, we get $\theta = 4.5 \times 10^{-4}$ radians and $y = 4.5 mm$. Light with shorter wavelengths leads to narrower interference patterns.

Example: A screen is 2 meters away from the double-slit source. The double slits are separated by a distance of 1.5*mm*. A 1cm section of screen contains 12 alternating bright fringes. What is the wavelength of light?

Since one centimeter contains 12 fringes, n changes by 12 over the course of a centimeter. One centimeter corresponds to $\theta = 0.01m / 2m = 0.005$ radians. Thus, $d \sin (0.005) \approx 0.005d = 12\lambda$, or $\lambda = 0.005 \times (1.5 \times 10^{-3} m) / 12 = 6.25 \times 10^{-7} m = 625\ nm$.