Fast calibration of the Libor Market Model with Jacobi stochastic volatility model

11^{th} European Summer School in Financial Mathematics

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August 28^{th}, 2018

École des Ponts ParisTech

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$\rightsquigarrow$ Intensive calibrations of the models.

The DD-SV-LMM

Insurer's portfolio: mainly composed of *bonds*, but also derivatives on interest-rates

$\rightsquigarrow$ **Models dedicated to interest rates are decisive**, some may be complex to handle. An issue is the pricing of swaptions (call option on swap rate)

$$PS(0,K) = B^S(0)\mathbb{E}^S\big[(S_T^{(m,n)} - K)_+\big]$$

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- **Displaced Diffusion with Stochastic Volatility Libor Market Model** (*Normal* version): under the probability measure $\mathbb{P}^S$

$$\mathrm{d}S_t^{(m,n)} = \mathrm{d}(S_t^{(m,n)} + \delta) = \sqrt{V_t}\sum_{j=m}^{n-1}\gamma_j(t)\cdot\mathrm{d}\mathbf{W}_t^S$$

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- The stochastic factor is modelled through a 'quasi-**CIR**':

$$\mathrm{d}V_t = \kappa(\theta - \xi(t)V_t)\mathrm{d}t + \epsilon\sqrt{V_t}\mathrm{d}Z_t^S$$

Overview: calibration procedures

- Heston type computations can be adapted to calibrate the model. Essentially, **the moment generating function is analytically known** (affine model).

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- Basic issue: this procedure is very long $\Rightarrow$ need of an innovative method for calibration.
- [ABBD2017] proposes to perform an **approximation of the density** of $S_T^{(m,n)}$ $\Rightarrow$ computation of approximated prices

$$\begin{aligned}\pi := \mathbb{E}^S[(S_T^{(m,n)} - K)_+] &= \int_{\mathbb{R}} (s-K)_+ f_T(s)\mathrm{d}s \\ &\approx \int_{\mathbb{R}} (s-K)_+ f_T^{(N)}(s)\mathrm{d}s =: \pi^{(N)}\end{aligned}$$

Gram-Charlier expansion

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Theorem (Cramèr [C1926])

If f is of finite variation in every finite interval and is such that

$$\int_{\mathbb{R}} |f(s)| e^{s^2/4} \mathrm{d}s < \infty$$

then $f^{(N)}(x) = g(x) \sum_{n=0}^{N} c_n H_n(x) \xrightarrow[N \to +\infty]{} f(x)$ *at every continuity point* x *of* f.

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- H_n polynomial function (n-th degree), explicitly known (**Hermite polynomials**)
- The coefficients c_n are **linear combination of moments** of the unknown density f: need a way to compute the needed moments

Gram-Charlier and stochastic volatility ? (1/2)

$$X := \sqrt{V} \times G$$

with $G \sim \mathcal{N}(0, \sigma^2)$ and $V \sim \chi^2(d)$, G and V being **independent**.

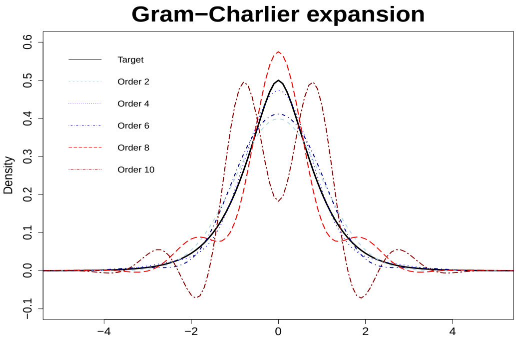

Figure 1: Gram-Charlier expansion of the density of X up to order 10 - $\sigma^2 = 0.25$ & $d = 4$

Gram-Charlier and stochastic volatility ? (2/2)

$$X^{(M)} := \sqrt{\min(V, M)} \times G$$

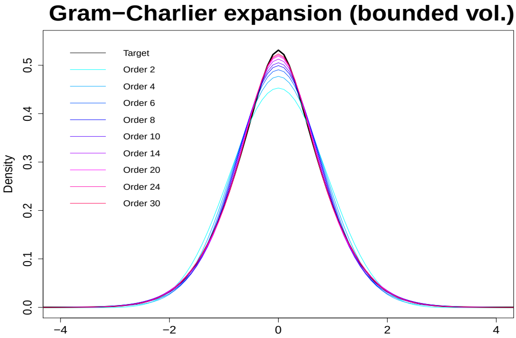

Figure 2: Gram-Charlier expansion of the density of $X^{(M)}$ up to order 30 - $M = 4$

Condition: $\sigma^2 M < 2$

Introduction of the Jacobi model

Based on a work of D. Ackerer, D. Filipović and S. Pulido ([AFP2018]), we introduce the Jacobi dynamic:

$$\begin{cases} \mathrm{d}S_t^{(m,n)} & = \rho\lambda(t)\sqrt{Q(V_t)}\mathrm{d}Z_t^S + \sqrt{V_t - \rho^2 Q(V_t)}\sum_{j=m}^{n-1}\gamma_j(t)\cdot \mathrm{d}\mathbf{Z}_t^{S,\perp} \\ \mathrm{d}V_t & = \kappa\left(\theta - \xi(t)V_t\right)\mathrm{d}t + \epsilon\sqrt{Q(V_t)}\mathrm{d}Z_t^S \end{cases}$$

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- **Polynomial model**: we can compute the moments of $S_T^{(m,n)}$

References

[ABBD2017] P.-E. Arrouy, and P. Bonnefoy, A. Boumezoued, L. Devineau, *Fast calibration of the Libor Market Model with Stochastic Volatility and Displaced Diffusion* (2017).

[AFP2018] D. Ackerer, D. Filipović and S. Pulido, *The Jacobi stochastic volatility model* (2018).

[C1926] H. Cramèr, *On some classes of series used in mathematical statistics* (1926).

Thank You !