JOURNAL OF COMBINATORIAL THEORY, Series A **78**, 280–291 (1997)
ARTICLE NO. TA972765

Quasi-Symmetric Designs and Codes Meeting the Grey–Rankin Bound

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Communicated by the Managing Editors

Received May 20, 1996

We give a characterization of codes meeting the Grey–Rankin bound. When the

1. INTRODUCTION

A binary (n, M, d) code C is a subset of $\{0, 1\}^n$ with M elements such that any two elements of C differ in at least d coordinates. A code C is said to be *self-complementary* whenever $x \in C$ implies $\bar{x} \in C$, where $\bar{x}$ denotes the complement of the binary vector x, obtained by replacing each 0 in x by 1, and each 1 by 0. For example, any binary linear code containing the all ones vector is a self-complementary code.

For any n and d, the Grey–Rankin bound is an upper bound for M. It states that

$$M \leqslant \frac{8d(n-d)}{n-(n-2d)^2}$$

for any (n, M, d) self-complementary code, provided the right-hand side is positive. We present a combinatorial proof of this bound in Section 2. Our main result, proved in Section 3, is the following theorem.

THEOREM A. *Suppose n and d satisfy $n-\sqrt{n}<2d<n$. Then*:

(i) *If n is odd, there exists a self-complementary code meeting the Grey–Rankin bound if and only if there exists a Hadamard matrix of size $n+1$.*

0097-3165/97 $25.00

(ii) *If n is even, there exists a self-complementary code meeting the Grey–Rankin bound if and only if there exists a quasi-symmetric $2-(n, d, \lambda)$ design with block intersection sizes $d/2$ and $(3d-n)/2$, where $\lambda = d(d-1)/(n-(n-2d)^2)$.*

Part (i) of Theorem A implies that there are no self-complementary codes of length $n \equiv 1 \pmod 4$ meeting the Grey–Rankin bound, since a Hadamard matrix (of size >2) has size divisible by 4.

We discuss linearity and find the parameters of all linear codes meeting the bound in Section 4.

2. PROOF OF THE GREY–RANKIN BOUND

We now give a combinatorial proof of the bound. The bound is originally due to Rankin [R], who proved it in a different context. Grey [G] observed that Rankin's result could be restated as Theorem 1. Delsarte [D3] (see also [MS, p. 544]) gave a proof via linear programming. The argument given here is due to R. M. Wilson. Recall that a code is said to form an orthogonal array of strength t if the projection of the code onto any t coordinates contains every t-tuple the same number of times.

THEOREM 1 (Grey–Rankin bound). *Let C be a binary (n, M, d) self-complementary code and suppose that $n - \sqrt{n} < 2d < n$. Then*

$$M \leqslant \frac{8d(n-d)}{n-(n-2d)^2}.$$

Equality holds if and only if the distances between codewords in C are all in $\{0, d, n-d, n\}$ and the codewords form an orthogonal array of strength 2.

Proof (Wilson). Let C be a binary self-complementary (n, M, d) code. Let C' be any subcode of C consisting of one codeword from each complementary pair of codewords, so C' has $M/2$ elements and all distances in C' are between d and $n-d$. We say $x \in C'$ "agrees" in coordinates i and j if x has the same digit in coordinates i and j.

Let $\{x, y\}$ denote an unordered pair of distinct elements of C', and let $\{i, j\}$ denote an unordered pair of distinct coordinates. Count the number N of ordered pairs $(\{x, y\}, \{i, j\})$ so that either x agrees in coordinates i and j but y does not, OR so that y agrees in coordinates i and j but x does not. Given a pair $\{x, y\}$ at distance t, the number of such pairs $\{i, j\}$ is

$t(n-t) \geqslant d(n-d)$ (the property is invariant under translation so one can assume $x = \mathbf{0}$ where this is easy to see). So

$$N \geqslant \binom{M/2}{2} d(n-d).$$

On the other hand, if we choose an unordered pair of coordinates $\{i, j\}$, the maximum number of ways to choose $\{x, y\}$ will occur when there are $M/8$ of each of 00, 01, 10, 11 appearing in these coordinates in C', in which case this number is $(M/4)^2$. Thus,

$$N \leqslant \binom{n}{2} (M/4)^2.$$

We conclude that

$$\binom{n}{2} (M/4)^2 \geqslant \binom{M/2}{2} d(n-d),$$

which implies that

$$n(n-1)\, M \geqslant 4(M-2)\, d(n-d)$$

and the result follows.

It is clear from this argument that equality holds if and only if any pair $\{x, y\}$ has distance either d or $n-d$ and the code is a strength 2 orthogonal array. ∎

Remarks. **(1)** The conditions for equality can also be derived from the linear programming proof of Delsarte (see [MS p. 544]) and the theorem of complementary slackness characterizing optimal programs in a linear programming problem [D2, Theorem 3.4; MS, p. 537].

3. A CHARACTERIZATION OF EQUALITY

We now investigate codes meeting the bound. Throughout this section we assume that C is a self-complementary code satisfying the Grey–Rankin bound with equality, so $M = 8d(n-d)/(n-(n-2d)^2)$. The proof of Theorem A for n even will be broken up into a few lemmata.

Let C be a self-complementary binary (n, M, d) code which satisfies the Grey–Rankin bound with equality; w.l.o.g. $\mathbf{0}, \mathbf{1} \in C$. If $d(x, y)$ denotes the Hamming distance between x and y, and $w(x)$ denotes the Hamming weight of x, we will use the elementary fact that

$$d(x, y) = w(x-y) = w(x+y) = w(x) + w(y) - 2w(x \cap y), \tag{1}$$

where $w(x \cap y)$ denotes the number of coordinates where x and y both have a 1.

LEMMA 2. *C is distance invariant.*

Proof. A typical calculation will suffice; let x be a codeword of weight d. There is one codeword at distance 0 from x (itself) and one codeword at distance n (its complement, $\bar{x}$). If R is the number of codewords of weight d at distance d from x, then the other $(M-2)/2-1-R$ codewords of weight d have distance $n-d$ from x (by Theorem 1). The complements of these two sets of codewords have the complementary distances from x, so there are (counting the zero codeword) $1+R+(M-2)/2-1-R=(M-2)/2$ codewords at distance d from x. ∎

COROLLARY 3. *If* $(A_0, A_1, ..., A_n)$ *denotes the distribution vector of* C, *then*

$$A_0 = A_n = 1, \qquad A_d = A_{n-d} = \frac{M-2}{2}, \qquad A_i = 0 \qquad \textit{otherwise.}$$

The dual distribution $(A'_0, A'_1, ..., A'_n)$ is given by

$$MA'_k = \sum_{i=0}^{n} A_i P_k(i),$$

where $P_k(x)$ is the binary Krawtchouk polynomial of degree k. By definition, the dual distance d' is the smallest nonzero i such that $A'_i \neq 0$.

LEMMA 4. *The codewords of C form an orthogonal array of strength* 3.

Proof. Using the properties that $P_k(i) = (-1)^k P_k(n-i)$ and $P_k(0) = \binom{n}{k}$, we see that

$$\begin{aligned} MA'_k &= \binom{n}{k} + \left(\frac{M-2}{2}\right) P_k(d) + \left(\frac{M-2}{2}\right) P_k(n-d) + P_k(n) \\ &= \begin{cases} 2\binom{n}{k} + (M-2)\, P_k(d), & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

This proves that the dual distance d' is at least 4, because by [D1, Theorem 4.5] the codewords of C form an orthogonal array of strength $d'-1$, and the dual distance d' is at least 3 since we know the codewords form an orthogonal array of strength 2. ∎

Remarks. **(2)** Note that $MA'_n = 2 + (M-2)\, P_n(d) = 2 + (M-2)(-1)^d = M$ if d is even, and so $A'_n = 1$. Also we have $A'_k = A'_{n-k}$. The expression for A'_4 simplifies to

$$A'_4 = \frac{1}{24}\, n(n-1)(n-2-(n-2d)^2)$$

which is zero iff $d = n/2 - \sqrt{n-2}/2$. That this never happens will follow later from some divisibility conditions; see Remark 8.

(3) Delsarte's inequalities [D2] say that $A'_k \geqslant 0$ for all k. From $k=4$ we get $d \geqslant n/2 - \sqrt{n-2}/2$ as a necessary condition, which is slightly stronger than the initial requirement on d.

(4) In terms of Delsarte's four fundamental parameters [D1], we have

$$d = d, \qquad s = 3, \qquad d' = 4, \qquad s' = \frac{n+1}{2} - 3.$$

The next lemma follows from some general theorems (MS, Chapt. 6, Section 4), but we include the proof because it is short and it gives the value of λ.

LEMMA 5. *The codewords of weight d (and $n-d$) in C form a 2-design.*

Proof. Let $k = d$ or $n-d$, and let $\lambda_k = \lambda_k(u)$ denote the number of codewords of weight k covering a fixed pair of coordinates, u. It will follow from this argument that λ_k does not depend on u. Since the codewords form an orthogonal array of strength 2 we have

$$\lambda_d + \lambda_{n-d} = \frac{M}{4} - 1.$$

Counting in two ways the number of pairs (v, c), where v is a vector of weight 3 covering u and $c \in C - \{\mathbf{0}, \mathbf{1}\}$ is a codeword covering v, we have

$$\binom{n-2}{1}\left(\frac{M}{8} - 1\right) = \lambda_d \binom{d-2}{1} + \lambda_{n-d} \binom{n-d-2}{1};$$

the left-hand side follows from the fact that the codewords form an orthogonal array of strength 3.

Solving these two equations gives

$$\lambda_k = \frac{k(k-1)}{n-(n-2d)^2}$$

for $k = d, n-d$. We have shown that the codewords of weight d form a $2-(n, d, \lambda_d)$ design, and the codewords of weight $n-d$ form the $2-(n, n-d, \lambda_{n-d})$ complementary design. ∎

EXAMPLE. The parameter set (24, 140, 10) gives equality in the Grey–Rankin bound, but $\lambda_{10} = 90/8$ is not an integer, so a self-complementary (24, 140, 10) code does not exist.

We now complete the proof of Theorem A. Recall that a 2-design with exactly two block intersection sizes is called a quasi-symmetric design.

THEOREM A. *Suppose n and d satisfy* $n - \sqrt{n} < 2d < n$. *Then*:

(i) *If n is odd, there exists a self-complementary* (n, M, d) *code with* $M = 8d(n-d)/(n-(n-2d)^2)$ *if and only if there exists a Hadamard matrix of size* $n+1$.

(ii) *If n is even, there exists a self-complementary* (n, M, d) *code with* $M = 8d(n-d)/(n-(n-2d)^2)$ *if and only if d is even and there exists a quasi-symmetric* $2-(n, d, \lambda)$ *design with block intersection sizes* $d/2$ *and* $(3d-n)/2$, *where* $\lambda = d(d-1)/(n-(n-2d)^2)$.

Proof. (i) Suppose n is odd. Any two of the $(M-2)/2$ codewords of weight d must differ in either d coordinates, if d is even, or $n-d$ coordinates if d is odd, using (1). In either case, these codewords (or their complements) give rise to a spherical 1-distance set in $\mathbf{R}^n$, i.e., a set of points on the unit sphere in $\mathbf{R}^n$ such that all points in the set are at distance 1 from all other points in the set. It is well known that such a set has cardinality at most n.

Hence $(M-2)/2 \leqslant n$, and so $M \leqslant 2n+2$. It is not hard to see that the only way this can happen is if $M = 2n+2$ and $n = 2d+1$. Assuming this, C is a self-complementary $(n, 2n+2, (n-1)/2)$ code with n codewords of each weight $(n \pm 1)/2$.

Such a code C exists if and only if there exists a Hadamard matrix of size $n+1$.

(ii) Suppose n is even, and let C be an (n, M, d) code meeting the Grey–Rankin bound. If d is odd, by (1) the only possibility is $C = \{\mathbf{0}, \mathbf{1}, x, \bar{x}\}$, and solving the equation $M = 4$ leads to $(n+d)^2 = 2n^2 - n$ which is impossible by parity.

Hence we may assume d is even. By Lemma 5, the codewords of weight d form a $2-(n, d, \lambda)$ design with $\lambda = d(d-1)/(n-(n-2d)^2)$. Besides 0 and n, there are only two distances in the code, namely d and $n-d$, so using (1) there can be only two block intersection sizes in the design, namely $d/2$ and $(3d-n)/2$.

Conversely, given the design, taking the rows of the incidence matrix (with blocks indexing rows) as codewords of a code, we get a $(n, (M-2)/2, d)$

TABLE I

n	d	$\frac{8d(n-d)}{n-(n-2d)^2}$	λ	$d/2$	$(3d-n)/2$	Existence	Comments
6	2	32	1	1	0	Yes	Remark 9
10	4	32	2	2	1	Yes	Remark 10
20	8	192	14	4	2	No	[CF]
28	12	128	11	6	4	Yes	Remark 11
36	16	128	12	8	6	Yes	Remark 11
42	18	576	51	9	6	?	
66	30	288	29	15	12	?	

constant weight code. Adding in the complements of these codewords, together with **0** and **1**, gives a $(n, 8d(n-d)/(n-(n-2d)^2), d)$ code which is closed under taking complements. ∎

Remarks. **(5)** In Table I we give all the possible parameter sets for the design/code satisfying $n \leqslant 70$, n and d both even, $n-\sqrt{n} < 2d < n$, and where the parameters satisfy all divisibility conditions. Entries for $n \geqslant 20$ are taken from [C], which is an extension (and update) of a table in [N].

(6) If $n \equiv 2 \pmod 4$, by Theorem A(i) there does not exist a $(n-1, 2n, (n-2)/2)$ self-complementary code, although if a conference matrix of size n exists then there is a code with these parameters [MS, p. 57]. For example, there is no self-complementary (17, 36, 8) code, although there is a conference matrix code with these parameters.

(7) In the $n=20$, $d=8$ case, we would have a quasi-symmetric $2-(20, 8, 14)$ design with block intersection sizes 4 and 2. Such a design does not exist; see [CF].

(8) Some divisibility conditions arise from computing the parameters of the design. A further condition may be derived from consideration of the eigenvalues of the adjacency matrix of the strongly regular block graph defined by the design [GS]. If the block intersection sizes are s_1 and s_2, the result is that $s_2 - s_1$ divides $d - s_1$. In our case this implies that $n-2d$ divides $n-d$. It follows that $n-2d$ divides d, and hence also n.

This rules out the possibility that $d = n/2 - \sqrt{n-2}/2$ as mentioned earlier in Remark 2, because $n - 2d = \sqrt{n-2}$ and this cannot divide n unless $n=6$.

4. LINEARITY

In all the cases in Table I that a code/design is known to exist, the number of codewords is a power of 2. This raises the question: are these codes linear? The binary code of a design is the binary code generated by the rows of the incidence matrix of the design. If the code C giving equality in the Grey–Rankin bound were linear, this would imply that the binary code of the corresponding design *is* C and consists of the blocks, their complements, and **0**, **1**.

In this section we consider the linearity of codes meeting the Grey–Rankin bound. Examples of linear and nonlinear codes meeting the bound are already known; see Remark 11.

Remarks. **(9)** In the case $n = 6$, the code C consists of all even weight vectors and is obviously linear. The blocks of the design are all 2-subsets of a 6-set. Neumaier in [N] classifies quasi-symmetric designs into four classes and then some exceptional designs. This design falls into Class 3 in [N].

(10) In the case $n = 10$, the design falls into Class 4 in [N], namely the design is the residual of a symmetric $2-(16, 6, 2)$ design. A symmetric $2-(n, d, \lambda)$ design with $\lambda = 2$ is also known as a biplane of order $d-2$. There are three nonisomorphic biplanes of order 4, and they may be distinguished by the mod 2 ranks of their incidence matrices, which are 6, 7, and 8. The biplane of 2-rank 6 has a residual of 2-rank 5, but the biplane of 2-rank 7 has a residual of 2-rank 6. Hence the answer to the question of linearity is no, in general.

(11) One of the exceptional designs in [N] is a $2-(28, 12, 11)$ design with block intersection sizes 4 and 6 constructed by Cameron (the reference in [N] is a personal communication). This gives us a $(28, 128, 12)$ self-complementary code giving equality in the Grey–Rankin bound. There is also a $2-(36, 16, 12)$ design, giving a $(36, 128, 16)$ code.

These designs are part of an infinite family constructed from the symplectic group $Sp(2m, 2)$. The designs on 28 and 36 points correspond to the $m = 3$ case. One considers either the hyperbolic or elliptic quadratic forms which polarize to a given symplectic form, and constructs the incidence matrix where rows (blocks) are indexed by nonzero points x in the symplectic space $GF(2)^{2m}$, and columns (points) are indexed by either hyperbolic or elliptic forms Q, and (x, Q) are incident if $Q(x) = 0$ (or maybe 1).

These designs are on $n = 2^{2m-1} \pm 2^{m-1}$ points, depending on whether one takes hyperbolic or elliptic forms. In the elliptic case the parameters (n, d, λ) are

$$(2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1}, 2^{2m-2} - 2^{m-1} - 1) \tag{2}$$

and $b = 2^{2m} - 1$, $r = n - 1$, and the block intersection sizes are $s_1 = 2^{2m-3} - 2^{m-1}$ and $s_2 = 2^{2m-3} - 2^{m-2}$. For each $m \geqslant 3$, by Theorem A we get a code meeting the Grey–Rankin bound, with parameters

$$(n, M, d) = (2^{2m-1} - 2^{m-1}, 2^{2m+1}, 2^{2m-2} - 2^{m-1}); \tag{3}$$

similarly for the hyperbolic case, where

$$(n, d, \lambda) = (2^{2m-1} + 2^{m-1}, 2^{2m-2}, 2^{2m-2} - 2^{m-1}) \tag{4}$$

and the code meeting the Grey–Rankin bound has parameters

$$(n, M, d) = (2^{2m-1} + 2^{m-1}, 2^{2m+1}, 2^{2m-2}). \tag{5}$$

By putting the incidence matrices of the elliptic and hyperbolic designs side-by-side, we obtain the incidence matrix of a symmetric design. This symmetric design has parameters

$$(2^{2m}, 2^{2m-1} - 2^{m-1}, 2^{2m-2} - 2^{m-1}) \tag{6}$$

and may be called a symplectic design; see [CV, Chapt. 5]. These designs were first introduced by Block [B]; see also [CS; K1]. The elliptic and hyperbolic designs above may also be constructed by starting with this symplectic design and by taking the derived design with respect to a block (elliptic), or the residual design with respect to a block (hyperbolic).

In fact Kantor [K1] studied a special class of designs with parameters (6), called SDP designs. These are designs with minimal 2-rank, namely $2m + 2$. The classical example of an SDP design is the symplectic design mentioned in the previous paragraph. Kantor [K2] showed that the number of nonisomorphic symmetric SDP designs with parameters (6) grows exponentially with m.

A residual or derived design of an SDP design has 2-rank equal to $2m + 1$, and it is seen from the actual construction that this is indeed the case for both the elliptic and hyperbolic designs mentioned above. This 2-rank is clearly minimal among designs with parameters (2) and (4), since there are $b = 2^{2m} - 1$ blocks. There are designs with the same parameters having a higher 2-rank; see [JT] or [LTT].

In [JT] it is shown that designs with parameters (2) or (4) which are derived or residual designs of nonisomorphic SDP designs are themselves nonisomorphic. From Kantor's result mentioned above, it follows that the number of nonisomorphic designs with parameters (2) or (4) and 2-rank $2m + 1$ grows exponentially with m. Each of these designs provides a linear code meeting the Grey–Rankin bound. For $m = 3$ it is shown in [JT] that there are at least four nonisomorphic designs of 2-rank 7, and in [T] (and also in [DEK]) it is shown that there are exactly four.

In [LTT] the $m=3$ case is examined, and the authors find (using a computer) a bunch of designs with 2-rank greater than $2m+1$. More specifically, they find eighteen $2-(28, 12, 11)$ designs, and twenty-eight $2-(36, 16, 12)$ designs, all with 2-rank greater than 7. Each of these designs provides a nonlinear code meeting the Grey–Rankin bound.

We conclude that the codes meeting the Grey–Rankin bound in Table I may be linear, but there are also nonlinear codes with the same parameters.

The following theorem characterizes the parameters of linear codes meeting the Grey–Rankin bound.

THEOREM B. *Let C be a linear self-complementary code meeting the Grey–Rankin bound. Then*:

(i) *If n is odd, the parameters of C are* $(n, M, d)=(2^s-1,\ 2^{s+1},\ 2^{s-1}-1)$ *for some* $s\geqslant 2$, *and the corresponding Hadamard matrix is of Sylvester type.*

(ii) *If n is even, the parameters of C are either* (3) *or* (5).

Proof. (i) By Theorem A, the code C is obtained from the rows and their complements of a normalized Hadamard matrix of size $n+1$ by replacing $+1$ by 0, -1 by 1, and deleting the first column. If the code is linear the size of the Hadamard matrix must be a power of 2, and furthermore the code is linear if and only if the Hadamard matrix is of Sylvester type (see [MS, p. 49]).

(ii) Let n be even, and suppose C is a linear self-complementary (n, M, d) code meeting the Grey–Rankin bound, so $M=8d(n-d)/(n-(n-2d)^2)$. Shorten C to obtain a linear $(n-1, M/2, d)$ code D with weight enumerator

$$X^{n-1}+(M/2-r-1)\,X^{n-d-1}Y^d+rX^{d-1}Y^{n-d},$$

where $r=d(n-1)/(n-(n-2d)^2)$ is the number of blocks in the design containing a given point, and the coefficient of $X^{n-1-i}\,Y^i$ is the number of codewords of weight i.

Since D has two weights, and $D^\perp$ has minimum weight 3 by the proof of Lemma 4 and Remark 2, we conclude that $D^\perp$ is uniformly packed by a theorem of Goethals and van Tilborg (see [V, p. 110]). The parameters of all uniformly packed codes are known, so the result follows from a list of such codes [V, p. 116]. ∎

COROLLARY C. *Any linear code of even length meeting the Grey–Rankin bound is the binary code of a derived or residual design of a symmetric SDP design.*

Proof. By Theorem B, such a code C has parameters (3) or (5). By Theorem A, the code gives rise to a design with parameters (2) or (4), and this design must have 2-rank equal to $2m+1$ since C is linear. A theorem of Tonchev [T] states that any such design is a derived or residual design of a symmetric SDP design. ∎

Remarks. **(12)** Even in the nonlinear case, from the proof of Theorem B we get a distance invariant two-weight code with dual distance 3.

ACKNOWLEDGMENTS

The author thanks R. M. Wilson for communicating the proof of Theorem 1 and H. N. Ward for helpful discussions. He also thanks the referee for pointing out Corollary C and providing some references.

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