Automatic Control Basic Course

Exercises 2022

LUND
UNIVERSITY

Department of Automatic Control

Compendium
ISSN 0280–5316

Department of Automatic Control
Lund University
Box 118
SE-221 00 LUND
Sweden

Printed in Sweden.
Lund 2022

Contents

Part I

Exercises

Exercises 1

Model Building and Linearization

1.1 Many everyday situations may be described and analyzed using concepts from automatic control. Analyze the scenarios below and try to give a description that captures the relevant properties of the system:

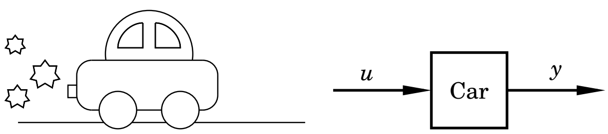
- What should be controlled?
- What control signals are used?
- What measurement signals are available?
- Is the system affected by any disturbances?
- Is feedback or feedforward used for control?
- Draw a block diagram that describes the system. The block diagram should show how the measurement signals, control signals, and the disturbances are connected to the human (which here is the controller), and to the process.

a. You take a shower and try to get desired temperature and flow of the water.

b. You drive a car.

c. You boil potatoes on the stove.

1.2 A car drives on a flat road and we assume that friction and air resistance are negligible. We want to study how the car is affected by the gas pedal position u. We assume that u varies between 0 and 1, and that the acceleration of the car is proportional to the gas pedal position, $a = ku$.



a. Write down the differential equation that describes the relation between the gas pedal position u and the velocity v of the car.

b. Let instead the position p of the car be the output signal, $y = p$. Introduce the states $x_1 = v$ and $x_2 = p$, and write the system on state-space form.

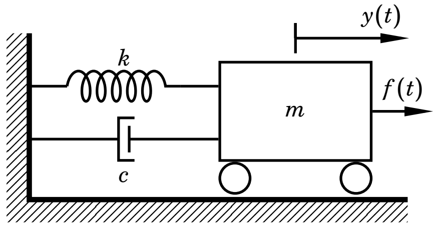
c. Assume that the car is affected by air resistance that gives a counter force that is proportional to the square of the velocity of the car. With the gas pedal position as control signal and the velocity of the car as measurement signal, the system may now be written as

$$\begin{aligned} \dot{x} &= -mx^2 + ku \\ y &= x \end{aligned}$$

The system is no longer linear (why?). Let $k = 1$ and $m = 0.001$. Find the stationary velocity y^0 that corresponds to the gas pedal being 10% down, $u^0 = 0.1$.

d. Linearize the system around the stationary point in **c.**

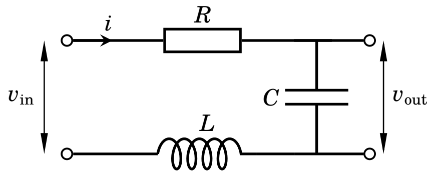
1.3 In the right figure, a mass m is attached to a wall with a spring and a damper. The spring has a spring constant k and the damper has a damping constant c. It is assumed that $k > c^2/4m$. An external force f is acting on the mass. We denote the translation of the mass from its equilibrium position by y. Further, we let $f(t)$ be the input signal and $y(t)$ be the output signal. The force equation gives



$$m\ddot{y} = -ky - c\dot{y} + f$$

Introduce the states $x_1 = y$ and $x_2 = \dot{y}$ and write down the state space representation of the system.

1.4 In the RLC circuit to the right, the input and output voltages are given by $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$, respectively. By means of Kirchhoff's voltage law we see that



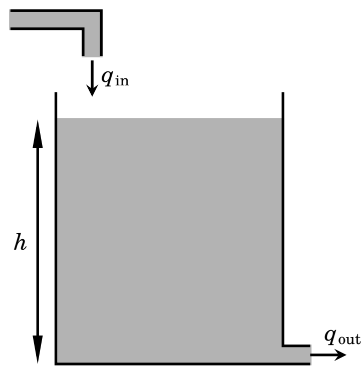
$$v_{\text{in}} - Ri - v_{\text{out}} - L\frac{di}{dt} = 0$$

For the capacitor, we additionally have

$$C\dot{v}_{\text{out}} = i$$

Introduce the states $x_1 = v_{\text{out}}$ and $x_2 = \dot{v}_{\text{out}}$ and give the state space representation of the system.

1.5 A cylindrical water tank with cross section A has an inflow q_{in} and an outflow q_{ut}. The outlet area is a. Under the assumption that the outlet area is small in comparison to the cross section of the tank, Torricelli's law $v_{\text{out}} = \sqrt{2gh}$ is valid and gives the outflow rate.



a. What would be a suitable state variable for this system? Determine a differential equation, which tells how the state variable depends on the inflow q_{in}.

b. Asume that measurement signal y is given by level h. Give a state-space representation of the system.

c. Let the inflow be constant, $q_{\text{in}} = q^0_{\text{in}}$. Determine the corresponding constant tank level h^0 and outflow q^0_{ut}. Linearize the system around this stationary point.

1.6 Give the state space representation of the system

$$\dddot{y} + 3\ddot{y} + 2\dot{y} + y = u$$

where $u(t)$ and $y(t)$ are the input and output, respectively. Choose states $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = \ddot{y}$.

1.7 A process with output $y(t)$ and input $u(t)$ is described by the differential equation

$$\ddot{y} + \sqrt{y} + y\dot{y} = u^2$$

a. Introduce states $x_1 = y$, $x_2 = \dot{y}$ and give the state space representation of the system.

b. Find all stationary points (x_1^0, x_2^0, u^0) of the system.

c. Linearize the system around the stationary point corresponding to $u^0 = 1$.

1.8 Linearize the system

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 + \sqrt{2}\sin u && (= f_1(x_1, x_2, u)) \\ \dot{x}_2 &= x_1 x_2^2 + \sqrt{2}\cos u && (= f_2(x_1, x_2, u)) \\ y &= \arctan\frac{x_2}{x_1} + 2u^2 && (= g(x_1, x_2, u)) \end{aligned}$$

around the stationary point $u^0 = \pi/4$.

1.9 A simple model of a satellite, orbiting the earth, is given by the differential equation

$$\ddot{r}(t) = r(t)\omega^2 - \frac{\beta}{r^2(t)} + u(t)$$

where r is the satellite's distance to the earth and ω is its angular acceleration, see Figure 1.1. The satellite has an engine, which can exert a radial force u.

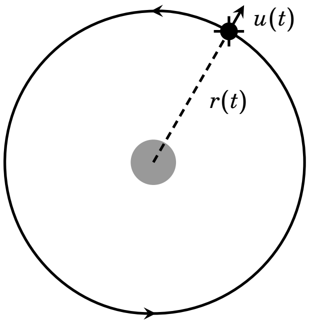


Figure 1.1 Satellite orbiting the earth.

a. Introduce the state vector

$$x(t) = \begin{pmatrix} r(t) \\ \dot{r}(t) \end{pmatrix}$$

and write down the nonlinear state space equations for the system.

b. Linearize the state space equations around the stationary point

$$(r, \dot{r}, u) = (r^0, 0, 0)$$

Consider r as the output and give the state space representation of the linear system. Express r^0 in β and ω.

Exercises 2

Dynamical Systems

2.1 A dynamical system may be described in various ways – with a transfer function, with a differential equation, and with a system of differential equations on state-space form. In this problem we transfer between the different representations for four examples of dynamical systems, from biology, mechanics, electronics, and economics.

a. A model for bacterial growth in a bioreactor is given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 10 & 1 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

where u is the inflow of glucose to the reactor, and y is the bio mass. Determine the transfer function from u to y, and a differential equation that determines the relation between the input and output signals of the system.

b. A simple model of a telescope is given by

$$J\frac{d^2y}{dt^2} + D\frac{dy}{dt} = u$$

where $y$is the angle of the telescope to the earth surface, and u is the torque from the motor that controls the telescope. Determine the transfer function from u to y and write the system on state-space form.

c. An electronic low pass filter is used at recordings to attenuate high frequency noise. The input u is the original noisy signal, and the output y is the recorded signal. The filter is given on state-space form as

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{k}x + \frac{1}{k}u \\ y &= x \end{aligned}$$

Determine the transfer function from u to y.

d. The transfer function for a model that describes economical growth is given by

$$G(s) = \frac{\gamma}{s^3 + \alpha s^2 + \beta s}$$

where the input u is the difference between savings and investments in the economy, and the output y is GDP. Write the system on state-space form.

2.2 Determine the transfer functions and give differential equations, describing the relation between input and output for the following systems, respectively.

a.

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 5 \\ 2 \end{pmatrix} u \\ y &= \begin{pmatrix} -1 & 1 \end{pmatrix} x + 2u \end{aligned}$$

b.

$$\dot{x} = \begin{pmatrix} -7 & 2 \\ -15 & 4 \end{pmatrix} x + \begin{pmatrix} 3 \\ 8 \end{pmatrix} u$$
$$y = \begin{pmatrix} -2 & 1 \end{pmatrix} x$$

c.

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} x + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x + 5u$$

d.

$$\dot{x} = \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 2 \end{pmatrix} x + 3u$$

2.3 Determine the impulse and step responses of the systems in assignment 2.2.

2.4 Derive the formula $G(s) = C(sI - A)^{-1}B + D$ for a general system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

2.5 Consider the system

$$G(s) = \frac{1}{s^2 + 4s + 3}$$

a. Calculate the poles and zeros of the system. Is the system stable?

b. What is the static gain of the system?

c. Calculate the initial value and final value of the step response of the system.

d. Calculate the initial value and final value of the impulse response of the system.

e. Calculate the initial derivative of the step response of the system.

2.6 Consider the system

$$G(s) = \frac{0.25}{s^2 + 0.6s + 0.25}$$

a. Calculate the poles and zeros of the system.

b. What is the static gain of the system?

c. Calculate and sketch the step response of the system.

2.7 Determine the transfer function and poles of the oscillating mass in assignment 1.3. Explain how the poles move if one changes k and c, respectively. Can the poles end up in the right half plane?

2.8 Determine the transfer function of

a. the RLC circuit in assignment 1.4,

b. the linearized tank in assignment 1.5.

2.9 Consider the linear time invariant system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & -1 \end{pmatrix} x$$

a. Is the system asymptotically stable?

b. Is the system stable?

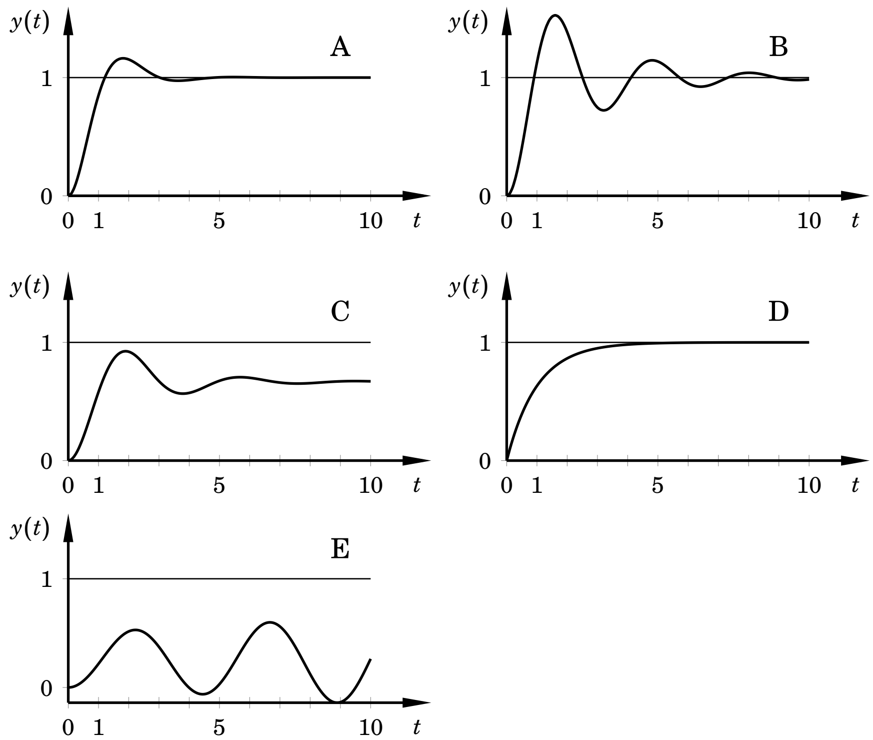
2.10 Does the transfer function

$$G(s) = \frac{s+4}{s^3 + 2s^2 + 3s + 7}$$

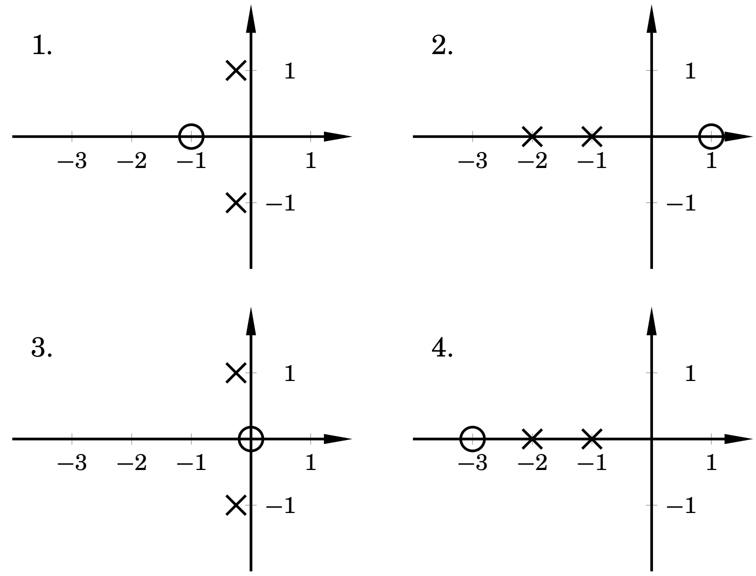
have any poles in the right half plane?

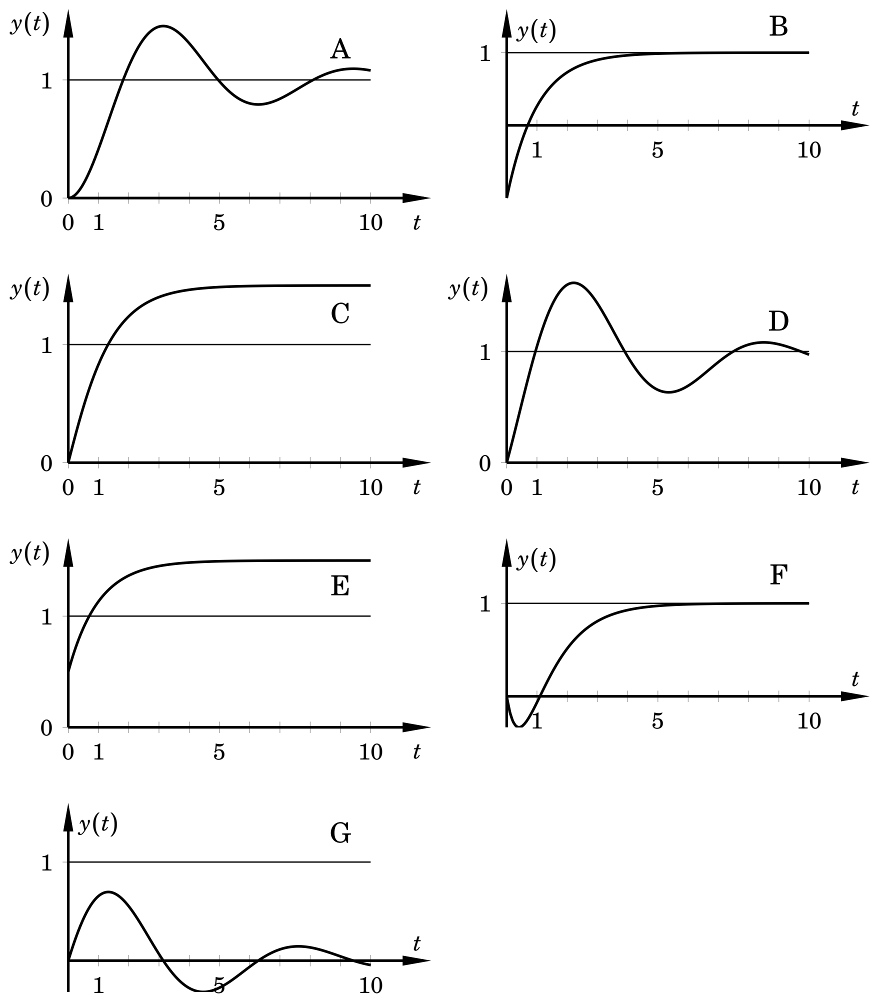
2.11 Determine which five of the following transfer functions correspond to the step responses A–E below.

$$G_1(s) = \frac{0.1}{s+0.1} \qquad G_2(s) = \frac{4}{s^2+2s+4}$$
$$G_3(s) = \frac{0.5}{s^2-0.1s+2} \qquad G_4(s) = \frac{-0.5}{s^2+0.1s+2}$$
$$G_5(s) = \frac{1}{s+1} \qquad G_6(s) = \frac{4}{s^2+0.8s+4}$$
$$G_7(s) = \frac{2}{s^2+s+3}$$

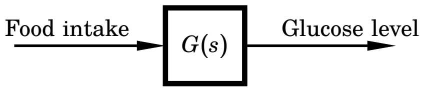


2.12 Pair each of the four pole-zero plots with the corresponding step responses A–G.





2.13 Glycemic index (GI) is a measure of how fast carbohydrates in food are processed by the body. To obtain the glycemic index, the system below is studied, where $G(s)$ is different for different types of carbohydrates.

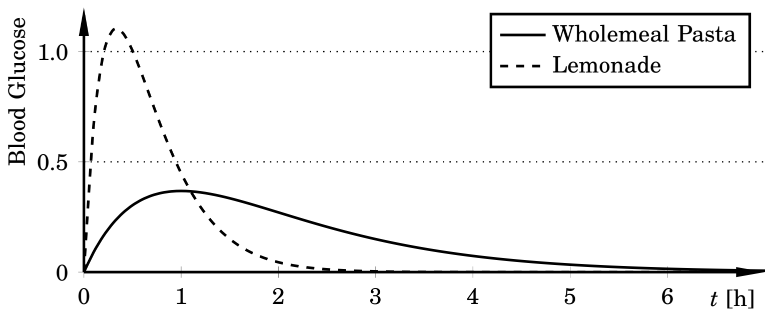


a. The figure below shows the impulse response from food intake to glucose level for two types of food: whole grain pasta with low GI (solid line) and lemonade with high GI (dashed line). Which of the following transfer functions may be used to model the uptake of whole grain pasta and lemonade, respectively?

$$G_1(s) = \frac{1}{s+1} \qquad G_2(s) = \frac{1}{s/3+1}$$

$$G_3(s) = \frac{1}{(s+1)^2} \qquad G_4(s) = \frac{1}{(s/3+1)^2}$$

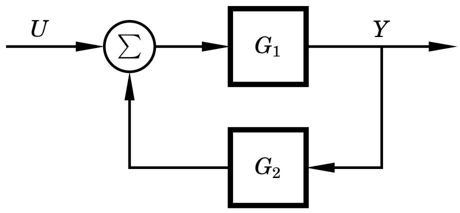
$$G_5(s) = \frac{1}{s(s+1)} \qquad G_6(s) = \frac{1}{s(s/3+1)}$$



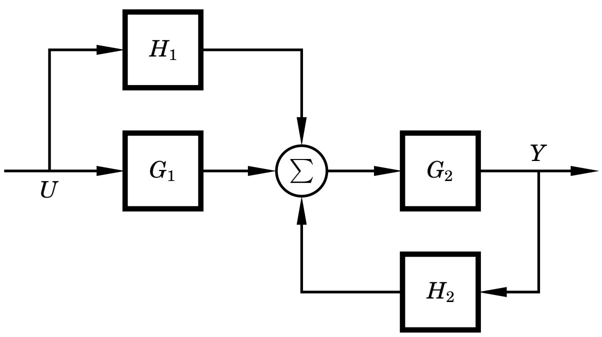
b. Why is it more relevant to look at the impulse response rather than the step response for this application?

2.14 Determine the transfer function from U to Y for the systems below.

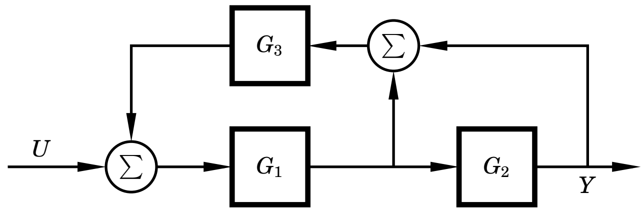
a.



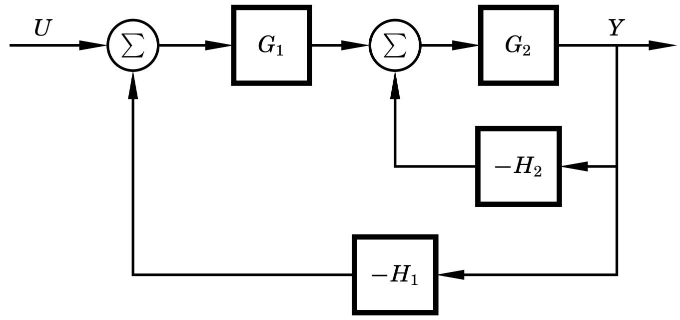
b.



c.



d.



2.15 The block diagram in Figure 2.1 describes temperature control in a room. The measurement signal y is the temperature in the room. The control signal u is the power of the radiators. The reference value r is the desired temperature. A controller $G_R(s)$ controls the power of the radiators based on the difference of the desired and measured temperature, e. The temperature of the room is also affected by the outdoor temperature, which may be seen as a disturbance, d.

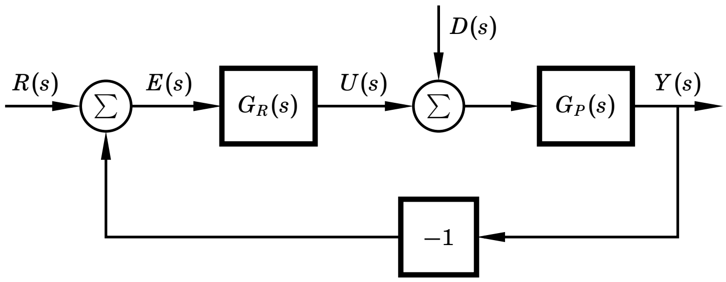


Figure 2.1

a. Determine the transfer function from $R(s)$ to $Y(s)$.

b. Determine the transfer function from $D(s)$ to $Y(s)$.

c. Determine the transfer function from $R(s)$ to $E(s)$.

d. Determine the transfer function from $D(s)$ to $U(s)$.

2.16 Consider the transfer function

$$G(s) = \frac{s^2 + 6s + 7}{s^2 + 5s + 6}$$

Write the system on

a. diagonal form,

b. controllable canonical form,

c. observable canonical form.

Exercises 3

Frequency Analysis

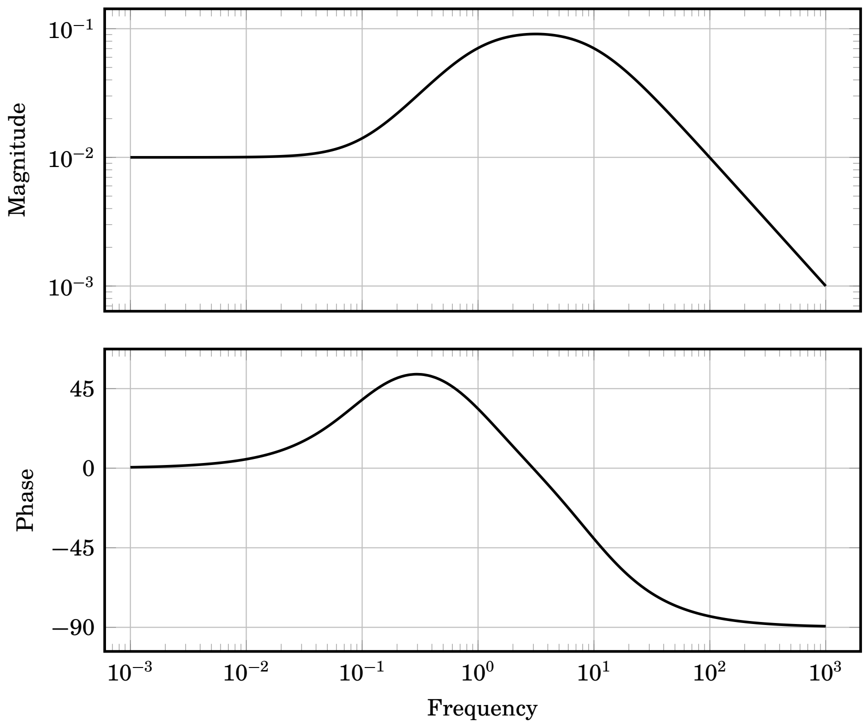
3.1 Assume that the system

$$G(s) = \frac{0.01(1+10s)}{(1+s)(1+0.1s)}$$

is subject to the input $u(t) = \sin 3t, \quad -\infty < t < \infty$

a. Determine the output $y(t)$.

b. The Bode plot of the system is shown below. Determine the output $y(t)$ by using the Bode plot instead.



3.2 We analyze the two systems in Figure 3.1; the sea water in Öresund and the water in a small garden pool. The input signal to the systems is the air temperature and the output is the water temperature.

a. Figure 3.2 shows two Bode diagrams. Which diagram corresponds to which system?

b. We assume that the air temperature has sinusoidal variations with a period time $T = 1$ year. The greatest temperature in the summer is 19°C and the lowest temperature in the winter is −5°C. What is the difference between the greatest and lowest sea water temperature over the year? Use the Bode diagram.

c. During a summer day we assume that the air temperature has sinusoidal variations with a period time $T = 1$ day. The greatest temperature of the day (at 13.00) is 27°C, and the lowest temperature (at 01.00) is 14°C. At what time during the day is the water in the garden pool the warmest?

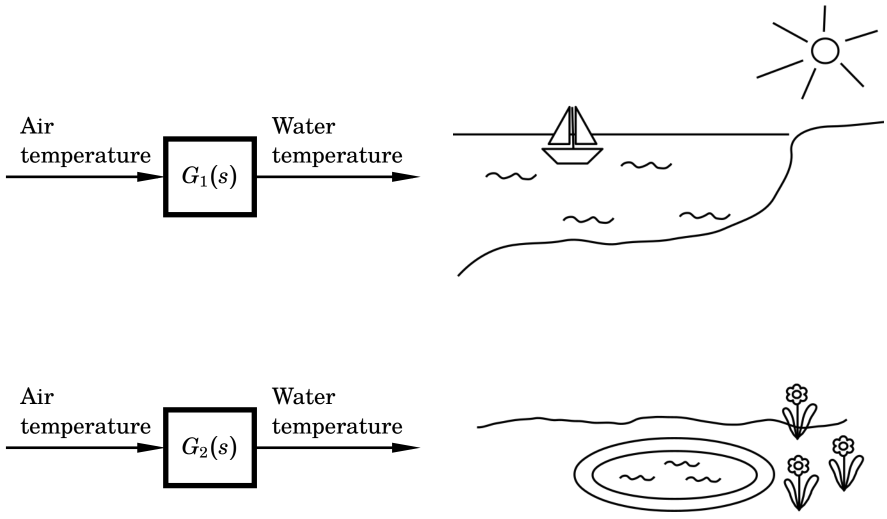


Figure 3.1

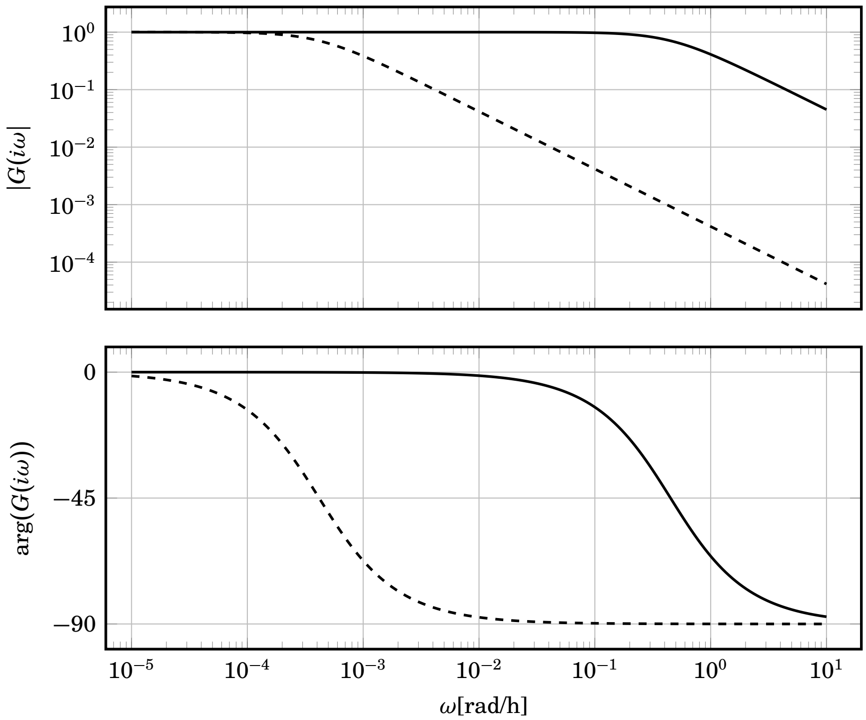


Figure 3.2 Bode diagram in problem 3.2.

3.3 Assume that the oscillating mass in assignment 1.3 has $m = 0.1$ kg, $c = 0.05$ Ns/cm and $k = 0.1$ N/cm. The transfer function is then given by

$$G(s) = \frac{10}{s^2 + 0.5s + 1}$$

a. Let the mass be subject to the force $f = \sin \omega t$, $-\infty < t < \infty$. Calculate the output for $\omega = 0.2$, 1 and 30 rad/s.

b. Instead, use the Bode plot of the system in Figure 3.3 to determine the output for $\omega = 0.2$, 1 and 30 rad/s.

3.4 Draw the Bode plots corresponding to the following transfer functions

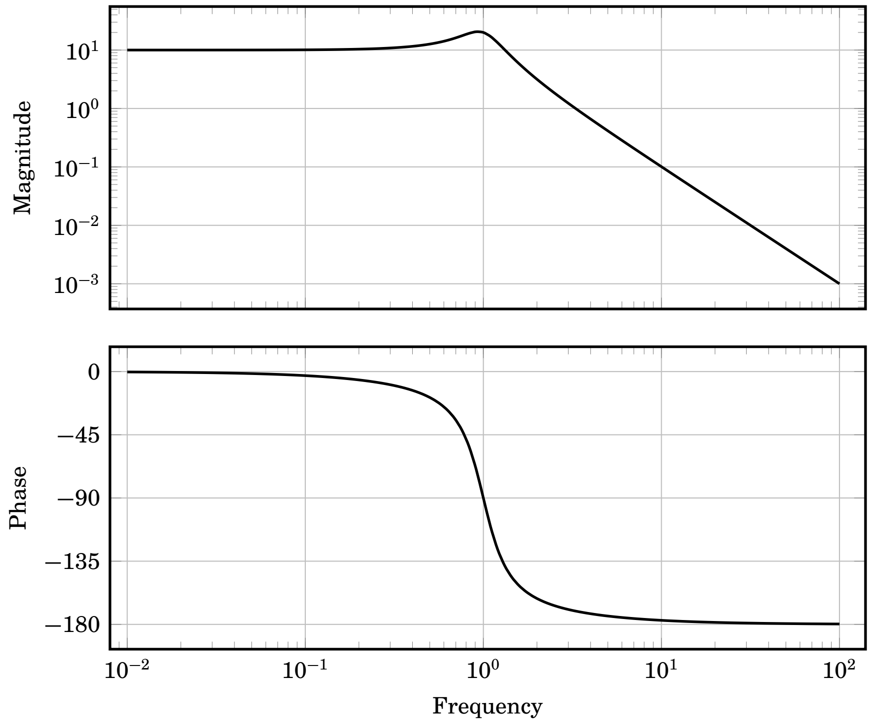


Figure 3.3 The Bode plot of the oscillating mass in assignment 3.3.

a.
$$G(s) = \frac{3}{1 + s/10}$$

b.
$$G(s) = \frac{10}{(1 + 10s)(1 + s)}$$

c.
$$G(s) = \frac{e^{-s}}{1 + s}$$

d.
$$G(s) = \frac{1 + s}{s(1 + s/10)}$$

e.
$$G(s) = \frac{2(1 + 5s)}{s(1 + 0.2s + 0.25s^2)}$$

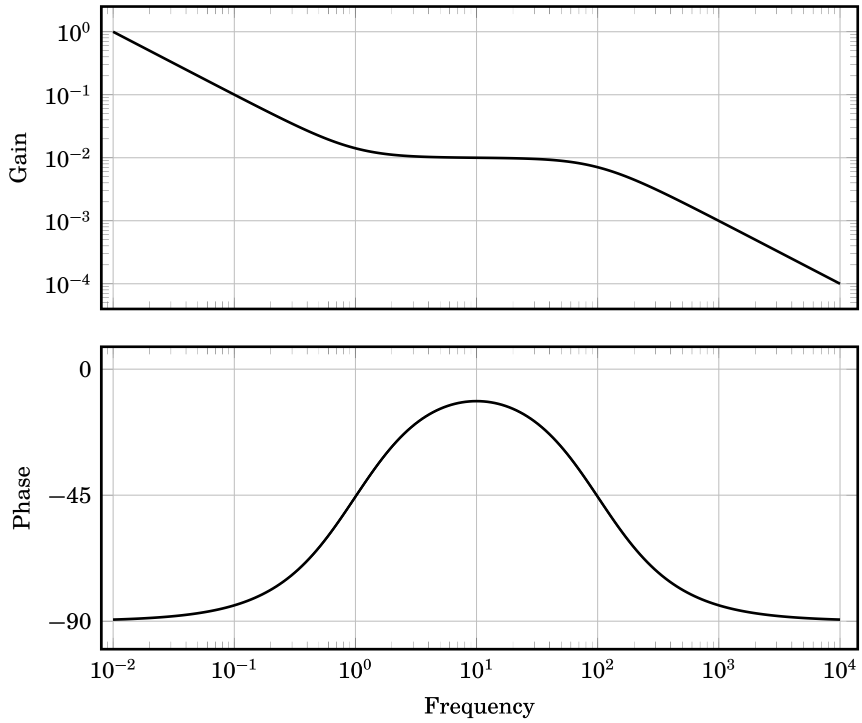
3.5 Exploit the results from the previous assignment in order to draw the Nyquist curves of

a.
$$G(s) = \frac{3}{1 + s/10}$$

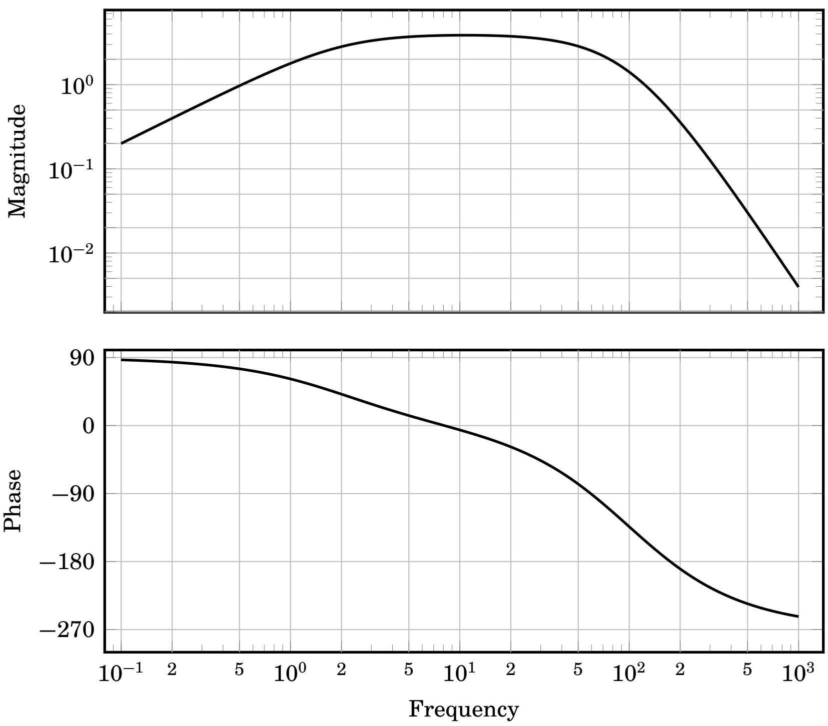
b.
$$G(s) = \frac{10}{(1 + 10s)(1 + s)}$$

c.
$$G(s) = \frac{e^{-s}}{1 + s}$$

3.6 The Bode plot below was obtained by means of frequency response experiments, in order to analyze the dynamics of a stable system. What is the transfer function of the system?



3.7 Measurements resulting in the Bode plot below have been conducted in order to analyze the dynamics of an unknown system. Use the Bode plot to determine the transfer function of the system.



Exercises 4

Feedback Systems

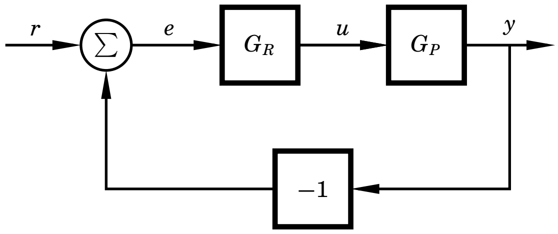
4.1 Assume that the air temperature y inside an oven is described by the differential equation

$$\dot{y}(t) + 0.01y(t) = 0.01u(t)$$

where u is the temperature of the heating element.

a. Let u be the input and y the output and determine the transfer function $G_P(s)$ of the oven.

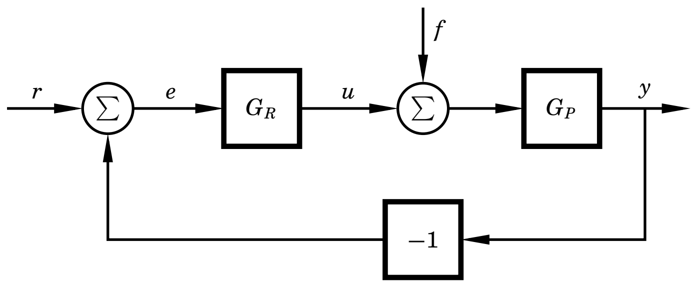
b. The oven is to be controlled by a P controller, $G_R(s) = K$, according to the block diagram below. Write down the transfer function of the closed loop system.



c. Choose K such that the closed loop system obtains the characteristic polynomial

$$s + 0.1$$

4.2 The below figure shows a block diagram of a hydraulic servo system in an automated lathe.



The measurement signal $y(t)$ represents the position of the tool head. The reference tool position is $r(t)$, and the shear force is denoted $f(t)$. G_R is the transfer function of the position sensor and signal amplifier, while G_P represents the dynamics of the tool mount and hydraulic piston

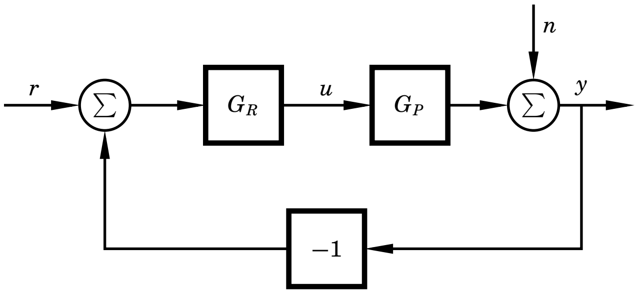
$$G_P(s) = \frac{1}{ms^2 + ds}$$

where m is the mass of the piston and tool mount, and d is the viscous damping of the tool mount. In the assignment it is assumed that $r(t) = 0$.

a. How large does the deviation $e(t) = r(t) - y(t)$ between the reference- and measured tool head position become in stationarity if the shear force $f(t)$ is a unit step? The controller is assumed to have a constant gain $G_R(s) = K$.

b. How is this error changed if the amplifier is replaced by a PI controller with transfer function $G_R(s) = K_1 + K_2/s$?

4.3 A process is controlled by a P controller according to the figure below. It is assumed that $r = 0$.

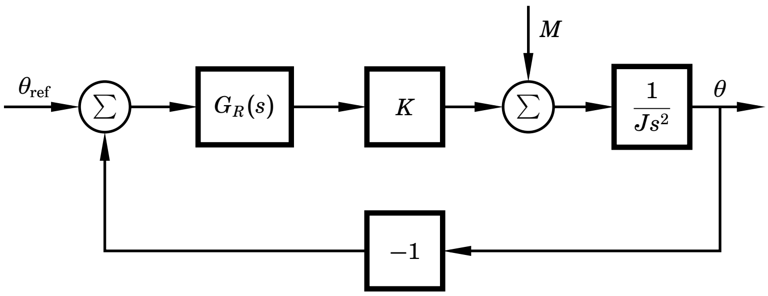


a. Measurements of the process output indicate a disturbance n. Calculate the transfer functions from n to y and n to u.

b. Let $G_P(s) = \frac{1}{s+1}$ and assume that the disturbance consists of a sinusoid $n(t) = A \sin \omega t$. What will u and y become, after the decay of transients?

c. Assume that $K = 1$ and $A = 1$ in the previous sub-assignment. Calculate the amplitude of oscillation in u and y for the cases $\omega = 0.1$ and 10 rad/s, respectively.

4.4 The below figure shows a block diagram of a gyro stabilized platform. It is controlled by an motor which exerts a momentum on the platform. The angular position of the platform is sensed by a gyroscope, which outputs a signal proportional to the platform's deviation from the reference value. The measurement signal is amplified by an amplifier with transfer function G_R.



It is desired that step changes in the reference θ_{ref} or the disturbance momentum M on the platform do not result in persisting angular errors. Give the *form* of the transfer function G_R, which guarantees that the above criteria hold. Hint: Postulate $G_R(s) = Q(s)/P(s)$

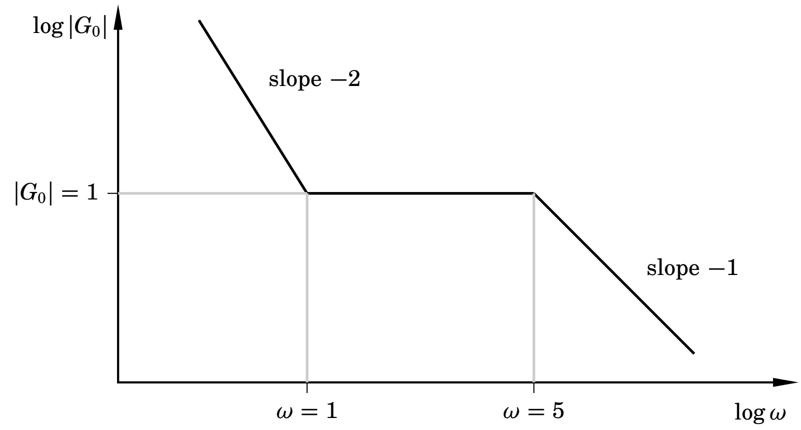
4.5 When heating a thermal bath, one can assume that the temperature increases linearly with 1°C/s. The temperature is measured by means of a thermocouple with transfer function

$$G(s) = \frac{1}{1 + sT}$$

with time constant $T = 10$ s.

After some initial oscillations, a stationary state, in the sense that the temperature measurement increases with constant rate, is reached. At a time instant, the temperature measurement reads 102.6°C. Calculate the actual temperature of the bath.

4.6 Consider the system $G_0(s)$ with the following asymptotic gain curve. Assume that the system lacks delays and right half plane zeros.



Further assume that the system is subject to negative feedback and that the closed loop system is stable. Which of the following setpoints can be tracked by the closed loop system, without a stationary error?

Assume $r(t) = 0$ for $t < 0$, and that the constants a, b and $c \neq 0$.

a. $r(t) = a$

b. $r(t) = bt$

c. $r(t) = ct^2$

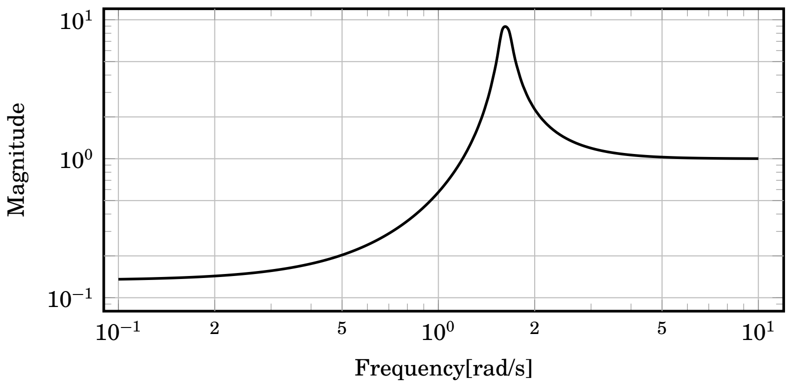
d. $r(t) = a + bt$

e. $r(t) = \sin(t)$

4.7 In a simple control circuit, the process and controller are given by $G_P(s) = \dfrac{1}{(s+1)^3}$ and $G_R(s) = 6.5$, respectively.

a. Determine the sensitivity function $S(s)$.

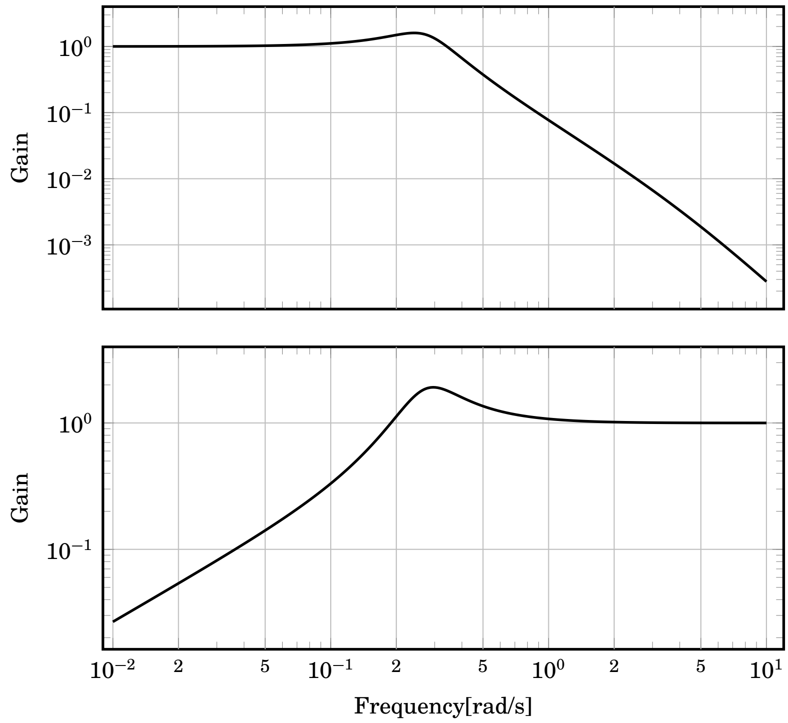
The gain plot of the sensitivity function is given below.



b. How much are low-frequency load disturbances damped by the control circuit in closed loop, as compared to open loop?

c. At which angular frequency does the control circuit exhibit the largest sensitivity towards disturbances and by how much are disturbances amplified at most?

4.8 The below figure shows the gain curves of the sensitivity function S and complementary sensitivity function T for a normal control circuit.



a. Determine which curve corresponds to the sensitivity function and complementary sensitivity function, respectively.

b. Give the frequency range where disturbances are amplified by the feedback loop, and the frequency range where they are damped by the feedback loop. What is the maximum gain of disturbance amplification?

c. Give the frequency ranges where the output exhibits good tracking of the reference signal.

d. What is the minimal distance between the Nyquist curve of the open loop system and the point -1 in the complex plane? What does this say about the gain margin?

4.9 In a simple control loop, the open loop transfer function is given by

$$G_o(s) = G_R(s)G_P(s) = \frac{K}{s(s+2)}$$

Draw the root locus of the characteristic equation of the closed loop system, with respect to the gain parameter K.

4.10 A simple control loop has the open loop transfer function

$$G_o(s) = G_R(s)G_P(s) = \frac{K(s+10)(s+11)}{s(s+1)(s+2)}$$

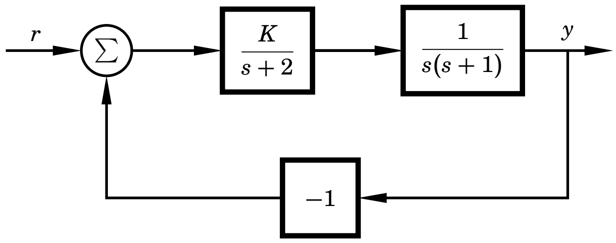
a. minor tick outside major Which values of K yield a stable closed loop system?

b. Sketch the characteristics of the root locus.

4.11 The figure below shows the block diagram of a printer.

a. Which values of the gain K yield an asymptotically stable system?

b. The goal is to track a reference which increases linearly with rate 0.1 V/s, and guarantee a stationary error of less than 5 mV. Can this be achieved by adequate tuning of the gain K?



4.12 Consider the Nyquist curves in Figure 4.1. Assume that the corresponding systems are controlled by the P controller

$$u = K(r - y)$$

In all cases the open loop systems lack poles in the right half plane. Which values of K yield a stable closed loop system?

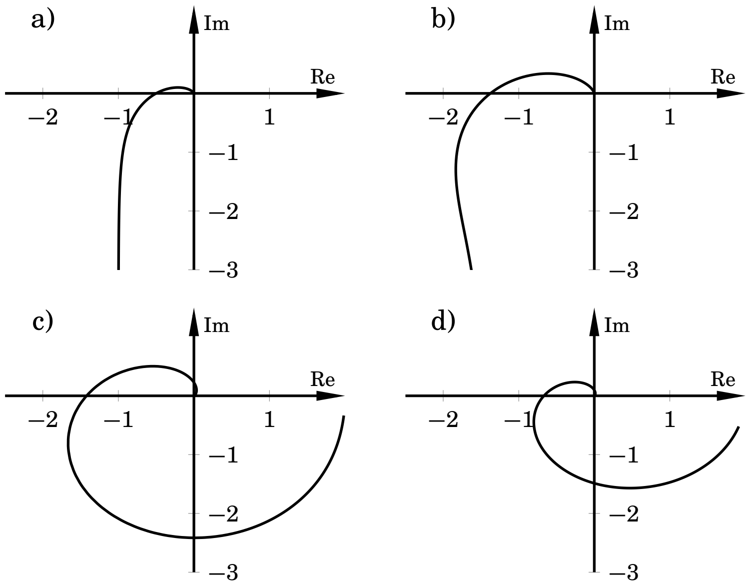


Figure 4.1 Nyquist curves in assignment 4.12.

4.13 The transfer function of a process is given by

$$G_p(s) = \frac{1}{(s+1)^3}$$

The loop is closed through proportional feedback

$$u = K(r - y)$$

Use the Nyquist criterion to find the critical value of the gain K (i.e. the value for which the system transits from stability to instability).

4.14 The Nyquist curve of a system is given in Figure 4.2. The system is stable, i.e. lacks poles in the right half plane.

Assume that the system is subject to proportional feedback

$$u = K(r - y)$$

Which values of the gain K result in a stable closed loop system?

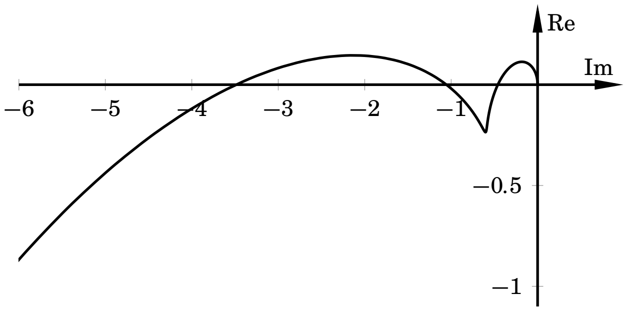
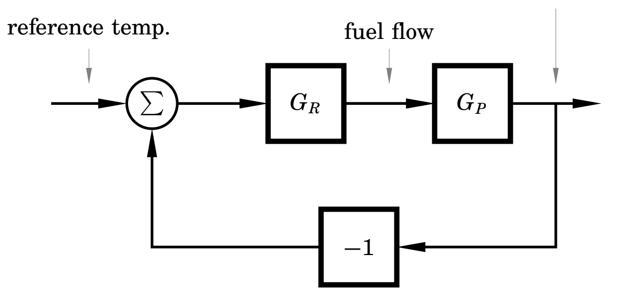


Figure 4.2 Nyquist curve of the system in assignment 4.14.

4.15 In order to obtain constant product quality in a cement kiln, it is crucial that the burn zone temperature is held constant. This is achieved by measuring the burn zone temperature and controlling the fuel flow with a proportional controller. A block diagram of the system is shown below.



Find the maximal value of the controller gain K, such that the closed loop system remains stable? The transfer function from fuel flow to burn zone temperature is given by

$$G_P(s) = \frac{e^{-9s}}{(1+20s)^2}$$

4.16 In a distillation column, the transfer function from supplied energy to liquid phase concentration of a volatile component is

$$G_P(s) = \frac{e^{-sL}}{1+10s}$$

where time is measured in minutes. The process is controlled by a PI controller with transfer function

$$G_R(s) = 10\left(1 + \frac{1}{2s}\right)$$

What is the maximal permitted transportation delay L, yielding at least a 10° phase margin?

4.17 A process with transfer function $G_P(s)$ is subject to feedback according to Figure 4.3.

All poles of $G_P(s)$ lie in the left half plane and the Nyquist curve of G_P is shown in Figure 4.4. It is assumed that $G_P(i\omega)$ does not cross the real axis at other points than shown in the figure.

Which of the below alternatives are true? Motivate!

a. The gain margin $A_m < 2$ for $K = 1$.

b. The phase margin $\varphi_m < 45^\circ$ for $K = 1$.

c. The phase margin decreases with decreasing gain K.

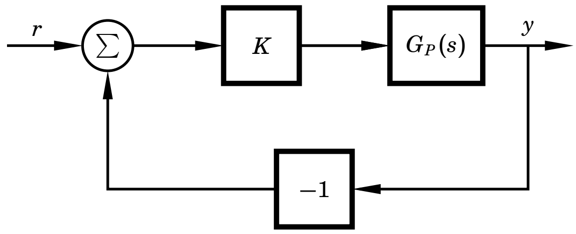


Figure 4.3 The closed loop system in assignment 4.17.

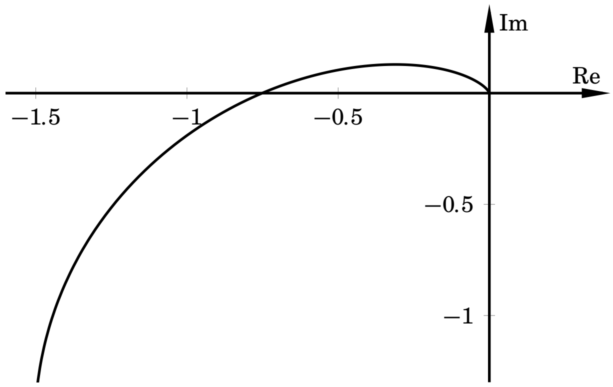


Figure 4.4 Nyquist curve of the process $G_P(s)$ in assignment 4.17.

d. For $K = 2$ the closed loop system becomes unstable.

4.18 The Bode plot of the open loop transfer function, $G_o = G_R G_P$, is shown in Figure 4.5. Assume that the system is subject to negative feedback.

a. How much can the the gain of the controller or process be increased without making the closed loop system unstable?

b. How much additional negative phase shift can be introduced at the cross-over frequency without making the closed loop system unstable?

4.19 A Bode plot of the open loop transfer function of the controlled lower tank in the double tank process is shown in Figure 4.6. What is the delay margin of the system?

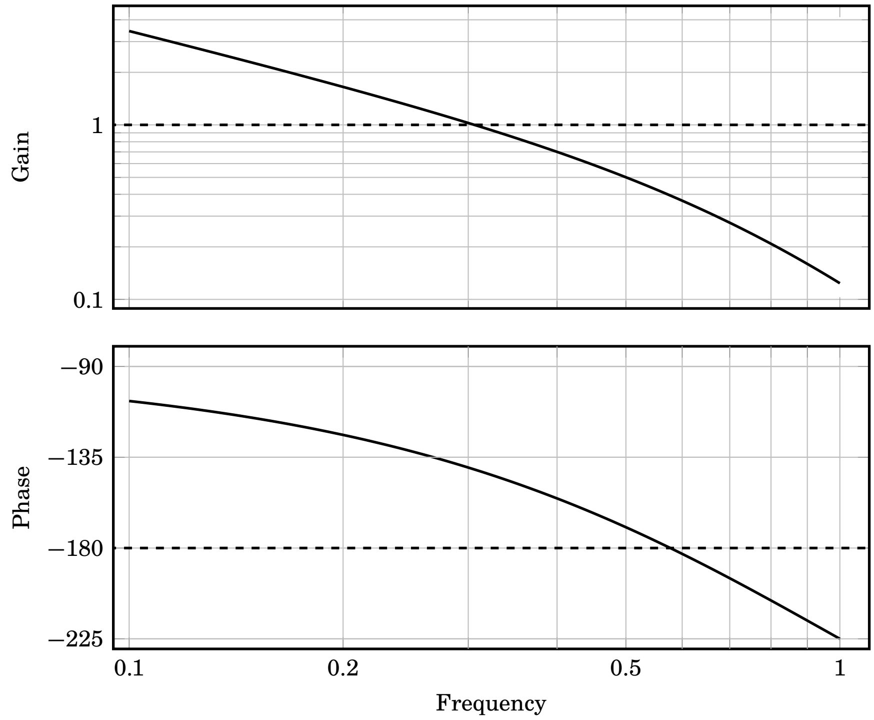


Figure 4.5 Bode plot of the open loop system in Figure 4.18.

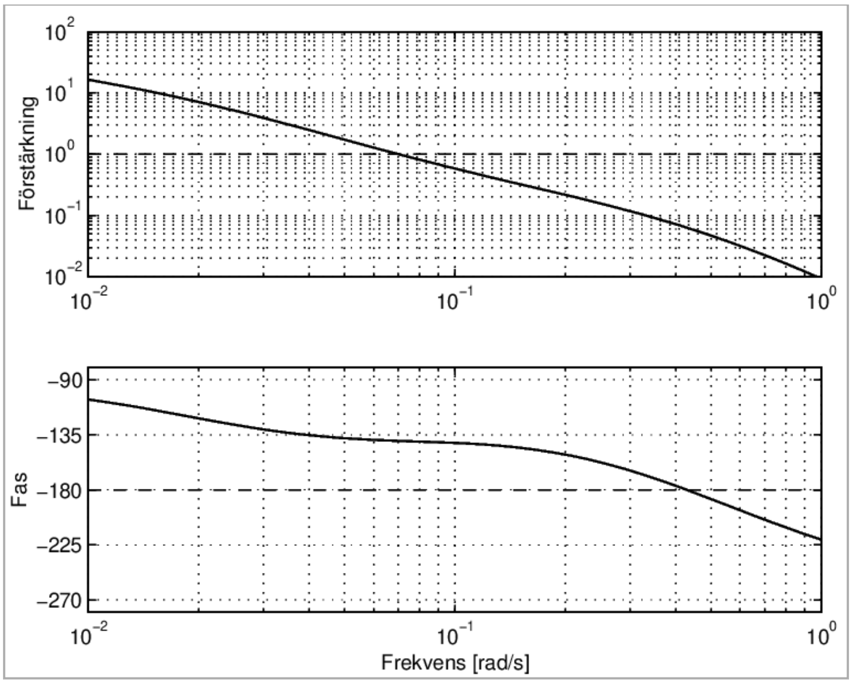


Figure 4.6 Bode plot of the open loop transfer function of the controlled lower tank in the double tank process in problem 4.19.

Exercises 5

State Feedback and Kalman Filtering

5.1 A linear system is described by the matrices

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} \beta \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & \gamma \end{pmatrix} \qquad D = 0$$

a. For which values of β is the system controllable?

b. For which values of γ is the system observable?

5.2 A linear system is described by the matrices

$$A = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

Find the set of controllable states.

5.3 Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

Is it observable? If not, find the set of unobservable states.

5.4 Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Which of the states $\begin{pmatrix} 3 & 0.5 \end{pmatrix}^T$, $\begin{pmatrix} 5 & 5 \end{pmatrix}^T$, $\begin{pmatrix} 0 & 0 \end{pmatrix}^T$, $\begin{pmatrix} 10 & 0.1 \end{pmatrix}^T$ or $\begin{pmatrix} 1 & -0.5 \end{pmatrix}^T$ can be reached in finite time?

5.5 Consider the following system:

$$\frac{dx}{dt} = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 3 & 7 \end{pmatrix} x$$

Is it controllable?

5.6 A dynamic system is described by the state space model below

$$\dot{x} = \begin{pmatrix} -2 & 2 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 5 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

a. Is the system controllable? Which states can be reached in finite time from the initial state $x(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$?

b. Calculate the transfer function of the system.

c. Can the same input-output relation be described with fewer states? Write down such a representation, if possible.

5.7 A linear dynamical system with transfer function $G(s)$ is given. The system is controllable. Which of the following statements are unquestionably true?

a. The poles of the closed loop system's transfer function can be arbitrarily placed by means of feedback from all states.

b. The zeros of the closed loop system's transfer function can be arbitrarily placed by means of feedback from all states.

c. If the state variables are not available for measurements, they can always be estimated by diffrentiating the system output.

d. If the state vector is estimated by a Kalman filter

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

one can obtain an arbitrarily fast convergence of the estimate $\hat{x}$ towards the actual state vector x, by choice of the matrix L.

5.8 Determine a control law $u = k_r r - Kx$ for the system

$$\frac{dx}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

such that the poles of the closed loop system are placed in -4 and the stationary gain is 1.

5.9 The position of a hard drive head is described by the state space model

$$\frac{dx}{dt} = \begin{pmatrix} -0.5 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 3 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

a. Determine a state feedback

$$u = -Kx + k_r r$$

which places the poles of the closed loop system in $s = -4 \pm 4i$ and results in static gain 1 from reference to output.

b. Determine a Kalman filter

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

for the system. Briefly motivate necessary design choices.

5.10 Figure 5.1 shows the lunar lander LEM of the Apollo project. We will study a possible system for controlling its horizontal movement above the moon surface.

Assume that the lander floats some distance above the moon surface by means of the rocket engine. If the angle of attack (the angle of the craft in relation to the normal of the moon surface) is nonzero, a horizontal force component appears, yielding an acceleration along the moon surface.

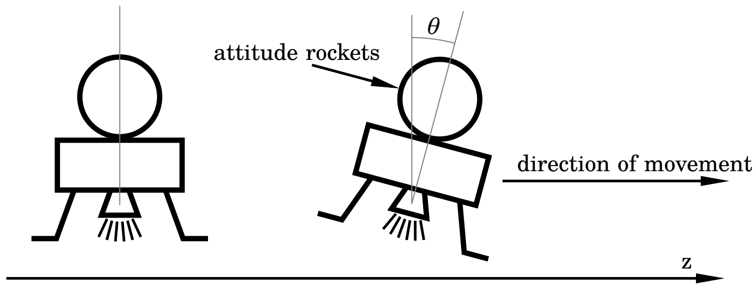


Figure 5.1 The lunar lander in assignment 5.10.

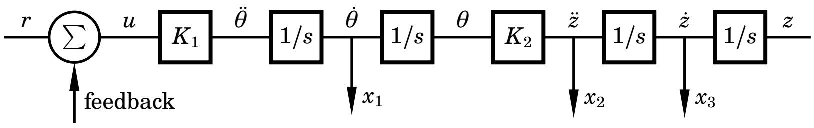


Figure 5.2 Block diagram of the lander dynamics along the z-axis.

Study the block diagram in Figure 5.2 showing the relation between the control signal u of the rocket engine, the angle of attack, θ, and the position z.

The craft obeys Newton's law of motion in both the θ and z directions. The transfer function from the astronaut's control signal u to the position z is

$$G_z(s) = \frac{l_1 l_2}{s^4}$$

and it is quite impossible to manually maneuver the craft. To facilitate the astronaut's maneuvering task, we alter the craft dynamics by introducing internal feedback loops. This means that the astronaut's control lever is not directly connected to the motors, but works as a joystick that decides the desired velocity of the craft.

A controller should then convert the movement of the control lever to a control signal to the steer rockets. We are in possession the following measurement signals:

- The time derivative of the attack angle, $\dot{\theta}$, measured by a rate gyro.
- The acceleration in the z direction, $\ddot{z}$, measured by accelerometers mounted on a gyro-stabilized platform.
- The speed in the z direction, $\dot{z}$, measured by Doppler radar.

a. Introduce the states

$$x_1 = \dot{\theta}$$
$$x_2 = \ddot{z}$$
$$x_3 = \dot{z}$$

and write the system on state-space form. Let the velocity in the z direction be the output signal of the system.

b. Determine a feedback controller which utilizes the three measurements, and results in a closed-loop system with three poles in $s = -0.5$, and lets the control signal of the astronaut be the *speed* reference in the z direction. You don't have to calculate the gain k_r.

5.11 A conventional state feedback law does note guarantee integral action. The following procedure is a way of introducing integral action. Let the nominal system be

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

Augment the state vector with an extra component

$$x_{n+1} = \int^t e(s)\,ds = \int^t (r(s) - y(s))\,ds$$

The obtained system is described by

$$\frac{dx_e}{dt} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix} x_e + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

where

$$x_e = \begin{pmatrix} x \\ x_{n+1} \end{pmatrix}$$

A state feedback law for this system results in a control law of the form

$$u = -Kx - k_{n+1}x_{n+1} = -K_e x_e$$

This controller, which steers y towards r, obviously has integral action. Use this methodology in order to determine a state feedback controller with integral action for the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

such that the closed loop system obtains the characteristic polynomial

$$(s+\alpha)\left(s^2 + 2\zeta\omega s + \omega^2\right) = 0$$

5.12 Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

One wishes to estimate the state variables by means of the model

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

Determine L such that the poles of the Kalman filter are placed in $s = -4$.

5.13 Consider the dynamical system

$$\frac{dx}{dt} = \begin{pmatrix} -4 & -3 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 3 \end{pmatrix} x$$

One desires a closed loop system with all poles in -4.

a. Assign feedback gains to all states such that the closed loop system obtains the desired feature.

b. Assume that only the output y is available for measurement. In order to use state feedback, the state x must be first be estimated by means of e.g. a Kalman filter, yielding the estimate $\hat{x}$. Subsequently, the control law $u = -K\hat{x}$ can be applied.

Is it possible to determine a Kalman filter for which the estimation error decreases according to the characteristic polynomial $(s+6)^2$?

c. Is it possible to determine a Kalman filter for which the estimation error decreases according to the characteristic polynomial $(s+3)^2$?

Briefly comment the obtained results.

Exercises 6

Design methods

6.1 A PID controller has the transfer function

$$G_R(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right)$$

a. Determine the gain and phase shift of the controller at an arbitrary frequency ω.

b. At which frequency does the controller have its minimal gain? What is the gain and phase shift for this frequency?

6.2 The process

$$G(s) = \frac{1}{(s+1)^3}$$

is controlled by a PID controller with $K = 2$, $T_i = 2$ and $T_d = 0.5$. In order to investigate the effect of changing the PID parameters, we will change K, T_I and T_d by a certain factor, one at a time. We will observe how this affects both the step response (from reference and load disturbance) and the Bode plot of the controlled open loop system. The reference is a unit step at $t = 0$ whereas the load disturbance is a negative unit step.

a. We start by studying what happens when the parameters are quadrupled, one at a time. Figure 6.1 shows the nominal case $(K, T_i, T_d) = (2, 2, 0.5)$ (solid black curves) together with the cases $(8, 2, 0.5)$, $(2, 8, 0.5)$ and $(2, 2, 2)$. Pair the three Bode plots and the step responses of Figure 6.1 with the three cases.

b. We now study what happens when each parameter is decreased by a factor 2. The nominal case $(K, T_i, T_d) = (2, 2, 0.5)$ (solid curves) is shown in Figure 6.2 together with the cases $(1, 2, 0.5)$, $(2, 1, 0.5)$ and $(2, 2, 0.25)$. Pair the three Bode plots and the three step responses in Figure 6.2 with these three cases.

6.3 The steer dynamics of a ship are approximately described by

$$J\frac{dr}{dt} + Dr = C\delta$$

where r is the yaw rate [rad/s] and δ is the rudder angle [rad]. Further, J [kgm^2] is the momentum of inertia wrt the yaw axis of the boat, D [Nms] is the damping constant and C [Nm/rad] is a constant describing the rudder efficiency. Let the rudder angle δ be the control signal. Give a PI controller for control of the yaw rate, such that the closed loop system obtains the characteristic equation

$$s^2 + 2\zeta\omega s + \omega^2 = 0$$

6.4 An electric motor can approximately be described by the differential equation

$$J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} = k_i I$$

where J is the moment of inertia, D is a damping constant and k_i is the current constant of the motor. Further, θ denotes the turning angle and I the current through the motor. Let θ be the measurement signal and I the control signal.

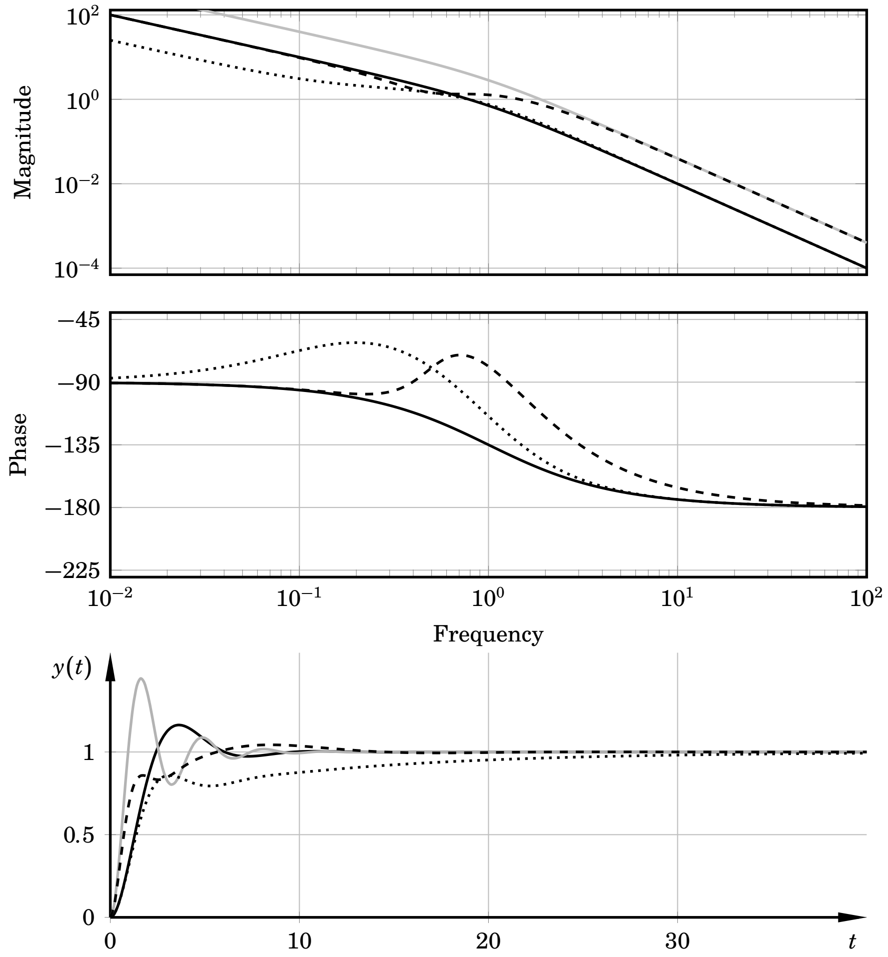


Figure 6.1 Bode plot and step response for the case when the PID parameters in subassignment 6.2b have been multiplied by four. The solid curves correspond to the nominal case.

Determine the parameters of a PID controller such that the closed loop system obtains the characteristic equation

$$(s + a)(s^2 + 2\zeta\omega s + \omega^2) = 0$$

Discuss how the parameters of the controller depend on the desired specifications on a, ζ and ω.

6.5 **a.** Draw the Bode plot of a PI controller (let $K = 1$ and $T_i = 1$).

b. Draw the Bode plot of a PD controller (let $K = 1$ and $T_d = 15$).

6.6 A cement kiln consists of a long, inclined, rotating cylinder. Sediment is supplied into its upper end and clinkers emerge from its lower end. The cylinder is heated from beneath by an oil burner. It is essential that the combustion zone temperature is kept constant, in order to obtain an even product quality. This is achieved by measuring the combustion zone temperature and controlling the fuel flow with a PI controller. A block diagram of the system is shown in Figure 6.3.

The transfer function from fuel flow to combustion zone temperature is given by

$$G_P(s) = \frac{e^{-9s}}{(1 + 20s)^2}$$

and the transfer function of the controller is

$$G_R(s) = K(1 + \frac{1}{sT_i})$$

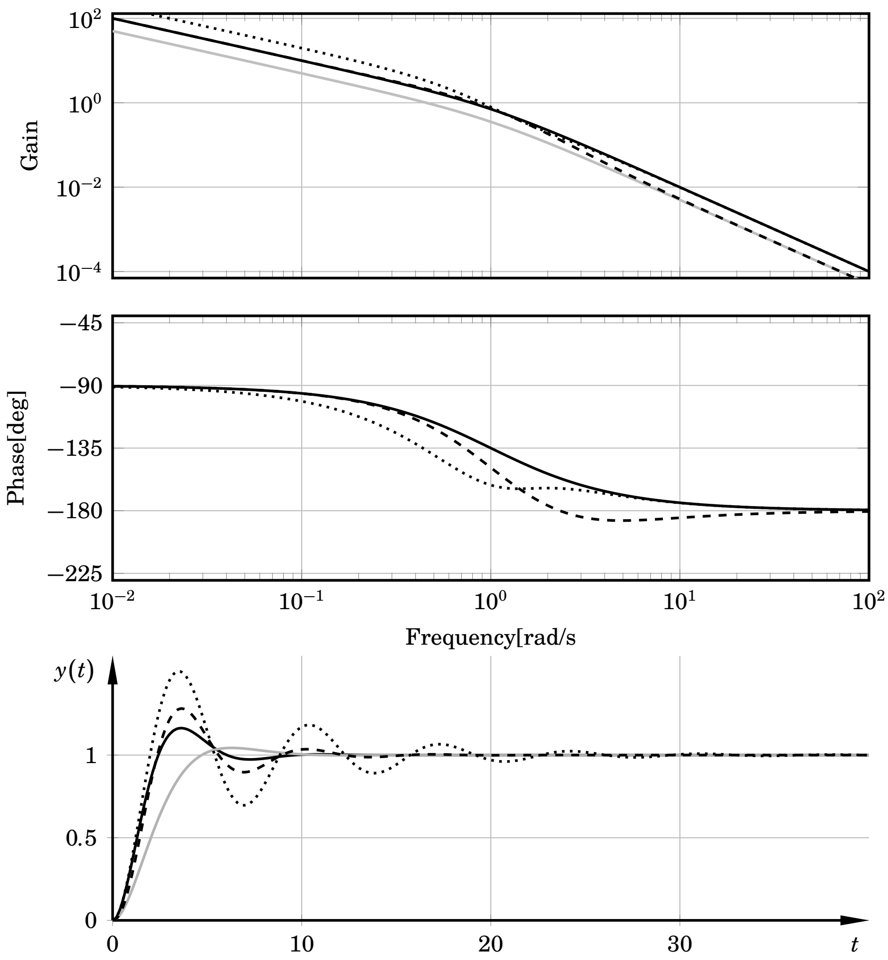


Figure 6.2 Bode plot and step response for the case when the PID parameters in subassignment 6.2b have been divided by two. The solid black curves correspond to the nominal case.

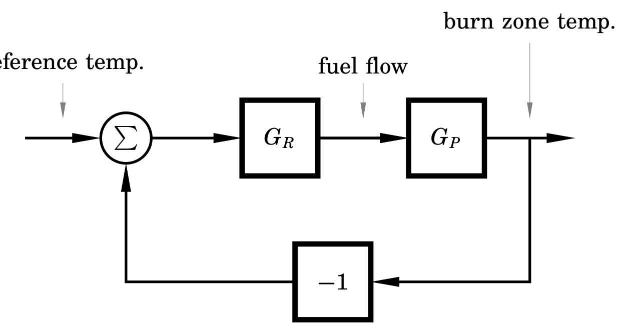


Figure 6.3 Block diagram of a cement kiln with temperature controller.

Use Ziegler-Nichol's frequency method to determine the parameters of the controller.

6.7 Martin has heard that the optimal effect from training is obtained when the pulse is 160 beats per minute (bpm). By feeding back the signal from his heart rate monitor to a treadmill, he wants to control the speed such that the pulse is exactly at the optimal value.

a. Suppose the dynamics in Martins body can approximately be described by the

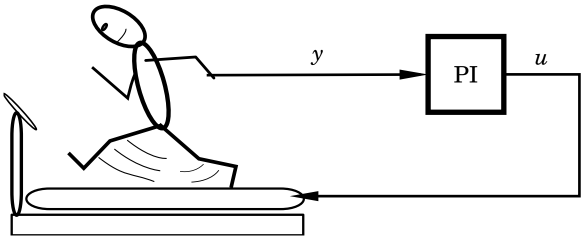


Figure 6.4 Treadmill

linear system

$$\begin{aligned} \dot{x} &= -\frac{1}{30}x + \frac{1}{15}u \\ y &= x \end{aligned} \tag{6.1}$$

where u is the speed of the treadmill, and x is the pulse in bpm. Design a PI controller such that both closed loop system poles are in -0.1.

b. The model in (6.1) is not available to Martin. Thus, he decides to tune his PI controller using Ziegler-Nichols frequency method. The Bode diagram for a more accurate model is shown in Figure 6.5. What controller parameters does Martin obtain? Use the Bode diagram.

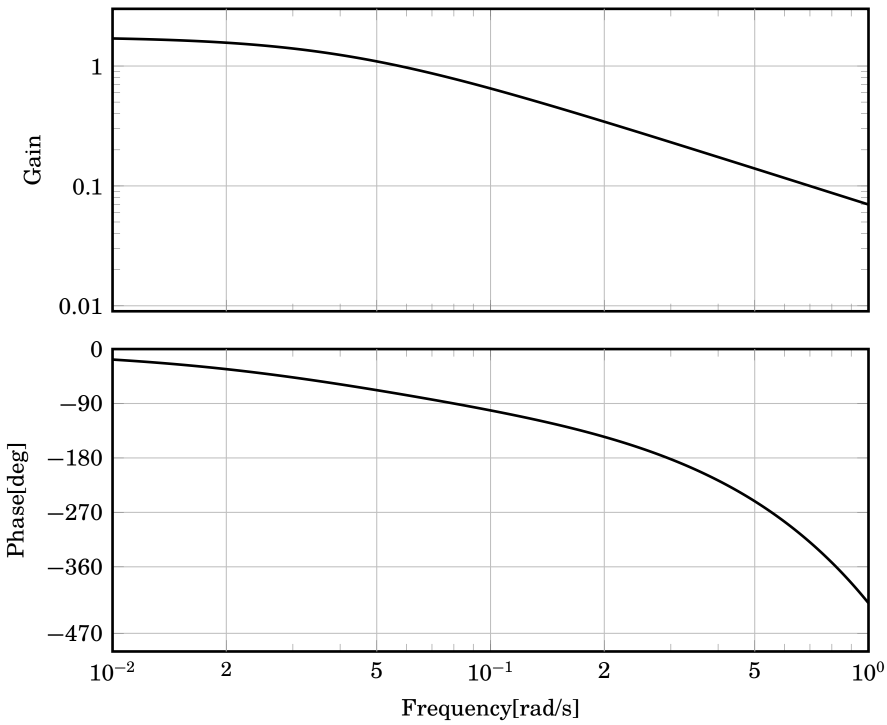


Figure 6.5 Bode diagram for assignment 6.7

6.8 Use Ziegler-Nichol's step response and frequency method, to determine the parameters of a PID controller for a system with the step response and Nyquist curve given in Figure 6.6. Also, determine a PI and a PID controller using the Lambda method with $\lambda = T$.

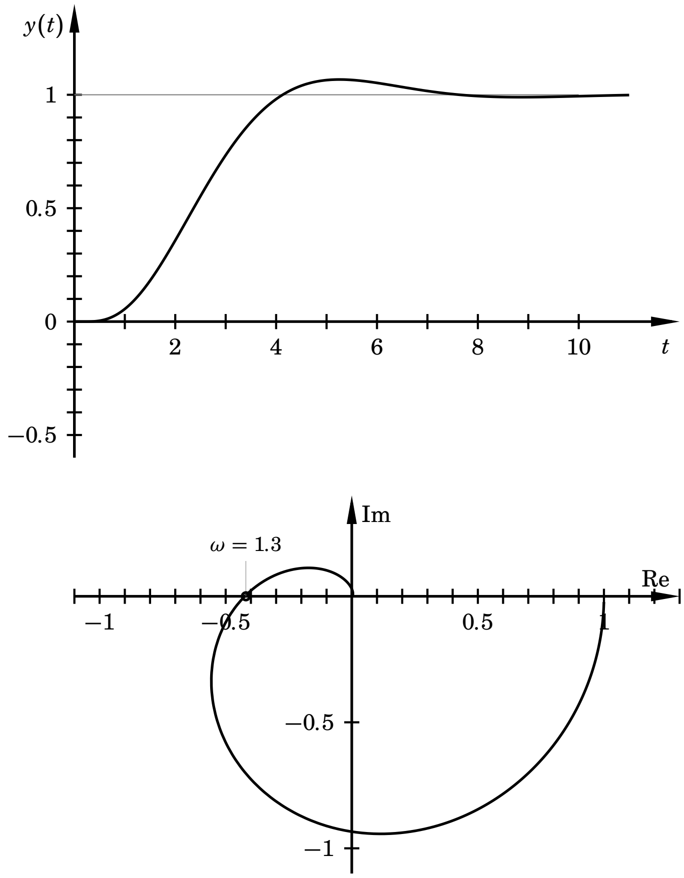


Figure 6.6 Step response and Nyquist curve of the system in assignment 6.8.

6.9 Consider a system with the transfer function

$$G(s) = \frac{1}{s+1} e^{-s}$$

a. Draw the step response of the system and use Ziegler-Nichol's step response method to determine the parameters of a PID controller. Write down the values of the obtained controller parameters K, T_i and T_d.

b. Use Ziegler-Nichol's frequency method to determine the parameters of a PID controller.

c. Use the Lambda method with $\lambda = T$ to determine the parameters of a PID controller.

6.10 A process is to be controlled by a PID controller obtained through Ziegler-Nichol's methods.

a. Use the step response method for the process with the black step response curve in Figure 6.7.

b. The Nyquist curve of the same system is shown in Figure 6.8. The point marked 'o' corresponds to the frequency $\omega = 0.429$ rad/s. Apply the frequency method to the process.

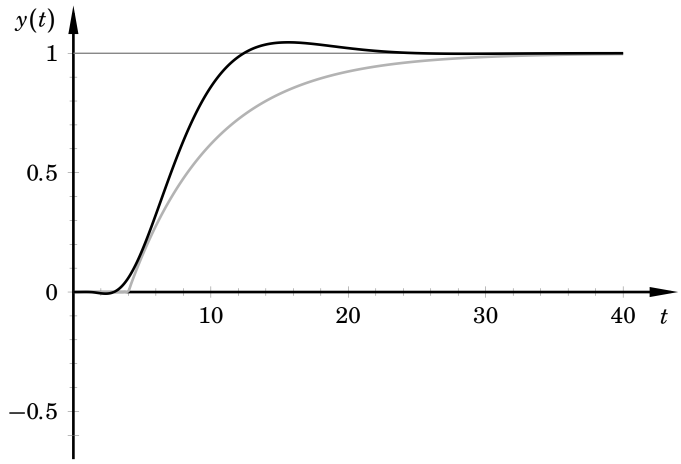


Figure 6.7 Step response for the process in Problem 6.10

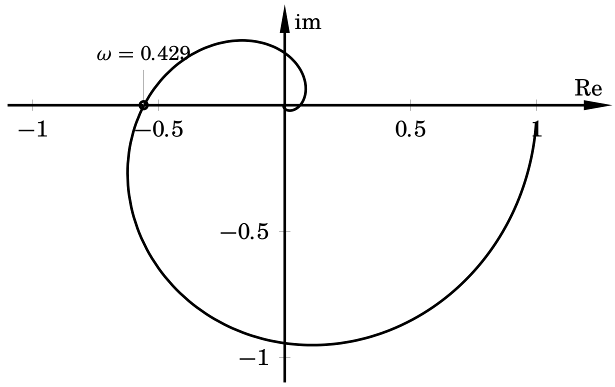


Figure 6.8 Nyquist curve for the process in Problem 6.10

c. Unfortunately the step response method results in an unstable closed loop system. The frequency method yields a stable but poorly damped system. The reason why the step response method works so badly, is that it tries to approximate the process with a delayed first order system (the gray step response in Figure 6.7). By exploiting the Nyquist curve, one can obtain PID parameters yielding the solid curve step response in Figure 6.9. The dashed and dotted curves were obtained through the step response method.

How do you think K has changed in the third method, as compared to the Ziegler-Nichol's methods (increase or decrease)?

6.11 A second order system has the Bode plot shown in Figure 6.10. We would like to connect a link G_K in series with the system, in order to increase the speed of the closed loop system. The cross-over frequency, ω_c, (the angle for which $|G_0| = 1$) is used as a measure of the system's speed. Which of the following G_K-candidates yield a faster system?

A $G_K = K, \ K > 1$

B $G_K = \dfrac{1}{s+1}$

C $G_K = \dfrac{s+1}{s+2}$

D $G_K = e^{-sL}, \ L > 0$

6.12 A system has the transfer function

$$G_P(s) = \frac{1}{s(s+1)(s+2)}$$

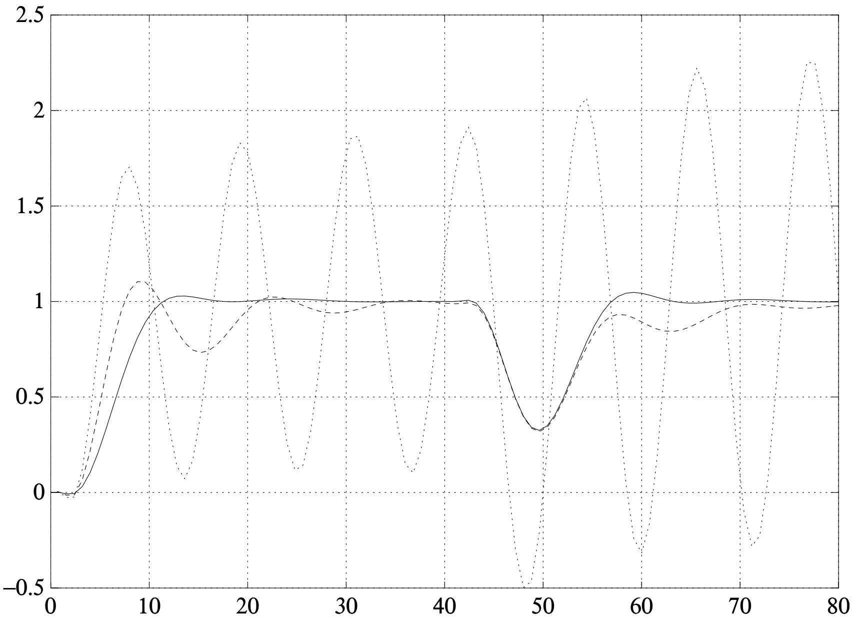


Figure 6.9

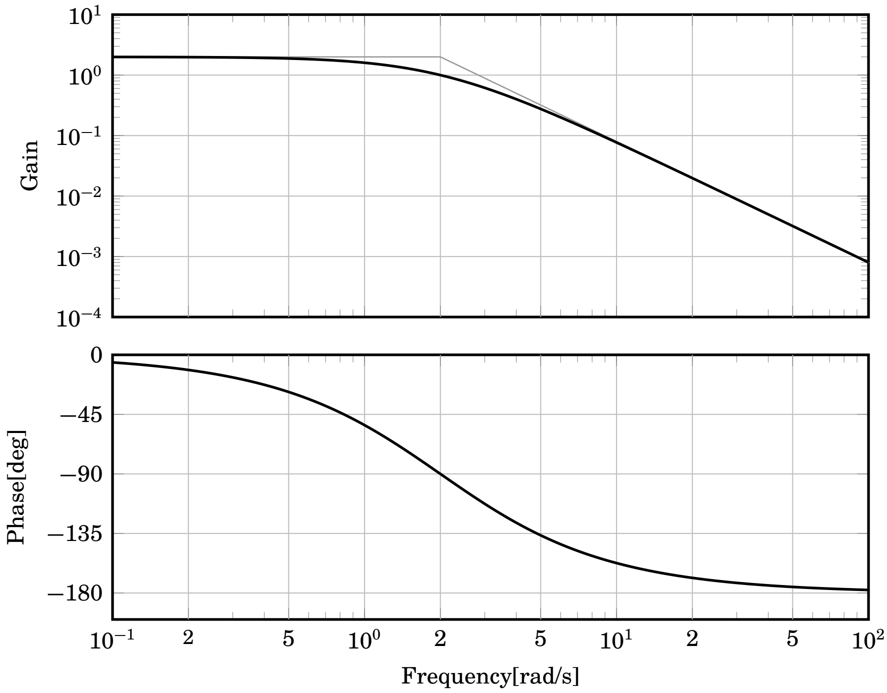


Figure 6.10 Bode plot of the system in assignment 6.11.

The system is part of a feedback loop together with a proportional controller with gain $K = 1$. The control error of the resulting closed loop system exhibits the following behavior: $e(t) \to 0$, $t \to \infty$ when the setpoint is a step and $e(t) \to 2$, $t \to \infty$ when the setpoint is a ramp.

Design a compensation link $G_k(s)$ which together with the proportional controller decreases the ramp error to a value less than 0.2. Also, the phase margin must not decrease by more than 6°.

6.13 Consider a system with the following transfer function

$$G_P(s) = \frac{1.1}{s(s+1)}$$

A proportional controller with gain $K = 1$ is used to close the loop. However, the closed loop system becomes too slow. Design a compensation link, $G_k(s)$, that roughly doubles the speed of the closed loop system without any change in robustness, i.e. the crossover frequency ω_c should be doubled and the phase margin φ_m should not decrease.

6.14 Consider the system

$$G_1(s) = \frac{1}{s(s+1)(s+2)}$$

If controlled by a proportional controller with gain $K = 1$, the stationary error of the closed loop system is $e = 0$ for a step input ($r = 1$, $t > 0$) and $e = 2$ for a ramp input ($r = t$, $t > 0$). One wants to increase the speed of the system by a factor 3, without compromising its phase margin or the ability to eliminate stationary errors. Device a compensation link $G_k(s)$ that fulfils the above criteria.

6.15 A servo system has the open loop transfer function

$$G_o(s) = \frac{2.0}{s(s+0.5)(s+3)}$$

The system is subject to simple negative feedback and has a step response according to Figure 6.11. As seen from the figure, the system is poorly damped and has a significant overshoot. The speed however, is satisfactory. The stationary error of the closed loop system with a ramp input is $e_1 = 0.75$.

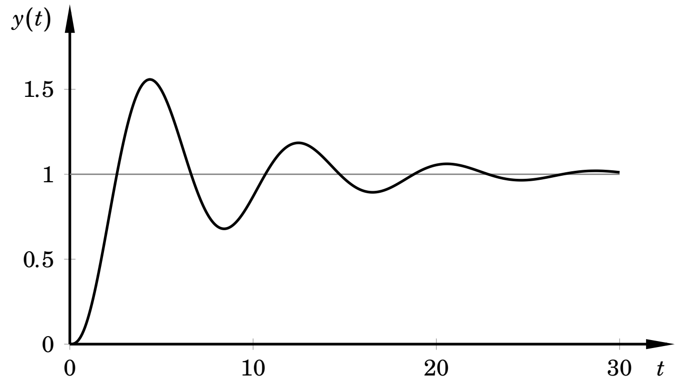


Figure 6.11 Step response of the closed loop servo system in assignment 6.15.

Design a compensation link that increases the phase margin to $\varphi_m = 50°$ without affecting the speed of the system. ($\varphi_m = 50°$ yields a relative damping $\zeta \approx 0.5$ which corresponds to an overshoot $M \approx 17\%$.) The stationary ramp error of the compensated system must not be greater than $e_1 = 1.5$.

6.16 Consider a system with the open loop transfer function

$$G_1(s) = \frac{1.5}{s(s^2+2s+2)}$$

The system is subject to simple negative feedback. The settling time (5%) is $T_s = 8.0$ s, the overshoot is $M_o = 27\%$, and the stationary ramp error ($r(t) = t$) is $e_1 = 1.33$.

Device a phase lag compensation link

$$G_k(s) = K\frac{s+a}{s+a/M}$$

such that the stationary ramp error of the closed loop system is decreased to $e_1 = 0.1$, while speed and damping (robustness) are virtually sustained.

Exercises 7

Controller Structures

7.1 Figure 7.1 shows a block diagram of the temperature control system in a house. The reference temperature (the thermostat set point) is given by r, the output y is the indoor temperature and the disturbance d is due to the outdoor temperature. The transfer function $G_1(s)$ represents the dynamics of the heating system and $G_2(s)$ represents the dynamics of the air inside the house. The controller G_R is a P controller with gain $K = 1$.

Assume that the influence d of the outdoor temperature can be exactly measured. Determine a feedforward link H, such that the indoor temperature becomes independent of the outdoor temperature. What is required in order to obtain a good result from the feedforward?

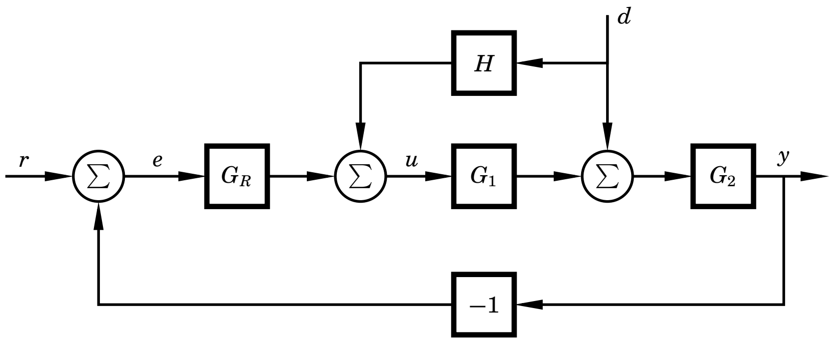


Figure 7.1 Block diagram of the temperature control system in a house.

7.2 Figure 7.2 shows a block diagram of a level control system for a tank. The inflow $x(t)$ of the tank is determined by the valve position and the outflow $v(t)$ is governed by a pump. The cross section of the tank is $A = 1\ m^2$. The assignment is to control the system so that the level h in the tank is held approximately constant despite variations in the flow v. This is done by adjusting the valve at the outflow from the buffer tank.

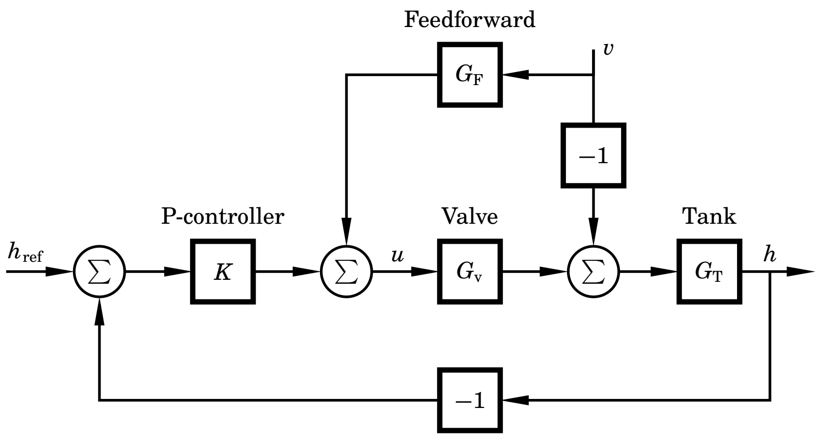


Figure 7.2 Block diagram of the level control system in assignment 7.2.

The transfer function of the valve from position to flow is

$$G_v(s) = \frac{1}{1 + 0.5s}$$

The tank dynamics can be determined through a simple mass balance.

a. Assume that $G_F = 0$, i.e. that we don't have any feedforward. Design a P controller such that the closed loop system obtains the characteristic polynomial $(s + \omega)^2$. How large does ω become? What stationary level error is obtained after a 0.1 step in $v(t)$?

b. Design a PI controller which eliminates the stationary control error otherwise caused by load disturbances. Determine the controller parameters so that the closed loop system obtains the characteristic polynomial $(s + \omega)^3$. How large does ω become?

c. To further decrease the influence of load disturbances, we introduce feedforward based on measurements of $v(t)$. Design a feedforward controller G_F that eliminates the influence of outflow variations by making corrections to $x(t)$.

As all variables describe deviations from the operation point, the reference value for the level h can be set to zero.

7.3 Consider the system in Figure 7.3. The transfer function of the process is given by

$$G_P(s) = \frac{1}{s + 3}$$

and $G_R(s)$ is a PI controller with transfer function

$$G_R(s) = K(1 + \frac{1}{ST_i})$$

K_f is a constant feedforward from the reference signal r.

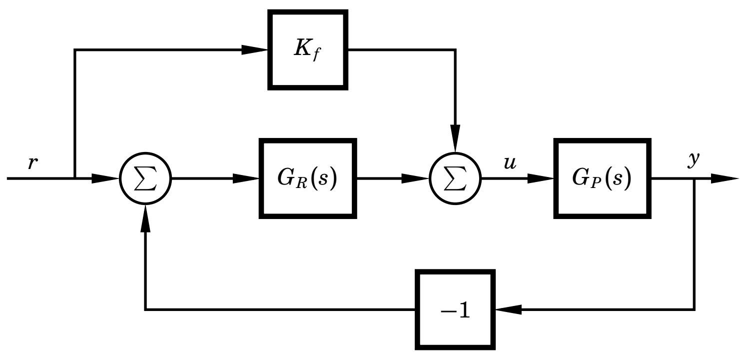


Figure 7.3 Block diagram showing assignment 7.3.

a. Let $K_f = 0$ and determine K and T_i such that the poles of the closed loop system are placed in $-2 \pm 2i$, which is assessed to supress disturbances well.

b. Discuss the influence of the feedforward on the system's response to reference changes.

The closed loop transfer function of the system has one zero. Eliminate it by choosing an appropriate constant feedforward K_f.

7.4 The system in assignment 7.3 can be described by an equivalent block diagram, according to Figure 7.4. Write down the transfer functions $H_{\text{ff}}(s)$ and $H_{\text{fb}}(s)$. Discuss the result and consider the effect of the feedforward when the controller contains a D term.

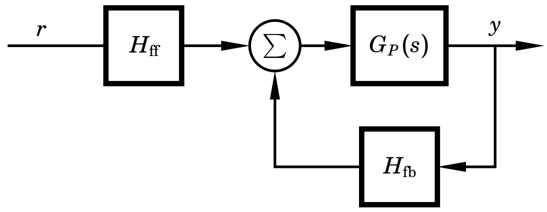


Figure 7.4 Equivalent block diagram in assignment 7.3.

7.5 The block diagram in Figure 7.5 shows cascade control of a tank. The transfer function G_1 describes a valve whereas the transfer function G_2 describes the dynamics of the tank. The objective is to control the tank level y. This is done by controlling the valve G_1 in an inner control loop, whereas y is controlled by an outer control loop. Both the control loops are cascaded so that the reference of the inner loop is the output of the controller in the outer loop. It is presumed that G_1 and G_2 have no poles or zeros in the origin.

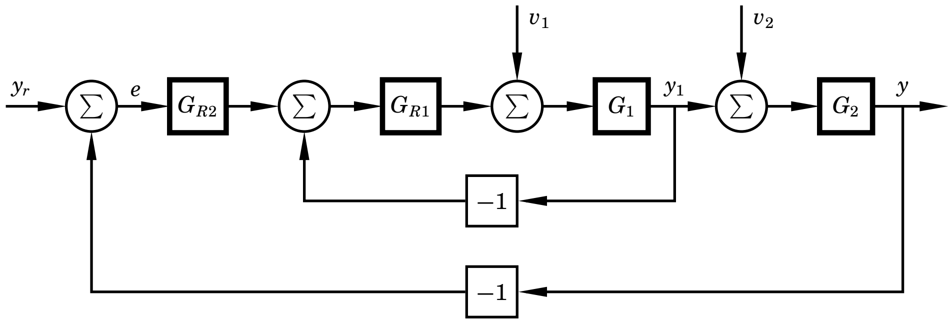


Figure 7.5 The cascade in assignment 7.5.

There are two disturbances in the system, namely the disturbance flow v_2, which is added to the controlled flow y_1 and pressure variations v_1 in the flow before the valve. Discuss the choice of controller (P or PI) in the inner and outer loop, respectively, with respect to elimination of stationary control errors at step changes in disturbances v_1 and v_2.

7.6 Consider Figure 7.5 and assume that $G_1(s) = \frac{2}{s+2}$ describes a valve whereas $G_2(s) = \frac{1}{s}$ describes a tank.

a. Determine a P controller $G_{R1}(s) = K_1$ such that the inner control loop becomes 5 times faster than the uncontrolled valve.

b. Design a PI controller$G_{R2}(s) = K_2(1 + \frac{1}{T_i s})$ for the outer loop, which gives closed loop poles a factor 10 closer to the origin than the pole for the inner control loop. Approximate the inner loop by $G_{\text{inner}}(s) \approx G_{\text{inner}}(0)$.

7.7 In a certain type of steam boiler, a dome is used to separate the steam from the water (see Figure 7.6). It is essential to keep the dome level constant after load changes. The dome can be described by the model

$$Y(s) = \frac{10^{-3}}{s} M(s) + \frac{s - 0.01}{s(s+0.1)} 10^{-3} F(s)$$

where Y is the dome level [m], M is the feed water flow [kg/s] and F is the steam flow [kg/s].

a. Assume a constant steam flow. Design a P controller, controlling the feed water flow by measuring the dome level. Choose the controller parameters such that the control error caused by a step in the dome level goes down to 10 % of its initial value after 10 seconds.

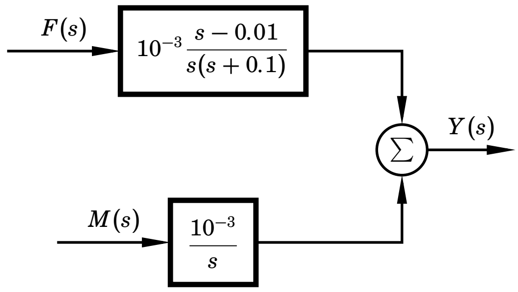


Figure 7.6 Block diagram of steam boiler with dome.

b. Consider the closed loop system. Write down the stationary level error Y caused by a step disturbance of 1 kg/s in the steam flow F.

c. Consider the initial system. Determine a feedforward link $H(s)$ from steam flow $F(s)$ to feed water flow $M(s)$, such that the level Y becomes independent of changes in the steam flow.

7.8 Assume that a servo motor

$$G_P(s) = \frac{1}{s(s+1)}$$

is controlled by the P controller $G_R(s) = 2$. What is the delay margin of the system?

7.9 Consider the same process and controller as in the previous assignment. Now the process is controlled over a very slow network which introduces a one second delay in the control loop. In order to deal with this problem a Smith predictor is utilized, see Figure 7.7.

a. Assume that the model and the process are identical. What are the transfer functions for the blocks (*Controller*, *Process*, *Model*, *Model with no delay*) in our example?

b. The block diagram of the Smith predictor can be redrawn according to Figure 7.8. What is the transfer function of the Smith predictor (from e to u) in our example?

c. Use the approximation $e^x \approx 1 + x$ in order to simplify the transfer function of the controller. Compare the controller to compensation links.

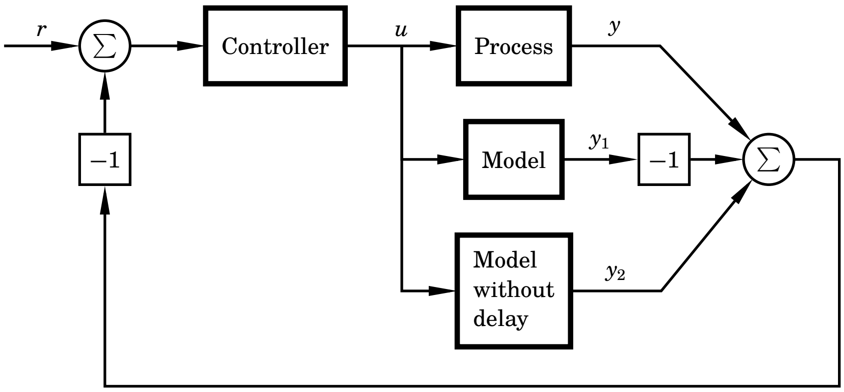


Figure 7.7 Working principle of the Smith predictor.

7.10 Figure 7.9 shows the result of a frequency analysis carried out on the beam (a part of the 'ball on the beam' process). One sees that the process dynamics can be well approximated by an integrator, for low frequencies. One also sees that

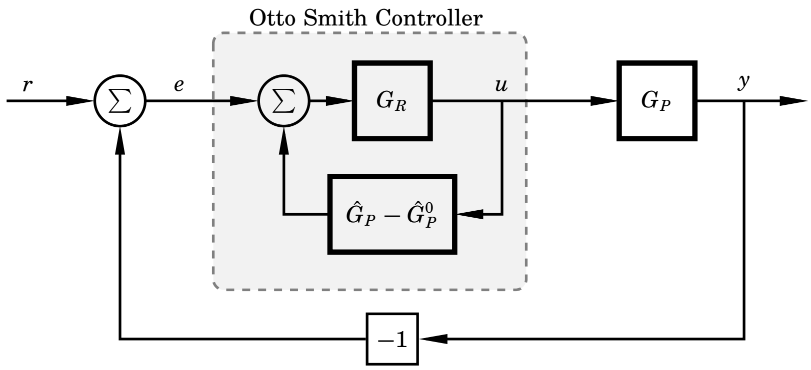


Figure 7.8 Block diagram equivalent to Figure 7.7.

for high frequencies, the phase curve diverges in a way which resembles a delay. Consequently, it would be possible to describe the process by

$$G(s) = \frac{k}{s} e^{-sL}$$

Use the Bode plot in order to determine approximate values of the gain k and delay L.

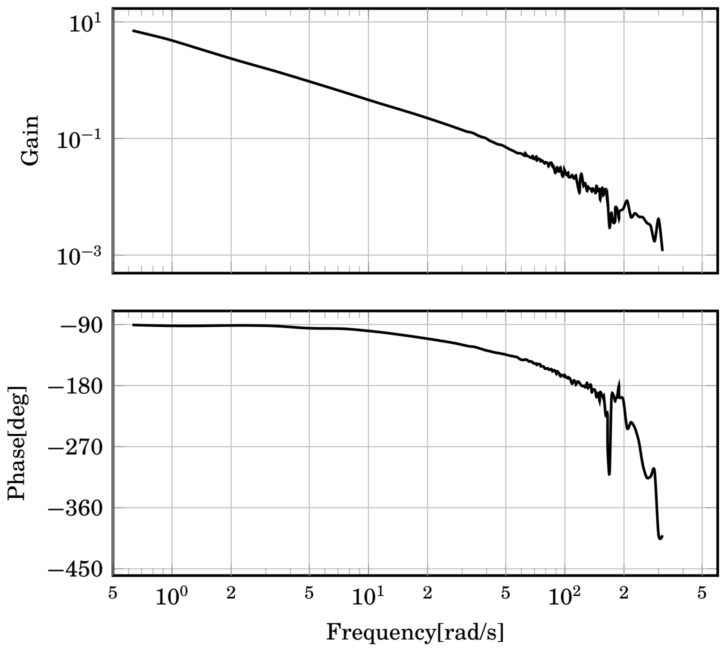


Figure 7.9 Measured Bode plot of the beam.

Exercises 8

Design Examples

8.1 Depth Control of Submarine

Purpose

This assignment deals with depth control of a submarine from the nineteen-forties. Two control methods are tested — PD and state feedback. The latter method was used in reality.

Background

Depth control of submarines can be achieved by means of varying the rudder angle β according to Figure 8.1. The depth h is measured by means of a manometer. By

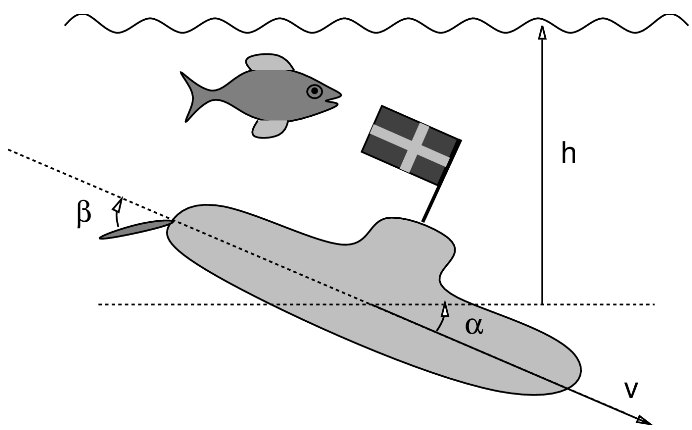


Figure 8.1 Depth control of the submarine in assignment 8.1.

manually generating a sinusoidal rudder angle β (by means of a table and watch — don't forget that this was the end of the nineteen-forties) one can use frequency analysis to estimate the transfer function $G(s)$ from β to h (for a constant speed v). The resulting Bode plots for three different speeds are shown in Figure 8.2.

Specifications

In this case no specifications were given except "Make it as good as you can".

Problem Formulation

Assume that the speed is $v = 3$ knots. The problem lies in computing a control law which gives a satisfactory settling of the depth h for the given speed. This does not guarantee equally satisfactory results at other speeds.

In an initial approach one wanted to control the depth h of the submarine, solely based on measurements of h.

a. What is the maximal allowed gain K in order to achieve a stable closed loop system with a P controller $\beta = K(h_{\text{ref}} - h)$. Use the Bode plot in Figure 8.2?

b. The desired cross-over frequency is $\omega_c = 0.03$ rad/s, using a PD controller $G_r(s) = K(1+T_D s)$. How should K and T_D be chosen in order to obtain a 60° phase margin ϕ_m?

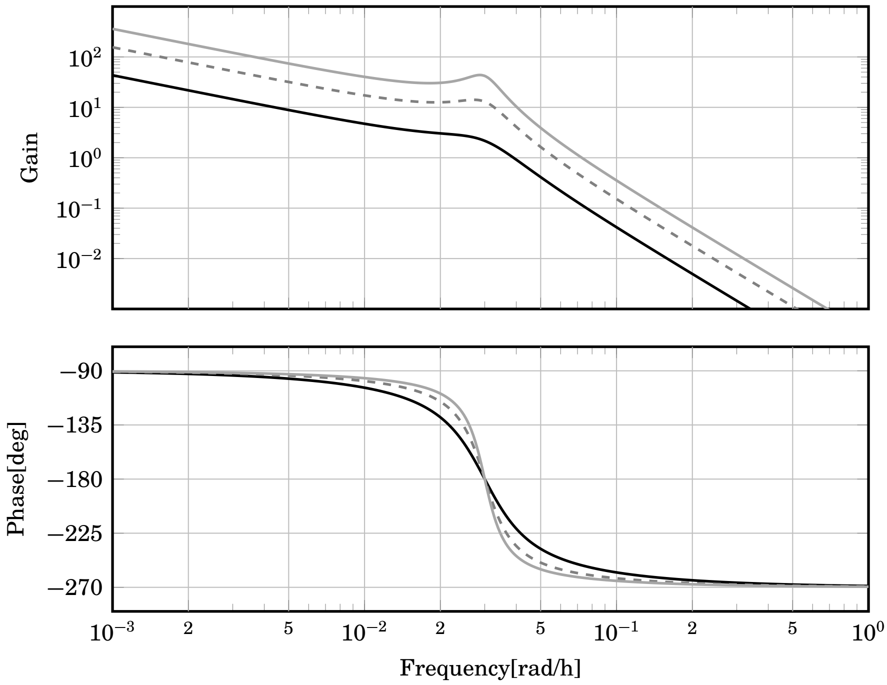


Figure 8.2 Bode plot of the estimated transfer function $G(s)$ from β [deg] to h [m] in assignment 8.1 for the speeds $v = 3$ (solid black curves), 5 (dashed curves) and 7 knots (gray curves).

c. How is the stability of the closed loop system in (b) affected if the speed is increased from 3 to 7 knots? Suggest different ways in which speed variations can be taken into consideration.

For angular frequencies above 0.05 rad/s one can use the approximation

$$\begin{cases} G_{\alpha\beta}(s) = \dfrac{k_v}{s^2} \\ G_{h\alpha}(s) = \dfrac{v}{s} \end{cases} \tag{8.1}$$

where $G_{\alpha\beta}(s)$ and $G_{h\alpha}(s)$ are the transfer functions from β to α and from α to h, respectively (see Figure 8.3). The constant k_v depends on the speed v.

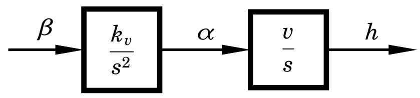


Figure 8.3 Block diagram of a submarine model which is valid for $\omega > 0.05$ rad/s.

d. Determine k_v by means of the Bode plot in Figure 8.2. (1 knot $\approx$ 1.852 km/h = 1.852/3.6 $\approx$ 0.514 m/s.)

e. Assume that the approximate model

$$G_{h\beta}(s) = \frac{k_v v}{s^3}$$

is under P control $\beta = K(h_{ref} - h)$. Determine which values of K that yield an asymptotically stable system. Does this concur with the results obtained in sub-assignment a?

One can improve the performance of the control system by utilizing additional feedback from the trim angle α and its derivative $d\alpha/dt$.

f. Introduce the states $x_1 = d\alpha/dt$, $x_2 = \alpha$ and $x_3 = h$ together with the input $u = \beta$. Use the control law $u = u_r - k_1x_1 - k_2x_2 - k_3x_3 = u_r - Kx$ and determine K such that the characteristic equation of the closed loop system becomes

$$(s + \gamma\omega_0)(s^2 + 2\zeta\omega_0 s + \omega_0^2) = 0$$

g. The reference h_{ref} for the depth h is introduced according to

$$u_r = K_r h_{\text{ref}}$$

How should K_r be chosen in order to obtain $h = h_{\text{ref}}$ in stationarity?

It was decided to choose $\zeta = 0.5$ and $\gamma = 0.2$ which was considered to give an adequately damped step response. However, the choice of ω_0 requires some further thought. It should not be chosen too low, since the approximate model (8.1) is only valid for $\omega > 0.05$ rad/s. On the other hand, choosing ω_0 too high would result in large rudder angles caused by the large values of the coefficients l_j, $j = 1, 2, 3$.

h. How large can ω_0 be chosen if a step disturbance in the manometer signal corresponding to $\Delta h = 0.1m$ should not give rise to larger rudder angles than 5°?

In the actual case $\omega_0 = 0.1$ rad/s was chosen. A semi-automatic system was evaluated first. The signal $u = u_r - k_1x_1 - k_2x_2 - k_3x_3$ was displayed to an operator, who manually tried to keep the signal zero by means of the ordinary rudder servo. The control action was very satisfactory. Settling times of 30–60 s were obtained throughout the speed range. The complete automatic system was then evaluated on the Swedish submarine 'Sjöborren' (The Sea Urchin). The accuracy during cruising in calm weather was ±0.05 m.

8.2 Control of Elastic Servo

Purpose

The aim of the assignment is to control the angular speed of a flywheel which is connected to another flywheel by a weak axis. The second flywheel is driven by a motor. Different control strategies are evaluated and compared with respect to performance.

Background

Figure 8.4 shows a simplified model of an elastic servo. It could also constitute a model of a weak robot arm or an elastic antenna system mounted on a satellite. The turn angles of the flywheels are denoted φ_1 and φ_2, respectively, whereas

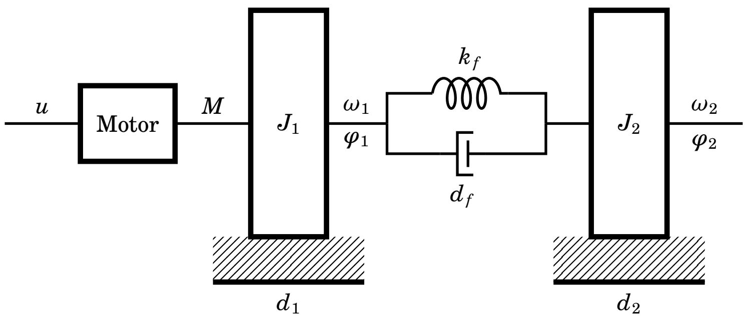


Figure 8.4 Model of the elastic servo in assignment 8.2.

$\omega_1 = \dot\varphi_1$ and $\omega_2 = \dot\varphi_2$ denote the corresponding angular speeds. The flywheels have moments of inertia J_1 and J_2, respectively. They are connected by an axle with spring constant k_f and damping constant d_f. The system is subject to bearing friction, which is represented by the damping constants d_1 and d_2. One of the flywheels is driven by a DC motor, which is itself driven by a current-feedback amplifier. The motor and amplifier dynamics are neglected. The momentum of the motor is proportional to the input voltage u of the amplifier, according to

$$M = k_m \cdot I = k_m k_i u$$

where I is the current through the rotor coils. Momentum equilibrium about the flywheel yields the following equations

$$\begin{cases} J_1\dot{\omega}_1 = -k_f(\varphi_1 - \varphi_2) - d_1\omega_1 - d_f(\omega_1 - \omega_2) + k_m k_i u \\ J_2\dot{\omega}_2 = +k_f(\varphi_1 - \varphi_2) - d_2\omega_2 + d_f(\omega_1 - \omega_2) \end{cases}$$

We introduce the state variables

$$\begin{cases} x_1 = \omega_1 \\ x_2 = \omega_2 \\ x_3 = \varphi_1 - \varphi_2 \end{cases}$$

and consider the angular speed ω_2 as the output, i.e.

$$y = k_{\omega_2} \cdot \omega_2$$

This gives us the following state space model of the servo.

$$\dot{x} = Ax + Bu = \begin{pmatrix} -\dfrac{d_1 + d_f}{J_1} & \dfrac{d_f}{J_1} & -\dfrac{k_f}{J_1} \\ \dfrac{d_f}{J_2} & -\dfrac{d_f + d_2}{J_2} & \dfrac{k_f}{J_2} \\ 1 & -1 & 0 \end{pmatrix} x + \begin{pmatrix} \dfrac{k_m k_i}{J_1} \\ 0 \\ 0 \end{pmatrix} u$$

$$y = Cx = \begin{pmatrix} 0 & k_{\omega_2} & 0 \end{pmatrix} x$$

The following values of constants and coefficients have been measured and estimated for a real lab process.

$$\begin{aligned} J_1 &= 22 \cdot 10^{-6}\,\mathrm{kgm}^2 \\ J_2 &= 65 \cdot 10^{-6}\,\mathrm{kgm}^2 \\ k_f &= 11.7 \cdot 10^{-3}\,\mathrm{Nm/rad} \\ d_f &= 2e^{-5} \\ d_1 &= 1 \cdot 10^{-5}\,\mathrm{Nm/rad/s} \\ d_2 &= 1 \cdot 10^{-5}\,\mathrm{Nm/rad/s} \\ k_m &= 0.1\,\mathrm{Nm/A} \\ k_i &= 0.027\,\mathrm{A/V} \\ k_{\omega_1} = k_{\omega_2} &= 0.0167\,\mathrm{V/rad/s} \end{aligned}$$

Problem Formulation

The input is the voltage u over the motor and we want to control the angular speed ω_2 of the outer flywheel.

It is desired to quickly be able to change ω_c, while limiting the control system's sensitivity against load disturbances and measurement noise. The system also requires active damping, in order to avoid an excessively oscillative settling phase.

Specifications

1. The step response of the closed loop system should be fairly well damped and have a rise time of 0.1-0.3 s. The settling time to ±2% shall be at most 0.5 s. A graphical specification of the step response is given in Figure 8.5.
2. Load disturbances must not give rise to any static errors.
3. Noise sensitivity should not be excessive.

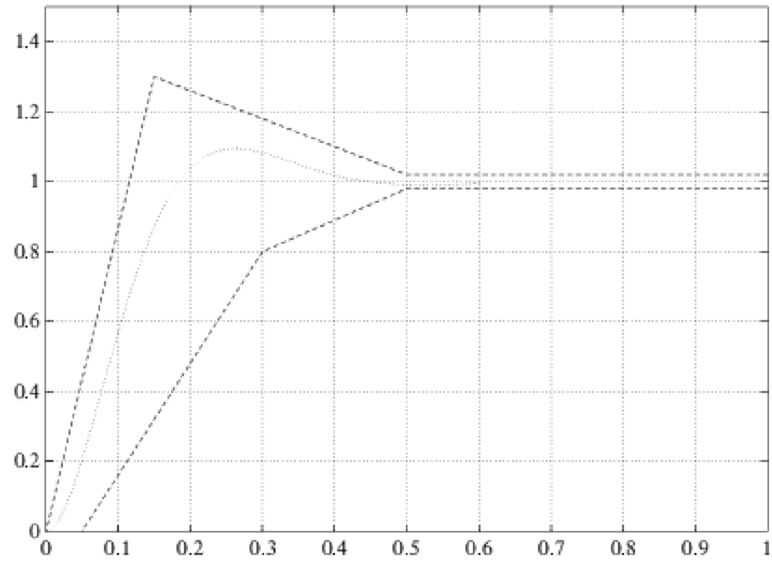


Figure 8.5 The step response of the closed loop system shall lie between the dashed lines.

Ziegler-Nichols Method

The Bode plot of the transfer function from u to ω_2 is shown in Figure 8.6.

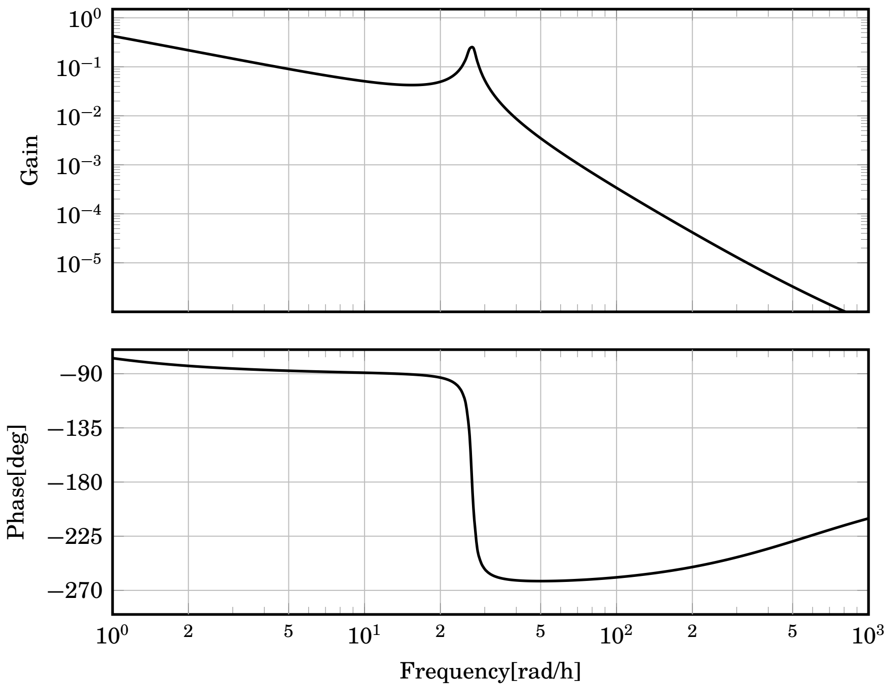


Figure 8.6 Bode plot of the servo process.

a. Use Ziegler-Nichols frequency method in order to determine suitable PID parameters.

Ziegler-Nichols method often gives a rather oscillative closed loop system. However, the obtained parameters are often a reasonable starting point for manual tuning.

State Feedback and Kalman Filtering

If it is possible to measure all states, the poles of the closed loop system can be arbitrarily placed through the feedback control law

$$u(t) = -Kx(t) + K_r y_r(t)$$

if the system is also controllable.

b. Determine the gain K_r so that the stationary gain of the closed loop system becomes 1, i.e. $y = y_r$ in stationarity.

In order to meet specification 2, one must introduce integral action in the controller. One way to achieve this it thorough the control law

$$u(t) = -Kx(t) + K_r y_r(t) - K_i \int_{-\infty}^{t} (y(s) - y_r(s))ds$$

This can be interpreted as feedback from an 'extra' state x_i according to

$$\begin{cases} \dot{x}_i = y - y_r \\ u = -Kx + K_r y_r - K_i x_i \end{cases}$$

Figure 8.7 shows a block diagram of the entire system

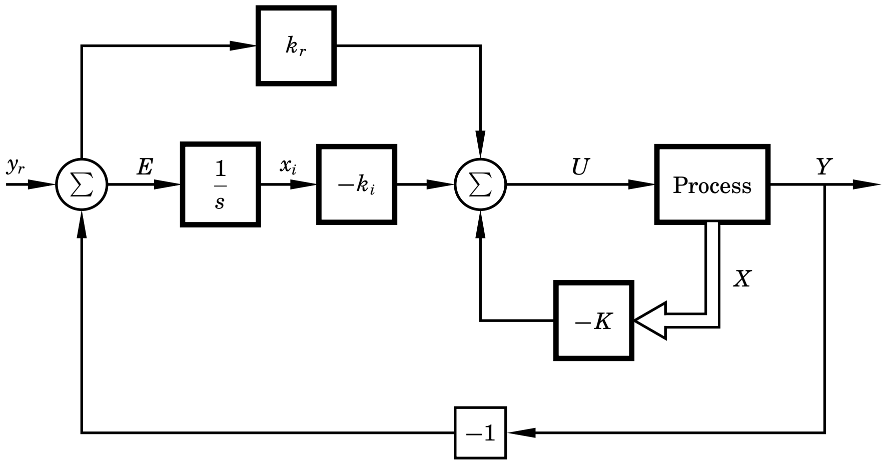


Figure 8.7 Block diagram of the state feedback control system in assignment 8.2.

c. How does the augmented state space model look like? Introduce the notion x_e for the augmented state vector.

Since the states are not directly measurable, they must be reconstructed in some way. A usual way is to introduce a Kalman filter

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

and then close the loop from the estimated states $\hat{x}$

$$u = -K\hat{x} + K_r y_r - K_i x_i$$

It is, however, unnecessary to estimate x_i since we have direct access to this state. The block diagram of the entire system is shown in Figure 8.8. Let K' denote the augmented row matrix $\begin{pmatrix} K & K_i \end{pmatrix}$ and call the augmented system matrices A' and B', respectively. The problem consists in finding suitable K, K_i and L by placing the eigenvalues of $A' - B'K'$ and $A - LC$. Since the both eigenvalue problems are of a bit too high dimension for enjoyable hand calculations, we use Matlab to investigate a few choices of pole placements.

In order not to end up with too many free parameters, we place the poles in a Butterworth pattern. I.e. the poles are equally distributed on a half circle in the left half plane. We place the eigenvalues of $A' - B'K'$ on a half circle with radius ω_m, whereas the eigenvalues of $A - LC$ are placed on a half circle with radius ω_o (see Figure 8.9). A suitable ω_m can be obtained from Specification 1, i.e. that the

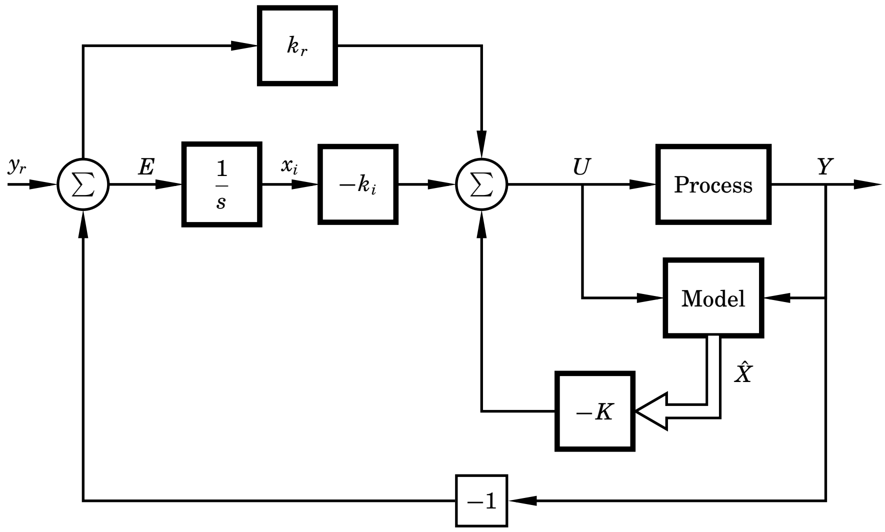


Figure 8.8 Block diagram showing the Kalman filter and state feedback in assignment 8.2.

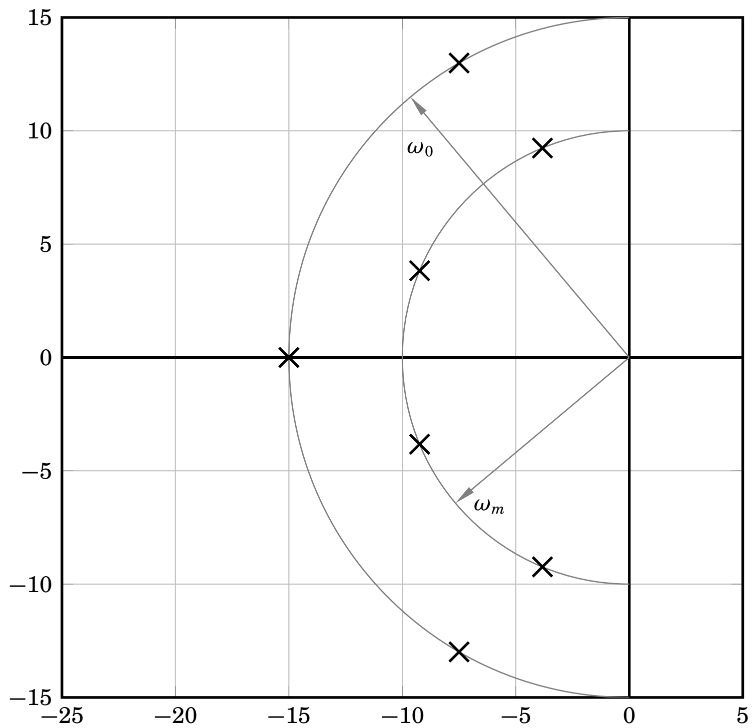


Figure 8.9 The pole placement in assignment 8.2.

settling time T_s to reach within 2% of the stationary value must be less than 0.5 s. A coarse estimation of T_s for a second order system with relative damping ζ and natural frequency ω is given by

$$T_s \approx -\frac{\ln \epsilon}{\zeta \omega}$$

where ϵ is the maximal deviation from the final value. Since we have a 4^{th} degree system, we cannot use this approximation directly. If we, however, only consider the least damped pole pair ($\zeta = 0.38$ and $\omega = \omega_m$) in Figure 8.9 we obtain

$$\omega_m \approx -\frac{\ln \epsilon}{T_s \zeta} \tag{8.2}$$

d. Which value of ω_m is obtained from the formula (8.2)?

We let $\omega_m = 20$ which yields $T_s < 0.5$ s. We can let $K_r = 0$ since we have integral action in the controller and thus stationary closed loop gain 1. Figure 8.10 shows the step response of the closed loop system for $K_r = 0$ and K_r chosen according to sub-assignment b, respectively. By letting $K_r = 0$ the step response overshoot is sufficiently decreased to fulfill the specification.

We now fix ω_m and vary ω_o. The following test shall be used to evaluate the control performance. At time $t = 0$ there is a unit step in the reference value y_r followed by a load disturbance $d = -1$ in the control signal at $t = 1$ and the introduction of measurement noise (in y) at $t = 3$. The variance of the noise is 0.01. The result is shown in Figure 8.11.

e. Which value of ω_o seems to be best when it comes to elimination of load disturbances? Which ω_o is best when it comes to suppressing measurement noise?

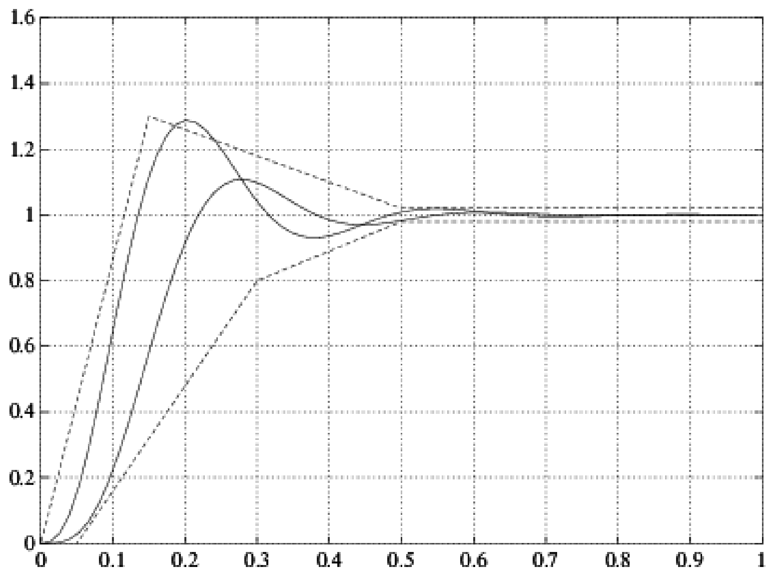


Figure 8.10 Step response of the closed loop system for $\omega_m = 20$ rad/s. Choosing K_r according to sub-assignment b yields the system with the larger overshoot. The other curve is the step response corresponding to $K_r = 0$.

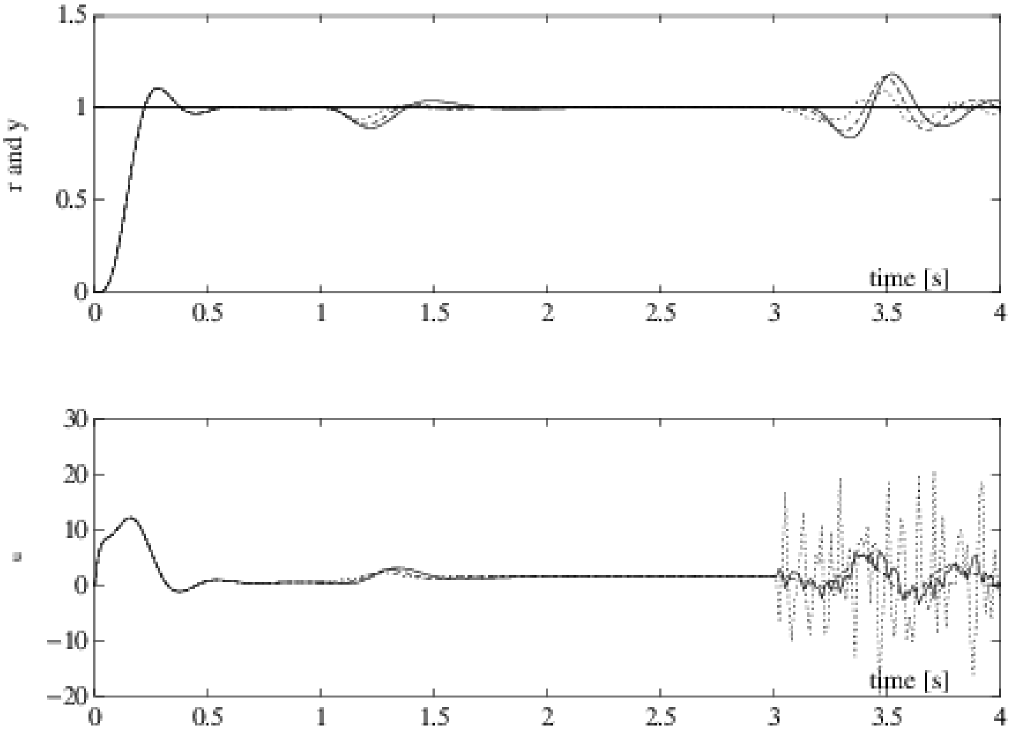


Figure 8.11 Evaluation of control with $\omega_m = 20$ and $\omega_o = 10$ (solid curves), 20 (dashed curves) and 40 (dotted curves).

Exercises 9

Interactive Comparison Between Model Descriptions

There are many ways to describe the dynamics of processes and control systems, e.g., step responses, transfer functions, state-space descriptions, pole-zero diagrams, Bode diagrams, and Nyquist diagrams. A good way of learning the correspondence between different descriptions is to use interactive tools. On the web page

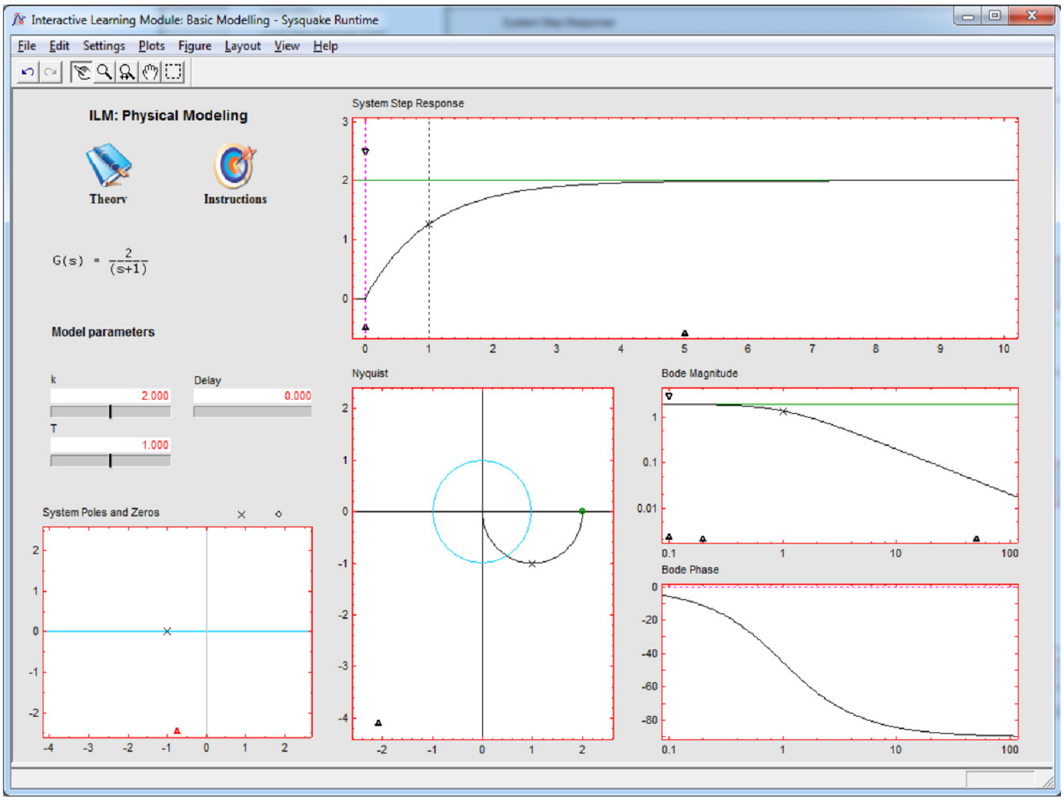
`http://aer.ual.es/ilm/`

there are several interactive programs that may be downloaded for free. The module *Modeling* is convenient to use to study model descriptions. The interface of this module is shown in the figure below.

The model structure that you want to study is entered by "dragging" poles and zeros in the pole-zero diagram. Parameters and dead time may then be changed by dragging points or lines in the different diagrams, or by entering numerical values for the transfer function. The best way of learning to use the tool and examine its possibilities is to try things out by experimenting with the different menus. More information is available on the web page.

The gradation of the axes in the different diagrams can be changed by clicking at the small triangles by the gradations. It is also possible to zoom in and out.

Under the tab *File* you find the useful command “Reset data”, which resets all values to default.



9.1 Study the transfer function

$$G(s) = \frac{K}{1+sT}$$

Let the starting poing be $K = T = 1$.

a. First, vary the gain K and note how the pole, step response, Nyquist diagram and Bode diagram are affected. How can K be determined from the step response, Nyquist diagram, and Bode diagram?

b. Set $K = 1$ and vary T. How are the different representations affected? Why is the shape of the Nyquist diagram not affected?

c. Set $K = T = 1$ and add a dead time L such that the transfer function becomes

$$G(s) = \frac{K}{1+sT}e^{-Ls}$$

Vary L and obseve how that affects the representations. Explain what happens with the step response. Why does the Nyquist curve look like it does? Why is the gain curve of the Bode diagram not affected?

9.2 Study the transfer function

$$G(s) = \frac{K}{(1+sT_1)(1+sT_2)}$$

Let the starting point be $K = T_1 = T_2 = 1$.

a. First, vary the gain K and note how poles, step response, Nyquist diagram, and Bode diagram are affected.

b. Set $K = 1$ and vary T_1 and T_2. Note the difference between the cases $T_1 \approx T_2$ and $T_1 \gg T_2$. How can $G(s)$ be approximated in the case $T_1 \gg T_2$?

c. Set $K = T_1 = T_2 = 1$ and add a zero, such that the transfer function becomes

$$G(s) = \frac{K(1+sT_3)}{(1+sT_1)(1+sT_2)}$$

Vary T_3 and observe how that affects the representations. What happens when $T_3 < 0$? What does this mean if you want to control such a process? Try to explain the phenomenon.

9.3 Study the transfer function

$$G(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

Let the starting point be $\zeta = 0.7$ och $\omega = 1$.

a. First, vary the frequency ω and note how poles, step response, Nyquist diagram, and Bode diagram are affected.

b. Set $\omega = 1$ and vary ζ. Note how the representations change.

Part II

Solutions to Exercises

Solutions to Exercises 1

Model Building and Linearization

1.1 **a.** A block diagram of the system is shown in Figure S1.1. The person that takes a shower senses the temperature and the flow (measurement signals), and adjusts the shower handle (control signal) to get desired temperature and flow. Feedback is primarily used. Disturbances may be variations in water pressure and temperature in the water pipes.

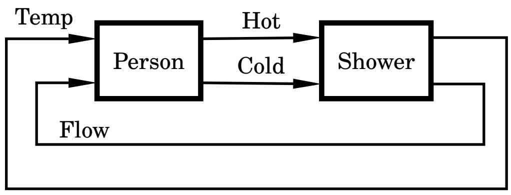


Figure S1.1 Block diagram of person taking a shower.

b. A car driver uses several control signals: the gas pedal, the brake, the steering wheel. The driver wants to control the car such that it keeps on the road with desired velocity, and keeps safe distance to other vehicles. Measurement signals are the speedometer, and visual feedback of how the car turns, the distance to the car in front, and other road conditions. The uses feedforward, e.g., to adjust the velocity in advance before a curve, but also feedback, by looking at the speedometer to keep desired velocity. A block diagram of the system is shown in Figure S1.2.

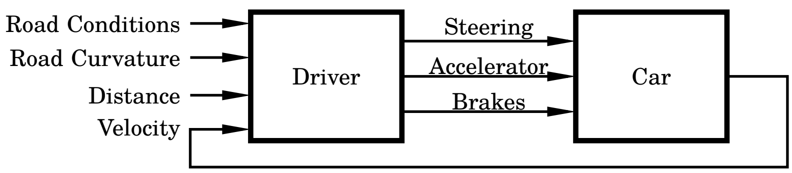


Figure S1.2 Block diagram of car driving.

c. The control signal is the heat from the stove plate. Measurement signals from the systems are obtained by observing how intensively the water boils, and sensing how soft the potatoes are. Feedback is used, e.g., when adjusting the power of the stove plate when the water boils too intensively. Feedforward is used when following a given recipe, e.g., ”the potatoes are done after 20 minutes” or ”when the water boils, decrease the heat of the stove plate to half of full power”. A block diagram is shown in Figure S1.3.

1.2 **a.**

$$\dot{v} = ku$$

The system is linear.

b. Now, we get an additional differential equation

$$\dot{p} = v$$

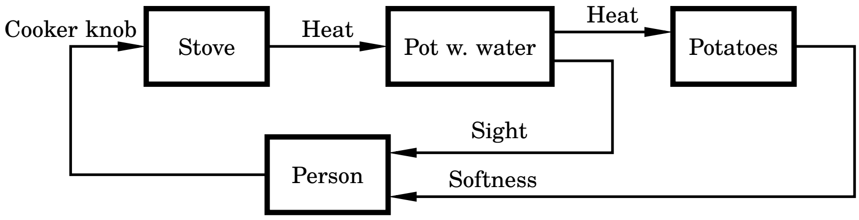


Figure S1.3 Block diagram of potato boiling.

Choosing the states as $x_1 = v$, $x_2 = p$, we get

$$\begin{aligned} \dot{x}_1 &= ku \\ \dot{x}_2 &= x_1 \\ y &= x_2 \end{aligned}$$

One can also write the system on matrix form

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} k \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x \end{aligned}$$

c. The term mx^2 makes the system nonlinear. The stationary point is given by

$$-0.001(x^0)^2 + u^0 = 0$$

which for $u_0 = 0.1$ gives $x^0 = \pm 10$. The stationary velocity thus becomes $y^0 = 10$ m/s.

d. The system is given by

$$\begin{aligned} \dot{x} &= f(x,u) = -mx^2 + ku \\ y &= g(x,u) = x \end{aligned}$$

We differentiate f and g with respect to x and u and get

$$\begin{aligned} \frac{\partial f}{\partial x} &= -2mx \\ \frac{\partial f}{\partial u} &= k \\ \frac{\partial g}{\partial x} &= 1 \\ \frac{\partial g}{\partial u} &= 0 \end{aligned}$$

We insert the stationary point $(x^0, u^0, y^0) = (10, 0.1, 10)$ in the expressions for the derivatives, and introduce the new variables $\Delta x = x - x^0$, $\Delta y = y - y^0$, $\Delta u = u - u^0$. We get the linear system

$$\begin{aligned} \frac{d\Delta x}{dt} &= -0.02\Delta x + \Delta u \\ \Delta y &= \Delta x \end{aligned}$$

1.3 With $x_1 = y$ and $x_2 = \dot{y}$ the system is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} f$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

1.4 With states $x_1 = v_{\text{out}}$ and $x_2 = \dot{v}_{\text{out}}$, the system is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{LC} \end{pmatrix} v_{\text{in}}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

1.5 **a.** We can choose e.g. the height h as state variable. The volume change in the tank is given by

$$A\dot{h} = q_{\text{in}} - q_{\text{ut}}$$

and from Torricelli's law we obtain $q_{\text{ut}} = a\sqrt{2gh}$. The sought differential equation becomes

$$\dot{h} + \frac{a}{A}\sqrt{2gh} = \frac{1}{A}q_{\text{in}}$$

b.

$$\begin{aligned} \dot{h} &= -\frac{a}{A}\sqrt{2gh} + \frac{1}{A}q_{\text{in}} && (= f(h, q_{\text{in}})) \\ y &= h && (= g(h, q_{\text{in}})) \end{aligned}$$

c. The outlflow must equal the inflow $q^0_{\text{ut}} = q^0_{\text{in}}$. The level is calculated by letting $\dot{h} = 0$, which yields

$$h^0 = \frac{1}{2g}\left(\frac{q^0_{\text{in}}}{a}\right)^2$$

We determine the partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial h} &= -\frac{a}{A}\sqrt{\frac{g}{2h}} && \frac{\partial f}{\partial q_{\text{in}}} = \frac{1}{A} \\ \frac{\partial g}{\partial h} &= 1 && \frac{\partial g}{\partial q_{\text{in}}} = 0 \end{aligned}$$

By inserting $h = h^0$ above and introducing variables which denote deviations from the operating point: $\Delta h = h - h^0$, $\Delta q_{\text{in}} = q_{\text{in}} - q^0_{\text{in}}$, $\Delta y = \Delta h$ the linearized system is

$$\begin{aligned} \dot{\Delta h} &= -\frac{a}{A}\sqrt{\frac{g}{2h^0}}\Delta h + \frac{1}{A}\Delta q_{\text{in}} \\ \Delta y &= \Delta h \end{aligned}$$

1.6

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

1.7 **a.**

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sqrt{x_1} - x_1x_2 + u^2 \\ y &= x_1 \end{aligned}$$

b. A stationary point implies $\dot{x}_1 = \dot{x}_2 = 0$. From the first equation we directly obtain $x_2 = 0$. Subsequently, the second equation yields $\sqrt{x_1} = u^2$. Hence there are infinitely many stationary points and they can be parametrized through t as $(x^0_1, x^0_2, u^0) = (t^4, 0, t)$.

c. $u^0 = 1$ gives the stationary point $(x_1^0, x_2^0, u^0) = (1, 0, 1)$. We let

$$\begin{aligned} f_1(x_1, x_2, u) &= x_2 \\ f_2(x_1, x_2, u) &= -\sqrt{x_1} - x_1x_2 + u^2 \\ g(x_1, x_2, u) &= x_1 \end{aligned}$$

and compute the partial derivatives

$$\begin{array}{lll} \dfrac{\partial f_1}{\partial x_1} = 0 & \dfrac{\partial f_1}{\partial x_2} = 1 & \dfrac{\partial f_1}{\partial u} = 0 \\ \dfrac{\partial f_2}{\partial x_1} = -\dfrac{1}{2\sqrt{x_1}} - x_2 & \dfrac{\partial f_2}{\partial x_2} = -x_1 & \dfrac{\partial f_2}{\partial u} = 2u \\ \dfrac{\partial g}{\partial x_1} = 1 & \dfrac{\partial g}{\partial x_2} = 0 & \dfrac{\partial g}{\partial u} = 0 \end{array}$$

At the stationary point we have

$$\begin{array}{lll} \dfrac{\partial f_1}{\partial x_1} = 0 & \dfrac{\partial f_1}{\partial x_2} = 1 & \dfrac{\partial f_1}{\partial u} = 0 \\ \dfrac{\partial f_2}{\partial x_1} = -\dfrac{1}{2} & \dfrac{\partial f_2}{\partial x_2} = -1 & \dfrac{\partial f_2}{\partial u} = 2 \\ \dfrac{\partial g}{\partial x_1} = 1 & \dfrac{\partial g}{\partial x_2} = 0 & \dfrac{\partial g}{\partial u} = 0 \end{array}$$

After a variable substitution, the linearized system can be written

$$\begin{aligned} \begin{pmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Delta u \\ \Delta y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \end{aligned}$$

1.8 At the sought operating point it holds that

$$\begin{aligned} 0 &= x_1^2x_2 + 1 \\ 0 &= x_1x_2^2 + 1 \\ y &= \arctan\frac{x_2}{x_1} + \frac{\pi^2}{8} \end{aligned}$$

which yields $x_1^0 = -1$, $x_2^0 = -1$ and $y^0 = \frac{\pi}{4} + \frac{\pi^2}{8}$. Computation of the partial derivatives now yields

$$\begin{array}{lll} \dfrac{\partial f_1}{\partial x_1} = 2x_1x_2 & \dfrac{\partial f_1}{\partial x_2} = x_1^2 & \dfrac{\partial f_1}{\partial u} = \sqrt{2}\cos u \\ \dfrac{\partial f_2}{\partial x_1} = x_2^2 & \dfrac{\partial f_2}{\partial x_2} = 2x_1x_2 & \dfrac{\partial f_2}{\partial u} = -\sqrt{2}\sin u \\ \dfrac{\partial g}{\partial x_1} = \dfrac{-x_2}{x_1^2 + x_2^2} & \dfrac{\partial g}{\partial x_2} = \dfrac{x_1}{x_1^2 + x_2^2} & \dfrac{\partial g}{\partial u} = 4u \end{array}$$

With the variable substitution

$$\begin{aligned} \Delta u &= u - \frac{\pi}{4} \\ \Delta x_1 &= x_1 + 1 \\ \Delta x_2 &= x_2 + 1 \\ \Delta y &= y - \frac{\pi}{4} - \frac{\pi^2}{8}. \end{aligned}$$

the linearized system becomes

$$\begin{pmatrix} \dot{\Delta x_1} \\ \dot{\Delta x_2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Delta u$$

$$\Delta y = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \pi \Delta u$$

1.9 a. The nonlinear state space equations are

$$\begin{aligned} \dot{x}_1 &= x_2 && = f_1(x_1, x_2, u) \\ \dot{x}_2 &= \omega^2 x_1 - \frac{\beta}{x_1^2} + u && = f_2(x_1, x_2, u) \\ y &= x_1 && = g(x_1, x_2, u) \end{aligned}$$

b. At stationarity it holds that

$$\ddot{r}(t) = \omega^2 r_0 - \frac{\beta}{r_0^2} + 0 = 0$$

i.e. $r_0^3 = \beta/\omega^2$. We now compute the partial derivatives

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial u} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \omega^2 + 2\beta/r_0^3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 3\omega^2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The linear system hence becomes

$$\frac{d\Delta x}{dt} = \begin{pmatrix} 0 & 1 \\ 3\omega^2 & 0 \end{pmatrix} \Delta x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Delta u$$

$$\Delta y = \begin{pmatrix} 1 & 0 \end{pmatrix} \Delta x$$

Solutions to Exercises 2

Dynamical Systems

2.1 **a.** The transfer function for a linear system on state-space form is given by

$$\begin{aligned}G(s) &= C(sI - A)^{-1}B + D \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 10 & 1 \\ -1 & -1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{(s-10)(s+1)+1}\end{aligned}$$

The transfer function gives the relation

$$Y(s) = \frac{1}{(s-10)(s+1)+1} U(s)$$

which may be written as

$$s^2 Y(s) - 9sY(s) - 9Y(s) = U(s)$$

Inverse Laplace transform gives the differential equation

$$\ddot{y} - 9\dot{y} - 9y = u$$

b. Laplace transform of the differential equation gives

$$Js^2 Y(s) + DsY(s) = U(s)$$

We get

$$Y(s) = \frac{1}{Js^2 + Ds} U(s)$$

and thus the transfer function is

$$G(s) = \frac{1}{Js^2 + Ds}$$

When writing the system on state-space form, the states may be chosen as $x_1 = y$, $x_2 = \dot{y}$. This gives the equations

$$\begin{aligned}\dot{x}_1 &= \dot{y} = x_2 \\ \dot{x}_2 &= \ddot{y} = \frac{1}{J}(-D\dot{y} + u) = \frac{1}{J}(-Dx_2 + u)\end{aligned}$$

The system on state-space form is given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -\frac{D}{J} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{J} \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

c. The transfer function is given by

$$G(s) = C(sI - A)^{-1}B + D = 1(s - -1/k)^{-1}1/k = \frac{1}{1 + sk}$$

d. The transfer function gives the following relation between $Y(s)$ and $U(s)$

$$(s^3 + \alpha s^2 + \beta s)Y(s) = \gamma U(s)$$

Inverse Laplace transform gives

$$\dddot{y} + \alpha \ddot{y} + \beta \dot{y} = \gamma u$$

Choosing the states

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \end{aligned}$$

gives the state-space description

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\beta & -\alpha \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x \end{aligned}$$

We may also use any of the "standard forms" for state-space descriptions given in the collection of formulae (diagonal form, observable canonical form, controllable canonical form). To write the system on diagonal form we must factorize the transfer function, which is hard when we don't know the values of α and β. Thus, the controllable or observable canonical forms are more convenient to choose. We compare the coefficients in our transfer function to the structure in the collection of formulae and get

$$a_1 = \alpha, \quad a_2 = \beta, \quad a_3 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = \gamma$$

A state-space description on observable canonical form is thus given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -\alpha & 1 & 0 \\ -\beta & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x \end{aligned}$$

The model is presented and analyzed in the paper *Complex dynamics in low-dimensional continuous-time business cycle models: the Sil'nikov case, H.W. Lorenz, System Dynamics Review, vol.8 no.3, 1992.*

2.2 **a.** The transfer function is

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} s+2 & 0 \\ 0 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + 2 \\ &= \frac{2s^2 + 7s + 1}{s^2 + 5s + 6}. \end{aligned}$$

From the transfer function it is easy to determine the differential equation

$$\begin{aligned} Y(s) &= G(s)U(s) \\ (s^2 + 5s + 6)Y(s) &= (2s^2 + 7s + 1)U(s) \\ \ddot{y} + 5\dot{y} + 6y &= 2\ddot{u} + 7\dot{u} + u \end{aligned}$$

b. The transfer function is

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} s+7 & -2 \\ 15 & s-4 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \\ &= \frac{2s+3}{s^2+3s+2}. \end{aligned}$$

The differential equation becomes

$$\begin{aligned} Y(s) &= G(s)U(s) \\ (s^2+3s+2)Y(s) &= (2s+3)U(s) \\ \ddot{y} + 3\dot{y} + 2y &= 2\dot{u} + 3u \end{aligned}$$

c. $G(s) = \dfrac{5s+8}{s+1}, \quad \dot{y} + y = 5\dot{u} + 8u$

d. $G(s) = \dfrac{3s^2+7s+18}{s^2+2s+5}, \quad \ddot{y} + 2\dot{y} + 5y = 3\ddot{u} + 7\dot{u} + 18u$

2.3 **a.** Partial fraction expansion of the transfer function yields

$$G(s) = 2 + \frac{2}{s+3} - \frac{5}{s+2}$$

and by applying the inverse Laplace transform, one obtains the impulse response

$$h(t) = \mathcal{L}^{-1}G(s) = 2\delta(t) + 2e^{-3t} - 5e^{-2t}, \quad t \geq 0.$$

Because the system matrix was given in diagonal form, another possibility would have been to compute the impulse response as

$$h(t) = Ce^{At}B + D\delta(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0e^{-3t} & \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + 2\delta(t), \quad t \geq 0.$$

The step response is computed by e.g. integrating the impulse response

$$\begin{aligned} y(t) &= \int_0^t h(\tau)d\tau = \int_0^t \left(2\delta(\tau) + 2e^{-3\tau} - 5e^{-2\tau}\right)d\tau \\ &= 2 + \left[\frac{5}{2}e^{-2\tau} - \frac{2}{3}e^{-3\tau}\right]_0^t \\ &= \frac{1}{6} + \frac{5}{2}e^{-2t} - \frac{2}{3}e^{-3t}, \quad t \geq 0. \end{aligned}$$

b. The transfer function has the partial fraction expansion

$$G(s) = \frac{1}{s+1} + \frac{1}{s+2}$$

and the impulse response becomes

$$h(t) = \mathcal{L}^{-1}G(s) = e^{-t} + e^{-2t}, \quad t \geq 0.$$

The step response is thus given by

$$y(t) = \int_0^t h(\tau)d\tau = \frac{3}{2} - e^{-t} - \frac{1}{2}e^{-2t}, \quad t \geq 0.$$

c. $h(t) = 5\delta(t) + 3e^{-t}, \quad y(t) = 8 - 3e^{-t}, \quad t \geq 0$

d. $h(t) = 3\delta(t) + e^{-t}\sin 2t + e^{-t}\cos 2t = 3\delta(t) + \sqrt{2}e^{-t}\sin\left(2t + \frac{\pi}{4}\right)$
$y(t) = 3 + \frac{1}{5}e^{-t}\left(3 + \sin 2t - 3\cos 2t\right), \quad t \geq 0$

2.4 After the Laplace transform, one obtains

$$\begin{aligned} sX &= AX + BU \\ Y &= CX + DU \end{aligned}$$

Solve for X

$$\begin{aligned} (sI - A)X &= BU \\ X &= (sI - A)^{-1}BU \end{aligned}$$

This gives

$$Y = C(sI - A)^{-1}BU + DU = \Big(C(sI - A)^{-1}B + D\Big)U$$

2.5 **a.** The poles are the solutions of the characteristic equation $s^2+4s+3 = 0$, i.e. $s = -1$ and $s = -3$. The system lacks zeros. Since all poles have negative real part, the system is stable.

b. The static gain is $G(0) = \frac{1}{3}$.

c. At a step response, the input signal $u(t)$ is a step, that has the Laplace transform $U(s) = \frac{1}{s}$. The output becomes

$$Y(s) = G(s)U(s) = \frac{1}{s^2 + 4s + 3}\frac{1}{s}$$

The final value can be calculated using the final value theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{1}{s^2 + 4s + 3}\frac{1}{s} = \frac{1}{3}$$

The final value theorem may be used only if we know that the final value exists (i.e., that $y(t)$ does not go to infinity). Since we have shown that the system is stable, we know that the final value exists. The initial value may, in the same manner, be computed using the initial value theorem

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{1}{s^2 + 4s + 3}\frac{1}{s} = 0$$

d. At an impulse response, the Laplace transform of the input signal is $U(s) = 1$. The output signal becomes

$$Y(s) = \frac{1}{s^2 + 4s + 3}$$

We use the final and initial value theorems

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{1}{s^2 + 4s + 3} = 0$$

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{1}{s^2 + 4s + 3} = 0$$

e. The step response of the system is given by

$$Y(s) = \frac{1}{s^2 + 4s + 3}\frac{1}{s}$$

from a previous subproblem. The derivative of a signal is in the Laplace domain given by multiplication of s. We may denote the derivative of the step response $z(t)$ and get

$$Z(s) = sY(s) = \frac{1}{s^2 + 4s + 3}$$

We see that the derivative of the step response is the same as the impulse response, and from the previous subproblem we thus get

$$\lim_{t\to 0} z(t) = 0$$

2.6 **a.** The poles are the solutions the characteristic equation $s^2 + 0.6s + 0.25 = 0$, i.e. $s = -0.3 \pm 0.4i$. The system lacks zeros.

b. The static gain is $G(0) = 1$.

c. The input (a step) has the Laplace transform $U(s) = \frac{1}{s}$. The output becomes

$$Y(s) = G(s)U(s) = \frac{0.25}{s(s^2 + 0.6s + 0.25)}$$

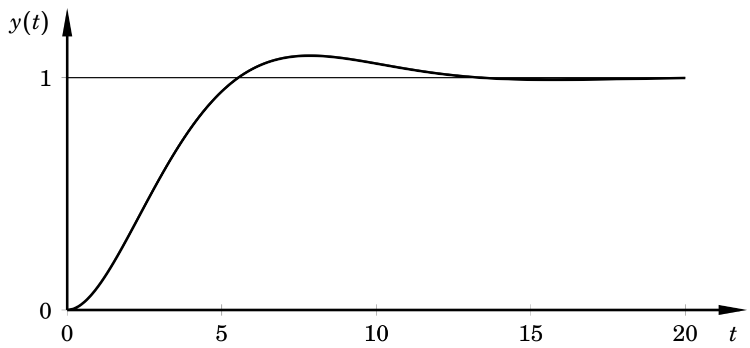
Because this system has complex poles, we first rewrite it as

$$Y(s) = \frac{\omega^2}{s(s^2 + 2\zeta\omega s + \omega^2)}$$

where $\omega = 0.5$ and $\zeta = 0.6$. We then utilize the inverse Laplace transformation (transform no. 28) and obtain

$$y(t) = 1 - 1.25e^{-0.3t}\sin(0.4t + 0.9273)$$

The step response is shown below.



2.7 Laplace transformation of the differential equation $m\ddot{y} + c\dot{y} + ky = f$ yields

$$(ms^2 + cs + k)Y = F$$

and the transfer function is hence

$$G(s) = \frac{1}{ms^2 + cs + k}.$$

The poles are $s = -c/2m \pm i\sqrt{k/m - c^2/4m^2}$. A change in k implies a change of the imaginary part of the poles. A change in c affects both the real and imaginary parts.

The poles cannot end up in the right half plane due to physical reasons, since $c \geq 0$ and $m > 0$.

2.8 **a.** $G(s) = \dfrac{1}{LCs^2 + RCs + 1}$

b. $G(s) = \dfrac{1}{Ts + 1}, \quad T = \dfrac{A}{a}\sqrt{\dfrac{2h^0}{g}}$

2.9 **a.** To be asymptotically stable, all eigenvalues of the system matrix A must lie strictly within the left half plane (LHP). I.e. $\text{Re}(\lambda_i) < 0 \ \forall \ i$.

The eigenvalues of A are given by the characteristic equation

$$\det(\lambda I - A) = 0$$

which in this case has two solutions, $\lambda_1 = -i$ and $\lambda_2 = i$. Since the eigenvalues do not lie strictly within the LHP, the system is not asymptotically stable.

b. If all the eigenvalues of A lie strictly within the LHP, we are guaranteed stability. If any eigenvalue lies strictly in the RHP we have an unstable system. If, on the other hand, there are eigenvalues on the imaginary axis, the system can be either stable or unstable.

In this example there are no eigenvalues in the RHP. Additionally, all eigenvalues on the imaginary axis are distinct. This tells us that the system is stable.

2.10 The characteristic equation is

$$s^3 + 2s^2 + 3s + 7 = 0$$

The transfer function is stable if all coefficients are positive, which is the case, and if the product of the s^2- and s^1 coefficients is greater than the s^0 coefficient. The transfer function is therefore not stable, since $2 \cdot 3 < 7$.

2.11 First and second order systems are stable if and only if the coefficients in the denominator polynomial of the transfer function are positive. All transfer functions except for G_3 are stable. All step responses except for E corresponds to stable systems. Thus, $G_3 = E$.

All transfer functions except G_3, G_4, and G_7 have a static gain of $G_i(0) = 1$, which means that the final value of the step response is one. G_7 has a static gain of $G_7(0) = 2/3$, which gives $G_7 = C$.

Step response D has a derivative that is not equal to 0 at $t = 0$. Checking the initial derivative of the transfer functions gives that this only is true for G_1 and G_5. G_1 has a time constant of 10s while G_5 has a time constant of 1s. Thus, $G_5 = D$.

Now two step responses remain, A and B, which corresponds to second order systems with complex poles and a static gain of one. The transfer functions that fulfills these specifications are G_2 and G_6. The relative damping ζ is less for G_6 than for G_2, which gives $G_2 = A$ and $G_6 = B$.

2.12 PZ1. The system has the poles in $-1/4 \pm i$ and a zero in -1. The transfer function is thus

$$G(s) = K\frac{s+1}{(s+\frac{1}{4})^2+1} \approx K\frac{s+1}{s^2+\frac{1}{2}s+1}.$$

The initial value, initial derivative and final value become

$$\begin{aligned} y(0) &= \lim_{s\to+\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to+\infty} sG(s) = K \neq 0 \\ \lim_{t\to+\infty} y(t) &= G(0) = K \neq 0 \end{aligned}$$

The step response is oscillating with period $T = 2\pi/1 \approx 6$. This must be step response D.

PZ2. The system has poles in -1 and -2 and a zero in 1. The transfer function is

$$G(s) = K\frac{s-1}{(s+1)(s+2)}$$

The initial value, initial derivative and final value become

$$\begin{aligned} y(0) &= \lim_{s\to+\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to+\infty} sG(s) = K \neq 0 \\ \lim_{t\to+\infty} y(t) &= G(0) = -\frac{K}{2} \neq 0 \end{aligned}$$

We see that the initial derivative and the final value have different signs. This fits step response F.

PZ3. The system has poles in $-1/4 \pm i$ and a zero in 0. The transfer function is

$$G(s) = K\frac{s}{(s+\frac{1}{4})^2+1} \approx K\frac{s}{s^2+\frac{1}{2}s+1}$$

The initial value, initial derivative and final value become

$$\begin{aligned} y(0) &= \lim_{s\to+\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to+\infty} sG(s) = K \neq 0 \\ \lim_{t\to+\infty} y(t) &= G(0) = 0 \end{aligned}$$

The step response is oscillating with period $T = 2\pi/1 \approx 6$. This is step response G.

PZ4. The system has poles in -1 and -2 and a zero in -3. The transfer function is

$$G(s) = K\frac{s+3}{(s+1)(s+2)}.$$

The initial value, initial derivative and final value become

$$\begin{aligned} y(0) &= \lim_{s\to+\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to+\infty} sG(s) = K \neq 0 \\ \lim_{t\to+\infty} y(t) &= G(0) = \frac{3K}{2} \neq 0 \end{aligned}$$

The initial derivative and final value have the same sign. The only nonoscillative step response which suits these criteria is C.

2.13 a. For the impulse response, $u(t) = \delta(t)$ (Dirac function). We have

$$U(s) = 1$$
$$Y(s) = G(s)U(s) = G(s)$$

Initial value:

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} sG(s)$$

Final value:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)$$

All transfer functions have stable poles, and thus we may use the final and initial value theorems.

	$y(0)$	$\lim_{t\to\infty} y(t)$
$G_1(s)$	1	0
$G_2(s)$	3	0
$G_3(s)$	0	0
$G_4(s)$	0	0
$G_5(s)$	0	1
$G_6(s)$	0	1

Both impulse responses in the figure have initial value 0 and final value 0. Thus, they correspond to $G_3(s)$ and $G_4(s)$. The poles of $G_3(s)$ are located in $s = -1$, and the poles of $G_4(s)$ in $s = -3$. $G_4(s)$ is thus a faster system, which corresponds to the impulse response for lemonade. $G_3(s)$ is slower, which corresponds to the impulse response for whole grain pasta.

b. Normally, you eat a certain amount of food in a relatively short time, and then do not eat for a longer period of time. Thus, you may model food intake as an impulse, that occurs instantaneously compared to the time it takes for the body to digest the food.

A step response in food intake would correspond to eating continuously during a longer time. Feeding through intravenous therapy could be described by a step response.

2.14 a.

$$\begin{aligned} Y &= G_1(U + G_2Y) \\ Y(1 - G_1G_2) &= G_1U \\ Y &= \frac{G_1}{1 - G_1G_2}U \end{aligned}$$

b.

$$\begin{aligned} Y &= G_2(H_1U + G_1U + H_2Y) \\ Y(1 - G_2H_2) &= (G_2H_1 + G_2G_1)U \\ Y &= \frac{G_2H_1 + G_2G_1}{1 - G_2H_2}U \end{aligned}$$

c. Introduce the auxiliary variable Z, being the output of G_1

$$\begin{aligned} Z &= G_1(U + G_3(Z + G_2Z)) \\ Z(1 - G_1G_3 - G_1G_3G_2) &= G_1U \\ Z &= \frac{G_1}{1 - G_1G_3 - G_1G_3G_2}U \\ Y &= \frac{G_2G_1}{1 - G_1G_3 - G_1G_3G_2}U \end{aligned}$$

d.

$$\begin{aligned} Y &= G_2(-H_2Y + G_1(U - H_1Y)) \\ Y(1 + G_2H_2 + G_2G_1H_1) &= G_2G_1U \\ Y &= \frac{G_2G_1}{1 + G_2H_2 + G_2G_1H_1}U \end{aligned}$$

2.15 a. From the block diagram we get

$$\begin{aligned} Y(s) &= G_P(s)(U(s) + D(s)) \\ U(s) &= G_R(s)E(s) \\ E(s) &= R(s) - Y(s) \end{aligned}$$

Solve for $Y(s)$:

$$Y(s) = \frac{G_P(s)G_R(s)}{1 + G_P(s)G_R(s)}R(s) + \frac{G_P(s)}{1 + G_P(s)G_R(s)}D(s)$$

The system has two input signals, $R(s)$ and $D(s)$. The transfer function from $R(s)$ to $Y(s)$ is

$$G_{yr}(s) = \frac{G_P(s)G_R(s)}{1 + G_P(s)G_R(s)}$$

b. The transfer function from $D(s)$ to $Y(s)$ is

$$G_{yd}(s) = \frac{G_P(s)}{1 + G_P(s)G_R(s)}$$

c. Solve for $E(s)$:

$$E(s) = \frac{1}{1 + G_P(s)G_R(s)}R(s) - \frac{G_P(s)}{1 + G_P(s)G_R(s)}D(s)$$

The transfer function from $R(s)$ to $E(s)$ becomes

$$G_{er}(s) = \frac{1}{1 + G_P(s)G_R(s)}$$

d. Solve for $U(s)$:

$$U(s) = \frac{G_R(s)}{1 + G_P(s)G_R(s)}R(s) - \frac{G_P(s)G_R(s)}{1 + G_P(s)G_R(s)}D(s)$$

The transfer function from $D(s)$ to $U(s)$ becomes

$$G_{ud}(s) = -\frac{G_P(s)G_R(s)}{1 + G_P(s)G_R(s)}$$

2.16 a. Partial fraction expansion yields

$$G(s) = \frac{s^2 + 6s + 7}{s^2 + 5s + 6} = \frac{s + 1}{s^2 + 5s + 6} + 1 = \frac{-1}{s + 2} + \frac{2}{s + 3} + 1$$

One has a certain freedom when choosing the coefficients of the B and C matrices. However, the products b_ic_i remain constant. Let e.g. $b_1 = b_2 = 1$. This enables us to immediately write the system in diagonal form:

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} -1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \end{pmatrix} u \end{aligned}$$

b. First rewrite the system as

$$G(s) = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2} + d = \frac{s+1}{s^2 + 5s + 6} + 1$$

The controllable canonical form can be directly read from the transfer function

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -5 & -6 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \end{pmatrix} u \end{aligned}$$

c. The observable canonical form is obtained in the same manner

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -5 & 1 \\ -6 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \end{pmatrix} u \end{aligned}$$

Solutions to Exercises 3

Frequency Analysis

3.1

a. The output is given by

$$y(t) = |G(3i)| \sin\Big(3t + \arg G(3i)\Big)$$

where

$$|G(i\omega)| = \frac{0.01\sqrt{1+100\omega^2}}{\sqrt{1+\omega^2}\sqrt{1+0.01\omega^2}}$$

and

$$\arg G(i\omega) = \arctan 10\omega - \arctan \omega - \arctan 0.1\omega$$

For $\omega = 3$ one obtains $|G(i\omega)| = 0.0909$ and $\arg G(i\omega) = -0.003$ which gives

$$y(t) = 0.0909 \sin(3t - 0.003)$$

b. Reading from the plot yields $|G(3i)| \approx 0.09$ and $\arg G(3i) \approx 0$. We obtain

$$y(t) = 0.09 \sin 3t$$

3.2 **a.** The air temperature will affect the water temperature much faster in the small garden pool than in the sea. Faster influence means that the gain is greater for higher frequencies. Thus, the solid line represents the garden pool – $G_2(s)$, and the dashed line the sea water – G_1(s).

b. The time scale in the Bode diagram is hours. Thus, we convert the period time to hours:

$$1 \text{ year} = 365 \cdot 24 \text{ h} = 8760 \text{ h}$$

The angular frequency of the oscillation thus becomes

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8760} \text{ rad/h} = 7 \cdot 10^{-4} \text{ rad/h.}$$

We read off the dashed amplitude curve at this frequency and get

$$|G(i\omega)| \approx 0.5$$

This means that the oscillation in the output (the water temperature) has half as big amplitude as the oscillation in the input (the air temperature) at this frequency. The difference between the maximum and minimum temperature in the air is $\Delta T_{luft} = 19°\text{C} - (-5°\text{C}\ ) = 24°\text{C}$. The difference between the maximum and minimum temperature in the water thus becomes $\Delta T_{vatten} = 0.5 \cdot 24°\text{ C} = 12°\text{C}$.

c. The angular frequency for the oscillation is now

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24} \text{ rad/h} = 0.26 \text{ rad/h.}$$

We read off the solid phase curve at this frequency and get

$$\arg(G(i\omega)) \approx -30°.$$

This means that the peak of the output signal will be $\frac{30°}{360°} \approx 0.08$ period $= 0.08 \cdot 24$ h $= 1.92$ h later than the peak of the input signal. Thus, the pool is as warmest around 15.00.

3.3 **a.** The output is given by

$$y(t) = |G(i\omega)| \sin\Big(\omega t + \arg G(i\omega)\Big)$$

where

$$|G(i\omega)| = \left| \frac{10}{(i\omega)^2 + 0.5i\omega + 1} \right| = \frac{10}{\sqrt{(1-\omega^2)^2 + (0.5\omega)^2}}$$

and

$$\arg G(i\omega) = \arg \frac{10}{(i\omega)^2 + 0.5i\omega + 1} = -\arg\big((1-\omega^2) + 0.5\omega i\big)$$
$$= \begin{cases} -\arctan \frac{0.5\omega}{1-\omega^2}, & \omega < 1 \\ -\pi/2, & \omega = 1 \\ -\arctan \frac{0.5\omega}{1-\omega^2} - \pi, & \omega > 1 \end{cases}$$

The output becomes

$$10.4 \sin(0.2t - 5.9^\circ), \quad 20.0 \sin(t - 90.0^\circ), \quad 0.011 \sin(30t - 179.0^\circ)$$

b. For $\omega = 0.2$ one reads $|G(i\omega)| \approx 10$ and $\arg G(i\omega) \approx -5^\circ$. For $\omega = 1$ one reads $|G(i\omega)| \approx 20$ and $\arg G(i\omega) \approx -90^\circ$. For $\omega = 30$ one reads $|G(i\omega)| \approx 0.01$ and $\arg G(i\omega) \approx -180^\circ$. The output is approximately

$$10 \sin(0.2t - 5^\circ), \quad 20 \sin(t - 90^\circ), \quad 0.01 \sin(30t - 180^\circ)$$

3.4 We use the following general approach to draw Bode plots

- Factor the transfer function of the system.
- Determine the low frequency asymptote (small s).
- Determine the corner frequencies (i.e. the magnitude of the poles and zeros of the system.)
- Draw the asymptotes of the gain curve from low to high frequencies, aided by the following rules of thumb
  - A pole decreases the slope of the gain curve by 1 at the corner frequency.
  - A zero increases the slope of the gain curve by 1 at the corner frequency.
- Draw the asymptotes of the phase curve from low to high frequencies, aided by the following rules of thumb
  - A (stable) pole decreases the value of the phase curve by 90° at the corner frequency.
  - A (stable) zero increases the value of the phase curve by 90° at the corner frequency.
- Draw the real gain- and phase curves, aided by the asymptotes and sample curves in the collection of formulae.

a. The transfer function can be written

$$G(s) = 3 \cdot \frac{1}{1 + s/10}$$

Low frequency asymptote: $G(s) \approx 3$.

Corner frequency: $\omega = 10$ rad/s (pole).

The gain curve starts with slope 0 and value 3. The slope decreases by 1 at $\omega = 10$ rad/s, due to the pole, and thus ends being -1.

The phase curve starts at 0°. The phase is decreased by 90° at $\omega = 10$ rad/s, due to the pole, and thus ends being $-90°$.

The asymptotes and the finished Bode plots are shown in Figure S3.1.

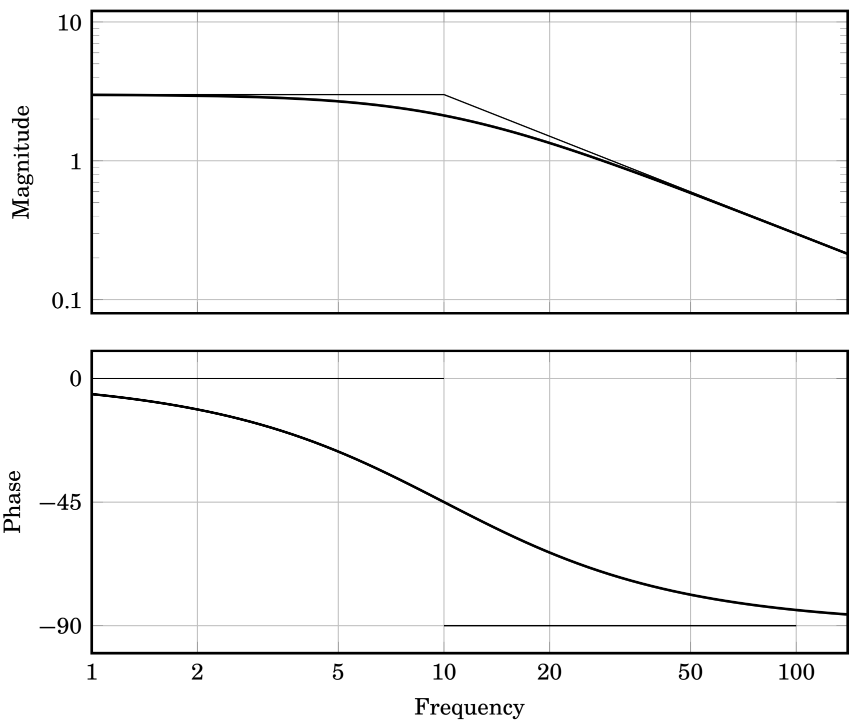


Figure S3.1 The Bode plot of $G(s) = \frac{3}{1+s/10}$.

b. The transfer function can be written

$$G(s) = 10 \cdot \frac{1}{1+10s} \cdot \frac{1}{1+s}$$

Low frequency asymptote: $G(s) \approx 10$.

Corner frequencies: $\omega = 0.1$ rad/s (pole), $\omega = 1$ rad/s (pole).

The gain curve starts with slope 0 and value 10. The slope is decreased by 1 at $\omega = 0.1$ rad/s, due to the first pole, and by 1 at $\omega = 1$ rad/s, due to the second pole. Thus, the final slope becomes -2.

The phase curve starts at 0°. The phase is decreased by 90° at $\omega = 0.1$ rad/s, due to the first pole, and by 90° at $\omega = 1$ rad/s, due to the second pole. Thus, the final phase is $-180°$.

The asymptotes and the finished Bode plot are shown in Figure S3.2.

c. The transfer function can be written

$$G(s) = e^{-s} \cdot \frac{1}{1+s}$$

Low frequency asymptote: $G(s) \approx 1$.

Corner frequency: $\omega = 1$ rad/s (pole).

The delay (e^{-s}) does not affect the gain curve, which starts with slope 0 and value 1. The slope is decreased by 1 at $\omega = 1$ rad/s, due to the pole, and the final slope is thus -1.

The phase curve is harder to sketch. One approach is to draw the asymptotes of the system without the delay and superposition it with the phase curve of e^{-s}, which can be obtained from the collection of formulae or by computing some points and interpolating between these.

Anyway, we see that the phase curve starts at 0° and that the phase then decreases both due to the pole (at $\omega = 1$ rad/s) and the delay. The delay causes the phase to approach $-\infty$ for large ω.

The finished plot is shown in Figure S3.3.

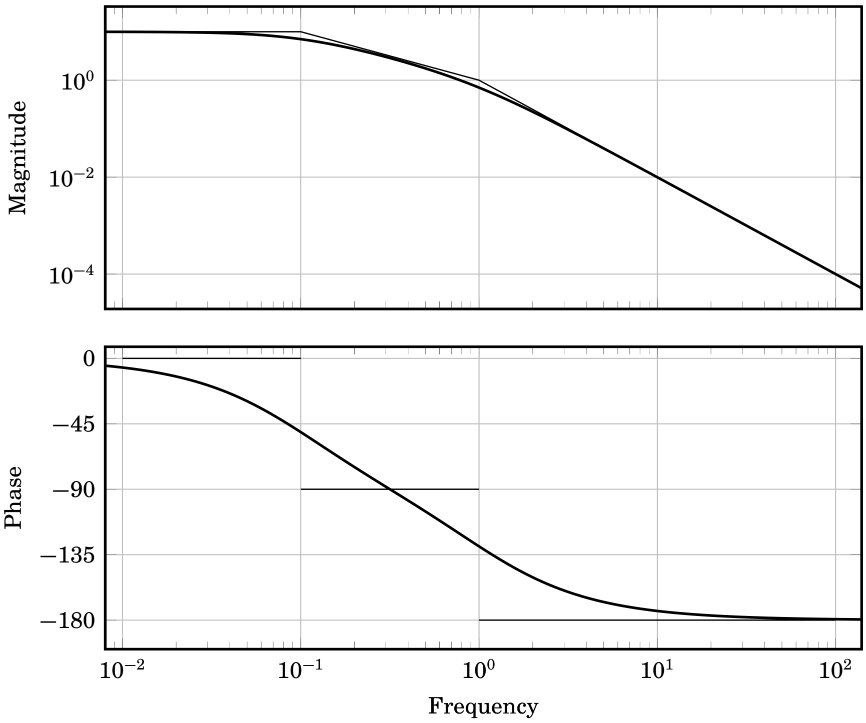


Figure S3.2 Bode plot of $G(s) = \dfrac{10}{(1+10s)(1+s)}$.

d. The transfer function can be written

$$G(s) = \frac{1}{s} \cdot (1+s) \cdot \frac{1}{1+s/10}$$

Low frequency asymptote: $G(s) \approx \dfrac{1}{s}$.

Corner frequencies: $\omega = 1$ rad/s (zero), $\omega = 10$ rad/s (pole).

The gain curve starts with slope -1. The slope increases by 1 at $\omega = 1$ rad/s, due to the zero, and at $\omega = 10$ rad/s the slope decreases by 1, due to the pole. Consequently, the final slope is -1.

The phase curve starts at $-90°$. The phase increases by $90°$ at $\omega = 1$ rad/s, due to the zero, and decreases by $90°$ at $\omega = 10$ rad/s, due to the pole. Consequently, the final phase is $-90°$.

The finished plot is shown in Figure S3.4.

e. The transfer function can be written

$$G(s) = 2 \cdot \frac{1}{s} \cdot (1+5s) \cdot \frac{1}{1 + 2\zeta(s/2) + (s/2)^2}$$

where $\zeta = 0.2$.

Low frequency asymptote: $G(s) \approx \dfrac{2}{s}$.

Corner frequencies: $\omega = 0.2$ rad/s (zero), $\omega = 2$ rad/s (complex conjugated pole pair).

The gain curve starts with slope -1. The slope is increased by 1 at $\omega = 0.2$ rad/s, due to the zero, and decreased by 2 at $\omega = 2$ rad/s, due to the pole pair. Consequently, the final slope is -2.

The phase curve starts at $-90°$. The phase is increased by $90°$ at $\omega = 0.2$ rad/s, due to the zero, and decreased by $180°$ at $\omega = 2$ rad/s, due to the pole pair. Consequently, the final phase is $-180°$.

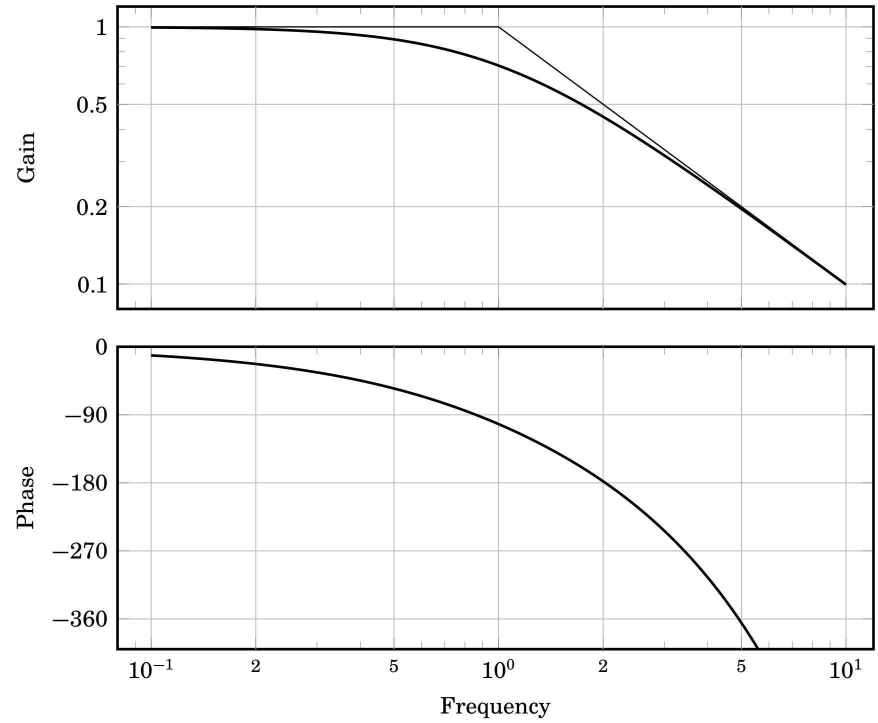


Figure S3.3 Bode plot of $G(s) = \frac{e^{-s}}{1+s}$.

The low damping ($\zeta = 0.2$) of the complex conjugated pole pair gives the gain curve a resonance peak at $\omega = 2$ rad/s. Additionally, the phase decreases rapidly at this frequency, cf. the sample curves in the collection of formulae. The finished plot is shown in Figure S3.5.

3.5 **a.** The Nyquist curve start in 3 (the static gain) for $\omega = 0$ rad/s. Both the gain and phase are strictly decreasing, which makes the curve turn clockwise while its distance to the origin decreases. The gain and phase approach 0 and $-90°$, respectively, for large values of ω. The curve is thus bound to the fourth quadrant and approaches the origin along the negative imaginary axis as $\omega \to \infty$.

Aided by this analysis, one can now sketch the Nyquist curve by choosing a few frequencies (e.g. $\omega = 1$, 10 and 100 rad/s) and drawing the corresponding points in the complex plane. The finished curve is shown in Figure S3.6.

b. The Nyquist curve starts in 10 (the static gain) for $\omega = 0$ rad/s. Both the gain and phase are strictly decreasing, which makes the curve turn clockwise while its distance to the origin decreases. The gain and phase approach 0 and $-180°$, respectively, for large values of ω. The curve will thus go from the fourth to the third quadrant, approaching the origin along the negative real axis as $\omega \to \infty$.

The intersection with the negative imaginary axis can be drawn by reading off the magnitude when the phase is $-90°$. One can now sketch the Nyquist curve by choosing a few additional frequencies (e.g. $\omega = 0.1$, 1 rad/s) and drawing the corresponding points in the complex plane. The finished curve is shown in Figure S3.7.

c. The Nyquist curve starts in 1 (the static gain) for $\omega = 0$ rad/s. Both the gain and phase are strictly decreasing, which makes the curve turn clockwise while its distance to the origin decreases. The gain and phase approach 0 and $-\infty$, respectively, for large values of ω. The curve will thus rotate infinitely many times as it approaches the origin. The first intersections with the axis can be drawn by reading off the magnitude when the phase is $-90°$, $-180°$, $-270°$ and $-360°$, respectively. The finished curve is shown in Figure S3.8.

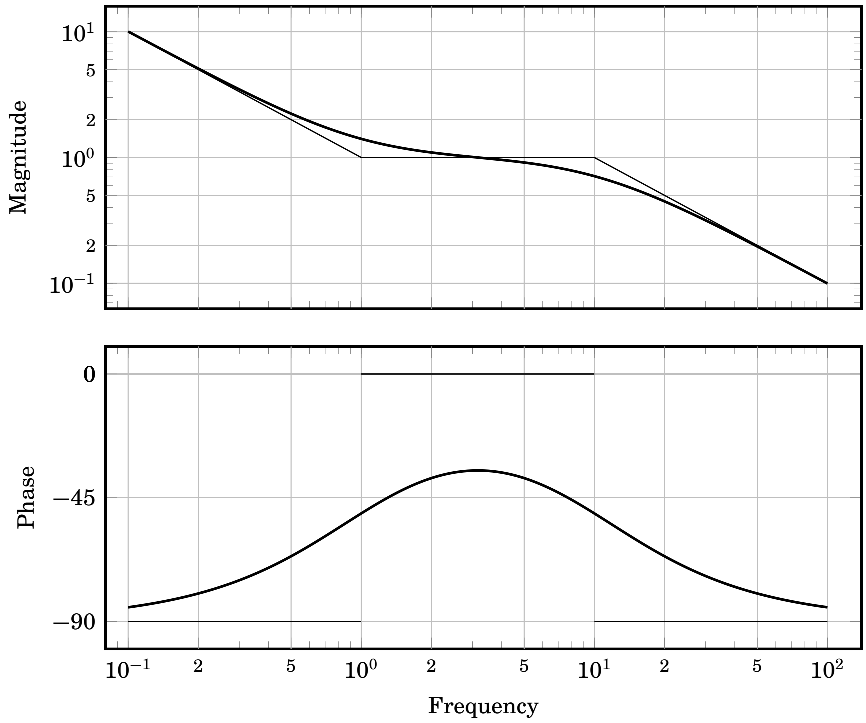


Figure S3.4 Bode plot of $G(s) = \dfrac{1+s}{s(1+s/10)}$.

3.6 Let the sought transfer function be $G(s)$. The gain curve starts with slope -1, which indicates that $G(s)$ contains a factor $\frac{1}{s}$ (an integrator). We observe that there are two corner frequencies: $\omega_1 = 1$ and $\omega_2 = 100$ rad/s. The gain curve breaks upwards once at ω_1 and downward once at ω_2. Hence, the nominator hosts a factor $1 + s$, whereas the denominator contains a factor $1 + s/100$. In addition, $G(s)$ contains a constant gain K. We thus have

$$G(s) = \frac{K(1+s)}{s(1+s/100)}$$

We evaluate the low frequency asymptote of the gain curve at e.g. $\omega = 0.01$ rad/s, in order to determine K. This yields

$$|G(0.01i)| = \tfrac{K}{0.01} = 1 \quad \Rightarrow \quad K = 0.01$$

Finally we verify that the phase curve matches this system.

3.7 Let the sought transfer function be $G(s)$. The gain curve has two corner frequencies: $\omega_1 = 2$ and $\omega_2 = 100$ rad/s. The gain curve breaks downwards once at ω_1 and three times at ω_2. Thus the denominator of $G(s)$ contains the factors $(1 + \frac{s}{2})$ and $(1 + \frac{s}{100})^3$. The slope of the low frequency asymptote is 1. Thus $G(s)$ has a factor s in the nominator. Additionally, $G(s)$ contains a constant gain K. We have

$$G(s) = \frac{Ks}{(1+\frac{s}{2})(1+\frac{s}{100})^3}$$

The factor K is computed by determining a point on the LF asymptote, e.g. $G_{LF}(s) = Ks$

$$|G_{LF}(i\omega)| = K\omega = 1$$

for $\omega = 0.5$ rad/s. This gives

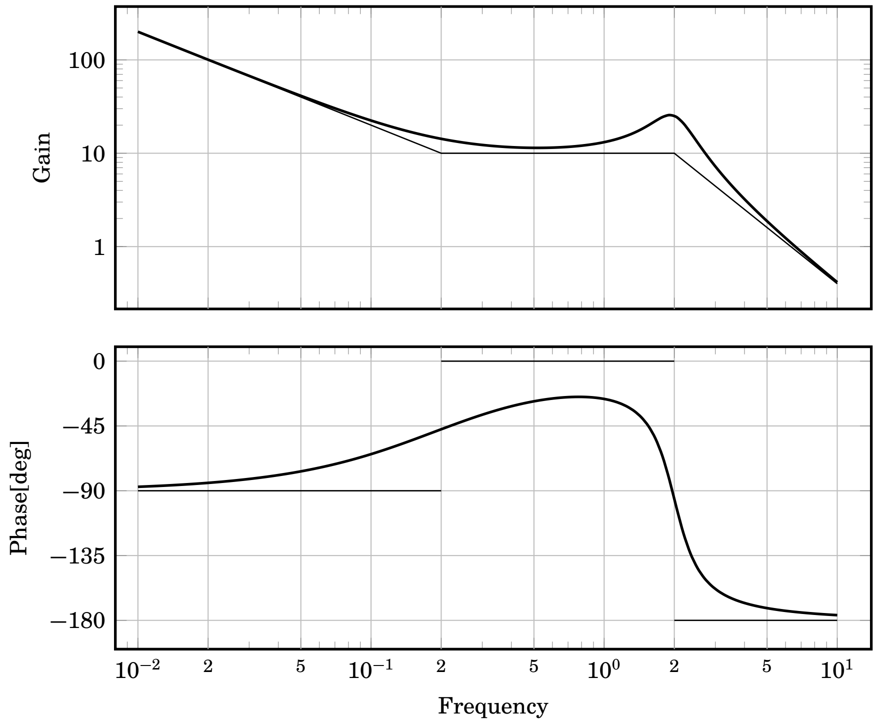


Figure S3.5 Bode plot of $G(s) = \dfrac{2(1+5s)}{s(1+0.2s+0.25s^2)}$.

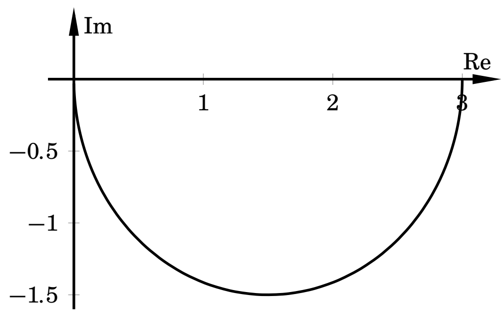


Figure S3.6 Nyquist curve of $G(s) = \dfrac{3}{1+s/10}$.

$$K = 2$$

Finally we verify by checking that the phase curve matches this system.

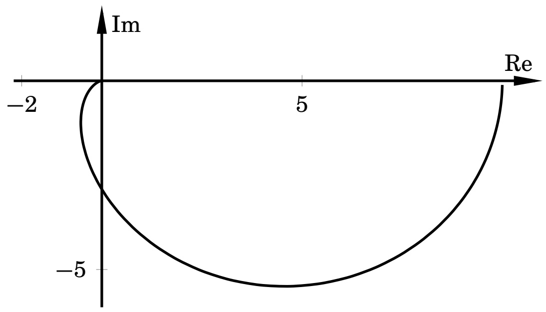


Figure S3.7 Nyquist curve of $G(s) = \dfrac{10}{(1+10s)(1+s)}$.

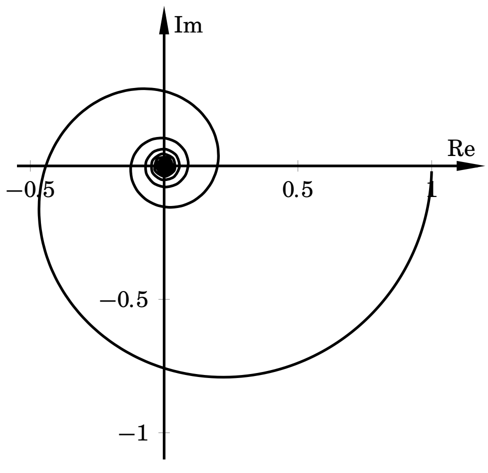


Figure S3.8 Nyquist curve of $G(s) = \dfrac{e^{-s}}{1+s}$.

Solutions to Exercises 4

Feedback Systems

4.1 **a.** Laplace transformation of the differential equation yields

$$sY(s) + 0.01Y(s) = 0.01U(s)$$

The transfer function $G_P(s)$ is thus given by

$$Y(s) = G_P(s)U(s) = \frac{0.01}{s+0.01}U(s)$$

b. The transfer function of the closed loop system becomes

$$G(s) = \frac{G_P(s)G_R(s)}{1+G_P(s)G_R(s)} = \frac{\frac{0.01}{s+0.01}K}{1+\frac{0.01}{s+0.01}K} = \frac{0.01K}{s+0.01+0.01K}$$

c. The desired and actual characteristic polynomials are the same if all their coefficients match. Identification of coefficients yields

$$0.1 = 0.01 + 0.01K \quad \Leftrightarrow \quad K = 9$$

4.2 Since $r(t) = 0$, the control error becomes $e(t) = -y(t)$.

$$Y(s) = G_P(s)(F(s) - G_R(s)Y(s)) \quad \Leftrightarrow \quad Y(s) = \frac{G_P(s)}{1+G_R(s)G_P(s)}F(s)$$

If $f(t)$ is a unit step, we have $F(s) = \frac{1}{s}$.

a. Seek $y(\infty)$ for $G_R = K$

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{1}{(ms^2+ds+K)}\frac{1}{s} = \frac{1}{K}$$

The function $sY(s)$ has all poles in the left-half plane when the parameters m, d and K are positive.

b. The same assignment, but with $G_R(s) = K_1 + K_2/s$. This yields

$$\begin{aligned} y(\infty) &= \lim_{s\to 0} s\frac{1}{(ms^2+ds+K_1+\frac{K_2}{s})}\frac{1}{s} \\ &= \lim_{s\to 0} \frac{s}{ms^3+ds^2+K_1s+K_2} = 0 \end{aligned}$$

under the assumption of stability, which is the case for $m > 0$, $d > 0$ and $K_1 > \frac{m}{d}K_2 > 0$. Rule: If the disturbance is a step, one needs at least one integrator before the point in the block diagram where the disturbance is introduced, in order to make the stationary error zero.

4.3 **a.** For the closed loop system it holds, when $R = 0$, that

$$U(s) = K(0 - Y(s)) = -K(G_P(s)U(s) + N(s))$$

from which one obtains

$$\begin{aligned} U(s) &= \frac{-K}{1+KG_P(s)}N(s) \\ Y(s) &= G_P(s)U(s) + N(s) = \frac{1}{1+KG_P(s)}N(s) \end{aligned} \tag{4.1}$$

b. Inserting $G_P(s) = \frac{1}{s+1}$ into (4.1) yields the relations

$$U(s) = \frac{-K(s+1)}{s+1+K} N(s)$$
$$Y(s) = G_P(s)U(s) + N(s) = \frac{s+1}{s+1+K} N(s) =: G_{yn}(s)N(s)$$

In stationarity it holds that

$$\begin{aligned} y(t) &= A|G_{yn}(i\omega)| \sin(\omega t + \arg G_{yn}(i\omega)) \\ &= A\frac{\sqrt{1+\omega^2}}{\sqrt{(K+1)^2+\omega^2}} \sin\left(\omega t + \arctan\omega - \arctan\frac{\omega}{K+1}\right) \\ u(t) &= -Ky(t) \\ &= -KA\frac{\sqrt{1+\omega^2}}{\sqrt{(K+1)^2+\omega^2}} \sin\left(\omega t + \arctan\omega - \arctan\frac{\omega}{K+1}\right) \end{aligned}$$

c. With $A = 1$ and $K = 1$ the amplitudes of the oscillations in u and y become

$$A_u = \sqrt{\frac{1+\omega^2}{4+\omega^2}}$$
$$A_y = \sqrt{\frac{1+\omega^2}{4+\omega^2}}$$

For $\omega = 0.1$ rad/s the amplitudes become

$$A_u \approx 0.5$$
$$A_y \approx 0.5$$

while $\omega = 10$ rad/s yields

$$A_u \approx 1$$
$$A_y \approx 1$$

4.4 With $G_P(s) = 1/(Js^2)$ we obtain

$$\begin{aligned} E(s) &= \theta_{ref}(s) - \theta(s) \\ &= \theta_{ref}(s) - G_P(s)(M(s) + KG_R(s)E(s)) \\ \Rightarrow E(s) &= \frac{1}{1+KG_P(s)G_R(s)}\theta_{ref}(s) - \frac{G_P(s)}{1+KG_P(s)G_R(s)}M(s) \end{aligned}$$

Assume step changes in disturbance momentum M_d^0 and reference θ_{ref}^0. We postulate $G_R(s) = Q(s)/P(S)$, which gives

$$\begin{aligned} E(s) &= \frac{1}{1+K\dfrac{Q(s)}{Js^2P(s)}} \cdot \frac{\theta_{ref}^0}{s} - \frac{\dfrac{1}{Js^2}}{1+K\dfrac{Q(s)}{Js^2P(s)}} \cdot \frac{M_d^0}{s} \\ &= \frac{s^2JP(s)}{s^2JP(s)+KQ(s)} \cdot \frac{\theta_{ref}^0}{s} - \frac{P(s)}{s^2JP(s)+KQ(s)} \cdot \frac{M_d^0}{s} \end{aligned}$$

The stationary error becomes

$$\begin{aligned} e_\infty &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\ &= 0 - \frac{P(0)}{KQ(0)}M_d^0 = -\frac{P(0)}{KQ(0)}M_d^0 \end{aligned}$$

where we have assumed that $Q(0) \neq 0$ and that the conditions for the final value theorem are fulfilled. We see that $P(0) = 0$ yields $e_\infty = 0$. In order to eliminate persistent angular errors caused by disturbance momenta, it is consequently required to utilize a controller $G_R(s)$ with at least one pole in the origin ($P(0) = 0$).

4.5 The input of the thermocouple is the temperature $u(t)$ of the bath, which gives

$$u(t) = t \quad \Rightarrow \quad U(s) = \frac{1}{s^2}$$

The output $y(t)$ is the reading of the temperature sensor. Thus

$$Y(s) = G(s)U(s) = \frac{1}{1+sT} \cdot \frac{1}{s^2}$$

For the error $e(t) = u(t) - y(t)$ it holds that

$$E(s) = U(s) - Y(s) = \frac{1}{s^2}\left[1 - \frac{1}{1+sT}\right] = \frac{sT}{1+sT} \cdot \frac{1}{s^2}$$

The stationary error is obtained by means of the final value theorem

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^2T}{1+sT}\frac{1}{s^2} = T = 10$$

The thermocouple measurement is hence 10° less than the actual temperature. I.e. the actual temperature of the bath is 102.6°C + 10°C = 112.6°C.

Observe that the error in this has a bounded limit, despite the fact that both $u(t)$ and $y(t)$ lack (bounded) limits as $t \to \infty$. It is the *difference* between u and y which converges to a constant value.

4.6 The low frequency asymptote is

$$G_{LF}(s) = \frac{K}{s^2}$$

where the constant K is given by

$$|G_{LF}(i\omega)| = \frac{K}{\omega^2}; \quad |G_{LF}(i)| = 1 \quad \Rightarrow \quad K = 1$$

At the corner frequency $\omega_1 = 1$ rad/s the slope changes from -2 to 0, and at $\omega_2 = 5$ rad/s it changes from 0 to -1. The transfer function for the open loop system is thus

$$G_o(s) = \frac{(1+sT_1)^2}{s^2(1+sT_2)}$$

where $T_1 = 1/\omega_1 = 1$ and $T_2 = 1/\omega_2 = 0.2$.

The transfer function of the closed loop system becomes

$$G(s) = \frac{G_o(s)}{1+G_o(s)}$$

The output is $Y(s) = G(s)R(s)$ and the error $E(s)$ becomes

$$E(s) = R(s) - Y(s) = \frac{1}{1+G_o}R(s) = \frac{s^2(1+0.2s)}{s^2(1+0.2s)+(1+s)^2}R(s)$$

a.

$$R(s) = \frac{a}{s} \quad \Rightarrow \quad e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{as^2(1+0.2s)}{s^2(1+0.2s)+(1+s)^2} = 0$$

The system can thus track inputs $r(t) = a$ without a stationary error.

b.

$$R(s) = \frac{b}{s^2} \quad \Rightarrow \quad e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{bs(1+0.2s)}{s^2(1+0.2s)+(1+s)^2} = 0$$

The system can also track inputs $r(t) = bt$ without a stationary error.

c.

$$R(s) = \frac{2c}{s^3} \quad \Rightarrow \quad e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{2c(1+0.2s)}{s^2(1+0.2s)+(1+s)^2} = 2c \neq 0$$

The input $r(t) = ct^2$, however, yields a stationary error.

d. Superposition can be used, since the closed loop system is linear and time invariant (LTI). Here, the input is the sum of the inputs in sub-assignments a and b. The total (superpositioned) stationary error thus becomes $e_\infty = 0 + 0 = 0$.

e. The input $r(t) = \sin(t)$ yields

$$R(s) = \frac{1}{1+s^2} \quad \Rightarrow \quad \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s \frac{s^2(1+0.2s)}{(s^2(1+0.2s)+(1+s)^2)(1+s^2)} = 0$$

but the input $r(t) = \sin(t)$ yields the output $y(t) = y_o \sin(t+\phi)$, where

$$y_o = |G(i)|\,, \quad \phi = arg\ G(i)$$

once transients have decayed. The error $e(t) = r(t) - y(t)$ is thus also a sinusoid and the limit

$$\lim_{t\to\infty} e(t)$$

does not exist. This shows that the final value theorem should not be used without caution. It is only valid for cases where a limit really exists. The criterion is that all poles of $sE(s)$ must have negative real parts. (The factor s^2+1 in the denominator yields two poles on the imaginary axis.)

4.7 **a.** The sensitivity function is given by

$$S(s) = \frac{1}{1+G_P(s)G_R(s)} = \frac{1}{1+\frac{6.5}{(s+1)^3}} = \frac{s^3+3s^2+3s+1}{s^3+3s^2+3s+7.5}$$

b. For $\omega = 0$ rad/s we have $|S(i\omega)| = 1/7.5$. Low-frequency load disturbances are thus damped by a factor 7.5.

c. The sensitivity functions has its maximum value $|S(i\omega)| =\approx 10$ at $\omega \approx 1.6$ rad/s.

4.8 **a.** The top curve shows the complementary sensitivity function, whereas the sensitivity function is given by the bottom curve.

b. The disturbances at various frequencies are amplified according to the gain curve of the sensitivity function. Disturbances below 0.2 rad/s are hence reduced, disturbances in the range 0.2 to 2 rad/s are amplified and disturbances above 2 rad/s pass straight through. The worst case gain, 2, is obtained at the frequency 0.55 rad/s.

c. The complementary sensitivity function, corresponding to the closed loop transfer function from r to y, lies close to 1 up to approximately 0.7 rad/s.

d. The maximal magnitude of the sensitivity function equals the inverse of the minimal distance between the Nyquist curve and the point -1. The minimal distance is thus $1/2 = 0.5$. The distance to -1, as the Nyqist curve intersects the negative real axis, must hence be at least 0.5. This implies that the gain margin is at least 2.

4.9 The closed loop transfer function is given by

$$G(s) = \frac{G_o}{1+G_o} = \frac{K}{s^2+2s+K}$$

The poles of the closed loop system are given by the characteristic equation

$$s^2+2s+K=0 \quad \Rightarrow \quad s=-1\pm\sqrt{1-K}$$

For $K=0$ the roots $s_{1,2}=0,-2$, i.e. the poles of the open loop system, are obtained. The closed loop system $G(s)$ has a double pole in $s=-1$ for $K=1$. And as $K\to\infty$ the roots become

$$s_{1,2}=-1\pm i\infty$$

The root locus, i.e. the roots of the characteristic equation as K varies, is shown in Figure S4.1 .

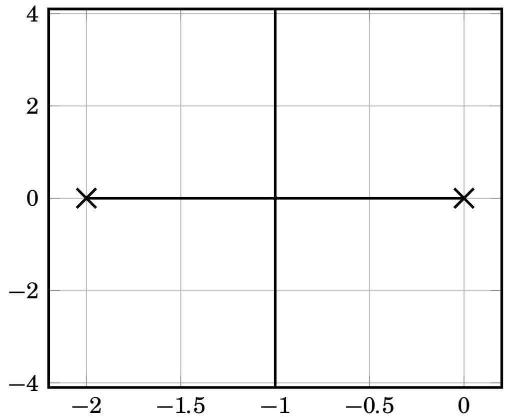


Figure S4.1 Root locus of the system in assignment 4.9.

4.10 The open loop transfer function of the system is

$$G_o(s) = \frac{K(s+10)(s+11)}{s(s+1)(s+2)} = K\frac{Q(s)}{P(s)}$$

The closed loop system becomes

$$G(s) = \frac{G_o(s)}{1+G_o(s)} = \frac{KQ(s)}{P(s)+KQ(s)}$$

The characteristic equation is thus

$$\begin{aligned} & P(s)+KQ(s)=0 \\ \Leftrightarrow & s(s+1)(s+2)+K(s+10)(s+11)=0 \\ \Leftrightarrow & s^3+(3+K)s^2+(2+21K)s+110K=0 \end{aligned}$$

a. The criterion for stability is that all coefficients of the characteristic polynomial

$$s^3+(3+K)s^2+(2+21K)s+110K$$

are positive and that

$$(3+K)(2+21K) > 110K$$

The inequality yields

$$K^2-\frac{15}{7}K+\frac{2}{7}>0$$

It is fulfilled for $K > 2$ and $K < 1/7$. The closed loop system is hence stable for

$$0 < K < \frac{1}{7}$$

and

$$K > 2$$

b. Find the root locus for the characteristic equation, $P(s) + KQ(s) = 0$

$$s(s+1)(s+2) + K(s+10)(s+11) = 0 \tag{4.2}$$

Let n = the degree of $P(s)$ and m = the degree of $Q(s)$. The root locus has a maximum of $max(n, m) = 3$ branches.

Starting points:

$$P(s) = 0 \quad \Rightarrow \quad s = 0, -1, -2$$

Ending points:

$$Q(s) = 0 \quad \Rightarrow \quad s = -10, -11$$

The third branch will approach infinity.

To the right of each real point of the root locus, there must exists an odd number of zeros of $P(s)$ and $Q(s)$. The points x, which fulfill this are

$$x < -11 \quad -10 < x < -2 \quad -1 < x < 0$$

The root locus has $|n - m| = 1$ asymptote. This is the negative real axis, since the range $x < -11$ on the real axis belongs to the root locus.

The intersection with the imaginary axis is obtained by introducing $s = i\omega$ (4) above. This yields

$$-(3+K)\omega^2 + 110K + i(-\omega^3 + (2+21K)\omega) = 0$$

The resulting equation has a solution $\omega = K = 0$ and

$$\begin{cases} -(3+K)\omega^2 + 110K = 0 \\ \omega^2 - (2+21K) = 0 \end{cases}$$

gives $K = 1/7, \quad \omega = \pm\sqrt{5}$ or $K = 2, \quad \omega = \pm\sqrt{44}$.

We know from sub-assignment a that the closed loop system is unstable for $1/7 < K < 2$. Consequently, the root locus lies in the right half plane for these values of K. The principal shape of the root locus is shown in Figure S4.2.

4.11 a. The open loop transfer function of the system is

$$G_0(s) = \frac{K}{s(s+1)(s+2)}$$

The closed loop transfer function is thus

$$G_{cl}(s) = \frac{G_0(s)}{1 + G_0(s)} = \frac{K}{s(s+1)(s+2) + K}$$

The system is asymptotically stable if all zeros of the characteristic polynomial

$$s(s+1)(s+2) + K = s^3 + 3s^2 + 2s + K$$

have negative real parts. This is the case if all coefficients are positive and if

$$3 \cdot 2 > K$$

The system is thus asymptotically stable if $0 < K < 6$.

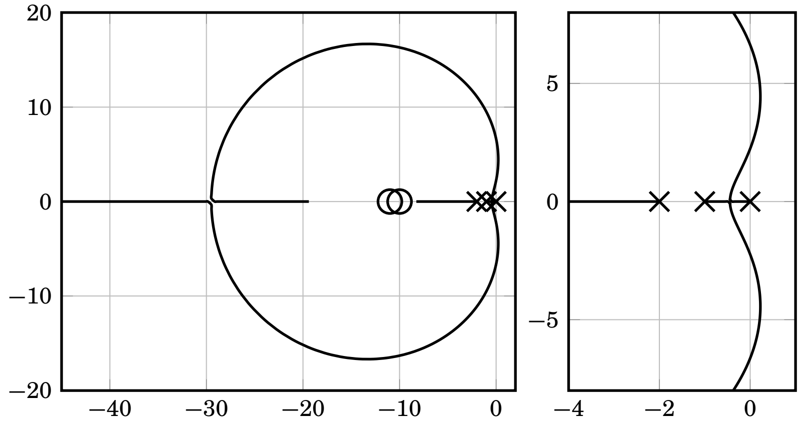


Figure S4.2 Root locus of the system in assignment 4.10. The right hand figure is a magnification of the area close to the origin.

b. Now we want to study the dependence of K on the stationary error, as the reference increases as a linear function of time. The Laplace transform of the control error is given by

$$E(s) = \frac{1}{1+G_0}R(s) = \frac{s(s+1)(s+2)}{s(s+1)(s+2)+K}R(s)$$

With $r(t) = 0.1t$, i.e. $R(s) = 0.1/s^2$, we obtain

$$E(s) = \frac{0.1(s+1)(s+2)}{s(s(s+1)(s+2)+K)}$$

The signal $sE(s)$ has all poles in the left-half plane when $0 < K < 6$, according to sub-assignment a. For this case we can utilize the final value theorem

$$e(\infty) = \lim_{s\to 0} s\frac{0.1(s+1)(s+2)}{s(s(s+1)(s+2)+K)} = \frac{0.2}{K}$$

In order to obtain a stationary error less than 5 mV for the given reference, it is required that $K > 40$. For such large values of K the system is, however, not stable. It it hence impossible to meet the specification.

4.12 According to the Nyquist theorem, the closed loop system is stable exactly for those $K > 0$, which are also

a. $K < 2$

b. $K < 1/1.5 = 2/3$

c. $K < 1/1.5 = 2/3$

d. $K < 1/(2/3) = 1.5$

4.13 The Nyquist curve intersects the negative real axis when $\arg(G_0(i\omega)) = -\pi$, i.e. when

$$-3\arctan(\omega) = -\pi$$

This is fulfilled when

$$\omega = \tan\frac{\pi}{3} = \sqrt{3}$$

The intersection point is given by

$$|G_0(i\sqrt{3})| = \frac{1}{8}$$

This means that the system is stable for $K < 8$.

4.14 The system is stable for

$$0 < K < \frac{1}{3.5} \quad \Leftrightarrow \quad 0 < K < 0.29$$

as well as

$$1 < K < \frac{1}{0.5} \quad \Leftrightarrow \quad 1 < K < 2$$

4.15 The easiest way to solve the problem is through the Nyquist theorem. The transfer function of the process is

$$G_P(s) = \frac{e^{-9s}}{(1+20s)^2}$$

The phase shift of the process is

$$\arg G_P(i\omega) = -9\omega - 2\arctan(20\omega)$$

We seek the frequency for which the phase shift is $-180°$. It is obtained by solving the equation

$$-9\omega - 2\arctan(20\omega) = -\pi$$

The equation lacks an analytic solution. However, it can be solved numerically in several ways. The solution is

$$\omega_0 \approx 0.1$$

The next step is to determine the process gain for the given frequency.

$$|G(i\omega_0)| = \frac{1}{1+400\omega_0^2} = 0.2$$

This gives us the gain margin

$$A_m = \frac{1}{0.2} = 5$$

The gain $K = 5$ is thus the maximal admissible gain.

4.16 The loop transfer function is

$$G_P(s)G_R(s) = e^{-sL} \cdot \frac{10(1+\frac{1}{2s})}{(1+10s)} = e^{-sL} \cdot \frac{5(1+2s)}{s(1+10s)}$$

The cross-over frequency is the frequency where the magnitude of the loop transfer function is equal to one.

$$|G_0(i\omega_c)| = \frac{5\sqrt{1+4\omega_c^2}}{\omega_c\sqrt{1+100\omega_c^2}} = 1$$

The equation can be solved numerically or analytically

$$\begin{aligned}
25(1+4\omega_c^2) &= \omega_c^2(1+100\omega_c^2) \\
\omega_c^4 - 0.99\omega_c^2 - 0.25 = 0 & \\
\omega_c^2 &\approx 1.199 \\
\omega_c &\approx 1.1 \quad \text{rad/min}
\end{aligned}$$

The phase at this frequency is

$$\arg G_0(i\omega_c) = \arctan 2\omega_c - \arctan 10\omega_c - 90° - \omega_c L$$

The requirement of the phase margin, $\varphi_m \geq 10°$, gives

$$\begin{aligned}
\varphi_m &= 180° + \arg G_0(i\omega_c) \\
&= 180° + \arctan 2\omega_c - \arctan 10\omega_c - 90° - \omega_c L \\
&\approx 70° - \omega_c L \geq 10°
\end{aligned}$$

This gives the following limit for the time delay L:

$$L \leq \frac{60}{\omega_c} \cdot \frac{\pi}{180} = 1\,\mathrm{min}$$

The time delay must therefore be less than one minute.

4.17 a. True. $A_m = 1/|KG_P(i\omega_0)|$, where ω_0 is the frequency for which the Nyquist curve intersects the negative real axis.

b. True. $\varphi_m = \pi + \arg G_P(i\omega)$ for $|G_P(i\omega)| = 1$.

c. False. As K is decreased, all points on the Nyquist curve move closer to the origin. Thus the phase margin increases as K is decreased.

d. True. The system is stable for $K = 1$ and all poles of $G_P(s)$ lie in the left half plane. Consequently, the simplified Nyquist criterion can be applied. For $K = 2$, the point -1 lies to the right of the Nyquist curve, when it is traversed as ω increases. The closed loop system is thus stable.

4.18 a. This is the definition of the gain margin. From the plot one sees that the phase $-180°$ corresponds to the gain ~ 0.4. This yields the gain margin $1/0.4 = 2.5$.

b. This is the definition of the phase margin. From the plot one sees that for gain 1, the phase is approximately $-140°$. This yields a phase margin of approximately $180° - 140° = 40°$.

4.19 The cross-over frequency and phase margin are read to be $\omega_c = 0.07$ and $\varphi_m = 40°$, respectively. The delay margin becomes

$$L_m = \frac{\varphi_m}{\omega_c} = \frac{40° \cdot \frac{\pi}{180°}}{0.07} = 10$$

Solutions to Exercises 5

State Feedback and Kalman Filtering

5.1 **a.** From the controllability matrix

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} \beta & 1-\beta \\ 1 & -2 \end{pmatrix}$$

we obtain $\det W_s = -\beta - 1$, i.e. controllability for all $\beta \neq -1$.

b. The observability matrix

$$W_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & \gamma \\ 0 & -2\gamma \end{pmatrix}$$

has zero determinant independent of γ, i.e. the system is not observable for any value of γ.

5.2 The controllability matrix is given by

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 4 & -8 \\ -2 & 4 \end{pmatrix}$$

The controllable states are determined by the columns of W_s and are given as $\alpha(2, -1)^T$, where α is a scalar.

5.3

$$W_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

We see that W_o is singular ($\det W_o = 0$). The state x is non-observable if and only if (iff)

$$W_o x = 0$$

We obtain a non-observable x iff $x_1 + x_2 = 0$. The non-observable states are thus given by

$$x = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where α is a number $\neq 0$.

5.4 The controllability matrix

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

is singular, yielding an uncontrollable system. We can, however, conduct a more detailed investigation. The system can be written

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 + u, & x_1(0) = 1 \\ \dfrac{dx_2}{dt} = -2x_2, & x_2(0) = 1 \end{cases}$$

Hence $x_2(t) = x_2(0)e^{-2t} = e^{-2t}$, independent of the applied control signal u. On the contrary, x_1 can be controlled by means of u, to take on any desired value. Thus, $x_2 \to 0$ as $t \to \infty$. The states (x_1, x_2) which can be reached in finite time $t < \infty$ make up the band $0 < x_2 < 1$ in Figure S5.1.

As a consequence, only the states $\begin{pmatrix} 3 \\ 0.5 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 0.1 \end{pmatrix}$ can be reached in finite time.

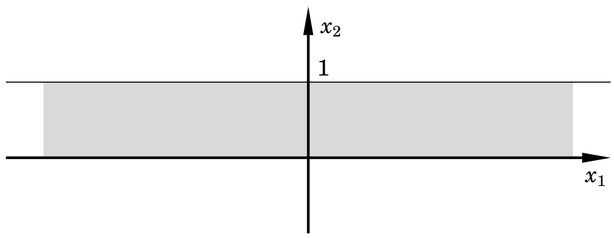


Figure S5.1 Reachable states in assignment 2.9.

5.5 The system is controllable, since

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -7 \end{pmatrix}$$

has full rank.

5.6 **a.**

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 5 & -10 \\ 0 & 0 \end{pmatrix}$$

W_s has rank 1, i.e. the system is not controllable. The states which can be reached in finite time from the origin are determined by the columns of W_s. The controllable states can be parametrized by t as $x = t \cdot (1 \quad 0)^T$.

b. $G(s) = C(sI - A)^{-}1B + D = 5/(s+2)$.

c. The following is a minimal state space representation of $G(s)$

$$\begin{cases} \dot{x} = -2x + 5u \\ y = x \end{cases}$$

5.7 **a.** True. Since the system is controllable, one can place the poles of the closed loop system arbitrarily by means of linear feedback from all state variables.

b. False. A linear state feedback does not affect the zeros of the closed loop system.

c. True if the system is observable.

d. True if the system is observable.

5.8 The closed loop system becomes

$$\begin{cases} \dot{x} = (A - BK)x + Bk_r r \\ y = Cx \end{cases}$$

The characteristic equation is thus

$$\det(sI - A + BK) = s^2 + (3 + k_1 + 2k_2)s + 2(1 + k_1 + k_2) = 0$$

We need $(s+4)^2 = s^2 + 8s + 16 = 0$. Identification of coefficients yields $k_1 = 9, \quad k_2 = -2$. The closed loop transfer function is $G(s) = C(sI - A + BK)^{-1}Bk_r$. The stationary gain is $G(0)$ is unity if

$$G(0) = C(-A + BK)^{-1}Bk_r = \frac{k_r}{4} = 1$$

yielding $k_r = 4$.

5.9 **a.** The characteristic polynomial of the closed loop system is given by

$$\det(sI - (A - BK)) = \begin{vmatrix} s + 0.5 + 3k_1 & 3k_2 \\ -1 & s \end{vmatrix} = s^2 + (0.5 + 3k_1)s + 3k_2$$

The desired characteristic polynomial is

$$(s + 4 + 4i)(s + 4 - 4i) = s^2 + 8s + 32$$

Identification of coefficients yields

$$K = \begin{pmatrix} 5/2 & 32/3 \end{pmatrix} = \begin{pmatrix} 2.5 & 10.7 \end{pmatrix}$$

The closed loop system transfer function is $G_{yr}(s) = C(sI - A + BK)^{-1}Bk_r$. The stationary gain is unity if

$$G_{yr}(0) = C(-A + BK)^{-1}Bk_r = \frac{3}{32}k_r = 1$$

which yields $k_r = 32/3$.

b. According to a rule of thumb, the observer poles shall be chosen 1.5–2 times faster than the state feedback. Place the poles of the Kalman filter so that the distance from the origin is twice as large, leading to the following characteristic polynomial

$$(s + 8 + 8i)(s + 8 - 8i) = s^2 + 16s + 128$$

The characteristic polynomial of the Kalman filter is given by

$$\det(sI - (A - LC)) = \begin{vmatrix} s + 0.5 & l_1 \\ -1 & s + l_2 \end{vmatrix} = s^2 + (0.5 + l_2)s + 0.5l_2 + l_1$$

Identification of coefficients yields

$$L = \begin{pmatrix} 120.25 \\ 15.5 \end{pmatrix}$$

5.10 **a.** From the block diagram we see that

$$\begin{aligned} X_1 &= \frac{1}{s}K_1 U \\ X_2 &= K_2 \frac{1}{s} X_1 \\ X_3 &= \frac{1}{s} X_2 \end{aligned}$$

which gives

$$\begin{aligned} \dot{x}_1 &= K_1 u \\ \dot{x}_2 &= K_2 x_1 \\ \dot{x}_3 &= x_2 \\ y &= x_3 \end{aligned}$$

In matrix form we get

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 & 0 \\ K_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} K_1 \\ 0 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x \end{aligned}$$

b. The feedback law is given by

$$u = k_r r - Kx$$

The closed-loop system becomes

$$\dot{x} = Ax + B(k_r r - Kx) = (A - BK)x + Bk_r r$$

The poles of the closed-loop system are given by the eigenvalues of $A - BK$, i.e., the roots of the closed loop characteristic equation

$$\begin{aligned} \det(sI - (A - BK)) &= \det \begin{pmatrix} s + K_1 k_1 & K_1 k_2 & K_1 k_3 \\ -K_2 & s & 0 \\ 0 & -1 & s \end{pmatrix} \\ &= s^3 + K_1 k_1 s^2 + K_1 K_2 k_2 s + K_1 K_2 k_3 = 0 \end{aligned}$$

The poles of the closed loop system are given by the eigenvalues of $A - BK$, i.e. the roots of the closed loop characteristic equation

$$(s + 0.5)^3 = s^3 + 1.5s^2 + 0.75s + 0.125 = 0$$

One immediately obtains the solution

$$\begin{aligned} k_1 &= \frac{1.5}{K_1} \\ k_2 &= \frac{0.75}{K_1 K_2} \\ k_3 &= \frac{0.125}{K_1 K_2} \end{aligned}$$

5.11 The augmented system becomes

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r \\ &= \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{A_e} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x_e} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{B_e} u + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{B_r} r \end{aligned}$$

We seek $K_e = \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix}$ such that

$$\det(sI - (A_e - B_e K_e)) = (s + \alpha)(s^2 + 2\zeta\omega s + \omega^2)$$

Insertion of A_e, B_e and K_e into the above expression yields

$$s^3 + k_2 s^2 + k_1 s - k_3 \equiv s^3 + (\alpha + 2\zeta\omega)s^2 + (\omega^2 + 2\zeta\omega\alpha)s + \alpha\omega^2$$

Identifications of coefficients now yields

$$\begin{aligned} k_1 &= \omega^2 + 2\zeta\omega\alpha \\ k_2 &= \alpha + 2\zeta\omega \\ k_3 &= -\alpha\omega^2 \end{aligned}$$

5.12 The estimation error $\tilde{x}$ fulfills $\dot{\tilde{x}} = (A - LC)\tilde{x}$. where $L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^T$. The characteristic equation of the estimation error becomes

$$\det(sI - (A - LC)) = s^2 + (4 + l_2)s + l_1 + 2l_2 + 3 = 0$$

The desired characteristic equation is

$$(s + 4)^2 = s^2 + 8s + 16 = 0$$

Identification of coefficients yields $l_1 = 5$, $l_2 = 4$.

5.13 a. The characteristic equation of the closed loop system is given by

$$\det\big(sI - (A - BK)\big) = \begin{vmatrix} s+4+k_1 & 3+k_2 \\ -1 & s \end{vmatrix} = s^2 + (4+k_1)s + 3 + k_2 = 0$$

The desired characteristic equation is

$$(s+4)^2 = s^2 + 8s + 16 = 0$$

Which yields $k_1 = 4$ and $k_2 = 13$. The control law becomes

$$u = -k_1 x_1 - k_2 x_2 = -4x_1 - 13x_2$$

b. The states shall be estimated by means of a Kalman filter, i.e.

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

For $\tilde{x}$ we have

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}$$

Determine L such that all eigenvalues of the matrix $A - LC$ are placed in $\lambda = -6$.

$$\begin{aligned} \det(\lambda I - A + LC) &= \lambda^2 + (4 + l_1 + 3l_2)\lambda + 3 + 3l_1 + 9l_2 \\ &= (\lambda+6)^2 = \lambda^2 + 12\lambda + 36 \end{aligned}$$

Identify the coefficients and solve for l_1 and l_2:

$$\begin{cases} 4 + l_1 + 3l_2 = 12 \\ 3 + 3l_1 + 9l_2 = 36 \end{cases} \quad \Rightarrow \quad \begin{cases} l_1 + 3l_2 = 8 \\ l_1 + 3l_2 = 11 \end{cases}$$

The system of equations lacks solution, see the comment below.

c. The states are to be estimated by a Kalman filter, for which the eigenvalues of $A - KC$ shall be chosen such that

$$\lambda^2 + (4 + l_1 + 3l_2)\lambda + 3 + 3l_1 + 9l_2 = (\lambda+3)^2 = \lambda^2 + 6\lambda + 9$$

Identifying coefficients and solving for l_1 and l_2 yields

$$\begin{cases} 4 + l_1 + 3l_2 = 6 \\ 3 + 3l_1 + 9l_2 = 9 \end{cases} \quad \Rightarrow \quad \begin{cases} l_1 + 3l_2 = 2 \\ l_1 + 3l_2 = 2 \end{cases}$$

This leaves only one equation, which implies that there exists infinitely many solutions, e.g. $l_1 = 2$, $l_2 = 0$ or $l_1 = 0$, $l_2 = \frac{2}{3}$ etc.

The drawback of the proposed observer pole placement is that it yields an estimation slower than the closed loop system. This does not affect the response to reference changes, which is governed by the poles of the closed loop system. However, the handling of process disturbances becomes slower.

Comment

An inspection of the system's observability shows that

$$\det W_o = \begin{vmatrix} C \\ CA \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & -3 \end{vmatrix} = 0$$

I.e. the system is not observable. The transfer function of the system is given by

$$G(s) = C(sI - A)^{-1}B = \frac{s+3}{s^2 + 4s + 3} = \frac{1}{s+1}$$

The eigenvalue -3 corresponds to a non-observable mode. The mode is, however, controllable, which follows from the canonical controllable form realization of the system. The characteristic equation of $A - LC$ can be written

$$\det(\lambda I - A + LC) = (\lambda + 3)(\lambda + l_1 + 3l_2 + 1)$$

This means that the Kalman filter has to estimate the non-observable mode with its own speed. I.e. (at least) one of the eigenvalues of $A - LC$ must be placed in -3. This explains the failure to compute a Kalman filter when the eigenvalues were to be placed in -6 and a success when they were to be placed in -3. Note that in cases such as this one, the result is either that there does not exist a solution L to the Kalman filter problem, or that it exists infinitely many solutions. When the system is observable, there exists a unique solution L to the Kalman filter.

Solutions to Exercises 6

Design methods

6.1 **a.** The frequency function of the controller is given by

$$G_R(i\omega) = K\left(1 + i(\omega T_d - \frac{1}{\omega T_i})\right)$$

The gain and phase shift for a frequency ω are directly obtained from the gain function $A(\omega)$ and phase function $\varphi\omega$ of the controller, respectively.

$$A(\omega) = |G_R(i\omega)| = K\sqrt{1 + (\omega T_d - \frac{1}{\omega T_i})^2}$$
$$\varphi(\omega) = \arg G_R(i\omega) = \arctan(\omega T_d - \frac{1}{\omega T_i})$$

b. We immediately realize that the gain function $A(\omega)$ has a unique minimum for Im $A(i\omega) = 0$, which means that

$$\omega_{\min} = \frac{1}{\sqrt{T_i T_d}}$$

At this frequency the gain and phase shift are given by

$$A(\omega_{\min}) = K$$
$$\varphi(\omega_{\min}) = 0$$

Note that the phase shift is negative for $\omega < \omega_{\min}$ (phase lag) and positive for $\omega > \omega_{\min}$ (phase lead).

6.2 **a.** The gray gain curve is identical to the nominal one, except that it is raised by a factor 4. This is thus the case where K has been multiplied by 4. Observe that the gray phase curve is not visible in the plot since it coincides with the solid black phase curve. The dotted gain curve differs from the nominal (solid black) curve at low frequencies, for which it is lower. This indicates that T_i has been increased, resulting in decreased low frequency gain. Also note that the phase curve has been raised for low frequencies. The last (dashed) curve apparently corresponds to the case where T_d has been increased. This is further indicated by the factor 4 raise of the gain curve for high frequencies. Also for this case, one can notice a certain increase in the phase, although for somewhat higher frequencies.

The gray step response is faster and less damped than the nominal (solid black) one. This is a characteristic sign of an increased gain K. The corresponding Bode plot confirmingly shows that the cutoff frequency, ω_c, has increased (faster), while the phase margin has decreased (less damped). The dotted step response features a slow mode both in the reference- and load disturbance responses. Observe the relatively fast increase in the reference response to approximately 0.8, followed by a slow convergence to 1. This must be due to decreased integral action. The integral time T_i has thus *increased* in this case. The corresponding Bode plot shows that ω_c is virtually unchanged. This is seen in the step response by the fact that the first part has approximately the same speed as the nominal case, whereas the following slow settling is due to the decreased low frequency gain. The last (dashed) step response obviously corresponds to an increase of the derivative time T_d. The reference response is subject to an fast initial increase, followed by a somewhat slower settling. This is seen in the Bode plot by the fact that the high frequency gain has increased, while the low frequency gain has remained unchanged. The load response is somewhat slower and more damped than in the nominal case.

b. The gray gain curve is lowered by a factor 2 in comparison to the nominal (solid black) one, corresponding to a decreased value of K. The gray phase curve consequently coincides with the nominal (solid black) case. The dotted gain curve has been increased for low frequencies, i.e. T_i has *decreased*. The dashed gain curve has been lowered at high frequencies, i.e. T_d has decreased.

The gray step response is slower and more damped than the nominal (solid black) one. This indicates that K has decreased, since neither a decrease in T_i nor T_d would yield a more damped step response. This is further confirmed by the Bode plot. The only case where ω_c has decreased is when K has decreased. It is also the only case for which the phase margin has increased. The two remaining step responses are both less damped than the nominal one. In order to determine which of these corresponds to a decrease of T_i, we look at the corresponding Bode plot (the dotted one). This shows that the cutoff frequency ω_c has increased somewhat compared to the other nominal case. The dashed Bode plot, however, shows that the decrease of T_d has not changed ω_c. The dotted step response is initially somewhat faster than the nominal (solid black) one, whereas the dashed one is initially approximately as fast as the nominal one. This implies that the dotted step response corresponds to a decrease in T_i, while the dashed one corresponds to a decrease in T_d.

6.3 The transfer function of the process is given by

$$G_P = \frac{C}{Js + D}$$

and the transfer function of the PI controller is given by

$$G_R = K\left(1 + \frac{1}{sT_i}\right)$$

We can now write down the closed loop transfer function G_{cl} as

$$G_{cl} = \frac{G_R G_P}{1 + G_R G_P}$$

The characteristic polynomial is the denominator of G_{cl}

$$s^2 + \frac{D + CK}{J}s + \frac{CK}{JT_i}$$

and the desired characteristic polynomial is

$$s^2 + 2\zeta\omega s + \omega^2$$

Identification of coefficients yields

$$\begin{cases} K = \frac{2\zeta\omega J - D}{C} \\ T_i = \frac{2\zeta\omega J - D}{\omega^2 J} \end{cases}$$

6.4 The transfer function of the process, G_p, is given by

$$\Theta = G_P I = \frac{k_i}{Js^2 + Ds} I$$

and the transfer function of the PID controller, G_R, is given by

$$I = G_R(\Theta_{ref} - \Theta) = K\left(1 + \frac{1}{sT_i} + sT_d\right)(\Theta_{ref} - \Theta)$$

where Θ_{ref} is the Laplace transform of the reference value of θ. The transfer function of the closed loop system, G, is thus given by

$$\Theta = G\Theta_{ref} = \frac{G_R G_P}{1 + G_R G_P}\Theta_{ref}$$

The characteristic polynomial is given by the denominator of G and is

$$s^3 + \frac{D + Kk_i T_d}{J}s^2 + \frac{Kk_i}{J}s + \frac{Kk_i}{JT_i}$$

We hence arrive at the polynomial equation

$$(s+\alpha)(s^2 + 2\zeta\omega s + \omega^2) = s^3 + (\alpha + 2\zeta\omega)s^2 + (2\alpha\zeta\omega + \omega^2)s + \alpha\omega^2$$

Identification of coefficients yields the equations

$$\begin{cases} \frac{D+Kk_iT_d}{J} = \alpha + 2\zeta\omega \\ \frac{Kk_i}{J} = 2\alpha\zeta\omega + \omega^2 \\ \frac{Kk_i}{JT_i} = \alpha\omega^2 \end{cases}$$

from which one can calculate the sought controller parameters

$$\begin{cases} K = \frac{J}{k_i}(2\alpha\zeta\omega + \omega^2) \\ T_i = \frac{2\zeta}{\omega} + \frac{1}{\alpha} \\ T_d = \frac{\alpha + 2\zeta\omega - D/J}{2\alpha\zeta\omega + \omega^2} \end{cases}$$

6.5 **a.** The transfer function of the controller is

$$G_r(s) = K(1 + \frac{1}{sT_i}) = (1 + \frac{1}{s}) = \frac{s+1}{s}$$

The low frequency asymptote becomes

$$G_r(s) \approx \frac{1}{s}$$

I.e. the gain curve is a straight line with slope = -1 and $\arg G_r(i\omega) = -90°$. The slope of the gain curve increases to 0 at the corner frequency $\omega_1 = 1$.

The high frequency asymptote is $G_r(s) \approx 1$ with $|G_r(i\omega)| = 1$, i.e. slope = 0 and $\arg G_r(i\omega) = 0$. The corresponding Bode plot is shown in Figure S6.1.

b. The transfer function of the controller is

$$G_r(s) = K(1 + T_d s) = 1 + s$$

The low frequency asymptote becomes $G_r(s) \approx 1$, i.e. the gain curve is a straight line with magnitude 1 and slope = 0 and the phase curve is described by $\arg G_r(i\omega) = 0°$. The slope of the gain curve increased to $+1$ at the corner frequency $\omega_1 = 1$.

The high frequency asymptote is given by $G_r(s) \approx s$, i.e. the slope of the gain curve is $+1$ and the phase curve is described by $\arg G_r(i\omega) = +90°$. The corresponding Bode plot is shown in Figure S6.2.

6.6 With the Ziegler-Nichols frequency method, an oscillation is generated at the frequency ω_0, where the phase shift of the process is $-180°$. This frequency is given by

$$-2\arctan 20\omega_0 - 9\omega_0 = -\pi$$

Numerical solution gives $\omega_0 \approx 0.1$ rad/s.

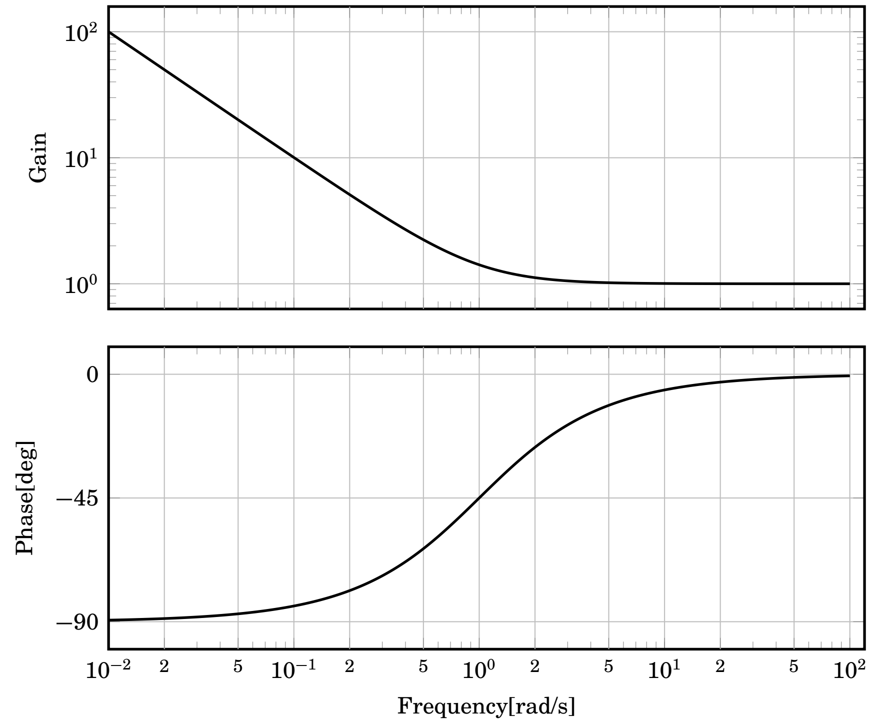


Figure S6.1 Bode plot of a PI controller with $K = 1$ and $T_i = 1$.

The process gain at ω_0 is

$$\frac{1}{1 + 20^2\omega_0^2} = 0.2$$

The critical gain K_c becomes

$$K_c = \frac{1}{0.2} = 5$$

and the period time is

$$T_o = \frac{2\pi}{\omega_o} = 63$$

The controller parameters become

$$\begin{cases} K = 0.45K_c = 2.25 \\ T_i = T_o/1.2 = 53 \end{cases}$$

6.7 **a.** Laplace transform of Martins dynamics give

$$\begin{aligned} sX &= -\frac{1}{30}X + \frac{1}{15}U \\ Y &= X \end{aligned}$$

and the transfer function $Y = G_P U$ becomes

$$G_P = \frac{2}{30s + 1}$$

The process is controlled by a PI controller, $U = G_R(R - Y)$ where

$$G_R = K(1 + \frac{1}{T_i s}) = K\frac{T_i s + 1}{T_i s}$$

The controller parameters K and T_i should be chosen such that the closed loop transfer function $Y = G_{cl}R$ has its poles in -0.1. The transfer function from R to

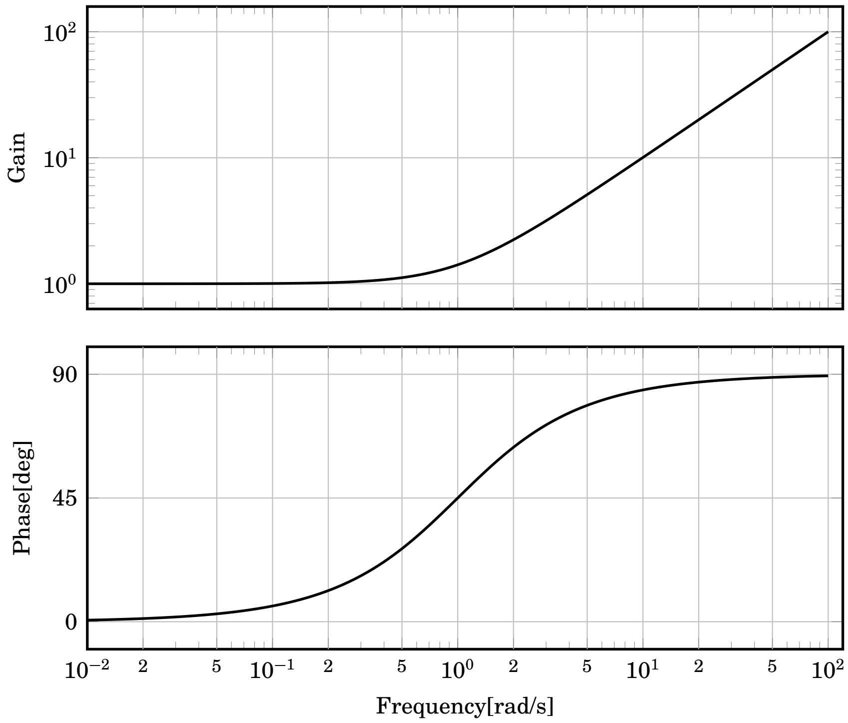


Figure S6.2 Bode plot of a PD controller with $K = 1$ and $T_d = 1$.

Y is

$$G_{cl} = \frac{G_P G_R}{1 + G_P G_R} = \frac{K\dfrac{T_i s + 1}{T_i s} \cdot \dfrac{2}{30s + 1}}{1 + K\dfrac{T_i s + 1}{T_i s} \cdot \dfrac{2}{30s + 1}}$$

$$= \frac{K(T_i s + 1)2}{T_i s(30s + 1) + K(T_i s + 1)2}$$

where the denominator polynomial has the same roots as

$$s^2 + \frac{2K + 1}{30}s + \frac{K}{T_i 15}$$

These roots should be -0.1, i.e., the same as for

$$(s + 0.1)^2 = s^2 + 0.2s + 0.01$$

This is true if the coefficients for the polynomials are equal, i.e.,

$$\frac{2K + 1}{30} = 0.2$$
$$\frac{K}{T_i 15} = 0.01$$

The controller parameters thus become

$$K = 2.5$$
$$T_i = 16.7$$

b. When using Ziegler-Nichols frequency method, an experiment is performed using feedback with a P controller. The gain is increased until the system starts oscillating (stability boundary). The Bode diagram of the process can be thought of as the Bode diagram for the loop transfer function, when the controller is a P controller with a gain of 1. When the gain is increased, the gain curve in the Bode diagram will move upwards, whereas the phase will remain unaffected. The system will start to oscillate when the phase margin becomes 0. In the figure it can be seen that this will happen when the cutoff frequency is 0.3 rad/s, since the phase curve crosses $-180°$ at this frequency. This is the resonant frequency, with which the system will oscillate. For this frequency, the gain is equal to 0.2. Thus, a gain of 5 is needed in order for the system to start oscillating with the resonant frequency. According to the collection of formulae, we have

$$K = 0.45K_0$$
$$T_i = T_0/1.2$$

where K_0 is the gain and T_0 the period time of the oscillation. With

$$K_0 = 5$$
$$T_0 = \frac{2\pi}{0.3} = 20.9$$

we get the controller parameters $K = 2.25$ and $T_i = 17.4$.

6.8 See fig. S6.3.

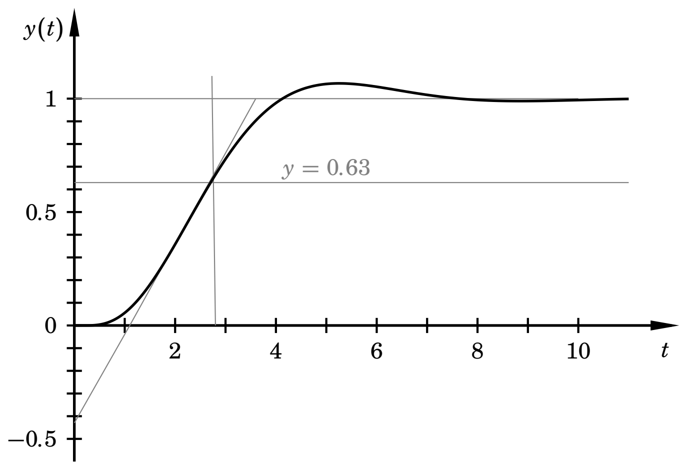


Figure S6.3 Step response for the system in assignment 6.8 with some interesting lines included.

Ziegler-Nichols step response method: With the customary notion we obtain $a = 0.4$ and $b = 1.1$. The controller parameters become $K = 1.2/a = 3$, $T_i = 2b = 2.2$ and $T_d = b/2 = 0.55$.

Ziegler-Nichols frequency method: The Nyquist curve intersects the negative real axis in -0.4 for $\omega = 1.3$ rad/s, which yields $T_0 = 2\pi/\omega = 4.8$ and $K_0 = 2.5$. The controller parameters become $K = 0.6K_0 = 1.5$, $T_i = T_0/2 = 2.4$ and $T_d = T_0/8 = 0.6$.

The Lambda method: By drawing the tangent to the step response, an approximation of the dead time is obtained, $L \approx 1.1s$. The step response has reached 63% of its final value after approximately 2.7s. The time constant thus becomes $T = 2.7 - 1.1 = 1.6s$. The static gain is $K_P = 1$. With $\lambda = T$, we get the PI controller parameters

$$K = \frac{1}{K_p}\frac{T}{L+\lambda} \approx 0.6$$
$$T_i = T = 1.6\text{s}$$

For a PID controller, we get the parameters

$$K = \frac{1}{K_p}\frac{L/2+T}{L/2+\lambda} = 1$$
$$T_i = T + L/2 = 2.15\text{s}$$
$$T_d = \frac{TL}{L+2T} = 0.4\text{s}$$

6.9 **a.** The step response of the system is shown in Figure S6.4.

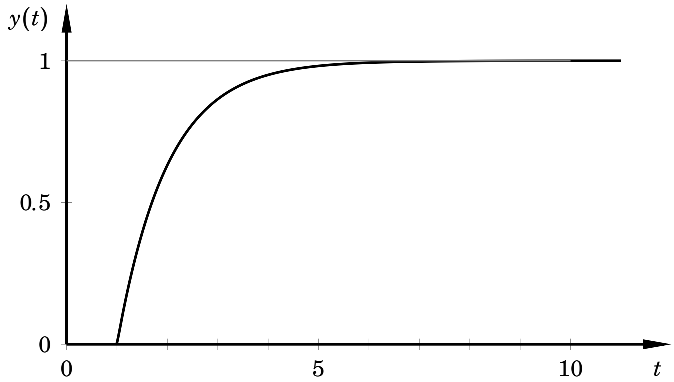


Figure S6.4 Step response of $G(s) = e^{-s}/(s+1)$.

From the figure we obtain (with the customary notion) $a = b = 1$. This yields the controller parameters $K = 1.2/a = 1.2$, $T_i = 2b = 2$ and $T_d = b/2 = 0.5$.

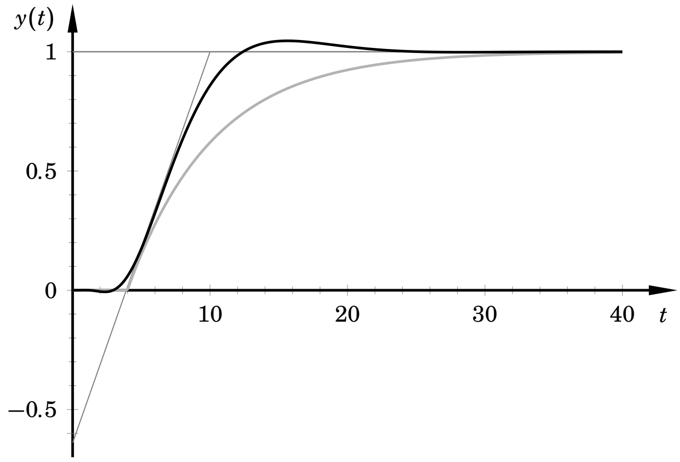
b. The resonance frequency is determined by $\arg G(i\omega_0) = -\arctan\omega_0 - \omega_0 = -\pi$. Numerical solution yields $\omega_0 \approx 2.03$, resulting in $T = 2\pi/\omega_0 = 3.1$.

Further, $K_0 = 1/|G(i\omega_0)| = 2.26$, which gives $K = 1.4$, $T_i = 1.5$ and $T_d = 0.39$.

c. From the step response in subproblem a, we get the process parameters $K_P = 1$, $L = 1$ and $T = 1$. This gives the controller parameters

$$K = \frac{1}{K_p}\frac{L/2+T}{L/2+\lambda} = 1$$
$$T_i = T + L/2 = 1.5$$
$$T_d = \frac{TL}{L+2T} \approx 0.33$$

6.10 a. The figure does not allow for any greater precision. Draw the tangent of the step response where the derivative attains a maximum and study the intersection of the tangent and the two coordinate axis. The parameter a is given by the distance between 0 and the intersection with the vertical axis, whereas b is given by the distance between 0 and the intersection with the horizontal axis. In our example we have $a = 0.65$ and $b = 4$. From the table we obtain the following controller parameters: $K = 1.9$, $T_i = 8$ and $T_d = 2$.



b. The critical gain K_c is the gain which causes the Nyquist curve to pass through -1. In our case we have $K_c = 1/0.55 = 1.8$. The critical period T_0 corresponds to the frequency at 'o', i.e. $T_0 = 2\pi/\omega = 14.6$. This yields the controller parameters: $K = 1.1$, $T_i = 7.3$ and $T_d = 1.8$.

c. The value of K obtained from the last method is smaller than the values obtained through Ziegler-Nichol's methods.

6.11 Generally it is required that $|G_K(i\omega_c)| > 1$ in order for ω_c to increase.

A The speed of the system increases, but simultaneously its robustness is reduced since the phase margin decreases.

B $|G_K| < 1$ for all ω, resulting in decreased cross-over frequency and speed.

C Cf. B.

D $|G_K| = 1$ for all ω, leaving the cross-over frequency unaffected.

6.12 The process is connected in a feedback loop with a proportional controller. By adding a compensation link one wants to decrease the ramp error of the compensated system by a factor 10. Simultaneously, a small decrease in robustness (phase margin) is accepted, resulting in a certain decrease of the system's transient behavior.

We can affect the ramp error by introducing a phase lag compensation link

$$G_k(s) = M\frac{s+a}{sM+a}$$

Assuming that the system is stable, the resulting ramp error becomes

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\cdot\frac{1}{1+G_k(s)G_P(s)}\cdot\frac{1}{s^2}\\ &= \lim_{s\to 0}\frac{(sM+a)(s+1)(s+2)}{s(sM+a)(s+1)(s+2)+KM(s+a)} = \frac{2}{KM}\end{aligned}$$

By choosing $M = 10$ ($K = 1$) the ramp error is reduced to 0.2.

Now it remains to decide a value for a. The phase lag link contributes to a phase lag in the open loop. The phase lag is largest around the frequency $\omega = a/\sqrt{M}$. In order not to compound the transient behavior of the closed loop system excessively, a must be chosen such that the phase around the cross-over frequency is left unaffected. This can be achieved by choosing a adequately small. However, a overly small value of a results in a long time before the ramp error decreases to 0.2. Let ω_c denote the cross-over frequency of the uncompensated system. At this frequency, the compensation link has a phase contribution

$$\arg G_k(i\omega_c) = \arctan \frac{\omega_c}{a} - \arctan \frac{M\omega_c}{a}$$

A simple rule of thumb is to choose $a = 0.1\omega_c$. In our example it means that the compensation link contributes with a phase shift of

$$\arg G_k(i\omega_c) = \arctan 10 - \arctan 100 \approx -5.1^\circ$$

The crossover frequency may be determined numerically according to

$$|G_P(i\omega_c)| = \frac{1}{\omega_c\sqrt{1+\omega_c^2}\sqrt{4+\omega_c^2}} = 1$$

which gives $\omega_c \approx 0.45$-

The compensation link thus becomes

$$G_k(s) = 10\frac{s+0.045}{10s+0.045} = \frac{s+0.045}{s+0.0045}$$

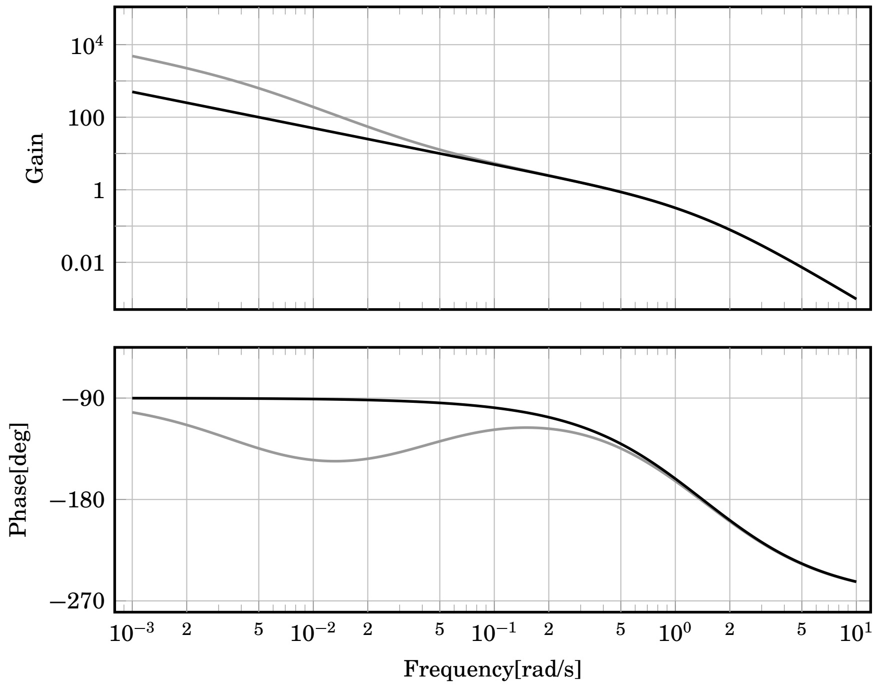


Figure S6.5 Bode plot of the uncompensated open loop system (black line) and compensated open loop system (gray line) in assignment 6.12. $K = 1$ for both cases.

In Figure S6.5 Bode plots for both the uncompensated open loop system $KG_P(s)$ and the compensated open loop system $KG_k(s)G_P(s)$ are shown.

The compensation link alters the transient behavior of the system. Figure S6.6 shows how the overshoot of the step response has increased, compared to the

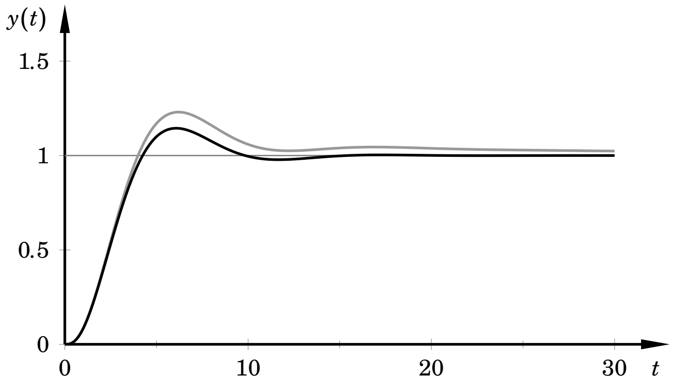


Figure S6.6 Step responses of the uncompensated closed loop system (solid line) and the compensated closed loop system (dashed line) in assignment 6.12.

uncompensated system. Also the settling time has increased, partly due to the slow mode in the compensation link. The purpose of introducing the compensation link was to decrease the ramp error. Figure S6.7 shows the error $e = r - y$ for both the uncompensated and compensated systems, with $r = t$.

As seen from the figure, the compensated system fulfills the criterion of a ramp error less than 0.2.

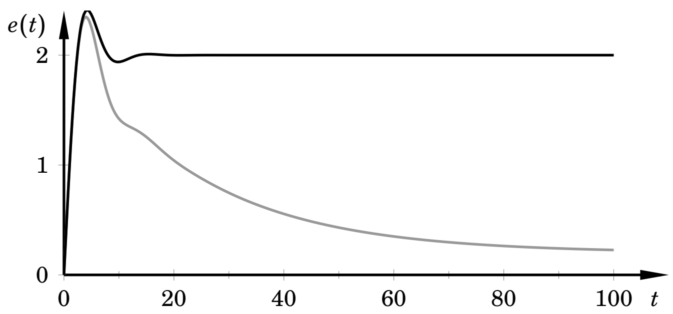


Figure S6.7 Ramp error of the uncompensated system (solid line) as well as the compensated system (dashed line) in assignment 6.12.

6.13 Use a phase lag compensation link

$$G_k(s) = KN\frac{s+b}{s+bN}$$

The crossover frequency ω_c of the uncompensated system can be read from the Bode plot in Figure S6.8.

One can also determine ω_c from the equation

$$|G_P(i\omega_c)| = \frac{1.1}{\omega_c\sqrt{\omega_c^2+1}} = 1$$

This yields $\omega_c = 0.84$. The new cross-over frequency is chosen to be $\omega_c^* = 1.68$. The phase shift of the uncompensated system at the frequency ω_c is

$$\arg G_P(i\omega_c) = -90^o - \arctan(0.84) = -130^o$$

In order not to decrease the phase margin, it must hold that

$$\arg(G_k(i\omega_c^*)G_P(i\omega_c^*)) \geq \arg G_P(i\omega_c)$$

We have

$$\arg G_P(i\omega_c^*) = -90^o - \arctan(1.68) = -149^o$$

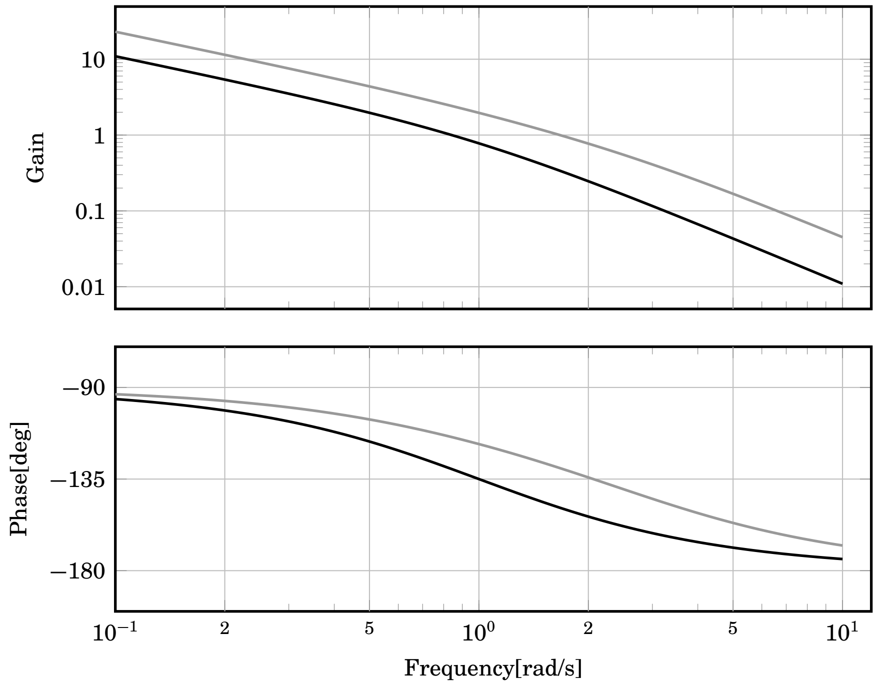


Figure S6.8 Bode plot of the uncompensated open loop system (black line) as well as the compensated open loop system (gray line) in assignment 6.13.

For the compensation link it must hence hold that

$$\arg G_k(i\omega_c^*) \geq 19^o$$

From the collection of formulae we find that $N = 2$ is adequate. The compensation link has its maximal phase shift at the frequency $b\sqrt{N}$. This shall occur at the new cross-over frequency, i.e.

$$\omega_c^* = b\sqrt{N} \quad \Rightarrow \quad b = \frac{\omega_c^*}{\sqrt{N}} = 1.2$$

Now choose K such that ω_c^* becomes the actual cross-over frequency (observe that $|G_k(i\omega_c^*)| = K\sqrt{N}$)

$$|G_k(i\omega_c^*)G_P(i\omega_c^*)| = 1 \quad \Rightarrow \quad K = 2.1$$

We thus obtain the compensation link

$$G_k(s) = 4.2\frac{s+1.2}{s+2.4}$$

Figure S6.8 shows the Bode plot of the uncompensated open loop system $G_P(s)$ as well as the compensated open loop system $G_k(s)G_P(s)$. Figure S6.9 shows the step responses of the uncompensated and compensated systems.

6.14 We choose a phase lead link

$$G_k(s) = K_K \cdot N\frac{s+b}{s+bN}$$

The specification implies that the low frequency gain shall not decrease (which would increase the stationary error). The cross-over frequency shall increase by a factor 3 and the phase margin shall remain unchanged.

The open loop transfer function is

$$G_0(s) = G_k(s)G_1(s) = K_K \cdot N\frac{s+b}{s+bN} \cdot \frac{1}{s(s+1)(s+2)}$$

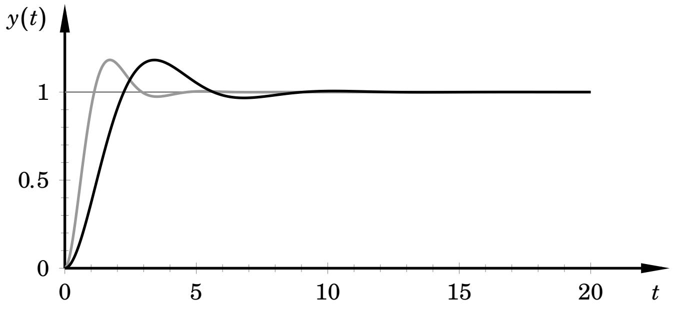


Figure S6.9 Step response of the uncompensated closed loop system (black line) as well as the compensated closed loop system (gray line) in assignment 6.13.

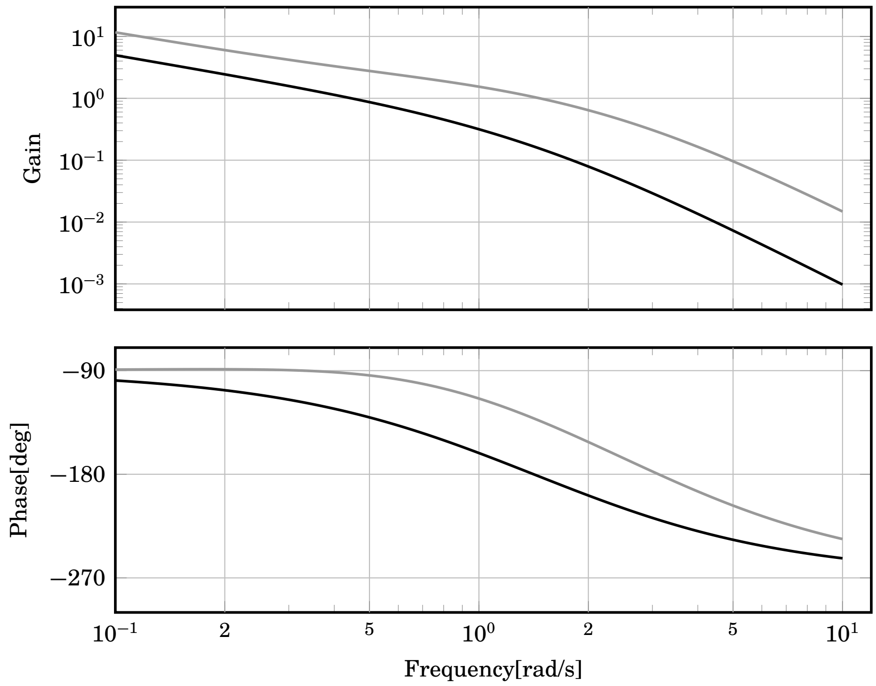


Figure S6.10 Bode plot of the uncompensated system G_1 (solid line) and compensated system G_kG_1 (dash-dotted line) in assignment 6.14.

The Bode plot of G_1 is presented in Figure S6.10. From this, or from numerical calculations, the cross-over frequency is determined to $\omega_c = 0.45$ rad/s and the phase margin is $\varphi_m = 53°$. The new cross-over frequency shall thus be $\omega_c^* = 3 \cdot \omega_c = 1.35$ rad/s with unchanged phase margin. Since $\arg G_1(i\omega_c^*) \approx -180°$, the phase curve must be raised approximately 50° by G_k.

From the collection of formulae it is obtained that $N = 8$ gives a maximal phase lead of approximately 50°. The phase lead is maximal at the frequency $b\sqrt{N} = \omega_c^*$, yielding $b = 0.48$. The gain shall be unity at the new cross-over frequency ω_c^*, i.e.

$$|G_k(i\omega_c^*)| \cdot |G_1(i\omega_c^*)| = 1$$

The magnitude of the compensator is $|G_k(i\omega_c^*)| = K_K\sqrt{N}$. Numerical calculations give $|G_1(i\omega_c^*)| = 0.183$. Hence

$$K_K = \frac{1}{\sqrt{N} \cdot 0.183} = 1.9$$

The step response of the uncompensated and compensated systems, respectively, are shown in Figure S6.11 and the ramp response is shown in Figure S6.12. Since $K_K > 1$ the stationary errors become smaller than before, thus fulfilling the specifications.

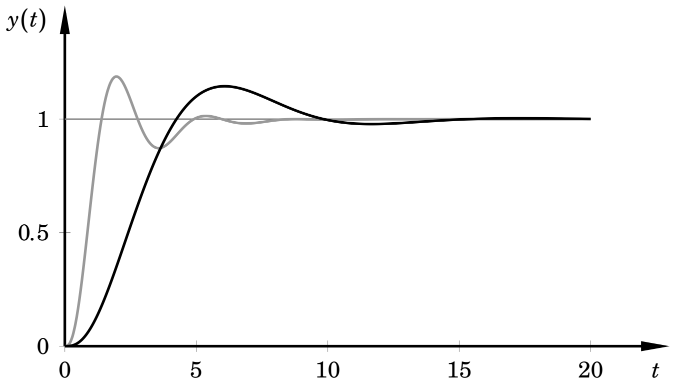


Figure S6.11 The step response of the uncompensated closed loop system (black line) as well as the compensated closed loop system (gray line) in assignment 6.14.

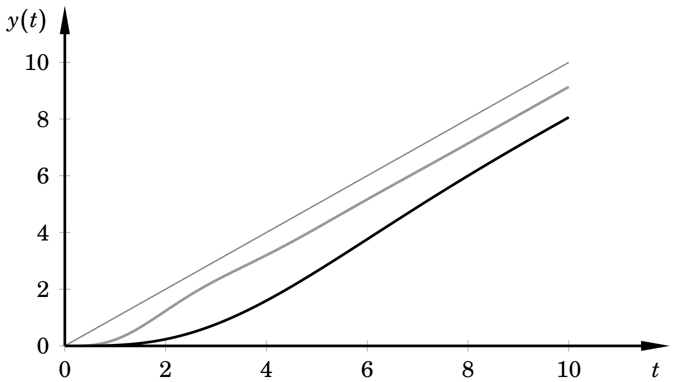


Figure S6.12 The ramp response of the uncompensated system (solid line) as well as the compensated system (dashed line) in assignment 6.14.

6.15 From the Bode plot of $G_o(s)$ (see Figure S6.13) we obtain $\varphi_m = 20°$ and $\omega_c = 0.7$ rad/s. Unchanged speed necessitates a compensation link which does not affect the cross-over frequency. We hence need a phase lead of $\Delta\varphi = 30°$ at $\omega = \omega_c = 0.7$ rad/s. We utilize a phase lead compensation link

$$G_k(s) = KN\frac{s+b}{s+bN}$$

1. The sample curves in the collection of formulae yield $N = 3$.

2. $b\sqrt{N} = \omega_c \quad \Rightarrow \quad b = \frac{0.7}{\sqrt{3}} = 0.40$

3. $|G_k(i\omega_c)G_o(i\omega_c)| = K\sqrt{N}\cdot 1$ gives $K = \dfrac{1}{\sqrt{N}} = 0.58$

The compensation link thus becomes

$$G_k(s) = 0.58 \cdot 3\frac{s+0.4}{s+1.2}$$

The system is stable, so the resulting stationary error becomes

$$E(s) = \frac{1}{1+G_kG_o}R(s) = \frac{s(s+0.5)(s+3)(s+bN)}{s(s+0.5)(s+3)(s+bN)+2KN(s+b)}R(s)$$

With $R(s) = 1/s^2$ the stationary ramp error becomes

$$\lim_{s\to 0} sE(s) = \frac{1.5}{2K} = 1.29$$

which fulfills the specification. Figure S6.14 shows the step response of the system before and after the compensation. The ramp error is shown in Figure S6.15. The fact that the ramp error is increased by the compensation is due to $K < 1$.

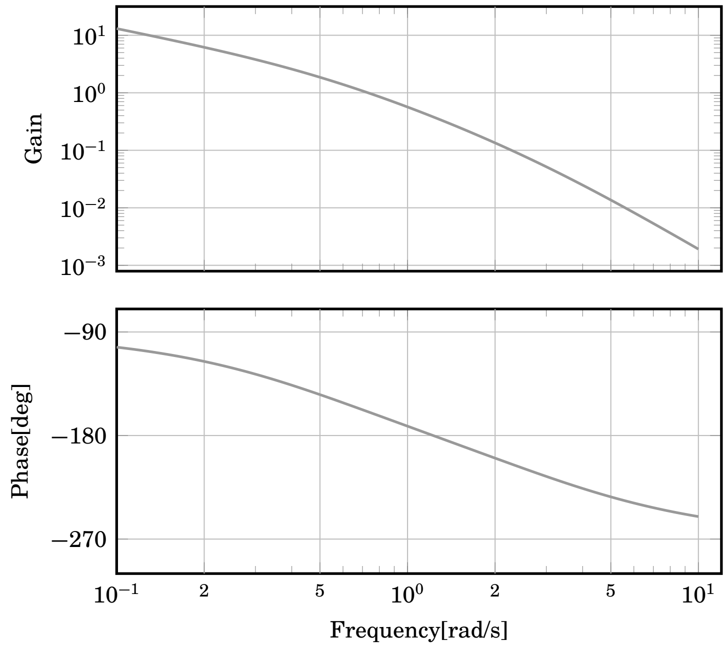


Figure S6.13 Bode plot of $G_o(s)$ in assignment 6.15.

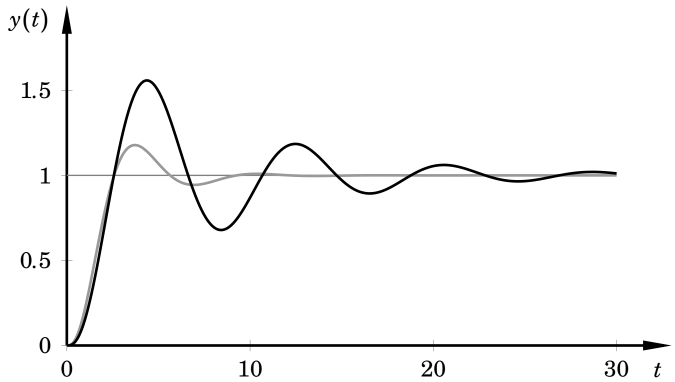


Figure S6.14 Step response of the uncompensated closed loop system (black line) and compensated system (gray line) in assignment 6.15.

6.16 We know that a phase lag compensation link dimensioned according to the rules of thumb will decrease the phase margin by approximately 6°, which yields a certain decrease of robustness. In order not to obtain an excessive overshoot, we start by decreasing the gain of the process in order to increase the phase margin.

From the Bode plot of the process (see Figure S6.16) we find that $G-1$ has a phase shift of $-133°$ at the cross-over frequency $\omega_c = 0.7$. At the frequency $\omega_c^* = 0.6$ we have the phase shift $-133° + 6° = -127°$ and the gain $|G_1(\omega_c^*)| = 1.2$.

By decreasing the open loop gain by a factor 1.2 we obtain the new cross-over frequency ω_c^* and a phase margin increase of 6°. Since we cannot affect the process gain directly, we equivalently let $K = 1/1.2 = 0.83$ in the compensation link.

The main problem is to decrease the stationary ramp error to $e_1 \leq 0.1$. The final value theorem gives

$$e(\infty) = \lim_{s\to 0} sU(s)\frac{1}{1+G_k(s)G_1(s)} =$$
$$= \lim_{s\to 0} s\frac{1}{s^2}\frac{(s+a/M)s(s^2+2s+2)}{(s+a/M)s(s^2+2s+2)+1.5K(s+a)} = \frac{2}{1.5KM} \leq 0.1$$

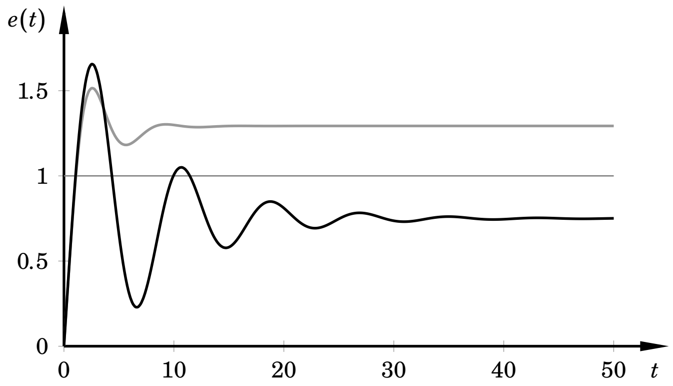


Figure S6.15 Ramp error of the uncompensated closed loop system (black line) as well as the compensated closed loop system (gray line) in assignment 6.15.

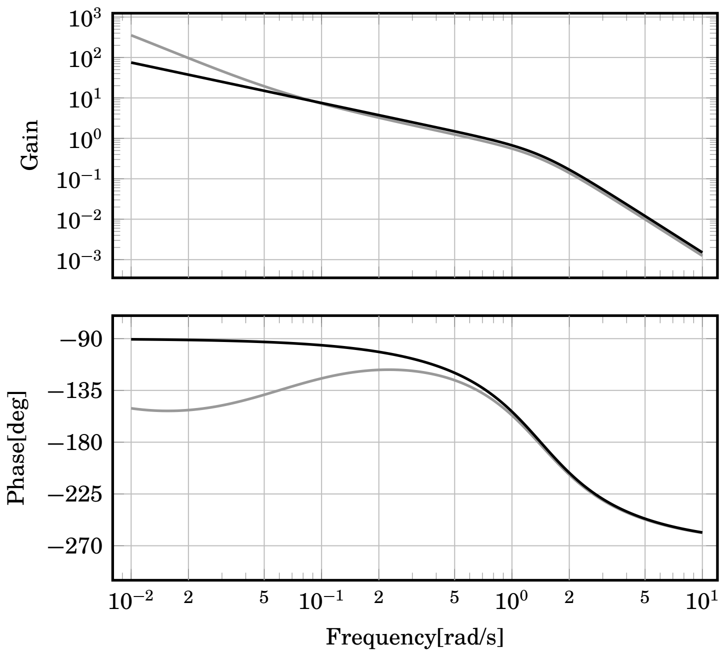


Figure S6.16 Bode plot of the uncompensated open loop system (black line) as well as the compensated open loop system (gray line) in assignment 6.16.

which yields $M \geq 16$. Choose $M = 16$. According to the rule of thumb we let $a = 0.1\omega_c^* = 0.06$. The chosen compensation link thus becomes

$$G_k(s) = 0.83\frac{s + 0.06}{s + 0.00375}$$

Figure S6.17 shows the step response before and after the compensation. The ramp errors of the uncompensated closed loop system and the compensated closed loop system are shown in Figure S6.18.

Comment:

Since we have decreased the open loop gain we obtain a decreased cross-over frequency and hence a somewhat slower system. In Figure S6.17 one especially notes the slow mode which appears as the process settles. It is caused by the slow pole of the controller in combination with the low gain. The rise time and damping are, however, virtually unaffected. An alternative to decreasing the open loop gain, in order to maintain the desired phase margin, would be to introduce a phase lead compensation link.

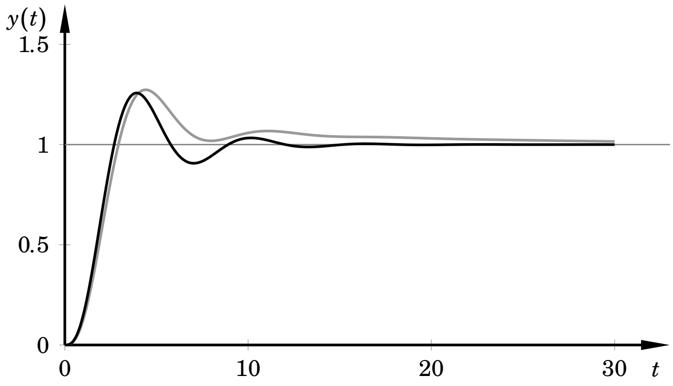


Figure S6.17 Step response of the uncompensated closed loop system (black line) as well as the compensated closed loop system (gray line) in assignment 6.16.

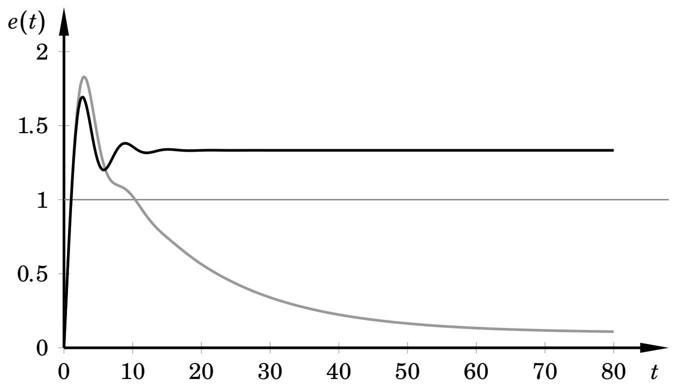


Figure S6.18 Ramp error of the uncompensated closed loop system (black line) as well as the compensated closed loop system (gray line) in assignment 6.16.

Solutions to Exercises 7

Controller Structures

7.1 The disturbance d does obviously lack influence if

$$G_1(s)H(s) + 1 = 0 \quad \Leftrightarrow \quad H(s) = -\frac{1}{G_1(s)}$$

To be a practically useful control law it is required that the disturbance can be measured, that *the model* $G_1(s)$ of the heating system is a "good" description of reality and that the inverse transfer function $1/G_1(s)$ is practically realizable. This means that $H(s)$ must not contain derivatives of the signal d. The realization of $H(s)$ can also be problematic if $G_1(s)$ lacks a stable inverse (i.e. if $G_1(s)$ has right half plane zeros, which is equivalent to being a non-minimum phase system). Further, we cannot invert processes with low pass characteristics more than at low frequencies and delays can obviously not be inverted.

7.2 A block diagram for the system is shown in Figure S7.1. Mass balance for the tank

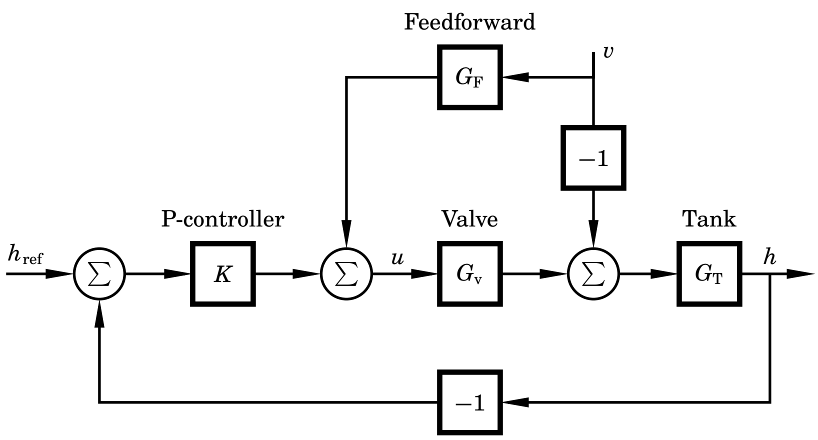


Figure S7.1 Block diagram of the level controlling system in assignment 7.2.

yields

$$A\frac{dh}{dt} = x(t) - v(t)$$

Laplace transformation gives ($A = 1\ m^2$)

$$H(s) = \frac{1}{s}(X(s) - V(s))$$

The transfer function of the tank is thus

$$G_T(s) = \frac{1}{s}$$

a. The closed loop transfer function becomes

$$G(s) = \frac{G_T G_V K}{1 + G_T G_V K} = \frac{K}{0.5s^2 + s + K}$$

The characteristic polynomial is hence

$$s^2 + 2s + 2K$$

The desired characteristic polynomial is

$$(s + \omega)^2 = s^2 + 2\omega s + \omega^2$$

Identification of coefficients yields

$$\begin{cases} \omega = 1 \\ K = \frac{1}{2} \end{cases}$$

The transfer function form $v(t)$ to $h(t)$ is given by

$$H(s) = -\frac{G_T}{1 + G_T G_V K} V(s) = -\frac{1 + 0.5s}{s(1 + 0.5s) + K} V(s)$$

If $v(t)$ is a step of amplitude 0.1 we obtain $V(s) = 0.1/s$. The final value theorem gives

$$h(\infty) = \lim_{s\to 0} sH(s) = -\frac{0.1}{K}$$

The theorem may be used since $sH(s)$ is of second order with positive coefficients in the denominator.

b. A PI controller has the transfer function

$$G_R(s) = K(1 + \frac{1}{sT_i})$$

The closed loop transfer function becomes

$$G(s) = \frac{G_T G_V G_R}{1 + G_T G_V G_R} = \frac{K(1 + sT_i)}{s(1 + 0.5s)sT_i + K(1 + sT_i)}$$

The characteristic polynomial becomes

$$s^3 + 2s^2 + 2Ks + \frac{2K}{T_i}$$

The desired characteristic polynomial is

$$(s + \omega)^3 = s^3 + 3\omega s^2 + 3\omega^2 s + \omega^3$$

Identification of coefficients yields

$$\begin{cases} \omega = \frac{2}{3} \\ K = \frac{2}{3} \\ T_i = \frac{9}{2} \end{cases}$$

c. The relation between the flow disturbance v and the level h is given by

$$H(s) = \frac{G_T(G_V G_F - 1)}{1 + G_T G_V G_R} V(s)$$

To eliminate the influence of v, we choose

$$G_F(s) = \frac{1}{G_V} = 1 + 0.5s$$

Note that G_F is not realizable. One can either cancel the derivative term or add a low-pass filter so that the infinite gain at high frequencies is avoided.

7.3 The closed loop system has the transfer function

$$\frac{(G_R + K_f)G_P}{1 + G_P G_R} = \frac{(K + K_f)s + K/T_i}{s^2 + (3 + K)s + K/T_i}$$

a. The characteristic equation of the closed loop system is

$$s^2 + (3 + K)s + K/T_i = 0$$

The desired characteristic equation is

$$(s + 2 - 2i)(s + 2 + 2i) = s^2 + 4s + 8 = 0$$

Identification of coefficients yields $K = 1$ and $T_i = 1/8$.

b. The feedforward K_f affects the zeros of the closed loop system, but leaves the poles unaffected. The poles can be placed by means of the controller H in order to obtain adequate supression of disturbances, cf. sub-assignment a above. One can subsequently translate the zeros by means of K_f in order to e.g. reach a desired overshoot in the reference step responses. The zero of the closed loop system is eliminated by choosing $K_f = -K$. With the pole placement in sub-assignment a, which corresponds to a relative damping $\zeta = 1/\sqrt{2} \approx 0.7$, the overshoot of the closed loop system becomes approximately 5%.

7.4 The block diagram in assignment 7.3 can be re-drawn according to Figure S7.2. By comparing to the block diagram in assignment 7.3 we see that $H_{ff} = H + K_f$ and $H_{fb} = H$. Observe that manipulation of K_f offers the possibility to neutralize the derivation in H, i.e. achieve a controller which derivates the output, but not the reference value.

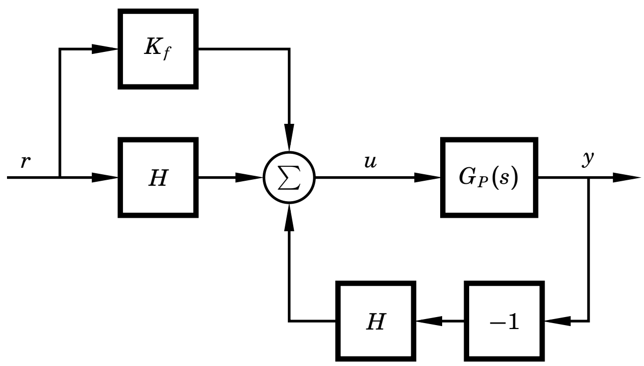


Figure S7.2 Modified block diagram in assignment 7.3.

7.5 The system has three inputs: the reference y_r and the two disturbances v_1 and v_2. The transfer functions between these three signals and the output y are given by

$$Y = \frac{G_1 G_2 G_{R1} G_{R2}}{1 + G_1 G_{R1} + G_1 G_2 G_{R1} G_{R2}} Y_r + \frac{G_1 G_2}{1 + G_1 G_{R1} + G_1 G_2 G_{R1} G_{R2}} V_1$$
$$+ \frac{(1 + G_1 G_{R1}) G_2}{1 + G_1 G_{R1} + G_1 G_2 G_{R1} G_{R2}} V_2$$

Let us name the three transfer functions G_{yr}, G_{v1} and G_{v2}, respectively. Ideally we would have $G_{yr} = 1$ and $G_{v1} = G_{v2} = 0$ for all frequencies. This is, however, not achievable. Nonetheless, we can assure that it holds in stationarity, i.e. for $s = 0$. For a P controller we have $G_R(0) = K$, where K is the gain of the controller. For a PI controller it holds that $G_R(0) = \infty$.

The transfer function G_{yr} becomes unity if G_{R2} is a PI controller. The transfer function G_{v1} becomes 0 if any of the controllers are PI. The transfer function G_{v2}, however, is only zero if G_{R2} is a PI controller.

Consequently G_{R2} must contain an integral part in order to guarantee 0 stationary control error. The controller G_{R1} can then be chosen to be a P controller. (If we furthermore want the internal signal y_1 to coincide with its reference, also this controller would need an integral part.)

7.6 **a.** The closed loop transfer function is given by

$$G_{\text{inner}}(s) = \frac{K_1 G_1(s)}{1 + K_1 G_1(s)} = \frac{2K_1}{s + 2 + 2K_1}$$

In order to make the system 5 times as fast, the pole of the closed loop system must be placed in $s = -10$, calling for $K_1 = 4$.

b. The approximation $G_{\text{inner}}(s) \approx G_{\text{inner}}(0) = 0.8$ yields

$$G_{\text{outer}}(s) = \frac{G_{R2}(s)G_2(s)G_{\text{inner}}(0)}{1 + G_{R2}(s)G_2(s)G_{\text{inner}}(0)} = \frac{(K_2 s + \frac{K_2}{T_i})0.8}{s^2 + 0.8K_2 s + 0.8\frac{K_2}{T_i}}$$

The specification of a system 10 times slower than the inner loop calls for a pole in $s = -1$. Since we deal with a second order system, we choose to locate both poles in $s = -1$ (somewhat slower than the single pole case). This yields $K_2 = 2.5$ and $T_i = 2$.

A general rule when cascading controllers is to make the inner loop 5–10 times faster than the outer loop in order to enable separation of the controller calculations for the two loops. The actual closed loop system (without approximations) becomes

$$G_{\text{outer}}(s) = \frac{10(2s + 1)}{s^3 + 10s^2 + 20s + 10}$$

and has poles in approximately -7.516, -1.702 and -0.7815 where the slower pole ($s = -0.7815$) will be the one essentially determining the speed of the system. This corresponds fairly well to the specified speed.

7.7 **a.** Since the steam flow is assumed to be constant, we can let $F = 0$, which yields the following description of the dome

$$Y(s) = \frac{10^{-3}}{s} M(s)$$

Since the controller is of P type we have $M(s) = K(Y_r - Y)$, where Y_r denotes the reference dome level. This yields

$$Y(s) = \frac{K}{K + 10^3 s} Y_r(s)$$

Since the system is linear and subject to negative feedback, a step disturbance in the level gives rise to the same transient behavior as a step disturbance in the reference. Hence let $Y_r(s) = \frac{1}{s}$. Inverse transformation of $Y(s)$ yields

$$y(t) = 1 - e^{-K10^{-3}t}$$

The specification on the settling time of the system now yields

$$y(10) = 1 - e^{-K10^{-2}} = 0.9 \quad \Rightarrow \quad K = 230$$

b. The dome and P controller are described by

$$Y(s) = \frac{K}{K + 1000s} Y_r(s) + \frac{s - 0.01}{(s + 0.1)(1000s + K)} F(s)$$

Let $Y_r(s) = 0$. A step disturbance in the steam flow $F(s) = \frac{1}{s}$ thus gives

$$Y(s) = \frac{s - 0.01}{(s + 0.1)(1000s + K)} \frac{1}{s}$$

The final value theorem yields

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s \frac{s - 0.01}{(s + 0.1)(1000s + K)} \frac{1}{s} = \frac{-0.1}{K}$$

and stationary error becomes

$$e = y_r - y = -y = \frac{0.1}{K}$$

c. Determine a feedforward link $H(s)$ from steam flow $F(s)$ to feed water flow $M(s)$ for the initial system, such that the level $Y(s)$ becomes independent of changes in the steam flow.

The system with feedforward $H(s)F(s)$ is described by

$$\begin{aligned} Y(s) &= \frac{10^{-3}}{s}(M(s) + H(s)F(s)) + \frac{s - 0.01}{s(s + 0.1)} 10^{-3} F(s) \\ &= \frac{10^{-3}}{s} M(s) + \frac{10^{-3}}{s} \left(\frac{s - 0.01}{s + 0.01} + H(s) \right) F(s) \end{aligned}$$

We want the influence from $F(s)$ to be zero. Therefore choose $H(s)$ so that the expression in front of $F(s)$ becomes zero. This criterion is fulfilled when

$$H(s) = -\frac{s - 0.01}{s + 0.1}$$

which gives the desired feedforward.

7.8 The delay margin is given by

$$L_m = \frac{\varphi_m}{\omega_c}$$

First we compute the cross-over frequency ω_c as

$$|G_0(i\omega_c)| = |G_P(i\omega_c)G_R(i\omega_c)| = \left| \frac{2}{i\omega_c(i\omega_c + 1)} \right| = \frac{2}{\omega_c \sqrt{\omega_c^2 + 1}} \equiv 1$$

$$\Leftrightarrow \quad \omega_c^4 + \omega_c^2 - 4 = 0 \quad \Leftrightarrow \quad \omega_c = \sqrt{\frac{-1 + \sqrt{17}}{2}} = 1.25$$

Then we calculate the the phase margin φ_m

$$\varphi_m = \pi + \arg G_0(i\omega_c) = \pi - \frac{\pi}{2} - \arctan \omega_c = 0.675$$

We thus obtain $L_m = \varphi_m/\omega_c = 0.54$.

7.9 **a.** The one second delay e^{-s} is considered part of the process.

Controller	$G_R(s) = K$
Process	$G_P(s) = \frac{1}{s(s+1)} e^{-s}$
Model	$\hat{G}_P(s) = G_P(s) = \frac{1}{s(s+1)} e^{-s}$
Model without delay	$\hat{G}_P^0(s) = \frac{1}{s(s+1)}$

b. According to the block diagram the control signal is given by

$$U(s) = G_R(s)\Big(E(s) + \hat{G}_P(s)U(s) - \hat{G}_P^0(s)U(s)\Big)$$

The transfer function of the controller becomes

$$\begin{aligned} U(s) &= \frac{G_R(s)}{1 - G_R(s)\hat{G}_P(s) + G_R(s)\hat{G}_P^0(s)} E(s) \\ &= \frac{2}{1 - \frac{2}{s(s+1)}e^{-s} + \frac{2}{s(s+1)}} E(s) = \frac{2s(s+1)}{s(s+1) + 2 - 2e^{-s}} E(s) \end{aligned}$$

The Bode plot of the controller is shown in Figure S7.3. One notes that the Smith predictor gives a large phase lead at the cross-over frequency of the initial system.

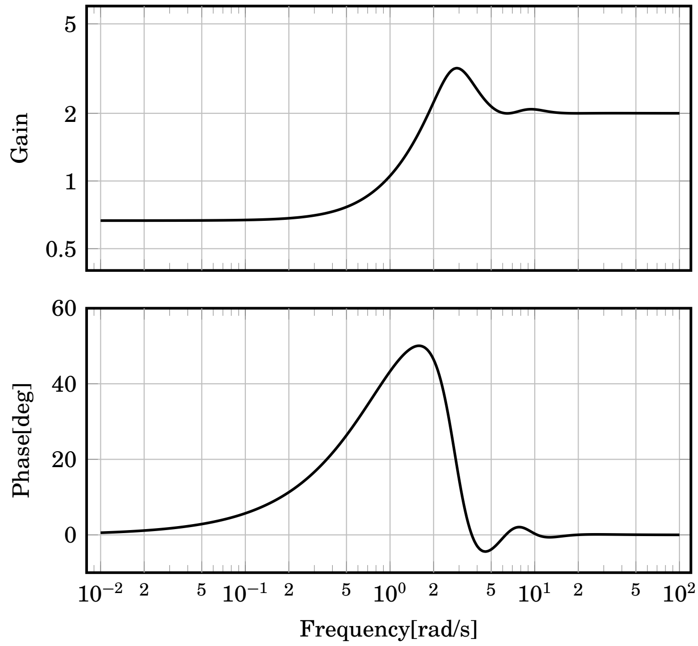


Figure S7.3 Bode plot of the Smith predictor.

c.

$$\begin{aligned} U(s) &= \frac{2s(s+1)}{s(s+1) + 2 - 2e^{-s}} E(s) \approx \frac{2s(s+1)}{s(s+1) + 2 - 2(1-s)} E(s) \\ &= \frac{2(s+1)}{s+3} E(s) \end{aligned}$$

This is a phase lead link with $N = 3$.

7.10 The gain curve of the system is given by

$$|G(i\omega)| = \frac{k}{\omega}$$

It is sufficient to read the value of the gain curve at a single frequency in order to determine k. The gain is e.g. 1 at approximately $\omega = 4.5$. This yields

$$1 = \frac{k}{4.5} \quad \Leftrightarrow \quad k = 4.5$$

The phase curve of the system is given by

$$\arg G(i\omega) = -\pi/2 - \omega L$$

Now, it is sufficient to read the value of the phase curve at a single frequency in order to determine L. The phase is e.g. $-\pi$ at approximately $\omega = 120$. This yields

$$-\pi = -\pi/2 - 120L \quad \Leftrightarrow \quad L = 0.013$$

Solutions to Exercises 8

Design Examples

8.1 **a.** The phase curve for $v = 3$ knots cuts $-180°$ at $\omega_o \approx 0.03$ rad/s. At this frequency we have $|G(i0.03)| \approx 2$. The gain K must hence be smaller than 0.5 in order to yield a stable closed loop system.

b. In order to acquire the cross-over frequency ω_c and phase margin φ_m it is required that

$$\begin{aligned} |G_r(i\omega_c)G(i\omega_c)| &= 1 \\ \arg G_r(i\omega_c)G(i\omega_c) &= \varphi_m - 180° \end{aligned}$$

where $G_r(s) = K(1 + T_D s)$. This leads to the equations

$$\begin{aligned} K|G(i\omega_c)|\sqrt{1 + T_D^2\omega_c^2} &= 1 \\ \arg G(i\omega_c) + \arctan T_D\omega_c &= \varphi_m - 180° \end{aligned}$$

With $\omega_c = 0.03$ rad/s, $\varphi_m = 60°$, $|G(i\omega_c)| \approx 2$ and $\arg G(i\omega_c) \approx -180°$ we obtain

$$\begin{aligned} T_d &= \frac{\tan 60°}{0.03} = \frac{\sqrt{3}}{0.03} \approx 57.7 \\ K &= \frac{1}{|G(i\omega_c)|\sqrt{1 + T_d^2\omega_c^2}} \approx \frac{1}{2 \cdot 2} = 0.25 \end{aligned}$$

c. If the speed suddenly increases from 3 to 7 knots, we have to turn to the gray Bode plots in Figure 8.2. The most drastic change is that the gain curve has been raised by a factor of 20. Additionally, the phase curve has decreased for frequencies above 0.03 rad/s. This results in heavily reduced phase- and gain margins. A more thorough examination shows that this in fact leads to instability of the closed loop system. This can be seen in the Bode plot in Figure 8.2, which shows both the nominal case $v = 3$ knots and the case $v = 7$ knots.

One way to avoid this problem is to instead choose $v = 7$ knots as the nominal case for the calculation of the PD controller. This, however, means that one has to accept a slower settling time for the slowest speed $v = 3$ knots. A better way is to let K and T_d depend on the speed v. This method is known as *gain scheduling*.

d. The transfer function from β to h can be approximated by

$$G_{h\beta}(s) = \frac{k_v v}{s^3}$$

From the Bode plot one can see that $|G_{h\beta}(i \cdot 0.1)| \approx 0.04$ for $v = 3\,\text{knots} = 3 \cdot 1.852/3.6 \approx 0.5144 \cdot 3\,\text{m/s}$, which yields

$$k_v \approx \frac{0.1^3 \cdot 0.04}{3 \cdot 0.5144} \approx 2.6 \cdot 10^{-5}$$

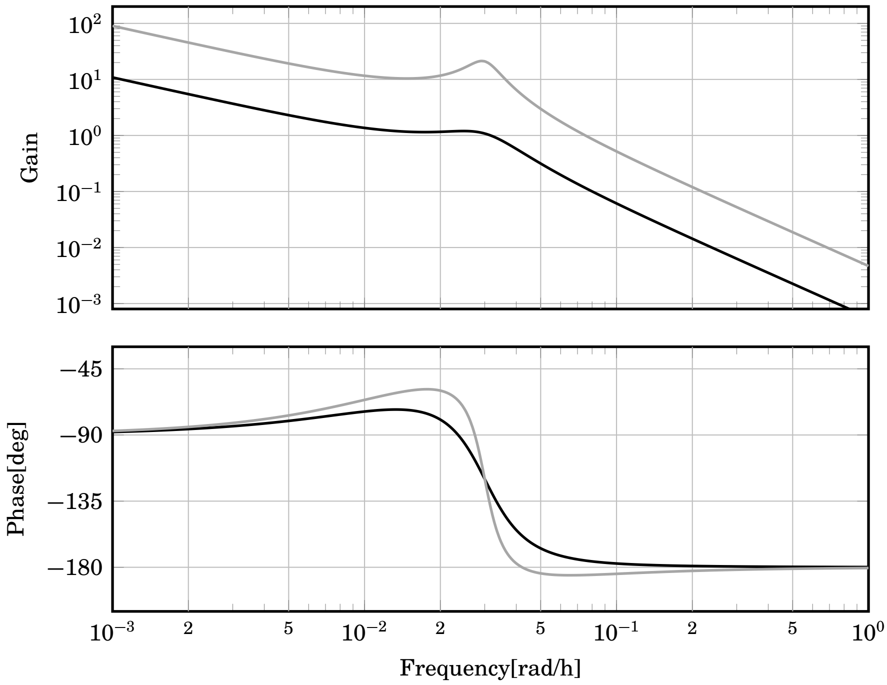


Figure S8.1 Bode plot of the PD compensated open loop system in assignment 8.1. The black curves show the case $v = 3$ knots (nominal case), while the gray curves show the case $v = 7$ knots. Note that the latter case yields an unstable closed loop system.

e. The characteristic equation of the closed-loop is given by

$$s^3 + Kk_v v = 0$$

Since not all coefficients are positive, the closed-loop system is not asymptotically stable for any value of K. In sub-assignment a it was concluded through the measured frequency response that the closed loop system was stable for $K < 0.5$. The explanation to this apparent contradiction is found in the Bode plot which was used in sub-assignment a: The approximation only holds for high frequencies ($\omega > 0.05$). For low gains, such as $K < 0.5$, the cross-over frequency $\omega_c < 0.03$ lies outside the valid range of the model.

For e.g. $\omega < 0.03$ the Bode plot shows a phase above $-180°$ while the simplified model features the phase $-270°$ for *all* frequencies.

f. If $x = (\dot{\alpha},\ \alpha,\ h)^T$ and $u = \beta$ we obtain the state space equations

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & v & 0 \end{pmatrix} x + \begin{pmatrix} k_v \\ 0 \\ 0 \end{pmatrix} u$$

With the state feedback $u = -Kx + u_r$ the characteristic polynomial of the closed loop system becomes

$$p(s) = \det(sI - A + BK) = s^3 + k_v k_1 s^2 + k_v k_2 s + k_v v k_3$$

the desired characteristic polynomial is

$$p(s) = (s + \gamma\omega_0)(s^2 + 2\zeta\omega_0 s + \omega_0^2) = s^3 + (\gamma + 2\zeta)\omega_0 s^2 + (2\gamma\zeta + 1)\omega_0^2 s + \gamma\omega_0^3$$

Direct comparison gives

$$\begin{cases} k_1 = \dfrac{(\gamma + 2\zeta)\omega_0}{k_v} \\ k_2 = \dfrac{(2\gamma\zeta + 1)\omega_0^2}{k_v} \\ k_3 = \dfrac{\gamma\omega_0^3}{k_v v} \end{cases}$$

g. Here stationarity means constant height, $h = h_{\text{ref}}$. This in turns mean that all derivatives of h must be zero, i.e. $\alpha = 0$ and $\dot{\alpha} = 0$. When the height has reached its correct value the control signal $u = \beta$ must also be zero since the submarine would otherwise continue to rise. Thus K_r is obtained from the equation

$$0 = K_r h_{\text{ref}} - k_1 \cdot 0 - k_2 \cdot 0 - k_3 h_{\text{ref}}$$

For $v = 3$ knots, we end up with the following result

$$K_r = k_3 = \frac{\gamma\omega_0^3}{k_v v} \approx \frac{\gamma\omega_0^3 \cdot}{2.6 \cdot 10^{-5} \cdot 3 \cdot 0.5144} \approx \frac{\gamma\omega_0^3 \cdot}{4.0 \cdot 10^{-5}}$$

h. At a momentary disturbance $\Delta h = 0.1$ m the rudder angle becomes

$$\Delta\beta = k_3 \cdot \Delta h = \frac{\gamma\omega_0^3}{v k_v} \cdot 0.1 \approx \frac{0.2\omega_0^3}{3 \cdot 0.5144 \cdot 2.6 \cdot 10^{-5}}$$

Since $\Delta\beta \leq 5°$, we must have

$$\omega_0 \leq \left(\frac{5 \cdot 3 \cdot 0.5144 \cdot 2.6 \cdot 10^{-5}}{0.2}\right)^{\frac{1}{3}} \approx 0.1$$

8.2 **a.** The oscillation frequency $\omega_o \approx 27$ rad/s and critical gain $K_c \approx 3.6$ can be read from the Bode plot. The oscillation period is hence $T_o = 2\pi/\omega_o \approx 0.23$. This yields the PID parameters $K = 0.6K_c \approx 2.2$, $T_i = T_o/2 \approx 0.12$ and $T_d = T_o/8 \approx 0.03$. The step response of the closed loop system is shown in Figure S8.2. The specifications

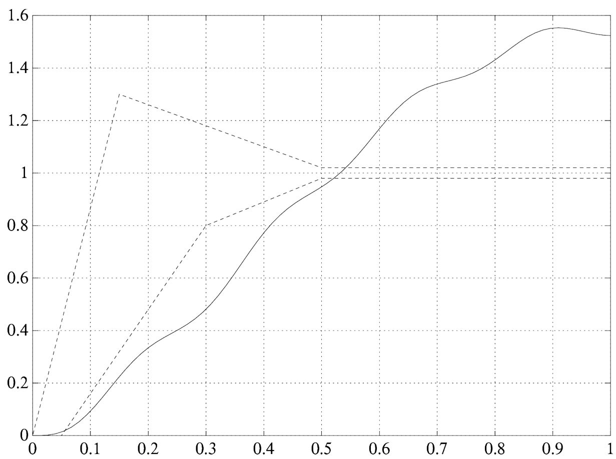


Figure S8.2 The step response with PID control according to Ziegler-Nichols.

are apparently not fulfilled. A PID controller (with filter factor) can be considered a second order controller with integral action. As a matter of fact, the specifications can be met, using a more general second order controller with integral action (see Figure S8.3). If one tries to interpret it as a PID controller, one would end up with a negative derivative time T_d.

b. In stationarity all derivatives of the states must be zero $\dot{x} = 0$. It hence holds that

$$\begin{cases} 0 = Ax^o + Bu^o = (A - BK)x^o + BK_r y_r \\ y_r = y^o = Cx^o \end{cases}$$

This yields

$$K_r = -\frac{1}{C(A - BK)^{-1}B}$$

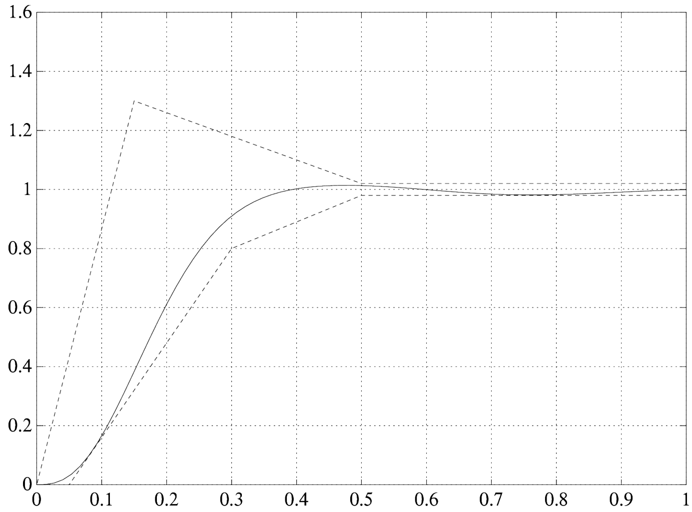


Figure S8.3 The step response of the closed loop system with a second order integrating controller.

c. With x augmented to $x_e = (x_1, x_2, x_3, x_i)^T$ we obtain

$$\dot{x}_e = \begin{pmatrix} -\dfrac{d_1+d_f}{J_1} & \dfrac{d_f}{J_1} & -\dfrac{k_f}{J_1} & 0 \\ \dfrac{d_f}{J_2} & -\dfrac{d_f+d_2}{J_2} & \dfrac{k_f}{J_2} & 0 \\ 1 & -1 & 0 & 0 \\ 0 & k_{\omega_2} & 0 & 0 \end{pmatrix} x_e + \begin{pmatrix} \dfrac{k_m k_i}{J_1} \\ 0 \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} y_r$$

$$y = \begin{pmatrix} 0 & k_{\omega_2} & 0 & 0 \end{pmatrix} x_e$$

where the reference y_r has been introduces as an extra input.

d. The approximate value of ω_m becomes

$$\omega_m \approx -\frac{\ln 0.02}{0.5 \cdot 0.38} \approx 20$$

e. When it comes to load disturbances, the fast Kalman filter ($\omega_o = 40$) has the best performance. It is also best when it comes to suppressing the influence of of measurement noise. However, it is the worst choice when it comes to suppressing the influence of noise in the control signal. The two cases $\omega_o = 10$ and $\omega_o = 20$ feature approximately the same noise sensitivity, while $\omega_o = 10$ is slower when it comes to eliminating load disturbances. A satiable choice is thus $\omega_o = 20$.

Solutions to Exercises 9

Interactive Comparison Between Model Descriptions

9.1

a. The amplitude of the step response is affected, but not the time constant. The pole is not affected. The Nyquist diagram keeps its shape, but each point on the curve moves radially from the origin. In the Bode diagram, the gain curve changes in the y direction while the phase curve is unchanged.

Since the control signal is a unit step, K is given by the stationary value of the measurement signal in the step response. In the Nyquist diagram, K is given from the starting point on the positive real axis. In the Bode diagram, K is given from the gain curve where $\omega \to 0$.

b. The amplitude of the step response is not affected, but the time constant is given by T. The pole is in $s = -1/T$, which means that a large time constant gives a pole close to the origin, whereas a short time constant gives a pole far away from the origin. In the Bode diagram, the corner frequencty is $1/T$, and it thereby varies when T varies. No change is visible in the Nyquist diagram, but the frequency varies along the curve.

Suppose that we have two processes with different values of T. Then, we can always find two frequencies such that

$$G(i\omega_1 T_1) = \frac{K}{1 + i\omega_1 T_1} = G(i\omega_2 T_2) = \frac{K}{1 + i\omega_2 T_2}$$

i.e., all points that are on the first Nyquist curve are also on the second one, but at another frequency.

c. A variation in L corresponds to a translational movement of the step response. We can not represent a dead time in a singularity diagram. The gain curve in the Bode diagram is not affected, since $|e^{-i\omega L}| = 1$, but the phase is reduced. For each frequency point in the Nyquist diagram, the distance to the origin remains unchanged, but the phase decreases. Since the phase goes towards $-\infty$ when $\omega \to \infty$, the Nyquist curve has the spiral shape.

9.2

a. The changes are the same as in problem 1a.

b. The changes are analogous to the ones in problem 1b. When $T_1 \gg T_2$, the step response is similar to the one in problem 1a, with $T \approx T_1$. The Bode and Nyquist diagrams are also similar to the ones in problem 1a for low frequencies. Thus, you can in many cases approximate the transfer function by

$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)} \approx \frac{K}{(1 + sT_1)}$$

c. If the zero is far away from the origin, the representations are not significantly affected. If the zero is negative and is close to the origin, there is a large overshoot in the step response. If the zero is positive and close to the origin, the step response will initially go in the wrong direction. If the zero is positive it will give a positive contribution to the phase.

When $T_3 < 0$, e.g., when the zero is in the right half plane, the process is hard to control. You can imagine that it is hard for a controller to act in the right way when a control signal change makes the measurement signal go in the wrong direction initially. The phenomenon could be understood by writing the transfer function in the following way

$$G(s) = \frac{K(1+sT_3)}{(1+sT_1)(1+sT_2)} = \frac{K}{(1+sT_1)(1+sT_2)} + \frac{sKT_3}{(1+sT_1)(1+sT_2)}$$

Thus, the transfer function consists of two terms, one that is the transfer function that we had in problem 2a, and one that is the same transfer function, but multiplied with sT_3. Thus, the second term is proportional to the derivative of the measurement signal we would have obtained if we did not have any zero. If $T_3 < 0$, this term will give a negative contribution, which explains that the step response initially goes in the wrong direction.

9.3

a. The frequency ω affects the speed of the system, but not the shape of the step response. Variations in ω moves the poles radially from the origin. In the Bode diagram, ω does not affect the shape, but only the location of the corner frequency. The shape of the Nyquist curve is not affected, but the frequencies are moved along the curve.

b. The relative damping ζ does not affect the speed of the step response, but the shape. A small value of ζ gives an oscillatory and poorly damped response. A small value of ζ gives a large resonance peak in the Bode diagram. In the Nyquist diagram, you get a big increase of gain and fast phase shift around ω.